

# THE DYNAMICS AND GEOMETRY OF SEMI-HYPERBOLIC RATIONAL SEMIGROUPS

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ABSTRACT. We deal with various classes of finitely generated semi-hyperbolic rational semigroups  $G$  acting on the Riemann sphere  $\hat{\mathbb{C}}$ . Our primary tool, and also an object of interest on its own, is the associated skew product map  $\tilde{f}$ . For every real number  $t \geq 0$  we define the topological pressure  $P(t)$  ascribed to the (possibly) unbounded potential  $-t \log |\tilde{f}'|$ . We show that for some non-degenerate open (in  $[0, +\infty)$ ) interval containing 0 and the Hausdorff dimension of the Julia set  $J(G)$ , the function  $t \mapsto P(t) \in \mathbb{R}$  is real-analytic.

We further show that for all  $t$  in such an interval there exist a unique  $t$ -conformal measure  $m_t$  for the map  $\tilde{f}$  and a unique Borel probability  $\tilde{f}$ -invariant measure  $\mu_t$  absolutely continuous with respect to  $m_t$ . We then show that  $\mu_t$  and  $m_t$  are equivalent measures and that the dynamical system  $(\tilde{f}, \mu_t)$  is metrically exact (thus ergodic), and that for Hölder continuous observables  $(f, \mu_t)$  satisfies the Central Limit Theorem, the Law of Iterated Logarithm and the Exponential Decay of Correlations. We prove a version of the Variational Principle for the potentials  $-t \log |\tilde{f}'|$ , and we characterize measures  $\mu_t$  as their unique equilibrium states.

In order to prove these theorems we, among others, provide a thorough understanding of “holomorphic” inverse branches for the skew product maps, we do a detailed analysis of the behavior of conformal measures  $m_t$  along the orbits of critical points, we apply Martens’ method of constructing  $\sigma$ -finite invariant measures, we use Herbert Federer’s general Vitaly classes and density theorems from [6], we develop, and heavily utilize, the method of  $K(V)$  sets originated in [4], and we push forward and develop the method of nice sets previously used for single rational functions and maps of an interval. Nice sets naturally give rise to countable alphabet iterated function systems, and this in particular enables us to make use of the thermodynamic formalism with a countable alphabet as provided in [19] and then also to apply the method of Lai-Sang Young’s towers.

Leaving dynamics and passing to geometry, we first introduce the notion of the Nice Open Set Condition, we derive several of its consequences and we prove that each \*semi-hyperbolic rational semigroup satisfying this condition is of finite type, meaning that the set of critical points of  $\tilde{f}$  lying in the Julia set of  $\tilde{f}$  is finite. Armed with this concept, we introduce the class of non-exceptional semigroups and perform a full multifractal analysis of equilibrium states  $\mu_t$ . In particular, we show that the corresponding multifractal spectrum is non-trivial for every non-exceptional totally non-recurrent rational semigroup satisfying the Nice Open Set Condition. We show here that exceptional semigroups are exceptional indeed. We provide their full characterization proving, among others, that each element of an exceptional semigroup must be a critically finite rational function with a parabolic orbifold.

Still dealing with geometry, we exhibit a striking dichotomy between equilibrium measures  $\mu_t$  and generalized Hausdorff measures in the spirit of [27].

Finally, we settle a long standing problem in the theory of rational semigroups by proving that for our class of semigroups the Hausdorff dimension of each fiber Julia set is strictly smaller than the Hausdorff dimension of the global Julia set of the semigroup.

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## 1. INTRODUCTION

A rational semigroup  $G$  is a semigroup generated by a family of non-constant rational maps  $g : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ , where  $\hat{\mathbb{C}}$  denotes the Riemann sphere, with the semigroup operation

being functional composition. For a rational semigroup  $G$ , we set

$$F(G) := \left\{ z \in \hat{\mathbb{C}} : G \text{ is normal in a neighborhood of } z \right\}$$

and

$$J(G) := \hat{\mathbb{C}} \setminus F(G).$$

$F(G)$  is called the Fatou set of  $G$  and  $J(G)$  is called the Julia set of  $G$ . If  $G$  is generated by a family  $\{f_i\}_i$ , then we write  $G = \langle f_1, f_2, \dots \rangle$ .

The work on the dynamics of rational semigroups was initiated by Hinkkanen and Martin ([11], [12]), who were interested in the role of the dynamics of polynomial semigroups while studying various one–complex–dimensional moduli spaces for discrete groups, and by F. Ren’s group ([57]), who studied such semigroups from the perspective of random complex dynamics. The theory of the dynamics of rational semigroups on  $\hat{\mathbb{C}}$  has developed in many directions since the 1990s ([11, 57, 12, 29, 30, 31, 32, 33, 34, 35, 36, 44, 37, 38, 39, 40, 41]).

Since the Julia set  $J(G)$  of a rational semigroup  $G$  generated by finitely many elements  $f_1, \dots, f_u$  has backward self–similarity, i.e.,

$$(1.1) \quad J(G) = f_1^{-1}(J(G)) \cup \dots \cup f_u^{-1}(J(G))$$

(see [31]), it can be viewed as a significant generalization and extension of both, the theory of iteration of rational maps (see [22]), and conformal iterated function systems (see [19]). For example, the Sierpiński gasket can be regarded as the Julia set of a rational semigroup. The theory of the dynamics of rational semigroups borrows and develops tools from both of these theories. It has also developed its own unique methods, notably the skew product approach (see [31, 32, 33, 34, 37, 38, 39, 40, 44], and [45]). We remark that by (1.1), the analysis of the Julia sets of rational semigroups somewhat resembles “backward iterated functions systems”, however since each map  $f_j$  is not in general injective (critical points), some qualitatively different extra effort in the cases of semigroups is needed.

The theory of the dynamics of rational semigroups is somehow related to that of the random dynamics of rational maps. Especially, if one looks at separated fiber Julia sets  $J_\omega$ . Similarities and differences of these two theories become particularly transparent in Section 16 where we compare the Hausdorff dimension of those fiber Julia sets  $J_\omega$  and the Hausdorff dimension of the global Julia set  $J(G)$ .

In this paper, we investigate the dynamics and geometry of  $t$ –conformal measures  $m_t$  and equilibrium states  $\mu_t$ , equivalent to  $m_t$ , of the skew product map

$$\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \longrightarrow \Sigma_u \times \hat{\mathbb{C}}$$

and geometric potentials on the Julia set  $J(\tilde{f})$ , i.e. ones of the form

$$J(\tilde{f}) \ni \xi \longmapsto -t \log |\tilde{f}'(\xi)| \in \mathbb{R}$$

where  $t \geq 0$  is a fixed parameter. Our starting point is the paper [46] and we go far beyond. We introduce the class of totally non–recurrent (TNR) C–F balanced rational semigroups of finite type; in particular these satisfy the Nice Open Set Condition of [46]. We develop the ergodic theory, stochastic properties, and geometry of measures  $m_t$  and  $\mu_t$ .

Our key concept is the one of nice sets, originally introduced in [28] and extensively used, among others in [25]. It permits us to build sufficiently rich symbolic dynamics of the skew

product map  $\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \rightarrow \Sigma_u \times \hat{\mathbb{C}}$  to utilize extensively the results and methods of the countable alphabet theory of subshifts of finite type, also known as topological Markov chains. Most notably those developed in [19], [17], and [54] and [55]. As in [46] our main technical tool is that of “holomorphic” inverse branches; in quotations because they live on the Cartesian product  $\Sigma_u \times \hat{\mathbb{C}}$ . We however define them more carefully and more precisely than in [46], and we make a more refined use of them than in [46]. In particular they are instrumental in forming nice families of  $\tilde{f}$  and are members of the graph directed Markov system (in the sense of [19]) generated by these families.

We deal with  $t$ -conformal measures  $m_t$  and equilibrium states  $\mu_t$  by bringing up and elaborating on the refined tool of Vitali relations due to Federer (see [6]). This tool is needed basically because, unlike for true conformal maps, the “holomorphic” inverse branches of iterates of  $\tilde{f}$ , enormously distort the balls in the product space  $\Sigma_u \times \hat{\mathbb{C}}$ . In fact, they are more like affine maps with two different contracting factors. We do rely here heavily on deep results from [6]. Another tool, already employed in [51] and subsequent papers of the third named author, is the Marco–Martens method of producing  $\sigma$ -finite invariant measures absolutely continuous with respect to a given quasi-invariant measure. It was considerably refined and generalized in [46]. We apply and develop this to construct the  $\tilde{f}$ -invariant measures  $\mu_t$ . These are first constructed as merely  $\sigma$ -finite measures and proven to be finite only much later after preparing the machinery of nice families and symbolic representation.

Using the, already several times mentioned, symbolic dynamics we prove that the  $\tilde{f}$ -invariant measures  $\mu_t$  are the unique equilibrium states of the potentials  $-t \log |\tilde{f}'(\xi)|$  in a very classical sense. These are the only measures that maximize the free energy functions associated to these potentials (see below in this introduction for a more precise statement) and the supremum is equal to the topological pressure  $P(t)$ .

The methods we develop in this paper are sufficiently strong and refined to allow us to solve a long standing open problem in the theory of iteration of rational semigroups asking about the relation of the Hausdorff dimension of the fiberwise Julia sets  $J_\omega$  and the global Julia set  $J(G)$ . We show that for C–F balanced TNR rational semigroups of finite type the Hausdorff dimension of every fiberwise Julia set  $J_\omega$  is smaller than the Hausdorff dimension of the global Julia set  $J(G)$ . We prove a little bit more in this respect.

Now we shall describe the main results of [46] and the current paper. We start with [46]. Its main results are comprised in the following.

**Theorem 1.1.** *Let  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  be a multi-map. Let  $G = \langle f_1, \dots, f_u \rangle$ . Suppose that*

- (1) *There exists an element  $g$  of  $G$  such that  $\deg(g) \geq 2$ ,*
- (2) *Each element of  $\text{Aut}(\hat{\mathbb{C}}) \cap G$  (if this is not empty) is loxodromic,*
- (3)  *$G$  is semi-hyperbolic, and*
- (4)  *$G$  satisfies the Nice Open Set Condition.*

*Then, we have the following.*

- (a)  *$J(G) \cap \text{PCV}(G)$  is nowhere dense in  $J(G)$  and, for each  $t \geq 0$ , the function*

$$z \mapsto P_z(t) \in \mathbb{R}$$

is constant throughout a neighborhood of  $J(G) \setminus \text{PCV}(G)$  in  $\hat{\mathbb{C}}$ . Denote this constant by  $P(t)$ .

(b) The function

$$[0, +\infty) \ni t \mapsto P(t) \in \mathbb{R}$$

has a unique zero. This zero is denoted by  $h = h_f$ .

(c) There exists a unique  $|\tilde{f}'|^h$ -conformal measure  $\tilde{m}_h$  for the map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ .

(d) Let  $m_h := \tilde{m}_h \circ p_2^{-1}$ . Then there exists a constant  $C \geq 1$  such that

$$C^{-1} \leq \frac{m_h(B_s(z, r))}{r^h} \leq C$$

for all  $z \in J(G)$  and all  $r \in (0, 1]$ .

(e)

$$h_f = \text{HD}(J(G)) = \text{PD}(J(G)) = \text{BD}(J(G)),$$

where HD, PD, BD denote the Hausdorff dimension, packing dimension, and box dimension, respectively, with respect to the spherical distance in  $\hat{\mathbb{C}}$ .

We denote this common value by  $h_G$ ; it depends only on the semigroup  $G$  and is independent of the set of generators (satisfying conditions (1)–(4) above) used to for the skew product map  $\tilde{f}$ . Moreover, for each  $z \in J(G) \setminus \text{PCV}(G)$ , we have

$$h_G = T_f(z) = t_0(f) = S_G(z) = s_0(G).$$

(f) For every  $t \geq 0$  there exists a  $|\tilde{f}'|^t$ -conformal measure  $m_t$  for the map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ .

(g) Let  $H^h$  and  $P^h$  be the  $h$ -dimensional Hausdorff measure and  $h$ -dimensional packing measure respectively. Then, all the measures

$$H^h, P^h, \text{ and } m_h$$

are mutually equivalent with Radon–Nikodym derivatives uniformly separated away from zero and infinity.

(h)

$$0 < H^h(J(G)), P^h(J(G)) < \infty.$$

(i) There exists a unique Borel probability  $\tilde{f}$ -invariant measure  $\tilde{\mu}_h$  on  $J(\tilde{f})$  which is absolutely continuous with respect to  $\tilde{m}_h$ . The measure  $\tilde{\mu}_h$  is metrically exact, hence ergodic, and equivalent with  $\tilde{m}_h$ .

**Definition 1.2.** A rational semigroup  $G$  satisfying conditions (1)–(3) from Theorem 1.1 is called *\*semi-hyperbolic*.

We want to emphasize now, and will repeat it within the main body of our manuscript, that being semi-hyperbolic and \*semi-hyperbolic does not depend on the choice of generators but on the semigroup alone.

We will now describe the main results of the current paper. As we have already mentioned, we introduce in it the class of totally non-recurrent (TNR) C–F balanced rational semigroups of finite type. Loosely speaking these adjectives respectively mean that the closure of the postcritical set is disjoint from the critical set and the part of the postcritical

set lying in the Fatou set is at a positive distance from the Julia set. In Section 17 we give some examples of totally non-recurrent (TNR) C-F balanced rational semigroups of finite type. Many of our main results need only some parts of these assumptions but we do not discern them here for the sake of ease of exposition. We refer the reader to actual theorems in the body of the paper for most adequate assumptions.

We prove that there exists an open interval  $\Delta_G^* \supset [0, h]$  for which, among others, the following theorems hold.

**Proposition 1.3.** *If  $G$  is a \*semi-hyperbolic rational semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ , then the function  $t \mapsto P(t)$ ,  $t \geq 0$ , has the following properties.*

- (a) *For every  $t \geq 0$  we have that  $P(t) \in (-\infty, +\infty)$  and  $P(0) \geq \log 2 > 0$ .*
- (b) *The function  $[0, +\infty) \mapsto P(t)$  is strictly decreasing and Lipschitz continuous. More precisely:*
- (c) *If  $0 \leq s \leq t < +\infty$ , then*

$$-\log \|\tilde{f}'\|_\infty(t-s) \leq P(t) - P(s) \leq -\alpha(t-s),$$

*where the constant  $\alpha > 0$  comes from the Exponential Shrinking Property (Theorem 3.5).*

- (d)  $\lim_{t \rightarrow +\infty} P(t) = -\infty$ .

Making a substantial use of Marco Martens's method, which originated in [15] and was explored for example in [46], we prove the following.

**Theorem 1.4.** *Let  $G$  be a \*semi-hyperbolic rational semigroup. If  $t \in \Delta_G$  then there exists a unique, up to a multiplicative constant, Borel  $\sigma$ -finite  $\tilde{f}$ -invariant measure  $\mu_t$  on  $J(\tilde{f})$  which is absolutely continuous with respect to  $m_t$ . In addition, the measure  $\mu_t$  is weakly metrically exact and equivalent to  $m_t$ , in particular it is ergodic.*

In Section 6, entitled Nice Sets (Families), we explore in detail one of the most important tools for us in the current paper. It is commonly referred to as nice sets or nice families. It has been introduced in [28], and extensively used, among others in [25]. We adopt this concept to the setting of rational semigroups. The absolutely first fact needed about them is their existence. It is by no means obvious and we devote the whole Section 6 for this task. In the existing proofs for ordinary conformal systems, i.e. cyclic semigroups, the concept of connectivity of the phase space, usually  $\mathbb{C}$  or  $\hat{\mathbb{C}}$ , plays a substantial role. In our present setting of the skew product map

$$\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \longrightarrow \Sigma_u \times \hat{\mathbb{C}},$$

the phase space is “highly” not connected. In order to overcome this difficulty we define the concept of connected families of arbitrary sets. These have sufficiently many properties of ordinary connected sets, for example one can speak of connected components of any family of sets, to allow for a proof of the existence of nice families. As a matter of fact, we do not even use the topological concept of connected subsets of the Riemann sphere  $\hat{\mathbb{C}}$ . Our main theorem of this section is the following.



**Theorem 1.5.** *Let  $G = \langle f_1, \dots, f_u \rangle$  be a TNR semigroup. Fix  $R \in (0, R_*(G))$ . Fix also  $\kappa \in (1, 2)$ . Let*

$$\text{Crit}_*(f) \subset S \subset J(G) \setminus B(\text{PCV}(G), 8R)$$

*be a finite aperiodic set. Then for every  $r \in (0, R]$  small enough there exists*

$$\mathcal{U}_S(\kappa, r) = \{U_s(\kappa, r)\}_{s \in S},$$

*a nice family of sets for  $\tilde{f}$ , associated to the set  $S$ , such that*

(A)

$$\Sigma_u \times B_2(s, r) \subset U_s(\kappa, r) \subset \Sigma_u \times B_2(s, \kappa r)$$

*for each  $s \in S$ .*

(B) *If  $a, b \in S$ ,  $\rho \in \text{HIB}(U_b(\kappa, r))$ , and  $\tilde{f}_\rho^{-\|\rho\|}(U_b(\kappa, r)) \subset U_a(\kappa, r)$ , then*

$$\left| (f_\rho^{-1})'(z) \right| \leq \frac{1}{4}$$

*for all  $z \in B_2(b, 2R) \supset p_2(U_b(\kappa, r))$ .*

The first consequence of this theorem, which is the gate to all of its other consequences, is that it gives rise to the existence of sufficiently rich ‘‘conformal–holomorphic’’ maximal graph directed Markov system in the sense of [19]. More precisely, it gives the following.

**Theorem 1.6** (for the notation and details see Section 6 and Theorem 6.9). *If  $G$  is a TNR rational semigroup generated by a multi–map  $(f_1, \dots, f_u) \in \text{Rat}^u$  and  $\mathcal{U} = \{U_s\}_{s \in S}$  is a nice family of sets for  $\tilde{f}$ , then the family*

$$\mathcal{S}_{\mathcal{U}} := \{\tilde{f}_\tau^{-\|\tau\|} : X_{t(\tau)} \longrightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_{\mathcal{U}}}$$

*forms a graph directed system in the sense of [19]. Furthermore,*

(a) *The corresponding incidence matrix  $A(\mathcal{U})$  is then determined by the condition that*

$$A_{\tau\omega}(\mathcal{U}) = 1$$

*if and only if  $t(\tau) = i(\omega)$ .*

(b) *The limit set  $\tilde{J}_{\mathcal{U}}$  of the system  $\mathcal{S}_{\mathcal{U}}$  is contained in  $J(\tilde{f})$  and contains  $U \cap \text{Trans}(\tilde{f})$ , where  $\text{Trans}(\tilde{f})$  is the set of transitive points of  $\tilde{f} : J(\tilde{f}) \longrightarrow J(\tilde{f})$ , i.e. the set of points  $z \in J(\tilde{f})$  such that the set  $\{\tilde{f}^n(z) : n \geq 0\}$  is dense in  $J(\tilde{f})$ .*

(c) *The graph directed system  $\mathcal{S}_{\mathcal{U}}$  is finitely primitive.*

We denote by  $\mathcal{D}_{\mathcal{U}}^\infty$  the symbol space  $(\mathcal{D}_{\mathcal{U}})_{A(\mathcal{U})}^\infty$  generated by the matrix  $A(\mathcal{U})$ ; as in the case of  $\Sigma_u$  its elements (infinite sequences) start with coordinates labeled by the integer 1. Likewise  $\mathcal{D}_{\mathcal{U}}^*$  and  $\mathcal{D}_{\mathcal{U}}^n$ ,  $n \in \mathbb{N}$ , abbreviate respectively  $(\mathcal{D}_{\mathcal{U}})_{A(\mathcal{U})}^*$  and  $(\mathcal{D}_{\mathcal{U}})_{A(\mathcal{U})}^n$ ,  $n \in \mathbb{N}$ .

In addition, we denote by  $\phi_e$ ,  $e \in \mathcal{D}_{\mathcal{U}}$ , all the elements of  $\mathcal{S}_{\mathcal{U}}$ .



The next section, Section 7, entitled The Behavior of Absolutely Continuous Invariant Measures  $\mu_t$  Near Critical Points, is very technical and devoted to study the behavior of conformal measures  $m_t$  and their invariant versions  $\mu_t$  near critical points of the skew product map

$$\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \longrightarrow \Sigma_u \times \hat{\mathbb{C}}.$$

Its main outcome is Proposition 7.7 which gives a quantitative strengthening of quasi-invariance of conformal measures  $m_t$ . This is the first and only place where the hypothesis of finite type of the semigroup  $G$  is explicitly needed; it demands that the set of critical points of  $\tilde{f}$  lying in the Julia set  $J(\tilde{f})$  of  $\tilde{f}$  is finite.

Section 8, Small Pressure  $P_V^\Xi(t)$ , is still technical. We prove that the (ordinary) topological pressure of the potentials  $-t \log |\tilde{f}'|$ ,  $t \in \Delta_G^*$ , with respect to the dynamical system  $\tilde{f}$  restricted to the compact  $\tilde{f}$ -invariant set of all points whose forward iterates avoid an open neighborhood of the set of critical points of  $f$ , is smaller than  $P(t)$ . This inconspicuous looking fact is instrumental, one could even say, indispensable, in many further proofs. It intervenes for example in the proof of Lemma 8.3 which in plain words asserts that the measure  $m_t$  of the “tails” of the maximal graph directed Markov system generated by a (sufficiently good) nice family, decays exponentially fast. This fact is in turn instrumental in Sections 12.1 and 12.2, making application of Young’s Towers possible. This fact also makes the proof of Variational Principle in Section 11, so simple. It is also used in Section 16 to show that the Hausdorff dimension of fiber Julia sets  $J_\omega$  is smaller than the Hausdorff dimension of the global Julia set  $J(G)$ .

Section 9, Symbol Space Thermodynamic Formalism associated to Nice Families; Real Analyticity of the Original Pressure  $P(t)$ , brings up full fledged fruits of the existence of nice families. It forms a symbolic representation (subshift of finite type with a countable infinite alphabet) of the map generated by a nice family and develops the thermodynamic formalism of the potentials  $\zeta_{t,s}$  resulting from those of the form  $-t \log |\tilde{f}'|$  and the “first return time”  $\|\tau_1\|$ . The first most transparent of its consequences, already possible to be stated and proved in this section, is the following.

**Theorem 1.7** (Theorem 9.9). *If  $G$  be a C-F balanced TNR rational semigroup of finite type generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ , then the topological pressure function*

$$P : \Delta_G^* \longrightarrow \mathbb{R}$$

*is real-analytic.*

In Section 10, Invariant Measures:  $\mu_t$  versus  $\tilde{\mu}_t \circ \pi_U^{-1}$ ; Finiteness of  $\mu_t$ , we link the measures  $\tilde{m}_t$  and  $\tilde{\mu}_t$  of the previous section living on the symbol space with the conformal and invariant measures  $m_t$  and  $\mu_t$  living on the Julia set  $J(\tilde{f})$ . This link is given by Lemma 10.7. We translate here many results of the previous sections, expressed in the symbolic language, to the language of the actual map  $\tilde{f}$ . We eventually prove here, see Theorem 10.13, that all the measures  $\mu_t$ ,  $t \in \Delta_G^*$ , are finite, thus probability measures after normalization.

In Section 11, Variational Principle; The Invariant Measures  $\mu_t$  are the Unique Equilibrium States, we prove a full version of the classical Variational Principle for potentials

$-t \log |\tilde{f}'|$ ,  $t \in \Delta_G^*$  with respect to the dynamical system  $\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \rightarrow \Sigma_u \times \hat{\mathbb{C}}$  and we identify measures  $\mu_t$  as the only equilibrium states. More precisely, we prove the following.

**Theorem 1.8.** *If  $G = \langle f_1, \dots, f_u \rangle$  is a C–F balanced TNR rational semigroup of finite type and  $t \in \Delta_G^*$ , then*

$$\begin{aligned} \sup \left\{ h_\mu(\tilde{f}) - t \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu : \mu \in M(\tilde{f}) \right\} &= \\ &= \sup \left\{ h_\mu(\tilde{f}) - t \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu : \mu \in M_e(\tilde{f}) \right\} = P(t), \end{aligned}$$

and

$$h_{\mu_t}(\tilde{f}) - t \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu_t = P(t),$$

while

$$h_\mu(\tilde{f}) - t \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu < P(t)$$

for every measure  $\mu \in M(\tilde{f})$  different from  $\mu_t$ .

In Section 12.1, Stochastic Laws on the Symbol Space for the Shift Map Generated by Nice Families, making use of the link with symbol thermodynamic formalism of Section 9, we embed the symbol space  $\mathcal{D}_U^\infty$ , along with the shift map acting on it, into an abstract Young’s Tower (see [54] and [55]) as its first return map. We then prove the fundamental stochastic laws such as the Law of Iterated Logarithm, the Central Limit Theorem, and exponential decay of correlations, in such abstract setting.

In Section 12.2, Stochastic Laws for the Dynamical System  $(\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f}), \mu_t)$ , making use of the previous section, via the natural projection from the abstract Young’s Tower to the Julia set  $J(\tilde{f})$ , we prove in Theorem 12.6 the fundamental stochastic laws for dynamical systems  $(\tilde{f}, \mu_t)$ ,  $t \in \Delta_G^*$ , such as the Law of Iterated Logarithm, the Central Limit Theorem, and exponential decay of correlations.

Part 3 of our manuscript is devoted to study finer fractal and geometrical properties of the fiber Julia sets  $J_\omega$  and the global Julia set  $J(G)$ . Throughout this whole section we assume that  $G$  is a C–F balanced TNR rational semigroup satisfying the Nice Open Set Condition. We would like to mention that we prove at the end of Section 13 that each TNR rational semigroup satisfying the Nice Open Set Condition is of finite type.

In Section 13, entitled Nice Open Set Condition, we formulate this condition and thoroughly study it at length preparing all the tools based on this condition that we need in further sections. We would like to note that our formulation of this condition and its treatment somewhat differ from that of [46].

In Section 14, entitled Multifractal Analysis of Invariant Measures  $\mu_t$ , we provide a full multifractal analysis of equilibrium measures  $\mu_t$ ,  $t \in \Delta_G^*$ . More precisely, for every point  $(\omega, z) \in J(\tilde{f})$ , we denote

$$\bar{\chi}(\omega, z) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log |(\tilde{f}^n)'(\omega, z)| \quad \text{and} \quad \underline{\chi}(\omega, z) := \liminf_{n \rightarrow \infty} \frac{1}{n} \log |(\tilde{f}^n)'(\omega, z)|$$

and call them respectively the upper and lower Lyapunov exponents at the point  $(\omega, z)$ . If  $\underline{\chi}(\omega, z) = \bar{\chi}(\omega, z)$ , we denote the common value by  $\chi(\omega, z)$  and call it the Lyapunov exponent at  $(\omega, z)$ . Given  $\chi \geq 0$ , we define

$$K(\chi) := \left\{ (\omega, z) \in J(\tilde{f}) : \underline{\chi}(\omega, z) = \bar{\chi}(\omega, z) = \chi \right\},$$

which is actually the level set of the function  $J(\tilde{f}) \ni (\omega, z) \mapsto \chi(\omega, z)$  corresponding to its value  $\chi$ . For every  $t \in \Delta_G^*$ , let

$$D_t(\tilde{f}) := \{ \chi_{\mu_{t,q}} : q \in [0, 1] \},$$

where the measures  $\mu_{t,q}$  are introduced in the formula (14.22) by means of the temperature function  $T_t(q)$ , and  $\chi_{\mu_{t,q}}$  are the corresponding Lyapunov exponents. The main result of Section 14 is the following.

**Theorem 1.9** (Theorem 14.9). *If  $G$  is a C–F balanced TNR non-exceptional semigroup satisfying the Nice Open Set Condition, then for every  $t \in \Delta_G^* \setminus \{h_f\}$ , the set  $D_t(\tilde{f})$  is a non-degenerate interval with endpoints  $\chi_{\mu_{h_f}}$  and  $\chi_{\mu_t}$ , and the function*

$$D_t(\tilde{f}) \ni \chi \mapsto \text{HD}(p_2(K(\chi))) \in [0, 2]$$

*is real-analytic.*

Being a non-exceptional semigroup is a mild requirement meaning that either one of the conditions (a)–(g) from Proposition 14.8 holds. It is for example satisfied if  $J(\tilde{f})$  contains some non-exceptional critical points, see Proposition 14.15, in fact if and only if each element of  $G$  is a critically finite map with parabolic orbifold; see Theorem 14.14. As a preparatory result to the above theorem, which is however also interesting on its own, we prove in this section the following result.

**Theorem 1.10** (Theorem 14.2). *If  $G$  is a C–F balanced TNR semigroup satisfying the Nice Open Set Condition and  $t \in \Delta_G^*$ , then*

$$\text{HD}(\mu_t \circ p_2^{-1}) = \frac{h_{\mu_t}(\tilde{f})}{\chi_{\mu_t}} = t + \frac{P(t)}{\chi_{\mu_t}}.$$

Throughout the whole manuscript, given an integer  $q \geq 1$  and  $u > 0$  large enough, we mean by  $\log^q(u)$  the  $q$ th iteration of the natural logarithm applied to  $u$ ; for example:

$$\log^1(u) = \log(u), \quad \log^2(u) = \log(\log(u)), \quad \log^3(u) = \log(\log(\log(u))).$$

In Section 15, which is entitled Measures  $m_t$  and  $\mu_t$  versus Hausdorff Measures  $H_{u^\kappa}$  and  $H_{u^\kappa \exp(c\sqrt{\log(1/u)\log^3(1/u)})}$ , following the general scheme of [27] and [52] (see also [19]), we establish singularity and absolute continuity relations between measures  $\mu_t$  and  $m_t$  with respect to generalized measures  $H_{u^\kappa \exp(c\sqrt{\log(1/u)\log^3(1/u)})}$ , where the subscript is a gauge function. More precisely, denoting

$$g_{t,c}(u) := u^{\text{HD}(\mu_t \circ p_2^{-1})} \exp\left(c\sqrt{\log(1/u)\log^3(1/u)}\right),$$

as the ultimate theorem of this section, we prove the following.

**Corollary 1.11** (Corollary 15.5). *Let  $G$  be a C–F balanced TNR rational semigroup satisfying the Nice Open Set Condition. Assume that  $t \in \Delta_G^*$  and  $\tilde{\sigma}_t > 0$ . Then*

- (a) *If  $0 \leq c \leq 2\tilde{\sigma}_t\chi_{\mu_t}^{-1/2}$ , then  $\mu_t \circ p_2^{-1}$  and  $H_{g_{t,c}}$  on  $J(G)$  are mutually singular. In particular, the measures  $\mu_t \circ p_2^{-1}$  and  $H_{\ell^{\text{HD}(\mu_t \circ p_2^{-1})}}$  are mutually singular.*
- (b) *If  $c > 2\tilde{\sigma}_t\chi_{\mu_t}^{-1/2}$ , then  $\mu_t \circ p_2^{-1}$  is absolutely continuous with respect to  $H_{g_{t,c}}$  on  $J(G)$ . Moreover,  $H_{(\tilde{\ell}_c)_t}(E) = +\infty$  whenever  $E \subset J(G)$  is a Borel set such that  $\mu_t \circ p_2^{-1}(E) > 0$ .*

This result is a corollary for Theorem 15.4 involving the upper and lower class functions commonly used in probability theory.

In the final section of our paper, i.e. Section 16,  $\text{HD}(J(G))$  versus Hausdorff Dimension of Fiber Julia Sets  $J_\omega$ ,  $\omega \in \Sigma_u$ , we provided the solution to the long standing open problem in the theory of rational semigroups concerning the size of fiberwise Julia sets  $J_\omega$  versus the global Julia set  $J(G)$ . We show that for C–F balanced TNR rational semigroups of finite type the Hausdorff dimension of every fiberwise Julia set  $J_\omega$  is smaller than the Hausdorff dimension of the global Julia set  $J(G)$ . We also show that if  $G$  is expanding then the supremum of Hausdorff dimensions of all fiberwise Julia set  $J_\omega$ ,  $\omega \in \Sigma_u$  is smaller than the Hausdorff dimension of the global Julia set  $J(G)$ . In formal terms, we have the following.

**Theorem 1.12.** *If  $G$  is a C–F balanced TNR rational semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  and satisfying the Nice Open Set Condition, then*

$$\text{HD}(J_\omega) < h = \text{HD}(J(G))$$

for every  $\omega \in \Sigma_u$ . If in addition,  $G$  is expanding, then

$$\sup \{ \text{HD}(J_\omega) : \omega \in \Sigma_u \} < h = \text{HD}(J(G)).$$

## 2. GENERAL PRELIMINARIES ON RATIONAL SEMIGROUPS

Let  $u \in \mathbb{N} = \{1, 2, 3, \dots\}$ . Let  $\text{Rat}$  be the set of all rational maps on the Riemann sphere  $\hat{\mathbb{C}}$ . In this paper, an element of  $\text{Rat}^u$  is called a multi-map. Let  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  be a multi-map and let

$$G = \langle f_1, \dots, f_u \rangle$$

be the rational semigroup generated by  $\{f_1, \dots, f_u\}$ . We then also say that  $G$  is generated by the multi-map  $f = (f_1, \dots, f_u)$ . To be sure, this means that

$$G = \left\{ f_{\omega_n} \circ f_{\omega_{n-1}} \circ \dots \circ f_{\omega_2} \circ f_{\omega_1} : n \geq 1 \text{ and } (\omega_1, \omega_2, \dots, \omega_n) \in \{1, \dots, u\}^n \right\}.$$

Let

$$\Sigma_u := \{1, \dots, u\}^{\mathbb{N}}$$

be the space of all one-sided sequences of  $u$ -symbols endowed with the product (Tichonov) topology.  $\Sigma_u$  becomes then a compact metrizable space. There are plenty of metrics on  $\Sigma_u$  compatible with this topology. For our purposes in this paper we choose only one of

them, one which is quite natural. Indeed, fix  $\vartheta \in (0, 1)$  and denote by  $|\cdot, \cdot|_{\vartheta}$  the metric on  $\Sigma_u$  defined by the following formula.

$$(2.1) \quad |\omega, \tau|_{\vartheta} := \vartheta^{\min\{n \geq 1: \omega_n \neq \tau_n\}-1},$$

with the usual convention that  $\vartheta^{\infty} = 0$ . We then define the metric  $\| \cdot, \cdot \|_{\vartheta}$  on  $\Sigma_u \times \hat{\mathbb{C}}$  as equal to  $\max\{|\cdot, \cdot|_{\vartheta}, |\cdot - \cdot|\}$ ; precisely:

$$\|(\omega, z), (\tau, w)\|_{\vartheta} := \max\{|\omega, \tau|_{\vartheta}, |z - w|\}.$$

We will frequently omit the subscript  $\vartheta$  writing just  $\|(\omega, z), (\tau, w)\|$  for simplicity. We denote the ball with radius  $r > 0$  centered at some point  $\omega \in \Sigma_u$  and generated by the metric  $|\cdot, \cdot|_{\vartheta}$  as  $B_1(\omega, r)$ . Likewise, we denote the ball with radius  $r > 0$  centered at some point  $z \in \hat{\mathbb{C}}$  and generated by the metric  $|\cdot - \cdot|$  as  $B_2(z, r)$ . Then  $B((\omega, z), r)$  denotes the ball centered at  $(\omega, z)$  with radius  $r$  defined with respect to the metric  $\| \cdot, \cdot \|_{\vartheta}$ . Obviously

$$B((\omega, z), r) = B_1(\omega, r) \times B_2(z, r).$$

We will also need annuli. We recall their standard definition. Given  $w \in \hat{\mathbb{C}}$  and two radii  $0 < r \leq R$ , we set

$$A(w; r, R) := B_2(w, R) \setminus B_2(w, r) = \{z \in \hat{\mathbb{C}} : r \leq |z - w| < R\}.$$

In general, denote by  $\text{diam}(A)$  the diameter of a set  $A$  with respect to the metric on the space the set  $A$  is contained in. This is just the supremum of distances between points in  $A$ . Usually in this paper such metric space will be either  $\hat{\mathbb{C}}$ ,  $\Sigma_u$ , or  $\Sigma_u \times \hat{\mathbb{C}}$ . Also, for  $A, B$ , any two subsets of a metric space put

$$\text{dist}(A, B) := \inf\{\text{dist}(a, b) : a \in A, b \in B\}.$$

Let  $G$  be a rational semigroup and let  $F$  be a subset of  $\hat{\mathbb{C}}$ . We set

$$G(F) := \bigcup_{g \in G} g(F)$$

and

$$G^{-1}(F) := \bigcup_{g \in G} g^{-1}(F).$$

Moreover, we set

$$G^* := G \cup \{\text{Id}\},$$

where  $\text{Id}$  denotes here the identity map on  $\hat{\mathbb{C}}$ . We then define analogously the sets  $G^*(F)$  and  $(G^*)^{-1}(F)$ . In fact

$$G^*(F) = F \cup G(F) \quad \text{and} \quad (G^*)^{-1}(F) = F \cup G^{-1}(F).$$

Let  $\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \rightarrow \Sigma_u \times \hat{\mathbb{C}}$  be the skew product map associated with  $f = (f_1, \dots, f_u)$ . It is given by the formula

$$\tilde{f}(\omega, z) := (\sigma(\omega), f_{\omega_1}(z)),$$

where  $(\omega, z) \in \Sigma_u \times \hat{\mathbb{C}}$ ,  $\omega = (\omega_1, \omega_2, \dots)$ , and  $\sigma : \Sigma_u \rightarrow \Sigma_u$  denotes the one-sided shift map, i.e.

$$\sigma((\omega_n)_{n=1}^{\infty}) = (\omega_{n+1})_{n=1}^{\infty}.$$

We denote by  $p_1 : \Sigma_u \times \hat{\mathbb{C}} \rightarrow \Sigma_u$  the projection onto  $\Sigma_u$  and  $p_2 : \Sigma_u \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  the projection onto  $\hat{\mathbb{C}}$ . That is,

$$p_1(\omega, z) = \omega \quad \text{and} \quad p_2(\omega, z) = z.$$

Under the canonical identification  $p_1^{-1}\{\omega\} \cong \hat{\mathbb{C}}$ , each fiber  $p_1^{-1}\{\omega\}$  is a Riemann surface which is isomorphic to  $\hat{\mathbb{C}}$ .

