ON HYPERBOLIC DIMENSION GAP FOR ENTIRE FUNCTIONS

Abstract. Polynomials and entire functions whose hyperbolic dimension is strictly smaller than the Hausdorff dimension of their Julia set are known to exist but in all these examples the latter dimension is maximal, i.e. equal to two. In this paper we show that there exist hyperbolic entire functions f having Hausdorff dimension of the Julia set $\mathrm{HD}(\mathcal{J}_f) < 2$ and hyperbolic dimension $\mathrm{HypDim}(f) < \mathrm{HD}(\mathcal{J}_f)$.

1. Introduction

In this paper we consider some relations between the Hausdorff dimension $\mathrm{HD}(\mathcal{J}_f)$ and the hyperbolic dimension $\mathrm{HypDim}(f)$ of the Julia set $\mathcal{J}_f$ of an entire function $f : \mathbb{C} \to \mathbb{C}$, where $\mathbb{C}$, as usually, denotes the complex plane. More precisely, we show the following.

Theorem 1.1. *There exist hyperbolic entire functions f in the Eremenko–Lyubich class $\mathcal{B}$ such that*

$$\mathrm{HypDim}(f) < \mathrm{HD}(\mathcal{J}_f) < 2.$$

The concept $\mathrm{HypDim}(f)$ of hyperbolic dimension has been introduced by Shishikura in [16]. Given an entire function $f : \mathbb{C} \to \mathbb{C}$ it is defined to be the supremum of Hausdorff dimensions of all hyperbolic sets of f. We recall that a set $X \subset \mathbb{C}$ is hyperbolic if it is compact, forward–invariant under f and if there exist $c > 0$ and $\kappa > 1$ such that

$$|(f^n)'(z)| \geqslant c\kappa^n \quad \text{for every } z \in X \text{ and all } n \geqslant 1.$$

It is immediate from this definition that $X \subset \mathcal{J}_f$. For hyperbolic polynomials the whole Julia set is a hyperbolic set, whence there is no difference between the hyperbolic dimension and the Hausdorff dimension of the Julia set. In general, since hyperbolic sets of f are subsets of the Julia set of f, we have that

$$\mathrm{HypDim}(f) \leqslant \mathrm{HD}(\mathcal{J}_f). \tag{1.1}$$

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Examples of entire functions with strict inequality are known ([18], [20]). Quite recently Avila-Lyubich [1, 2] showed that there exist Feigenbaum polynomials having this property. But in all known examples with strict inequality in (1.1) the Hausdorff dimension of the Julia set is maximal, i.e. equal to two and Avila-Lyubich mention that for arbitrary polynomials f with $\mathrm{HD}(\mathcal{J}_f) < 2$ one should have equality. Here we show that this is not the case for entire functions even inside the Eremenko-Lyubich class $\mathcal{B}$ consisting of all entire functions having a bounded set of finite singularities.

In order to prove Theorem 1.1 we first need good candidates of entire functions whose Julia sets have Hausdorff dimension less than two. The first such examples where found by Gwyneth Stallard during 1990's. The interested reader can find an overview in her survey in [15]. These examples are entire functions having one single logarithmic tract over infinity (see Section 2.1 for the definition of the singularities of entire functions) and, as nowadays it is well known, the geometry of such a tract or the growth of the function in the tract does influence the size of the Julia set. Particularly interesting for the present work is her family of intermediate growth in [17]. The growth does depend on a parameter $p > 0$ and these functions are defined by the formula

$$E(z) := \frac{1}{2i\pi} \int_L \frac{\exp\left(e^{(\log \xi)^{1+p}}\right)}{\xi - z} d\xi \,, \tag{1.2}$$

where L is the boundary of the region

$$G = \left\{ x + iy \in \mathbb{C} : |y| < \frac{\pi x}{(1+p)(\log x)^p} \;,\; x > 3 \right\}, \tag{1.3}$$

oriented in the clockwise direction, for $z \in \mathbb{C} \backslash \overline{G}$ and by analytic continuation for $z \in \overline{G}$; details of the analytic extension are given in Section 2.2. The reader should have in mind that this function is close to

$$f(z) = \exp\left(e^{(\log z)^{1+p}}\right) \quad \text{for } z \in G \tag{1.4}$$

and is bounded elsewhere. Here $(\log z)^{1+p}$ is defined so that it gives real values for real $z > e$.

Consider then the family $\left(\mathbf{E}_l : \mathbb{C} \to \mathbb{C}\right)_{l \in \mathbb{C}}$ defined by the formula

$$\mathbf{E}_l(z) := E(z - l).$$

Shifting in this way the function E by a large $l > 0$ makes the logarithmic tract backward invariant and $J_{\mathbf{E}_l} \subset G$ so that only the dynamics of $\mathbf{E}_l$ in G, the domain on which $\mathbf{E}_l$ is close to the function $\mathbf{f}_l$ given by the formula

$$\mathbf{f}_l(z) := f(z - l),$$

is relevant for our purposes, details of this and the definition of the Julia set in the present setting are given in Section 2.1.

