EQUILATERAL TRIANGULATIONS AND THE POSTCRITICAL DYNAMICS OF MEROMORPHIC FUNCTIONS

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Abstract. We show that any dynamics on any planar set S discrete in some domain D can be realized by the postcritical dynamics of a function holomorphic in D, up to a small perturbation. A key step in the proof, and a result of independent interest, is that any planar domain D can be equilaterally triangulated with triangles whose diameters $\to 0$ (at any prescribed rate) near ∂D.

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1. Introduction

We begin by briefly introducing some conventions. Unless otherwise specified, we will always consider the spherical metric when measuring the distance between any two points in $\widehat{\mathbb{C}}$. If $D \subset \widehat{\mathbb{C}}$ is a domain, we will say a set $S \subset D$ is *discrete* in D if S has no accumulation points in D. We define the *singular values* of a holomorphic function $f : D \to \widehat{\mathbb{C}}$ to be the set $S(f)$ of critical values and *asympotic values* of f. A point $w \in \widehat{\mathbb{C}}$ is an asymptotic value of $f : D \to \widehat{\mathbb{C}}$ if there exists a curve

$$\gamma : [0, \infty) \to D \text{ with } \gamma(t) \xrightarrow{t\to\infty} \partial D \text{ and } f \circ \gamma(t) \xrightarrow{t\to\infty} w.$$

The *postsingular set* of f is defined by

$$P(f) := \{f^n(w) : w \in S(f) \text{ and } n \geq 0\}.$$

In the study of the dynamics of a holomorphic function $f : D \to \widehat{\mathbb{C}}$, a fundamental role is played by the sets $S(f)$, $P(f)$, and the behavior of f restricted to $P(f)$. For instance, the boundary of any Siegel disc of f is contained in $\overline{P(f)}$, and much more generally, any component in the Fatou set of f always necessitates a certain behavior for the orbit of a singular value of f (see for instance [Ber95], [Mil06]). Thus, the following question arises: which dynamics on which sets $S \subset D$ can be realized by the postsingular dynamics of a holomorphic function $f : D \to \widehat{\mathbb{C}}$? Our first result (Theorem A below) says that as long as $S \subset D$ is discrete, *any* dynamics on S can be realized, up to a small perturbation. Before stating this result more precisely, we need:

Definition 1.1. Let $\varepsilon > 0$ and X, $Y \subset \widehat{\mathbb{C}}$. We say a homeomorphism $\phi : X \to Y$ is an *ε-homeomorphism* if $\sup_{z\in X} d(\phi(z), z) < \varepsilon$. If a conjugacy ϕ between two dynamical systems is an ε-homeomorphism, we say ϕ is an *ε-conjugacy.*

Theorem A. *Let $D \subseteq \widehat{\mathbb{C}}$ be a domain, $S \subset D$ a discrete set with $|S| \geq 3$, $h : S \to S$ a map, and $\varepsilon > 0$. Then there exists an ε-homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and a holomorphic map $f : \phi(D) \to \widehat{\mathbb{C}}$ with no asymptotic values such that $P(f) \subset \phi(D)$ and $f : P(f) \to P(f)$ is ε-conjugate to $h : S \to S$.*

When $D = \widehat{\mathbb{C}}$, Theorem A is exactly Theorem 1.1 of [DKM20]. When $D = \mathbb{C}$, Theorem A is very similar to Theorem 1 of [BL19] (the difference being that functions in [BL19] have asymptotic values and there the conjugacy $P(f) \mapsto S$ may be taken tangent to the identity at ∞). The main technique in [DKM20] is iteration in Teichmüller space, whereas in [BL19] it is quasiconformal folding. The present manuscript provides a new approach that works simultaneously in both the settings $D = \widehat{\mathbb{C}}$, $\mathbb{C}$, as well as in the much more general setting. We remark that our techniques do not answer whether for particular S and $h : S \to S$ one can take $P(f) = S$ and $f|_{P(f)} = h$ (see Question 1.2 of [DKM20]). Related questions were also studied in [Bar01], [NS21]. We also remark that since the function f of Theorem A has no asymptotic values, the postsingular set $P(f)$ coincides with the postcritical set of f.

The proof of Theorem A proceeds by quasiconformally deforming a certain *Belyi function* on D: a holomorphic map $g : D \to \widehat{\mathbb{C}}$ branching only over the three values $\pm 1, \infty$. Given the existence of g, the main tools in the proof of Theorem A are the Measurable Riemann Mapping Theorem and an improvement of a fixpoint technique first introduced in [BL19] (see also [MPS20], [Laz21]). The existence of $g : D \to \widehat{\mathbb{C}}$, on the other hand, will follow from the existence of a particular *equilateral triangulation* of the domain D: a topological triangulation of D with the property that for any two adjacent triangles T, T', there is an anti-conformal map $T \mapsto T'$ which fixes pointwise the common edge. Indeed, this means that by the Schwarz reflection principle, any triangle T and any vertex-preserving conformal map $T \mapsto \mathbb{H}(-1, 1, \infty)$ defines a Belyi function. The connection between equilateral triangulations and Belyi functions was first described in [VS89]. The existence of the desired equilateral triangulation of D will follow from:

Theorem B. Let $D \subset \widehat{\mathbb{C}}$ be a domain. Suppose $\eta : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing, and $\eta(0) = 0$. Then there exists an equilateral triangulation $\mathcal{T}$ of D so that for every $z \in D$ and every triangle $T \in \mathcal{T}$ containing z we have

$$\text{diameter}(T) \leq \eta(\text{dist}(z, \partial D)). \tag{1.1}$$

Moreover, the degree of any vertex v is bounded, independently of v, D and η.

The existence of an equilateral triangulation of D is already implied by the recent result of [BR21]: that any non-compact Riemann surface can be equilaterally triangulated. In order to prove Theorem A, however, we will need to prove that the triangulation can also be taken to satisfy the condition (1.1).

Theorem B is a key step in the proof of Theorem A, but it is also of independent interest. As already partially alluded to, by [VS89] a Riemann surface X has an equilateral triangulation if and only if it has a Belyi function $g : X \to \widehat{\mathbb{C}}$, in which case $g^{-1}([-1, 1])$ is a so-called *dessin d'enfant* on X. There is an extensive literature on dessins d'enfants (see [LZ04] for an overview), and of recent interest is the question of which geometries on a given Riemann surface a dessin may achieve. For instance, [Bis14] shows that unicellular dessins d'enfants are dense in all planar continua. Condition (1.1) is equivalent to a certain geometry for the corresponding dessin, and it is likely the techniques used in proving (1.1) will be of use in the question of attainable geometries for a dessin d'enfant on a given Riemann surface.

We now briefly outline the paper. In Section 2 we will sketch the proofs of Theorems A, B. In Sections 3-7, we prove Theorem A by first assuming Theorem B, and in Sections 8-10 we prove Theorem B. Sections 8-10 may be read independently of Sections 3-7. We will give a more detailed outline of the paper after sketching the main proofs in Section 2.

2. Sketch of the Proofs

In this Section, we sketch the proofs of Theorems A, B. We begin with Theorem A, where the main ideas are already present in the case $D = \widehat{\mathbb{C}}$, and we discuss this case first.

Consider a sequence of equilateral triangulations $\mathcal{T}_n$ of $\widehat{\mathbb{C}}$ satisfying

$$\sup_{T \in \mathcal{T}_n} \text{diameter}(T) \xrightarrow{n \to \infty} 0. \tag{2.1}$$

The existence of $\mathcal{T}_n$ is trivial: see for instance Figure 1. As described above, any triangle $T \in \mathcal{T}_n$ and any vertex-preserving conformal map $T \mapsto \mathbb{H}(-1, 1, \infty)$ defines a holomorphic map $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

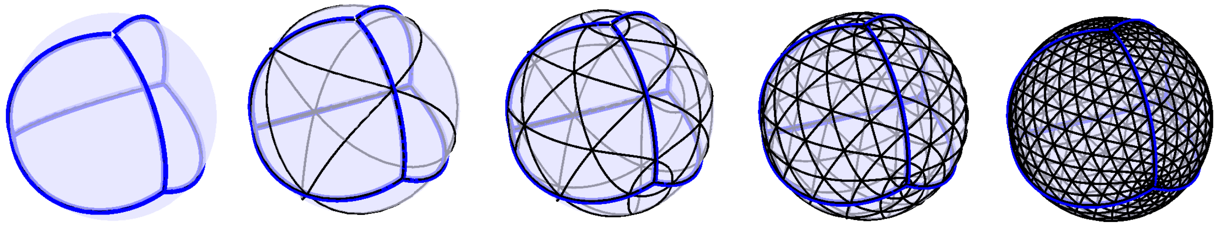

FIGURE 1. Illustrated is a sequence of triangulations $\mathcal{T}_n$ of $\widehat{\mathbb{C}}$. $\mathcal{T}_0$ is the tetrahedral subdivision of $\widehat{\mathbb{C}}$, and $\mathcal{T}_n$ is obtained from $\mathcal{T}_{n-1}$ by connecting the centers of each edge in each triangle in $\mathcal{T}_{n-1}$.

The critical points of g are precisely the vertices in the triangulation $\mathcal{T}_n$, and the critical values of g are $\pm 1, \infty$. For any vertex $v \in \mathcal{T}_n$, let $\mathcal{T}_{\{v\}}$ denote the union of triangles in $\mathcal{T}_n$ which have v as a vertex. We can change the definition of $g|_{\mathcal{T}_{\{v\}}}$ to a map $\tilde{g}|_{\mathcal{T}_{\{v\}}}$ by post-composing $g|_{\mathcal{T}_{\{v\}}}$ with a quasiconformal map of $\widehat{\mathbb{C}}$ which perturbs the critical value $g(v) \in \{\pm 1, \infty\}$ to a parameter $\tilde{g}(v) \in \widehat{\mathbb{C}}$, in such a way that $\tilde{g}|_{\partial \mathcal{T}_{\{v\}}} = g|_{\partial \mathcal{T}_{\{v\}}}$. Doing so over a sparse subset of vertices in $\mathcal{T}_n$, we call this new quasiregular map $\tilde{g} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

Given a discrete (finite) $S \subset \widehat{\mathbb{C}}$ and a map $h : S \to S$, we choose a vertex $v_s \in \mathcal{T}_n$ nearby each $s \in S$, and consider the family of mappings $\tilde{g}$ determined by a choice of $(\tilde{g}(v_s))_{s \in S}$. Each such choice $(\tilde{g}(v_s))_{s \in S}$ determines a holomorphic map $f := \tilde{g} \circ \phi^{-1}$, where ϕ is a quasiconformal mapping obtained from the Measurable Riemann Mapping theorem. In order to obtain the conjugacy between $f : P(f) \to P(f)$ and $h : S \to S$, the main idea (see also Figure 3) is to justify that we can choose $(\tilde{g}(v_s))_{s \in S}$ so that

$$\tilde{g}(v_s) = \phi(v_{h(s)}), \text{ for all } s \in S. \tag{2.2}$$

Indeed, suppose we have the relation (2.2), and assume for simplicity that h is onto. Then we would have

$$P(f) = \tilde{g}\left((v_s)_{s \in S}\right) = \phi\left((v_{h(s)})_{s \in S}\right) = \phi\left((v_s)_{s \in S}\right), \tag{2.3}$$

and the desired conjugacy between $f : P(f) \to P(f)$ and $h : S \to S$ would be defined by $\phi(v_s) \mapsto s$, since:

$$f\left(\phi(v_s)\right) = \tilde{g} \circ \phi^{-1} \circ \phi(v_s) = \tilde{g}(v_s) = \phi(v_{h(s)}). \tag{2.4}$$

That we can choose each $\tilde{g}(v_s)$ so that (2.2) holds is non-trivial. The dilatation of $\tilde{g}$, and hence the mapping ϕ, depends on the parameter $\tilde{g}(v_s)$ in a non-explicit manner (by solution of the Beltrami equation). Nevertheless, we can show the desired choice of $\tilde{g}(v_s)$ exists by application of a fixpoint theorem, where the variable is the set of parameters $\tilde{g}(v_s)$ and the output is the set of points $\phi(v_{h(s)})$. Moreover, if n is large, the triangulation $\mathcal{T}_n$ is fine by (2.1) and the dilatation of ϕ small, so that $\phi(v_s) \approx v_s \approx s$, and hence the conjugacy is close to the identity. Much of the technical work in Sections 3-7 is in setting up the parameters n, $\tilde{g}(v_s)$ so that the hypotheses of an appropriate fixpoint theorem hold.

The crucial property of the domain $D = \widehat{\mathbb{C}}$ that was used in the above sketch was the existence of the equilateral triangulations $\mathcal{T}_n$ of D. While this property is trivial in the cases $D = \widehat{\mathbb{C}}$, $D = \mathbb{C}$ and it is well known in many other cases, it is non-trivial in the general setting. This is the content of Theorem B. The main idea of the proof of Theorem B is as follows. Assume $\infty \in D$, and let $K := \partial D$. We consider sets Γ_k which are contours surrounding K (see Figure 10). The desired triangulation $\mathcal{T}$ is produced by an inductive procedure. Roughly speaking, at the k^{th} step we define the triangulation $\mathcal{T}_k$ to equal the previous triangulation $\mathcal{T}_{k-1}$ outside Γ_k and equal a Euclidean equilateral triangulation inside Γ_k. However, these two triangulations need to be merged in a very thin neighborhood of Γ_k (with a non-equilateral triangulation) and a quasiconformal correction is then applied to make the merged triangulation equilateral. The dilatation of the correction map is supported in a thin neighborhood of Γ_k, and is chosen so thin that so the correction map is close to the identity. The desired triangulation $\mathcal{T}$ is then the limit of the triangulations $\mathcal{T}_k$ as $k \to \infty$.

We now give a detailed outline of the rest of the paper. In Section 3 we describe how we will change the map $g|_{\mathcal{T}_{\{v\}}}$ to the map $\tilde{g}|_{\mathcal{T}_{\{v\}}}$, introducing the parameters $\tilde{g}(v_s)$. In Section 4, we deduce from Theorem B the only result (Theorem 4.4) about equilateral triangulations we will need in order to prove Theorem A. In Section 5, we introduce the family of mappings amongst which we will find our desired fixpoint, and prove some estimates about this family. In Sections 6 and 7, we conclude the proof of Theorem A (modulo the proof of Theorem B) by applying a fixpoint theorem. In Section 8 we introduce the regions in which we will merge equilateral triangulations, and we triangulate them in Section 9. In Section 10 we construct the contours Γ_k surrounding K and prove Theorem B.

3. Moving a Critical Value

In this short Section we set up the framework we will need in order to be able to perturb the critical values of the function g described in the Introduction. First we recall the definition of the spherical metric (see Section I.1.1 of [LV73]):

Definition 3.1. Two finite points $z_1, z_2 \in \mathbb{C}$ have spherical distance

$$d(z_1, z_2) := \arctan\left|\frac{z_1 - z_2}{1 + \overline{z_1}z_2}\right| \text{ where } 0 \leq d(z_1, z_2) \leq \pi/2, \tag{3.1}$$

and $d(z_1, \infty) = \arctan|1/z_1|$.

We will use the basic theory of quasiconformal mappings throughout this paper, for which we refer the reader to the standard references [Ahl06] and [LV73].