Let

$$\Sigma_u^* := \bigcup_{n=0}^{\infty} \{1, \dots, u\}^n$$

be the family of all finite words over the alphabet  $\{1, 2, \dots, u\}$  with the convention that  $\{1, \dots, u\}^0$  is the singleton consisting of the empty word denoted in the sequel by  $\emptyset$ . For every  $\tau \in \Sigma_u^*$ , we denote by  $|\tau|$  the only integer  $n \geq 0$  such that  $\tau \in \{1, \dots, u\}^n$ . For every  $\tau \in \Sigma_u$  we set  $|\tau| = \infty$ . In addition, for every  $\tau = (\tau_1, \tau_2, \dots) \in \Sigma_u^* \cup \Sigma_u$  and  $n \in \mathbb{N}$  with  $n \leq |\tau|$ , we set

$$\tau|_n := (\tau_1, \tau_2, \dots, \tau_n) \in \Sigma_u^*.$$

Furthermore, for every set  $\Gamma \subset \Sigma_u^* \cup \Sigma_u$ , we define

$$\Gamma|_n := \{\omega|_n : \omega \in \Gamma\}.$$

For every  $\tau \in \Sigma_u^*$ , we denote

$$\hat{\tau} = \tau|_{|\tau|-1}, \quad \tau_* := \tau|_{|\tau|},$$

and

$$(2.2) \quad [\tau] := \{\omega \in \Sigma_u : \omega|_{|\tau|} = \tau\}.$$

We call  $[\tau]$  the cylinder generated by  $\tau$ . Furthermore, for every  $\omega \in \Sigma_u^* \cup \Sigma_u$  and all  $a, b \in \mathbb{N}$  with  $a < b \leq |\omega|$ , we set

$$\omega_a^b := (\omega_a, \dots, \omega_b) \in \Sigma_u^*.$$

For any two words  $\omega, \tau \in \Sigma_u^*$ , we say that  $\omega$  and  $\tau$  are comparable if either

$$(1) \quad |\tau| \leq |\omega| \text{ and } \omega|_{|\tau|} = \tau; \text{ equivalently } \omega \in [\tau],$$

or

$$(2) \quad |\omega| \leq |\tau| \text{ and } \tau|_{|\omega|} = \omega; \text{ equivalently } \tau \in [\omega].$$

We say that  $\omega, \tau$  are incomparable if they are not comparable. By  $\tau\omega \in \Sigma_u^*$  we denote the concatenation of the words  $\tau$  and  $\omega$ . We may now rewrite formula (2.1) in the form.

$$(2.3) \quad |\omega, \tau|_{\vartheta} = \vartheta^{|\omega \wedge \tau|},$$

where  $\omega \wedge \tau$  is the longest initial common block of both  $\omega$  and  $\tau$ . For each  $\omega = (\omega_1, \dots, \omega_n) \in \Sigma_u^*$ , let

$$f_\omega := f_{\omega_n} \circ \dots \circ f_{\omega_1} : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}.$$

For each  $n \in \mathbb{N}$  and  $(\omega, z) \in \Sigma_u \times \hat{\mathbb{C}}$ , we set

$$(\tilde{f}^n)'(\omega, z) := f'_\omega(z).$$

For each  $\omega \in \Sigma_u$  we define

$$J_\omega := \left\{ z \in \hat{\mathbb{C}} : \{f_\omega\}_{\omega \in \Sigma_u^*} \text{ is not normal in any neighborhood of } z \right\},$$

and we then set

$$J(\tilde{f}) := \overline{\bigcup_{w \in \Sigma_u} \{\omega\} \times J_w},$$

where the closure is taken in the product space  $\Sigma_u \times \hat{\mathbb{C}}$ . By its very definition, the set  $J(\tilde{f})$  is compact. Furthermore, setting

$$E(G) := \left\{ z \in \hat{\mathbb{C}} : \# \bigcup_{g \in G} g^{-1}(\{z\}) < \infty \right\},$$

by Proposition 3.2 in [31], we have the following.

**Proposition 2.1.** *If  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  is a multi-map and  $G = \langle f_1, \dots, f_u \rangle$  is the rational semigroup generated by  $\{f_1, \dots, f_u\}$ , then*

- (a)  $J(\tilde{f})$  is completely invariant under  $\tilde{f}$ , meaning that  $\tilde{f}^{-1}(J(\tilde{f})) = J(\tilde{f}) = \tilde{f}(J(\tilde{f}))$ .
- (b) The map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  is open, meaning that it maps open sets onto open sets.
- (c) If

$$\#J(G) \geq 3 \quad \text{and} \quad E(G) \subset F(G),$$

then the skew product map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  is topologically exact meaning that for every non-empty open set  $U$  in  $J(\tilde{f})$  there exists an integer  $n \geq 0$  such that

$$\tilde{f}^n(U) = J(\tilde{f}).$$

In particular, the map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  is topologically transitive.

- (d)  $J(\tilde{f})$  is equal to the closure of the set of repelling periodic points of  $\tilde{f}$  provided that  $\#J(G) \geq 3$ , where a periodic point  $(\omega, z)$  of  $\tilde{f}$  with period  $n$  is said to be repelling if the absolute value of the multiplier  $|(\tilde{f}^n)'(\omega, z)|$  of  $f_\omega$  at  $z$  is strictly larger than 1.
- (e) Furthermore,

$$p_2(J(\tilde{f})) = J(G).$$

We now introduce further notation. A pair  $(c, j) \in \hat{\mathbb{C}} \times \{1, 2, \dots, u\}$  is called critical for the multi-map  $f$  if

$$f'_j(c) = 0.$$

The set of all critical pairs of  $f$  will be denoted by  $\text{CP}(f)$ . Let  $\text{Crit}(f)$  be the union

$$\text{Crit}(f) := \bigcup_{j=1}^u \text{Crit}(f_j).$$

For every  $c \in \text{Crit}(f)$  put

$$c_+ = \{f_j(c) : (c, j) \in \text{CP}(f)\}.$$

The set  $c_+$  is called the set of critical values of  $c$ . For any subset  $A$  of  $\text{Crit}(f)$  put

$$A_+ = \{c_+ : c \in A\}.$$



For each  $(c, j) \in \text{CP}(f)$  let  $q(c, j)$  be the local order of  $f_j$  at  $c$ . We define the direct postcritical set of  $G$  to be

$$G^*(\text{Crit}(f)_+).$$

We further consider the set

$$\overline{G^*(\text{Crit}(f)_+)}.$$

Note that this set does not depend on the choice of generators, and it is in fact equal to the closure of the set

$$\{g(c) : g \in G, g'(c) = 0\}.$$

We denote it by  $\text{PCV}(G)$  and call it the postcritical set of  $G$ <sup>1</sup>, i.e.

$$\text{PCV}(G) := \overline{G^*(\text{Crit}(f)_+)} = \overline{\{g(c) : g \in G, g'(c) = 0\}}.$$

We also set

$$\text{Crit}_*(f) := \bigcup_{i=1}^u \{c \in \text{Crit}(f_i) : f_i(c) \in J(G)\}.$$

Of course

$$\text{Crit}_*(f) \subset J(G) \cap \text{Crit}(G),$$

but equality need not to hold. Let

$$\text{Crit}(\tilde{f}) := \{\xi \in \Sigma_u \times \hat{\mathbb{C}} : \tilde{f}'(\xi) = 0\}.$$

Of course

$$\text{Crit}(\tilde{f}) = \{(\omega, c) \in \Sigma_u \times \hat{\mathbb{C}} : f'_{\omega_1}(c) = 0\}.$$

Put

$$\text{Crit}_*(\tilde{f}) := J(\tilde{f}) \cap \text{Crit}(\tilde{f}).$$

Note that

$$(2.4) \quad \text{Crit}_*(f) = p_2(\text{Crit}_*(\tilde{f})).$$

We call

$$\bigcup_{n=1}^{\infty} \tilde{f}^n(\text{Crit}(\tilde{f}))$$

the direct postcritical set of  $\tilde{f}$ , and its closure, i.e. the set

$$\text{PCV}(\tilde{f}) := \overline{\bigcup_{n=1}^{\infty} \tilde{f}^n(\text{Crit}(\tilde{f}))},$$

the postcritical set of  $\tilde{f}$ . We also define

$$\text{PCV}_*(\tilde{f}) := J(\tilde{f}) \cap \text{PCV}(\tilde{f}) = \overline{\bigcup_{n=1}^{\infty} \tilde{f}^n(\text{Crit}_*(\tilde{f}))}.$$

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<sup>1</sup>Some authors call the direct postcritical set just postcritical and give no short name for  $\text{PCV}(G)$  just calling it the closure of the postcritical set. Our choice of notation and terminology follows a well established tradition and makes exposition simpler.

Set

$$\text{Sing}(\tilde{f}) := \bigcup_{n \geq 0} \tilde{f}^{-n}(\text{Crit}(\tilde{f}))$$

and

$$\text{Sing}(f) := \bigcup_{g \in G^*} g^{-1}(\text{Crit}(f)).$$

Now we define and deal with holomorphic inverse branches of iterates of the skew product map  $\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \rightarrow \Sigma_u \times \hat{\mathbb{C}}$ . We start with the following.

**Lemma 2.2.** *For every integer  $n \geq 1$ , we have that*

$$\tilde{f}^n(\text{Crit}(\tilde{f}^n)) = \Sigma_u \times \left( \bigcup_{\tau \in \Sigma_u^n} f_\tau(\text{Crit}(f_\tau)) \right).$$

*Proof.* Assume that  $(\omega, z) \in \tilde{f}^n(\text{Crit}(\tilde{f}^n))$ . This means that there exists  $(\gamma, x) \in \text{Crit}(\tilde{f}^n)$  such that

$$(\omega, z) = \tilde{f}^n(\gamma, x).$$

Then,

$$z = f_{\gamma|_n}(x),$$

and

$$f'_{\gamma|_n}(x) = (\tilde{f}^n)'(\gamma, x) = 0.$$

So,  $x \in \text{Crit}(f_{\gamma|_n})$  and

$$z \in f_{\gamma|_n}(\text{Crit}(f_{\gamma|_n})) \subset \bigcup_{\tau \in \Sigma_u^n} f_\tau(\text{Crit}(f_\tau)).$$

Thus,

$$(\omega, z) \in \Sigma_u \times \left( \bigcup_{\tau \in \Sigma_u^n} f_\tau(\text{Crit}(f_\tau)) \right),$$

and the inclusion

$$\tilde{f}^n(\text{Crit}(\tilde{f}^n)) \subset \Sigma_u \times \left( \bigcup_{\tau \in \Sigma_u^n} f_\tau(\text{Crit}(f_\tau)) \right)$$

is proven.

So, assume now that

$$(\omega, z) \in \Sigma_u \times \left( \bigcup_{\tau \in \Sigma_u^n} f_\tau(\text{Crit}(f_\tau)) \right).$$

So, there exists  $\tau \in \Sigma_u^n$  such that  $z \in f_\tau(\text{Crit}(f_\tau))$ . Hence, there exists  $x \in \text{Crit}(f_\tau)$  such that  $z = f_\tau(x)$ . So,  $f'_\tau(x) = 0$ . Then,

$$\tilde{f}^n(\tau\omega, x) = (\omega, f_\tau(x)) = (\omega, z),$$

and

$$(\tilde{f}^n)'(\tau\omega, x) = f'_\tau(x) = 0.$$

So,  $(\tau\omega, x) \in \text{Crit}(\tilde{f}^n)$ , and, in consequence,  $(\omega, z) \in \tilde{f}^n(\text{Crit}(\tilde{f}^n))$ . We have thus proved the inclusion

$$\Sigma_u \times \left( \bigcup_{\tau \in \Sigma_u^n} f_\tau(\text{Crit}(f_\tau)) \right) \subset \tilde{f}^n(\text{Crit}(\tilde{f}^n)).$$

Lemma 2.2 is thus proven.  $\square$

Denote by

$$\text{DPCV}(\tilde{f}) := \bigcup_{n=1}^{\infty} \tilde{f}^n(\text{Crit}(\tilde{f})) = \bigcup_{n=1}^{\infty} \tilde{f}^n(\text{Crit}(\tilde{f}^n))$$

the direct postcritical set of  $\tilde{f}$  and by

$$\text{DPCV}(G) := G^*(\text{Crit}(f)_+) = \bigcup_{\tau \in \Sigma_u^*} f_\tau(\text{Crit}(f_\tau))$$

the direct postcritical set of  $G$ . Then, as an immediate consequence of Lemma 2.2, we get the following.

**Lemma 2.3.**  $\text{DPCV}(\tilde{f}) = \Sigma_u \times \text{DPCV}(G)$ .

By applying the closures, we thus obtain:

**Lemma 2.4.**  $\text{PCV}(\tilde{f}) = \Sigma_u \times \text{PCV}(G)$ .

Fix  $\tau \in \Sigma_u^*$ ,  $x \in \hat{\mathbb{C}}$ , and  $n \in \mathbb{N}$ . Suppose that

$$f'_\tau(x) \neq 0,$$

which precisely means that

$$(\tilde{f}^{|\tau|})'(\omega, x) \neq 0$$

for every  $\omega \in [\tau]$ . Then if  $V \subset \hat{\mathbb{C}}$  is a sufficiently small, in the sense of diameter, non-empty, open, connected, simply connected set containing  $f_\tau(x)$ , more precisely if the connected component of  $f_\tau^{-1}(V)$  does not contain critical points of  $f_\tau$ , then there exists a unique holomorphic inverse branch

$$f_{\tau,x}^{-1} : V \longrightarrow \hat{\mathbb{C}}$$

mapping  $f_\tau(x)$  to  $x$ . Furthermore, we denote by

$$(2.5) \quad \tilde{f}_{\tau,x}^{-|\tau|} : \Sigma_u \times V \longrightarrow [\tau] \times \hat{\mathbb{C}}$$

the map defined by the formula

$$(2.6) \quad \tilde{f}_{\tau,x}^{-|\tau|}(\omega, y) := (\tau\omega, f_{\tau,x}^{-1}(y)),$$

and we call it the holomorphic inverse branch of  $\tilde{f}^{|\tau|}$  defined on  $\Sigma_u \times V$  which maps  $\Sigma_u \times \{f_\tau(x)\}$  onto  $[\tau] \times \{x\}$ .

Now keep  $\tau \in \Sigma_u^*$ , and suppose that  $V \subset \hat{\mathbb{C}}$  is a non-empty, open, connected, simply connected set such that

$$(2.7) \quad V \cap f_\tau(\text{Crit}(f_\tau)) = \emptyset.$$

Equivalently

$$(2.8) \quad (\Sigma_u \times V) \cap \tilde{f}^{|\tau|}([\tau] \times \text{Crit}(f^\tau)) = \emptyset.$$

Then, by the above, all holomorphic inverse branches of  $f_\tau$  are well defined on  $V$ . More precisely, for every  $x \in f_\tau^{-1}(V)$  there exists a unique holomorphic inverse branch

$$f_{\tau,x}^{-1} : V \longrightarrow \hat{\mathbb{C}}$$

mapping  $f_\tau(x)$  to  $x$ . The map  $\tilde{f}_{\tau,x}^{-|\tau|} : \Sigma_u \times V \longrightarrow [\tau] \times \hat{\mathbb{C}}$  has then the same meaning as in (2.5) and (2.6).

Now fix an integer  $n \geq 0$  and suppose that  $V \subset \hat{\mathbb{C}}$  is a non-empty, open, connected, simply connected set such that

$$V \cap \bigcup_{\tau \in \Sigma_u^n} f_\tau(\text{Crit}(f_\tau)) = \emptyset.$$

By Lemma 2.2, this means that

$$(\Sigma_u \times V) \cap \tilde{f}^n(\text{Crit}(\tilde{f}^n)) = \emptyset.$$

Then, by the above, for every  $\tau \in \Sigma_u^n$ , all holomorphic inverse branches of  $f_\tau$  are well defined on  $V$ . More precisely, for every  $x \in f_\tau^{-1}(V)$  there exists a unique holomorphic inverse branch

$$f_{\tau,x}^{-1} : V \longrightarrow \hat{\mathbb{C}}$$

mapping  $f_\tau(x)$  to  $x$ . As in the above, the map

$$\tilde{f}_{\tau,x}^{-n} : \Sigma_u \times V \longrightarrow [\tau] \times \hat{\mathbb{C}}$$

has then the same meaning as in (2.5) and (2.6). We denote by  $\text{HIB}_n(V)$  the family of all such inverse branches  $\tilde{f}_{\tau,x}^{-|\tau|} : \Sigma_u \times V \longrightarrow [\tau] \times \hat{\mathbb{C}}$ . Fixing any point  $\xi \in V$ , this family is equal to

$$\{\tilde{f}_{\tau,x}^{-|\tau|} : \Sigma_u \times V \longrightarrow [\tau] \times \hat{\mathbb{C}} \mid \tau \in \Sigma_u^n, x \in f_\tau^{-1}(\xi)\}.$$

For every  $\rho \in \text{HIB}_n(V)$  for the sake of naturality we write  $\rho$  as

$$(2.9) \quad \tilde{f}_\rho^{-n} : \Sigma_u \times V \longrightarrow [\tilde{\rho}] \times \hat{\mathbb{C}},$$

being given by the formula

$$(2.10) \quad \tilde{f}_\rho^{-n}(\omega, z) = (\tilde{\rho}\omega, f_\rho^{-1}(z)) \in [\tilde{\rho}] \times \hat{\mathbb{C}},$$

with a unique  $\tilde{\rho} \in \Sigma_u^n$  and a unique  $f_\rho^{-1} : V \rightarrow \hat{\mathbb{C}}$  being a holomorphic inverse branch of  $f_{\tilde{\rho}}$  defined on  $V$ . We set

$$(2.11) \quad \|\rho\| := |\tilde{\rho}| = n.$$

Lastly, let  $V \subset \hat{\mathbb{C}}$  be a non-empty, open, connected, simply connected set such that

$$V \cap \text{PCV}(G) = \emptyset.$$

By Lemma 2.3, this means that

$$(\Sigma_u \times V) \cap \text{PCV}(\tilde{f}) = \emptyset.$$

Then, by the above, all the families  $\text{HIB}_n(V)$  are well defined and we denote

$$\text{HIB}(V) := \bigcup_{n=0}^{\infty} \text{HIB}_n(V).$$

Note that if  $W$  is any non-empty, open, connected, simply connected subset of  $V$ , then

$$\text{HIB}(W) = \{\rho|_W : \rho \in \text{HIB}(V)\}.$$

We end these preliminaries with bringing up the classical and celebrated distortion theorems due to Koebe. We do it for the sake of completeness and since these form for us an indispensable truly powerful tool.

**Theorem 2.5** (Koebe's  $\frac{1}{4}$ -Theorem). *If  $z \in \mathbb{C}$ ,  $r > 0$  and  $H : B(z, r) \rightarrow \mathbb{C}$  is an arbitrary univalent analytic function, then*

$$H(B(z, r)) \supset B(H(z), 4^{-1}|H'(z)|r).$$

**Theorem 2.6** (Koebe's Distortion Theorem, I). *There exists a function  $k : [0, 1) \rightarrow [1, \infty)$  such that for any  $z \in \mathbb{C}$ ,  $r > 0$ ,  $t \in [0, 1)$  and any univalent analytic function  $H : B(z, r) \rightarrow \mathbb{C}$  we have that*

$$\sup\{|H'(w)| : w \in B(z, tr)\} \leq k(t) \inf\{|H'(w)| : w \in B(z, tr)\}.$$

We put  $K = k(1/2)$ .

The following is a straightforward consequence of these two distortion theorems.

**Lemma 2.7.** *Suppose that  $D \subset \mathbb{C}$  is an open set,  $z \in D$  and  $H : D \rightarrow \mathbb{C}$  is an analytic map which has an analytic inverse  $H_z^{-1}$  defined on  $B(H(z), 2R)$  for some  $R > 0$ . Then for every  $0 \leq r \leq R$ ,*

$$B(z, K^{-1}r|H'(z)|^{-1}) \subset H_z^{-1}(B(H(z), r)) \subset B(z, Kr|H'(z)|^{-1}).$$

We also use the following more geometric version of Koebe's Distortion Theorem involving moduli of annuli.

**Theorem 2.8** (Koebe's Distortion Theorem, II). *There exists a function  $w : (0, +\infty) \rightarrow [1, \infty)$  such that for any two open topological disks  $Q_1 \subset Q_2 \subset \mathbb{C}$  with  $\text{Mod}(Q_2 \setminus Q_1) \geq t$  and any univalent analytic function  $H : Q_2 \rightarrow \mathbb{C}$  we have*

$$\sup\{|H'(\xi)| : \xi \in Q_1\} \leq w(t) \inf\{|H'(\xi)| : \xi \in Q_1\}.$$

## Part 1. The Dynamics of \*Semi-hyperbolic Rational Semigroups

### 3. BASIC PROPERTIES OF SEMI-HYPERBOLIC AND \*SEMI-HYPERBOLIC RATIONAL SEMIGROUPS

In this section we define semi-hyperbolic and \*semi-hyperbolic rational semigroups, and we collect their dynamical properties, which will be needed in the sequel.

**Definition 3.1.** A rational semigroup  $G$  is called *semi-hyperbolic* if and only if there exist an  $N \in \mathbb{N}$  and a  $\delta > 0$  such that for each  $x \in J(G)$  and each  $g \in G$ ,

$$\deg(g : V \rightarrow B_s(x, \delta)) \leq N$$

for each connected component  $V$  of  $g^{-1}(B_s(x, \delta))$ .

From now on throughout the rest of the paper, we always assume the following.

**Assumption (\*):** If  $G$  is a rational semigroup, then the following three conditions are assumed to hold.

- There exists an element  $g$  of  $G$  such that  $\deg(g) \geq 2$ .
- Each element of  $\text{Aut}(\hat{\mathbb{C}}) \cap G$  is loxodromic.
- $F(G) \neq \emptyset$ .

**Definition 3.2.** Any *semi-hyperbolic rational semigroup satisfying Assumption (\*)* is called *\*semi-hyperbolic*.

It is immediate from these definitions that both concepts of semi-hyperbolicity and \*semi-hyperbolicity are independent of any set generators of the rational semigroup considered.

The crucial tool indispensable in developing the theory of \*semi-hyperbolic rational semigroups is given by the the following semigroup version of Mañé's Theorem proved in [32].

**Theorem 3.3.** If  $G = \langle f_1, \dots, f_u \rangle$  is a finitely generated rational semigroup satisfying Assumption (\*), then  $G$  is \*semi-hyperbolic if and only if all of the following conditions are satisfied.

- (a) For each  $z \in J(G)$  there exists a neighborhood  $U$  of  $z$  in  $\hat{\mathbb{C}}$  such that for any sequence  $\{g_n\}_{n=1}^{\infty}$  in  $G$ , any domain  $V$  in  $\hat{\mathbb{C}}$ , and any point  $\zeta \in U$ , the sequence  $\{g_n\}_{n=1}^{\infty}$  does not converge to  $\zeta$  locally uniformly on  $V$ .
- (b) If  $c \in \text{Crit}_*(f)$ , then

$$\text{dist}(c, G^*(c_+)) > 0.$$

Now the following characterization of \*semi-hyperbolic semigroups, more in terms of the skew product map  $\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \rightarrow \Sigma_u \times \hat{\mathbb{C}}$ , is immediate.

**Theorem 3.4.** If  $G = \langle f_1, \dots, f_u \rangle$  is a finitely generated rational semigroup satisfying Assumption (\*), then  $G$  is \*semi-hyperbolic if and only if all of the following conditions are satisfied.

- (a) For each  $z \in J(G)$  there exists a neighborhood  $U$  of  $z$  in  $\hat{\mathbb{C}}$  such that for any sequence  $\{g_n\}_{n=1}^{\infty}$  in  $G$ , any domain  $V$  in  $\hat{\mathbb{C}}$ , and any point  $\zeta \in U$ , the sequence  $\{g_n\}_{n=1}^{\infty}$  does not converge to  $\zeta$  locally uniformly on  $V$ .
- (b) For each  $\xi \in \text{Crit}_*(\tilde{f})$  we have that

$$\text{dist}(\xi, \{\tilde{f}^n(\xi) : n \geq 1\}) > 0.$$

The second named author proved in [32], as Corollary 2.16, the following theorem which is very important for us in the current paper.

**Theorem 3.5** (Exponential Shrinking Property). *Let  $G = \langle f_1, \dots, f_u \rangle$  be a  $\ast$ semi-hyperbolic finitely generated rational semigroup. Then there exist  $R > 0$ ,  $C > 0$ , and  $\alpha > 0$  such that if  $x \in J(G)$ ,  $\omega \in \Sigma_u^\ast$ , and  $V$  is any connected component of  $f_\omega^{-1}(B_2(x, R))$ , then*

- (a)  $V$  is simply connected  
and
- (b)

$$\text{diam}(V) \leq Ce^{-\alpha|\omega|}.$$

**Remark 3.6.** *Note that this property, i.e. Exponential Shrinking, does not in fact depend on the choice of finitely many generators. It is a property of finitely generated rational semigroups themselves. Only  $\alpha$  (and  $C$  too) may depend on the generators.*

We have proved in [46] the following lemma. We provide here its simple independent proof for the sake of completeness and convenience of the reader.

**Lemma 3.7.** *If  $G = \langle f_1, \dots, f_u \rangle$  is a finitely generated  $\ast$ semi-hyperbolic rational semigroup, then*

- (a)  $\overline{G^\ast(\text{Crit}_\ast(f))} \cap J(G)$  is a nowhere dense subset of  $J(G)$ ,  
and
- (b)  $\text{PCV}_\ast(\tilde{f})$  is a nowhere dense subset of  $J(\tilde{f})$ .

*Proof.* Proving item (a), suppose for a contradiction the set  $\overline{G^\ast(\text{Crit}_\ast(f))} \cap J(G)$  is not nowhere dense in  $J(G)$ . Since

$$\overline{G^\ast(\text{Crit}_\ast(f))} \cap J(G) = \bigcup_{c \in \text{Crit}_\ast(f)} J(G) \cap \overline{G^\ast(c_+)},$$

since the set  $\text{Crit}_\ast(f)$  is finite, and since each constituent of the above union is closed, it follows that there exists  $c \in \text{Crit}_\ast(f)$  such that

$$\text{Int}_{J(G)}(J(G) \cap \overline{G^\ast(c_+)}) \neq \emptyset.$$

It then follows from item (c) of Proposition 2.1 and forward  $G$ -invariance of the set  $\overline{G^\ast(c_+)}$  that  $\overline{G^\ast(c_+)} \supset J(G)$ . In particular

$$c \in \overline{G^\ast(c_+)},$$

contrary to item (b) of Theorem 3.3. Item (a) of our lemma is thus proved.

In order to prove its item (b) suppose for a contradiction that  $\text{PCV}_\ast(\tilde{f})$  is not nowhere dense in  $J(\tilde{f})$ .

This means that  $\text{PCV}_\ast(\tilde{f})$  has non-empty interior in  $J(\tilde{f})$ , and therefore, because of its forward invariance and topological exactness of the map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ , we conclude that

$$\text{PCV}_\ast(\tilde{f}) = J(\tilde{f}).$$



Hence,

$$J(G) = p_2(J(\tilde{f})) = p_2(\text{PCV}_*(\tilde{f})) \subset \overline{G^*(\text{Crit}_*(\tilde{f}))} \cap J(G),$$

contrary to, the already proved, item (a). Item (b) is thus proved and, simultaneously, the whole Lemma 3.7.  $\square$

Following the common tradition, given a point  $(\tau, z) \in \Sigma_u \times \hat{\mathbb{C}}$  we denote by  $\omega(\tau, z)$  the  $\omega$ -limit set of  $(\tau, z)$  with respect to the skew product map  $\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \rightarrow \Sigma_u \times \hat{\mathbb{C}}$ , i.e. the set of all accumulation points of the sequence  $(\tilde{f}^n(\tau, z))_{n=0}^\infty$ . For every  $(\tau, z) \in J(\tilde{f})$  put

$$\text{Crit}(\tau, z) := \text{Crit}(\tilde{f}) \cap \omega(\tau, z).$$

In [46] we proved the following whose proof we reproduce here for the sake of completeness and its importance for the further development of the present manuscript. It is convenient for the sake of this proof to introduce the following concept. If  $c \in \text{Crit}(G)$ , then by

$$\omega_G(c_+)$$

we denote the set of all accumulation points of the set  $G^*(c_+)$ . Of course

$$\overline{G^*(c_+)} = G^*(c_+) \cup \omega_G(c_+).$$

**Lemma 3.8.** *let  $G = \langle f_1, \dots, f_u \rangle$  be a finitely generated \*semi-hyperbolic rational semi-group. If  $(\tau, z) \in J(\tilde{f})$ , then*

$$p_2(\omega(\tau, z)) \not\subset \overline{G^*(p_2(\text{Crit}(\tau, z))_+)}$$

*Proof.* Suppose on the contrary that

$$(3.1) \quad p_2(\omega(\tau, z)) \subset \overline{G^*(p_2(\text{Crit}(\tau, z))_+)}.$$

Consequently,  $\text{Crit}(\tau, z) \neq \emptyset$ . Let  $(\tau^1, c_1) \in \text{Crit}(\tau, z)$ . This means that  $(\tau^1, c_1) \in \omega(\tau, z)$ , and it follows from (3.1) that there exists  $(\tau^2, c_2) \in \text{Crit}(\tau, z)$  such that

$$\text{either } c_1 \in \omega_G(c_{2+}) \quad \text{or} \quad c_1 = g_1(c_2)$$

for some  $g_1 \in G$  of the form  $f_\omega$  with  $f'_{\omega_1}(c_2) = 0$ . Iterating this procedure we obtain an infinite sequence  $((\tau^j, c_j))_{j=1}^\infty$  of points in  $\text{Crit}(\tau, z)$  such that for every  $j \geq 1$

$$\text{either } c_j \in \omega_G(c_{j+1+}) \quad \text{or} \quad c_j = g_j(c_{j+1})$$

for some  $g_j \in G$  of the form  $f_\rho$  with  $f'_{\rho_1}(c_{j+1}) = 0$ . Consider an arbitrary block

$$c_k, c_{k+1}, \dots, c_l$$

such that

$$c_j = g_j(c_{j+1})$$

for every  $k \leq j \leq l-1$ , and suppose that

$$l - (k - 1) \geq \#(\text{Crit}(f) \cap J(G)).$$

Then there are two indices  $k \leq a < b \leq l$  such that

$$c_a = c_b.$$

Hence,

$$g_a \circ g_{a+1} \circ \dots \circ g_{b-1}(c_b) = c_a = c_b \quad \text{and} \quad (g_a \circ g_{a+1} \circ \dots \circ g_{b-1})'(c_b) = 0.$$

This however contradicts our assumption that the Julia set,  $J(G)$ , of  $G$  contains no superstable fixed points. In consequence, the length of the block  $c_k, c_{k+1}, \dots, c_l$  is bounded above by  $\#(\text{Crit}(f) \cap J(G))$ . Therefore, there exists an infinite sequence  $(j_n)_{n=1}^\infty$  such that

$$c_{j_n} \in \omega_G(c_{j_n+1+})$$

for all  $n \geq 1$ . In conclusion, there would exist at least one point  $c \in \text{Crit}_*(f)$  such that

$$c \in \omega_G(c_+),$$

contrary to Theorem 3.3 (b). This finishes the proof.  $\square$

The following proposition, also proved in [46], will be used many times in the current paper. We therefore provide its proof here too.

**Proposition 3.9.** *If  $G = \langle f_1, \dots, f_u \rangle$  is a finitely generated  $*$ semi-hyperbolic rational semigroup, then for each point  $(\tau, z) \in J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$ , there exist a number  $\eta(\tau, z) > 0$ , an increasing sequence  $(n_j)_{j=1}^\infty$  of positive integers, and a point*

$$(\hat{\tau}, \hat{z}) \in \omega(\tau, z) \setminus p_2^{-1}(\overline{G^*(p_2(\text{Crit}(\tau, z)))_+})$$

with the following two properties.

- (a)  $\lim_{j \rightarrow \infty} \tilde{f}^{n_j}(\tau, z) = (\hat{\tau}, \hat{z})$ .
- (b) The connected component of the set  $f_{\tau|n_j}^{-1}(B(f_{\tau|n_j}(\tau, z), \eta(\tau, z)))$  containing  $z$  contains no critical points of the map  $f_{\tau|n_j} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ .

*Proof.* In view of Lemma 3.8 there exists a point  $(\hat{\tau}, \hat{z}) \in \omega(\tau, z)$  such that

$$\hat{z} \notin \overline{G^*(p_2(\text{Crit}(\tau, z)))_+}.$$

Let

$$\eta := \frac{1}{2} \text{dist}(\hat{z}, \overline{G^*(p_2(\text{Crit}(\tau, z)))_+}).$$

Then there exists an infinite increasing sequence  $(n_j)_{j=1}^\infty$  of positive integers such that

$$(3.2) \quad \lim_{j \rightarrow \infty} \tilde{f}^{n_j}(\tau, z) = (\hat{\tau}, \hat{z})$$

and

$$(3.3) \quad f_{\tau|n_j}(z) \notin B(\overline{G^*(p_2(\text{Crit}(\tau, z)))_+}, \eta)$$

for all  $j \geq 1$ . We claim that there exists  $\eta(\tau, z) > 0$  such that for all  $j \geq 1$  large enough

$$\text{Comp}(z, f_{\tau|n_j}, \eta(\tau, z)) \cap \text{Crit}(f_{\tau|n_j}) = \emptyset.$$

Indeed, otherwise we can find an increasing subsequence  $(j_i)_{i=1}^\infty$  and a decreasing to zero sequence of positive numbers  $\eta_i < \eta$  such that

$$\text{Comp}(z, f_{\tau|n_{j_i}}, \eta_i) \cap \text{Crit}(f_{\tau|n_{j_i}}) \neq \emptyset.$$

Let  $\tilde{c}_i \in \text{Comp}(z, f_{\tau|n_{j_i}}, \eta_i) \cap \text{Crit}(f_{\tau|n_{j_i}})$ . Then there exist  $0 \leq p_i \leq n_{j_i} - 1$  and

$$(3.4) \quad c_i \in \text{Crit}(f_{\tau_{p_i+1}})$$

such that  $c_i = f_{\tau|_{p_i}}(\tilde{c}_i)$ . Since  $\lim_{i \rightarrow \infty} \eta_i = 0$ , it follows from Theorem 3.5 that  $\lim_{i \rightarrow \infty} \tilde{c}_i = z$ . Since  $(\tau, z) \notin \bigcup_{n \geq 0} \tilde{f}^{-n}(\text{Crit}(\tilde{f}))$ , this implies that  $\lim_{i \rightarrow \infty} p_i = +\infty$ . But then, making use of Theorem 3.5 again and of the formula  $(\sigma^{p_i}(\tau), c_i) = f^{p_i}(\tau, \tilde{c}_i)$ , we conclude that the set of accumulation points of the sequence

$$((\sigma^{p_i}(\tau), c_i))_{i=1}^{\infty}$$

is contained in  $\omega(\tau, z)$ . Fix  $(\tau^\infty, c)$  to be one of these accumulation points. Since  $\text{Crit}(\tilde{f})$  is closed, we conclude that

$$(3.5) \quad (\tau^\infty, c) \in \text{Crit}(\tau, z).$$

Since that set  $\text{Crit}(f)$  is finite, passing to a subsequence, we may assume without loss of generality that  $(c_i)_{i=1}^{\infty}$  is a constant sequence, so equal to  $c$ . Since  $c = f_{\tau|_{p_i}}(\tilde{c}_i)$ , we get

$$\left| f_{\tau|_{n_{j_i}}}(z) - f_{\tau|_{p_i+1}}^{n_{j_i}}(c) \right| = \left| f_{\tau|_{n_{j_i}}}(z) - f_{\tau|_{n_{j_i}}}(\tilde{c}_i) \right| < \eta_i < \eta.$$

But, looking at (3.4) and (3.5), we conclude that  $f_{\tau|_{p_i+1}}^{n_{j_i}}(c) \in G^*(\text{Crit}(\tau, z)_+)$ . We have thus arrived at a contradiction with (3.3), and the proof is finished.  $\square$

#### 4. CONFORMAL AND INVARIANT MEASURES $m_t$ AND $\mu_t$ FOR $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$

##### 4.1. Conformal and Invariant Measures (and Topological Pressure) for $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ ; Preliminaries.

In this section we rigorously recall, generalize, and extend, the thermodynamic concepts introduced in [46]. Some of them have already been introduced and explored in [33], [34], and [45].

**Definition 4.1.** Let  $G = \langle f_1, \dots, f_u \rangle$  and let  $t \geq 0$ . For all  $\xi \in J(\tilde{f}) \setminus \text{PCV}_*(\tilde{f})$  define

$$P_\xi(t) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in \tilde{f}^{-n}(\xi)} |(\tilde{f}^n)'(x)|^{-t} \in [-\infty, +\infty],$$

where, if  $x = (\omega, a)$ , then  $|(\tilde{f}^n)'(x)|$  denotes the norm of the derivative of  $f_{\omega_n} \circ \dots \circ f_{\omega_1}$  at  $a$  with respect to the spherical metric.

The following proposition was proved in [46] as Lemma 7.3. We provide its proof here for the sake of completeness, convenience of the reader, its shortness, simplicity, crucial importance, and we expose the proof in terms of the skew product map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  rather than in terms of the semigroup  $G$  itself, as it was done in [46].

**Proposition 4.2.** If  $G$  is a  $*$ semi-hyperbolic rational semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ , then for each  $t \geq 0$ , the function

$$J(\tilde{f}) \setminus \text{PCV}_*(\tilde{f}) \ni \xi \mapsto P_\xi(t) \in [-\infty, +\infty]$$

is constant. Denote this constant by  $P(t)$  and call it the topological pressure of  $t$ .