Fact 1.2 (Stallard [19]). *Let $p > 0$. All the functions $\mathbf{E}_l$, $l \in \mathbb{C}$, belong to the Eremenko–Lyubich class $\mathcal{B}$ and there exists a constant $C_p > 0$ such that for all real $l > C_p$ we have that*

$$\mathrm{HD}(J_{\mathbf{E}_l}) = 1 + \frac{1}{1+p} < 2\,.$$

In the present note we analyze the hyperbolic dimension of these functions. In fact, we first work with the functions $\mathbf{f}_l$ and then transfer the results to the globally defined entire functions $\mathbf{E}_l$.

The key point is to employ the thermodynamic formalism of [10] and, in particular, the Bowen's Formula from this paper that determines hyperbolic dimension. We will see that $\lim_{l\to\infty} \mathrm{HypDim}(\mathbf{E}_l) = 1$ which clearly implies that $\mathrm{HypDim}(\mathbf{E}_l) < \mathrm{HD}(\mathcal{J}_{\mathbf{E}_l})$ provided that $l > C_p$ is large enough.

1.1. **Notation.** We use standard notation such as $\mathbb{D}(z, r)$ for the open disk in $\mathbb{C}$ with center $z \in \mathbb{C}$ and radius $r > 0$. When the center is the origin, we also use the simplified notation

$$\mathbb{D}_r := \mathbb{D}(0, r).$$

The complement of its closure will be denoted by

$$\mathbb{D}_r^* := \mathbb{C} \backslash \overline{\mathbb{D}}_r.$$

Frequently we deal with half–spaces. Let

$$\mathcal{H}_s := \left\{ z \in \mathbb{C} : \ \Re z > s \right\} \quad , \quad s \geqslant 0\,.$$

When $s = 0$, then we also write $\mathcal{H}$ for $\mathcal{H}_0$.

Many constants, especially those in Fact 2.3, depend on the parameter p of the definitions of the functions E and f. However, this will be fixed throughout the whole paper and we may ignore it.

We say that

$$A \preceq B$$

for non–negative real expressions A and B if and only if there exists a positive constant C independent of variable parameters involved in A and B such that $A \leqslant CB$. We then say that $A \succeq B$ if and only if $B \preceq A$. Finally, $A \asymp B$ if and only if $A \preceq B$ and $B \preceq A$.

2. Singularities, models and approximating entire functions

2.1. **General definitions.** Iversen's classification of singularities is explained in length in [9], see also [4]. An entire function $g : \mathbb{C} \to \mathbb{C}$ can have only two types of singular values. Firstly, a point $b \in \hat{\mathbb{C}}$ is a *critical value* of g if and only if $b = g(c)$ for some $c \in \mathbb{C}$ with $g'(c) = 0$. Secondly, a complex number $b \in \hat{\mathbb{C}}$ is an *asymptotical value* of g if and only if there exists a continuous function $\gamma : [0, +\infty) \to \mathbb{C}$ such that

$$\lim_{t\to+\infty} \gamma(t) = \infty \text{ and } \lim_{t\to+\infty} f(\gamma(t)) = b.$$

In this latter case for every $r > 0$ there exists an unbounded connected component Ω_r of $g^{-1}(\mathbb{D}(b,r))$ such that

$$\Omega_{r'} \subset \Omega_r$$

whenever $r' < r$ and

$$\bigcap_{r>0} \Omega_r = \varnothing.$$

Such a choice of components is called an asymptotic tract over b and it is called *logarithmic tract* in the case when the map $g : \Omega_r \to \mathbb{D}(b,r)\backslash\{b\}$ is a universal covering for some $r > 0$. The set of singular values of f is proved to consist of all critical and asymptotic values of f. Its intersection with $\mathbb{C}$ will be denoted by $S(g)$.

We consider functions belonging to the Eremenko–Lyubich class $\mathcal{B}$ that consists of all entire functions g for which $S(g)$ is a bounded set. These functions are also called of *bounded type.* If $g \in \mathcal{B}$, then there exists $r > 0$ such that $S(g) \subset \mathbb{D}_r$. Then $g^{-1}(\mathbb{D}_r^*)$ consists of mutually disjoint unbounded Jordan domains Ω_r with real analytic boundaries such that $g : \Omega \to \mathbb{D}_r^*$ is a covering map (see [8]). Thus, an entire function g in class $\mathcal{B}$ has only logarithmic singularities over infinity. As we already mentioned it, the connected components of $g^{-1}(\mathbb{D}_r^*)$ are called *tracts* or, more precisely, *logarithmic tracts*. Then there exist all holomorphic branches of the logarithm of g restricted to Ω_r. Fix one of them and denote it by τ. So,

$$g|_{\Omega_r} = \exp\circ\tau, \tag{2.1}$$

where

$$\varphi = \tau^{-1} : \mathcal{H}_{\log r} \to \Omega_r$$

is a conformal homeomorphism. In addition, φ extends continuously to ∞ and $\varphi(\infty) = \infty$.