Notation 3.2. If ϕ is a quasiconformal mapping, we will denote its Beltrami coefficient $\phi_{\overline{z}}/\phi_z$ by $\mu(\phi)$.

Definition 3.3. For $w \in \{\pm 1, \infty\}$, let I_w be the subarc of $\hat{\mathbb{R}}$ with endpoints in $\{\pm 1, \infty\} \setminus \{w\}$ which does not pass through w (so for instance, $I_{-1} = (1, \infty)$). Given $w \in \{\pm 1, \infty\}$ and $\zeta \in \widehat{\mathbb{C}}$ satisfying $d(\zeta, I_w) \geq \pi/12$, we will define a quasiconformal map $\phi_w^\zeta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as follows. Let

(1) $\phi_w^\zeta : B(w, \pi/24) \to B(\zeta, \pi/24)$ be the restriction to $B(w, \pi/24)$ of an isometry of $\hat{\mathbb{C}}$ mapping w to ζ,
(2) $\phi_w^\zeta(z) = z$ for $z \in I_w$,
(3) $\phi_w^\zeta(z)$ is a smooth interpolation between (1) and (2) on $\hat{\mathbb{C}} \setminus (I_w \cup B(w, \pi/24))$, and
(4) $\mu(\phi_w^\zeta)$ varies smoothly with respect to ζ.

Remark 3.4. The constant $\pi/12$ in Definition 3.3 is chosen because $\pi/6 = 2\pi/12$, and

$$\bigcup_{w \in \{\pm 1, \infty\}} \left\{ \zeta \in \widehat{\mathbb{C}} : d(\zeta, I_w) \geq \pi/6 \right\} = \widehat{\mathbb{C}}. \tag{3.2}$$

This fact will be important in the proof of Theorem A.

Proposition 3.5. *There exists* $0 < k_0 < 1$ *such that for any* $\zeta \in \widehat{\mathbb{C}}$*, there is* $w \in \{\pm 1, \infty\}$ *such that* $||\mu(\phi_w^\zeta)||_{L^\infty(\widehat{\mathbb{C}})} < k_0$.

Proof. Fix $w \in \{\pm 1, \infty\}$ and consider ζ satisfying $d(\zeta, I_w) \geq \pi/12$. We have that ϕ_w^ζ is a quasiconformal mapping, and moreover $\mu(\phi_w^\zeta)$ varies continuously with respect to ζ by (4) of Definition 3.3. Thus, as $||\mu(\phi_w^\zeta)||_{L^\infty(\widehat{\mathbb{C}})} < 1$ for each ζ satisfying $d(\zeta, I_w) \geq \pi/12$, we have that

$$\sup_{\zeta \in \{\zeta : d(\zeta, I_w) \geq \pi/12\}} ||\mu(\phi_w^\zeta)||_{L^\infty(\widehat{\mathbb{C}})} < 1.$$

The result now follows from (3.2).

□

4. Equilateral Triangulations

In this Section, we will deduce from Theorem B the only result (Theorem 4.4) we will need about equilateral triangulations in order to prove Theorem A. First we fix our definitions and some notation:

Definition 4.1. Let $D \subset \widehat{\mathbb{C}}$ be a domain. A *triangulation* of D is a countable and locally finite collection of closed topological triangles in D that cover D, such that two triangles intersect only in a full edge or at a vertex.

Definition 4.2. Let $D \subset \widehat{\mathbb{C}}$ be a domain, and $\mathcal{T}$ a triangulation of D. We say $\mathcal{T}$ is an *equilateral triangulation* if for any two triangles T, T' in $\mathcal{T}$ which share an edge e, there is an anti-conformal map of T onto T' which fixes pointwise the edge e and sends the vertex opposite e in T to the vertex opposite e in T'.

Notation 4.3. Given a subset $\mathcal{V}$ of vertices in a triangulation $\mathcal{T}$, we will denote by $\mathcal{T}_{\mathcal{V}}$ the union of those triangles in $\mathcal{T}$ with at least one vertex in $\mathcal{V}$. Unless otherwise specified, in what follows area will always refer to spherical area.

Theorem 4.4. *Let $D \subset \widehat{\mathbb{C}}$ be a domain and S a discrete set in D. Then there exists a sequence of equilateral triangulations $\{\mathcal{T}_n\}_{n=1}^{\infty}$ of D and a collection of pairwise non-adjacent triangles $\{T_s^n\}_{s\in S} \subset \mathcal{T}_n$ for each n satisfying:*

(1) *$s \in T_s^n$ for all $s \in S$ and $n \in \mathbb{N}$,*

(2) *For any choice of vertices $v_s^n \in T_s^n$ we have:*

$$\sum_{s\in S} \text{area}\left(\mathcal{T}_{\{v_s^n\}}\right) \xrightarrow{n\to\infty} 0. \tag{4.1}$$

(3) *Let $\varepsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that if $n \geq N$ and $s \in S$, then*

$$T_s^n \subset B(s, \varepsilon). \tag{4.2}$$

Proof of Theorem 4.4 assuming Theorem B. Label the elements of S as $\{s_k\}_{k=1}^{\infty}$ so that

$$\text{dist}(s_1, \partial D) \geq \text{dist}(s_2, \partial D) \geq \text{dist}(s_3, \partial D) \geq ... \tag{4.3}$$

Theorem 4.4 will quickly follow if we can prove Theorem 4.4 under the extra assumption that each $\geq$ in (4.3) is a $>$, and so we may assume without loss of generality that

$$\text{dist}(s_1, \partial D) > \text{dist}(s_2, \partial D) > \text{dist}(s_3, \partial D) > ....$$

We will build a sequence of continuous, strictly increasing functions $(\eta_n)_{n=1}^{\infty} : [0, \infty) \to [0, \infty)$ satisfying $\eta_n(0) = 0$ to which we will apply Theorem B. We start with η_1. Let $c_k := \text{dist}(s_k, \partial D)$, where we note that $c_k \to 0$. Define η_1 in a small neighborhood of each c_k so that

$$\eta_1(c_k) < \frac{\text{dist}(s_k, S \setminus \{s_k\})}{2} \text{ and } \eta_1(c_k + \eta_1(c_k)) < \frac{1}{2^k}. \tag{4.4}$$

Finish the definition of η_1 by setting $\eta_1(0) = 0$ and interpolating on the rest of $[0, \infty)$. We let

$$\eta_n := \eta_1/n. \tag{4.5}$$

Theorem B applied to $(\eta_n)_{n=1}^{\infty}$ yields a sequence of equilateral triangulations $\{\mathcal{T}_n\}_{n=1}^{\infty}$ of D. We define the collection $\{T_s^n\}_{s\in S} \subset \mathcal{T}_n$ by setting T_s^n to be any triangle in $\mathcal{T}_n$ containing s. By (1.1), (4.4) and (4.5), we have that if $s, s' \in S$ with $s \neq s'$, then T_s^n, $T_{s'}^n$ are non-adjacent for any n. Let v_s^n be any choice of vertex in T_s^n for each $s \in S$ and $n \in \mathbb{N}$. Since $v_s^n \in T_s^n$, we have by Theorem B that

$$\text{dist}(v_s^n, \partial D) < \text{dist}(v_s^n, s) + \text{dist}(s, \partial D) \leq \eta_n(c_k) + c_k.$$

Thus, again by Theorem B, we have that if T is a triangle with the vertex v_s^n, then

$$\text{diameter}(T) \leq \eta_n(c_k + \eta_n(c_k)).$$

Recalling that the maximal degree of a vertex in any of the triangulations $\mathcal{T}_n$ is bounded by a universal constant (call it d) by Theorem B, it follows from (4.4) and (4.5) that:

$$\sum_{s\in S} \text{area}\left(\mathcal{T}_{\{v_s^n\}}\right) \lesssim d \cdot \sum_{k\in\mathbb{N}} [\eta_n(c_k+\eta_n(c_k))]^2 \leq \frac{d}{n^2} \cdot \sum_{k\in\mathbb{N}} [\eta_1(c_k+\eta_1(c_k))]^2 \xrightarrow{n\to\infty} 0. \tag{4.6}$$

Thus Property (2) in the conclusion of the Theorem is proven, and Property (3) follows from Theorem B and the observation that

$$\sup_{k\in\mathbb{N}} \eta_n(c_k) \xrightarrow{n\to\infty} 0.$$

Property (1) holds by definition of T_s^n.

□

5. A Base Family of Mappings

Having proven Theorem 4.4, we now have the holomorphic function $g : D \to \widehat{\mathbb{C}}$ described in the Introduction (see Definition 5.3 below). In this Section, we introduce a family of quasiregular perturbations of g by moving critical values of g using the results of Section 3. The application we have in mind is roughly to prove Theorem A by finding a fixpoint in this family, and so we will need to establish certain technical estimates about this family which roughly correspond to verifying the hypotheses of an appropriate fixpoint theorem.

Remark 5.1. Throughout Section 5 we will fix a domain $D \subset \widehat{\mathbb{C}}$, a discrete set $S \subset D$, and equilateral triangulations $\mathcal{T}_n$ of D as given in Theorem 4.4.

Remark 5.2. A triangulation is called 3-colourable if its vertices may be coloured with three distinct colours in such a way that adjacent vertices have different colours. Any triangulation can be subdivided into a 3-colourable triangulation by barycentric subdivision (see Figure 2). Since barycentric subdivision preserves the properties of Theorem 4.4, we may assume that the triangulations $\mathcal{T}_n$ are 3-colourable. This allows us to define the following (see also Remark 2.8 of [BR21]):

Definition 5.3. We will define a sequence of holomorphic maps $g_n : D \to \widehat{\mathbb{C}}$ as follows. For any n, fix a triangle $T \in \mathcal{T}_n$, and let $g_n : T \to \mathbb{H}(-1, 1, \infty)$ be a conformal map such that the vertices of T map to $\pm 1, \infty$. The definition of g_n on D is then obtained by application of the Schwarz reflection principle.

Proposition 5.4. *The critical points of g_n are precisely the vertices of the triangles in $\mathcal{T}_n$. The only critical values of g_n are $\pm 1, \infty$.*

Proof. The maps g_n are locally univalent except at the vertices of triangles in $\mathcal{T}_n$. At a vertex v in $\mathcal{T}_n$, the map g_n is locally $m : 1$ where m is such that $2m$ edges of the triangulation $\mathcal{T}_n$ meet at v. The last statement follows since each vertex is sent to one of ± 1, ∞ by g_n. □

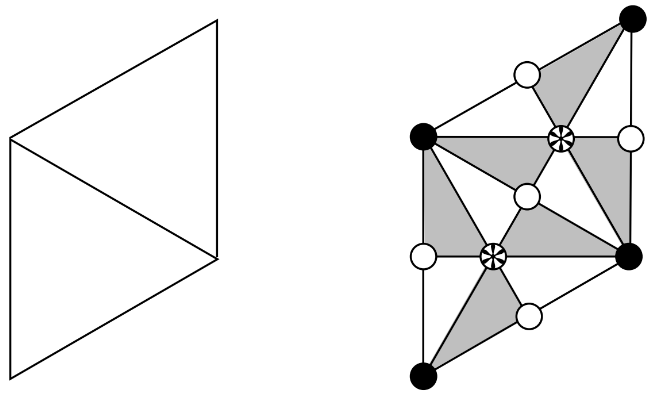

FIGURE 2. Illustrated is the process of barycentric subdivision. This figure is borrowed from [BR21].

Proposition 5.5. *Let $n > 0$, let $\mathcal{V}$ be a subset of pairwise non-adjacent vertices in $\mathcal{T}_n$, and suppose we have a mapping $\tilde{h} : \mathcal{V} \to \widehat{\mathbb{C}}$. If $d(\tilde{h}(v), I_{g_n(v)}) \geq \pi/12$ for each $v \in \mathcal{V}$, then there exists a quasiregular mapping $\tilde{g}_n : D \to \widehat{\mathbb{C}}$ such that:*

1. $\tilde{g}_n(v) = \tilde{h}(v)$ *for all* $v \in \mathcal{V}$,
2. $\tilde{g}_n \equiv g_n$ *on* $\mathcal{T}_n \setminus \mathcal{T}_{\mathcal{V}}$
3. $\mu(\tilde{g}_n)$ *is supported on* $\mathcal{T}_{\mathcal{V}}$, *and*
4. $||\mu(\tilde{g}_n)||_{L_\infty(D)} < k_0$.

Proof. We will abbreviate $g = g_n$, and assume as in the statement of the Proposition that $d(\tilde{h}(v), I_{g(v)}) \geq \pi/12$ for each $v \in \mathcal{V}$. Thus, the quasiconformal map $\phi_{g(v)}^{\tilde{h}(v)}$ of Definition 3.3 satisfies:

$$\phi_{g(v)}^{\tilde{h}(v)}(g(v)) = \tilde{h}(v) \text{ (by (1) of Definition 3.3)}, \tag{5.1}$$

and

$$\left|\left|\mu\left(\phi_{g(v)}^{\tilde{h}(v)}\right)\right|\right|_{L^\infty(\widehat{\mathbb{C}})} < k_0 \text{ (by Proposition 3.5)} \tag{5.2}$$

for all $v \in \mathcal{V}$. For any $v \in \mathcal{V}$, we define

$$\tilde{g}_n := \phi_{g(v)}^{\tilde{h}(v)} \circ g \text{ in } \mathcal{T}_{\{v\}}, \tag{5.3}$$

and

$$\tilde{g}_n := g \text{ in } \mathcal{T}_n \setminus \mathcal{T}_{\mathcal{V}}. \tag{5.4}$$

Note that (5.3) is well-defined since we have assumed no two vertices in $\mathcal{V}$ are adjacent. Moreover, since the boundary of $\mathcal{T}_{\{v\}}$ is mapped to $I_{g(v)}$, (2) of Definition 3.3 implies that the Definitions (5.3) and (5.4) coincide along $\partial\mathcal{T}_{\mathcal{V}}$. Thus, by removability of analytic arcs for quasiregular mappings, (5.3) and (5.4) define a quasiregular mapping on $\widehat{\mathbb{C}}$. Properties (1)-(4) in the statement of the Proposition now follow from (5.1)-(5.4). $\square$

Remark 5.6. Following the hypotheses of Proposition 5.5, we will call n, $\mathcal{V}$, $\tilde{h}$ *permissible* if $d(\tilde{h}(v), I_{g_n(v)}) \geq \pi/12$ for each $v \in \mathcal{V}$. We use the notation $\tilde{h}$ since this mapping will later be chosen to approximate the mapping h of Theorem A. The mapping $\tilde{g}_n$ is completely determined by a choice of permissible n, $\mathcal{V}$, $\tilde{h}$, so that a more precise (but more cumbersome) notation for $\tilde{g}_n$ would be $\tilde{g}_{n,\mathcal{V},\tilde{h}}$. Instead, we will usually omit all of these parameters and simply denote the mapping by $\tilde{g}$, with the dependence on n, $\mathcal{V}$, and $\tilde{h}$ understood.