*Proof.* Because of Lemma 2.4 for every  $\xi \in J(\tilde{f}) \setminus \text{PCV}_*(\tilde{f})$  there exists  $r_\xi > 0$  such that

$$B_2(p_2(\xi), 2r_\xi) \cap \text{PCV}(G) = \emptyset.$$

It then directly follows from Koebe's Distortion Theorem that the function

$$\gamma \longmapsto P_\gamma(t)$$

is constant on  $J(\tilde{f}) \cap (\Sigma_u \times B_2(p_2(\xi), r_\xi))$ . Now, fix  $\xi_1, \xi_2 \in J(\tilde{f}) \setminus \text{PCV}_*(\tilde{f})$ . By Proposition 2.1 (c) there exists an integer  $q \geq 1$  such that

$$\tilde{f}^q(J(\tilde{f}) \cap (\Sigma_u \times B_2(p_2(\xi_1), r_{\xi_1}))) \cap (J(\tilde{f}) \cap (\Sigma_u \times B_2(p_2(\xi_2), r_{\xi_2}))) \neq \emptyset.$$

Fix then a point  $\rho \in J(\tilde{f}) \cap (\Sigma_u \times B_2(p_2(\xi_1), r_{\xi_1}))$  such that

$$\tilde{f}^q(\rho) \in J(\tilde{f}) \cap (\Sigma_u \times B_2(p_2(\xi_2), r_{\xi_2})).$$

Then

$$\sum_{x \in \tilde{f}^{-(n+q)}(\tilde{f}^q(\rho))} |(\tilde{f}^{n+q})'(x)|^{-t} \geq |(\tilde{f}^q)'(\rho)|^{-t} \sum_{y \in \tilde{f}^{-n}(\rho)} |(\tilde{f}^n)'(y)|^{-t}.$$

Therefore,  $P_{\tilde{f}^q(\rho)}(t) \geq P_\rho(t)$ . Hence,  $P_{\xi_2}(t) \geq P_{\xi_1}(t)$ . Exchanging the roles of  $\xi_1$  and  $\xi_2$ , we get  $P_{\xi_1}(t) \geq P_{\xi_2}(t)$ , and we are done.  $\square$

We want to emphasize that topological pressure does depend on the choice of generators and not on the semigroup alone. Now, we shall prove the following basic properties of the function

$$[0, +\infty) \ni t \longmapsto P(t) \in [-\infty, +\infty].$$

**Proposition 4.3.** *If  $G$  is a  $*$ semi-hyperbolic rational semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ , then the function  $t \longmapsto P(t)$ ,  $t \geq 0$ , has the following properties.*

- (a) *For every  $t \geq 0$  we have that  $P(t) \in (-\infty, +\infty)$  and  $P(0) \geq \log 2 > 0$ .*
- (b) *The function  $[0, +\infty) \longmapsto P(t)$  is strictly decreasing and Lipschitz continuous. More precisely:*
- (c) *If  $0 \leq s \leq t < +\infty$ , then*

$$-\log \|\tilde{f}'\|_\infty(t-s) \leq P(t) - P(s) \leq -\alpha(t-s),$$

*where the constant  $\alpha > 0$  comes from the Exponential Shrinking Property (Theorem 3.5).*

- (d)  $\lim_{t \rightarrow +\infty} P(t) = -\infty$ .

*Proof.* The inequality

$$(4.1) \quad \log 2 \leq P(0) < +\infty$$

follows immediately from degree considerations. Assuming that  $P(s) \in \mathbb{R}$ , the left-hand side of item (c) follows immediately, while its right-hand side follows from the Exponential Shrinking Property (Theorem 3.5) in conjunction with Koebe's Distortion Theorem. Having this and (4.1) we deduce that items (a) and (c) hold. Item (b) is a direct consequence of (c) while item (d) is now an immediate consequence of the right-hand side of (c).  $\square$

It follows from items (a), (b), and (d) of this proposition that there exists a unique  $t \in [0, +\infty)$ , we denote it by  $h_f$ , such that

$$(4.2) \quad P(h_f) = 0.$$

We now recall the concept of  $t$ -conformal measures for the skew product map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ . A Borel probability measure  $m$  on  $J(\tilde{f})$  is called  $t$ -conformal for  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  if and only if

$$m(\tilde{f}(A)) = e^{P(t)} \int_A |\tilde{f}'|^t dm$$

whenever  $A \subset J(\tilde{f})$  is a Borel set such that the restricted map  $\tilde{f}|_A$  is 1-to-1. In order to discern it from “truly” conformal measures, i.e.  $h_f$ -conformal measures (for which the factor  $e^{P(t)}$  disappears in the above displayed formula), any  $e^{P(t)}|\tilde{f}'|^t$ -conformal measure will also be frequently called a generalized  $t$ -conformal measure. We have proved in [46] the following fact of crucial importance for us in the current paper. We provide its proof here for the sake of completeness, convenience of the reader, its crucial importance, and because the construction of conformal measures given in this proof (coming from [46]) will be needed later in the paper.

**Proposition 4.4.** *If  $G$  is a  $*$ semi-hyperbolic rational semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ , then for every  $t \geq 0$  there exists an  $e^{P(t)}|\tilde{f}'|^t$ -conformal measure  $m_t$  for the skew product map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ .*

*Proof.* The construction of generalized  $t$ -conformal measures starts as follows. Fix  $\xi \in J(\tilde{f}) \setminus \text{PCV}(\tilde{f})$ . Observe that the critical parameter for the series

$$SP_t(\xi, s) := \sum_{n=1}^{\infty} e^{-sn} \sum_{x \in \tilde{f}^{-n}(\xi)} |(\tilde{f}^n)'(x)|^{-t}, \quad s \geq 0,$$

is equal to the topological pressure  $P(t)$ , i.e.  $SP_t(\xi, s) = +\infty$  if  $s < P(t)$  and  $SP_t(\xi, s) < +\infty$  if  $s > P(t)$ . In the latter case define a Borel probability measure  $\nu_{t,s}$  on  $J(\tilde{f})$  by the following formula:

$$(4.3) \quad \nu_{t,s}(A) := \frac{1}{SP_t(\xi, s)} \sum_{n=1}^{\infty} e^{-sn} \sum_{x \in A \cap \tilde{f}^{-n}(\xi)} |\tilde{f}'(x)|^{-t}.$$

Now we want to take weak\* limits as  $s \searrow P(t)$ . For this we need the concept of the Perron–Frobenius operator. Given a function  $g : J(\tilde{f}) \rightarrow \mathbb{C}$  let  $L_t g : J(\tilde{f}) \rightarrow \mathbb{C}$  be defined by the following formula:

$$(4.4) \quad L_t g(y) = \sum_{x \in \tilde{f}^{-1}(y)} |\tilde{f}'(x)|^{-t} g(x).$$

$L_t g(y)$  is finite if and only if  $y \notin \text{Crit}(\tilde{f})$ . Otherwise  $L_t g(y)$  is declared to be  $\infty$ . Although  $L_t$  is not actually an operator since it does not preserve the class of finite valued functions,

it is still referred to as the Perron–Frobenius operator associated to the parameter  $t \geq 0$ . Iterating formula (4.4), we get for all integers  $n \geq 1$  that

$$L_t^n g(y) = \sum_{x \in \tilde{f}^{-n}(y)} |(\tilde{f}^n)'(x)|^{-t} g(x).$$

For every  $\sigma$ -finite Borel measure  $m$  on  $J(\tilde{f})$  let the  $\sigma$ -finite Borel measure  $L_t^{*n}m$  be given by the formula

$$L_t^{*n}m(A) = m(\mathcal{L}_t^n \mathbf{1}_A),$$

where  $A \subset J(\tilde{f})$  is a Borel set and  $m(g) := \int g dm$ . Notice that if  $(\tau, \xi) \in J(\tilde{f}) \setminus \text{DPCV}(\tilde{f})$ , then for all Borel sets  $A \subset J(\tilde{f})$  we have

$$L_t^{*n} \delta_{(\tau, \xi)}(A) = \delta_{(\tau, \xi)}(\mathcal{L}_t^n \mathbf{1}_A) = L_t^n \mathbf{1}_A(\tau, \xi) = \sum_{|\omega|=n} \sum_{x \in A \cap f_\omega^{-1}(\xi)} |f'_\omega(x)|^{-t} \leq \mathcal{L}_t^n \mathbf{1}(\xi) < \infty.$$

In particular,

$$L_t^{*n} \delta_{(\tau, \xi)}(J(\tilde{f})) \leq L_t^n \mathbf{1}(\xi) < +\infty.$$

Observing that the formula (4.3) can be now expressed in the form

$$(4.5) \quad \nu_{t,s} = SP_t(\xi, s)^{-1} \sum_{n=1}^{\infty} e^{-sn} \mathcal{L}_t^{*n} \delta_{(\tau, \xi)},$$

a direct straightforward calculation shows that

$$(4.6) \quad e^{-s} L_t^* \nu_{t,s} = SP_t(\xi, s)^{-1} \sum_{n=1}^{\infty} e^{-s(n+1)} L_t^{*(n+1)} \delta_{(\tau, \xi)} = \nu_{t,s} - SP_t(\xi, s)^{-1} (e^{-s} L_t^* \delta_{(\tau, \xi)}).$$

If  $SP_t(\xi, P(t)) = +\infty$ , then  $\lim_{s \searrow P(t)} S_s(\xi) = +\infty$  and it follows from the formula (4.6) that any weak limit  $m_t$  of  $\nu_{t,s}$  when  $s \searrow P(t)$  is the required generalized  $t$ -conformal measure. If  $SP_t(\xi, P(t)) < +\infty$  then we apply the usual modifications involving slowly varying functions whose details can be found for example in [4].  $\square$

We have proved in [46] the following. We will need it in the sequel and we provide its short computational proof taken from [46].

**Lemma 4.5.** *Let  $G$  be a \*semi-hyperbolic rational semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ . Fix  $t \geq 0$ . If  $s > P(t)$ , then*

$$\nu_{t,s}(A) = e^{-s} \int_{\tilde{f}(A)} |(\tilde{f}|_A^{-1})'(x)|^t d\nu_{t,s}(x) + \begin{cases} 0 & \text{if } A \cap \tilde{f}^{-1}(\xi) = \emptyset \\ e^{-s} SP_t(\xi, s)^{-1}(\xi) |f'(y)|^{-t} & \text{if } A \cap \tilde{f}^{-1}(\xi) = \{y\} \end{cases}$$

whenever  $A \subset J(\tilde{f})$  is a Borel set such that the map  $\tilde{f}|_A$  is 1-to-1.

*Proof.* We calculate:

$$\begin{aligned}
\nu_{t,s}(A) &= e^{-s} L_t^* \nu_{t,s}(\mathbb{1}_A) + S_s^{-1}(\xi) e^{-s} L_t^* \delta_{(\tau,\xi)}(\mathbb{1}_A) \\
&= e^{-s} \int L_t(\mathbb{1}_A) d\nu_{t,s} + e^{-s} S_s^{-1}(\xi) \int L_t(\mathbb{1}_A) d\delta_{(\tau,\xi)} \\
&= e^{-s} \int \sum_{y \in \tilde{f}^{-1}(x)} |\tilde{f}'|^{-t} \mathbb{1}_A(y) d\nu_{t,s}(x) + e^{-s} S_s^{-1}(\xi) L_t(\mathbb{1}_A)(\tau, \xi) \\
&= e^{-s} \int_{\tilde{f}(A)} |(\tilde{f}|_A^{-1})'(x)|^t d\tilde{\nu}_s(x) + \begin{cases} 0 & \text{if } A \cap \tilde{f}^{-1}(\tau, \xi) = \emptyset \\ e^{-s} S_s^{-1}(\xi) |\tilde{f}'(y)|^{-t} & \text{if } A \cap \tilde{f}^{-1}(\tau, \xi) = \{y\}. \end{cases}
\end{aligned}$$

□

#### 4.2. Ergodic Theory of Conformal and Invariant Measures for $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ .

We begin this section with some abstract auxiliary facts about measures. Frequently, in order to denote that a Borel measure  $\mu$  is absolutely continuous with respect to  $\nu$ , we write  $\mu \prec \nu$ . However, we do not use any special symbol to record the equivalence of measures. We use some notations from [1]. Let  $(X, \mathcal{F}, \mu)$  be a  $\sigma$ -finite measure space and let  $T : X \rightarrow X$  be a measurable almost everywhere defined transformation.  $T$  is said to be nonsingular if and only if for any  $A \in \mathcal{F}$ ,

$$\mu(T^{-1}(A)) = 0 \Leftrightarrow \mu(A) = 0.$$

The map  $T$  is said to be ergodic with respect to  $\mu$ , or  $\mu$  is said to be ergodic with respect to  $T$ , if and only if either

$$\mu(A) = 0 \text{ or } \mu(X \setminus A) = 0$$

whenever the measurable set  $A$  is  $T$ -invariant, meaning that  $T^{-1}(A) = A$ . For a nonsingular transformation  $T : X \rightarrow X$ , the measure  $\mu$  is said to be conservative with respect to  $T$  or  $T$  is said to be conservative with respect to  $\mu$  if and only if for every measurable set  $A$  with  $\mu(A) > 0$ ,

$$\mu \left( \left\{ z \in A : \sum_{n=0}^{\infty} \mathbb{1}_A \circ T^n(z) < +\infty \right\} \right) = 0.$$

Note that by [1, Proposition 1.2.2], for a nonsingular transformation  $T : X \rightarrow X$ ,  $\mu$  is ergodic and conservative with respect to  $T$  if and only if for any  $A \in \mathcal{F}$  with  $\mu(A) > 0$ ,

$$\mu \left( \left\{ z \in X : \sum_{n=0}^{\infty} \mathbb{1}_A \circ T^n(z) < +\infty \right\} \right) = 0.$$

Finally, the measure  $\mu$  is said to be  $T$ -invariant, or  $T$  is said to preserve the measure  $\mu$  if and only if  $\mu \circ T^{-1} = \mu$ . It follows from Birkhoff's Ergodic Theorem that every finite ergodic  $T$ -invariant measure  $\mu$  is conservative, for infinite measures this is no longer true. Finally, two ergodic invariant measures defined on the same  $\sigma$ -algebra are either singular or they coincide up to a multiplicative constant.



**Definition 4.6.** *Suppose that  $(X, \mathcal{F}, \nu)$  is a probability space and  $T : X \rightarrow X$  is a measurable map such that  $T(A) \in \mathcal{F}$  whenever  $A \in \mathcal{F}$ . The map  $T : X \rightarrow X$  is said to be weakly metrically exact provided that*

$$\limsup_{n \rightarrow \infty} \mu(T^n(A)) = 1$$

whenever  $A \in \mathcal{F}$  and  $\mu(A) > 0$ .

We need the following two facts about weak metrical exactness, the first being straightforward (see the argument in [1, page 15]), the latter more involved (see [23] and [14]).

**Fact 4.7.** *If a nonsingular measurable transformation  $T : X \rightarrow X$  of a probability space  $(X, \mathcal{F}, \nu)$  is weakly metrically exact, then it is ergodic and conservative.*

**Fact 4.8.** *A measure-preserving transformation  $T : X \rightarrow X$  of a probability space  $(X, \mathcal{F}, \mu)$  is weakly metrically exact if and only if it is exact, which means that*

(a)

$$\lim_{n \rightarrow \infty} \mu(T^n(A)) = 1$$

whenever  $A \in \mathcal{F}$  and  $\mu(A) > 0$ , or equivalently,

(b) *The  $\sigma$ -algebra  $\bigcap_{n \geq 0} T^{-n}(\mathcal{F})$  consists of sets of measure 0 and 1 only.*

Note that if  $T : X \rightarrow X$  is exact, then the Rokhlin's natural extension  $(\tilde{T}, \tilde{X}, \tilde{\mu})$  of  $(T, X, \mu)$  is  $K$ -mixing.

We now pass to our  $*$ semi-hyperbolic semigroup  $G$  and investigate in detail generalized conformal measures.

**Definition 4.9.** *Let  $G$  be a  $*$ semi-hyperbolic rational semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ . Then,*

- *A real number  $t \geq 0$  is called a parameter of continuity for the  $*$ semi-hyperbolic semigroup  $G$  if and only if*

$$m_t(\text{Crit}(\tilde{f})) = 0.$$

- *The conformal measure  $m_t$  is then called continuous.*
- *The set of all parameters of continuity for  $G$  will be denoted by  $\Delta_G$ .*

As an immediate consequence of this definition and generalized conformality of measures  $m_t$ , we get the following.

**Observation 4.10.** *If  $G$  is a  $*$ semi-hyperbolic semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  and  $t \in \Delta_G$ , then*

$$m_t(\text{Sing}(\tilde{f})) = 0.$$

Given a point  $(\tau, z) \in J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$  let

(4.7)

$$B_j(\tau, z) := \tilde{f}_{\tau|_{n_j}, z}^{-n_j} \left( \Sigma_u \times B_2 \left( f_{\tau|_{n_j}}(z), \frac{1}{4} \eta(\tau, z) \right) \right) = [\tau|_{n_j}] \times f_{\tau|_{n_j}, z}^{-1} \left( B_2 \left( f_{\tau|_{n_j}}(z), \frac{1}{4} \eta(\tau, z) \right) \right)$$

and

$$(4.8) \quad B_j^*(\tau, z) := B_j(\tau, z) \setminus \text{Sing}(\tilde{f}),$$

where  $\eta(\tau, z) > 0$  and  $n_j := n_j(\tau, z) \geq 1$  are the integers produced in Proposition 3.9. We now shall recall the concept of Vitali relations defined on the page 151 of Federer's book [6]. Let  $X$  be an arbitrary set. By a covering relation on  $X$  one means a subset of

$$\{(x, S) : x \in S \subset X\}.$$

If  $C$  is a covering relation on  $X$  and  $Z \subset X$ , one puts

$$C(Z) = \{S \subset X : (x, S) \in C \text{ for some } x \in Z\}.$$

One then says that  $C$  is fine at  $x$  if

$$\inf\{\text{diam}(S) : (x, S) \in C\} = 0.$$

If in addition  $X$  is a metric space and a Borel measure  $\mu$  is given on  $X$ , then a covering relation  $V$  on  $X$  is called a Vitali relation if

- (a) All elements of  $V(X)$  are Borel sets,
- (b)  $V$  is fine at each point of  $X$ ,
- (c) If  $C \subset V$ ,  $Z \subset X$  and  $C$  is fine at each point of  $Z$ , then there exists a countable disjoint subfamily  $\mathcal{F}$  of  $C(Z)$  such that  $\mu(Z \setminus \cup \mathcal{F}) = 0$ .

Now, given  $(\tau, z) \in J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$ , let

$$\mathcal{B}_{(\tau, z)} = \left\{ \left( (\tau, z), B_j^*(\tau, z) \right) \right\}_{j=1}^{\infty},$$

where the sets  $B_j(\tau, z)$  are defined by formula (4.7). Let

$$\mathcal{B} = \bigcup_{(\tau, z) \in J(\tilde{f}) \setminus \text{Sing}(\tilde{f})} \mathcal{B}_{(\tau, z)}$$

and, following notation from Federer's book [6], let

$$\mathcal{B}_2 := \mathcal{B}(J(\tilde{f}) \setminus \text{Sing}(\tilde{f})) = \{B_j^*(\tau, z) : (\tau, z) \in J(\tilde{f}) \setminus \text{Sing}(\tilde{f}), j \geq 1\}.$$

We shall prove the following.

**Lemma 4.11.** *If  $G$  is a \*semi-hyperbolic semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ , then for every  $t \in \Delta_G$  the family  $\mathcal{B}$  is a Vitali relation for the measure  $m_t$  on the set  $J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$ .*

*Proof.* Fix  $1 < \kappa < e$  (a different notation for  $1 < \tau < +\infty$  appearing in Theorem 2.8.17 from [6]). Fix  $(\tau, z) \in J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$ . Since

$$(4.9) \quad \lim_{j \rightarrow \infty} n_j = +\infty,$$

we have that

$$\lim_{j \rightarrow \infty} \text{diam}(B_j^*(\tau, z)) = 0.$$

This means that the relation  $\mathcal{B}$  is fine at the point  $(\tau, z)$ . Aiming to apply Theorem 2.8.17 from [6], we set

$$\delta(B_j^*(\omega, x)) = \exp(-n_j)$$

for every  $B_j^*(\omega, x) \in \mathcal{B}_2$ . With the notation from page 144 in [6] ( $\mathcal{F} = \mathcal{B}_2$ ) we have

$$\hat{B}_j^*(\tau, z) = \bigcup \mathcal{B}_j^*(\tau, z),$$

where

$$\mathcal{B}_j^*(\tau, z) := \{B : B \in \mathcal{B}_2, B \cap B_j^*(\tau, z) \neq \emptyset, \delta(B) \leq \kappa \delta(B_j^*(\tau, z))\}.$$

Fix a  $B$  from the above family. Then there exists  $(\omega, x) \in J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$  and an integer  $i \geq 1$  such that

$$B = B_i^*(\omega, x).$$

Since  $\delta(B_i(\omega, x)) \leq \kappa \delta(B_j(\tau, z))$ , we have that  $e^{-n_i} \leq \kappa e^{-n_j}$ . Equivalently:  $n_j \leq n_i + \log \kappa$ . Since both  $n_j$  and  $n_i$  are integers and since  $\log \kappa < 1$ , this yields

$$(4.10) \quad n_i \geq n_j.$$

Since

$$(4.11) \quad B_i^*(\omega, x) \cap B_j^*(\tau, z) \neq \emptyset,$$

we have that  $[\omega|_{n_i}] \cap [\tau|_{n_j}] \neq \emptyset$ . In conjunction with (4.10) this yields

$$\omega|_{n_j} = \tau|_{n_j}.$$

Then

(4.12)

$$\begin{aligned} B_i^*(\omega, x) \cap B_j^*(\tau, z) &= \\ &= \left( \left( [\tau|_{n_j}] \times f_{\tau|_{n_j}, z}^{-1} \left( B \left( f_{\tau|_{n_j}}(z), \frac{1}{4} \eta(\tau, z) \right) \right) \right) \setminus \text{Sing}(\tilde{f}) \right) \cap \\ &\cap \left( \left( [\tau|_{n_j} \sigma^{n_j}(\omega)]|_{n_i - n_j} \times f_{\tau|_{n_j}, x}^{-1} \circ f_{\sigma^{n_j}(\omega)|_{n_i - n_j}, f_{\tau|_{n_j}}(x)}^{-1} \left( B \left( f_{\omega|_{n_i}}(x), \frac{1}{4} \eta(\omega, x) \right) \right) \right) \setminus \text{Sing}(\tilde{f}) \right) \\ &\subset \left( [\tau|_{n_j}] \times f_{\tau|_{n_j}, z}^{-1} \left( B \left( f_{\tau|_{n_j}}(z), \frac{1}{4} \eta(\tau, z) \right) \right) \cap \right. \\ &\quad \left. \cap f_{\sigma^{n_j}(\omega)|_{n_i - n_j}, f_{\tau|_{n_j}}(x)}^{-1} \left( B \left( f_{\omega|_{n_i}}(x), \frac{1}{4} \eta(\omega, x) \right) \right) \right) \setminus \text{Sing}(\tilde{f}). \end{aligned}$$

Now, since  $\hat{B}_j^*(\tau, z)$  is a separable metrizable topological space and  $\mathcal{B}_j^*(\tau, z)$  is its cover consisting of open sets, it follows from Lindelöf Theorem that there are countably many points  $(\omega^{(s)}, x_s) \in J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$ ,  $s = 1, 2, \dots$ , along with positive integers  $i_s$ ,  $s = 1, 2, \dots$ , such that  $B_{i_s}(\omega^{(s)}, x_s) \in \mathcal{B}_j^*(\tau, z)$  for every  $s \geq 1$  and

$$\bigcup_{s=1}^{\infty} B_{i_s}^*(\omega^{(s)}, x_s) = \hat{B}_j^*(\tau, z).$$

Define then the sets  $F_s$ ,  $s = 1, 2, \dots$ , inductively as follows:

$$F_1 := \left( B \left( f_{\tau|_{n_j}}(z), \frac{1}{4} \eta(\tau, z) \right) \cap f_{\sigma^{n_j}(\omega^{(1)})|_{n_{i_1} - n_j}, f_{\tau|_{n_j}}(x_1)}^{-1} \left( B \left( f_{\omega^{(1)}|_{n_{i_1}}}(x_1), \frac{1}{4} \eta(\omega^{(1)}, x_1) \right) \right) \right) \setminus \text{Sing}(\tilde{f})$$

and

$$F_{s+1} := \left( \left( B \left( f_{\tau|n_j}(z), \frac{1}{4}\eta(\tau, z) \right) \cap \right. \right. \\ \left. \left. \cap f_{\sigma^{n_j}(\omega^{(s+1)})|n_{i_{s+1}-n_j}, f_{\tau|n_j}(x_{s+1})} \left( B \left( f_{\omega^{(s+1)}|n_{i_{s+1}}}(x_{s+1}), \frac{1}{4}\eta(\omega^{(s+1)}, x_{s+1}) \right) \right) \setminus \text{Sing}(\tilde{f}) \right) \right) \setminus \\ \setminus (F_1 \cup F_2 \cup \dots \cup F_s).$$

By virtue of (4.12) we have that

$$\hat{B}_j^*(\tau, z) \subset \bigcup_{s=1}^{\infty} [\tau|n_j] \times f_{\tau|n_j, z}^{-1}(F_s) = \bigcup_{s=1}^{\infty} \tilde{f}_{\tau|n_j, z}^{-n_j}(\Sigma_u \times F_s),$$

and, by the very definition of the sets  $F_s$ , the sets forming this union are mutually disjoint. Using Koebe's Distortion Theorem, we therefore get

$$\begin{aligned} m_t(\hat{B}_j^*(\tau, z)) &\leq \sum_{s=1}^{\infty} m_t \left( \tilde{f}_{\tau|n_j, z}^{-n_j}(\Sigma_u \times F_s) \right) \leq K^t e^{-P(t)n_j} |f'_{\tau|n_j}(z)|^{-t} \sum_{s=1}^{\infty} m_t(\Sigma_u \times F_s) \\ &= K^t e^{-P(t)n_j} |f'_{\tau|n_j}(z)|^{-t} m_t \left( \Sigma_u \times \bigcup_{s=1}^{\infty} F_s \right) \leq K^t e^{-P(t)n_j} |f'_{\tau|n_j}(z)|^{-t} \\ &\leq K^{2t} \frac{m_t \left( [\tau|n_j] \times f_{\tau|n_j, z}^{-1} \left( B \left( f_{\tau|n_j}(z), \frac{1}{4}\eta(\tau, z) \right) \right) \right)}{m_t \left( \Sigma_u \times B \left( f_{\tau|n_j}(z), \frac{1}{4}\eta(\tau, z) \right) \right)} \\ &= K^{2t} \frac{m_t(B_j^*(\tau, z))}{m_t \left( \Sigma_u \times B \left( f_{\tau|n_j}(z), \frac{1}{4}\eta(\tau, z) \right) \right)} \\ &\leq K^{2t} M(\tau, z)^{-1} m_t(B_j^*(\tau, z)), \end{aligned}$$

where

$$M(\tau, z) := \inf \{ m_t \circ p_2^{-1} (B(\xi, \eta(\tau, z)/4)) : \xi \in J(G) \} > 0$$

since  $J(G)$  is a compact set and  $m_t \circ p_2^{-1}$  is positive on non-empty open subsets of  $J(G)$ . Therefore,

$$\overline{\lim}_{j \rightarrow \infty} \left( \delta(B_j^*(\tau, z)) + \frac{m_t(\hat{B}_j^*(\tau, z))}{m_t(B_j^*(\tau, z))} \right) \leq \lim_{j \rightarrow \infty} (e^{-n_j} + K^{2t} M(\tau, z)^{-1}) = K^{2t} M(\tau, z)^{-1} < +\infty.$$

Thus, all the hypothesis of Theorem 2.8.17 in [6], p. 151 are verified and the proof of our lemma is complete.  $\square$

As an immediate consequence of this lemma and Theorem 2.9.11, p. 158 in [6] we get the following.

**Proposition 4.12.** *Let  $G$  be a  $*$ semi-hyperbolic rational semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ . If  $t \in \Delta_G$ , then for every Borel set  $A \subset J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$  with*

$$A_t := \left\{ (\tau, z) \in A : \lim_{j \rightarrow \infty} \frac{m_t(A \cap B_j(\tau, z))}{m_t(B_j(\tau, z))} = 1 \right\},$$

then  $m_t(A_t) = m_t(A)$ .

Now we are in position to prove the following.

**Lemma 4.13.** *Let  $G$  be a  $*$ semi-hyperbolic rational semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ . If  $t \in \Delta_G$ , then every generalized  $t$ -conformal measure  $\nu$  for the skew product map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  is equivalent to  $m_t$ .*

*Proof.* Fix an integer  $v \geq 1$  and let

$$I_v := \{(\tau, z) \in J(\tilde{f}) \setminus \text{Sing}(\tilde{f}) : \eta(\tau, z) \geq 1/v\},$$

where  $\eta(\tau, z) > 0$  is the number produced in Proposition 3.9. We may assume without loss of generality that  $\eta(\tau, z) \leq 1$ . Let also  $(\hat{\tau}, \hat{z})$  and  $(n_j)_{j=1}^\infty$  be the objects produced in this proposition. Fix  $(\tau, z) \in I_v$ . Disregarding finitely many values of  $j$ , we may assume without loss of generality that

$$|f_{\tau|n_j}(z) - \hat{z}| < \frac{1}{8}\eta(\tau, z).$$

Keep  $B_j(\tau, z)$  and  $B_j^*(\tau, z)$  respectively given by formulas (4.7) and (4.8). By Koebe's Distortion Theorem and Proposition 3.9 we get that,

$$\begin{aligned} \nu(B_j(\tau, z)) &= \nu\left(\tilde{f}_{\tau|n_j, z}^{-n_j}\left(\Sigma_u \times B\left(f_{\tau|n_j}(z), \frac{1}{4}\eta(\tau, z)\right)\right)\right) \\ &\asymp e^{-P(t)n_j} |f'_{\tau|n_j}(z)|^{-t} \nu\left(\Sigma_u \times B\left(f_{\tau|n_j}(z), \frac{1}{4}\eta(\tau, z)\right)\right) \\ &= e^{-P(t)n_j} |f'_{\tau|n_j}(z)|^{-t} \nu \circ p_2^{-1}\left(B\left(f_{\tau|n_j}(z), \frac{1}{2}\eta(\tau, z)\right)\right). \end{aligned}$$

Hence,

$$\nu(B_j(\tau, z)) \preceq e^{-P(t)n_j} |f'_{\tau|n_j}(z)|^{-t}$$

and

$$\begin{aligned} \nu(B_j(\tau, z)) &\succeq e^{-P(t)n_j} |f'_{\tau|n_j}(z)|^{-t} \nu \circ p_2^{-1}\left(B\left(f_{\tau|n_j}(z), \frac{1}{4v}\right)\right) \\ &\succeq e^{-P(t)n_j} |f'_{\tau|n_j}(z)|^{-t} \inf\{\nu \circ p_2^{-1}(B(w, 1/(4v))) \mid w \in J(G)\} > 0. \end{aligned}$$

In conclusion

$$(4.13) \quad M_v^{-1} e^{-P(t)n_j} |f'_{\tau|n_j}(z)|^{-t} \leq \nu(B_j(\tau, z)), m_t(B_j(\tau, z)) \leq M_v e^{-P(t)n_j} |f'_{\tau|n_j}(z)|^{-t},$$

with some constant  $M_v \in (0, +\infty)$  depending only on  $v$ . Now suppose that  $F \subset J(\tilde{f})$  is an arbitrary Borel set contained with  $m_t(F) > 0$ . Since  $t \in \Delta_G$  and since  $J(\tilde{f}) \setminus \text{Sing}(\tilde{f}) = \bigcup_{v=1}^\infty I_v$ , there thus exists  $v \geq 1$  such that

$$m_t(F \cap I_v) > 0.$$

By regularity of  $m_t$  there then exists a compact set  $E \subset F \cap I_v$  such that

$$m_t(E) > 0.$$

Fix also  $\varepsilon \in (0, m_t(E)/2)$ . By outer regularity of  $\nu$  and compactness of  $E$  there now exists  $\delta > 0$  such that

$$(4.14) \quad \nu(B(E, \delta)) \leq \nu(E) + \varepsilon.$$

Of course the family

$$\{((\tau, z), B_j^*(\tau, z)) : (\tau, z) \in E \text{ and } B_j^*(\tau, z) \subset B(E, \delta/2)\}$$

is fine at each point of  $E$ . Therefore, because of Lemma 4.11, property (c) of the definition of Vitali relations yields the existence of some countable set  $Y \subset E$  and some function  $j : Y \rightarrow \mathbb{N}$  such that the countable family  $\{B_{j(y)}(y)\}_{y \in Y}$  consists of mutually disjoint sets,

$$B_{j(y)}^*(y) \subset B(E, \delta/2)$$

for every  $y \in Y$ , and also

$$m_t \left( E \setminus \bigcup_{y \in Y} B_{j(y)}^*(y) \right) = 0.$$

But then also

$$m_t \left( E \setminus \bigcup_{y \in Y} B_{j(y)}(y) \right) = 0,$$

and

$$B_{j(y)}(y) \subset B(E, \delta)$$

for every  $y \in Y$ , and, as the set  $J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$  is dense in  $J(\tilde{f})$  and all the sets  $B_j^*(\tau, z)$  are open relative to  $J(\tilde{f})$ , the countable family  $\{B_{j(y)}(y)\}_{y \in Y}$  consists of mutually disjoint sets too. Therefore,

$$\begin{aligned} \nu(F) &\geq \nu(E) \geq \nu(B(E, \delta)) - \varepsilon \geq \nu \left( \bigcup_{y \in Y} B_{j(y)}(y) \right) - \varepsilon \\ &\geq M_v^{-1} \sum_{(\tau, z) \in Y} e^{-P(t)n_j(\tau, z)} |f'_{\tau|_{n_j(\tau, z)}}(z)|^{-t} - \varepsilon \\ &\geq M_v^{-2} \sum_{(\tau, z) \in Y} m_t(B_{j(\tau, z)}(\tau, z)) - \varepsilon \\ &= M_v^{-2} m_t \left( \bigcup_{y \in Y} B_{j(y)}(y) \right) - \varepsilon \\ &\geq M_v^{-2} m_t(E) - \varepsilon. \end{aligned}$$

Letting  $\varepsilon \searrow 0$ , we thus get

$$\nu(F) \geq M_v^{-2} m_t(E) > 0.$$

This shows that

$$m_t|_{J(\tilde{f}) \setminus \text{Sing}(\tilde{f})} \prec \nu|_{J(\tilde{f}) \setminus \text{Sing}(\tilde{f})}.$$

Since in addition  $m_t(\text{Sing}(\tilde{f})) = 0$  (as  $t \in \Delta_G$ ), we thus get that

$$(4.15) \quad m_t \prec \nu.$$

By symmetry we also have that

$$m_t|_{J(\tilde{f}) \setminus \text{Sing}(\tilde{f})} \asymp \nu|_{J(\tilde{f}) \setminus \text{Sing}(\tilde{f})}.$$

Thus, in order to complete the proof it suffices to show that

$$\nu(\text{Sing}(\tilde{f})) = 0.$$

To do this, suppose on the contrary that  $\nu(\text{Sing}(\tilde{f})) > 0$ . Since  $\tilde{f}'$  vanishes on  $\text{Crit}(\tilde{f})$ , the measure

$$\nu_0 = (\nu(\text{Sing}(\tilde{f})))^{-1} \nu|_{\text{Sing}(\tilde{f})},$$

is generalized  $t$ -conformal for  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ . But then (4.15) would be true with  $\nu$  replaced by  $\nu_0$ . We would thus have  $m_t(J(\tilde{f}) \setminus \text{Sing}(\tilde{f})) = 0$ . Since  $m_t(\text{Sing}(\tilde{f})) = 0$ , we would get  $m_t(J(\tilde{f})) = 0$ . This contradiction finishes the proof.  $\square$

Now, we shall prove the following.

**Proposition 4.14.** *Let  $G$  be a  $*$ semi-hyperbolic rational semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ . If  $t \in \Delta_G$ , then the measure  $m_t$  is weakly metrically exact for the skew product map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ . In particular, it is ergodic and conservative.*

*Proof.* Fix a Borel set  $F \subset J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$  with  $m_t(F) > 0$ . Proposition 4.12 there exists at least one point  $(\tau, z) \in F_t$ . Our first goal is to show that

$$(4.16) \quad \lim_{j \rightarrow \infty} \frac{m_t(\tilde{f}^{n_j}(F) \cap p_2^{-1}(B(f_{\tau|n_j}(z), \eta/2)))}{m_t(p_2^{-1}(B(f_{\tau|n_j}(z), \eta/2)))} = 1,$$

where, we recall  $\eta = \eta(\tau, z) > 0$  is the number produced in Proposition 3.9 and  $(n_j)_{j=1}^{\infty}$  is the corresponding sequence produced there. Indeed, suppose for the contrary that

$$\kappa = \frac{1}{2} \liminf_{j \rightarrow \infty} \frac{m_t(p_2^{-1}(B(f_{\tau|n_j}(z), \eta/2)) \setminus \tilde{f}^{n_j}(F))}{m_t(p_2^{-1}(B(f_{\tau|n_j}(z), \eta/2)))} > 0.$$

Then, disregarding finitely many  $j$ 's, we may assume that

$$\frac{m_t(p_2^{-1}(B(f_{\tau|n_j}(z), \eta/2)) \setminus \tilde{f}^{n_j}(F))}{m_t(p_2^{-1}(B(f_{\tau|n_j}(z), \eta/2)))} \geq \kappa > 0$$

for all  $j \geq 1$  and some  $\kappa > 0$ . But

$$\begin{aligned} \tilde{f}_{\tau|n_j, z}^{-n_j} \left( p_2^{-1}(B(f_{\tau|n_j}(z), \eta/2)) \setminus \tilde{f}^{n_j}(F) \right) &\subset \left( [\tau|n_j] \times B \left( z, \frac{1}{2} K \eta \left| f'_{\tau|n_j}(z) \right|^{-1} \right) \right) \setminus F \\ &= B_j(\tau, z) \setminus F \end{aligned}$$

and

$$\begin{aligned}
m_t(\tilde{f}_{\tau|n_j, z}^{-n_j}(p_2^{-1}(B(f_{\tau|n_j}(z), \eta/2)) \setminus f^{n_j}(F))) &\geq \\
&\geq K^{-t} e^{-P(t)n_j} \left| f'_{\tau|n_j}(z) \right|^{-t} m_t(p_2^{-1}(B(f_{\tau|n_j}(z), \eta/2)) \setminus f^{n_j}(F)) \\
&\geq \kappa K^{-t} e^{-P(t)n_j} \left| f'_{\tau|n_j}(z) \right|^{-t} m_t(p_2^{-1}(B(f_{\tau|n_j}(z), \eta/2))) \\
&= \kappa K^{-h} e^{-P(t)n_j} \left| f'_{\tau|n_j}(z) \right|^{-t} m_t \circ p_2^{-1}(B(f_{\tau|n_j}(z), \eta/2)) \\
&\geq \kappa K^{-t} Q_{\eta/2} e^{-P(t)n_j} \left| f'_{\tau|n_j}(z) \right|^{-t},
\end{aligned}$$

where

$$Q_{\eta/2} := \inf \{ m_t \circ p_2^{-1}(B(w, 1/\eta/2)) \mid w \in J(G) \} > 0$$

as  $\text{supp}(m_t \circ p_2^{-1}) = J(G)$ . Hence, making use of Koebe's Distortion Theorem and generalized conformality of  $m_t$ , we obtain, we obtain

$$\begin{aligned}
m_t(B_j(\tau, z) \setminus F) &\geq \kappa K^{-t} Q_{\eta/2} e^{-P(t)n_j} \left| f'_{\tau|n_j}(z) \right|^{-t} \\
&\geq \kappa K^{-2t} Q_{\eta/2} m_t(B_j(\tau, z)).
\end{aligned}$$

Thus,

$$\frac{m_t(B_j(\tau, z) \setminus F)}{m_t(B_j(\tau, z))} \geq \kappa K^{-2t} Q_{\eta/2} > 0.$$

Letting  $j \rightarrow \infty$  this contradicts the fact that  $(\tau, z) \in F_t$  and finishes the proof of (4.16). Now since  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  is topologically exact, there exists  $q \geq 0$  such that  $\tilde{f}^q(p_2^{-1}(B(w, \eta/2))) \supset J(\tilde{f})$  for all  $w \in J(G)$ . It then easily follows from (4.16) and conformality of  $m_t$  that

$$\limsup_{k \rightarrow \infty} m_t(\tilde{f}^k(F)) \geq \limsup_{j \rightarrow \infty} m_t(\tilde{f}^{q+n_j}(F)) = 1.$$

Since  $m_t(\text{Sing}(\tilde{f})) = 0$  (as  $t \in \Delta_G$ ), we are therefore done.  $\square$

As an immediate consequence of this proposition we get the following.