Keeping this notation, if we restrict g to the tracts over infinity then it is now standard, especially since the appearance of the papers [5, 6] by Chris Bishop, to call the map

$$g_{|g^{-1}(\mathbb{D}_r^*)} : g^{-1}(\mathbb{D}_r^*) \to \mathbb{D}_r^*$$

a model function. We will see that the functions considered in our current paper have only one single tract over infinity. This is the reason why we use the following simplified definition of a model function. This is in the spirit of the definition in [13], see [5, 6] for the general one.

Definition 2.1. *A model is any holomorphic map*

$$g = e^\tau : \Omega_r \to \mathbb{D}_r^*,$$

where

(1) $r \in [1, +\infty)$,
(2) Ω_r *is a simply connected unbounded domain in* $\mathbb{C}$*, called a tract, such that* $\partial\Omega_r$ *is a connected subset of* $\mathbb{C}$ *and*
(3) $\tau : \Omega_r \to \mathcal{H}_{\log r}$ *is a conformal homeomorphism fixing infinity; the latter more precisely meaning that*

$$\tau(z) \to \infty \text{ as } z \to \infty.$$

The tract Ω_r may or may not intersect the disk $\mathbb{D}_r$. The later case has important dynamical consequences.

Definition 2.2. *If* f *is a model or an entire function of bounded type and if there exists* $r > 0$ *such that*

$$S(f) \subset \mathbb{D}_r \quad \text{and} \quad \overline{f^{-1}(\mathbb{D}_r^*)} \subset \mathbb{D}_r^* \tag{2.2}$$

then f *is called of disjoint type.*

If f is such a disjoint type model or entire function then the *Julia set* of f is defined to be

$$\mathcal{J}_f := \left\{z \in \mathbb{D}_r^* : f^n(z) \in \mathbb{D}_r^* \text{ for all } n \geqslant 1\right\}.$$

For disjoint type entire functions this definition conicides with the usual one, see Proposition 2.2 in [14].

2.2. **Elementary properties of the functions E and f.** We now discuss elementary properties of the functions introduced in the introduction and we examine how they behave with respect to the above definitions.

To start with we recall that these functions have been introduced and studied by Stallard and her paper [17, Section 3] lists elementary properties of E and f. We recall now some necessary facts from this paper.

Let's denote by

$$G_{x_0,\kappa} := \left\{ z = x + iy \in \mathbb{C} : x > x_0 \text{ and } |y| < \kappa \frac{\pi x}{(1+p)(\log x)^p} \right\}$$

and abbreviate $G_{x_0} = G_{x_0,1}$ so that the set G of (1.3) is $G_3 = G_{3,1}$. Let $n \geqslant 3$ and σ_{n+1} be the boundary of $G \backslash \overline{G}_{n+1}$. The orientation of σ_{n+1} as well of all following boundary curves are always understood in the clockwise direction. Cauchy's Integral Formula shows that

$$\frac{1}{2i\pi} \int_{\sigma_n^+} \frac{f(\xi)}{\xi - z} d\xi = 0 \quad \text{for every } z \notin \overline{G}.$$

Therefore, still for $z \notin \overline{G}$,

$$E(z) = \frac{1}{2i\pi} \int_{\partial G} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2i\pi} \int_{\partial G_{n+1}} \frac{f(\xi)}{\xi - z} d\xi\,.$$

It follows that the right hand integral gives the holomorphic extension of E to the domain $\mathbb{C} \backslash \overline{G}_{n+1}$.

Consider now a point $z \in G \backslash \overline{G}_{n+1}$. Then Cauchy's Residue Theorem shows that

$$\begin{aligned} E(z) =& \frac{1}{2i\pi} \int_{\partial G_{n+1}} \frac{f(\xi)}{\xi - z} d\xi \\ =& -\frac{1}{2i\pi} \int_{\partial (G_n \backslash \overline{G}_{n+1})} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2i\pi} \int_{\partial G_n} \frac{f(\xi)}{\xi - z} d\xi \\ =& f(z) + \frac{1}{2i\pi} \int_{\partial G_n} \frac{f(\xi)}{\xi - z} d\xi \end{aligned}$$

Starting with this observation, one can get the following fact which is contained in Lemma 3.1 in [17] along with its proof.