Remark 5.7. We recall the definition of an *asymptotic value*. A value $w \in \widehat{\mathbb{C}}$ is an asymptotic value of a holomorphic function $f : D \to \widehat{\mathbb{C}}$ if there exists a curve $\gamma : [0, \infty) \to D$ with $\gamma(t) \to \partial D$ as $t \to \infty$ such that $f \circ \gamma(t) \to w$ as $t \to \infty$. As mentioned in the Introduction, the function f of Theorem A has no asymptotic values, and hence the postcritical set and postsingular set of f coincide. This will follow from the following Proposition (see also the proof of Theorem 7.2):

Proposition 5.8. *Let n, $\mathcal{V}$, $\tilde{h}$ be permissible. Then the only branched values of $\tilde{g}$ are $\{\pm 1, \infty\} \cup \tilde{h}(\mathcal{V})$. Moreover, if $\gamma : [0, \infty) \to D$ is a curve with $\gamma(t) \to \partial D$ as $t \to \infty$, then $\tilde{g} \circ \gamma(t)$ does not converge as $t \to \infty$.*

Proof. By Proposition 5.4, the only branched values of g are $\pm 1, \infty$, so it follows from (5.1) and (5.3) that the only branched values of $\tilde{g}$ are $\{\pm 1, \infty\} \cup \tilde{h}(\mathcal{V})$.

Let $\gamma : [0, \infty) \to D$ be a curve with $\gamma(t) \to \partial D$ as $t \to \infty$. Suppose by way of contradiction that there exists $w \in \widehat{\mathbb{C}}$ such that $\tilde{g} \circ \gamma(t) \to w$ as $t \to \infty$. By Definition 4.1 and (2) of Proposition 5.5, $\gamma([0, \infty))$ must cross infinitely many edges e of the triangulation $\mathcal{T}_n$ such that $\tilde{g}(e) \subset \hat{\mathbb{R}}$. Thus we must have $w \in \hat{\mathbb{R}}$. On the other hand, consider any Jordan curve Γ passing through $\pm 1, \infty$ with $\Gamma \cap \hat{\mathbb{R}} = \{\pm 1, \infty\}$. Then we similarly see $\gamma([0, \infty))$ must cross infinitely many edges of the triangulation $\tilde{g}^{-1}(\Gamma)$, and so $w \in \Gamma \cap \hat{\mathbb{R}} = \{\pm 1, \infty\}$. But

$$\tilde{g}^{-1}\left(\bigcup_{w \in \{\pm 1, \infty\}} B(w, \pi/12) \right) \tag{5.5}$$

is a disconnected subset of D, and so there can not be $w \in \{\pm 1, \infty\}$ such that $\tilde{g}(\gamma(t)) \in B(w, \pi/12)$ for all sufficiently large t.

$\square$

Theorem 5.9. *Let $h : S \to S$ and $\varepsilon > 0$. Then for all sufficiently large n, there exists a set of pairwise non-adjacent vertices $\mathcal{V}_n \subset \mathcal{T}_n$ such that:*

(1) *There exists an ε-bijection $\psi_n : S \to \mathcal{V}_n$,*
(2) $\text{area}(\bigcup_{s \in S} \mathcal{T}_{\{\psi_n(s)\}}) \to 0$ *as $n \to \infty$,*

(3) *If* $\tilde{h} : \mathcal{V}_n \to \widehat{\mathbb{C}}$ *is such that* $\sup_{v\in\mathcal{V}_n} d(\tilde{h}(v), h \circ \psi_n^{-1}(v)) \leq \pi/12$*, then* n*,* $\mathcal{V}_n$*,* $\tilde{h}$ *are permissible.*

Proof. Let $h : S \to S$ and $\varepsilon > 0$. Recall the triangles $\{T_s^n\}_{s\in S}$ of Theorem 4.4. By Theorem 4.4, there exists N such that we have $T_s^n \subset B(s, \varepsilon)$ for all $n \geq N$ and $s \in S$. We henceforth assume $n \geq N$, and prove the conclusions of Theorem 5.9 hold for such n.

We first define $\mathcal{V}_n$ and the bijection $\psi_n : S \to \mathcal{V}_n$. Let $s \in S$. We will define $\psi_n(s)$ to be one of the three vertices of the triangle T_s^n: in order to determine which vertex, we first consider $h(s)$. By (3.2), there is $w \in \{\pm 1, \infty\}$ such that

$$d(h(s), I_w) \geq \pi/6. \tag{5.6}$$

We define $\psi_n(s)$ to be the vertex v of T_s^n satisfying $g_n(v) = w$. This defines ψ_n and $\mathcal{V}_n := \psi_n(S)$, where we note ψ_n is a bijection onto $\mathcal{V}_n$ since T_s^n, $T_{s'}^n$ are non-adjacent for distinct s, s'. That ψ_n is an ε-bijection follows from (4.2). Moreover, property (2) in the conclusion of Theorem 5.9 now also follows from property (2) of Theorem 4.4.

We will now prove property (3). Let $s \in S$. Note that by our choice of $\psi_n(s)$ and the relation (5.6) we have that

$$d(h(s), I_{g_n\circ\psi_n(s)}) \geq \pi/6.$$

Thus, if ζ is such that $d(\zeta, h(s)) \leq \pi/12$, we have

$$d(\zeta, I_{g_n\circ\psi_n(s)}) \geq \pi/12.$$

Thus for any $\tilde{h} : \mathcal{V}_n \to \widehat{\mathbb{C}}$ such that

$$\sup_{v\in\mathcal{V}_n} d(\tilde{h}(v), h \circ \psi_n^{-1}(v)) \leq \pi/12,$$

we have

$$\inf_{v\in\mathcal{V}_n} d(\tilde{h}(v), I_{g_n(v)}) \geq \pi/12.$$

Thus as defined in Remark 5.6, we have that n, $\mathcal{V}_n$, $\tilde{h}$ are permissible.

□

Remark 5.10. The vertex set $\mathcal{V}_n$ in the conclusion of Theorem 5.9 is determined by a choice of n, h, ε. When we wish to emphasize this dependence, we will use the notation $\mathcal{V}(n, h, \varepsilon)$. We also remark that we will sometimes simply write ψ in place of ψ_n when n is understood from the context.

Remark 5.11. Recall that the mapping $\tilde{g}$ is determined by permissible n, $\mathcal{V}$, $\tilde{h}$. In particular, the parameters n, $\mathcal{V}$, $\tilde{h}$ also determine (by way of the Measurable Riemann Mapping Theorem) a unique quasiconformal mapping $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

(1) $\tilde{g} \circ \phi^{-1} : \phi(D) \to \widehat{\mathbb{C}}$ is holomorphic,
(2) ϕ fixes each of ± 1, ∞, and
(3) $\mu(\phi) = 0$ on $\widehat{\mathbb{C}} \setminus D$.

As for $\tilde{g}$, we will omit the dependence of ϕ on the parameters n, $\mathcal{V}$, $\tilde{h}$ in our notation.

Proposition 5.12. *Let $h : S \to S$, and $\varepsilon > 0$. For all sufficiently large n, we have that if $\tilde{h}$ is such that n, $\mathcal{V}(n, h, \varepsilon)$, $\tilde{h}$ are permissible, then*

$$\sup_{z\in\widehat{\mathbb{C}}} d(\phi(z), z) < \varepsilon. \tag{5.7}$$

Proof. Let $h : S \to S$, and $\varepsilon > 0$. Let N be sufficiently large so that $\mathcal{V}(N, h, \varepsilon)$ is defined, let $n \geq N$, and let $\tilde{h}$ be such that n, $\mathcal{V}(n, h, \varepsilon)$, $\tilde{h}$ are permissible. Then

$$\mathrm{supp}(\phi_{\overline{z}}) \subset \bigcup_{v\in\mathcal{V}(n,h,\varepsilon)} T_{\{v\}} = \bigcup_{s\in S} T_{\psi_n(s)} \tag{5.8}$$

Thus, by (2) of Theorem 5.9, we have

$$\mathrm{area}(\mathrm{supp}(\phi_{\overline{z}})) \xrightarrow{n\to\infty} 0. \tag{5.9}$$

Lastly, we recall that by (4) of Proposition 5.5, we have $||\mu(\phi)||_{L_\infty(\widehat{\mathbb{C}})} < k_0 < 1$, in other words ϕ is k_0-quasiconformal with k_0 independent of n, $\mathcal{V}(n, h, \varepsilon)$, $\tilde{h}$. The result now follows from the fact that there exists $\delta > 0$ such that if $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is any normalized k_0-quasiconformal mapping with $\mathrm{area}(\mathrm{supp}(\phi_{\overline{z}})) < \delta$, then (5.7) holds (see for instance Lemma 2.1 of [Bis14]). □

6. Continuity of a Fixpoint Map

In Section 7, we will prove Theorem A. As already described in the Introduction, the main strategy is to describe the desired function in the conclusion of the theorem as the fixpoint of a particular mapping we call Υ (see Definition 6.1 and Figure 3). The estimates proven in Section 5 will allow us to verify the appropriate continuity and contraction properties of Υ in order to apply a fixpoint theorem. Section 6 is dedicated to defining Υ and proving continuity.

Definition 6.1. Let D, S, h, ε be as in Theorem A and let n be sufficiently large so that $\mathcal{V}(n, h, \varepsilon/2)$ is defined (see Remark 5.10). We will define a map

$$\Upsilon : \prod_{t\in h(S)} \overline{B(t, \pi/12)} \to \prod_{t\in h(S)} \widehat{\mathbb{C}} \tag{6.1}$$

as follows. Let

$$(\zeta_t)_{t\in h(S)} \in \prod_{t\in h(S)} \overline{B(t, \pi/12)}.$$

Define a mapping

$$\tilde{h} : \mathcal{V}(n, h, \varepsilon/2) \to \widehat{\mathbb{C}} \text{ by } \tilde{h} \circ \psi(s) = \zeta_{h(s)} \text{ for all } s \in S,$$

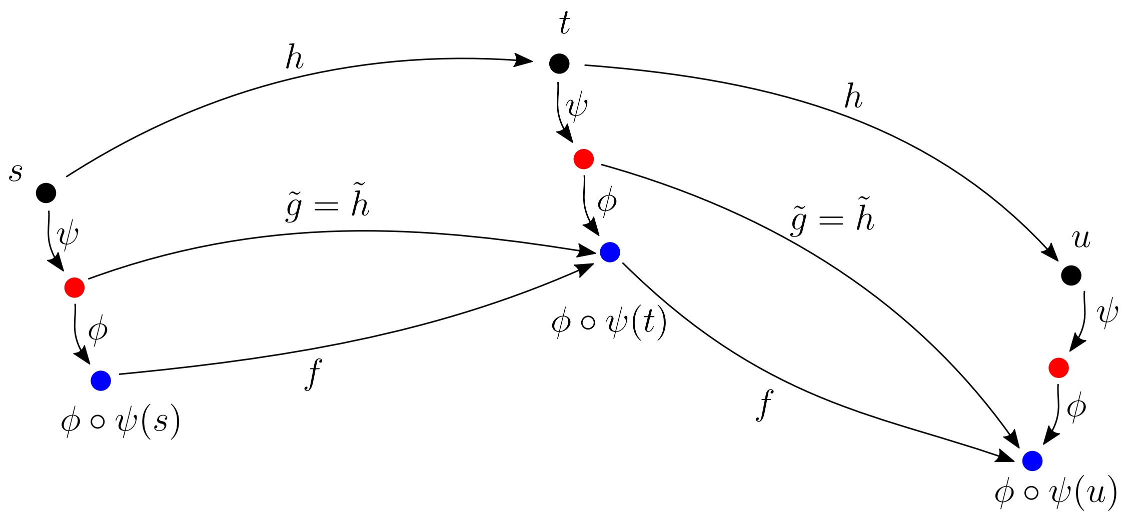

FIGURE 3. Illustrated is the behavior of a fixpoint of the mapping Υ. In black are points s, t, $u \in S$. In red are vertices of triangles containing s, t, u. In blue are the perturbations of these vertices under the correction mapping ϕ.

where $\psi = \psi_n$ is the bijection of Theorem 5.9. By (3) of Theorem 5.9, the triple n, $\mathcal{V}(n, h, \varepsilon/2)$, $\tilde{h}$ is permissible, and hence determines the mappings $\tilde{g}$, ϕ. We define:

$$\Upsilon\left((\zeta_t)_{t\in h(S)}\right) := (\phi \circ \psi(t))_{t\in h(S)} . \tag{6.2}$$

Remark 6.2. We will always consider any product space $\prod_{i\in I} X_i$ to be endowed with the standard product topology. Recall that this topology is generated by subsets of the form $\prod_{i\in I} U_i$ where each $U_i \subset X_i$ is open and $U_i = X_i$ except for finitely many i. With this topology, Tychonoff's Theorem says that any product of compact sets is compact. In particular, the domain of the mapping Υ is compact.

Theorem 6.3. *The mapping Υ of Definition 6.1 is continuous.*

Proof. Fix

$$(\zeta_t^0)_{t\in h(S)} = \zeta^0 \in \prod_{t\in h(S)} \overline{B(t, \pi/12)} \text{ and } (\xi_t^0)_{t\in h(S)} := \Upsilon(\zeta^0).$$

Let $V \subset \prod_{t\in h(S)} \widehat{\mathbb{C}}$ be an open set containing $\Upsilon(\zeta^0)$. Since V is open, there is an $\varepsilon' > 0$ such that

$$\prod_{t\in h(S)} B(\xi_t^0, \varepsilon') \subset V.$$

Thus, in order to prove the Theorem, it suffices to show that there exists $\delta > 0$ and a finite subset $\{t_1, ..., t_m\} \in h(S)$ such that if we define

$$U_t := B(\zeta_t^0, \delta) \text{ for } t \in \{t_1, ..., t_m\}, \tag{6.3}$$
$$U_t := \overline{B(t, \pi/12)} \text{ for } t \in h(S) \setminus \{t_1, ..., t_m\},$$

then $U := \prod_{t\in h(S)} U_t$ satisfies:

$$\Upsilon(U) \subset \prod_{t\in h(S)} B(\xi_t^0, \varepsilon'). \tag{6.4}$$

In fact, we will show something stronger than (6.4). For

$$\zeta \in \prod_{t\in h(s)} \overline{B(t, \pi/12)},$$

let $\phi^\zeta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ denote the quasiconformal mapping of Definition 6.1, and let $\phi_0 := \phi^{\zeta_0}$. We will show that there exists $\delta > 0$ so that for $U := \prod_{t\in h(S)} U_t$ defined as in (6.3), we have:

$$\sup_{z\in\widehat{\mathbb{C}}} d(\phi^\zeta(z), \phi_0(z)) < \varepsilon' \text{ for all } \zeta \in U. \tag{6.5}$$

Recall the constant $k_0 < 1$ of Proposition 3.5. We will use the following two facts:

($*$) There exists $\delta' > 0$ such that if $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is any normalized k_0-quasiconformal mapping with $\text{area}(\text{supp}(\phi_{\overline{z}})) < \delta'$, then

$$\sup_{z\in\widehat{\mathbb{C}}} d(\phi(z), z) < \varepsilon'/2. \tag{6.6}$$

($**$) There exists $\delta'' > 0$ such that if $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is any normalized δ''-quasiconformal mapping, then (6.6) holds.