**Corollary 4.15.** *Assume that  $G$  is a  $^*$ semi-hyperbolic rational semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ . If  $t \in \Delta_G$ , then  $m_t(\text{Trans}(\tilde{f})) = 1$ , where, we recall,  $\text{Trans}(\tilde{f})$  is the set of transitive points of  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ , i.e. the set of points  $z \in J(\tilde{f})$  such that the set  $\{f^n(z) : n \geq 0\}$  is dense in  $J(\tilde{f})$ .*

**Corollary 4.16.** *Let  $G$  be a  $^*$ semi-hyperbolic rational semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ . If  $t \in \Delta_G$ , then  $m_t$  is the only generalized  $t$ -conformal measure on  $J(\tilde{f})$  for the map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ .*

*Proof.* Let  $\nu$  be an arbitrary generalized  $t$ -conformal measure on  $J(\tilde{f})$  for the map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ . Since, by Lemma 4.13 the measure  $\nu$  is absolutely continuous with



respect  $m_t$ , it follows from Theorem 2.9.7 in [6], p. 155 and Lemma 4.11 that for  $m_t$ -a.e.  $(\tau, z) \in J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$ ,

$$\frac{d\nu}{dm_t}(\tilde{f}(\tau, z)) = \lim_{j \rightarrow \infty} \frac{\nu(B_j^*(\tilde{f}(\tau, z)))}{m_t(B_j^*(\tilde{f}(\tau, z)))}$$

and

$$\lim_{j \rightarrow \infty} \frac{\nu(\tilde{f}(B_j^*(\tau, z)))}{m_t(\tilde{f}(B_j^*(\tau, z)))} = \lim_{j \rightarrow \infty} \frac{\int_{B_j^*(\tau, z)} |\tilde{f}'|^t d\nu}{\int_{B_j^*(\tau, z)} |\tilde{f}'|^t dm_t} = \lim_{j \rightarrow \infty} \frac{\nu(B_j^*(\tau, z))}{m_t(B_j^*(\tau, z))} = \frac{d\nu}{dm_t}(\tau, z).$$

But, a straightforward calculation shows that  $\tilde{f}(B_j^*(\tau, z))$  is equal to the set  $B_j^*(\tilde{f}(\tau, z))$  obtained with the redefined functions  $\eta(\xi) := \eta(\tilde{f}(\xi))$  and the corresponding sequence  $n_{j-1}(\xi)$ . Therefore by the same token as above

$$\frac{d\nu}{dm_t}(\tilde{f}(\tau, z)) = \lim_{j \rightarrow \infty} \frac{\nu(\tilde{f}(B_j^*(\tau, z)))}{m_t(\tilde{f}(B_j^*(\tau, z)))}$$

for  $m_t$ -a.e.  $(\tau, z) \in J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$ . In consequence

$$\frac{d\nu}{dm_t}(\tilde{f}(\tau, z)) = \frac{d\nu}{dm_t}(\tau, z)$$

for  $m_t$ -a.e.  $(\tau, z) \in J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$ . Since, by Proposition 4.14, the measure  $m_t$  is ergodic, it follows that the Radon–Nikodym derivative  $\frac{d\nu}{dm_t}$  is  $m_t$ -almost everywhere constant. Thus  $\nu = m_t$ , and we are done.  $\square$

Now we pass to consider invariant measures absolutely continuous (and equivalent) to generalized conformal measures  $m_t$ . We first show their existence. Indeed, in order to prove the existence of a Borel probability  $\tilde{f}$ -invariant measure on  $J(\tilde{f})$  equivalent to  $m_t$ ,  $t \in \Delta_G$ , we will use Marco Martens's method which originated in [15]. This means that we shall first produce a  $\sigma$ -finite  $\tilde{f}$ -invariant measure equivalent to  $m_t$  (this is the Marco–Martens's method) and then we will prove this measure to be finite. The heart of Martens's method is the following theorem which is a generalization of Proposition 2.6 from [15] proved by us in [46] as Theorem 8.13. It is a generalization of Martens's original result in the sense that we neither assume our probability space  $(X, \mathcal{B}, m)$  below to be a  $\sigma$ -compact metric space, nor we assume that our map is conservative. Instead, we merely assume that item ((6) in Definition 4.17 holds.

**Definition 4.17.** *Suppose  $(X, \mathcal{B}, m)$  is a probability space. Suppose  $T : X \rightarrow X$  is a measurable mapping, such that  $T(A) \in \mathcal{B}$  whenever  $A \in \mathcal{B}$ , and such that the measure  $m$  is quasi-invariant with respect to  $T$ , meaning that  $m \circ T^{-1} \prec m$ . Suppose further that there exists a countable family  $\{X_n\}_{n=0}^{\infty}$  of subsets of  $X$  with the following properties.*

- (1) For all  $n \geq 0$ ,  $X_n \in \mathcal{B}$ .
- (2)  $m(X \setminus \bigcup_{n=0}^{\infty} X_n) = 0$ .

(3) For all integers  $m, n \geq 0$ , there exists an integer  $j \geq 0$  such that  $m(X_m \cap T^{-j}(X_n)) > 0$ .

(4) For all  $j \geq 0$  there exists a  $K_j \geq 1$  such that for all  $A, B \in \mathcal{B}$  with  $A, B \subset X_j$  and for all  $n \geq 0$ ,

$$m(T^{-n}(A))m(B) \leq K_j m(A)m(T^{-n}(B)).$$

(5)  $\sum_{n=0}^{\infty} m(T^{-n}(X_0)) = +\infty$ .

(6)

$$\lim_{l \rightarrow \infty} m \left( T \left( \bigcup_{j=l}^{\infty} Y_j \right) \right) = 0,$$

where  $Y_j := X_j \setminus \bigcup_{i < j} X_i$ .

Then the map  $T : X \rightarrow X$  is called a Marco–Martens map and  $\{X_j\}_{j=0}^{\infty}$  is called a Marco–Martens cover.

**Remark 4.18.** Let us record the following observations.

- (a) Of course, condition (2) follows from the stronger hypothesis that  $\bigcup_{n=0}^{\infty} X_n = X$ .
- (b) condition (3) imposes that  $m(X_n) > 0$  for all  $n \geq 0$ .
- (c) If  $T$  is conservative with respect to  $\mu$ , then condition (5) is fulfilled.
- (d) If the map  $T : X \rightarrow X$  is finite-to-one, then condition (6) is satisfied. For, if  $T$  is finite-to-one, then  $\bigcap_{l=1}^{\infty} T(\bigcup_{j=l}^{\infty} Y_j) = \emptyset$ .

**Theorem 4.19.** If  $(X, \mathcal{B}, m)$  is a probability space and  $T : X \rightarrow X$  is a Marco–Martens map with a Marco–Martens cover  $\{X_j\}_{j=0}^{\infty}$ , then

- There exists a  $\sigma$ -finite  $T$ -invariant measure  $\mu$  on  $X$  equivalent to  $m$ .
- In addition,  $0 < \mu(X_j) < +\infty$  for each  $j \geq 0$ .

The measure  $\mu$  is constructed in the following way: Let  $l_B : l^\infty \rightarrow \mathbb{R}$  be a Banach limit. For each  $A \in \mathcal{B}$ , set

$$m_n(A) := \frac{\sum_{k=0}^n m(T^{-k}(A))}{\sum_{k=0}^n m(T^{-k}(X_0))}.$$

If  $A \in \mathcal{B}$  and  $A \subset Y_j$  with some  $j \geq 0$ , then we obtain  $(m_n(A))_{n=1}^{\infty} \in l^\infty$ . We set

$$\mu(A) := l_B((m_n(A))_{n=1}^{\infty}).$$

For a general measurable subset  $A \subset X$ , set

$$\mu(A) := \sum_{j=0}^{\infty} \mu(A \cap Y_j).$$

In addition, if for a measurable subset  $A \subset X$ , the sequence  $(m_n(A))_{n=1}^{\infty}$  is bounded, then we have the following formula:

$$(4.17) \quad \mu(A) = l_B((m_n(A))_{n=1}^{\infty}) - \lim_{l \rightarrow \infty} l_B \left( \left( m_n \left( A \cap \bigcup_{j=l}^{\infty} Y_j \right) \right)_{n=0}^{\infty} \right).$$

In particular, if  $A \in \mathcal{B}$  is contained in a finite union of sets  $X_j$ ,  $j \geq 0$ , then

$$\mu(A) = l_B((m_n(A))_{n=1}^\infty).$$

Furthermore, if the measure-preserving transformation  $T : X \rightarrow X$  is ergodic (equivalently with respect to the measure  $m$  or  $\mu$ ), then the  $T$ -invariant measure  $\mu$  is unique up to a multiplicative constant.

Now we are ready to prove the following.

**Theorem 4.20.** *Let  $G$  be a  $*$ semi-hyperbolic rational semigroup. If  $t \in \Delta_G$ , then there exists a unique, up to a multiplicative constant, Borel  $\sigma$ -finite  $\tilde{f}$ -invariant measure  $\mu_t$  on  $J(\tilde{f})$  which is absolutely continuous with respect to  $m_t$ . In addition, the measure  $\mu_t$  is weakly metrically exact and equivalent to  $m_t$ , in particular it is ergodic.*

*Proof.* Since the topological support of  $m_t$  is equal to the Julia set  $J(\tilde{f})$  and since, by Lemma 3.7,  $\text{PCV}(\tilde{f})$  is a nowhere dense subset of  $J(\tilde{f})$ , we have that

$$(4.18) \quad m_t(\text{PCV}(\tilde{f})) < 1.$$

Since the set  $\text{PCV}(\tilde{f})$  is forward invariant under  $\tilde{f}$ , it thus follows from ergodicity and conservativity of  $m_t$  (see Proposition 4.14) that

$$(4.19) \quad m_t(\text{PCV}(\tilde{f})) = 0.$$

Now, because of Lemma 2.4, for every point  $(\omega, z) \in J(\tilde{f}) \setminus \text{PCV}(\tilde{f})$  there exists a radius  $r_{(\omega, z)} > 0$  such that

$$\Sigma_u \times B(z, r_{(\omega, z)}) \subset \Sigma_u \times B(z, 2r_{(\omega, z)}) \subset J(\tilde{f}) \setminus \text{PCV}(\tilde{f}).$$

Since  $J(\tilde{f}) \setminus \text{PCV}(\tilde{f})$  is a separable metrizable space, Lindelöf's Theorem yields the existence of a countable set  $\{(\omega^{(j)}, z_j)\}_{j=0}^\infty \subset J(\tilde{f}) \setminus \text{PCV}(\tilde{f})$  such that

$$(4.20) \quad \bigcup_{j=0}^\infty \Sigma_u \times B(z, r_{(\omega^{(j)}, z_j)}) \supset J(\tilde{f}) \setminus \text{PCV}(\tilde{f}).$$

For every  $j \geq 0$  set

$$X_j := \Sigma_u \times B(z, r_{(\omega^{(j)}, z_j)}).$$

Verifying the conditions of Definition 4.17 (with  $X := J(\tilde{f})$ ,  $T := \tilde{f}$ ,  $m := m_t$ ), we note that  $\tilde{f}$  is nonsingular because of generalized  $t$ -conformality of  $m_t$  and since  $t \in \Delta_G$ . We immediately see that condition (1) is satisfied, that (2) holds because of (4.18) and (4.20), and that (3) holds because of generalized  $t$ -conformality of  $m_t$  and topological exactness of the map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ . Condition (5) follows directly from ergodicity and conservativity of the measure  $m_t$ . Condition (6) follows since  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  is finite-to-one (see Remark 4.18).

Let us prove condition (4). For every  $j \geq 0$  put

$$B_j := B(z, r_{(\omega^{(j)}, z_j)}) \quad \text{and} \quad 2B_j := B(z, 2r_{(\omega^{(j)}, z_j)}).$$

Because of Koebe's Distortion Theorem and generalized  $t$ -conformality of the measure  $m_t$ , we have

$$\begin{aligned}
m_t \circ \tilde{f}^{-n}(A) &= m_t \left( \bigcup_{\rho \in \text{HIB}(2B_j)} \tilde{f}_\rho^{-n}(A) \right) = \sum_{\rho \in \text{HIB}(2B_j)} m_t(\tilde{f}_\rho^{-n}(A)) \\
&\leq \sum_{\rho \in \text{HIB}(2B_j)} K^t e^{-P(t)n} |(\tilde{f}_\rho^{-n})'(\omega^{(j)}, z_j)|^t m_t(A) \\
&= K^{2t} \frac{m_t(A)}{m_t(B)} \sum_{\rho \in \text{HIB}(2B_j)} K^{-t} e^{-P(t)n} |(f_\rho^{-n})'(\omega^{(j)}, z_j)|^t m_t(B) \\
&\leq K^{2t} \frac{m_t(A)}{m_t(B)} \sum_{\rho \in \text{HIB}(2B_j)} m_t(\tilde{f}_\rho^{-n}(B)) \\
&= K^{2t} \frac{m_t(A)}{m_t(B)} m_t \left( \bigcup_{\rho \in \text{HIB}(2B_j)} \tilde{f}_\rho^{-n}(B) \right) \\
&= K^{2t} m_t \circ \tilde{f}^{-n}(B) \frac{m_t(A)}{m_t(B)}.
\end{aligned}$$

Hence,

$$\frac{m_t \circ \tilde{f}^{-n}(A)}{m_t \circ \tilde{f}^{-n}(B)} \leq K^{2t} \frac{m_t(A)}{m_t(B)},$$

and consequently, condition (4) of Definition 4.17 is satisfied. Therefore, Theorem 4.19 produces a Borel  $\sigma$ -finite  $\tilde{f}$ -invariant measure  $\mu_t$  on  $J(\tilde{f})$ , equivalent to  $m_t$ . The proof is complete.  $\square$

## Part 2. The Dynamics of Totally Non-Recurrent Rational Semigroups

### 5. TOTALLY NON-RECURRENT AND C-F BALANCED RATIONAL SEMIGROUPS

In this short section we introduce various classes of rational semigroups with which we will deal in subsequent sections, and we establish their most basic properties.

**Definition 5.1.** *We say that a finitely generated rational semigroup  $G$  generated by a multi-map  $f := (f_1, \dots, f_u) \in \text{Rat}^u$  and satisfying the Assumption (\*) is totally non-recurrent (TNR) if and only if*

- (a) *For each  $z \in J(G)$  there exists a neighborhood  $U$  of  $z$  in  $\hat{\mathbb{C}}$  such that for any sequence  $\{g_n\}_{n=1}^\infty$  in  $G$ , any domain  $V$  in  $\hat{\mathbb{C}}$  and any point  $\zeta \in U$ , the sequence  $\{g_n\}_{n=1}^\infty$  does not converge to  $\zeta$  locally uniformly on  $V$*

*and*

- (b)

$$\text{Crit}_*(f) \cap \text{PCV}(G) = \emptyset.$$

**Remark 5.2.** *Note that if two multi-maps  $f$  and  $h$  generate the same rational semigroup  $G$ , then*

$$\text{Crit}_*(f) \subset G^{-1}(\text{Crit}_*(h)) \quad \text{and} \quad \text{Crit}_*(h) \subset G^{-1}(\text{Crit}_*(f)).$$

*Since also the set  $\text{PCV}(G)$  is forward invariant under  $G$ , i.e.  $G(\text{PCV}(G)) \subset \text{PCV}(G)$ , it follows that item (b) of Definition 5.1 holds for  $f$  if and only if it holds for  $h$ . In conclusion, the concept of being totally non-recurrent is independent of the choice of generators and depends on the semigroup alone.*

The following observations are immediate consequences of Theorem 3.3 and Theorem 3.4.

**Observation 5.3.** *Every TNR rational semigroup is  $*$ semi-hyperbolic.*

**Observation 5.4.** *Let  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  be a multi-map and let  $G = \langle f_1, \dots, f_u \rangle$ . If  $G$  is  $*$ semi-hyperbolic and  $\text{Crit}_*(f)$  has at most one element, then  $G$  is a TNR rational semigroup.*

Now we shall provide a characterization of TNR semigroups more in terms of the skew product map  $\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \rightarrow \Sigma_u \times \hat{\mathbb{C}}$ .

**Proposition 5.5.** *A finitely generated rational semigroup  $G$  generated by a multi-map  $f := (f_1, \dots, f_u) \in \text{Rat}^u$  and satisfying the Assumption (\*) is TNR if and only if the following conditions are satisfied.*

(a) *For each  $z \in J(G)$  there exists a neighborhood  $U$  of  $z$  in  $\hat{\mathbb{C}}$  such that for any sequence  $\{g_n\}_{n=1}^\infty$  in  $G$ , any domain  $V$  in  $\hat{\mathbb{C}}$  and any point  $\zeta \in U$ , the sequence  $\{g_n\}_{n=1}^\infty$  does not converge to  $\zeta$  locally uniformly on  $V$ .*

(b)

$$\text{Crit}_*(\tilde{f}) \cap \text{PCV}(\tilde{f}) = \emptyset.$$

*Proof.* We are to show that conditions (b) of Definition 5.1 and (b) of Proposition 5.5 are equivalent, and this follows immediately from Lemma 2.4. The proof is complete.  $\square$

**Definition 5.6.** *A rational semigroup  $G$  is called C-F balanced if*

$$D(G) := \text{dist}(J(G), \text{PCV}(G) \cap F(G)) > 0.$$

**Remark 5.7.** *Of course the number  $D(G)$  depends only on the semigroup  $G$  and not on any set of generators. The same is therefore true of the concept of being C-F balanced.*

Assume that  $G$  is finitely generated and let  $f := (f_1, \dots, f_u) \in \text{Rat}^u$  be a multi-map generating  $G$ . Given a point  $c \in \text{Crit}_*(f)$  we define

$$\Xi(c) := \left\{ \omega \in \Sigma_u^* : f'_{\omega_1}(c) = 0, f_\omega(c) \in F(G) \quad \text{and} \quad f_{\omega|_k}(c) \in J(G) \text{ for every } 0 \leq k \leq |\omega| - 1 \right\}.$$

It follows from Theorem 1.36 in [32] that the following holds.

**Observation 5.8.** *If  $G$  is  $*$ semi-hyperbolic rational semigroup generated by a multi-map  $f := (f_1, \dots, f_u) \in \text{Rat}^u$ , then  $G$  is C-F balanced if and only if*

$$\text{dist} \left( J(G), F(G) \cap \bigcup_{c \in \text{Crit}_*(f)} \{f_\omega(c) : \omega \in \Xi(c)\} \right) > 0.$$

**Definition 5.9.** A rational semigroup  $G$  generated by a multi-map  $f := (f_1, \dots, f_u) \in \text{Rat}^u$  is called J-F balanced if

$$\text{dist} \left( J(G), \bigcup_{j=1}^u f_j(J(G)) \cap F(G) \right) > 0.$$

**Remark 5.10.** It is easy to see that the concept of being J-F depends only on the semigroup  $G$  alone and not on any set of its generators.

Using again Theorem 1.36 in [32], we immediately have the following.

**Observation 5.11.** If  $G$  is a J-F balanced *\*semi-hyperbolic* rational semigroup, then  $G$  is C-F balanced.

## 6. NICE SETS (FAMILIES)

In this section 6, we explore in detail one of the most important tools for us in the current manuscript. It is commonly referred to as nice sets or nice families. It has been introduced in [28] in the context of rational functions (cyclic semigroup), and extensively used, among others in [25]. We adopt this concept to the setting of rational semigroups. The absolute first fact needed about them is their existence. It is by no means obvious and we devote the whole current section for this task. In the existing proofs for ordinary conformal systems, i.e. cyclic semigroups, the concept of connectivity of the phase space, usually  $\mathbb{C}$  or  $\hat{\mathbb{C}}$ , plays a substantial role. In our present setting of the skew product map

$$\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \longrightarrow \Sigma_u \times \hat{\mathbb{C}},$$

the phase space is “highly” not connected. In order to overcome this difficulty we define the concept of connected families of arbitrary sets. These have sufficiently many properties of ordinary connected sets, e.g. one can speak of connected components of any family of sets, to allow for a proof of the existence of nice families. As a matter of fact, we do not even use the topological concept of connected subsets of the Riemann sphere  $\hat{\mathbb{C}}$ .

We say that  $\mathcal{F}$ , a family of non-empty subsets of  $\Sigma_u \times \hat{\mathbb{C}}$ , is connected if for any two elements  $D, H \in \mathcal{F}$  there exist finitely many sets, call them  $F_1, F_2, \dots, F_n$  such that

- (c)  $F_1 = D$  and  $F_n = H$ , and
- (d)  $F_j \cap F_{j+1} \neq \emptyset$  for all  $j = 1, 2, \dots, n - 1$ .

From now on throughout this section we assume that  $f := (f_1, \dots, f_u) \in \text{Rat}^u$  and  $G = \langle f_1, \dots, f_u \rangle$ . We record the following immediate observations.

**Observation 6.1.** If  $\mathcal{F}$  is a connected family of subsets of  $\Sigma_u \times \hat{\mathbb{C}}$ , then so is the family

$$\tilde{f}(\mathcal{F}) := \{\tilde{f}(F) : F \in \mathcal{F}\}.$$

**Observation 6.2.** If  $\mathcal{F}$  and  $\mathcal{H}$  are two connected families of subsets of  $\Sigma_u \times \hat{\mathbb{C}}$  and  $F \cap H \neq \emptyset$  for some sets  $F \in \mathcal{F}$  and  $H \in \mathcal{H}$ , then the family  $\mathcal{F} \cup \mathcal{H}$  is also connected.

This latter observation enables us to speak about connected components of any given family  $\mathcal{F}$  of subsets of  $\Sigma_u \times \hat{\mathbb{C}}$ . These are, by definition, maximal connected subfamilies of  $\mathcal{F}$ . These components are mutually disjoint but even more, the unions of all their elements are mutually disjoint. In addition, the union of all connected components of  $\mathcal{F}$  is equal to  $\mathcal{F}$ .

Let  $V \subset \hat{\mathbb{C}}$  be a non-empty, open, connected, simply connected set such that

$$V \cap \text{PCV}(G) = \emptyset.$$

Given an integer  $n \geq 0$  we set

$$\text{HIB}_{\leq n}(V) := \bigcup_{k=0}^n \text{HIB}_k(V) = \{\rho \in \text{HIB}(V) : \|\rho\| \leq n\}.$$

We also define

$$\begin{aligned} \text{Pre}(V) &:= \{\tilde{f}_\rho^{-\|\rho\|}(V) : \rho \in \text{HIB}(V)\}, \\ \text{Pre}_n(V) &:= \{\tilde{f}_\rho^{-n}(V) : \rho \in \text{HIB}_n(V)\}, \\ \text{Pre}_{\leq n}(V) &:= \{\tilde{f}_\rho^{-\|\rho\|}(V) : \rho \in \text{HIB}_{\leq n}(V)\}. \end{aligned}$$

From now on throughout this section  $G$  is assumed to be a TNR rational semigroup generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . We can then take, and from now on fix, an arbitrary

$$R_*(G) \in \left(0, \frac{1}{16} \min \left\{ \text{dist}(\text{Crit}_*(f), \text{PCV}(G)), \min \{|c_2 - c_1| : c_1, c_2 \in \text{Crit}(G), c_1 \neq c_2\} \right\} \right)$$

so small as required in the Exponential Shrinking Property, i.e. in Theorem 3.5. Because of the Exponential Shrinking Property, i.e. Theorem 3.5, for every  $\varepsilon > 0$  there exists  $\delta_\varepsilon \in (0, \varepsilon/4)$  such that

$$(6.1) \quad \text{diam}(W) < \varepsilon$$

for every element  $g \in G$ , every  $\xi \in J(G)$ , and  $W$ , any connected component of  $g^{-1}(B(\xi, \delta_\varepsilon))$ . We shall prove the following.

**Lemma 6.3.** *Let  $G$  be a TNR rational semigroup generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Fix  $R > 0$ . Fix any  $r \in (0, \delta_R/4)$ . Fix a finite set  $S \subset J(G) \setminus B(\text{PCV}(G), 8R)$ . Suppose that  $\tilde{f}_\alpha^{-\|\alpha\|}(\Sigma_u \times B_2(a, r))$  and  $\tilde{f}_\beta^{-\|\beta\|}(\Sigma_u \times B_2(b, r))$  are two arbitrary elements of*

$$\text{Pre}(S, r) := \bigcup_{s \in S} \text{Pre}(B_2(s, r))$$

such that

$$(6.2) \quad \tilde{f}_\alpha^{-\|\alpha\|}(\Sigma_u \times B_2(a, 4r)) \cap \tilde{f}_\beta^{-\|\beta\|}(\Sigma_u \times B_2(b, 4r)) \neq \emptyset.$$

Assume without loss of generality that  $|\tilde{\beta}| \geq |\tilde{\alpha}|$ , with the “ $\sim$ ” operation defined by formulas (2.9)–(2.11). Then

$$(6.3) \quad \beta = \alpha\gamma,$$

where  $\gamma$  represents the holomorphic branch of  $\tilde{f}^{-(\|\beta\|-\|\alpha\|)}$  defined on  $\Sigma_u \times B_2(b, 4R)$ , in particular belonging to  $\text{HIB}(B_2(b, 4R))$ , and sending any point  $(\omega, z) \in \Sigma_u \times B_2(b, 4R)$  to the point  $(\tilde{\beta}|_{|\tilde{\beta}|-|\tilde{\alpha}|}\omega, f_{\tilde{\alpha}}(f_{\tilde{\beta}}^{-1}(z)))$ . Formula (6.3) more precisely means that

$$(6.4) \quad \tilde{f}_{\gamma}^{-\|\gamma\|}(\Sigma_u \times B_2(b, 4R)) \subset \Sigma_u \times B_2(a, 2R)$$

and

$$(6.5) \quad \tilde{f}_{\beta}^{-\|\beta\|}|_{\Sigma_u \times B_2(b, 4R)} = \tilde{f}_{\alpha}^{-\|\alpha\|} \circ \tilde{f}_{\gamma}^{-\|\gamma\|}|_{\Sigma_u \times B_2(b, 4R)}.$$

Furthermore,

(a) If (6.2) holds (and (6.3) does then too), then we say that  $\beta$  is an extension of  $\alpha$  or that  $\alpha$  is a restriction of  $\beta$ .

(b) Formula (6.3), i.e. (6.4) and (6.5) taken together, yields

$$(6.6) \quad \tilde{f}_{\alpha}^{-\|\alpha\|}(\Sigma_u \times B_2(a, 2R)) \supset \tilde{f}_{\beta}^{-\|\beta\|}(\Sigma_u \times B_2(b, 4R)).$$

*Proof.* All what we are to prove is that (6.2) entails (6.3). So assume that (6.2) holds. Then  $[\tilde{\alpha}] \cap [\tilde{\beta}] \neq \emptyset$  and (we assume that)  $|\tilde{\beta}| \geq |\tilde{\alpha}|$ . We thus conclude that  $\tilde{\beta}$  is an extension of  $\tilde{\alpha}$ , which means that

$$\tilde{\beta} = \tilde{\alpha} \tilde{\beta}|_{|\tilde{\beta}|-|\tilde{\alpha}|}.$$

Formula (6.2) also gives

$$(6.7) \quad f_{\alpha}^{-1}(B_2(a, 4R)) \cap f_{\beta}^{-1}(B_2(b, 4R)) \neq \emptyset.$$

Therefore

$$(6.8) \quad f_{\beta}^{-1} = f_{\alpha^*}^{-1} \circ f_{\gamma}^{-1},$$

where  $\alpha^*$  refers to the holomorphic branch of  $f_{\tilde{\alpha}}^{-1}$  defined on  $f_{\gamma}^{-1}(B_2(b, 4R))$  and sending  $f_{\gamma}^{-1}(b)$  to  $f_{\beta}^{-1}(b)$ . More precisely, it defines the map

$$\Sigma_u \times f_{\gamma}^{-1}(B_2(b, 4R)) \ni (\omega, z) \longmapsto \tilde{f}_{\alpha^*}^{-\|\alpha\|}(\omega, z) = \left( \tilde{\alpha}\omega, f_{\beta}^{-1} \left( f_{\tilde{\beta}|_{|\tilde{\beta}|-|\tilde{\alpha}|}}^{-1}(z) \right) \right).$$

By applying  $f_{\tilde{\alpha}}$  to (6.7), we get

$$B_2(a, 4R) \cap f_{\gamma}^{-1}(B_2(b, 4R)) \neq \emptyset.$$

Due to our choice of  $r$  we have that

$$\text{diam}(f_{\gamma}^{-1}(B_2(b, 4R))) < R.$$

Hence

$$B_2(a, r) \cup f_{\gamma}^{-1}(B_2(b, 4R)) \subset B_2(a, 2R) \subset B_2(a, 4R).$$

Therefore,  $f_{\alpha^*}^{-1}$  extends uniquely to  $B_2(a, 4R)$  as a holomorphic branch of  $f_{\tilde{\alpha}}^{-1}$ ; we can thus treat  $\alpha^*$  as an element of  $\text{HIB}(B_2(a, 4R))$ . In view of (6.8) and (6.7) this yields

$$\begin{aligned} f_{\alpha}^{-1}(B_2(a, 4R)) \cap f_{\alpha^*}^{-1}(B_2(a, 4R)) &\supset f_{\alpha}^{-1}(B_2(a, 4R)) \cap f_{\alpha^*}^{-1}(f_{\gamma}^{-1}(B_2(b, 4R))) \\ &= f_{\alpha}^{-1}(B_2(a, 4R)) \cap f_{\alpha^*}^{-1} \circ f_{\gamma}^{-1}(B_2(b, 4R)) \\ &= f_{\alpha}^{-1}(B_2(a, 4R)) \cap f_{\beta}^{-1}(B_2(b, 4R)) \neq \emptyset. \end{aligned}$$



Thus  $\alpha^* = \alpha$  and the proof of our lemma is complete.  $\square$

We call a set  $S \subset \hat{\mathbb{C}}$  aperiodic if

$$S \cap G(S) = \emptyset.$$

An immediate but important for us observation is this.

**Observation 6.4.** *If  $G = \langle f_1, \dots, f_u \rangle$  is a TNR semigroup, then  $\text{Crit}_*(f)$  is aperiodic.*

**Definition 6.5.** *Let  $G = \langle f_1, \dots, f_u \rangle$  be a TNR semigroup. Fix  $R > 0$ . Let*

$$\text{Crit}_*(f) \subset S \subset J(G) \setminus B(\text{PCV}(G), 128K^2R)$$

*be a finite aperiodic set. A collection  $\mathcal{U} = \{U_s\}_{s \in S}$  of open subsets of  $\Sigma_u \times \hat{\mathbb{C}}$  is called a nice family for  $\tilde{f}$  if the following conditions are satisfied.*

- (a) *The sets  $U_s$ ,  $s \in S$ , are mutually disjoint,*
- (b)  *$p_2(U_s) \subset B_2(s, 2R)$  for each  $s \in S$ ,*
- (c) *For any  $a, b \in S$  and any  $\rho \in \text{HIB}(U_b)$  we have either that*

$$U_a \cap \tilde{f}_\rho^{-\|\rho\|}(U_b) = \emptyset \quad \text{or} \quad \tilde{f}_\rho^{-\|\rho\|}(U_b) \subset U_a.$$

The main result of this section is the following.

**Theorem 6.6.** *Let  $G$  be a finitely generated TNR semigroup. Let  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  be a multi-map generating  $G$ . Fix  $R \in (0, R_*(G))$ . Fix also  $\kappa \in (1, 2)$ . Let*

$$\text{Crit}_*(f) \subset S \subset J(G) \setminus B(\text{PCV}(G), 8R)$$

*be a finite aperiodic set. Then for every  $r \in (0, R]$  small enough there exists*

$$\mathcal{U}_S(\kappa, r) = \{U_s(\kappa, r)\}_{s \in S},$$

*a nice family of sets for  $\tilde{f}$ , associated to the set  $S$ , such that*

(A)

$$\Sigma_u \times B_2(s, r) \subset U_s(\kappa, r) \subset \Sigma_u \times B_2(s, \kappa r)$$

*for each  $s \in S$ .*

(B) *If  $a, b \in S$ ,  $\rho \in \text{HIB}(U_b(\kappa, r))$ , and  $\tilde{f}_\rho^{-\|\rho\|}(U_b(\kappa, r)) \subset U_a(\kappa, r)$ , then*

$$\left| (f_\rho^{-1})'(z) \right| \leq \frac{1}{4}$$

*for all  $z \in B_2(b, 2R) \supset p_2(U_b(\kappa, r))$ ,*

**Remark 6.7.** *We would like to note right away that with  $r \in (0, R]$ , condition (A) of Theorem 6.6 alone entails conditions (a) and (b) of Definition 6.5.*

*Proof.* Fix  $R \in (0, R_*(G)]$  arbitrary. Then, by Theorem 3.5 and by Koebe's Distortion Theorem there exists an integer  $N \geq 1$  so large that

$$(6.9) \quad \left| (f_\rho^{-1})'(x) \right| \leq \min \left\{ \frac{1}{4}, \frac{\kappa - 1}{2\kappa} \right\}$$

for all  $s \in S$ , all  $\rho \in \text{HIB}(B_2(s, 4R))$  with  $\|\rho\| \geq N$ , and all  $x \in B(s, 4R_*(G))$ . Now, because the set  $S$  is aperiodic there exists  $r \in (0, R]$  so small that if  $a, b \in S$ ,  $\text{Id} \neq \rho \in \text{HIB}(B(b, 4R))$ , and

$$f_\rho^{-1}(B_2(b, 2r)) \cap B_2(a, 2r) \neq \emptyset,$$

then

$$(6.10) \quad \text{diam}(f_\rho^{-1}(B_2(b, r))) < R \quad \text{and} \quad \|\rho\| \geq N.$$

For every integer  $n \geq 0$  set

$$\begin{aligned} \text{Pre}_{\leq n}(S, r) &:= \bigcup_{s \in S} \text{Pre}_{\leq n}(B_2(s, r)) \\ &= \left\{ \tilde{f}_\rho^{-\|\rho\|}(\Sigma_u \times B_2(s, r)) : s \in S, \rho \in \text{HIB}(B_2(s, 4R)), \|\rho\| \leq n \right\}. \end{aligned}$$

Furthermore, for every  $s \in S$  denote by  $\mathcal{N}_n(s)$  the (unique) connected component of the set  $\text{Pre}_{\leq n}(S, r)$  containing  $\Sigma_u \times B_2(s, r)$ . Our first goal is to prove the following.