Fact 2.3. *Let $\check{L}, \widehat{L}$ be the boundary of $G_{D+1,\frac{5}{6}}, G_{D-1,\frac{7}{6}}$ respectively. Then there exist constants $C, D > 3$ such that the following hold.*

(1) *If $z \notin G_D$ then*

$$|E(z)| \leqslant C$$

and
if $z \in G_D$ then

$$|E(z) - f(z)| \leqslant C \quad \textit{as well as} \quad |E'(z) - f'(z)| \leqslant C.$$

(2)

$$E(z) = \frac{1}{2\pi i} \int_{\check{L}} \frac{f(t)}{t - z} dt \qquad \textit{for } z \notin G_D$$

and

$$E(z) = f(z) + \frac{1}{2\pi i}\int_{\hat{L}} \frac{f(t)}{t-z}dt \qquad \textit{for } z \in G_D.$$

(3) *If* $z \in G_{D,\frac{7}{6}} \backslash \mathrm{Int}(G_{D,\frac{5}{6}})$ *then*

$$|f(z)| \leqslant \exp\Big(-\frac{1}{2}e^{\frac{1}{2}(\log \Re z)^{1+p}}\Big).$$

Item (1) from this Fact 2.3 shows that $f^{-1}(\mathbb{D}_r^*) \subset G_D$ for every $r > 2C$. Elementary estimates based on the explicit representation of f show that $f^{-1}(\mathbb{D}_r^*)$ is a simply connected unbounded domain in $\mathbb{C}$. It turns out that the same is true for the approximating entire function E, details can be found in Proposition 2.2 of [13]. Thus, we have the following.

Fact 2.4. *Let C be given by Fact 2.3. Then there exists $r_0 > 4C$ such that*

$$S(E) \subset \mathbb{D}_{r_0/2}$$

and for every $r \geqslant r_0/2$ both sets $E^{-1}(\mathbb{D}_r^*)$ and $f^{-1}(\mathbb{D}_r^*)$ are simply connected unbounded domains in $\mathbb{C}$ contained in G_D. They will be respectively denoted by

$$\Omega_{E,r} := E^{-1}(\mathbb{D}_r^*) \quad \textit{and} \quad \Omega_{f,r} := f^{-1}(\mathbb{D}_r^*)\,.$$

From now on fix any

$$r \geqslant r_0/2, \tag{2.3}$$

where r_0 comes from Fact 2.4. Then the map $f : \Omega_{f,r} \to \mathbb{D}_r^*$ is of the form $f(z) = e^{\tau(z)}$ with $\tau : \Omega_{f,r} \to \mathcal{H}_{\log r}$ given by

$$\tau(z) := \exp((\log z)^{1+p}).$$

We have to know what the inverse conformal homeomorphism $\varphi = \tau^{-1} : \mathcal{H}_{\log r} \to \Omega_{f,r}$ looks like. Indeed, a straightforward calculation gives

$$\varphi(\xi) = \exp\Big((\log \xi)^{\frac{1}{1+p}}\Big) \tag{2.4}$$

where log is the principal branch of logarithm again, i.e. determined by the requirement that $\log 1 = 0$.

In conclusion

$$f_{|\Omega_{f,r}} = e^{\tau} : \Omega_{f,r} \to \mathbb{D}_r^* \tag{2.5}$$

is a model as defined in Definition 2.1 and Fact 2.3 explains how the entire function E approximates this model.

Lemma 2.5. *There exists a constant* $K \geqslant 1$ *such that*

$$\frac{1}{K} \leqslant \frac{|\varphi'(\xi + iy)|}{|\varphi'(\xi)|} \leqslant K$$

for every ξ *with* $\Re(\xi) \geqslant \log r_0$ *and every* $0 \leqslant y \leqslant 2\pi$.

Proof. The statement follows from Koebe's Distortion Theorem since the conformal map $\varphi = \mathcal{H}_{\log r_0} \to \Omega_{f,r_0}$ is in fact defined on the half space $\mathcal{H}_{\log(r_0/2)}$. □

2.3. **Disjoint Type Versions of E_l and f_l.** Given any $l \in \mathbb{C}$, the functions $\mathbf{f}_l = f \circ T_l$ and $\mathbf{E}_l = E \circ T_l$, where T_l is the translation $z \mapsto z - l$, have been defined in the introduction. We have that $\mathbf{E}_l \in \mathcal{B}$ since it is known, see [19], that $E \in \mathcal{B}$.