We will abbreviate $\mathcal{V} := \mathcal{V}(n, h, \varepsilon/2)$. Note that:

$$\text{supp}(\phi_{\overline{z}}^\zeta) \subset \bigcup_{v\in\mathcal{V}} T_v \text{ for all } \zeta \in \prod_{t\in h(s)} \overline{B(t, \pi/12)}. \tag{6.7}$$

Since

$$\sum_{v\in\mathcal{V}} \text{area}(T_v) < \text{area}(\widehat{\mathbb{C}}) < \infty, \tag{6.8}$$

there exist v_1, ..., $v_m \in \mathcal{V}$ such that

$$\sum_{v\in\mathcal{V}\setminus\{v_1,...,v_m\}} \text{area}(T_v) < \delta'/C, \tag{6.9}$$

where $C > 0$ is such that any normalized k_0-quasiconformal mapping ϕ satisfies $\text{area}(\phi(E)) \leq C \cdot \text{area}(E)$ for all measurable $E \subset \widehat{\mathbb{C}}$. In (6.3), we let

$$\{t_1, ..., t_m\} := \{h \circ \psi(v_1), ..., h \circ \psi(v_m)\}. \tag{6.10}$$

Denote $A := \cup_{1 \leq i \leq m} T_{v_i}$, and for $\zeta \in U$, let $\phi_1^\zeta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ denote the normalized integrating map for $\mathbb{1}_A \cdot \mu(\phi^\zeta)$. By (4) of Definition 3.3 and (6.3), there exists $\delta > 0$ so that

$$||\mu(\phi_1^\zeta \circ \phi_0^{-1})||_{L^\infty(A)} < \delta'' \text{ for } \zeta \in U. \tag{6.11}$$

Let $\phi_2^\zeta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be such that ϕ_2^ζ is conformal in $\widehat{\mathbb{C}} \setminus \phi_1^\zeta(D)$, and $\phi_2^\zeta \circ \phi_1^\zeta$ is the normalized integrating map for $\mu(\phi^\zeta)$, so that we have $\phi_2^\zeta \circ \phi_1^\zeta = \phi^\zeta$. Then

$$\text{supp}(\mu(\phi_2^\zeta)) \subset \phi_1^\zeta\bigg(\bigcup_{\mathcal{V} \setminus \{v_1, ..., v_n\}} T_v\bigg), \tag{6.12}$$

and so by (6.9), we have:

$$\text{area}(\text{supp}(\mu(\phi_2^\zeta))) < C \cdot \sum_{v \in \mathcal{V} \setminus \{v_1, ..., v_m\}} \text{area}(T_v) \leq \delta'. \tag{6.13}$$

Thus by combining $(*)$ and $(**)$ we have that for $\zeta \in U$:

$$\begin{aligned} &\sup_{z \in \widehat{\mathbb{C}}} d(\phi_2^\zeta \circ \phi_1^\zeta(z), \phi_0(z)) = \sup_{z \in \widehat{\mathbb{C}}} d(\phi_2^\zeta \circ \phi_1^\zeta \circ \phi_0^{-1}(z), z) \\ \leq &\sup_{z \in \widehat{\mathbb{C}}} d(\phi_2^\zeta \circ \phi_1^\zeta \circ \phi_0^{-1}(z), \phi_1^\zeta \circ \phi_0^{-1}(z)) + \sup_{z \in \widehat{\mathbb{C}}} d(\phi_1^\zeta \circ \phi_0^{-1}(z), z) < \varepsilon'/2 + \varepsilon'/2 = \varepsilon'. \end{aligned}$$

This is the relation (6.5) which we needed to show.

□

Remark 6.4. A map very similar to Υ was considered in [BL19] (see Lemma 14 there), however there the proof of continuity was considerably simpler than in the present context. The added difficulty in the present setting is due to the fact that the map

$$\prod_{t \in h(S)} \overline{B(t, \pi/12)} \mapsto L^\infty(\widehat{\mathbb{C}})$$

(given by considering the Beltrami coefficient of the quasiregular map generated by any element in the domain) is not continuous, whereas in [BL19] the domain of this map is different: it consists of a product of discs with radii $\to 0$ and hence there the map into $L^\infty(\widehat{\mathbb{C}})$ is continuous.

We conclude Section 6 by recording the statement of the classical Schauder-Tychonoff fixpoint theorem (see for instance Theorem 5.28 of [Rud91]) which we will apply in the proof of Theorem A:

Theorem 6.5. *Let V be a locally convex topological vector space. For any non-empty compact convex set X in V, any continuous function $f : X \to X$ has a fixpoint.*

7. Finding a Fixpoint

We now turn to the proof of Theorem A. It will be convenient to first prove a slightly modified version of the Theorem (see Theorem 7.2 below), where we assume $\pm 1, \infty \in h(S)$ and consider the map $h|_{h(S)}$ rather than h. We will also first assume the following condition holds:

Definition 7.1. Let $D \subseteq \widehat{\mathbb{C}}$ be a domain, $S \subset D$ a discrete set, $h : S \to S$ a map, and $\varepsilon > 0$. We say D, S, h, ε are *normalizably triangulable* if there exist arbitrarily large n such that the vertex set $\mathcal{V} = \mathcal{V}(n, h, \varepsilon/2)$ of Theorem 5.9 satisfies

(1) $\pm 1, \infty \in \mathcal{V}$, and
(2) $\psi(s) = s$ for $s \in \{\pm 1, \infty\}$.

As we will see, Theorem A will follow easily from the following Theorem:

Theorem 7.2. *Let $D \subseteq \widehat{\mathbb{C}}$ be a domain, $S \subset D$ a discrete set, $h : S \to S$ a map with ± 1, $\infty \in h(S)$, and $\varepsilon > 0$. Assume D, S, h, ε are normalizably triangulable. Then there exists an ε-homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and a holomorphic map $f : \phi(D) \to \widehat{\mathbb{C}}$ with no asymptotic values such that $P(f) \subset \phi(D)$ and $f : P(f) \to P(f)$ is ε-conjugate to $h|_{h(S)} : h(S) \to h(S)$.*

Proof. We let D, S, h, ε be as in the statement of Theorem 7.2. Fix $n > 0$ sufficiently large so that the conclusions of Theorem 5.9 and Proposition 5.12 hold for $h : S \to S$ and $\varepsilon/2$, and so that the vertex set $\mathcal{V} := \mathcal{V}(n, h, \varepsilon/2)$ is as in Definition 7.1. By (3) of Theorem 5.9 and Proposition 5.12, if $\tilde{h} : \mathcal{V} \to \widehat{\mathbb{C}}$ is any map such that

$$\sup_{v \in \mathcal{V}} d(\tilde{h}(v), h \circ \psi^{-1}(v)) \leq \pi/12, \tag{7.1}$$

then n, $\mathcal{V}$, $\tilde{h}$ are permissible and

$$\sup_{z \in \widehat{\mathbb{C}}} d(\phi(z), z) < \varepsilon/2. \tag{7.2}$$

Thus, given

$$(\zeta_t)_{t \in h(S)} \in \prod_{t \in h(S)} \overline{B(t, \pi/12)}, \tag{7.3}$$

we define $\tilde{h}$ as in Definition 6.1 by

$$\tilde{h} \circ \psi(s) := \zeta_t \text{ for all } t \in h(S) \text{ and } s \in h^{-1}(t), \tag{7.4}$$

which in turn defines the mappings $\tilde{g}$, ϕ, where ϕ satisfies (7.2).

Consider now the mapping Υ of Definition 6.1. By (7.2) and (1) of Theorem 5.9, we have for any $(\zeta_t)_{t\in h(S)} \in \prod_{t\in h(S)} \overline{B(t,\pi/12)}$ that:

$$d(\phi\circ\psi(t),t) \le d(\phi\circ\psi(t),\psi(t)) + d(\psi(t),t) < \varepsilon/2+\varepsilon/2 = \varepsilon. \tag{7.5}$$

Thus in fact Υ defines a map:

$$\Upsilon : \prod_{t\in h(S)} \overline{B(t,\pi/12)} \to \prod_{t\in h(S)} \overline{B(t,\varepsilon)}. \tag{7.6}$$

We claim that Υ has a fixpoint. Indeed, Υ is continuous by Proposition 6.3, and the domain of Υ is compact and convex, so Theorem 6.5 implies the existence of a fixpoint of Υ.

The fixpoint of Υ yields a choice of $\tilde{g}$, ϕ such that

$$\tilde{h}\circ\psi(s) = \phi\circ\psi(t), \text{ for all } t\in h(S) \text{ and } s\in h^{-1}(t). \tag{7.7}$$

Define the holomorphic map $f := \tilde{g}\circ\phi^{-1} : \phi(D)\to\widehat{\mathbb{C}}$. We will show that f, ϕ satisfy the conclusions of the Theorem. We have already proven (see (7.2)) that ϕ is an ε-homeomorphism. We claim that $\{\pm1,\infty\}\subset\tilde{h}(\mathcal{V})$. Indeed, if $t\in\{\pm1,\infty\}$ and $s\in h^{-1}(t)$ (here we are using the assumption that $\pm1,\infty\in h(S)$), then by (7.7) and (2) of Definition 7.1 we have $\tilde{h}\circ\psi(s) = \phi\circ\psi(t) = \phi(t) = t$. Thus by Proposition 5.8, we conclude that f has has no asymptotic values and

$$P(f) = \{\pm1,\infty\}\cup\tilde{h}(\mathcal{V}) = \tilde{h}(\mathcal{V}). \tag{7.8}$$

Also, by (7.7), we have:

$$\tilde{h}(\mathcal{V}) = \tilde{h}\circ\psi(S) = \phi\circ\psi(h(S)), \tag{7.9}$$

and since $\psi(h(S))\subset D$ (since ψ maps to vertices in a triangulation of D), we have $P(f) = \tilde{h}(\mathcal{V})\subset\phi(D)$. It remains to show that $f: P(f)\to P(f)$ and $h|_{h(S)} : h(S)\to h(S)$ are ε-conjugate. Indeed, we claim that $\phi\circ\psi : h(S)\to P(f)$ is the desired conjugacy. By (7.8) and (7.9) we have that $\phi\circ\psi : h(S)\to P(f)$ is onto and hence a bijection. By (7.5), we have that $\phi\circ\psi : h(S)\to P(f)$ is an ε-bijection. Lastly, for all $t\in h(S)$:

$$f\circ\phi\circ\psi(t) = \tilde{g}\circ\psi(t) = \tilde{h}\circ\psi(t) = \phi\circ\psi\circ h(t), \tag{7.10}$$

where the first $=$ is since $f := \tilde{g}\circ\phi^{-1}$, the second $=$ is (1) of Proposition 5.5, and the last $=$ is by (7.7).

□

Now we remove the hypothesis of Definition 7.1 from Theorem 7.2:

Theorem 7.3. *Let $D\subseteq\widehat{\mathbb{C}}$ be a domain, $S\subset D$ a discrete set, $h: S\to S$ a map with ±1, $\infty\in h(S)$, and $\varepsilon>0$. Then there exists an ε-homeomorphism $\phi:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ and a holomorphic map $f:\phi(D)\to\widehat{\mathbb{C}}$ with no asymptotic values such that $P(f)\subset\phi(D)$ and $f: P(f)\to P(f)$ is ε-conjugate to $h|_{h(S)} : h(S)\to h(S)$.*

Proof. We let D, S, h, ε be as in the statement of Theorem 7.3. Let $\varepsilon' > 0$, and recall the bijection $\psi = \psi_{n,h,\varepsilon'} : \mathcal{V}(n, h, \varepsilon') \to S$ of Theorem 5.9. Define a Möbius transformation $M = M_n$ by

$$M \circ \psi_{n,h,\varepsilon'}(s) = s \text{ for } s \in \{\pm 1, \infty\}. \tag{7.11}$$

Then, by fixing ε' sufficiently small, we have that M is an $\varepsilon/2$-homeomorphism for all sufficiently large n. We define

$$S' := M(S \setminus \{\pm 1, \infty\}) \cup \{\pm 1, \infty\}. \tag{7.12}$$

We define $h' : S' \to S'$ by a simple adjustment of the definition of h:

$$h'(s) := \begin{cases} M \circ h \circ M^{-1}(s) & \text{if } s, h(s) \notin \{\pm 1, \infty\} \\ h(s) & \text{if } s, h(s) \in \{\pm 1, \infty\} \\ M \circ h(s) & \text{if } s \in \{\pm 1, \infty\}, h(s) \notin \{\pm 1, \infty\} \\ h \circ M^{-1}(s) & \text{if } s \notin \{\pm 1, \infty\}, h(s) \in \{\pm 1, \infty\} \end{cases} \tag{7.13}$$

For $n > 0$, let $\mathcal{T}_n$ denote the triangulation of D of Theorem 5.9. Note that $M(\mathcal{T}_n)$ is a triangulation of $M(D)$, and moreover by (7.11) the vertex set $M(\mathcal{V}(n, h, \varepsilon/2)) \subset M(\mathcal{T}_n)$ contains $\pm 1, \infty$. Thus Theorem 7.2 applies to $M(D)$, S', h', $\varepsilon/2$ to yield an $\varepsilon/2$-homeomorphism $\tilde{\phi} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and a holomorphic map $f : \tilde{\phi} \circ M(D) \to \widehat{\mathbb{C}}$ with no asymptotic values such that $P(f) \subset \tilde{\phi} \circ M(D)$ and $f : P(f) \to P(f)$ is $\varepsilon/2$-conjugate to $h'|_{h'(S')} : h'(S') \to h'(S')$. We claim that $\phi := \tilde{\phi} \circ M$ and f satisfy the conclusions of Theorem 7.3.

Indeed, since M is an $\varepsilon/2$-homeomorphism, it follows that $\phi = \tilde{\phi} \circ M$ is an ε-homeomorphism. We have already justified that $P(f) \subset \tilde{\phi} \circ M(D)$. Lastly, by Definition (7.13), $h'|_{h'(S')} : h'(S') \to h'(S')$ is $\varepsilon/2$-conjugate to $h|_{h(S)} : h(S) \to h(S)$, and so $f : P(f) \to P(f)$ is ε-conjugate to $h|_{h(S)} : h(S) \to h(S)$. □

Next we remove the assumption that $\pm 1, \infty \in h(S)$.

Theorem 7.4. *Let $D \subseteq \widehat{\mathbb{C}}$ be a domain, $S \subset D$ a discrete set with $|h(S)| \geq 3$, $h : S \to S$ a map, and $\varepsilon > 0$. Then there exists an ε-homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and a holomorphic map $f : \phi(D) \to \widehat{\mathbb{C}}$ with no asymptotic values such that $P(f) \subset \phi(D)$ and $f : P(f) \to P(f)$ is ε-conjugate to $h|_{h(S)} : h(S) \to h(S)$.*

Proof. We let D, S, h, ε be as in the statement of Theorem 7.4. Let M be a Möbius transformation sending any three points of $h(S)$ to ± 1, ∞. Then applying Theorem 7.3 to $M(D)$, $M(S)$, $M \circ h \circ M^{-1}$, $\varepsilon(M)$ yields mappings we will denote by

$$\tilde{\phi} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \text{ and } \tilde{f} : \tilde{\phi} \circ M(D) \to \widehat{\mathbb{C}}. \tag{7.14}$$

It is straightforward to then check that the functions $\phi := M^{-1} \circ \tilde{\phi} \circ M$ and $f := M^{-1} \circ \tilde{f} \circ M$ satisfy the conclusions of Theorem 7.4 for aptly chosen $\varepsilon(M)$. □

In the case that h is onto, Theorem 7.4 is exactly Theorem A, and so all that remains is to consider the case that h is not onto:

Proof of Theorem A. We let D, S, h, ε be as in the statement of Theorem A. We augment the set S to a set $S' \supset S$ so that S' is still discrete in D, and such that we can define a mapping $h' : S' \to S'$ such that $h'(S') = S$ and $h'|_S = h$. Then since $h'|_{h'(S')} : h(S') \to h(S')$ is the same function as $h : S \to S$, applying Theorem 7.4 to D, S', h', ε yields the desired functions in the conclusion of Theorem A.