**Lemma 6.8.** *For every integer  $n \geq 0$  and every  $s \in S$ , we have that*

$$\bigcup \mathcal{N}_n(s) \subset \Sigma_u \times B_2(s, \kappa r).$$

*Proof.* For  $n = 0$  this is immediate as  $\mathcal{N}_0(s) = \{\Sigma_u \times B_2(s, r)\}$ . Proceeding further by induction suppose that our lemma holds for all  $s \in S$  and all  $0 \leq j \leq n - 1$  with some  $n \geq 1$ . We are to prove it for the integer  $n$ . Towards this aim, given  $s \in S$ , let  $\mathcal{M}$  be an arbitrary connected component of the family

$$\mathcal{N}_n(s) \setminus \{\Sigma_u \times B_2(s', r)\}_{s' \in S}.$$

From now on throughout this proof we let  $k \geq 0$  to be the minimal length of an element  $\rho \in \bigcup_{a \in S} \text{HIB}(B_2(a, 4R))$  such that

$$\tilde{f}_\rho^{-1}(\Sigma_u \times B_2(\xi, r)) \in \mathcal{M}$$

for some  $\xi \in S$ . By the very definition of  $k$  and  $\mathcal{M}$  we have that  $k \geq 1$ . Again by the definition of  $k$  we have that

$$\tilde{f}^k(\mathcal{M}) \subset \text{Pre}_{\leq n-k}(S, r).$$

But

$$\Sigma_u \times B_2(\xi, r) = \tilde{f}^k(\tilde{f}_\rho^{-\|\rho\|}(\Sigma_u \times B_2(\xi, r))) \in \tilde{f}^k(\mathcal{M}),$$

and, by Observation 6.1, the family  $\tilde{f}^k(\mathcal{M}(s))$  is connected. Thus

$$\tilde{f}^k(\mathcal{M}) \subset \mathcal{N}_{n-k}(\xi).$$

Hence, our inductive hypothesis yields

$$\tilde{f}^k\left(\bigcup \mathcal{M}\right) = \bigcup \tilde{f}^k(\mathcal{M}(s)) \subset \Sigma_u \times B_2(\xi, \kappa r).$$

Therefore,

$$(6.11) \quad \bigcup \mathcal{M} \subset \tilde{f}^{-k}(\Sigma_u \times B_2(\xi, \kappa r)) = \bigcup_{\alpha \in \text{HIB}_k(B_2(\xi, 4R))} \tilde{f}_\alpha^{-k}(\Sigma_u \times B_2(\xi, \kappa r)).$$

So, if  $\tilde{f}_\beta^{-\|\beta\|}(\Sigma_u \times B_2(b, r))$  is an arbitrary element in  $\mathcal{M}$ , then there must exist  $\alpha \in \text{HIB}_k(B(\xi, 4R))$  (remembering that  $\kappa < 2$ ) such that

$$(6.12) \quad \tilde{f}_\beta^{-\|\beta\|}(\Sigma_u \times B_2(b, r)) \cap \tilde{f}_\alpha^{-\|\alpha\|}(\Sigma_u \times B_2(\xi, 2r)) \neq \emptyset.$$

Hence, according to Lemma 6.3 (keeping in mind that  $\|\beta\| \geq k = \|\alpha\|$ ),

$$(6.13) \quad \tilde{f}_\beta^{-\|\beta\|} \Big|_{\Sigma_u \times B_2(b, 4r)} = \tilde{f}_\alpha^{-\|\alpha\|} \circ \tilde{f}_\gamma^{-\|\gamma\|} \Big|_{\Sigma_u \times B_2(b, 4r)}$$

with  $\gamma$  as specified in this lemma. Likewise if  $\omega \in \text{HIB}_k(B_2(\xi, 4R))$  is another, different from  $\alpha$ , element such that

$$\tilde{f}_\beta^{-\|\beta\|}(\Sigma_u \times B_2(b, r)) \cap \tilde{f}_\omega^{-\|\omega\|}(\Sigma_u \times B_2(\xi, 2r)) \neq \emptyset,$$

then

$$(6.14) \quad \tilde{f}_\beta^{-\|\beta\|} \Big|_{\Sigma_u \times B_2(b, 4r)} = \tilde{f}_\omega^{-\|\omega\|} \circ \tilde{f}_\delta^{-\|\delta\|} \Big|_{\Sigma_u \times B_2(b, 4r)}$$

with  $\delta$  as specified in this Lemma 6.3. It follows from (6.13) and (6.14) that  $\tilde{\alpha} = \tilde{\beta}|_k = \tilde{\omega}$ . So,

$$f_\alpha^{-1}(B_2(\xi, 4R)) \cap f_\omega^{-1}(B_2(\xi, 4R)) = \emptyset$$

as  $\alpha \neq \omega$ . But this contradicts (6.13) and (6.14), and shows that  $\alpha$  is the only element of  $\text{HIB}_k(B_2(\xi, 4R))$  such that (6.12) holds. Hence,

$$(6.15) \quad \tilde{f}_\beta^{-\|\beta\|}(\Sigma_u \times B_2(b, r)) \subset \tilde{f}_\alpha^{-\|\alpha\|}(\Sigma_u \times B_2(\xi, \kappa r)).$$

Therefore, since the family  $\{\tilde{f}_\rho^{-\|\rho\|}(\Sigma_u \times B_2(\xi, \kappa r)) : \rho \in \text{HIB}_k(B_2(\xi, 4R))\}$  consists of mutually disjoint sets and since the family  $\mathcal{M}$  is connected, we conclude that there exists a unique  $\tau \in \text{HIB}_k(B_2(\xi, 4R))$  such that

$$(6.16) \quad \bigcup \mathcal{M} \subset \tilde{f}_\tau^{-k}(\Sigma_u \times B_2(\xi, \kappa r)).$$

Again, since  $\mathcal{M}$  is a connected component of  $\mathcal{N}_n(s) \setminus \{\Sigma_u \times B_2(s', r)\}_{s' \in S}$  which is a proper subset of  $\mathcal{N}_n(s)$  (as it misses  $B(s, r)$ ), and since  $\mathcal{N}_n(s)$  is connected, there must exist  $\tilde{f}_\alpha^{-\|\alpha\|}(B(a, r)) \in \mathcal{M}$  and  $s' \in S$  such that

$$(6.17) \quad \tilde{f}_\alpha^{-\|\alpha\|}(\Sigma_u \times B_2(a, r)) \cap (\Sigma_u \times B_2(s', r)) \neq \emptyset.$$

Looking at this and (6.16), and remembering that  $\kappa < 2$ , we see that

$$(6.18) \quad \tilde{f}_\tau^{-k}(\Sigma_u \times B_2(\xi, 2r)) \cap (\Sigma_u \times B_2(s', r)) \supset \tilde{f}_\tau^{-k}(\Sigma_u \times B_2(\xi, \kappa r)) \cap B_2(s', r) \neq \emptyset.$$

Hence,  $k = \|\tau\| \geq N$  by virtue of (6.10). Also, of course (6.18) yields

$$(6.19) \quad f_\tau^{-1}(B_2(p_2(\xi), \kappa r)) \cap B_2(p_2(s'), r) \neq \emptyset.$$

Combining this and (6.9), we get that

$$(6.20) \quad \bigcup \mathcal{M} \subset \Sigma_u \times B_2\left(s', r + \frac{\kappa - 1}{2\kappa}(2\kappa r)\right) = \Sigma_u \times B_2(s', \kappa r).$$

Since  $\mathcal{N}_n(s)$  is a union of all such components  $\mathcal{M}$  along with some elements of  $\{B(a, r)\}_{a \in S}$ , and since  $\kappa > 1$ , we thus conclude that

$$\bigcup \mathcal{N}_n(s) \subset \bigcup_{a \in S} \Sigma_u \times B_2(a, \kappa r).$$

But since the sets  $\Sigma_u \times B_2(a, \kappa r)$ ,  $a \in S$ , are mutually disjoint and since  $\mathcal{N}(s)$  is a connected family, there must exist  $b \in S$  such that

$$\bigcup \mathcal{N}_n(s) \subset \Sigma_u \times B_2(b, \kappa r).$$

Since  $\Sigma_u \times B_2(s, r) \subset \bigcup \mathcal{N}_n(s)$ , we must have  $b = s$ , and in conclusion

$$\bigcup \mathcal{N}_n(s) \subset \Sigma_u \times B_2(s, \kappa r).$$

The proof of Lemma 6.8 is thus complete.  $\square$

Now we can complete the proof of Theorem 6.6. We shall show that the family

$$\mathcal{U}_S := \left\{ U_s = U_s(\kappa, r) := \bigcup_{n=0} \bigcup \mathcal{N}_n(s) \right\}_{s \in S}$$

has all the required properties, i.e. being a nice family of sets along with conditions (A) and (B) required in Theorem 6.6.

Firstly,  $U_s(\kappa, r)$ ,  $s \in S$ , are all open subsets of  $\Sigma_u \times \hat{\mathbb{C}}$  since all elements of  $\mathcal{N}_n(s)$  are open.

Secondly, for every  $n \geq 0$ ,  $B(s, r) \in \mathcal{N}_n(s)$ , so

$$(6.21) \quad \Sigma_2 \times B_2(s, r) \subset \bigcup \mathcal{N}_n(s) \subset U_s(\kappa, r)$$

for every  $s \in S$ .

Thirdly, it directly follows from Lemma 6.8 that

$$(6.22) \quad U_s(\kappa, r) \subset \Sigma_2 \times B_2(s, \kappa r)$$

for all  $s \in S$ . Along with (6.21) this means that property (A) of Theorem 6.6 holds. Therefore, by virtue of Remark 6.7 the properties (a) and (b) of Definition 6.5 (of a nice family of sets) also hold. In addition, condition (B) of Theorem 6.6 follows immediately from (6.9), the already proven property (b) of Definition 6.5, and the definition of  $r$ . We thus are only left to show that item (c) of Definition 6.5 is satisfied. In order to prove this item assume that

$$(6.23) \quad \tilde{f}_\alpha^{-k}(U_b) \cap U_a \neq \emptyset$$

for some  $a, b \in S$  and some  $\alpha \in \text{HIB}(U_b)$ , where  $k = \|\alpha\|$ . Fix an arbitrary integer  $n \geq 0$  such that

$$(6.24) \quad U_a \cap \tilde{f}_\alpha^{-k} \left( \bigcup \mathcal{N}_n(b) \right) \neq \emptyset.$$

Now consider  $\tilde{f}_\beta^{-l}(B(\xi, r))$ ,  $\xi \in S$ ,  $l = \|\beta\| \geq 1$ , an arbitrary element of  $\mathcal{N}_n(b)$  such that

$$U_a \cap \tilde{f}_\alpha^{-k} \left( \tilde{f}_\beta^{-l}(\Sigma_u \times B_2(\xi, r)) \right) \neq \emptyset.$$

But since  $(\mathcal{N}_n(a))_{n=0}^\infty$  is an ascending sequence of sets, there thus exists  $p \geq k+l$  such that

$$(6.25) \quad \bigcup \mathcal{N}_p(a) \cap \tilde{f}_\alpha^{-k} \circ \tilde{f}_\beta^{-l}(\Sigma_u \times B_2(\xi, r)) \neq \emptyset.$$

Denote by  $\alpha\beta$  the only element of  $\text{HIB}(B(b, 4R))$  such that

$$\tilde{f}_{\alpha\beta}^{-(k+l)}|_{\Sigma_u \times B_2(\xi, r)} = \tilde{f}_\alpha^{-k} \circ \tilde{f}_\beta^{-l}|_{\Sigma_u \times B_2(\xi, r)}.$$

But then (6.25) means that  $\tilde{f}_{\alpha\beta}^{-(k+l)}(\Sigma_u \times B_2(\xi, r))$  intersects at least one element of  $\mathcal{N}_p(a)$ , and therefore  $\tilde{f}_{\alpha\beta}^{-(k+l)}(\Sigma_u \times B_2(\xi, r)) \in \mathcal{N}_p(a)$ . Consequently,

$$\tilde{f}_\alpha^{-k}(\tilde{f}_\beta^{-l}(\Sigma_u \times B_2(\xi, r))) = \tilde{f}_{\alpha\beta}^{-(k+l)}(\Sigma_u \times B_2(\xi, r)) \subset U_a.$$

We thus have that

$$(6.26) \quad \tilde{f}_\alpha^{-k}\left(\bigcup \mathcal{N}_n^*(a, b)\right) \subset U_a,$$

where

$$\mathcal{N}_n^*(a, b) := \{\Gamma \in \mathcal{N}_n(b) : U_a \cap \tilde{f}_\alpha^{-k}(\Gamma) \neq \emptyset\}.$$

Now, if  $\Gamma_1 \in \mathcal{N}_n^*(a, b)$  and  $\Gamma_2 \in \mathcal{N}_n(b) \setminus \mathcal{N}_n^*(a, b)$ , then on the one hand  $\tilde{f}_\alpha^{-k}(\Gamma_1) \subset U_a$  and, on the other hand,  $\tilde{f}_\alpha^{-k}(\Gamma_2) \cap U_a = \emptyset$ . Hence,  $\Gamma_1 \cap \Gamma_2 = \emptyset$ . Since however  $\mathcal{N}_n(b)$  is a connected family, this implies that either  $\mathcal{N}_n^*(a, b) = \emptyset$  or  $\mathcal{N}_n(b) \setminus \mathcal{N}_n^*(a, b) = \emptyset$ . But by (6.24), we have that  $\mathcal{N}_n^*(a, b) \neq \emptyset$ , yielding  $\mathcal{N}_n^*(a, b) = \mathcal{N}_n(b)$ . Along with (6.26) this gives

$$(6.27) \quad \tilde{f}_\alpha^{-k}\left(\bigcup \mathcal{N}_n(b)\right) \subset U_a.$$

Since by (6.23), formula (6.24) holds for all  $n \geq 1$  sufficiently large, and since  $(\mathcal{N}_n(b))_{n=0}^\infty$  is an ascending sequence of sets, formula (6.27) yields

$$\tilde{f}_\alpha^{-k}(U_b) \subset U_a.$$

The proof of property (c) of Definition 6.5 is complete. This simultaneously finishes the proof of Theorem 6.6.  $\square$

The most important consequence of having a nice family of sets is that it gives rise to a graph directed system in the sense of [19] which has a sufficient degree of conformality. Namely, let

$$U := \bigcup_{s \in S} U_s,$$

and for every  $s \in S$  and any integer  $n \geq 0$  let

$$\mathcal{D}_n^*(s) = \mathcal{D}_n^*(G, s) := \{\tau \in \text{HIB}_n(U_s) : \tilde{f}_\tau^k(\tilde{f}_\tau^{-n}(U_s)) \cap U = \emptyset \text{ for all } 0 \leq k \leq n-1\},$$

and let

$$\mathcal{D}_n(s) = \mathcal{D}_n(G, s) := \{\tau \in \text{HIB}_n(U_s) : \tilde{f}_\tau^{-\|\tau\|}(U_s) \cap U \neq \emptyset \text{ and } \tilde{f} \circ \tilde{f}_\tau^{-\|\tau\|} \in \mathcal{D}_{n-1}^*(s)\}.$$

Finally let

$$\mathcal{D}_n = \mathcal{D}_n(G) := \bigcup_{s \in S} \mathcal{D}_n(s) \text{ and } \mathcal{D}_\mathcal{U} = \mathcal{D}_\mathcal{U}(G) := \bigcup_{n=1}^\infty \mathcal{D}_n.$$

We will usually skip the indication of these sets on  $G$  since we only very rarely deal with more than one rational semigroup at a time. This will be however the case for example in Section 14, Multifractal Analysis of Invariant Measures  $\mu_t$ , in the context of non-exceptional rational semigroups.

We now note that for every element  $\tau \in \mathcal{D}_U$  there are a unique element  $t(\tau) \in S$  such that  $\tau \in \text{HIB}(B(t(\tau), 4R))$  and a unique element  $i(\tau) \in S$  such that  $\tilde{f}_\tau^{-\|\tau\|}(U_{t(\tau)}) \subset U_{i(\tau)}$ . For every element  $s \in S$  let

$$X_s := \overline{U}_s.$$

As an immediate consequence of Definition 6.5, Theorem 6.6 (particularly its part (B)) and topological exactness of the dynamical system  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  (implying item (c) below), we get the following.

**Theorem 6.9.** *If  $G$  is a TNR rational semigroup generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$  and  $\mathcal{U} = \{U_s\}_{s \in S}$  is a nice family of sets for  $\tilde{f}$  produced in Theorem 6.6, then the family*

$$\mathcal{S}_U = \mathcal{S}_U(G) := \{\tilde{f}_\tau^{-\|\tau\|} : X_{t(\tau)} \rightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_U}$$

*forms a graph directed system in the sense of [19]. Furthermore,*

- (a) *The corresponding incidence matrix  $A(\mathcal{U}) = A(G, \mathcal{U})$  is then determined by the condition that*

$$A_{\tau\omega}(\mathcal{U}) = 1$$

*if and only if  $t(\tau) = i(\omega)$ .*

- (b) *The limit set  $\tilde{J}_U$  of the system  $\mathcal{S}_U$  is contained in  $J(\tilde{f})$  and contains  $U \cap \text{Trans}(\tilde{f})$ , where, we recall,  $\text{Trans}(\tilde{f})$  is the set of transitive points of  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ , i.e. the set of points  $z \in J(\tilde{f})$  such that the set  $\{\tilde{f}^n(z) : n \geq 0\}$  is dense in  $J(\tilde{f})$ .*

- (c) *The graph directed system  $\mathcal{S}_U$  is finitely primitive.*

We denote by  $\mathcal{D}_U^\infty$  the symbol space  $(\mathcal{D}_U)_{A(\mathcal{U})}^\infty$  generated by the matrix  $A(\mathcal{U})$ ; as in the case of  $\Sigma_u$  its elements (infinite sequences) start with coordinates labeled by the integer 1. Likewise  $\mathcal{D}_U^*$  and  $\mathcal{D}_U^n$ ,  $n \in \mathbb{N}$ , abbreviate respectively  $(\mathcal{D}_U)_{A(\mathcal{U})}^*$  and  $(\mathcal{D}_U)_{A(\mathcal{U})}^n$ ,  $n \in \mathbb{N}$ .

In addition, we denote by  $\phi_e$ ,  $e \in \mathcal{D}_U$ , all the elements of  $\mathcal{S}_U$ .

## 7. THE BEHAVIOR OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES $\mu_t$ NEAR CRITICAL POINTS

This section is very technical and devoted to study the behavior of conformal measures  $m_t$  and their invariant versions  $\tilde{\mu}_t$  near critical points of the skew product map

$$\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \rightarrow \Sigma_u \times \hat{\mathbb{C}}.$$

Its main outcome is Proposition 7.7 which gives a quantitative strengthening of quasi-invariance of conformal measures  $m_t$ . This is the first and only place where the hypothesis of finite type of the semigroup  $G$  is explicitly needed; it demands that the set of critical points of  $\tilde{f}$  lying in the Julia set  $J(\tilde{f})$  of  $\tilde{f}$  is finite.

Let  $G$  be a \*semi-hyperbolic rational semigroup generated by  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Let  $V \neq \emptyset$  be an open subset of  $\Sigma_u \times \hat{\mathbb{C}}$  containing  $\text{Crit}_*(\tilde{f})$ . Define

$$K(V) := \bigcap_{n=0}^{\infty} \tilde{f}^{-n}(J(\tilde{f}) \setminus V) = \{z \in J(\tilde{f}) : \tilde{f}^n(z) \notin V \forall n \geq 0\}.$$

Of course  $K(V)$  is a closed subset of  $J(\tilde{f})$ ,

$$(7.1) \quad \tilde{f}(K(V)) \subset K(V),$$

and

$$(7.2) \quad K(V) \cap \text{Crit}(\tilde{f}) = \emptyset.$$

So, we can consider the dynamical system  $\tilde{f}|_{K(V)} : K(V) \rightarrow K(V)$ . Because of (7.2) for every  $t \in \mathbb{R}$  the potential  $-t \log |\tilde{f}'| : K(V) \rightarrow \mathbb{R}$  is continuous, and because of Koebe's Distortion Theorem it is Hölder continuous. Because of Exponential Shrinking Property, Theorem 3.5, and Koebe's Distortion Theorem used again, we have the following.

**Lemma 7.1.** *Let  $G$  be a \*semi-hyperbolic rational semigroup generated by  $(f_1, \dots, f_u) \in \text{Rat}^u$ . If  $V \neq \emptyset$  is an open subset of  $\Sigma_u \times \hat{\mathbb{C}}$  containing  $\text{Crit}_*(\tilde{f})$ , then*

$$\tilde{f}|_{K(V)} : K(V) \longrightarrow K(V)$$

*is an infinitesimally expanding map. This precisely means that there exists an integer  $n \geq 1$  such that*

$$|(\tilde{f}^n)'(\xi)| \geq 2$$

*for every  $\xi \in K(V)$ . In addition, there exists  $R_2 > 0$  such that for every  $\xi \in K(V)$  and every integer  $k \geq 0$  there exists a unique continuous inverse branch  $\tilde{f}_\xi^{-k} : B(\tilde{f}^k(\xi), 2R_2) \rightarrow \Sigma_u \times \hat{\mathbb{C}}$  of  $\tilde{f}^k$  sending  $\tilde{f}^k(\xi)$  back to  $\xi$ .*

*Proof.* Let  $R > 0$  come from the Exponential Shrinking Property (Theorem 3.5). Fix  $R_1 \in (0, R]$  so small that

$$(7.3) \quad B(\text{Crit}_*(\tilde{f}), 2R_1) \subset V$$

and

$$(7.4) \quad B(J(\tilde{f}), 2R_1) \cap (\text{PCV}(\tilde{f}) \setminus \text{PCV}_*(\tilde{f})) = \emptyset.$$

Now fix an integer  $q \geq 1$  so large that

$$(7.5) \quad \vartheta^q + Ce^{-\alpha q} < R_1,$$

where both  $C$  and  $\alpha$  come from the Exponential Shrinking Property (Theorem 3.5). Now take  $R_2 \in (0, R_1/4)$  so small that

$$\text{diam}([\tau] \times W) < R_1$$

for every integer  $0 \leq n \leq q$ , every  $\tau \in \Sigma_u^n$ , every  $z \in J(G)$  and any connected component  $W$  of  $f_\tau^{-1}(B_2(z, 2R_2))$ . In conjunction with the Exponential Shrinking Property (Theorem 3.5) and (7.5) this gives that

$$\text{diam}([\tau] \times W) < R_1$$

for every integer  $n \geq 0$ , every  $\tau \in \Sigma_u^n$ , every  $z \in J(G)$  and any connected component  $W$  of  $f_\tau^{-1}(B_2(z, 2R_2))$ . Making also use of (7.3), (7.4), and the definition of  $K(V)$ , it thus follows that if  $\xi = (\omega, z) \in K(V)$ , then

$$([\omega|_n] \times W(\xi, n)) \cap \text{Crit}(\tilde{f}^n) = \emptyset$$

for every integer  $n \geq 0$  and  $W(\xi, n)$ , the connected component of  $f_{\omega|_n}^{-1}(B_2(z, 2R_2))$  containing  $z$ . Hence, the map  $f_{\omega|_n}|_{W(\xi, n)} : W(\xi, n) \rightarrow B_2(z, 2R_2)$  is a conformal homeomorphism. Thus Koebe's Distortion Theorem applies to give

$$|(\tilde{f}^n)'(\xi)| = |(f_{\omega|_n})'(z)| \geq K^{-1} \frac{R_2}{C e^{-\alpha n}} = \frac{R_2}{KC} e^{\alpha n} \geq 2$$

for all  $n \geq \frac{1}{\alpha} \log(2K/R_2)$ . The proof is complete.  $\square$

As an immediate consequence of this lemma, we get the following.

**Corollary 7.2.** *Let  $G$  be a  $\ast$ semi-hyperbolic rational semigroup generated by  $(f_1, \dots, f_u) \in \text{Rat}^u$ . If  $V \neq \emptyset$  is an open subset of  $\Sigma_u \times \hat{\mathbb{C}}$  containing  $\text{Crit}_*(\tilde{f})$ , then*

$$\tilde{f}|_{K(V)} : K(V) \longrightarrow K(V)$$

*is a distance expanding map with respect to the metric in the sense of [26]. This precisely means that there exist  $\eta > 0$  and an integer  $n \geq 1$  such that*

$$\|\tilde{f}^n(w), \tilde{f}^n(z)\|_{\vartheta} \geq 2\|w, z\|_{\vartheta}$$

*whenever  $w, z \in K(V)$  and  $\|w, z\|_{\vartheta} \leq \eta$ .*

Since  $G$  is TNR there exists  $V$ , an open neighborhood of  $\text{Crit}_*(\tilde{f})$ , such that

$$V \cap \text{PCV}_*(\tilde{f}) = \emptyset.$$

Hence,

$$\text{PCV}_*(\tilde{f}) \subset K(V).$$

Therefore, as an immediate consequence of Lemma 7.1, we get the following.

**Lemma 7.3.** *Let  $G$  be a C-F balanced TNR rational semigroup generated by  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Then there exist an integer  $b \geq 1$ , and constants  $C > 0$  and  $\beta > 1$  such that*

$$|(\tilde{f}^b)'(\xi)| \geq 2$$

*and*

$$|(\tilde{f}^n)'(\xi)| \geq C\beta^n$$

*for every  $\xi \in \text{PCV}_*(\tilde{f})$  and every integer  $n \geq 0$ . In addition, there exists  $R_2 > 0$  such that for every  $\xi \in \text{PCV}_*(\tilde{f})$  and every integer  $n \geq 0$  there exists a unique continuous inverse branch  $\tilde{f}_\xi^{-n} : B(\xi, 2R_2) \rightarrow \Sigma_u \times \hat{\mathbb{C}}$  of  $\tilde{f}^n$ , sending  $\tilde{f}^n(\xi)$  back to  $\xi$ .*



For every point  $\xi \in \text{Crit}(\tilde{f})$  we put

$$\chi(\xi) := \lim_{k \rightarrow \infty} \frac{1}{k} \log \inf_{n \geq 1} \left\{ |(\tilde{f}^k)'(\tilde{f}^n(\xi))| \right\} \quad \text{and} \quad \chi_{\tilde{f}} := \min \left\{ \frac{\chi(\xi)}{q_\xi} : \xi \in \text{Crit}_*(\tilde{f}) \right\},$$

where  $q_\xi$  is the order of the critical point  $p_2(\xi)$  with respect to the holomorphic map  $f_{p_1(\xi)_1}$ . Now we define

$$(7.6) \quad \Delta_G^* := \{t \geq 0 : P(t) > -\chi_{\tilde{f}} t\}.$$

Although we will not really need for a long time to know that the set  $\Delta_G^*$  is nonempty, we remark already at the moment that this follows immediately from continuity of the pressure function  $t \mapsto P(t)$  and is formally stated in Proposition 7.8 in a much stronger form describing the structure of the set  $\Delta_G^*$ .

**Definition 7.4.** *A rational semigroup  $G$  generated by a multi-map  $f := (f_1, \dots, f_u) \in \text{Rat}^u$  is said to be of finite type if the set  $\text{Crit}_*(\tilde{f})$ , i.e. the set of all critical points of  $\tilde{f}$  lying in the Julia set  $J(\tilde{f})$ , is finite.*

**Remark 7.5.** *It is easy to see that the concept of being of finite type depends only on the semigroup  $G$  alone and not on any set of its generators.*

**Remark 7.6.** *We would like to mention that the Nice Open Set Condition, the one assumed in [46], treated at length in Section 13, and indispensable for the whole Part 3, entails, see Lemma 13.11, the semigroup  $G$  to be of finite type.*

The main result of this section is the following technical proposition.

**Proposition 7.7.** *Let  $G = \langle f_1, f_2, \dots, f_u \rangle$  be a C-F balanced TNR rational semigroup of finite type. Then for every  $b \in \Delta_G^*$  there exist  $\eta > 0$  and an integer  $l \geq 1$  such that  $(b - \eta, b + \eta) \subset \Delta_G$ , and if  $t \in (b - \eta, b + \eta) \cap \Delta_G^*$ , then*

$$(7.7) \quad m_t(\tilde{f}^{-1}(A)) \leq C m_t^{1/l}(A)$$

for every Borel sets  $A \subset J(\tilde{f})$  with some constant  $C > 0$  independent of  $t$ .

*Proof.* For every  $c \in \text{Crit}_*(f)$  let

$$I(c) := \{i \in \{1, 2, \dots, u\} : c \in J_{i\omega} \text{ for some } \omega \in \Sigma_u \text{ and } f'_i(c) = 0\}.$$

Fix  $i \in I(c)$ . Let  $q \geq 2$  be the order of the critical point  $c$  with respect to the map  $f_i$ . Fix  $\theta > 0$  so small that there exists a constant  $Q \in [1, +\infty)$  such that if  $f_{i,c}^{-1}(B(f_i(c), \theta))$  denotes the connected component of  $f_i^{-1}(B(f_i(c), \theta))$  containing  $c$ , then

$$(7.8) \quad Q^{-1} \leq \frac{|f_i(z) - f_i(c)|}{|z - c|^q} \leq Q, \quad Q^{-1} \leq \frac{|f'_i(z)|}{|z - c|^{q-1}} \leq Q, \quad Q^{-1} \leq \frac{|f'_i(z)|}{|f_i(z) - f_i(c)|^{\frac{q-1}{q}}} \leq Q$$

for all  $z \in f_{i,c}^{-1}(B(f_i(c), \theta))$ . For every set  $A \subset B(f_i(c), \theta)$  we put

$$f_{i,c}^{-1}(A) := f_i^{-1}(A) \cap f_{i,c}^{-1}(B(f_i(c), \theta)).$$

Furthermore, for every set  $A \subset \Sigma_u \times B(f_i(c), \theta)$ , put

$$\tilde{f}_{i,c}^{-1}(A) := \{(i\omega, z) \in \Sigma_u \times \hat{\mathbb{C}} : (\omega, f_i(z)) \in A \text{ and } z \in f_{i,c}^{-1}(B(f_i(c), \theta))\}.$$

Now we take  $\delta > 0$  so small that  $8Q^2\|\tilde{f}'\|_\infty\delta \leq \theta$ . We will in fact need  $\delta > 0$  to be even smaller as specified later in the course of this proof. Fix  $\xi \in \tilde{f}^{-1}(\text{Crit}(\tilde{f}))$  arbitrary. Fix also  $\gamma > P(t)$  arbitrary. We shall first prove that

$$(7.9) \quad \nu_{t,\gamma}(\tilde{f}_{i,c}^{-1}(A)) \preceq \nu_{t,\gamma}^{1/l}(A)$$

for every Borel set  $A \subset J(\tilde{f}) \cap p_2^{-1}(B_2(f_i(c), \delta))$  and some integer  $l \geq 1$  independent of  $t, \gamma$ , and  $A$ ; the comparability constant of (7.9) does not depend on them either. To that end, let

$$\Gamma_i(c) := \{\omega \in \Sigma_u : f_i(c) \in J_\omega\}.$$

Of course

$$\text{Crit}_*(\tilde{f}) = \bigcup_{b \in \text{Crit}_*(f)} \bigcup_{j \in I(b)} \{(j\omega, b) : \omega \in \Gamma_j(b)\}.$$

Since  $\text{Crit}_*(\tilde{f})$  is finite, each set  $\Gamma_j(b)$ ,  $b \in \text{Crit}_*(f)$ ,  $j \in I(b)$ , is finite. Therefore, there exists an integer  $p \geq 1$  so large that the function  $\Gamma_i(c) \ni \omega \mapsto \omega|_p \in \Sigma_u^*$  is 1-to-1. Equivalently, the family

$$\{[\omega|_p] : \omega \in \Gamma_i(c)\}$$

consists of mutually disjoint sets. From now on throughout this section we fix one such  $p \geq 1$  arbitrary. Note that then

$$(7.10) \quad (\Gamma_i(c) \cap [\omega|_p])|_n = \{\omega|_n\}$$

for all  $n \geq p$ . Clearly, in order to show that (7.9) holds, it suffices to prove that

$$(7.11) \quad \nu_{t,\gamma}(\tilde{f}_{i,c}^{-1}(A)) \preceq \nu_{t,\gamma}^{1/l}(A)$$

for every  $\omega \in \Gamma_i(c)$  and for every Borel set  $A \subset J(\tilde{f}) \cap ([\omega|_p] \times B_2(f_i(c), \delta))$ ; in fact for  $A \subset J(\tilde{f}) \cap ([\omega|_p] \times B_2(f_i(c), \delta)) \setminus \{(\omega, f_i(c))\}$  as  $\nu_{t,\gamma}(\{i\omega, c\}) = 0$ . Fix such respective  $\omega$  and  $A$  arbitrary. For every  $n \geq 0$  and every  $\omega \in \Gamma_i(c)$  let

$$\lambda_n(\omega) := |(\tilde{f}^n)'(\omega, f_i(c))| \quad \text{and} \quad \hat{\lambda}_n(\omega) := \max \{\lambda_k(\omega) : 0 \leq k \leq n\}.$$

Because of Lemma 7.3, we have that

$$(7.12) \quad \lambda_n(\omega) \leq \hat{\lambda}_n(\omega) \preceq \lambda_n(\omega).$$

The structure of the proof consists of the following six steps (a), (b), (c), (d), (e), and (f). First, we will prove that

(a)

$$\nu_{t,\gamma}([i\omega|_n] \times B_2(c, (Q\delta\lambda_n^{-1}(\omega))^{1/q})) \preceq \lambda_n^{-t/q}(\omega)e^{-P(t)n}$$

for all integers  $n \geq 1$ ,

and

(b)

$$[i\tau] \times f_{i,c}^{-1}(B_2(f_i(c), \delta\hat{\lambda}_n^{-1}(\omega))) \subset F(\tilde{f})$$

for all integers  $n \geq p + 1$  and all  $\tau \in \Sigma_u^n \setminus \{\omega|_n\}$  such that  $\tau|_p = \omega|_p$ .

(c) Having in mind the task of item (d) ahead, fixing an integer  $s \geq p + 1$ , we will partition the ball  $B_2(f_i(c), \delta)$  into suitable annuli and define the stopping time

$$k := \sup \left\{ n \geq 0 : \nu_{t,\gamma} \left( A \cap ([\omega]_p \times A(f_i(c); \delta \hat{\lambda}_{sn}^{-1}(\omega), \delta)) \right) \leq \hat{\lambda}_{sn}^{-t}(\omega) e^{-P(t)sn} \right\}.$$

Of course 0 belongs to the set whose supremum is being taken above to define  $k$  and we will show that  $k$  is a finite number. Then combining this along with (a) and (b), we will be able to prove that

(d)

$$\nu_{t,\gamma}(\tilde{f}_{i,c}^{-1}(A)) \preceq \lambda_{sk}^{-t}(\omega) e^{-P(t)sk}.$$

Finally, we will prove the following two facts which will finish the proof of our proposition.

(e)

$$\lambda_{sk}^{-t/q}(\omega) e^{-P(t)sk} \leq (\lambda_{sk}^{-t}(\omega) e^{-P(t)sk})^{1/l}$$

for some  $l > 0$  and every  $s \geq 1$  sufficiently large.