Obviously,

$$\Omega_{\mathbf{f}_l,r} := \mathbf{f}_l^{-1}(\mathbb{D}_r^*) = f^{-1}(\mathbb{D}_r^*) + l = \Omega_{f,r} + l, \tag{2.6}$$

and also

$$\Omega_{\mathbf{E}_l,r} = \mathbf{E}_l^{-1}(\mathbb{D}_r^*) = E^{-1}(\mathbb{D}_r^*) + l. \tag{2.7}$$

By Fact 2.4, for all $r \geqslant r_0/2$ and $l \in [0, +\infty)$, all these tracts are contained in respective sets $G_D + l$. So, setting

$$l_r := \max\{0, r - D\}, \tag{2.8}$$

we have that

$$\Omega_{\mathbf{f}_l,r}\ ,\ \Omega_{\mathbf{E}_l,r} \subset \mathbb{D}_r^* \tag{2.9}$$

for all $r \geqslant r_0/2$ and all $l \geqslant l_r$. Consequently, all the functions $\mathbf{f}_l, \mathbf{E}_l$, $l \geqslant l_r$, are of disjoint type and for their Julia sets we have that

$$\mathcal{J}_{\mathbf{f}_l}, \mathcal{J}_{\mathbf{E}_l} \subset \mathbb{D}_r^* \tag{2.10}$$

for all $r \geqslant r_0/2$ and all $l \geqslant l_r$.

Recall that for the model f we have the expression (2.5). The analogous expression for $\mathbf{f}_l$ is

$$\mathbf{f}_{l|\Omega_{\mathbf{f}_l,r}} = e^{\tau_l} : \Omega_{\mathbf{f}_l,r} \to \mathbb{D}_r^* \tag{2.11}$$

where $\tau_l(z) = \tau(z - l)$ so that the inverse of τ_l is

$$\varphi_l = \varphi + l : \mathcal{H}_{\log r} \to \Omega_{\mathbf{f}_l,r} \tag{2.12}$$

where φ is still the conformal map defined by (2.4)

3. Thermodynamical formalism

Our ultimate goal is to determine the hyperbolic dimension of the functions $\mathbf{E}_l$ which can be done under certain conditions by employing the methods of thermodynamic formalism. The hyperbolic dimension is then given by the zero of the topological pressure, the fact that goes back to Bowen [7]. In the present context, namely for disjoint type models and entire functions of bounded type, such a theory has been developed in [10].

Let $g := \mathbf{f}_l$ or $g := \mathbf{E}_l$. Given $t > 0$, the transfer operator for the map g and for the parameter t is defined by the formula

$$\mathcal{L}_{g,t}h(w) := \sum_{g(z)=w} |g'(z)|_1^{-t} h(z) \quad \text{for every } w \in \mathbb{D}_r^* \tag{3.1}$$

where

$$|g'(z)|_1 := \frac{|g'(z)|}{|g(z)|}|z|$$

is the logarithmic derivative of g evaluated at the point z and where h is a function belonging to $\mathcal{C}_b(\mathbb{D}_r^*)$, that is, the Banach space of all complex–valued bounded continuous functions defined on $\mathbb{D}_r^*$ endowed with the supremum norm.

We are to find out for which parameters $t > 0$ the following two crucial properties hold:

$$\|\mathcal{L}_{g,t}\mathbb{1}\|_\infty < +\infty \quad \text{and} \quad \lim_{w\to\infty} \mathcal{L}_{g,t}\mathbb{1}(w) = 0. \tag{3.2}$$

Indeed, since our map g is of disjoint type, once (3.2) is verified then, following [10, Section 8], we deduce that the whole thermodynamic formalism, along with all its applications obtained in [10], holds. Especially Bowen's Formula does. This formula involves topological pressure which for the disjoint type map g is given at a parameter $t \in (0, +\infty)$ by the formula

$$\mathrm{P}(g,t) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_{g,t}^n \mathbb{1}(w) \tag{3.3}$$

where $w \in \mathbb{D}_r^*$ is any arbitrarily chosen point; the limit does exist.

3.1. Estimates for the Transfer operators of the Model Functions $\mathbf{f}_l$.

Proposition 3.1. *Let $\mathcal{L}_{\mathbf{f}_l,t}$ be the transfer operator of $\mathbf{f}_l$, $l \geqslant 0$, with a parameter $t > 0$. Fix $r \geqslant r_0$. Let $w_0 \in \mathbb{D}_r^*$. Then*

$$\mathcal{L}_{\mathbf{f}_l,t}\mathbb{1}(w_0) < \infty \quad \textit{if and only if} \quad t > 1\,.$$

Moreover, if $t > 1$ then (3.2) holds for $g = \mathbf{f}_l$.

Proof. Having $w_0 \in \mathbb{D}_r^*$ and $t > 0$, let us start exactly as in the proof of Theorem 4.1 in [10]. If $z_l \in \mathbf{f}_l^{-1}(w_0)$ then, using (2.11), the logarithmic derivative can be expressed as follows:

$$|\mathbf{f}_l'(z_l)|_1 = |\tau_l'(z_l)z_l| = \frac{|\varphi_l(\xi)|}{|\varphi_l'(\xi)|} = |(\log\varphi_l)'(\xi)|^{-1}$$

where $\xi = \tau_l(z_l)$ and where $\varphi_l = \varphi + l$ is the map of (2.12). Notice that $\xi = u + iv$ does not depend on l where $u = \log|w_0|$. From this together with Lemma 2.5, we get that

$$\mathcal{L}_{\mathbf{f}_l,t}1\!\!1(w_0) = \sum_{\exp(\xi)=w_0} |(\log\varphi_l)'(\xi)|^t \asymp \int_{\mathbb{R}} |(\log\varphi_l)'(\log|w_0| + iv)|^t dv\,.$$