8. Conformal Grid Annuli

In Sections 8-10, we turn our attention to the proof of Theorem B. As mentioned in the Introduction, Sections 8-10 may be read independently of Sections 3-7. We begin by studying the annuli in which we will interpolate between two different triangulations, as described in Section 2.

An *equilateral grid polygon* is a simple closed polygon that lies on the edges of an Euclidean equilateral triangulation of the plane. An *equilateral grid annulus* is a topological annulus in $\mathbb{R}^2$ so that the two boundary components are both equilateral grid polygons (on the same grid). Note that the boundary of either an equilateral grid polygon or annulus is marked by a finite number of points corresponding to the vertices of the triangulation. A boundary triangle is any grid triangle inside the grid polygon or annulus that intersects the boundary. The minimum number N of grid triangles needed to connect the two boundary components of an equilateral grid annulus will be denoted the *thickness* of the annulus. For any topological annulus A in the plane, we let $\partial_o A$ and $\partial_i A$ denote the outer and inner connected components of ∂A, i.e., $\partial_o A$ separates A from ∞. Recall that any planar topological annulus with non-degenerate boundary components can be conformally mapped to a round annulus of the form $B = \{1 < |z| < 1 + \delta\}$ and this map is unique up to rotations. We will be concerned primarily with the case where δ is small.

We wish to consider conformal images of equilateral grid annuli, but also a slightly more general class of annuli where each boundary component has a one-sided neighborhood that is a conformal image of a equilateral grid annulus. More precisely, we shall call A a *conformal grid annulus* if it has a finite set V of points on its boundary (called the *vertices* of A) so that the following conditions hold. Assume there are two conformal maps f_o, f_i on A and equilateral grid annuli A_o, A_i so that $f_o(A)$ is an topological annulus so that

(1) $A_o \subset f_o(A)$,
(2) $\partial_o(f_o(A)) = \partial_o A_o$,
(3) $f_o(V \cap \partial_o A)$ equals the grid vertices of $\partial_o A_o$.

We also assume the analogous conditions of the inner boundary, i.e.,

(1) $A_i \subset f_i(A)$,
(2) $\partial_i(f_i(A)) = \partial_i A_i$,
(3) $f_i(V \cap \partial_i A)$ equals the grid vertices of $\partial_i A_i$.

If $f_o = f_i$ and $A_o = A_i$, these conditions just say that A is the conformal image of a single equilateral grid annulus and the marked points are the images of the corresponding grid vertices.

Given a conformal grid annulus A we define

$$\mathrm{inrad}(A) = \sup_{z \in A} \mathrm{dist}(z, \partial A),$$

to be the in-radius of A, and

$$\mathrm{gap}(A) = \max_k \{\mathrm{diameter}(\gamma_k)\}$$

to be the maximum diameter of the connected components of $\partial A \setminus V$, i.e., the subarcs $\{\gamma_k\}$ defined by the vertex set on ∂A. Later we will find triangulations of A whose elements have diameters controlled by these quantities.

If T is an outer boundary triangle of A_o, the topological triangle $f_o^{-1}(T)$ is called a boundary triangle of A. Similarly for A_i. In our main application, the inner boundary of A will be an equilateral grid polygon and the f_i will be the identity map. The associated boundary triangles of A are then Euclidean equilateral. The outer boundary of A will be the image of an equilateral grid polygon under a map f_o that extends conformally past $\partial_o A$. Thus the boundary triangles of A along its outer boundary will be small, smooth perturbations of equilateral triangles.

Below we shall use several standard properties of conformal modulus. This is a well known conformal invariant whose basic properties are discussed in many sources such as [Ahl06] or [GM05]. We briefly recall the basic definitions. Suppose Γ is a family of locally rectifiable paths in a planar domain Ω and ρ is a non-negative Borel function on Ω. We say ρ is admissible for Γ (and write $\rho \in \mathcal{A}(\Gamma)$) if

$$\ell(\Gamma) = \ell_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \int_\gamma \rho ds \geq 1,$$

and define the modulus of Γ as

$$\mathrm{Mod}(\Gamma) = \inf_\rho \int \rho^2 dxdy,$$

where the infimum is over all admissible ρ for Γ. We shall frequently use the extension rule: if Γ, Γ' are path families so that every element $\gamma' \in \Gamma'$ equals or contains as a subarc an element $\gamma \in \Gamma$ then $M(\Gamma) \leq M(\Gamma')$ (if ρ is admissible for Γ, it is also admissible for Γ' so the infimum for Γ is over a smaller set of metrics). The modulus of the path family connecting the two boundary components of $\{1 < |z| < R\}$ is $2\pi/\log R$, and the extension rule implies that any path family where every curve crosses such an annulus has modulus $\leq 2\pi/\log R$. We shall use this basic fact later.

Lemma 8.1. *Suppose A is a conformal grid annulus and that there are at least four marked points on each component of ∂A. Suppose $f : A \to B = \{z : 1 < |z| < 1 + \delta\}$ is a conformal* *map of A onto a round annulus. This sends the sub-arcs on ∂A to sub-arcs on ∂B. Then there is a $M < \infty$, independent of A, so that any two adjacent sub-arcs on ∂B have lengths comparable to within a factor M, and every sub-arc in B has length $\leq M\delta$.*

Proof. Suppose J is a sub-arc of ∂A and I, K are the two adjacent sub-arcs. Let Γ be the path family in A that connects I to K. If I, J, K are in the outer boundary of A we let $f = f_o$ and $A' = A_o$ and otherwise we set $f = f_i$ and $A' = A_i$. In either case we let I', J', K' be the corresponding line segments on the boundary of A' and Γ' the path family connecting I' to K' in A'. Let U be the union of all the boundary triangles of A' that touch the boundary arc $\gamma' = I' \cup J' \cup K'$. Note that there are only finitely many shapes γ' can have, and only finitely many shapes for U (up to Euclidean similarity).

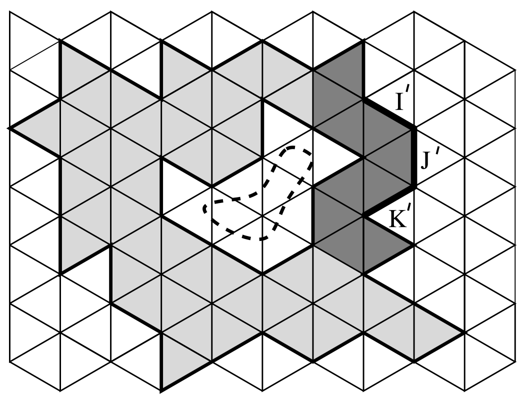

FIGURE 4. Here we assume that the outer boundary of A maps to the outer boundary of a equilateral grid annulus A' (shaded). The inner boundary of $f(A)$ (dashed) need not coincide with the inner boundary of A'. Given three segments I', J' and K' on the outer boundary of A' we let U be the union of all grid triangles in A' that touch one of these segments (darker shading). Since I' and K' don't touch each other and there are only finitely many possible shapes for U, the modulus of the path family connecting them in A' is uniformly bounded.

The path family Γ' need not be the image of Γ if $f(A) \neq A'$. However, since f is conformal and $A' \subset f(A)$ we have, by the extension rule that $M(\Gamma) \leq M(\Gamma')$. Again, $M(\Gamma')$ is one of a finite number of positive possibilities, so $M(\Gamma)$ is bounded uniformly from above.

We claim that $M(\Gamma)$ is also bounded uniformly from below. Let σ be the union of the three line segments I', J', K' and let $\Omega = \mathbb{C} \setminus \sigma$. Again using basic properties of modulus, $M(\Gamma)$ is bounded below by the modulus of the path family connecting I' to K' in Ω, because $f(A) \subset \Omega$. Again, this modulus is one of a finite number of positive possibilities, so $M(\Gamma)$ is bounded uniformly from below.

The modulus of the path family in A connecting J to the component of ∂A not containing J is bounded above by the analogous path family for J' in A'. This is bounded above by the modulus of the path family connecting J' to $\partial U \setminus \gamma'$. There are only a finite number of possible configurations of U and γ', and each gives a finite modulus, so the maximum of these values is also bounded above, independent of A.

Thus for each arc on one component of ∂B, the path family connecting it to the other component is bounded uniformly above. This implies the length of the arc is $O(\delta)$ as $\delta \to 0$.

Similarly, the path family Γ'' in B connecting arcs I, K that are both adjacent to an arc J has modulus bounded uniformly above and below. Recall that we have proven diameter$(J) = O(\delta)$. Thus, if we suppose by way of contradiction that diameter$(J) \neq O($diameter$(I))$ as $\delta \to 0$, we would deduce that $M(\Gamma'')$ degenerates, a contradiction. We conclude that diameter$(J) = O($diameter$(I))$ as $\delta \to 0$. Since the roles of I and J may be exchanged we deduce that the two arcs have comparable lengths. □

Lemma 8.2. *For every $\epsilon > 0$, there is an N so that if A is a conformal grid annulus with A_o, A_i each having thickness at least N, then in the conclusion of Lemma 8.1 each subarc on ∂B has length at most $\epsilon \cdot \delta$.*

Proof. In this case, the path family connecting J' to the opposite boundary component must connect points in J' to points outside a disk of radius $\simeq N \cdot$ diameter(J') centered on J'. The extension rule and the modulus calculation for annuli then imply this path family has modulus tending to zero as N increases to infinity. This implies the arc has small length compared to the width of B. □

For a rectifiable arc γ, we let $\ell(\gamma)$ denote the length of γ. A homeomorphism $f : \gamma \to \sigma$ between rectifiable curves is said to multiply lengths if for any subarc $\gamma' \subset \gamma$ we have $\ell(f(\gamma')) = \ell(\gamma') \cdot \ell(\sigma)/\ell(\gamma)$.

A rectifiable curve is γ is called M-chord-arc if for any two points $x, y \in \gamma$ the shorter sub-arc connecting x and y has length at most $M|x - y|$. A map f is L-biLipschitz if

$$\frac{1}{L} \leq \frac{|f(x) - f(y)|}{|x - y|} \leq L,$$

for all x, y in its domain, $x \neq y$. Bi-Lipschitz maps between planar domains are automatically quasiconformal with dilatation at most $K = L^2$. A closed curve is chord-arc if and only if it is the bi-Lipschitz image of a circle. A length multiplying map between two M-chord-arc curves is necessarily M-bi-Lipschitz, Moreover, an L-biLipschitz map between M-chord-arc curves has an K-biLipschitz extension between the interiors, where K only depends on L and M. See e.g., [Tuk81] by Tukia or [Mac95] by MacManus.

Lemma 8.3. *In Lemma 8.1, if A is a conformal grid annulus and each boundary triangle T of A is an L-biLipschitz image of a Euclidean equilateral triangle, then there is a K-quasiconformal map $\psi : A \to B$ so that*

(1) *ψ equals f on A minus the boundary triangles of A,*
(2) *ψ equals f on the boundary vertices of A,*
(3) *ψ multiplies arclength on each boundary arc of A.*
(4) *K depends only on the biLipschitz constant L.*

Proof. It is enough to consider the boundary corresponding to A_o; the argument for the inner boundary is the same.

Let $f : A \to B$ be the conformal map of the conformal grid annulus A to the round annulus B given in Lemma 8.1. Consider a boundary triangle T' of the equilateral grid annulus A_o and the corresponding boundary triangle $T = f_o^{-1}(T')$ of A. Then $g_T = f \circ f_o^{-1}$ is a conformal map of T' into B. Recall that the boundary of A_o is a grid polygon, so it has fixed side lengths (which we may assume are all unit length) and every angle is in $\{\frac{\pi}{6}, \frac{\pi}{3}, \dots \frac{5\pi}{6}\}$. Thus at each vertex v of $\partial_o A_o$, the Schwarz reflection principle implies there is an $\alpha \in \{3, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}\}$ so that mapping $g_T((z-v)^\alpha))$ has a conformal extension to $D(v, \frac{1}{2})$. This, together with the distortion theorem for conformal maps (e.g., Theorem I.4.5 of [GM05]) implies that each edge of $f(T) = g_T(T')$ is an analytic arc with uniform bounds, meeting the other two at angles bounded uniformly away from zero (at interior verticies all angles are $\pi/6$ and at boundary vertices the angles are π/k where k vertices meet, and at most 5 triangles can meet a boundary vertex of a equilateral grid polygon). Thus the image topological triangle $f(T)$ is a chord-arc curve with uniform bounds. Define a map ψ_T on the boundary of T by making ψ_T length multiplying on any edge lying on ∂A and on any edge in common with another boundary triangle, and let $\psi_T = f$ on any other edges (necessarily an edge shared with a non-boundary triangle). This is a bi-Lipschitz map from ∂T to $f(\partial T)$ between chord-arc curves and hence it has a bi-Lipschitz extension (which is also a quasiconformal extension) between the interiors with uniform bounds. So if we replace f in each boundary triangle T by the map ψ_T, we get a quasiconformal map $\psi : A \to B$ that satisfies all the desired properties. $\square$

Lemma 8.4. *Suppose Γ is a equilateral grid polygon bounding a region Ω and $\gamma \subset \Omega$ is a equilateral grid polygon (on the same grid as Γ) so that the annulus between γ and Γ has thickness at least* 10. *Let $\Omega' \subset \Omega$ be the region bounded by γ. Suppose f is conformal on Ω. Then there is K-quasiconformal map g on Ω' so that*

(1) *$g = f$ off the triangles touching γ,*
(2) *$g = f$ on the vertices of γ,*
(3) *g is length multiplying on the edges of γ.*
(4) *K is absolute (just depending on the thickness* 10*, and tending to* 1 *if the thickness is increased towards infinity).*

Proof. For each boundary triangle T of γ, f is conformal on a disk centered at the center of T with radius $\geq 4 \cdot \text{diameter}(T)$. Therefore the image $T' = f(T)$ consists of analytic arcs meeting at $60°$. Thus for any subset of the three edges of T we can define a biLipschitz map $g : T \to T'$ that agrees with f on this subset of edges, also agrees with f at all three vertices, and is length multiplying on the remaining edges. As above, this is a biLipschitz map between chord-arc curves so it has a biLipschitz (and hence quasiconformal) extension between the interiors, with constants that are uniformly bounded, say by K. On any non-boundary triangle in Ω' we set $g = f$. For each boundary triangle we take g as above that is length multiplying on the edges of T on γ or shared with another boundary triangle, and so that $g = f$ on edges of T that are shared with a non-boundary triangle.

If the thickness is very large, then $f(T)$ is close to an equilateral triangle, and it is clear that the maps defined above can be taken close to isometries, i.e., the quasiconformal dilatation is close to 1. □

9. Triangulating Annuli

In Section 9 we triangulate the conformal grid annuli introduced in Section 8. We do this by pulling back a triangulation of a conformally equivalent annulus by a certain quasiconformal mapping. We begin with a discussion of decomposition of domains into dyadic squares.