(f)

$$\lambda_{sk}^{-t}(\omega) e^{-P(t)sk} \preceq \nu_{t,\gamma} \left( A \cap ([\omega]_p \times A(f_i(c); \delta \hat{\lambda}_{s(k+1)}^{-1}(\omega), \delta)) \right) (\leq \nu_{t,\gamma}(A)).$$

**Proof of (a).** Fix an integer  $n \geq 1$ . Assume  $\delta > 0$  to be so small that  $8Q^2\delta < R_2$ , where  $R_2 > 0$  comes from Lemma 7.3. It then follows from this lemma that for every integer  $n \geq 0$  there exists a unique continuous inverse branch  $\tilde{f}_{(\omega, f_i(c))}^{-n} : B(\tilde{f}^n(\omega, f_i(c)), 8Q^2\delta) \rightarrow \Sigma_u \times \hat{\mathbb{C}}$  sending  $\tilde{f}^n(\omega, f_i(c))$  back to  $(\omega, f_i(c))$ . Also because of this same lemma there exists an integer  $s \geq 1$  so large that

$$(7.13) \quad \lambda_{j+s}(\omega) > \lambda_j(\omega)$$

for all integers  $j \geq 0$ . We furthermore require this integer  $s \geq 1$  to be so large that if  $\tau \in \Sigma_u$  and  $\rho \in \Sigma_u^*$ , then

$$(7.14) \quad \rho B_1(\tau, 8Q^2\delta) \supset [\rho]_{j+s},$$

where  $j = |\rho|$ . Of course, it suffices to prove the inequality required in (a) for all integers  $n \geq s$ . Due to our choice of  $\xi$  and  $\delta$ , the upper part of Lemma 4.5 is then applicable to the inverse branch  $\tilde{f}_{(\omega, f_i(c))}^{-(n-s)} : B(\tilde{f}^{n-s}(\omega, f_i(c)), 8Q^2\delta) \rightarrow \Sigma_u \times \hat{\mathbb{C}}$ . Applying also to it  $\frac{1}{4}$ -Koebe's Distortion Theorem, making use of both (7.13) and (7.14), and then applying Koebe's Distortion Theorem, we get first

$$\tilde{f}_{(\omega, f_i(c))}^{-(n-s)}(B(\tilde{f}^{n-s}(\omega, f_i(c)), 4Q^2\delta)) \supset [\omega]_n \times B_2(f_i(c), Q^2\delta \lambda_n^{-1}(\omega)),$$

and then

$$(7.15) \quad \begin{aligned} \nu_{t,\gamma}([\omega]_n \times B_2(f_i(c), Q^2\delta \lambda_n^{-1}(\omega))) &\leq \nu_{t,\gamma} \left( \tilde{f}_{(\omega, f_i(c))}^{-(n-s)}(B(\tilde{f}^{n-s}(\omega, f_i(c)), 4Q^2\delta)) \right) \\ &\leq K^t \lambda_{n-s}^{-t}(\omega) e^{-\gamma n} \\ &\leq K^t \|\tilde{f}'\|_{\infty}^{st} \lambda_n^{-t}(\omega) e^{-\gamma n} \\ &\leq K^t \|\tilde{f}'\|_{\infty}^{st} \lambda_n^{-t}(\omega) e^{-P(t)n}. \end{aligned}$$

Using this and (7.8), keeping in mind (7.13), and writing uniquely  $n = n^*s+r$ ,  $0 \leq r \leq s-1$ , we obtain

$$\begin{aligned}
\nu_{t,\gamma} \left( [(i\omega)|_{n+1}] \times B_2(c, (Q\delta\lambda_n^{-1}(\omega))^{1/q}) \right) &= \\
&= \sum_{j=n^*}^{\infty} \nu_{t,\gamma} \left( ([(i\omega)|_{js+r+1}] \setminus [(i\omega)|_{(j+1)s+r+1}]) \times A(c; (Q\delta\lambda_{(j+1)s+r}^{-1}(\omega))^{1/q}, (Q\delta\lambda_{js+r}^{-1}(\omega))^{1/q}) \right) \\
&= \sum_{j=n^*}^{\infty} \nu_{t,\gamma} \left( [(i\omega)|_{js+r+1}] \times A(c; (Q\delta\lambda_{(j+1)s+r}^{-1}(\omega))^{1/q}, (Q\delta\lambda_{js+r}^{-1}(\omega))^{1/q}) \right) \\
&\leq \sum_{j=n^*}^{\infty} \nu_{t,\gamma} \left( [(i\omega)|_{js+r+1}] \times f_{i,c}^{-1} \left( A(f_i(c); \delta\lambda_{(j+1)s+r}^{-1}(\omega), Q^2\delta\lambda_{js+r}^{-1}(\omega)) \right) \right) \\
&\preceq \sum_{j=n^*}^{\infty} \lambda_{js+r}^{(1-\frac{1}{q})t}(\omega) e^{-\gamma} \nu_{t,\gamma} \left( [\omega]_{js+r} \times A(f_i(c); \delta\lambda_{(j+1)s+r}^{-1}(\omega), Q^2\delta\lambda_{js+r}^{-1}(\omega)) \right) \\
&\leq e^{-\gamma} \sum_{j=n^*}^{\infty} \lambda_{js+r}^{(1-\frac{1}{q})t}(\omega) \nu_{t,\gamma} \left( [\omega]_{js+r} \times B_2(f_i(c), Q^2\delta\lambda_{js+r}^{-1}(\omega)) \right) \\
&\preceq \sum_{j=n^*}^{\infty} \lambda_{js+r}^{(1-\frac{1}{q})t}(\omega) \lambda_{js+r}^{-t}(\omega) e^{-\gamma(js+r)} \\
&= \sum_{j=n^*}^{\infty} \lambda_{js+r}^{-t/q}(\omega) e^{-\gamma(js+r)} \\
&\leq \sum_{j=n^*}^{\infty} \lambda_{js+r}^{-t/q}(\omega) e^{-P(t)(js+r)} \\
&\asymp \lambda_n^{-t/q}(\omega) e^{-P(t)n} \left( 1 + \sum_{j=n^*+1}^{\infty} \left( \frac{\lambda_{js+r}(\omega)}{\lambda_{n^*s+r}(\omega)} \right)^{-t/q} e^{-P(t)s(j-n^*)} \right).
\end{aligned}$$

Hence, all we are left to do in order to prove (a) is to show that the sum in the above parentheses is uniformly bounded above. Of course it is enough to do this for all  $n \geq 1$  large enough. Since  $b \in \Delta_G^*$ , we have that

$$\chi(i\omega, c) > -q_{(i\omega, c)} \frac{P(b)}{b}.$$

Since, by Lemma 7.4 in [46], the function  $t \mapsto P(t)/t$  is continuous, there thus exist  $\kappa > 0$ ,  $\eta_1 > 0$ , and some integer  $s \geq 1$  so large that

$$\frac{1}{j} \log |(\tilde{f}^j)'(\tilde{f}^n(i\omega, c))| \geq -\frac{qP(t)}{t} + \kappa$$

for all  $t \in (b - \eta_1, b + \eta_1)$ , all  $j \geq s$ , and all  $n \geq 1$ , and where we have abbreviated  $q_{(i\omega, c)}$  by  $q$ . Thus for every  $j \geq n^* + 1$ , we have

$$\begin{aligned} \log \left( \frac{\lambda_{js+r}(\omega)}{\lambda_{n^*s+r}(\omega)} \right) &= \sum_{p=n^*}^{j-1} (\log \lambda_{s(p+1)+r}(\omega) - \log \lambda_{sp+r}(\omega)) = \sum_{p=n^*}^{j-1} \log \left( \frac{\lambda_{s(p+1)+r}(\omega)}{\lambda_{sp+r}(\omega)} \right) \\ &= \sum_{p=n^*}^{j-1} \log |(\tilde{f}^s)'(\tilde{f}^{sp+r}(i\omega, c))| \\ &\geq \left( -\frac{qP(t)}{t} + \kappa \right) s(j - n^*). \end{aligned}$$

Therefore,

$$\begin{aligned} \left( \frac{\lambda_{js+r}(\omega)}{\lambda_{n^*s+r}(\omega)} \right)^{-t/q} e^{-P(t)s(j-n^*)} &\leq \exp \left( -(-qP(t) + t\kappa) \frac{s(j-n^*)}{q} - P(t)s(j-n^*) \right) \\ &= \exp \left( -\frac{\kappa st}{q} (j-n^*) \right). \end{aligned}$$

Hence,

$$\sum_{j=n^*+1}^{\infty} \left( \frac{\lambda_{js+r}(\omega)}{\lambda_{n^*s+r}(\omega)} \right)^{-t/q} e^{-P(t)s(j-n^*)} \leq \sum_{j=1}^{\infty} \exp \left( -\frac{\kappa st}{q} j \right) < +\infty,$$

and the proof of item (a) is complete.  $\square$

**Proof of (b).** In order to prove it, note that  $f_\tau(f_i(c)) \in F(G)$ . Since  $\tau \neq \omega|_n$  we have that  $|\omega \wedge \tau| \leq n - 1$ , and recall that  $l := |\omega \wedge \tau| \geq p$ . Then  $f_{i\omega|_l}(c) \in J(G)$ , and by virtue of the formula

$$J(G) = \cup_{\rho \in \Sigma_u} J_\rho$$

along with (7.10), we get that  $f_{i\omega|_l\tau_{l+1}}(c) \in F(G)$ . Now we take  $\delta > 0$  so small that  $4\|\tilde{f}'\|_\infty\delta < D(G)$ , where  $D(G)$  is the number coming from Definition 5.6 of C-F balanced semigroups. We then have that

$$B_2(f_{i\omega|_l\tau_{l+1}}(c), 4\|\tilde{f}'\|_\infty\delta) \subset F(G).$$

Since  $f_{i\omega|_l}(c) \in \text{PCV}(G) \cap J(G)$  and  $G$  is a TNR semigroup, we will have both that  $f_{i\omega|_l}(c) \notin B_2(\text{Crit}(G), 4\|\tilde{f}'\|_\infty\delta)$  by taking  $\delta > 0$  small enough, depending on the generators of  $G$  only, and that  $f_{\tau_{l+1}|_W}$  is 1-to-1, where  $W$  is the unique connected component of  $f_{\tau_{l+1}}^{-1} \left( B_2(f_{i\omega|_l\tau_{l+1}}(c), 4\|\tilde{f}'\|_\infty\delta) \right)$  containing  $f_{i\omega|_l}(c)$ . So,

$$f_{\omega|_l\tau_{l+1}, f_i(c)}^{-1} := f_{\omega|_l, f_i(c)}^{-1} \circ (f_{\tau_{l+1}|_W})^{-1} : B_2(f_{i\omega|_l\tau_{l+1}}(c), 4\|\tilde{f}'\|_\infty\delta) \rightarrow \hat{\mathbb{C}}$$

is a unique holomorphic branch of  $f_{\omega|_l\tau_{l+1}}$  defined on  $B_2(f_{i\omega|_l\tau_{l+1}}(c), 4\|\tilde{f}'\|_\infty\delta)$  and sending  $f_{\omega|_l\tau_{l+1}}(c)$  to  $f_i(c)$ ,

$$[\omega|_l\tau_{l+1}] \times f_{\omega|_l\tau_{l+1}, f_i(c)}^{-1} \left( B_2(f_{i\omega|_l\tau_{l+1}}(c), 4\|\tilde{f}'\|_\infty\delta) \right) \subset F(\tilde{f}),$$

and, by the  $\frac{1}{4}$  Koebe's Distortion Theorem,

$$\begin{aligned} f_{\omega|_{l\tau_{l+1}, f_i(c)}}^{-1} \left( B_2(f_{i\omega|_{l\tau_{l+1}}}(c), 4\|\tilde{f}'\|_{\infty}\delta) \right) &\supset B_2\left(f_i(c), \lambda_l^{-1}(\omega)|f'_{\tau_{l+1}}(f_{i\omega|_l}(c))|^{-1}\|\tilde{f}'\|_{\infty}\delta\right) \\ &\supset B_2\left(f_i(c), \delta\lambda_l^{-1}(\omega)\right) \\ &\supset B_2\left(f_i(c), \delta\hat{\lambda}_n^{-1}(\omega)\right). \end{aligned}$$

Therefore,

$$[i\omega|_{l\tau_{l+1}}] \times f_{i,c}^{-1} \left( B_2(f_i(c), \delta\hat{\lambda}_n^{-1}(\omega)) \right) \subset F(\tilde{f}).$$

Since also

$$[i\tau] \times f_{i,c}^{-1} \left( B_2(f_i(c), \delta\hat{\lambda}_n^{-1}(\omega)) \right) \subset [i\omega|_{l\tau_{l+1}}] \times f_{i,c}^{-1} \left( B_2(f_i(c), \delta\hat{\lambda}_n^{-1}(\omega)) \right),$$

item (b) is therefore proved.  $\square$

**Proof of (c).** Keep  $s \geq p + 1$  as determined in the above proof of item (a). Since

$$\lim_{n \rightarrow \infty} \nu_{t,\gamma} \left( A \cap ([\omega|_p] \times A(f_i(c); \delta\hat{\lambda}_{sn}^{-1}(\omega), \delta)) \right) = \nu_{t,\gamma} \left( A \cap ([\omega|_p] \times B(f_i(c), \delta)) \right) = \nu_{t,\gamma}(A) > 0,$$

in order to see that the stopping time  $k$  is finite, it is sufficient to show that

$$(7.16) \quad \lim_{n \rightarrow \infty} \lambda_{sn}^{-t}(\omega) e^{-P(t)sn} = 0.$$

If  $P(t) \geq 0$ , this is immediate as  $\lim_{n \rightarrow \infty} \lambda_n(\omega) = +\infty$ . If  $P(t) < 0$ , then the inequality  $t\chi_{(i\omega,c)} > -qP(t)$ , holding since  $t \in \Delta_G^*$ , implies that

$$t \log |(\tilde{f}^n)'(\tilde{f}(i\omega, c))| > -nqP(t)$$

for all  $n \geq 1$  large enough. Hence

$$\lambda_n^{-t}(\omega) e^{-P(t)n} < e^{P(t)(q-1)n}.$$

Since  $q \geq 2$ , formula (7.16) thus follows. The proof of item (c) is complete.  $\square$

**Proof of (d).** If  $k = 0$ , the required inequality is trivial as its right-hand side is equal to 1. So, we may, and we do, assume that  $k \geq 1$ . Then applying (a), (b), and (c) we estimate

as follows.

$$\begin{aligned}
\nu_{t,\gamma}(\tilde{f}_{i,c}^{-1}(A)) &= \\
&= \nu_{t,\gamma} \left( \tilde{f}_{i,c}^{-1} \left( A \cap ([\omega]_{sk} \times B_2(f_i(c), \delta \hat{\lambda}_{sk}^{-1}(\omega))) \right) \cup \right. \\
&\quad \left. \cup \tilde{f}_{i,c}^{-1} \left( \bigcup_{\tau \in ([\omega]_p]_{sk} \cap (\Sigma_u^{sk} \setminus \{\omega\}_{sk})} \left( A \cap ([\tau] \times B_2(f_i(c), \delta \hat{\lambda}_{sk}^{-1}(\omega))) \right) \right) \cup \right. \\
&\quad \left. \cup \tilde{f}_{i,c}^{-1} \left( A \cap ([\omega]_p \times A(f_i(c); \delta \hat{\lambda}_{sk}^{-1}(\omega), \delta)) \right) \right) \\
&= \nu_{t,\gamma} \left( \tilde{f}_{i,c}^{-1} \left( A \cap ([\omega]_{sk} \times B_2(f_i(c), \delta \hat{\lambda}_{sk}^{-1}(\omega))) \right) \right) + \\
&\quad + \sum_{\tau \in ([\omega]_p]_{sk} \cap (\Sigma_u^{sk} \setminus \{\omega\}_{sk}))} \nu_{t,\gamma} \left( \tilde{f}_{i,c}^{-1} \left( A \cap ([\tau] \times B_2(f_i(c), \delta \hat{\lambda}_{sk}^{-1}(\omega))) \right) \right) + \\
&\quad + \nu_{t,\gamma} \left( \tilde{f}_{i,c}^{-1} \left( A \cap ([\omega]_p \times A(f_i(c); \delta \hat{\lambda}_{sk}^{-1}(\omega), \delta)) \right) \right) \\
&= \nu_{t,\gamma} \left( \tilde{f}_{i,c}^{-1} \left( A \cap ([\omega]_{sk} \times B_2(f_i(c), \delta \hat{\lambda}_{sk}^{-1}(\omega))) \right) \right) + \\
&\quad + \nu_{t,\gamma} \left( \tilde{f}_{i,c}^{-1} \left( A \cap ([\omega]_p \times A(f_i(c); \delta \hat{\lambda}_{sk}^{-1}(\omega), \delta)) \right) \right) \\
&\preceq \nu_{t,\gamma} \left( \left( [i\omega]_{sk} \times B_2(c, (Q\delta \hat{\lambda}_{sk}^{-1}(\omega))^{1/q}) \right) \right) + \\
&\quad + (\delta \hat{\lambda}_{sk}^{-1}(\omega))^{(\frac{1}{q}-1)t} e^{-\gamma} \nu_{t,\gamma} \left( A \cap ([\omega]_p \times A(f_i(c); \delta \hat{\lambda}_{sk}^{-1}(\omega), \delta)) \right) \\
&\leq \nu_{t,\gamma} \left( \left( [i\omega]_{sk} \times B_2(c, (Q\delta \lambda_{sk}^{-1}(\omega))^{1/q}) \right) \right) + \\
&\quad + (\delta \hat{\lambda}_{sk}^{-1}(\omega))^{(\frac{1}{q}-1)t} e^{-P(t)} e^{-P(t)sk} \hat{\lambda}_{sk}^{-t}(\omega) \\
&\preceq \lambda_{sk}^{-t/q}(\omega) e^{-P(t)sk} + \hat{\lambda}_{sk}^{-t/q}(\omega) e^{-P(t)sk} \\
&\asymp \lambda_{sk}^{-t/q}(\omega) e^{-P(t)sk}.
\end{aligned}$$

The proof of item (d) is thus complete.  $\square$

**Proof of (e).** If  $k = 0$ , then both sides of the required inequality are equal to 1 and we are then done. So, we assume now that  $k \geq 1$ . Starting the proof we again recall that

$$\frac{1}{n} \log |(\tilde{f}^n)'(\omega, f_i(c))| \geq -\frac{qP(t)}{t} + \kappa$$

for all  $\omega \in \Gamma_i(c)$  and all  $n \geq s$ . Hence, if we take  $l \geq q + 1$  so large that

$$\frac{P(b)q(1-q)}{b(l-q)} < \kappa/2.$$

It then follows from the continuity of the function  $t \mapsto P(t)/t$ , holding because of Proposition 4.3 (b), that there exists  $\eta \in (0, \eta_1]$  such that

$$\frac{P(t)q(1-q)}{t(l-q)} < \kappa$$

for all  $t \in (b-\eta, b+\eta)$ . This in turn implies that for such parameters  $t$  and all  $n \geq s$ , we have

$$(l-q) \frac{\log \lambda_n(\omega)}{n} \geq \left( -\frac{qP(t)}{t} + \kappa \right) (l-q) \geq \frac{qP(t)}{t} (1-l).$$

An elementary rearrangement then gives

$$\lambda_n^{-t/q}(\omega) e^{-P(t)n} \leq (\lambda_n^{-t}(\omega) e^{-P(t)n})^{1/l}.$$

Hence,

$$\lambda_{sk}^{-t/q}(\omega) e^{-P(t)sk} \leq (\lambda_{sk}^{-t}(\omega) e^{-P(t)sk})^{1/l}$$

if  $s \geq p$  is taken to be large enough. Thus item (e) is proved.  $\square$

**Proof of (f).** The finiteness of  $k \geq 0$  and formula (7.12) yield

$$\begin{aligned} \nu_{t,\gamma} \left( A \cap ([\omega|_p] \times A(f_i(c); \delta \hat{\lambda}_{s(k+1)}^{-1}(\omega), \delta)) \right) &\geq \hat{\lambda}_{s(k+1)}^{-t}(\omega) e^{-P(t)s(k+1)} \\ &\geq \lambda_{s(k+1)}^{-t}(\omega) e^{-P(t)s(k+1)} \\ &\geq e^{-P(t)s} \|\tilde{f}'\|_\infty^{-st} \lambda_{sk}^{-t}(\omega) e^{-P(t)sk}. \end{aligned}$$

The proof of item (f) is thus complete.  $\square$

In conclusion, the conjunction of (d), (e), and (f), completes the proof of (7.11). Hence, (7.9) follows. Because the semigroup  $G$  is of finite type, we thus have that

$$(7.17) \quad \nu_{t,\gamma}(\tilde{f}^{-1}(A)) \leq C \nu_{t,\gamma}^{1/l}(A)$$

for every  $\gamma > P(t)$ , every Borel set  $A \subset J(\tilde{f})$  and some constant  $C$  independent of  $t$  and  $\gamma$ . Since  $m_t$  is a weak\* limit of the measures  $\nu_{t,\gamma}$  when  $\gamma \searrow P(t)$ , with the help of standard consideration involving outer and inner regularity of measures, the formula (7.7) of Proposition 7.7 follows. Also, by the weak\* convergence argument we conclude from (7.15) that

$$m_t([\omega|_n] \times B_2(f_i(c), Q^2 \delta \lambda_n^{-1}(\omega))) \leq K^t \lambda_n^{-t}(\omega) e^{-P(t)n}$$

for every integer  $n \geq p$ . As  $t \in \Delta_G^*$ , this formula along with (7.16), yield  $m_t(\omega, f_i(c)) = 0$ . This in turn, in conjunction with formula (7.7) entails  $m_t(i\omega, c) = 0$ . Hence,  $t \in \Delta_G$ , and the proof of Proposition 7.7 is complete.  $\square$

The following proposition enlightens the structure of the sets  $\Delta_G$  and  $\Delta_G^*$ .

**Proposition 7.8.** *If  $G$  is a C-F balanced TNR rational semigroup of finite type generated by a multi-map  $f = (f_1, f_2, \dots, f_u) \in \text{Rat}^u$ , then  $\Delta_G^*$  is an open subset of  $[0, +\infty)$  and there exists  $\eta > 0$  such that*

$$[0, h_f + \eta) \subset \Delta_G^* \subset \Delta_G.$$



*Proof.* The inclusion  $\Delta_G^* \subset \Delta_G$  and openness of  $\Delta_G^*$  in  $[0, +\infty)$  both follow directly from Proposition 7.7, while the inclusion  $\Delta_G^* \supset [0, h_f + \eta)$  with some  $\eta > 0$  is an immediate consequence of the definition of  $\Delta_G^*$  and continuity of the pressure function  $[0, +\infty) \ni t \mapsto \mathbb{R}$  proved in Proposition 4.3 (b).  $\square$

### 8. SMALL PRESSURE $P_V^\Xi(t)$

Let  $G$  be a \*semi-hyperbolic rational semigroup generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ . Let  $V \subset \Sigma_u \times \hat{\mathbb{C}}$  be a non-empty open neighborhood of  $\text{Crit}_*(\tilde{f})$ . Let  $\Xi$  be a non-empty finite subset of  $K(V)$ . For every  $t \geq 0$  we define

$$(8.1) \quad P_V^\Xi(t) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\xi \in (\tilde{f}|_{K(V)})^{-n}(\Xi)} |(\tilde{f}^n)'(\xi)|^{-t}.$$

The key technical result of this section is the following.

**Lemma 8.1.** *Let  $G$  be a \*semi-hyperbolic rational semigroup generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Let  $t \in \Delta_G$ . If  $V$  is an open neighborhood of  $\text{Crit}_*(\tilde{f})$  such that  $V \cap J(\tilde{f}) \neq \emptyset$ , and  $\Xi$  is a non-empty finite subset of  $K(V)$ , then*

$$P_V^\Xi(t) < P(t).$$

*Proof.* Note that the map  $\tilde{f}|_{K(V)} : K(V) \rightarrow K(V)$  has no critical points and all its points of non-openness, i.e. points  $\xi \in K(V)$  that have no local base of topology (consisting of open sets relative to  $K(V)$ ) whose images under  $\tilde{f}|_{K(V)}$  are also open relative to  $K(V)$ , is contained in  $\partial V$ , the boundary of  $V$  in  $\Sigma_u \times \hat{\mathbb{C}}$ . Defining the sets

$$E_n := (\tilde{f}|_{K(V)})^{-n}(\Xi), \quad n \geq 0,$$

and applying to them Lemmas 8.2.6 and 8.2.7 of [14] with the function  $\phi := -t \log |(\tilde{f}^n)'(\xi)|$ , we conclude that there exists  $m_{V,t}^\Xi$ , a Borel probability measure on  $K(V)$ , with the following properties:

(a)

$$m_{V,t}^\Xi(\tilde{f}(A)) \geq e^{P_V^\Xi(t)} \int_A |\tilde{f}'(\xi)|^t dm_{V,t}^\Xi$$

for every Borel set  $A \subset K(V)$  such that the map  $\tilde{f}|_A$  is 1-to-1,  
and

(b)

$$m_{V,t}^\Xi(\tilde{f}(A)) = e^{P_V^\Xi(t)} \int_A |\tilde{f}'(\xi)|^t dm_{V,t}^\Xi$$

if in addition  $A \cap \partial V = \emptyset$ .

Now, seeking contradiction suppose that

$$P_V^\Xi(t) = P(t).$$

Fix a point  $\xi = (\tau, z) \in K(V)$  arbitrary. Applying formula (a) above consecutively, we would obtain that

$$(8.2) \quad m_{V,t}^{\Xi}(B_j^*(\tau, z)) \leq K^t \exp(-P_V^{\Xi}(t)n_j(\xi)) |(f^{n_j(\xi)})'(\xi)|^{-t},$$

where the sets  $B_j^*(\tau, z)$  and integers  $n_j(\xi)$ ,  $j \geq 1$ , are defined by formulas (4.8) and (4.7). We also know that

$$(8.3) \quad m_t(B_j^*(\tau, z)) \geq C_t^{-1} \exp(-P(t)n_j(\xi)) |(\tilde{f}^{n_j(\xi)})'(\xi)|^{-t}$$

with some constant  $C_t \geq 1$ . Therefore,

$$(8.4) \quad m_{V,t}^{\Xi}(B_j^*(\tau, z)) \leq K^t C_t m_t(B_j^*(\tau, z)).$$

Now, the same proof (with  $m_t$  replaced by  $m_{V,t}^{\Xi}$ ) as that of Lemma 4.11, gives us that the family  $\mathcal{B}$  restricted to  $K(V)$  is a Vitali relation for the measure  $m_{V,t}^{\Xi}$  on the set  $K(V) \subset J(\tilde{f}) \setminus \text{Sing}(\tilde{f})$ . In conjunction with (8.4) this yields the measure  $m_{V,t}^{\Xi}$  to be absolutely continuous with respect to  $m_t$ , even more we see that  $\frac{dm_{V,t}^{\Xi}}{dm_t} \leq K^t C_t$ . But the set  $K(V)$  and so, also all its forward iterates under  $\tilde{f}$ , are disjoint from  $V$ . Hence,  $K(V) \cap \text{Trans}(\tilde{f}) = \emptyset$ , where, we recall,  $\text{Trans}(\tilde{f})$  is the set of transitive points of  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ . By virtue of Corollary 4.15 this entails  $m_t(K(V)) = 0$ . Thus also  $m_{V,t}^{\Xi}(K(V)) = 0$ , and this contradiction finishes the proof.  $\square$

**Remark 8.2.** *The proofs of Lemmas 8.2.6 and 8.2.7 from [14] are rather minor improvements of those from Section 3 in [3].*

From now on throughout this section  $G$  is a TNR rational semigroup generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Fix  $R \in (0, R_*(\tilde{f}))$ . Let  $\text{Crit}_*(\tilde{f}) \subset S \subset J(\tilde{f}) \setminus B(\text{PCV}(\tilde{f}), 8R)$  be an arbitrary finite aperiodic set. Let  $\mathcal{U} = \{U_s\}_{s \in S}$  be a nice family of sets, the existence of which is guaranteed by Theorem 6.6. Recall that

$$U = \bigcup_{s \in S} U_s.$$

We now also recall that for every  $s \in S$  and every  $n \geq 1$  we have denoted:

$$\mathcal{D}_n^*(s) := \{\tau \in \text{HIB}_n(U_s) : \tilde{f}^k(\tilde{f}_\tau^{-n}(U_s)) \cap U = \emptyset \text{ for all } 0 \leq k \leq n-1\}.$$

We put

$$\mathcal{D}_n^*(S) := \bigcup_{s \in S} \mathcal{D}_n^*(s).$$

Recall that

$$X_s = \overline{U}_s$$

for every  $s \in S$ . We shall prove the following.

**Lemma 8.3.** *Let  $G$  be a TNR rational semigroup generated by  $(f_1, \dots, f_u) \in \text{Rat}^u$ . If  $t \in \Delta_G^*$  and  $\mathcal{U}_S = \{U_s\}_{s \in S}$  is a nice family of sets, produced in Theorem 6.6, then there*

exist  $\gamma > 0$  and a finite set  $\Xi \subset K(B(S, \gamma))$  such that for every  $\varepsilon > 0$  there exists  $C_{\mathcal{U}}(\varepsilon) > 0$  such that for every integer  $n \geq 1$  we have that

$$m_t \left( \bigcup_{s \in S} \bigcup_{\tau \in \mathcal{D}_n^*(s)} \tilde{f}_\tau^{-n}(X_s) \right) \leq C_{\mathcal{U}}(\varepsilon) \exp((P_{B(S, \gamma)}^\Xi(t) - P(t) + \varepsilon)n)$$

with the number  $P(t) - P_{B(S, \gamma)}^\Xi(t)$  being positive.

*Proof.* Since periodic points of  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  are dense in  $J(\tilde{f})$ , for every  $s \in S$  there exists  $\xi_s$ , a periodic point of  $\tilde{f}$  belonging to  $U_s \setminus S$ , whose forward orbit under  $\tilde{f}$  is disjoint from  $S$ . Then there exists  $\gamma > 0$  so small that

$$(8.5) \quad B(S, \gamma) \subset U$$

and

$$B(S, \gamma) \cap \bigcup_{n=0}^{\infty} \tilde{f}^n(\{\xi_s : s \in S\}) = \emptyset.$$

This latter formula is equivalent to saying that

$$\Xi := \{\xi_s : s \in S\} \subset K(B(S, \gamma)),$$

which in the conjunction with (8.5), gives that

$$\bigcup \left\{ \tilde{f}_\tau^{-n}(\xi_s) : s \in S, \tau \in \mathcal{D}_n^*(s) \right\} \subset K(B(S, \gamma))$$

for every integer  $n \geq 0$ . It therefore follows from the definition of  $P_{B(S, \gamma)}^\Xi(t)$  that

$$\frac{1}{n} \log \sum_{s \in S} \sum_{\tau \in \mathcal{D}_n^*(s)} \left| \left( \tilde{f}_\tau^{-n} \right)'(\xi_s) \right|^t \leq \frac{1}{n} \log \sum_{\xi \in (\tilde{f}|_{K(V)})^{-n}(\Xi)} \left| (\tilde{f}^n)'(\xi) \right|^{-t} \leq P_{B(S, \gamma)}^\Xi(t) + \frac{\varepsilon}{2}$$

for all  $n \geq 1$  large enough. Invoking Koebe's Distortion Theorem, we further see that for all  $n \geq 1$  large enough, say  $n \geq N$ , we have

$$\frac{1}{n} \log \sum_{s \in S} \sum_{\tau \in \mathcal{D}_n^*(s)} \left\| \left( \tilde{f}_\tau^{-n} \right)' \Big|_{X_s} \right\|_\infty^t \leq P_{B(S, \gamma)}^\Xi(t) + \varepsilon.$$

Equivalently, for all  $n \geq N$ :

$$\frac{1}{n} \log \sum_{s \in S} \sum_{\tau \in \mathcal{D}_n^*(s)} \left\| \left( \tilde{f}_\tau^{-n} \right)' \Big|_{X_s} \right\|_\infty^t e^{-P(t)n} \leq P_{B(S, \gamma)}^\Xi(t) - P(t) + \varepsilon,$$

and note that the number  $P(t) - P_{B(S, \gamma)}^\Xi(t)$  is positive because of Lemma 8.1. But

$$m_t(\tilde{f}_\tau^{-n}(X_s)) \leq \left\| \left( \tilde{f}_\tau^{-n} \right)' \Big|_{X_s} \right\|_\infty^t e^{-P(t)n},$$

whence

$$\sum_{s \in S} \sum_{\tau \in \mathcal{D}_n^*(s)} m_t(\tilde{f}_\tau^{-n}(X_s)) \leq \exp((P_{B(S, \gamma)}^\Xi(t) - P(t) + \varepsilon)n).$$

Therefore, our lemma follows with appropriate constant  $C_{\mathcal{U}}(\varepsilon) > 0$  determined by the values of the above sum for  $ns$  ranging from 1 up to  $N - 1$ .  $\square$

As an immediate consequence of this lemma and Proposition 7.7 we get the following.

**Proposition 8.4.** *Let  $G$  be a C-F balanced TNR rational semigroup of finite type generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . If  $\mathcal{U} = \{U_s\}_{s \in S}$  is a nice family of sets produced in Theorem 6.6, and*

$$\mathcal{S}_{\mathcal{U}} = \{\tilde{f}_{\tau}^{-\|\tau\|} : X_{t(\tau)} \longrightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_{\mathcal{U}}}$$

is the corresponding graph directed system, then there exist  $\gamma > 0$  and a finite set  $\Xi \subset K(B(S, \gamma))$  such that for every  $b \in \Delta_G^*$  there exists  $\eta > 0$  such that for every  $t \in \Delta_G^* \cap (b - \eta, b + \eta)$  and every  $\varepsilon > 0$

$$m_t \left( \bigcup_{s \in S} \bigcup_{\tau \in \mathcal{D}_n(s)} \tilde{f}_{\tau}^{-n}(X_s) \right) \leq C_{\varepsilon} \exp \left( \left( \frac{P_{B(S, \gamma)}^{\Xi}(t) - P(t) + \varepsilon}{l} \right) n \right)$$

for every integer  $n \geq 1$  and some constant  $C_{\varepsilon} \in (0, +\infty)$  depending on  $\varepsilon$ . Also,

$$P(t) - P_{B(S, \gamma)}^{\Xi}(t) > 0.$$

## 9. SYMBOL SPACE THERMODYNAMIC FORMALISM ASSOCIATED TO NICE FAMILIES; REAL ANALYTICITY OF THE ORIGINAL PRESSURE $P(t)$

This section brings up the full fledged fruits of the existence of nice families. It forms a symbolic representation (subshift of finite type with a countable infinite alphabet) of the map generated by a nice family and develops the thermodynamic formalism of the potentials  $\zeta_{t,s}$  resulting from those of the form  $-t \log |\tilde{f}'|$  and the “first return time”  $\|\tau_1\|$ .

Throughout this section  $G$  is a TNR rational semigroup of finite type generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Let

$$\mathcal{U} = \{U_s\}_{s \in S}$$

be a nice family of sets coming from Theorem 6.6. Let  $\mathcal{S}_{\mathcal{U}}$  be the corresponding graph directed system described in Theorem 6.9. Let

$$\pi_{\mathcal{U}} : \mathcal{D}_{\mathcal{U}}^{\infty} \longrightarrow \Sigma_u \times \hat{\mathbb{C}}$$

be the canonical projection from  $\mathcal{D}_{\mathcal{U}}^{\infty}$  to  $\Sigma_u \times \hat{\mathbb{C}}$ , determined by the system  $\mathcal{S}_{\mathcal{U}}$ ; see [19] for details. We recall (see also [19]) that  $J_{\mathcal{U}}$ , the limit set of  $\mathcal{S}_{\mathcal{U}}$ , is defined as  $\pi_{\mathcal{U}}(\mathcal{D}_{\mathcal{U}}^{\infty})$ .

Given two real numbers  $t, s \geq 0$  we define the function  $\zeta_{t,s} : \mathcal{D}_{\mathcal{U}}^{\infty} \longrightarrow \mathbb{R}$  by the following formula:

$$\zeta_{t,s}(\tau) := -t \log |(\tilde{f}^{\|\tau_1\|})'(\pi_{\mathcal{U}}(\sigma(\tau)))| - s \|\tau_1\|.$$

The functions  $\zeta_{t,s}$ ,  $t, s \geq 0$ , will be frequently referred to as potentials. Let  $d_{\vartheta}$  be the metric defined on  $\mathcal{D}_{\mathcal{U}}^{\infty}$  by the formula

$$d_{\vartheta}(\alpha, \beta) := \vartheta^{\min\{n \geq 0 : \alpha_{n+1} \neq \beta_{n+1}\}}.$$

Because of Property B of Theorem 6.6, we immediately get the following.

**Lemma 9.1.** *Let  $G$  be a TNR rational semigroup of finite type generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . If  $\mathcal{U} = \{U_s\}_{s \in S}$  is a nice family of sets produced in Theorem 6.6, and  $\mathcal{S}_{\mathcal{U}} = \{\tilde{f}_{\tau}^{-\|\tau\|} : X_{t(\tau)} \rightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_{\mathcal{U}}}$  is the corresponding graph directed system, then the projection map*

$$\pi_{\mathcal{U}} : \mathcal{D}_{\mathcal{U}}^{\infty} \longrightarrow J_{\mathcal{U}}$$

*is Lipschitz continuous if  $\mathcal{D}_{\mathcal{U}}^{\infty}$  is endowed with the metric  $d_{\mathfrak{g}}$ .*

Applying Koebe's Distortion Theorem, this lemma yields the following.

**Proposition 9.2.** *Let  $G$  be a TNR rational semigroup of finite type generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . If  $\mathcal{U} = \{U_s\}_{s \in S}$  is a nice family of sets produced in Theorem 6.6, and  $\mathcal{S}_{\mathcal{U}} = \{\tilde{f}_{\tau}^{-\|\tau\|} : X_{t(\tau)} \rightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_{\mathcal{U}}}$  is the corresponding graph directed system, then for all  $t, s \geq 0$  the function*

$$\zeta_{t,s} : \mathcal{D}_{\mathcal{U}}^{\infty} \longrightarrow \mathbb{R}$$

*is Lipschitz continuous.*

Let  $P(t, s)$  be the topological pressure of the potential  $\zeta_{t,s} : \mathcal{D}_{\mathcal{U}}^{\infty} \rightarrow \mathbb{R}$  with respect to the shift map  $\sigma : \mathcal{D}_{\mathcal{U}}^{\infty} \rightarrow \mathcal{D}_{\mathcal{U}}^{\infty}$ . Given any function  $g : \mathcal{D}_{\mathcal{U}}^{\infty} \rightarrow \mathbb{R}$  and any integer  $n \geq 1$ , let

$$S_n^{\sigma}(g) := \sum_{j=0}^{n-1} g \circ \sigma^j$$

be the  $n$ th Birkhoff's sum of  $g$  with respect to the dynamical system  $\sigma : \mathcal{D}_{\mathcal{U}}^{\infty} \rightarrow \mathcal{D}_{\mathcal{U}}^{\infty}$ . If

$$\sum_{e \in \mathcal{D}_{\mathcal{U}}} \|\phi'_e\|_{\infty}^t e^{-s\|e\|} < +\infty,$$

then we call the function  $\zeta_{t,s}$  summable; we then also call the parameter  $(t, s)$  summable. Denote by  $\Omega(\mathcal{U})$  the set of all summable parameters  $(t, s) \in [0, +\infty) \times [0, +\infty)$ . Invoking Proposition 9.2, by virtue of Corollary 2.7.5 (a), (b), and then (c) in [19], we respectively obtain the following two theorems.

**Theorem 9.3.** *Let  $G$  be a TNR rational semigroup of finite type generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Let  $\mathcal{U} = \{U_s\}_{s \in S}$  be a nice family of set produced in Theorem 6.6 and let  $\mathcal{S}_{\mathcal{U}} = \{\tilde{f}_{\tau}^{-\|\tau\|} : X_{t(\tau)} \rightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_{\mathcal{U}}}$  the corresponding graph directed system.*

*If  $(t, s) \in \Omega(\mathcal{U})$ , then there exists a unique Borel probability measure  $\tilde{m}_{t,s}$  on  $\mathcal{D}_{\mathcal{U}}^{\infty}$  such that*

$$\tilde{m}_{t,s}(eF) = e^{-P(t,s)} \int_F A_{e\omega_1}(\mathcal{U}) |\phi'_e(\pi_{\mathcal{U}}(\omega))|^t e^{-s\|e\|} d\tilde{m}_{t,s}(\omega)$$

*for every  $e \in \mathcal{D}_{\mathcal{U}}$  and every Borel set  $F \subset \mathcal{D}_{\mathcal{U}}^{\infty}$ . We then have a stronger property than the displayed formula above, namely*

$$\tilde{m}_{t,s}(\tau F) = e^{-P(t,s)|\tau|} \int_F A_{\tau_k\omega_1}(\mathcal{U}) |\phi'_{\tau}(\pi_{\mathcal{U}}(\omega))|^t e^{-s\|\tau\|} d\tilde{m}_{t,s}(\omega)$$

*for every  $\tau \in \mathcal{D}_{\mathcal{U}}^*$  and every Borel set  $F \subset \mathcal{D}_{\mathcal{U}}^{\infty}$ , where  $k = |\tau|$ , and*

$$\|\tau\| := \sum_{j=0}^k \|\tau_j\|.$$

**Theorem 9.4.** *Let  $G$  be a TNR rational semigroup of finite type generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Let  $\mathcal{U} = \{U_s\}_{s \in S}$  be a nice family of sets produced in Theorem 6.6, and let  $\mathcal{S}_{\mathcal{U}} = \{\tilde{f}_{\tau}^{-\|\tau\|} : X_{t(\tau)} \rightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_{\mathcal{U}}}$  be the corresponding graph directed system.*

*If  $(t, s) \in \Omega(\mathcal{U})$ , then there exists a unique  $\sigma$ -invariant Gibbs state  $\tilde{\mu}_{t,s}$  for the potential  $\zeta_{t,s} : \mathcal{D}_{\mathcal{U}}^{\infty} \rightarrow \mathbb{R}$  with respect to the shift map  $\sigma : \mathcal{D}_{\mathcal{U}}^{\infty} \rightarrow \mathcal{D}_{\mathcal{U}}^{\infty}$ . This means that  $\tilde{\mu}_{t,s}$  is  $\sigma$ -invariant and at least one (equivalently all) of the following hold:*

(a)

$$C^{-1} \leq \frac{\tilde{\mu}_{t,s}([\tau|_n])}{|\phi'_{\tau|_n}(\pi_{\mathcal{U}}(\tau))|^t e^{-s\|\tau|_n\|}} \leq C$$

*with some constant  $C \geq 1$ , all integers  $n \geq 1$ , and all  $\tau \in \mathcal{D}_{\mathcal{U}}^{\infty}$ .*

(b)

$$C^{-1} \leq \frac{\tilde{m}_{t,s}([\tau|_n])}{\|\phi'_{\tau|_n}\|_{\infty}^t e^{-s\|\tau|_n\|}} \leq C$$

*with some constant  $C \geq 1$ , all integers  $n \geq 1$ , and all  $\tau \in \mathcal{D}_{\mathcal{U}}^{\infty}$ .*

(c)  $\tilde{\mu}_{t,s}$  is absolutely continuous with respect to  $\tilde{m}_{t,s}$ .(d)  $\tilde{\mu}_{t,s}$  is equivalent to  $\tilde{m}_{t,s}$ .(e)  $\tilde{\mu}_{t,s}$  is equivalent to  $\tilde{m}_{t,s}$  and the Radon–Nikodym derivative  $\frac{d\tilde{\mu}_{t,s}}{d\tilde{m}_{t,s}}$  is a log bounded Hölder continuous function.