Now, since $\varphi_l = \varphi + l$ and since we have the explicit expression (2.4) for φ, we can calculate as follows:

$$|(\log\varphi_l)'(\xi)| = \left|\frac{\varphi(\xi)}{\varphi(\xi)+l}\right| \frac{1}{1+p} \frac{1}{|\xi||\log\xi|^{\frac{p}{1+p}}} \asymp \left|\frac{\varphi(\xi)}{\varphi(\xi)+l}\right| \frac{1}{|\xi|(\log|\xi|)^{\frac{p}{1+p}}}.$$

since $\arg(\xi) \in (-\pi/2, \pi/2)$. Therefore,

$$\mathcal{L}_{\mathbf{f}_l,t}1\!\!1(w_0) \asymp \int_{\mathbb{R}} \left|\frac{\varphi(\xi)}{\varphi(\xi)+l}\right|^t \frac{1}{|\xi|^t(\log|\xi|)^{\frac{tp}{1+p}}} dv. \tag{3.4}$$

Since $\lim_{|v|\to+\infty} \varphi(\log|w_0| + iv) = \infty$, we have that

$$\frac{2}{3} \leqslant \left|\frac{\varphi(\xi)}{\varphi(\xi)+l}\right| \leqslant 2$$

whenever $|v| = |\Im(\xi)|$ is sufficiently large. Thus we get from (3.4) that $\mathcal{L}_{\mathbf{f}_l,t}1\!\!1(w_0)$ is finite if and only if $t > 1$.

The uniform bound of $\|\mathcal{L}_{\mathbf{f}_l,t}\|_\infty < \infty$ also follows from (3.4). Indeed, let $w = e^\xi \in \mathbb{D}_r^*$. Then $z = \varphi(\xi) \in G_D$, whence $x = \Re(z) > 0$. Thus,

$$\left|\frac{\varphi(\xi)}{\varphi(\xi)+l}\right|^2 = \frac{x^2+y^2}{(x+l)^2+y^2} \leqslant 1. \tag{3.5}$$

It follows from this that

$$\mathcal{L}_{\mathbf{f}_l,t}1\!\!1(w) \preceq \int_{\mathbb{R}} \frac{1}{|\xi|^t(\log|\xi|)^{\frac{tp}{1+p}}} dv = \frac{1}{2}\int_{\mathbb{R}} \frac{1}{(u^2+v^2)^{\frac{t}{2}}(\log(u^2+v^2))^{\frac{tp}{1+p}}} dv.$$

Since for every for $w \in \mathbb{D}_r^*$ we have $u \geqslant u_r = \log r$ it follows that

$$\sup_{w\in\mathbb{D}_r^*} \mathcal{L}_{\mathbf{f}_l,t}1\!\!1(w) \preceq C := \int_{\mathbb{R}} \frac{1}{(u_r^2+v^2)^{\frac{t}{2}}(\log(u_r^2+v^2))^{\frac{tp}{1+p}}} dv < +\infty. \tag{3.6}$$

Finally, if $t > 1$ then $\delta = (t-1)/2 > 0$, whence

$$\mathcal{L}_{\mathbf{f}_l,t}\mathbb{1}(w) \preceq \frac{1}{u^\delta}\int_{\mathbb{R}} \frac{1}{|u_r + iv|^{1+\delta}}dv \preceq \frac{1}{(\log|w|)^\delta}. \tag{3.7}$$

This shows that $\lim_{w\to\infty} \mathcal{L}_{\mathbf{f}_l,t}\mathbb{1}(w) = 0$. □

The next result gives an estimate for the topological pressure. More precisely, it shows that for a given $t > 1$ the pressure $\mathrm{P}(\mathbf{f}_l, t) < 0$ for all sufficiently large values of l.

Proposition 3.2. *Let $t > 1$. Fix $r \geqslant r_0$. Then, for every $\varepsilon > 0$ there exists $l_{\varepsilon,r,t} \geqslant l_r$ such that*

$$\mathcal{L}_{\mathbf{f}_l,t}\mathbb{1}(w) \leqslant \varepsilon \quad \textit{for every } l \geqslant l_{\varepsilon,r,t} \textit{ and every } w \in \mathbb{D}_r^* .$$