A dyadic interval $I \subset \mathbb{R}$ is one of the form $I = [j2^{-n}, (j+1)2^{-n}]$ for some integers j, n. A dyadic square in the plane is a product of dyadic intervals of equal length, i.e., $Q = [j2^{-n}, (j+1)2^{-n}] \times [k2^{-n}, (k+1)2^{-n}]$ for some integers j, k, n. We let $\ell(Q) = 2^{-n} = \text{diameter}(Q)/\sqrt{2}$ denote the side length of Q. Two dyadic squares either have disjoint interiors or one is contained in the other one. Given a domain D, we can therefore take the set of maximal dyadic dyadic squares $\mathcal{W} = \{Q_j\}$ so that $3Q_j \subset D$. Then

$$\ell(Q_j) \leq \text{dist}(Q_j, \partial D) \leq 3\sqrt{2}\ell(Q_j). \tag{9.1}$$

This is an example of a Whitney decomposition of D. Note that if Q and Q' are adjacent squares in the Whitney decomposition above, with $\ell(Q') < \ell(Q)$, then

$$\ell(Q') \geq \frac{1}{3\sqrt{2}} \text{dist}(Q', \partial D) \geq \frac{1}{3\sqrt{2}}[\text{dist}(Q, \partial D) - \sqrt{2}\ell(Q')]$$

which implies $\ell(Q') \geq \frac{1}{4\sqrt{2}}\ell(Q) > \frac{1}{8}\ell(Q)$. Since the side lengths are dyadic, we must have $\ell(Q') \geq \frac{1}{4}\ell(Q)$. Thus adjacent squares differ in size by at most a factor of 4.

Lemma 9.1. *Suppose $S = \{x + iy : 0 < y < 2\}$ is an infinite strip and the top and bottom edges are partitioned into segments of length $\leq 1/8$ and that adjacent edges have lengths comparable to within a factor of M. Then there is a locally finite triangulation of the strip using only the given boundary vertices and so that every angle of every triangle is $\geq \theta > 0$ where θ only depends on M. Thus the triangulation has "bounded degree" depending only on M, i.e., the number of triangles meeting at any vertex is uniformly bounded above by $2\pi/\theta$. If both partitions are L-periodic (under horizontal translations) for some $L \geq 1$, then the triangulation is also L-periodic.*

Proof. By splitting the strip into two parallel strips and rescaling, it suffices to consider the case when the top side is divided into unit segments (we triangular the top and bottom halves separately and join them along a unit partition running down the center of the strip). The following argument is adapted from the proof of Theorem 3.4 in [BR21].

If $\cdots < x_{-1} < x_0 < x_1 < \dots$ are the partition points on the bottom edge define

$$D_k = \min(|x_k - x_{k+1}|, |x_k - x_{k-1}|),$$

By assumption, any two adjacent values of D_k are comparable within a factor of $1 \le M < \infty$, and $\sup D_k \le 1/8$. Thus $0 < D_k/(16M) \le 1/128$ is contained in a dyadic interval of the form $(2^{-j-1}, 2^{-j}]$ for some $j \ge 6$ (these half-open intervals form a disjoint cover of $(0, \infty)$). Let $y_k = \frac{3}{4} \cdot 2^{-j}$ be the center of this interval. Note that y_k and $D_k/(16M)$ are comparable within a factor of $\frac{3}{2} < 2$, so $y_k < D_k/(8M) \le \min(\frac{1}{64}, D_k/8)$.

Let $z_k = x_k + iy_k$, $k \in \mathbb{Z}$ and consider the infinite polygonal arc σ with these vertices. Note that σ stays within $1/64$ of the bottom edge of the strip and every segment has slope between $-1/8$ and $1/8$: the heights of the endpoints above x_k, x_{k+1} are each less than

$$\max(y_k, y_{k+1}) \le \frac{1}{8}\max(D_k, D_{k+1}) \le \frac{1}{8}|x_k - x_{k+1}|,$$

so

$$\frac{|y_{k+1} - y_k|}{|x_{k+1} - x_k|} \le \frac{\max(y_{k+1}, y_k)}{|x_{k+1} - x_k|} \le \frac{1}{8}.$$

Tile the top half of S by unit squares. Below this place a row of squares of side length $1/2$. Continue in this way, as illustrated in Figure 5. We call this our decomposition of S into dyadic squares. (This corresponds to the restriction of a Whitney decomposition of a half-plane to the strip.)

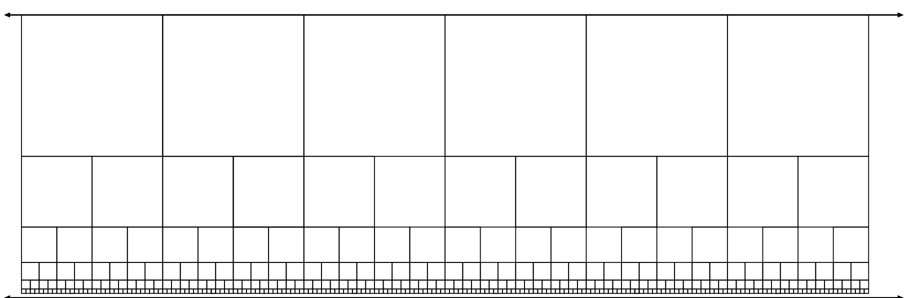

FIGURE 5. The decomposition of S into dyadic squares.

For each k, choose a square Q_k from our decomposition of the strip S that contains z_k. There is at least one decomposition square containing z_k since these squares cover S, and there are at most two, since by our choice of y_k, z_k cannot lie on the top or bottom edge of any such Q_k (y_k was chosen to be halfway between these heights). See Figure 6. Let I_k denote the vertical projection of Q_k onto the bottom edge of S. Since the segments of σ have slope $\le 1/8$, the height of σ can change by at most $\ell(Q_k)/8$ over I_k and since it contains a point z_k that is distance $\ell(Q_k)/2$ from both the top and bottom edges of Q_k, σ cannot hit these edges of Q_k. Similarly, it cannot hit the top or bottom edges of the adjacent dyadic squares of the same size as Q_k that share the left and right edges of Q_k. In fact, it takes at least horizontal distance $4\ell(Q_k)$ for σ to reach the height of the top or bottom of Q_k, so σ does not hit the top or bottom of the squares that are up to three positions to the left or right on Q_k. This implies that σ does not hit the "parent" square $Q_k^\uparrow$ of Q_k (the square of twice the size lying directly above Q_k), nor does it hit the left or right neighbors of $Q_k^\uparrow$. See Figure 6.

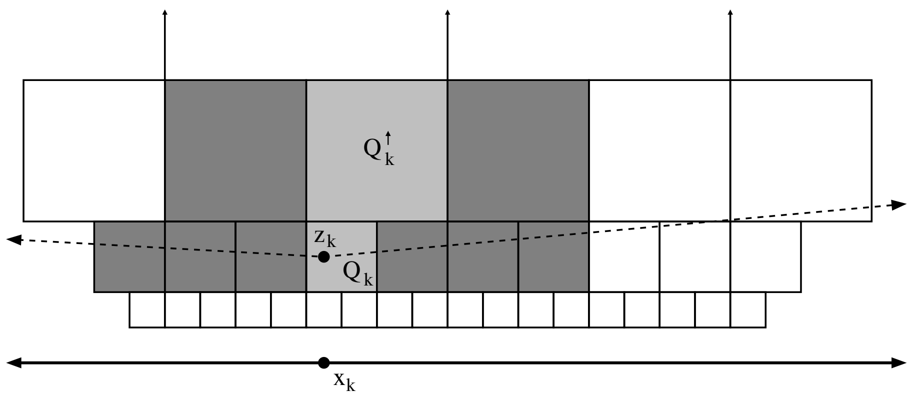

FIGURE 6. The point on the bottom edge is x_k, and above it is the corresponding z_k. The point z_k is contained in a square Q_k and above this is its "parent" $Q_k^\uparrow$ (both lightly shaded). The dashed curve is part of σ. Note that σ hits at least three squares to the left and right of Q_k (darker shading). This implies the "parent" square $Q_k^\uparrow$ does not hit σ, nor do the squares to the left or right of the parent (also dark shaded).

Now remove all the squares whose interiors hit σ or that lie below σ. The set of remaining squares contains the whole top row of unit squares. Since σ has small slope, if a square Q is above σ, so is its parent (and by induction, all its ancestors). Let γ denote the lower boundary of union See the top of Figure 7. of remaining squares; this is a locally polygonal curve made up of horizontal and vertical segments. A vertex of γ is any corner of a decomposition square that lies on γ, and a corner of γ is a vertex where a horizontal and vertical edge of γ meet. Let W denote the infinite region bounded above by γ and below by the bottom edge of S (shaded region in top picture of Figure 7).

Let γ_k be the subarc of γ that projects onto $[x_k, x_{k+1}]$. By construction, each x_k lies below the parent of Q_k, and the squares to the left and right of the parent are also above σ, so x_k is at least distance $2\ell(Q_k)$ from the vertical projection of any corner of γ. Connect x_k to a vertex w_k of γ whose vertical projection is closest to x_k, or to either one in case of a tie. Note that w_k is a vertex on the bottom edge of $Q_k^\uparrow$; a tie occurs only if w_k is the midpoint

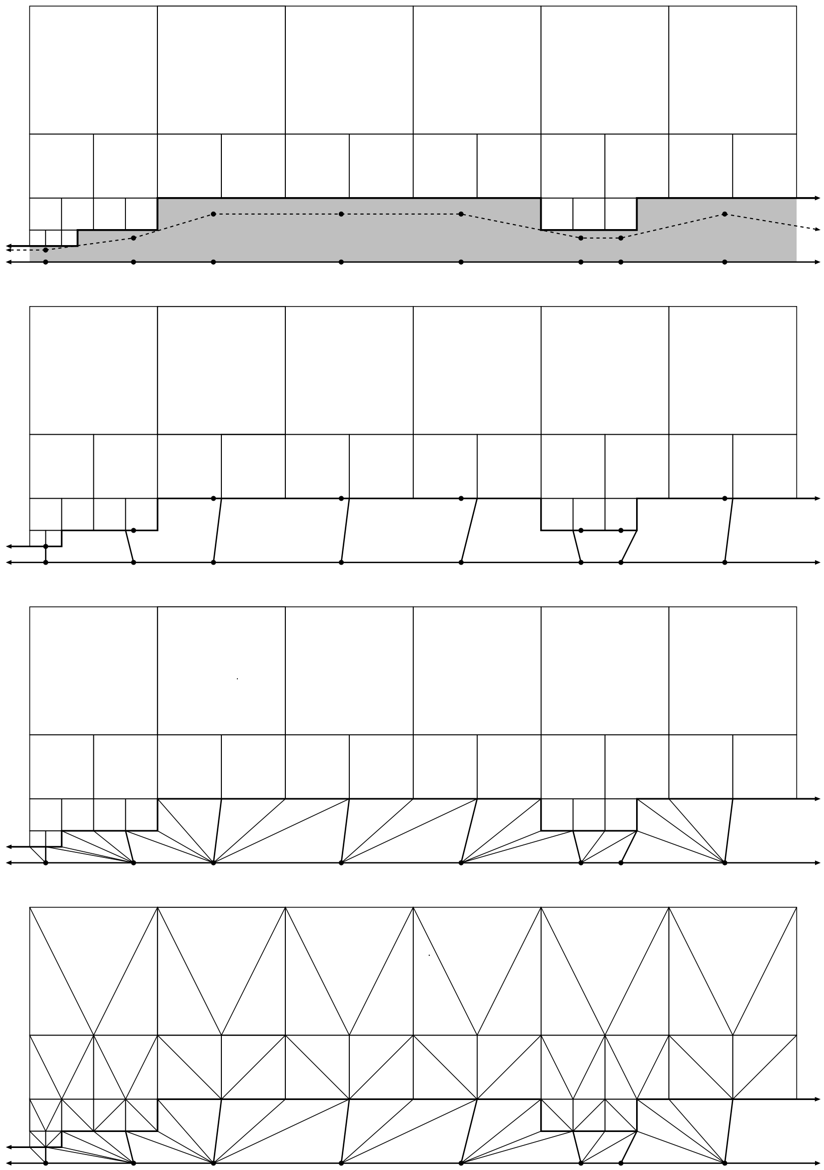

FIGURE 7. The top figure shows the region W (shaded) below γ. The second figure divides W into quadrilaterals by connecting each x_k to a vertex of γ that is closest to being "above" x_k. We then triangulate the quadrilaterals by connecting all vertices of γ to either the lower left or lower right corner, depending on whether γ is decreasing or increasing between x_k and x_{k+1}. The bottom picture shows the squares above γ triangulated in the obvious way.

of this bottom edge. Adding the segments from x_k to w_k divides W into quadrilaterals. See the second figure in Figure 7.

Over the interval (x_k, x_{k+1}), the polygonal curve γ is either a horizontal segment, a decreasing stair-step or an increasing stair-step. In the first two cases, connect every vertex of γ between w_k and w_{k+1} (including these points) to x_k. In the third case, connect them all to x_{k+1}. In either case, this triangulates W with triangles so that all three edges have comparable lengths and no angle is close to $180°$, so by the Law of Sines, all the angles are bounded uniformly away from 0 (the bound depends on M, the constant of comparability between adjacent arcs on the boundary of S). $\square$

The following simple lemma will allow us to build equilateral triangulations from topological triangulations that are "close to" equilateral in a precise sense.

Lemma 9.2. *Suppose $K < \infty$ and $\mathcal{T}$ is a topological triangulation of Ω and for each triangle $T \in \mathcal{T}$, there is a K-quasiconformal map f_T sending T to an Euclidean equilateral triangle and that is length multiplying on each boundary edge. Let μ_T be the dilatation of f_T. If f is a quasiconformal map on Ω with dilatation μ_T on T, then $f(\mathcal{T})$ is an equilateral triangulation of $f(\Omega)$.*

Proof. We use the characterization of equilateral triangulations given in Lemma 2.5 of [BR21]: a triangulation of a Riemann surface is equilateral iff given any two triangles T, T' that share an edge e, there is an anti-holomorphic homeomorphism $T \to T'$ that fixes e pointwise, and maps the vertex v opposite e in T to the vertex v' opposite e in T'.

For any two triangles T_1, T_2 in $f(\mathcal{T})$ that are adjacent along an edge e, define $g = \iota_k \circ f_{Tk} \circ f^{-1}$ on T_k, $k = 1, 2$, where ι is an appropriately chosen similarity of the plane to make the image triangles match up along the segment I that is the image of e. By the length multiplying property of the maps f_T, g is continuous across e. Then $g^{-1} \circ R \circ g$, where R is reflection across I, is the anti-holomorphic maps that swaps T_1 and T_2 as required. $\square$

The image triangulation $\mathcal{T}'$ will be close to $\mathcal{T}$ if the dilatation μ is close to zero in an appropriate sense. For our applications below, this will mean that the dilatation of $|\mu|$ is uniformly bounded below 1 and that the support of μ has small area. As the area tends to zero, f can be taken to uniformly approximate the identity, and so $\mathcal{T}'$ approximates $\mathcal{T}$ as closely as we wish.

The following is elementary and left to the reader. See Figure 8 for a hint.