*In addition, the measure  $\tilde{\mu}_{t,s}$  is ergodic with respect to the shift map  $\sigma : \mathcal{D}_{\mathcal{U}}^{\infty} \rightarrow \mathcal{D}_{\mathcal{U}}^{\infty}$ .*

Now we intend to establish some relationships between the measures  $m_t$  and  $\tilde{m}_{t,s}$ . We first obtain a straightforward (by now) fact about the formers. Indeed, as an immediate consequence of Corollary 4.15, of the fact that  $\text{supp}(m_t) = J(\tilde{f})$ , and of the fact that  $J_{\mathcal{U}} \supset (U \cap \text{Trans}(\tilde{f}))$  (see Theorem 6.9), we get the following.

**Lemma 9.5.** *Let  $G$  be a TNR rational semigroup of finite type generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Let  $\mathcal{U} = \{U_s\}_{s \in S}$  be a nice family of sets produced in Theorem 6.6, and let  $\mathcal{S}_{\mathcal{U}} = \{\tilde{f}_{\tau}^{-\|\tau\|} : X_{t(\tau)} \rightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_{\mathcal{U}}}$  be the corresponding graph directed system. If  $t \in \Delta_G$ , then  $m_t(J_{\mathcal{U}}) > 0$ ; furthermore*

$$m_t(U_s \cap J_{\mathcal{U}}) > 0$$

*for all  $s \in S$ .*

The announced link between the measures  $m_t$  and  $\tilde{m}_{t,s}$  is given by the following.

**Proposition 9.6.** *Let  $G$  be a TNR rational semigroup of finite type generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Let  $\mathcal{U} = \{U_s\}_{s \in S}$  be a nice family of sets produced in Theorem 6.6, and let  $\mathcal{S}_{\mathcal{U}} = \{\tilde{f}_{\tau}^{-\|\tau\|} : X_{t(\tau)} \rightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_{\mathcal{U}}}$  be the corresponding graph directed system.*

*If  $t \in \Delta_G$ , then  $(t, P(t)) \in \Omega(\mathcal{U})$  and*

$$(9.1) \quad P(t, P(t)) = 0.$$

Furthermore,

$$(9.2) \quad \tilde{m}_{t,P(t)}([\tau]) \asymp m_t(\phi_\tau(U_{t(\tau)})) \asymp m_t(\phi_\tau(X_{t(\tau)}))$$

for all  $\tau \in \mathcal{D}_U^*$ .

*Proof.* Taking  $\varepsilon > 0$  small enough, it immediately follows from Proposition 8.4 that  $(t, P(t)) \in \Omega(\mathcal{U})$ . By the very definition of both measures  $m_t$  and  $\tilde{m}_{t,P(t)}$ , along with Koebe's Distortion Theorem, we have that

$$\begin{aligned} m_t(\phi_\tau(U_{t(\tau)})) &= \int_{U_{t(\tau)}} e^{-P(t)\|\tau\|} |\phi'_\tau(z)|^t dm_t(z) \asymp e^{-P(t)\|\tau\|} \|\phi'_\tau\|_\infty^t m_t(U_{t(\tau)}) \\ &\asymp e^{-P(t)\|\tau\|} \|\phi'_\tau\|_\infty^t \\ &= e^{P(t,P(t))\|\tau\|} (e^{-P(t,P(t))\|\tau\|} e^{-P(t)\|\tau\|} \|\phi'_\tau\|_\infty^t) \\ &\asymp e^{P(t,P(t))\|\tau\|} \tilde{m}_{t,P(t)}([\tau]). \end{aligned}$$

Also, the same formula holds with  $U$  replaced everywhere by  $X$ . Therefore, if on the one hand  $P(t, P(t)) > 0$ , then by applying the standard covering argument with sets of the form  $[\tau]$  and  $U_{t(\tau)}$  with  $\|\tau\|$  diverging to  $+\infty$ , we would conclude that  $\tilde{m}_{t,P(t)}(\mathcal{D}_U^\infty) = 0$ , which is a contradiction. If on the other hand,  $P(t, P(t)) < 0$ , then by the same token  $m_t(J_U) = 0$ , which contradicts Lemma 9.5. Thus  $P(t, P(t)) = 0$ , and so formula (9.2) is also proved.  $\square$

From now on we denote

$$\tilde{m}_t := \tilde{m}_{t,P(t)} \quad \text{and} \quad \tilde{\mu}_t := \tilde{\mu}_{t,P(t)}.$$

**Lemma 9.7.** *Let  $G$  be a C–F balanced TNR rational semigroup of finite type generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Let  $\mathcal{U} = \{U_s\}_{s \in S}$  be a nice family of sets produced in Theorem 6.6 and let  $\mathcal{S}_\mathcal{U} = \{\tilde{f}_\tau^{-\|\tau\|} : X_{t(\tau)} \rightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_\mathcal{U}}$  the corresponding graph directed system.*

*If  $b \in \Delta_G^*$ , then there exists  $\delta > 0$  such that*

$$\Delta_G^* \cap ((b - \delta, b + \delta) \times (P(b) - \delta, P(b) + \delta)) \subset \Omega(\mathcal{U}).$$

*Proof.* Of course it suffices to show that

$$\Delta_G^* \cap ((b - \delta, b] \times (P(b) - \delta, P(b))) \subset \Omega(\mathcal{U})$$

for some  $\delta > 0$ . Fix a finite set  $S$  along with  $\eta > 0$ ,  $\gamma > 0$  and a finite set  $\Xi \subset K(B(S, \gamma))$ , all four of them coming from Proposition 8.4. Fix  $\varepsilon > 0$ . It follows from Proposition 8.4 and generalized conformality of  $m_t$  that, for  $t \leq b$  and  $P(b) - \delta \leq s \leq P(b)$  with  $t \in \Delta_G^*$

and  $\delta > 0$  to be determined in the course of the proof, we have

$$\begin{aligned}
\sum_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \|\phi'_\tau\|_\infty^t e^{-s\|\tau\|} &= \sum_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \|\phi'_\tau\|_\infty^t e^{-P(t)\|\tau\|} e^{(P(t)-s)n} = e^{(P(t)-s)n} \sum_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \|\phi'_\tau\|_\infty^t e^{-P(t)\|\tau\|} \\
&\asymp e^{(P(t)-s)n} \sum_{\tau \in \mathcal{D}_{\mathcal{U}}^n} m_\tau \left( \bigcup_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \phi_\tau(X_{t(\tau)}) \right) \\
&= e^{(P(t)-s)n} m_t \left( \bigcup_{a \in S} \bigcup_{\tau \in \mathcal{D}_n(a)} \phi_\tau(X_a) \right) \\
&\leq C_\varepsilon e^{(P(t)-s)n} \exp \left( \left( \frac{P_{B(S,\gamma)}^\Xi(t) - P(t) + \varepsilon}{l} \right) n \right) \\
&= C_\varepsilon \exp \left( (P(t) - s + l^{-1}(P_{B(S,\gamma)}^\Xi(t) - P(t) + \varepsilon))n \right) \\
&\leq C_\varepsilon \exp \left( (P(t) - P(b) + \delta + l^{-1}(P_{B(S,\gamma)}^\Xi(t) - P(t) + \varepsilon))n \right)
\end{aligned}$$

for every integer  $n \geq 1$ . Since the function  $[0, +\infty) \ni t \mapsto P_{B(S,\gamma)}^\Xi(t) \in \mathbb{R}$  is continuous and since, by Proposition 4.3 (b), the function  $[0, +\infty) \ni t \mapsto P(t)$  is also continuous, taking  $\delta > 0$  sufficiently small we will have both

$$|P(t) - P(b)| < \varepsilon/2 \quad \text{and} \quad |P_{B(S,\gamma)}^\Xi(t) - P_{B(S,\gamma)}^\Xi(b)| < \varepsilon/2.$$

Therefore,

$$\sum_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \|\phi'_\tau\|_\infty^t e^{-s\|\tau\|} \leq \exp \left( (\varepsilon + \delta + l^{-1}(P_{B(S,\gamma)}^\Xi(t) - P(t) + 2\varepsilon))n \right).$$

Since, by Lemma 8.1,  $P_{B(S,\gamma)}^\Xi(b) - P(b) < 0$ , taking both  $\delta > 0$  and  $\varepsilon > 0$  small enough, we will have  $\varepsilon + \delta + l^{-1}(P_{B(S,\gamma)}^\Xi(t) - P(t) + 2\varepsilon) < -\beta$  for some  $\beta > 0$ . Hence,

$$\sum_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \|\phi'_\tau\|_\infty^t e^{-s\|\tau\|} \leq e^{-\beta n}.$$

Thus

$$\sum_{n \geq 1} \sum_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \|\phi'_\tau\|_\infty^t e^{-s\|\tau\|} \leq \sum_{n \geq 1} e^{-\beta n} < +\infty.$$

The proof is complete.  $\square$

As an immediate consequence of this lemma, formula (4.2), and Proposition 9.6, we get the following.

**Corollary 9.8.** *Assume that  $G$  is a C-F balanced TNR rational semigroup of finite type generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ . If  $\mathcal{U} = \{U_s\}_{s \in S}$  is a nice family of sets, and  $\mathcal{S}_{\mathcal{U}} = \{\tilde{f}_\tau^{-\|\tau\|} : X_{t(\tau)} \rightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_{\mathcal{U}}}$  is the corresponding graph directed system, then there exists  $\delta > 0$  such that*

$$(h_f - \delta, h_f) \subset \Omega(\mathcal{U}),$$

and for all  $t \in (h_f - \delta, h_f)$ ,

$$0 < P(t, 0) < +\infty.$$



By analogy to the terminology of [19], this would mean in its language that the system  $\mathcal{S}_{\mathcal{U}}$  is strongly regular.

One of our main results, the last one in this section, is the following.

**Theorem 9.9.** *If  $G$  be a C–F balanced TNR rational semigroup of finite type generated by a multi–map  $(f_1, \dots, f_u) \in \text{Rat}^u$ , then the topological pressure function*

$$P : \Delta_G^* \longrightarrow \mathbb{R}$$

*is real–analytic.*

*Proof.* Using Lemma 9.7, the definition of  $\Delta_G^*$ , and applying Theorem 2.6.12 of [19], we see that for each  $\gamma \in \Delta_G^*$  there exists  $\delta > 0$  such that the function  $P$  is real–analytic on  $(\gamma - \delta, \gamma + \delta) \times (P(\gamma) - \delta, P(\gamma) + \delta)$  in both variables  $t$  and  $s$ . In order to prove that  $P$  is real–analytic on  $(\gamma - \delta, \gamma + \delta)$ , we thus may, and we will, employ the Implicit Function Theorem to show that  $P$  is the unique real–analytic function which satisfies  $P(t, P(t)) = 0$  for all  $t \in (\gamma - \delta, \gamma + \delta)$ . Since the equality  $P(t, P(t)) = 0$  holds because of Proposition 9.6, it is thus sufficient to prove that for all  $t \in (\gamma - \delta, \gamma + \delta)$  we have that

$$(9.3) \quad \left. \frac{\partial P(t, s)}{\partial s} \right|_{(t, P(t))} < 0.$$

But by Proposition 2.6.13 of [19] we have

$$(9.4) \quad \left. \frac{\partial P(t, s)}{\partial s} \right|_{(t, P(t))} = - \int_{\mathcal{D}_{\mathcal{U}}^{\infty}} \|\omega_1\| d\tilde{\mu}_t(\omega) < 0,$$

which completes the proof of Theorem 9.9. □

## 10. INVARIANT MEASURES: $\mu_t$ VERSUS $\tilde{\mu}_t \circ \pi_{\mathcal{U}}^{-1}$ ; FINITENESS OF $\mu_t$

In this section we link the measures  $\tilde{m}_t$  and  $\tilde{\mu}_t$  of the previous section, living on the symbol space, with the conformal and invariant measures  $m_t$  and  $\mu_t$  living on the Julia set  $J(\tilde{f})$ . This link is given by Lemma 10.7. We translate here many results of the previous sections, expressed in the symbolic language, to the one of the actual map  $\tilde{f}$ . We eventually prove here, see Theorem 10.13, that all of the measures  $\mu_t$ ,  $t \in \Delta_G^*$ , are finite, thus probability ones after normalization.

Throughout this section  $G$  is again a TNR rational semigroup of finite type generated by a multi–map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . As in previous sections let

$$\mathcal{U} = \{U_s\}_{s \in S}$$

be a nice family of sets coming from Theorem 6.6 and let  $\mathcal{S}_{\mathcal{U}}$  be the corresponding graph directed system described in Theorem 6.9. Given  $t \in \Delta_G^*$  let

$$(10.1) \quad \hat{\mu}_t := \tilde{\mu}_t \circ \pi_{\mathcal{U}}^{-1}.$$

We start with the following.

**Lemma 10.1.** *Let  $G$  be a TNR rational semigroup of finite type generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Let  $\mathcal{U} = \{U_s\}_{s \in S}$  be a nice family of sets produced in Theorem 6.6 and let*

$$\mathcal{S}_{\mathcal{U}} = \{ \tilde{f}_{\tau}^{-\|\tau\|} : X_{t(\tau)} \longrightarrow X_{i(\tau)} \}_{\tau \in \mathcal{D}_{\mathcal{U}}},$$

the corresponding graph directed system.

If  $t \in \Delta_G^*$ , then

$$\hat{\mu}_t \left( \bigcap_{n=1}^{\infty} \bigcup_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \phi_{\tau}(U_{t(\tau)}) \right) = 1.$$

*Proof.* Since  $U \cap J(\tilde{f})$  is a non-empty open set relative to  $J(\tilde{f})$  and since  $\text{Trans}(\tilde{f})$  is dense in  $J(\tilde{f})$ , we have that  $U \cap \text{Trans}(\tilde{f}) \neq \emptyset$ . It then follows from Theorem 6.9 (b) that there exists at least one point  $\xi \in \mathcal{J}_{\mathcal{U}} \cap U$ . Then  $\xi = \pi_{\mathcal{U}}(\tau)$  for some  $\tau \in \mathcal{D}_{\mathcal{U}}^{\infty}$ . Since  $U$  is open, there thus exists an integer  $n \geq 1$  such that  $\pi_{\mathcal{U}}([\tau]_n) \subset U$ . Equivalently,  $[\tau]_n \subset \pi_{\mathcal{U}}^{-1}(U)$ . Hence,

$$(10.2) \quad \tilde{\mu}_t \circ \pi_{\mathcal{U}}^{-1}(U) \geq \tilde{\mu}_t([\tau]_n) > 0.$$

But by item (c) of Definition 6.5 and by the definition of nice sets, we have that if  $\omega \in \pi_{\mathcal{U}}^{-1}(U)$ ,  $e \in \mathcal{D}_{\mathcal{U}}$  and  $A_{e\omega_1}(\mathcal{U}) = 1$ , then

$$\pi_{\mathcal{U}}(e\omega) = \phi_e(\pi_{\mathcal{U}}(\omega)) \in \phi_e(U_{t(e)}) \subset U_{i(e)} \subset U.$$

Therefore,  $e\omega \in \pi_{\mathcal{U}}^{-1}(U)$ . Hence,

$$\sigma^{-1}(\pi_{\mathcal{U}}^{-1}(U)) \subset \pi_{\mathcal{U}}^{-1}(U).$$

Since the measure  $\hat{\mu}_t$  is ergodic with respect to the shift map  $\sigma$ , it follows from this and from (10.2) that

$$(10.3) \quad \tilde{\mu}_t(\pi_{\mathcal{U}}^{-1}(U)) = 1.$$

Now fix an arbitrary integer  $n \geq 1$  and fix an  $\omega \in \sigma^{-n}(\pi_{\mathcal{U}}^{-1}(U))$ . This means that

$$(10.4) \quad \pi_{\mathcal{U}}(\sigma^n(\omega)) \in U.$$

Then

$$(10.5) \quad \pi_{\mathcal{U}}(\omega) = \phi_{\omega|_n}(\pi_{\mathcal{U}}(\sigma^n(\omega))) \in \phi_{\omega|_n}(X_{t(\omega)}) = \phi_{\omega|_n}(\overline{U}_{t(\omega)}).$$

So, since the sets  $\overline{U}_s$ ,  $s \in S$  are mutually disjoint, we conclude from (10.4) and (10.5) that

$$\pi_{\mathcal{U}}(\sigma^n(\omega)) \in U_{t(\omega)}.$$

Combining this with the first part of (10.5) we conclude that  $\pi_{\mathcal{U}}(\omega) \in \phi_{\omega|_n}(U_{t(\omega)})$ . Thus

$$\pi_{\mathcal{U}}(\omega) \in \bigcup_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \phi_{\tau}(U_{t(\tau)}).$$

Equivalently,

$$\omega \in \pi_{\mathcal{U}}^{-1} \left( \bigcup_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \phi_{\tau}(U_{t(\tau)}) \right).$$

Thus we have proved that

$$(10.6) \quad \sigma^{-n}(\pi_{\mathcal{U}}^{-1}(U)) \subset \pi_{\mathcal{U}}^{-1}\left(\bigcup_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \phi_{\tau}(U_{t(\tau)})\right).$$

Combining this along with (10.3) and shift invariance of the measure  $\mu_t$ , we conclude that

$$\hat{\mu}_t\left(\bigcup_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \phi_{\tau}(U_{t(\tau)})\right) \geq \tilde{\mu}_t(\sigma^{-n}(\pi_{\mathcal{U}}^{-1}(U))) = \tilde{\mu}_t(\pi_{\mathcal{U}}^{-1}(U)) = 1.$$

Thus,

$$\hat{\mu}_t\left(\bigcap_{n=1}^{\infty} \bigcup_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \phi_{\tau}(U_{t(\tau)})\right) = 1.$$

and the proof is complete.  $\square$

Keep the setting of Lemma 10.1. In particular  $t \in \Delta_G^*$ .

Let  $J_{\mathcal{U}}(\infty)$  be the set of all those points  $\xi \in J_{\mathcal{U}}$  for which there exists infinitely many integers  $n \geq 1$  such that  $\tilde{f}^n(\xi) \in J_{\mathcal{U}}$ .

Let also  $U(\infty)$  be the set of all those points  $\xi \in U$  for which there exists infinitely many integers  $n \geq 1$  such that  $\tilde{f}^n(\xi) \in U$ .

The Poincaré Recurrence Theorem asserts that

$$\mu_t(J_{\mathcal{U}}(\infty)) = \mu_t(J_{\mathcal{U}}).$$

For all points  $\xi \in J_{\mathcal{U}}(\infty)$  there is a well defined first return time to  $J_{\mathcal{U}}(\infty)$ , equal also to the first entrance time to  $J_{\mathcal{U}}$ , and defined as the least  $n \geq 1$  such that  $\tilde{f}^n(\xi) \in J_{\mathcal{U}}(\infty)$ . Denote this  $n$  by  $N_{J_{\mathcal{U}}}(\xi)$ . The first return map  $\tilde{f}_{J_{\mathcal{U}}} : J_{\mathcal{U}}(\infty) \rightarrow J_{\mathcal{U}}(\infty)$  is then defined as

$$\tilde{f}_{J_{\mathcal{U}}}(\xi) = \tilde{f}^{N_{J_{\mathcal{U}}}(\xi)}(\xi).$$

Now we shall prove the following. Of course in the above considerations  $J_{\mathcal{U}}$  could have been replaced by any measurable subset of  $\Sigma_u \times \hat{\mathbb{C}}$ .

**Lemma 10.2.** *With the same hypotheses as in Lemma 10.1, we have that*

$$J_{\mathcal{U}}^{\circ} := \bigcap_{n=1}^{\infty} \bigcup_{\tau \in \mathcal{D}_{\mathcal{U}}^n} \phi_{\tau}(U_{t(\tau)}) = U(\infty) \subset J_{\mathcal{U}} \subset \bar{U}$$

and

$$\mu_t(J_{\mathcal{U}}^{\circ}) = \mu_t(U) = \mu_t(\bar{U}) = \mu_t(J_{\mathcal{U}}) > 0.$$

*Proof.* Of course

$$J_{\mathcal{U}}^{\circ} \subset U(\infty) \subset J_{\mathcal{U}} \subset \bar{U}.$$

We will now prove the inclusion opposite to the first one, i.e.  $U(\infty) \subset J_{\mathcal{U}}^{\circ}$ . For this end it suffices to show that if  $\xi \in U(\infty)$  and  $\tilde{f}^n(\xi) \in \bar{U}$  for some  $n \geq 0$ , then  $\tilde{f}^n(\xi) \in U$ . So assume that

$$\xi \in U(\infty) \quad \text{and} \quad \tilde{f}^n(\xi) \in \bar{U}.$$

Then, there exists an integer  $q > n$  such that  $\tilde{f}^q(\xi) \in U$ . Hence, there exists  $s \in S$  such that  $\tilde{f}^q(\xi) \in U_s$ . Let  $\tilde{f}_\xi^{-(q-n)}$  be the unique continuous inverse branch of  $\tilde{f}^{q-n}$  defined on  $U_s$  and sending  $\tilde{f}^q(\xi)$  back to  $\tilde{f}^n(\xi)$ . Then  $\tilde{f}_\xi^{-(q-n)} = \phi_\omega$  with some  $\omega \in \mathcal{D}_U^*$ . Since  $\mathcal{U}$  is a nice family, this implies that  $\tilde{f}^n(\xi) \in \phi_\omega(U_s) \subset U_{i(\omega)} \subset U$ . Thus

$$\xi \in \bigcap_{n=1}^{\infty} \bigcup_{\tau \in \mathcal{D}_U^n} \phi_\tau(U_{t(\tau)}).$$

So, the first formula of our lemma is established.

Passing to the second formula of our lemma, note that its first two equality signs are now immediate. Note also that

$$0 < \mu_t(U) = \mu_t(U(\infty)) \leq \mu_t(J_U) \leq \mu_t(\bar{U}).$$

Thus, in order to conclude the proof, it suffices to show that

$$\mu_t(\bar{U} \setminus U) = 0.$$

But, since  $U$  is the union of all members of some nice family,  $U \cap \bigcup_{n=0}^{\infty} \tilde{f}^n(\bar{U} \setminus U) = \emptyset$ . Hence,  $(\bar{U} \setminus U) \cap \text{Trans}(\tilde{f}) = \emptyset$ , and therefore,  $\mu_t(\bar{U} \setminus U) = 0$  by virtue of Corollary 4.15 and Theorem 4.20. The proof is complete.  $\square$

As an immediate consequence of this lemma, we get the following.

**Corollary 10.3.** *With the same hypotheses as in Lemma 10.1, we have that for every  $\xi \in J_U^\circ$  the set*

$$\{n \geq 0 : \tilde{f}^n(\xi) \in J_U^\circ\}$$

*is infinite.*

Since, by virtue of Theorem 4.20, the measures  $m_t$  and  $\mu_t$  are equivalent, as an immediate consequence of Lemma 10.2 we get the following.

**Corollary 10.4.** *With the same hypotheses as in Lemma 10.1, we have that*

$$m_t(J_U^\circ) = m_t(U) = m_t(\bar{U}) = m_t(J_U) > 0.$$

Let

$$\mathcal{D}_U^\circ := \pi_U^{-1}(J_U^\circ).$$

As an immediate consequence of Lemma 10.1 and Lemma 10.2 we get the following.

**Corollary 10.5.** *With the hypothesis of Lemma 10.1 we have that*

- (a)  $\pi_U^{-1}(\xi)$  is a singleton in  $\mathcal{D}_U^\circ$  for every  $\xi \in J_U^\circ$ ,
- (b) The map  $\pi_U|_{\mathcal{D}_U^\circ} : \mathcal{D}_U^\circ \rightarrow J_U^\circ$  is bijective,
- (c)  $\tilde{\mu}_t(\mathcal{D}_U^\circ) = \hat{\mu}_t(J_U^\circ) = 1$ .

The following is an immediate consequence of Corollary 10.5.

**Corollary 10.6.** *With the hypothesis of Lemma 10.1 we have that*

(a) *For every  $\tau \in \mathcal{D}_U^*$  we have that*

$$\tilde{f}_{J_U} \circ \phi_\tau \Big|_{J_U^\circ \cap U_{t(\tau)}} = \text{Id} \Big|_{J_U^\circ \cap U_{t(\tau)}},$$

(b) *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{D}_U^\circ & \xrightarrow{\sigma} & \mathcal{D}_U^\circ \\ \pi_U \downarrow & & \downarrow \pi_U \\ J_U^\circ & \xrightarrow{\tilde{f}_{J_U}} & J_U^\circ. \end{array}$$

Now we are in position to prove the following.

**Lemma 10.7.** *With the same hypotheses as in Lemma 10.1, we have that*

$$\hat{\mu}_t = \frac{\mu_t}{\mu_t(J_U)} \Big|_{J_U}.$$

*Proof.* Put

$$m_t^\circ := \frac{m_t}{m_t(J_U)} \Big|_{J_U}, \quad \mu_t^\circ := \frac{\mu_t}{\mu_t(J_U)} \Big|_{J_U}$$

and also

$$\tilde{m}_t^\circ := m_t^\circ \circ \pi_U \Big|_{\mathcal{D}_U^\circ}, \quad \tilde{\mu}_t^\circ := \mu_t^\circ \circ \pi_U \Big|_{\mathcal{D}_U^\circ},$$

where the latter two are well defined since, by virtue of Corollary 10.5 (b), the map  $\pi_U \Big|_{\mathcal{D}_U^\circ}$  is 1-to-1. It then follows from formula (9.2) of Proposition 9.6 and from Corollary 10.4 that

$$\tilde{m}_t([\tau]) \asymp m_t(\phi_\tau(U_{t(\tau)})) = m_t(J_U) m_t^\circ(J_U^\circ \cap \phi_\tau(U_{t(\tau)})) = m_t(J_U) \tilde{m}_t^\circ([\tau]),$$

for every  $\tau \in \mathcal{D}_U^*$ . Therefore,  $\tilde{m}_t \asymp \tilde{m}_t^\circ$ . Hence

$$\tilde{\mu}_t \asymp \tilde{\mu}_t^\circ.$$

Hence

$$(10.7) \quad \hat{\mu}_t = \tilde{\mu}_t \circ \pi_U^{-1} \asymp \tilde{\mu}_t^\circ \circ \pi_U^{-1} \leq \mu_t^\circ.$$

Now, on the one hand,  $\mu_t^\circ$  is  $\tilde{f}_U$ -invariant because of its definition and since  $\mu_t$  is  $\tilde{f}$ -invariant. On the other hand,  $\hat{\mu}_t = \tilde{\mu}_t \circ \pi_U^{-1}$  is  $\tilde{f}_U$ -invariant because of shift invariance of  $\tilde{\mu}_t$  and because of item (b) of Corollary 10.6. Since in addition  $\mu_t^\circ$  is ergodic with respect to  $\tilde{f}_U$  (as  $\mu_t$  is ergodic with respect to  $\tilde{f}$ ), it follows from (10.7) that  $\hat{\mu}_t = \mu_t^\circ$ . The proof is complete.  $\square$

As an immediate consequence of this lemma, formula (9.2) of Proposition 9.6, item (e) of Theorem 9.4, and Proposition 8.4, we get the following.

**Proposition 10.8.** *Let  $G$  be a C–F balanced TNR rational semigroup of finite type generated by a multi–map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . If  $\mathcal{U} = \{U_s\}_{s \in S}$  is a nice family of sets produced in Theorem 6.6 and  $\mathcal{S}_{\mathcal{U}} = \{\tilde{f}_{\tau}^{-\|\tau\|} : X_{t(\tau)} \rightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_{\mathcal{U}}}$  is the corresponding graph directed system, then there exist  $\gamma > 0$  and a finite set  $\Xi \subset K(B(S, \gamma))$  such that for every  $b \in \Delta_G^*$  there exists  $\eta > 0$  such that for every  $t \in \Delta_G^* \cap (b - \eta, b + \eta)$  and every  $\varepsilon > 0$*

$$\mu_t \left( \bigcup_{s \in S} \bigcup_{\tau \in \mathcal{D}_n(s)} \tilde{f}_{\tau}^{-n}(X_s) \right) \leq C'_\varepsilon \exp \left( \left( \frac{P_{B(S, \gamma)}^\Xi(t) - P(t) + \varepsilon}{l} \right) n \right)$$

and

$$\tilde{m}_t \left( \bigcup_{\tau \in \mathcal{D}_n} [\tau] \right) \leq C'_\varepsilon \exp \left( \left( \frac{P_{B(S, \gamma)}^\Xi(t) - P(t) + \varepsilon}{l} \right) n \right)$$

for every integer  $n \geq 1$  and some constant  $C'_\varepsilon \in (0, +\infty)$  depending on  $\varepsilon$ . Also,

$$P(t) - P_{B(S, \gamma)}^\Xi(t) > 0.$$

In terms of return time, an equivalent reformulation of this proposition with regard to measure  $\mu_t$  is the following.

**Proposition 10.9.** *Let  $G$  be a C–F balanced TNR rational semigroup of finite type generated by a multi–map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . If  $\mathcal{U} = \{U_s\}_{s \in S}$  is a nice family of sets produced in Theorem 6.6 and  $\mathcal{S}_{\mathcal{U}} = \{\tilde{f}_{\tau}^{-\|\tau\|} : X_{t(\tau)} \rightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_{\mathcal{U}}}$  is the corresponding graph directed system, then there exist  $\gamma > 0$  and a finite set  $\Xi \subset K(B(S, \gamma))$  such that for every  $b \in \Delta_G^*$  there exists  $\eta > 0$  such that for every  $t \in \Delta_G^* \cap (b - \eta, b + \eta)$  and every  $\varepsilon > 0$*

$$\mu_t \left( \left\{ \xi \in J_{\mathcal{U}} : N_{J_{\mathcal{U}}}(\xi) = n \right\} \right) \leq C'_\varepsilon \exp \left( \left( \frac{P_{B(S, \gamma)}^\Xi(t) - P(t) + \varepsilon}{l} \right) n \right)$$

for every integer  $n \geq 1$  and some constant  $C'_\varepsilon \in (0, +\infty)$  depending on  $\varepsilon$ . Also,

$$P(t) - P_{B(S, \gamma)}^\Xi(t) > 0.$$

An immediate consequence of Proposition 10.8 and Proposition 10.9 along with item (e) of Theorem 9.4, is the following.

**Proposition 10.10.** *Let  $G$  be a C–F balanced TNR rational semigroup of finite type generated by a multi–map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . If  $\mathcal{U} = \{U_s\}_{s \in S}$  is a nice family of sets produced in Theorem 6.6, and  $\mathcal{S}_{\mathcal{U}} = \{\tilde{f}_{\tau}^{-\|\tau\|} : X_{t(\tau)} \rightarrow X_{i(\tau)}\}_{\tau \in \mathcal{D}_{\mathcal{U}}}$  is the corresponding graph directed system, then there exist  $\gamma > 0$  and a finite set  $\Xi \subset K(B(S, \gamma))$  such that for every  $b \in \Delta_G^*$  there exists  $\eta > 0$  such that for every  $t \in \Delta_G^* \cap (b - \eta, b + \eta)$  and every*

$$\varepsilon \in (0, P(t) - P_{B(S,\gamma)}^{\Xi}(t))$$

$$\begin{aligned} \mu_t \left( \{ \xi \in J_{\mathcal{U}} : N_{J_{\mathcal{U}}}(\xi) \geq n \} \right) &\leq C_{\varepsilon}'' \exp \left( \left( \frac{P_{B(S,\gamma)}^{\Xi}(t) - P(t) + \varepsilon}{l} \right) n \right), \\ \tilde{m}_t \left( \bigcup_{k \geq n} \bigcup_{\tau \in \mathcal{D}_k} [\tau] \right) &\leq C_{\varepsilon}'' \exp \left( \left( \frac{P_{B(S,\gamma)}^{\Xi}(t) - P(t) + \varepsilon}{l} \right) n \right), \quad \text{and} \\ \tilde{\mu}_t \left( \bigcup_{k \geq n} \bigcup_{\tau \in \mathcal{D}_k} [\tau] \right) &\leq C_{\varepsilon}'' \exp \left( \left( \frac{P_{B(S,\gamma)}^{\Xi}(t) - P(t) + \varepsilon}{l} \right) n \right) \end{aligned}$$

for every integer  $n \geq 1$  and some constant  $C_{\varepsilon}'' \in (0, +\infty)$  depending on  $\varepsilon$ . Also,

$$P(t) - P_{B(S,\gamma)}^{\Xi}(t) > 0.$$

An immediate consequence of this proposition is the following.

**Corollary 10.11.** *Let  $G$  be a C–F balanced TNR rational semigroup of finite type generated by a multi–map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . If  $\mathcal{U} = \{U_s\}_{s \in S}$  is a nice family of sets and  $t \in \Delta_G^*$ , then for every  $p > 0$  we have*

$$\int_{J_{\mathcal{U}}} N_{J_{\mathcal{U}}}^p(\xi) d\mu_t(\xi) < +\infty.$$

Now we can also prove the following.

**Lemma 10.12.** *Let  $G$  be a C–F balanced TNR rational semigroup of finite type generated by a multi–map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . If  $\mathcal{U} = \{U_s\}_{s \in S}$  is a nice family of sets and  $t \in \Delta_G^*$ , then the function  $\log \zeta_{1,0} : \mathcal{D}_{\mathcal{U}}^{\infty} \rightarrow (-\infty, 0)$  is integrable with respect to the measure  $\tilde{\mu}_t$ . Furthermore, all its positive moments are finite. More precisely:*

$$0 < \chi_{\tilde{\mu}_t} := \int_{\mathcal{D}_{\mathcal{U}}^{\infty}} \log |(f^{\|\tau_1\|})'(\pi_{\mathcal{U}}(\sigma(\tau)))| d\tilde{\mu}_t(\tau) < +\infty$$

and (note that the integrand of the above integral is, by Theorem 6.6, everywhere positive)

$$\int_{\mathcal{D}_{\mathcal{U}}^{\infty}} \log^p |(f^{\|\tau_1\|})'(\pi_{\mathcal{U}}(\sigma(\tau)))| d\tilde{\mu}_t(\tau) < +\infty$$

for every  $p > 0$ .

*Proof.* Taking  $\varepsilon > 0$  small enough and applying Proposition 10.10, we get that

$$\begin{aligned}
\int_{\mathcal{D}_u^\infty} \log^p |(\tilde{f}^{\|\tau_1\|})'(\pi_u(\sigma(\tau)))| d\tilde{\mu}_t(\tau) &\leq \sum_{e \in \mathcal{D}_u} \sup \left( \log |(\tilde{f}^{\|e\|})'| \Big|_{X_{t(e)}} \right)^p \tilde{\mu}_t([e]) \\
&= \sum_{n=1}^{\infty} \sum_{e \in \mathcal{D}_n} \sup \left( \log |(\tilde{f}^{\|e\|})'| \Big|_{X_{t(e)}} \right)^p \tilde{\mu}_t([e]) \\
&\leq \sum_{n=1}^{\infty} \sum_{e \in \mathcal{D}_n} \sup \left( \log (\|\tilde{f}'\|_\infty^n) \right)^p \tilde{\mu}_t([e]) \\
&\leq \log \|\tilde{f}'\|_\infty^p \sum_{n=1}^{\infty} n^p \sum_{e \in \mathcal{D}_n} \tilde{\mu}_t([e]) \\
&\leq \log \|\tilde{f}'\|_\infty^p C_\varepsilon'' \sum_{n=1}^{\infty} n^p \exp \left( \left( \frac{P_{B(S,\gamma)}^\Xi(t) - P(t) + \varepsilon}{l} \right) n \right) \\
&< +\infty.
\end{aligned}$$

The inequality  $\chi_{\tilde{\mu}_t} > 0$  follows immediately from item (B) of Theorem 6.6.  $\square$

In turn, as an immediate consequence of the last two results, i.e. Corollary 10.11 and Lemma 10.12 along with Kac's Lemma, we get the following.

**Theorem 10.13.** *Let  $G$  be a C–F balanced TNR rational semigroup of finite type generated by a multi–map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . If  $t \in \Delta_G^*$ , then the measure  $\mu_t$  is finite. From now on we normalize this measure so that it becomes a probability one. Furthermore, the function  $J(\tilde{f}) \ni z \mapsto \log |\tilde{f}'(z)| \in \mathbb{R}$  is integrable with respect to the measure  $\mu_t$  and*

$$\chi_{\mu_t} := \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu_t \in (0, +\infty).$$

*This number, i.e.  $\chi_{\mu_t}$ , is commonly called the Lyapunov exponent of the dynamical system  $(\tilde{f}, \mu_t)$ .*

Now, as a complement of Theorem 9.9, we can prove the following.