Proof. Let $t > 1$ and $\varepsilon > 0$. We are in the same situation as in the proof of Proposition 3.1. The first benefit we take out of this proof is that the convergence $\lim_{w\to\infty} \mathcal{L}_{\mathbf{f}_l,t}\mathbb{1}(w) = 0$ is uniform in $l \geqslant 0$; see (3.7). Therefore, there exists $r_\varepsilon \geqslant r$ such that

$$\mathcal{L}_{\mathbf{f}_l,t}\mathbb{1}(w) < \varepsilon \quad \text{whenever } |w| \geqslant r_\varepsilon \text{ and } l \geqslant 0.$$

Moreover, this proof shows that the integral

$$\int_{\mathbb{R}} \frac{1}{|\xi|^t (\log|\xi|)^{\frac{tp}{1+p}}}dv \qquad \xi = u + iv\,,$$

converges uniformly for $u \geqslant u_r = \log r$. Therefore, there exists $V = V_{\varepsilon,t}$ such that

$$\int_{|v|\geqslant V} \frac{1}{|\xi|^t (\log|\xi|)^{tp/1+p}}dv \leqslant \frac{\varepsilon}{2} \quad \text{for every } u \geqslant u_r.$$

So, by invoking now (3.4) and (3.5), we conclude that it remains to estimate the integral

$$\int_{|v|<V} \left|\frac{\varphi(\xi)}{\varphi(\xi)+l}\right|^t \frac{1}{|\xi|^t (\log|\xi|)^{\frac{tp}{1+p}}}dv$$

from above by $\varepsilon/2$ for all $l \geqslant 0$ large enough and all $w \in \mathbb{D}_r^* \backslash \mathbb{D}_{r_\varepsilon}^*$. Here we used again the notation $\xi = u + iv$, $u = \log|w|$. Notice that all points ξ that appear in this integral belong to the compact set

$$K = \{\xi = u + iv : \log r \leqslant u \leqslant \log r_\varepsilon \text{ and } |v| \leqslant V\}.$$

Since $M := \sup_{\xi\in K}\{|\varphi(\xi)|\} < +\infty$, we have that

$$\left|\frac{\varphi(\xi)}{\varphi(\xi)+l}\right| \leqslant \frac{M}{l-M} \quad ,$$

for every $l > M$ and all $\xi \in K$. Thus,

$$\int_{|v|<V} \left|\frac{\varphi(\xi)}{\varphi(\xi)+l}\right|^t \frac{1}{|\xi|^t(\log|\xi|)^{\frac{tp}{1+p}}} dv \leqslant C\frac{M}{l-M} \leqslant \frac{\varepsilon}{2},$$

where $C \in (0,+\infty)$ is the constant coming from (3.6) and the last inequality was written assuming that l is large enough. □

3.2. **Behavior of the Transfer Operators for Entire functions $\mathbf{E}_l$.** We now have sufficiently strong estimates for the transfer operators of the models f_l. Since ultimately we are after the entire functions $\mathbf{E}_l$, we have to carry over these estimates to the transfer operators of these functions $\mathbf{E}_l$. Since the entire functions approximate the models, i.e. since we have Fact 2.3, we are in a similar situation as in [11] where also the operators of some models and approximating entire functions have been compared. Following the approach of that paper we will prove the following.

Proposition 3.3. *There exist constants $\mathcal{K} \in [1,+\infty)$ and $r_1 \geqslant r_0$ such that for every $t > 1$, all $l \in \mathbb{C}$, and all $r \geqslant r_1$, we have that*

$$\frac{1}{\mathcal{K}^t} \leqslant \frac{\mathcal{L}_{\mathbf{E}_l,t}\mathbb{1}(w)}{\mathcal{L}_{\mathbf{f}_l,t}\mathbb{1}(w)} \leqslant \mathcal{K}^t \quad \textit{for all} \quad w \in \mathbb{D}_r^*.$$

In our proof of Proposition 3.3 we adapt here the approach of [11], particularly Section 7 of that paper. We will show that [11, Lemma 7.3] holds in the present setting if $r \geqslant r_0$ is large enough. This will suffice. We first shall prove the following.

Fact 3.4. *For all sufficiently large $r \geqslant r_0$, say $r \geqslant r_1 \geqslant r_0$, we have that*

$$\frac{1}{2} \leqslant \frac{|\mathbf{E}_l(z)|}{|\mathbf{f}_l(z)|} \leqslant 2 \quad \textit{and} \quad \frac{1}{2} \leqslant \frac{|\mathbf{E}_l'(z)|}{|\mathbf{f}_l'(z)|} \leqslant 2$$

for all $l \in \mathbb{C}$ and all $z \in \Omega_{\mathbf{f}_l,r}$.

Proof. The first inequality is a direct consequence of item (1) in Fact 2.3 combined with the inequality $r \geqslant r_0 > 4C$ established in Fact 2.4.