Lemma 9.3. *Any Euclidean triangle T can be uniquely mapped to a equilateral triangle T' by an affine map by specifying a distinct vertex of T' for each vertex of T. This map is K-quasiconformal where K depends only on the minimal angle of T.*

Lemma 9.4. *There is a constant $C < \infty$ so that the following holds. Suppose A is a conformal grid annulus as described above, and A is conformally equivalent to a round annulus $B = \{1 < |z| < 1 + \delta\}$, where $\delta \leq 1/100$. Suppose also that the image sub-arcs on B all*

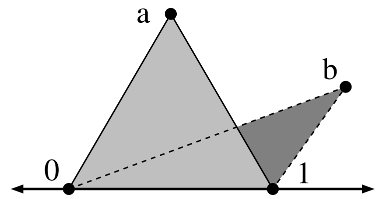

FIGURE 8. To compute the dilatation of affine maps between triangles, place both triangles with one edge $[0,1]$ that is fixed by the map, and opposite vertices a, b. The affine map has the form $z \to \alpha z + \beta \overline{z}$. Since $0, 1$ are fixed, we can solve for α, β and this gives $|\mu| = |\beta/\alpha| = |(b-a)/(b-\overline{a})|$. This is bounded below 1 iff the angles of the triangle with vertices $0, 1, b$ are bounded away from zero.

have length $\leq \delta/10$. Then A has a topological triangulation such that each triangle T in the triangulation can be mapped to a equilateral triangle by a C-quasiconformal map that multiplies arclength on each side of T.

Proof. Use the logarithm map (and a rescaling) lift the partition of ∂B to a partition of the ∂S where $S = \{x + iy : 0 < y < 2\}$. The resulting segments all have length $\leq 1/8$ and adjacent intervals have comparable lengths, so Lemma 9.1 applies. Since the partition of S is periodic, the triangulation of S is also periodic by Lemma 9.1, and hence defines a defines a smooth triangulation of the annulus B.

By Lemma 9.3, each triangle in our triangulation of the strip can be uniformly quasiconformally mapped to an equilateral triangle by a map that multiplies arclength on each edge. Thus for two triangles sharing an edge, and mapping to equilateral triangles that share the corresponding edges, the maps agree along the common edge. Pulling this periodic dilatation back to B via exponential map preserves the size of the dilatation (since the map is conformal). We then pull the triangulation back to A via the quasiconformal map $\psi : A \to B$ given by Lemma 8.3. This gives a smooth triangulation of A and a dilatation μ on A that is uniformly bounded (since the dilatation of ψ is) and that transforms the triangulation into an equilateral triangulation under any quasiconformal map of A that has dilatation μ on A by Lemma 9.2. $\square$

We will also want to bound the sizes of the triangles produced in the previous lemma. We will do this using estimates of harmonic measure and the hyperbolic metric.

Remark 9.5. If Ω is a simply connected domain then the hyperbolic metric in Ω satisfies the well known estimate

$$\frac{|dz|}{4 \cdot \operatorname{dist}(z, \partial\Omega)} \leq d\rho \leq \frac{|dz|}{\operatorname{dist}(z, \partial\Omega)}.$$

See, e.g., equation (I.4.15) of [GM05]. More generally, we have

$$d\rho \simeq \frac{|dz|}{\mathrm{dist}(z, \partial\Omega)}$$

for multiply connected domains with uniformly perfect boundaries. A set X is uniformly perfect if there is a constant $M < \infty$ so that every $0 < r <$ diameter(X) and every $x \in X$ there is a $y \in X$ with $r/M \le |x-y| \le r$. All round annuli $B = \{1 < |z| < 1+\delta\}$ considered here have this property with uniform M.

Lemma 9.6. *Suppose $S = \{x+iy : 0 < y < 1\}$ and I is an arc on the bottom edge of S with $\ell(I) \le 1/2$. Suppose $\epsilon > 0$ and $z = x+iy \in S$ with $\epsilon \cdot \mathrm{dist}(x, I) \le y \le \min(\frac{1}{2}, \ell(I)/\epsilon)$. Then the harmonic measure of I in S with respect to z satisfies $\omega(z, I, S) \ge \delta(\epsilon) > 0$.*

Proof. Let T be the right isosceles triangle with hypotenuse I. See Figure 9. Then the harmonic measure of I in S with respect to a point in T is greater than its harmonic measure in the square Q with base I, and the latter is easily checked to be $\ge 1/4$ in T. Moreover, our conditions imply z is a bounded hyperbolic distance (in S) from T, with a bound depending only on ϵ. Thus by Harnack's' inequality, the harmonic measure of I with respect z is comparable to $1/4$, e.g., is bounded uniformly away from zero in terms of ϵ. □

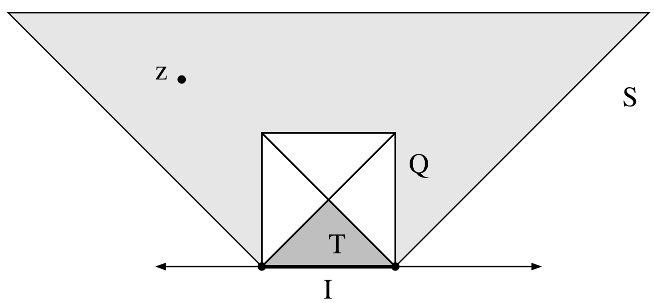

FIGURE 9. The harmonic measure of I in the square with base I is at least $1/4$ in all points of the shaded triangle. Hence it is at least $1/4$ in the strip containing the square. Thus it is $\simeq 1$ at any point within bounded hyperbolic distance of the shaded triangle.

Corollary 9.7. *The triangulation $\mathcal{T}$ of A given by Lemma 9.4 has the following properties. If $T \in \mathcal{T}$ does not touch ∂A, then*

$$\mathrm{diameter}(T) \le C \max\{\mathrm{dist}(z, \partial A) : z \in A\} = O(\mathrm{inrad}(A)),$$

for some fixed $C < \infty$. If $T \in \mathcal{T}$ has one side I on ∂A

$$\mathrm{diameter}(T) \le C\, \mathrm{diameter}(I) = O(\mathrm{gap}(A)).$$

This estimate also holds if $T \in \mathcal{T}$ has only one vertex on ∂A and this vertex is the endpoint of a sub-arc $I \subset \partial A$.

Proof. By the explicit construction given in the proof of Lemma 9.1, any interior triangle is contained in a Whitney square for the strip, and so has uniformly bounded hyperbolic diameter in the strip. Quasiconformal maps are quasi-isometries of the hyperbolic metric; for a sharp version of this, see Theorem 5.1 of [EMM04]. Therefore the hyperbolic diameter of the image triangle T in A is also uniformly bounded. Hence the standard estimate of hyperbolic metric discussed above (see Remark 9.5) shows that

$$\text{diameter}(f(T)) \leq C \, \text{dist}(T, \partial A) = O(\text{inrad}(A)).$$

On the other hand, our construction implies that if $T \subset S$ is associated to a sub-arc $I \subset \partial S$, in either of the two ways described in the current lemma, then by Lemma 9.6 we have $\omega(z, I, S) \geq \epsilon > 0$, i.e., the harmonic measure of I with respect to any point $z \in T$ is uniformly bounded above zero by a constant ϵ that only depends on the comparability constant M in the proof of Lemma 9.1. If we conformally map the strip S to the unit disk with z going to the origin, this means that I maps to an arc J on the unit circle whose length is bounded uniformly away from zero.

Now consider the path family of arcs in $\mathbb{D}$ with both endpoints on J that separate 0 from $\mathbb{T} \setminus J$. This has modulus that is bounded away from zero, since the length of J is bounded below. By the conformal invariance of modulus, the corresponding family in the strip S has modulus bounded away from zero, and by quasi-invariance so does the image of this family in A. Now suppose by way of contradiction that $\text{dist}(f(z), f(I)) \neq O(\text{diameter}(f(I)))$. Then the modulus of this family would be small: this can be seen by comparing it to the modulus of the paths connecting the two boundary components of a round annulus with inner boundary a circle of radius $\text{diameter}(f(I))$ and outer boundary a circle of radius $\text{dist}(f(z), f(I))$. This is a contradiction, and thus we conclude that $\text{dist}(z, f(I)) \leq M \cdot \text{diameter}(f(I))$ for some fixed $M < \infty$, as desired.

□

10. Triangulating Domains

In Section 10 we prove Theorem B following the inductive approach described in the Introduction. We start our construction of an equilateral triangulation of a planar domain D with the following lemma for surrounding a compact set with well separated contours.

Lemma 10.1. *Given a compact set $K \subset \mathbb{C}$, there are sets Γ_n so that for all $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ we have*

1. *each Γ_n is made up of a finite number of axis-parallel, simple polygons,*
2. *each Γ_n separates K from ∞ and separates Γ_{n+1} from ∞,*
3. $16^{-n} \leq \text{dist}(z, K) \leq 3 \cdot 16^{-n}$ *for every* $z \in \Gamma_n$,
4. $d_n = \text{dist}(\Gamma_n, \Gamma_{n+1}) \geq 13 \cdot 16^{-n-1}$,
5. *different connected components of* Γ_n *are at least distance* $2 \cdot 16^{-n-1}$ *apart.*

Proof. Let D be the unbounded connected component of $\mathbb{C} \setminus K$. This is an unbounded domain with compact boundary contained in K. For $n = 1, 2, 3, \dots$, let D_n be the union of all (closed) squares in $\mathcal{W}$ that intersect $\{z \in D : \mathrm{dist}(z, \partial D) \le 16^{-n}\}$. Each chosen square has distance $\le 16^{-n}$ from ∂D, so by (9.1), all the chosen squares have side lengths between 16^{-n-2} and 16^{-n}. Let $\Gamma_n = \partial D_n \cap D = \partial D_n \setminus \partial D$. Then Γ_n is a union of axis-parallel polygonal curves and each segment in Γ is on the boundary of a square not in D_n and therefore

$$16^{-n} \le \mathrm{dist}(z, \partial D)$$

for every $z \in \Gamma_n$. See Figure 10.

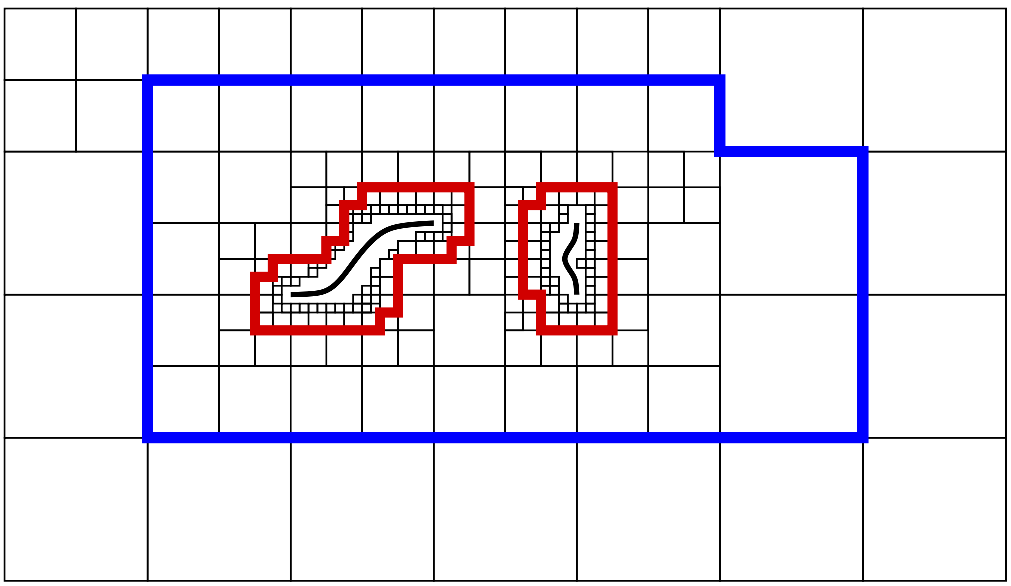

FIGURE 10. An example of a Whitney decomposition of the complement of a compact set K. By using boundaries of unions of Whitney boxes, we can create polygonal contours that surround K at approximately constant distance.

On the other hand, every segment in Γ is on the boundary of a square Q inside D_n, and hence for every $z \in \Gamma_n$ we have

$$\mathrm{dist}(z, \partial D) \le 16^{-n} + \mathrm{diameter}(Q) \le 16^{-n} + \sqrt{2} \cdot 16^{-n} < 3 \cdot 16^{-n}.$$

Thus (3) holds. To prove (4), note that

$$\mathrm{dist}(\Gamma_n, \Gamma_{n+1}) \ge 16^{-n} - 3 \cdot 16^{-n-1} \ge 13 \cdot 16^{-n-1}.$$

It remains to prove (5). If a connected component of Γ_n is not a simply polygon, is because there is a point $x \in \Gamma_n$ so that exactly two squares Q_1, Q_2 hitting $\{\mathrm{dist}(z, \partial D) = 16^{-n}\}$ both contain x as corners, but these two squares do not share edge, i.e., Γ_n looks like a cross at

x. We can replace the cross by two disjoint arcs passing through the centers of Q_1, Q_2, as shown in Figure 11. Doing this (at most finitely often) makes each connected component of Γ_n a simple polygon, every segment of which has length $\geq 2 \cdot 16^{-n-1}$.

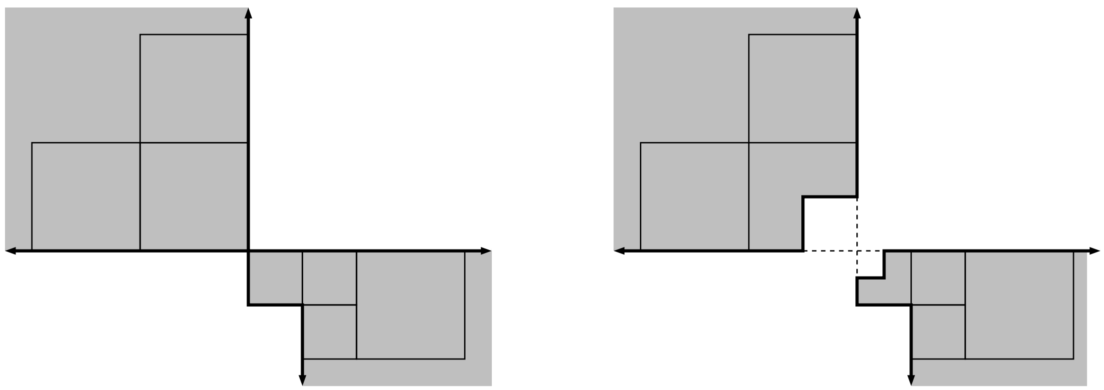

FIGURE 11. We can assume components of Γ_n are simple curves by removing any self-intersections at a point x as shown. The distance between the new curves is at least half the side length of the smaller square Q hitting x; by our estimates $\ell(Q) \geq \frac{1}{4}16^{-n}$.

Finally, any decomposition square that is adjacent to Γ_n contains a point at distance $\geq 16^{-n}$, for otherwise it would be contained in the interior of D_n and every surrounding square would hit D_n. Hence such a square has side length $\geq \frac{1}{4} \cdot 16^{-n}$. Since any two distinct components of Γ_n are separated by a collection of such squares, the two components are separated by at least $\frac{1}{4} \cdot 16^{-n}$. If the modification in the last paragraph creates two separate components, then these components are at least $\frac{1}{8} \cdot 16^{-n} = 2 \cdot 16^{-n-1}$ apart. □

We will build the desired triangulation using an inductive construction. The first step is given by the following lemma.