In order to proof the second inequality we also start with item (1) in Fact 2.3. It gives

$$\left|\frac{|E'(z)|}{|f'(z)|} - 1\right| \leqslant \frac{C}{|f'(z)|} \quad \text{for all } z \in \Omega_{f,r} \subset G_D.$$

This time we have to estimate $|f'(z)|$ and to show that there exists some $r \geqslant r_0$ such that

$$\frac{C}{|f'(z)|} \leqslant \frac{1}{2} \quad \text{for all } z \in \Omega_{f,r}. \tag{3.8}$$

Remember that $f(z) = e^{\tau(z)} = e^{\varphi^{-1}(z)}$ for every $z \in \Omega_{f,r}$. Thus,

$$f'(z) = \frac{f(z)}{\varphi'(\xi)} \quad \text{where } \xi = \varphi^{-1}(z) \in \mathcal{H}_{\log r}.$$

Obviously $|f(z)| > r$ but what about $|\varphi'(\xi)|$? From the formula (2.4) we get

$$\varphi'(\xi) = \frac{1}{1+p} \exp((\log \xi)^{\frac{1}{1+p}}) \frac{1}{(\log \xi)^{\frac{p}{1+p}} \xi}.$$

If $v := \log \xi$ then

$$|\varphi'(\xi)| \leqslant \left| \frac{\exp(v^{\frac{1}{1+p}})}{v^{\frac{p}{1+p}} e^v} \right| = \frac{\exp\left(\Re\left(v^{\frac{1}{1+p}} - v\right)\right)}{|v|^{\frac{p}{1+p}}}. \tag{3.9}$$

Since $\xi \in \mathcal{H}_{\log r}$, $\Re v > \log\log r$, and $|\Im v| < \pi/2$, so if we write $v = se^{i\alpha}$, then

$$s > \log\log r \quad \text{and} \quad |\alpha| < \frac{\pi/2}{\log\log r}.$$

Thus,

$$\Re\left(v^{\frac{1}{1+p}} - v\right) = -s\left(\cos\alpha - s^{-\frac{p}{1+p}} \cos\left(\frac{\alpha}{1+p}\right)\right) \leqslant -\frac{s}{2} \leqslant -\frac{\log\log r}{2}$$

provided r is sufficiently large. In this case we get from (3.9) that

$$|\varphi'(\xi)| \leqslant \frac{1}{\sqrt{\log r}(\log\log r)^{\frac{p}{1+p}}}.$$

This shows that (3.8) holds for all $r \geqslant r_0$ sufficiently large. Thus, 3.4 holds for f and E, i.e. if $l = 0$. It then holds for all $l \in \mathbb{C}$ because of (2.6).

□

Having established Fact 3.4, the proof of Proposition 7.4 in [11] applies word by word and shows that the required inequality in Proposition 3.3 holds.

4. Proof of Theorem 1.1

As it was explained in the Introduction, it suffices to show that

$$\lim_{l\to\infty} \mathrm{HypDim}(\mathbf{E}_l) = 1. \tag{4.1}$$

In order to do this fix $t > 1$. Fix also any $r \geqslant r_1$, for example $r = r_1$. By virtue of Proposition 3.2 we have that

$$\mathcal{L}_{f_l,t}\mathbb{1}(w) \leqslant \mathcal{K}^{-t}$$

for all $l \geqslant l_{\mathcal{K}^{-t},r,t}$ and all $w \in \mathbb{D}_r^*$. So, by Proposition 3.3,

$$\mathcal{L}_{E_l,t}\mathbb{1}(w) \leqslant 1$$

for all $l \geqslant l_{\mathcal{K}^{-t},r,t}$ and all $w \in \mathbb{D}_r^*$. In conjunction with (3.3) this gives that

$$\mathrm{P}(E_l, t) \leqslant 0$$

for all $l \geqslant l_{\mathcal{K}^{-t},r,t}$. So, if $X \subset \mathcal{J}_{E_l}$ is an arbitrary hyperbolic set for E_l, then

$$\mathrm{P}(E_l|_X, t) \leqslant 0.$$

The supremum over all hyperbolic sets of the left hand side of this inequality is the hyperbolic pressure $P_{hyp}(E_l, t)$ of E_l evaluated at t, so we have that

$$\mathrm{P}_{hyp}(E_l, t) = \sup\{\mathrm{P}(E_l|_X, t) : X \text{ is a hyperbolic set for } E_l\} \leqslant 0. \tag{4.2}$$

Now, we want to use the Bowen's Formula of [3]. Theorem B of this paper applies to the functions E_l and states that the hyperbolic dimension of the set E_l is equal to

$$\mathrm{HypDim}(\mathbf{E}_l) = \inf\{s > 0 : \mathrm{P}_{hyp}(E_l, s) \leqslant 0\}.$$

Combined with (4.2), we thus get that

$$\mathrm{HypDim}(\mathbf{E}_l) \leqslant t$$

for all $l \geqslant l_{\mathcal{K}^{-t},r,t}$. So, the formula (4.1) is established and the proof of Theorem 1.1 is complete.

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