Lemma 10.2. *For any $\epsilon > 0$ there is a (finite) equilateral triangulation $\mathcal{T}_0$ of the Riemann sphere so that*

1. *every triangle has spherical diameter $< \epsilon$,*
2. *the part of the triangulation contained in the unit disk is the conformal image of a Euclidean equilateral triangulation of some equilateral grid polygon under a conformal map f with $\frac{1}{2} \leq |f'| \leq 2$.*

Proof. The four sides of a equilateral tetrahedron give an equilateral triangulation of the sphere. By repeated dividing each Euclidean triangle into four smaller equilateral triangles, we may make every triangle on the sphere as small as we wish. If we normalize so that one side of the original tetrahedron covers a large disk around the origin, then the second condition above is also satisfied. See Figures 12 and 13. □

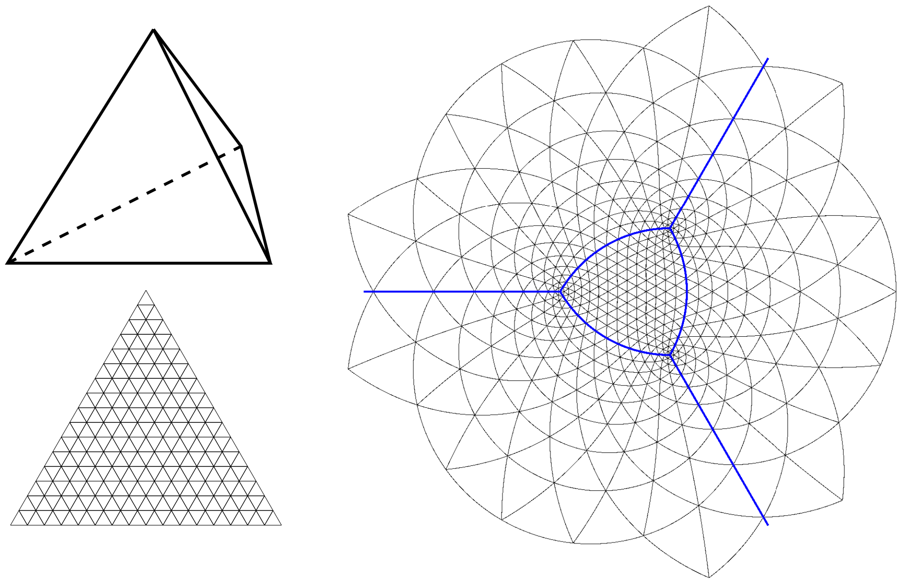

FIGURE 12. An equilateral tetrahedron with the flat metric on each side can be conformally mapped to the sphere by the uniformization theorem. Here we plot part of the image in the plane: the thick edges are the images of the edges of the tetrahedron, and the triangulation is invariant under reflection in these edges. The center region is a Reuleaux triangle with interior angles of 120° (each edge is a circular arc centered at the opposite vertex). See Figure 13 for the same triangulation drawn on a sphere.

Proof of Theorem B. Without loss of generality we may apply a Möbius transformation so that $\infty \in D$ and $K = \partial D$ is compact and contained inside $D(0, 1/16) \subset \mathbb{D}$. We are going to inductively create a sequence of (finite) equilateral triangulations $\{\mathcal{T}_n\}$ of the sphere so that $\mathcal{T}_n$ satisfies the estimate (1.1) for points in D that are distance $\geq 16^{-n}$ from ∂D. The desired triangulation of D will be the limit of these triangulations of the sphere.

In general, let $\{\Gamma_n\}_1^\infty$ be the polygonal contours inside D surrounding ∂D. Let Ω_n be the union of bounded complementary components of Γ_n, i.e., the points separated from ∞ by Γ_n. For $n \geq 1$, choose $a_n > 0$ so small that any C-quasiconformal map $\varphi : \mathbb{C} \to \mathbb{C}$ (with C as in Lemma 9.4) with dilatation supported on a set of area $\leq a_n$ and normalized to fix 0 and 1 satisfies $|\phi(z) - z| \leq 16^{-n-2}$ for all $z \in \mathbb{D}$. In particular it moves points near ∂D very little. We can do this since as the area of the support tends to zero, the map φ must tend uniformly on compact sets to a C-quasiconformal map with dilatation zero almost everywhere and fixing $0, 1$, i.e., it tends to the identity map. Next, choose $\epsilon_n > 0$ so small that the neighborhood

$$U_n = \{z : \mathrm{dist}(z, \Gamma_n) \leq N \cdot \epsilon_n\}$$

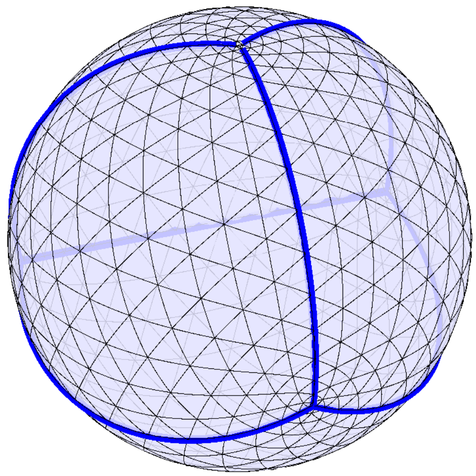

FIGURE 13. The equilateral triangulation from Figure 12 projected stereographically onto the sphere.

has area $\leq a_n$, where N will be a fixed number chosen below using Lemma 8.2. We will also assume that $\epsilon_n \ll \epsilon_{n-1}$ and $\epsilon_n < 16^{-n-2}$. At the end of the proof we may take ϵ_n even smaller, in order to ensure all the elements of our equilateral triangulation are sufficiently small.

We proceed by induction. Our induction hypothesis is that we have an equilateral triangulation $\mathcal{T}_{n-1}$ of the sphere, so that

(1) elements satisfy (1.1) for points $z \in D$ more than distance 16^{-n} from ∂D,
(2) the remaining triangles have diameter $\leq \epsilon_n$,
(3) the part of the triangulation within $Y_n = \{z \in \mathbb{C} : \mathrm{dist}(z, \Omega_n) < 16^{-n}\}$ (including the points inside Ω_n) is the image of a equilateral grid polygon with side lengths ϵ_n under a conformal map with derivative $\frac{1}{2} \leq |f'| \leq 2$.

Roughly speaking, the triangulation $\mathcal{T}_n$ equals the previous triangulation $\mathcal{T}_{n-1}$ outside Γ_n and equals an Euclidean equilateral triangulation inside Γ_n. However, these two triangulations need to be merged in a very thin neighborhood U_n of Γ_n and a quasiconformal correction is then applied to make the merged triangulation equilateral. The dilatation of the correction map is supported inside U_n. This neighborhood of Γ_n is chosen so thin that the correction map is close to the identity and conformal away from U_n. In particular, it is conformal and has derivative near 1 inside Y_{n+1}. The proof below will make this construction more precise.

To start the induction, we note that by Lemma 10.2, there exists a triangulation $\mathcal{T}_0$ satisfying (1)-(3) for $n = 1$: (1) and (2) are satisfied if we take ϵ in Lemma 10.2 to be at most $\min(\eta(\frac{1}{16}), \epsilon_1)$, and (3) holds since the unit disk, $\mathbb{D}$, contains all points within 1/32 of $\Omega_1 \subset D(0, 1/16)$.

Next we suppose $\mathcal{T}_{n-1}$ exists with the properties listed above. By the exterior region to U_n we mean the points not separated from ∞ by U_n, and by the interior region we mean the points that are separated from ∞ by U_n. Let V_n be the union of elements of the triangulation $\mathcal{T}_{n-1}$ that hit the exterior of U_n. Thus this triangulation covers the exterior of U_n.

Let $\mathcal{E}$ be some equilateral triangulation of the plane by triangles of side length 1. Let W_n be the union of triangles in $\epsilon_{n+1} \cdot \mathcal{E}$ that hit the interior region to U_n. Thus W_n covers the interior region to U_n. Note that

$$\mathrm{dist}(V_n, W_n) \simeq N\epsilon_n.$$

Thus the region between V_n and W_n consists of a union of topological annuli, one for each connected component γ of Γ_n. Suppose A is one of these annuli. We claim A is a conformal grid annulus.

The inner boundary of A is a boundary component γ' of W_n and is an equilateral grid polygon. Thus we may take f_i equal to the identity and A_i to the part of $\epsilon_{n+1} \cdot \mathcal{E}$ that lies between γ and γ'.

The outer boundary of A is a boundary component γ'' of V_n. Note that $U_n \subset Y_n$ and in fact

$$\mathrm{dist}(U_n, \partial Y_n) \geq 16^{-n} - N\epsilon_n > \max(\frac{1}{2} \cdot 16^{-n}, N\epsilon_n),$$

if ϵ_n is chosen small enough (depending on N). Thus the outer boundary γ'' of A is the conformal image of an equilateral grid polygon, and can take f_o to the inverse of this conformal map and take A_o to be the image of the triangles that lie outside γ. This completes the verification that A is a conformal grid annulus (and it is clear we may take the thickness to be as large as we wish by taking N large).

Note that the subarcs on the inner boundary of A have length exactly $\epsilon_{n+1} \ll \epsilon_n$. By induction hypothesis (3), the triangles along the outer boundary of A are images of equilateral triangles of side length ϵ_n under a map that is conformal inside Y_n and close to the identity. Thus these triangles have side length $\simeq \epsilon_n$ (the constant can be taken close to 1 if ϵ_n is small enough).

Moreover, if N in the definition of U_n is large enough, then Lemma 8.2 implies that the conformal map of A to the round annulus $B = \{1 < |z| < 1 + \delta\}$ sends the subarcs of ∂A to subarcs of ∂B that are very short compared to δ. By taking N large enough we may assume they are small enough to apply Lemma 9.4 to A.

Now we apply Lemma 9.4 to find a triangulation of A and a uniformly bounded dilation μ of A that converts the triangulation to an equilateral one. Because this triangulation uses precisely the given vertices on ∂A, doing this for every annulus between W_n and V_n gives

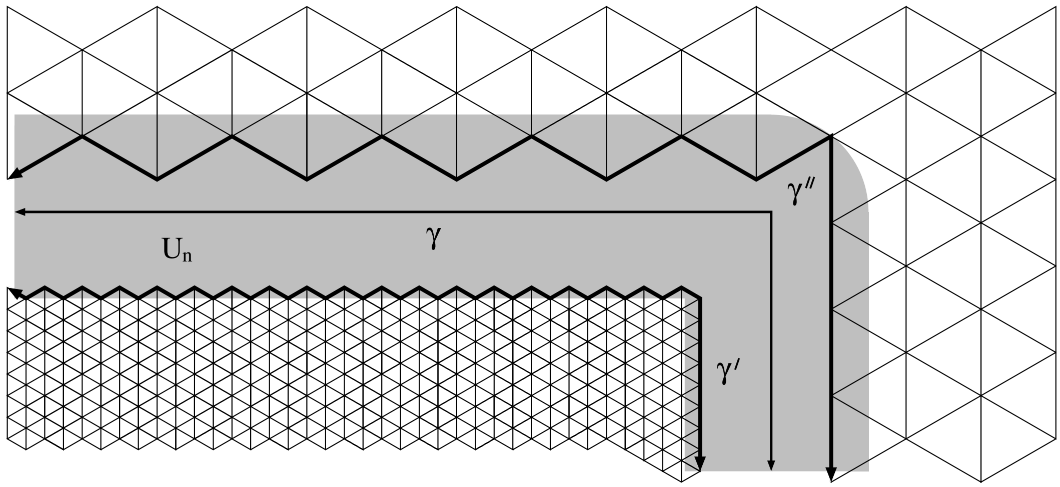

FIGURE 14. The shaded region represents part of U_n of Γ_n. The region interior to U_n is covered by a Euclidean equilateral triangulation with side lengths ϵ_{n+1} (the small triangles). The "exterior" region covered by elements of the triangulation $\mathcal{T}_{n-1}$ constructed at the previous stage (drawn here as larger equilateral triangles, but they need not be Euclidean triangles, only conformal images of such). The results of the previous section are used to triangulate the intervening region with the given boundary vertices, and then a quasiconformal correction will be applied to obtain an equilateral triangulation of the sphere. The correction map is conformal off U_n, and has derivative ≈ 1 at points more than $\frac{1}{2} \cdot 16^{-n}$ from U_n.

a triangulation of the sphere. Our triangulation has bounded degree since it has degree six except in the annular neighborhoods of Γ_n where the degree is bounded by Lemma 9.1.

Along the inner boundary of A, we are attaching the triangulation of A to a Euclidean equilateral triangulation, so the length multiplying property of the dilatation in A makes sure that the two triangulations "match up" correctly. On the outer boundary, we are matching the triangulation in A to the conformal image of a Euclidean triangulation. In this case, we can apply Lemma 8.4 to modify the conformal map to a quasiconformal map with dilatation ν in the boundary triangles outside A, to give the length multiplying property for these triangles. With this change, the triangulations on either side of the outer boundary of A match up as well. Now let $\varphi_n : \mathbb{C} \to \mathbb{C}$ be the quasiconformal map that fixes 0 and 1, and that has dilatation μ on A, has dilatation ν on the boundary triangles outside A and has dilatation zero elsewhere. Then φ_n maps this triangulation of the sphere to a equilateral triangulation of the sphere. By our choice of ϵ_n, φ_n is within $\ll 16^{-n-2}$ of the identity map, and is conformal inside W_n. In particular, the Cauchy estimates imply that $\varphi_n' \simeq 1$ within a 16^{-n-2} neighborhood of Γ_{n+1}. This completes the inductive step.

In the limit, we obtain an equilateral triangulation of D. The final step is to estimate the sizes of the triangles. By Corollary 9.7 the triangles constructed at stage n inside U_n have diameters bounded by $O(\epsilon_n)$ for some uniform constant. By construction, the triangles in W_n all have side length ϵ_{n+1}. The corresponding triangles in the final triangulation are images of these under a C-quasiconformal map of the plane fixing 0 and 1, and since such a map is Hölder continuous with uniform bounds, the size of the triangles are less than some $\delta_n = M\epsilon_n^\alpha$ for some fixed constants $M < \infty$ and $\alpha > 0$. This also holds for all triangles created at later stages of the construction so that all triangles inside Γ_n have size $\le \delta_n$, where δ_n tends to zero as quickly as we wish. Using the fact that φ_m for $m \ge n$ moves Γ_n by less than $16^{-m-2} = \text{dist}(\Gamma_n, \partial D)/256$ and summing a geometric series, we see that triangles constructed outside Γ_n (and hence at least distance 16^{-n} from ∂D) are moved by less than $\frac{1}{2} \cdot 16^{-n}$ by later steps in the construction and hence remain at least distance $\frac{1}{2} \cdot 16^{-n}$ from ∂D. Thus any triangle within distance $\frac{1}{2} \cdot 16^{-n}$ of ∂D must have diameter $\le \delta_n$. So choosing the ϵ_n small enough, we get $\delta_n \le \eta(\frac{1}{2} \cdot 16^{-n})$, which completes the proof of the theorem. □

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