**Theorem 10.14.** *Let  $G$  be a C–F balanced TNR rational semigroup of finite type generated by a multi–map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . If  $t \in \Delta_G^*$ , then*

$$(10.8) \quad P'(t) = -\chi_{\mu_t} < 0,$$

and

$$(10.9) \quad P''(t) = \sigma_{\mu_t}^2(-t \log |\tilde{f}'|) \geq 0,$$

where

$$\sigma_{\mu_t}^2(-t \log |\tilde{f}'|) := \lim_{n \rightarrow \infty} \frac{1}{n} \int_{J(\tilde{f})} S_n^2(-t \log |\tilde{f}'| + \chi_{\mu_t}) d\mu_t,$$

*is commonly called the asymptotic variance of the function  $-t \log |\tilde{f}'|$  with respect to the dynamical system  $(\tilde{f}, \mu_t)$ .*



*Proof.* Because of Lemma 10.7, formula (10.1), the definition of the measure  $\tilde{\mu}_t$ , and Propositions 2.6.13 from [19] along with Kac's Lemma, by differentiating formula (9.1) of Proposition 9.6, we get that

$$\begin{aligned} 0 &= \frac{\partial P(t, s)}{\partial t} \Big|_{(t, P(t))} + \frac{\partial P(t, s)}{\partial s} \Big|_{(t, P(t))} P'(t) = -\chi_{\tilde{\mu}_t} - P'(t) \int_{\mathcal{D}_{\mathcal{U}}^{\infty}} \|\omega_1\| d\tilde{\mu}_t(\omega) \\ &= -\chi_{\tilde{\mu}_t} - P'(t) \int_{J_{\mathcal{U}}} N_{J_{\mathcal{U}}} d\tilde{\mu}_t \\ &= -\chi_{\tilde{\mu}_t} - \frac{P'(t)}{\mu_t(J_{\mathcal{U}})}. \end{aligned}$$

Therefore, using the refined version of Kac's Lemma, we get

$$P'(t) = -\mu_t(J_{\mathcal{U}})\chi_{\tilde{\mu}_t} = -\chi_{\mu_t} < 0,$$

where the last inequality follows from Theorem 10.13. Passing to the second derivative of  $P(t)$ , by employing Propositions 2.6.14 from [19], we calculate that

$$P''(t) = \sigma_{\mu_t}^2(-t \log |\tilde{f}'|) \geq 0,$$

which finishes the proof.  $\square$

## 11. VARIATIONAL PRINCIPLE; THE INVARIANT MEASURES $\mu_t$ ARE THE UNIQUE EQUILIBRIUM STATES

Throughout this section we always assume that  $G = \langle f_1, \dots, f_u \rangle$  is a C-F balanced TNR rational semigroup of finite type. Our main goal in this section is to prove a variational principle for the potentials  $-t \log |\tilde{f}'|$ ,  $t \in \Delta_G^*$ , and the dynamical system  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ , and to show that the measures  $\mu_t$  are the only equilibrium states for these potentials. We start with the following technical auxiliary result.

**Lemma 11.1.** *If  $0 < r < s$ , then there exists a finite set  $\Xi \subset K(B(\text{Crit}_*(f), r))$  such that*

$$P \left( \tilde{f} \Big|_{K(B(\text{Crit}_*(f), s))}, -t \log |\tilde{f}'| \right) \leq P_{B(\text{Crit}_*(f), r)}^{\Xi}(t) < P(t)$$

for every  $t \in \Delta_G$ .

*Proof.* For the ease of notation, for every  $u > 0$ , put

$$(11.1) \quad B_u := B(\text{Crit}_*(f), u), \quad K_u := K(B_u),$$

$$(11.2) \quad P_u(t) := P \left( \tilde{f} \Big|_{K_u}, -t \log |\tilde{f}'| \right), \text{ and } P_u^{\Xi}(t) := P_{B(\text{Crit}_*(f), u)}^{\Xi}(t).$$

Let  $R_2 > 0$  be the constant produced in Lemma 7.1 for the set  $V = B_r$ . Let  $\Xi \subset K_r$  be a finite  $(R_2/4)$ -spanning set for the set  $K_r$ . Let  $n \geq 1$  be the integer produced in Lemma 7.1 for  $V = B_r$ . In view of this lemma and Corollary 7.2, we may assume  $R_2 > 0$  to be so small that

$$(11.3) \quad \|\tilde{f}_{\xi}^{-nj}(x), \tilde{f}_{\xi}^{-nj}(y)\|_{\vartheta} \leq 2^{-j} \|x, y\|_{\vartheta}$$

for all  $\xi \in K_r$ , all  $j \geq 0$ , and all  $x, y \in B(\tilde{f}^j(\xi), 2R_2)$ , where  $\tilde{f}_\xi^{-k} : B(\tilde{f}^k(\xi), 2R_2) \rightarrow \Sigma_u \times \hat{\mathbb{C}}$ ,  $k \geq 0$ , are the inverse branches of  $\tilde{f}^k$  produced in Lemma 7.1. Let  $\varepsilon \in (0, R_2/4)$  and fix an integer  $q \geq 1$  so large that

$$(11.4) \quad 2^{-q}R_2 < \varepsilon.$$

Fix an integer  $k \geq 1$ . Let  $F_\varepsilon \subset K_s$  be a maximal  $(k, \varepsilon)$ -separated set for the dynamical system  $\tilde{f}^n : K_s \rightarrow K_s$ . Then for every  $x \in F_\varepsilon$  there exists an element  $\hat{x} \in \Xi$  such that

$$\tilde{f}^{n(q+k)}(x) \in B(\hat{x}, R_2/4).$$

Let

$$\tilde{x} := \tilde{f}_x^{-n(q+k)}(\hat{x})$$

where  $\tilde{f}_x^{-n} : B(\tilde{f}^{n(q+k)}(x), 2R_2) \rightarrow \Sigma_u \times \hat{\mathbb{C}}$  is the inverse branch of  $\tilde{f}^{kn}$  produced in Lemma 7.1  $\xi = \tilde{f}^{n(q+k)}(x)$ . Then

$$\left| \left( \tilde{f}^{nk} \right)'(\tilde{x}) \right| \leq K^2 \left| \left( \tilde{f}^{nk} \right)'(x) \right|.$$

Equivalently,

$$\left| \left( \tilde{f}^{nk} \right)'(x) \right|^{-1} \leq K^2 \left| \left( \tilde{f}^{nk} \right)'(\tilde{x}) \right|^{-1}.$$

Therefore,

$$(11.5) \quad \begin{aligned} P_\varepsilon(k, t) &:= \frac{1}{k} \log \sum_{x \in F_\varepsilon} \left| \left( \tilde{f}^{nk} \right)'(x) \right|^{-t} \leq \frac{2 \log K}{k} + \frac{1}{k} \log \sum_{x \in F_\varepsilon} \left| \left( \tilde{f}^{nk} \right)'(\tilde{x}) \right|^{-t} \\ &\leq \frac{2 \log K}{k} + \frac{qn}{k} \log \|f'\|_\infty + n \frac{1}{nk} \log \sum_{x \in F_\varepsilon} \left| \left( \tilde{f}^{n(q+k)} \right)'(\tilde{x}) \right|^{-t}. \end{aligned}$$

We now claim that the function

$$(11.6) \quad F_\varepsilon \ni x \mapsto \tilde{x} \in J(\tilde{f})$$

is 1-to-1. Indeed, suppose that  $x, y \in F_\varepsilon$  and  $\tilde{x} = \tilde{y}$ . Then also  $\hat{x} = \tilde{f}^{n(q+k)}(\tilde{x}) = \tilde{f}^{n(q+k)}(\tilde{y}) = \hat{y}$ , and

$$x = \tilde{f}_z^{-n(q+k)}(\tilde{f}^{n(q+k)}(x)), \quad y = \tilde{f}_z^{-n(q+k)}(\tilde{f}^{n(q+k)}(y)),$$

where  $z := \tilde{x} = \tilde{y}$ . Since also

$$\|\tilde{f}^{n(q+k)}(x), \tilde{f}^{n(q+k)}(y)\|_\vartheta \leq \|\tilde{f}^{n(q+k)}(x), \hat{x}\|_\vartheta + \|\hat{y}, \tilde{f}^{n(q+k)}(y)\|_\vartheta \leq \frac{R_2}{4} + \frac{R_2}{4} = \frac{R_2}{2},$$

we conclude from (11.3) and (11.4) that

$$\begin{aligned} \|\tilde{f}^{nj}(x), \tilde{f}^{nj}(y)\|_\vartheta &\leq 2^{j-(q+k)} \|\tilde{f}^{n(q+k)}(x), \tilde{f}^{n(q+k)}(y)\|_\vartheta \\ &\leq 2^{-q} \|\tilde{f}^{n(q+k)}(x), \tilde{f}^{n(q+k)}(y)\|_\vartheta \\ &< 2^{-q}R_2 < \varepsilon, \end{aligned}$$

for all  $j = 0, 1, \dots, k$ . Since the set  $F_\varepsilon$  is  $(k, \varepsilon)$ -separated with respect to the map  $\tilde{f}|_{K_r} : K_r \rightarrow K_r$ , we thus conclude that  $x = y$ , and injectivity of the map from (11.6) is established. Having this and using also Lemma 8.1, we can continue (11.5) as

$$P_\varepsilon(t) := \overline{\lim}_{k \rightarrow \infty} P_\varepsilon(k, t) \leq n \overline{\lim}_{k \rightarrow \infty} \frac{1}{nk} \log \sum_{y \in \tilde{f}^{-n(q+k)}(\Xi)} \left| (f^{n(q+k)})'(y) \right|^{-t} \leq nP_{B_s^\Xi}(t) < nP(t).$$

Hence,

$$P_s(t) = \frac{1}{n} P \left( \tilde{f}|_{K_s}^n, -t \log |(f^n)'| \right) = \frac{1}{n} \overline{\lim}_{\varepsilon \rightarrow 0} P_\varepsilon(t) \leq P_{B_s^\Xi}(t) < P(t).$$

□

We shall prove in this section the following main and only theorem.

**Theorem 11.2.** *If  $G = \langle f_1, \dots, f_u \rangle$  is a C-F balanced TNR rational semigroup of finite type and  $t \in \Delta_G^*$ , then the integrals  $\int_{J(\tilde{f})} \log |\tilde{f}'| d\mu$ ,  $\mu \in M(\tilde{f})$ , are well defined and*

$$(11.7) \quad -t \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu > -\infty.$$

$$\begin{aligned} \sup \left\{ h_\mu(\tilde{f}) - t \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu : \mu \in M(\tilde{f}) \right\} &= \\ &= \sup \left\{ h_\mu(\tilde{f}) - t \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu : \mu \in M_e(\tilde{f}) \right\} = P(t), \end{aligned}$$

and

$$h_{\mu_t}(\tilde{f}) - t \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu_t = P(t),$$

while

$$h_\mu(\tilde{f}) - t \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu < P(t)$$

for every measure  $\mu \in M(\tilde{f})$  different from  $\mu_t$ .

*Proof.* First, note that since

$$\log |f'(\omega, z)| \leq \log \|f'\|_\infty$$

for all  $(\omega, z) \in J(\tilde{f})$ , all the integrals  $\int_{J(\tilde{f})} \log |\tilde{f}'| d\mu$ ,  $\mu \in M(\tilde{f})$ , are well defined and formula (11.7) follows. Hence, all the sums

$$h_\mu(\tilde{f}) - t \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu, \quad \mu \in M(\tilde{f}),$$

are well-defined. Using the definitions of the measures  $\tilde{\mu}_t$  and  $\hat{\mu}_t$ , Corollary 10.5, Corollary 10.6, Lemma 10.7, Abramov's Formula, and (the refined version of) Kac's Formula,

we get

$$\begin{aligned}
h_{\mu_t}(\tilde{f}) - \mu_t(t \log |\tilde{f}'|) &= \mu_t(J_{\mathcal{U}}) \left( h_{\hat{\mu}_t}(\tilde{f}_{J_{\mathcal{U}}}) - \hat{\mu}_t(t \log |\tilde{f}'_{J_{\mathcal{U}}}|) \right) \\
&= \mu_t(J_{\mathcal{U}}) (h_{\tilde{\mu}_t}(\sigma) + \tilde{\mu}_t(\zeta_{t,0})) \\
&= \mu_t(J_{\mathcal{U}}) \left( h_{\tilde{\mu}_t}(\sigma) + \tilde{\mu}_t(\zeta_{t,P(t)}) + P(t) \int_{\mathcal{D}_{\mathcal{U}}^{\infty}} \|\tau_1\| d\tilde{\mu}(\tau) \right).
\end{aligned}$$

Now using Proposition 9.6, Theorem 2.2.9 of [19] (for the potential  $\zeta_{t,P(t)}$ ), Corollaries 10.5, 10.6 again, and the ordinary version of Kac's Lemma, along with Lemma 10.7, we further get

$$\begin{aligned}
(11.8) \quad h_{\mu_t}(\tilde{f}) - \mu_t(t \log |\tilde{f}'|) &= \mu_t(J_{\mathcal{U}}) (P(t, P(t)) + P(t)\hat{\mu}_t(N_{J_{\mathcal{U}}})) \\
&= \mu_t(J_{\mathcal{U}})P(t) \frac{1}{\mu_t(J_{\mathcal{U}})} \\
&= P(t).
\end{aligned}$$

Now, let  $\mu \in M_e(\tilde{f})$  be arbitrary. Since  $f(\text{PCV}(\tilde{f})) \subset \text{PCV}(\tilde{f})$ , we have that either

$$\mu(\text{PCV}(\tilde{f})) = 1 \quad \text{or} \quad \mu(\text{PCV}(\tilde{f})) = 0.$$

Consider first the case where

$$\mu(\text{PCV}(\tilde{f})) = 1.$$

Since  $\text{PCV}(\tilde{f})$  is a compact set, since  $\tilde{f}(\text{PCV}(\tilde{f})) \subset \text{PCV}(\tilde{f})$  again, and since  $\text{PCV}(\tilde{f}) \subset K_s$  (see (11.1)) for every  $s > 0$  small enough, we conclude from Lemma 11.1 and the ordinary Variational Principle that

$$(11.9) \quad h_{\mu_t}(\tilde{f}) - \mu_t(t \log |\tilde{f}'|) \leq P\left(\tilde{f}|_{K_s}, -t \log |\tilde{f}'|\right) < P(t).$$

Now, suppose in turn that

$$\mu(\text{PCV}(\tilde{f})) = 0.$$

Then, taking  $R \in (0, R_*(\tilde{f}))$  sufficiently small, we will find a finite aperiodic set

$$\text{Crit}_*(\tilde{f}) \subset S \subset J(\tilde{f}) \setminus B(\text{PCV}(\tilde{f}), 8R)$$

such that

$$S \cap \text{supp}(\mu) \neq \emptyset.$$

Hence, if  $\mathcal{U}_S$  is a nice set produced in Theorem 6.6, then

$$(11.10) \quad \mu(\mathcal{U}) > 0.$$

Having this, the same proof as that of Lemma 10.2, gives that

$$\mu(J_{\mathcal{U}_S}^{\circ}) > 0.$$

Let

$$(11.11) \quad \hat{\mu} := \frac{\mu}{\mu(J_{\mathcal{U}_S}^{\circ})} \Big|_{J_{\mathcal{U}_S}^{\circ}}$$

be the corresponding conditional measure on  $J_{\mathcal{U}_S}^\circ$ . Because of Corollary 10.5, there exists a unique probability measure  $\tilde{\mu}$  on  $\mathcal{D}_{\mathcal{U}}^\infty$  such that

$$(11.12) \quad \hat{\mu} = \tilde{\mu} \circ \pi_{\mathcal{U}_S}^{-1}$$

and  $\pi_{\mathcal{U}_S} : \mathcal{D}_{\mathcal{U}}^\infty \rightarrow J_{\mathcal{U}_S}^\circ$  is a measure–theoretic isomorphism. Since

$$\hat{\mu}(N_{J_{\mathcal{U}_S}}) = \frac{1}{\mu(J_{\mathcal{U}_S})},$$

it follows from the very first two claims of Theorem 11.2 and the refined version of Kac’s Lemma that  $\tilde{\mu}(\zeta_{t,0})$  and  $\tilde{\mu}(\zeta_{t,P(t)})$  are both well–defined and larger than  $-\infty$ . Therefore, using Theorem 2.1.7 in [19], Proposition 9.6, and also Abramov’s formula, we get that

$$(11.13) \quad \begin{aligned} h_\mu(\tilde{f}) - \mu(t \log |\tilde{f}'|) &= \mu(J_{\mathcal{U}_S}) \left( h_{\tilde{\mu}}(\tilde{f}|_{J_{\mathcal{U}_S}}) - \hat{\mu}(t \log |\tilde{f}'_{\mathcal{U}_S}|) \right) = \mu(J_{\mathcal{U}_S}) (h_{\tilde{\mu}}(\sigma) - \tilde{\mu}(\zeta_{t,0})) \\ &= \mu(J_{\mathcal{U}_S}) \left( h_{\tilde{\mu}}(\sigma) - \tilde{\mu}(\zeta_{t,P(t)}) + P(t) \int_{\mathcal{D}_{\mathcal{U}}^\infty} \|\tau_1\| d\tilde{\mu}(\tau) \right) \\ &\leq \mu(J_{\mathcal{U}_S}) P(t) \int_{\mathcal{D}_{\mathcal{U}}^\infty} \|\tau_1\| d\tilde{\mu}(\tau) = \mu(J_{\mathcal{U}_S}) P(t) \frac{1}{\mu(J_{\mathcal{U}_S})} \\ &= P(t). \end{aligned}$$

Now, assume in addition that

$$h_\mu(\tilde{f}) - \mu(t \log |\tilde{f}'|) = P(t).$$

It then follows from the above formula that

$$h_{\tilde{\mu}}(\sigma) - \tilde{\mu}(\zeta_{t,P(t)}) = 0.$$

Hence, invoking Proposition 9.6 and Theorem 2.2.9 of [19], we get that  $\tilde{\mu} = \tilde{\mu}_t$ . Therefore, applying (11.12) and formula (10.1), we get

$$\hat{\mu} = \tilde{\mu} \circ \pi_{\mathcal{U}_S}^{-1} = \tilde{\mu}_t \circ \pi_{\mathcal{U}_S}^{-1} = \hat{\mu}_t.$$

Finally, applying Lemma 10.7 and formula (11.11), we obtain

$$\mu = \mu_t.$$

Along with formulas (11.8), (11.9), and (11.13), this completes the proof of Theorem 11.2.  $\square$

## 12. DECAY OF CORRELATIONS, CENTRAL LIMIT THEOREMS, THE LAW OF ITERATED LOGARITHM; THE METHOD OF LAI–SANG YOUNG TOWERS

**12.1. Stochastic Laws on the Symbol Space for the Shift Map Generated by Nice Families.** In this subsection, making use of the link with the symbolic thermodynamic formalism of Section 9, we embed the symbol space  $\mathcal{D}_{\mathcal{U}}^\infty$ , along with the shift map acting on it, into an abstract Young’s Tower (see [54] and [55]) as its first return map, and we prove the fundamental stochastic laws such as the Law of Iterated Logarithm, the Central Limit Theorem, and exponential decay of correlations, in such an abstract setting.

Let  $T : X \rightarrow X$  be a measurable dynamical system preserving a probability measure  $\mu$  on  $X$ . We say that a  $\mu$ -integrable function  $g : X \rightarrow \mathbb{R}$  with  $\int g d\mu = 0$ , satisfies the Central Limit Theorem if there exists  $\sigma > 0$  such that

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g \circ T^j \longrightarrow \mathcal{N}(0, \sigma)$$

in distribution when  $n \rightarrow \infty$ .  $\mathcal{N}(0, \sigma)$  is here the normal (Gaussian) distribution with 0 mean and variance  $\sigma$ . More precisely, for every  $t \in \mathbb{R}$ ,

$$\mu \left( \left\{ x \in X : \frac{1}{\sqrt{n}} S_n g(x) \leq t \right\} \right) \longrightarrow \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^t \exp(-u^2/2\sigma^2) du.$$

We say the function  $g$  satisfies the Law of Iterated Logarithm if there exists a positive number  $A_g$  such that

$$\limsup_{n \rightarrow \infty} \frac{S_n g(x)}{\sqrt{n \log \log n}} = A_g.$$

for  $\mu$ -a.e.  $x \in X$ .

Another important stochastic feature of a dynamical system is the rate of decay of correlations it yields. Let  $\psi_1$  and  $\psi_2$  be real square  $\mu$ -integrable functions on  $X$ . For every positive integer  $n$ , the  $n$ -th correlation of the pair  $\psi_1, \psi_2$  is the number

$$(12.1) \quad C_n(\psi_1, \psi_2) := \int \psi_1 \cdot (\psi_2 \circ T^n) d\mu - \int \psi_1 d\mu_\phi \int \psi_2 d\mu,$$

provided the above integrals exist. Notice that, due to the  $T$ -invariance of  $\mu$ , we can also write

$$C_n(\psi_1, \psi_2) = \int (\psi_1 - \mu(\psi_1)) ((\psi_2 - \mu(\psi_2)) \circ T^n) d\mu.$$

Finally, we say that two functions  $g, h : X \rightarrow \mathbb{R}$  are homologous or, perhaps more adequately, cohomologous, in a class  $\mathcal{G}$  of real-valued functions defined on  $X$  if and only if there exists a function  $k \in \mathcal{G}$  such that

$$(12.2) \quad h - g = k - k \circ T.$$

As an immediate consequence of results from [21] and [47], both for the Law of Iterated Logarithm (LIL), Theorem 3.1 in [8] (Central Limit Theorem, CLT), Theorem 1.3 in [9] (exponential decay of correlations), (comp. [8], [54], and [54]), and Proposition 10.10, we get the following.

**Theorem 12.1.** *Let  $T : X \rightarrow X$  be a measurable dynamical system. Let  $G$  be a C-F balanced TNR rational semigroup of finite type generated by a multi-map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Let  $\mathcal{U}$  be a nice family of sets for  $\hat{f}$ . Assume the following.*

- $X$  contains  $\mathcal{D}_{\mathcal{U}}^\infty$  as its measurable subset such that  $\bigcup_{n \geq 0} T^n(\mathcal{D}_{\mathcal{U}}^\infty) = X$ .
- The first return map of  $T$  from  $\mathcal{D}_{\mathcal{U}}^\infty$  to  $\mathcal{D}_{\mathcal{U}}^\infty$  is equal to the shift map  $\sigma : \mathcal{D}_{\mathcal{U}}^\infty \rightarrow \mathcal{D}_{\mathcal{U}}^\infty$ .
- The corresponding first return time is equal to  $\|\omega_1\|$  for every  $\omega \in \mathcal{D}_{\mathcal{U}}^\infty$ .

If  $t \in \Delta_G^*$ , then

- (a) There exists a unique  $T$ -invariant probability measure  $\nu_t$  on  $X$  which conditioned on  $\mathcal{D}_{\mathcal{U}}^\infty$  coincides with  $\tilde{\mu}_t$ . In addition,  $\nu_t$  is ergodic with respect to the map  $T : X \rightarrow X$ .

For every function  $g : X \rightarrow \mathbb{R}$  let  $\hat{g} : \mathcal{D}_U^\infty \rightarrow \mathbb{R}$  be defined by the following formula:

$$\hat{g}(\omega) := \sum_{j=0}^{\|\omega_1\|-1} g(T^j(\omega)).$$

Assume that  $\hat{g} \in L^2(\tilde{\mu}_t)$  and that  $\hat{g} : \mathcal{D}_U^\infty \rightarrow \mathbb{R}$  is Hölder continuous. Assume in addition that  $\psi : X \rightarrow \mathbb{R}$  is a bounded measurable function. Then we have the following:

(b)

$$|C_n(\psi, g)| = \left| \int (\psi \circ T^n) g \, d\nu_t - \int \psi \, d\nu_t \int g \, d\nu_t \right| = O(\theta^n)$$

for some  $\theta \in (0, 1)$ .

- (c) The Central Limit Theorem holds for the function  $g : X \rightarrow \mathbb{R}$  with respect to the dynamical system  $(T, \nu_t)$  provided that  $g$  is not cohomologous to a constant in  $L^2(\nu_t)$ .
- (d) The Law of Iterated Logarithm holds for the function  $g : X \rightarrow \mathbb{R}$  with respect to the dynamical system  $(T, \nu_t)$  provided that  $g$  is not cohomologous to a constant in  $L^2(\nu_t)$ .

We shall now describe a canonical way, known as Young's Tower (see [54] and [55]), of embedding  $\mathcal{D}_U^\infty$  into a larger space  $X$  and to construct an appropriate map  $T : X \rightarrow X$  so that all the hypotheses of the above theorem are satisfied. Let

$$\hat{\mathcal{D}}_U^\infty := \{(\omega, n) \in \mathcal{D}_U^\infty \times \mathbb{N} \cup \{0\} : 0 \leq n < \|\omega_1\|\}$$

where each point  $\omega \in \mathcal{D}_U^\infty$  is identified with  $(\omega, 0) \in \hat{\mathcal{D}}_U^\infty$ . We refer to  $\hat{\mathcal{D}}_U^\infty$  as the tower induced by  $\mathcal{D}_U^\infty$ . The map  $T$  acts on  $\hat{\mathcal{D}}_U^\infty$  as follows.

$$(12.3) \quad T(\omega, n) := \begin{cases} (\omega, n+1) & \text{if } n+1 < \|\omega_1\| \\ (\sigma(\omega), 0) & \text{if } n+1 = \|\omega_1\| \end{cases}.$$

As an immediate consequence of Theorem 12.1 we get the following.

**Theorem 12.2.** *Let  $G$  be a C-F balanced TNR rational semigroup of finite type generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ . Let  $\mathcal{U}$  be a nice family of sets for  $\tilde{f}$ . Let  $T : \hat{\mathcal{D}}_U^\infty \rightarrow \hat{\mathcal{D}}_U^\infty$  be defined by formula (12.3). Fix  $t \in \Delta^*(G)$ . Then*

- (a) *There exists a unique  $T$ -invariant probability measure  $\nu_t$  on  $\hat{\mathcal{D}}_U^\infty$  which conditioned on  $\mathcal{D}_U^\infty$  coincides with  $\tilde{\mu}_t$ .*

For every function  $g : \hat{\mathcal{D}}_U^\infty \rightarrow \mathbb{R}$  let  $\hat{g} : \mathcal{D}_U^\infty \rightarrow \mathbb{R}$  be defined by the following formula:

$$\hat{g}(\omega) := \sum_{j=0}^{\|\omega_1\|-1} g(T^j(\omega)).$$

Assume that  $\hat{g} \in L^2(\tilde{\mu}_t)$  and that  $\hat{g} : \mathcal{D}_U^\infty \rightarrow \mathbb{R}$  is Hölder continuous. Assume in addition that  $\psi : \hat{\mathcal{D}}_U^\infty \rightarrow \mathbb{R}$  is a bounded measurable function. Then

(b)

$$|C_n(\psi, g)| = \left| \int (\psi \circ T^n)g \, d\nu_t - \int \psi \, d\nu_t \int g \, d\nu_t \right| = O(\theta^n)$$

for some  $\theta \in (0, 1)$ .

(c) *The Central Limit Theorem holds for the function  $g : \hat{\mathcal{D}}_{\mathcal{U}}^{\infty} \rightarrow \mathbb{R}$  with respect to the dynamical system  $(T, \nu_t)$  provided that  $g$  is not cohomologous to a constant in  $L^2(\nu_t)$ .*

(d) *The Law of Iterated Logarithm holds for the function  $g : \hat{\mathcal{D}}_{\mathcal{U}}^{\infty} \rightarrow \mathbb{R}$  with respect to the dynamical system  $(T, \nu_t)$  provided that  $g$  is not cohomologous to a constant in  $L^2(\nu_t)$ .*

**12.2. Stochastic Laws for the Dynamical System** ( $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f}), \mu_t$ ). In this subsection, making use of the previous section, via the natural projection from the abstract Young's Tower to the Julia set  $J(\tilde{f})$ , we prove in Theorem 12.6 the fundamental stochastic laws for dynamical systems  $(\tilde{f}, \mu_t)$ ,  $t \in \Delta_G^*$ , such as the Law of Iterated Logarithm, the Central Limit Theorem, and exponential decay of correlations.

So, we pass to the actual dynamics of  $\tilde{f}$  on  $J(\tilde{f})$ . Consider  $H : \hat{\mathcal{D}}_{\mathcal{U}}^{\infty} \rightarrow \mathbb{C}$ , the natural projection from the tower  $\hat{\mathcal{D}}_{\mathcal{U}}^{\infty}$  to the complex plane  $\mathbb{C}$ , given by the formula

$$H(\omega, n) = \tilde{f}^n(\pi_{\mathcal{U}}(\omega)).$$

Then

$$(12.4) \quad H \circ T = \tilde{f} \circ H.$$

Consequently, we immediately get the following.

**Proposition 12.3.** *Let  $G$  be a C-F balanced TNR rational semigroup of finite type generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ . Let  $\mathcal{U}$  be a nice family of sets for  $f$ .*

*If  $\nu_t$  is the measure produced in Theorem 12.2 (a), then the Borel probability measure*

$$\nu_t \circ H^{-1}$$

*on  $J(\tilde{f})$  is  $\tilde{f}$ -invariant.*

In order to proceed further we will need the following lemma from abstract ergodic theory. Its proof is standard and can be found in many textbooks on ergodic theory.

**Lemma 12.4.** *If  $T : X \rightarrow X$  is a measurable map preserving a probability measure  $\mu$  and if  $g \in L^2(\mu)$ , then the following two statements are equivalent.*

(a) *The function  $g$  is a coboundary, i.e.  $g = u - u \circ T$  for some  $u \in L^2(\mu)$ .*

(b) *The sequence  $(S_n g)_{n=1}^{\infty}$  is bounded in the Hilbert space  $L^2(\mu)$ .*

We are now in position to prove the following.



**Theorem 12.5.** *Let  $G$  be a C–F balanced TNR rational semigroup of finite type generated by a multi–map  $(f_1, \dots, f_u) \in \text{Rat}^u$ . Let  $\mathcal{U}$  be a nice family of sets for  $\tilde{f}$ . If  $t \in \Delta_G^*$  and  $\nu_t$  is the corresponding probability measure produced in Theorem 12.2 (a), then for the dynamical system  $(\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f}), \nu_t \circ H^{-1})$  the following hold.*

- (a) *Fix  $\alpha \in (0, 1]$  and a bounded function  $g : J(\tilde{f}) \rightarrow \mathbb{R}$  which is Hölder continuous with the exponent  $\alpha$ . Then for every bounded measurable function  $\psi : J(\tilde{f}) \rightarrow \mathbb{R}$ , we have that*

$$\left| \int \psi \circ \tilde{f}^n \cdot g d(\nu_t \circ H^{-1}) - \int g d(\nu_t \circ H^{-1}) \int \psi d(\nu_t \circ H^{-1}) \right| = O(\theta^n)$$

for some  $0 < \theta < 1$  depending on  $\alpha$ .

- (b) *The Central Limit Theorem holds for every Hölder continuous function  $g : J(\tilde{f}) \rightarrow \mathbb{R}$  that is not cohomologous to a constant in  $L^2(\nu_t \circ H^{-1})$ , i.e. for which there is no square integrable function  $\eta$  for which  $g = \text{const} + \eta \circ \tilde{f} - \eta$ . Precisely, we recall, this means that there exists  $\sigma > 0$  such that*

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g \circ \tilde{f}^j \rightarrow \mathcal{N}(0, \sigma)$$

in distribution.

- (c) *The Law of Iterated Logarithm holds for every Hölder continuous function  $g : J(\tilde{f}) \rightarrow \mathbb{R}$  that is not cohomologous to a constant in  $L^2(\nu_t \circ H^{-1})$ . This, we recall, means that there exists a real positive constant  $A_g$  such that such that  $\nu_t \circ H^{-1}$  almost everywhere*

$$\limsup_{n \rightarrow \infty} \frac{S_n g - n \int g d(\nu_t \circ H^{-1})}{\sqrt{n \log \log n}} = A_g.$$

*Proof.* We aim to employ Theorem 12.2. Let  $g : J(\tilde{f}) \rightarrow \mathbb{R}$  and  $\psi : J(\tilde{f}) \rightarrow \mathbb{R}$  be as in the hypotheses of our theorem. Define the functions

$$\tilde{g} := g \circ H : \hat{\mathcal{D}}_{\mathcal{U}}^{\infty} \rightarrow \mathbb{R} \quad \text{and} \quad \tilde{\psi} := \psi \circ H : \hat{\mathcal{D}}_{\mathcal{U}}^{\infty} \rightarrow \mathbb{R}.$$

In order to apply Theorem 12.2, all what we need to do is to check the hypotheses of this theorem pertaining to the functions  $\tilde{\psi}$  and  $\tilde{g}$ . For  $\tilde{\psi}$  this is immediate: of course this function is measurable and bounded. We shall prove the following.

**Claim 1:** The function  $\hat{g} : \mathcal{D}_{\mathcal{U}}^{\infty} \rightarrow \mathbb{R}$  is Hölder continuous.

Indeed, consider two arbitrary points  $\omega, \tau \in \mathcal{D}_{\mathcal{U}}^{\infty}$  with  $\omega_1 = \tau_1$ . Put  $\gamma := \omega_1 = \tau_1$  and  $l := \|\gamma\|$ . Put also  $\beta := \omega \wedge \tau$  and  $n := \|\beta\|$ . In particular,  $\beta = \gamma \sigma(\beta)$  and  $\|\sigma(\beta)\| \geq n - 1$ . For every  $0 \leq k \leq l - 1$ , we have that

$$\begin{aligned} |\tilde{g}(T^k(\tau, 0)) - \tilde{g}(T^k(\omega, 0))| &= |\tilde{g}(\tau, k) - \tilde{g}(\omega, k)| = |g \circ H(\tau, k) - g \circ H(\omega, k)| \\ &= |g(\tilde{f}^k(\pi_{\mathcal{U}}(\tau))) - g(\tilde{f}^k(\pi_{\mathcal{U}}(\omega)))| \\ (12.5) \quad &= |g(\tilde{f}^k(\pi_{\mathcal{U}}(\tau))) - g(\tilde{f}^k(\pi_{\mathcal{U}}(\omega)))| \\ &\leq H_g \rho^{\alpha}(\tilde{f}^k(\pi_{\mathcal{U}}(\tau)), \tilde{f}^k(\pi_{\mathcal{U}}(\omega))), \end{aligned}$$

where  $H_g$  is the Hölder constant of  $g$ . Moreover,

$$\tilde{f}^k(\pi_{\mathcal{U}}(\omega)), \tilde{f}^k(\pi_{\mathcal{U}}(\tau)) \in \tilde{f}^k(\phi_{\gamma\sigma(\beta)}(W_{t(\gamma)})) = \phi_{\sigma^k(\gamma)\sigma(\beta)}(W_{t(\gamma)}),$$

where the last two  $\sigma$ s denote two different shift maps with obvious meanings. Therefore, because of Theorem 3.5 (Exponential Shrinking Property), we have that

$$\begin{aligned} \rho(\tilde{f}^k(\pi_{\mathcal{U}}(\tau)), \tilde{f}^k(\pi_{\mathcal{U}}(\omega))) &\preceq \text{diam}(\phi_{\sigma^k(\gamma)\sigma(\beta)}(W_{t(\beta)})) \preceq \exp(-\theta(\|\sigma^k(\gamma)\sigma(\beta)\|)) \\ &= \exp(-\theta((l-k) + \|\sigma(\beta)\|)) \\ &\leq e^{-1} \exp(-\theta(l-k))e^{-\theta n}. \end{aligned}$$

Hence,

$$|\tilde{g}(T^k(\tau, 0)) - \tilde{g}(T^k(\omega, 0))| \preceq \exp(-\alpha\theta(l-k))e^{-\alpha\theta n}.$$

Finally,

$$\begin{aligned} |\hat{g}(\omega) - \hat{g}(\tau)| &= \left| \sum_{k=0}^{l-1} (\tilde{g}(T^k(\tau, 0)) - \tilde{g}(T^k(\omega, 0))) \right| \leq \sum_{k=0}^{l-1} |\tilde{g}(T^k(\tau, 0)) - \tilde{g}(T^k(\omega, 0))| \\ &\preceq \sum_{k=0}^{l-1} \exp(-\theta(l-k))e^{-\theta n} \\ &\asymp e^{-\theta n}. \end{aligned}$$

So, the function  $\hat{g} : \mathcal{D}_{\mathcal{U}}^{\infty} \rightarrow \mathbb{R}$  is Hölder continuous, and the proof of Claim 1 is complete.

**Claim 2:** The function  $\hat{g} : \mathcal{D}_{\mathcal{U}}^{\infty} \rightarrow \mathbb{R}$  is square integrable with respect to the measure  $\tilde{\mu}_t$ .

Indeed, for every  $\omega \in \mathcal{D}_{\mathcal{U}}^{\infty}$  we have that  $|\hat{g}(\omega)| \leq \|\tilde{g}\|_{\infty} \cdot \|\omega_1\| \leq \|g\|_{\infty} \|\omega_1\|$ . Hence,

$$|\hat{g}(\omega)|^2 \leq \|g\|_{\infty}^2 \|\omega_1\|^2.$$

Therefore, invoking Proposition 10.10, we can estimate as follows.

$$\begin{aligned} \int_{\mathcal{D}_{\mathcal{U}}^{\infty}} |\hat{g}|^2 d\tilde{\mu}_t &= \sum_{n=1}^{\infty} \int_{\bigcup_{|e|=n} [e]} |\hat{g}|^2 d\tilde{\mu}_t \leq \|g\|_{\infty}^2 \sum_{n=1}^{\infty} n^2 \tilde{\mu}_t \left( \bigcup_{e \in \mathcal{D}_n} [e] \right) \\ &\preceq \|g\|_{\infty}^2 \sum_{n=1}^{\infty} n^2 e^{-\eta n} \\ &< +\infty, \end{aligned}$$

with some constant  $\eta > 0$  resulting from Proposition 8.4. The proof of Claim 2 is complete.  $\square$

**Claim 3:** The function  $\tilde{g} : \hat{\mathcal{D}}_{\mathcal{U}}^{\infty} \rightarrow \mathbb{R}$  is not cohomologous to a constant in  $L^2(\nu_t)$ .

Indeed, assume without loss of generality that  $\nu_t(g) = 0$ . By virtue of Lemma 12.4 the fact that  $g : J(\tilde{f}) \rightarrow \mathbb{R}$  is not a coboundary in  $L^2(\nu_t \circ H^{-1})$  equivalently means that the sequence  $(S_n(g))_{n=1}^{\infty}$  is not uniformly bounded in  $L^2(\nu_t \circ H^{-1})$ . But obviously  $\|S_n(\tilde{g})\|_{L^2(\nu_t)} = \|S_n(g)\|_{L^2(\nu_t \circ H^{-1})}$ . So, the sequence  $(S_n(\tilde{g}))_{n=0}^{\infty}$  is not uniformly bounded in  $L^2(\nu_t)$ . Thus, by Lemma 12.4 again,  $\tilde{g}$  is not a coboundary in  $L^2(\nu_t)$ .

Having these two claims, all items, (a), (b), and (c), now follow immediately from Theorem 12.2 and formula (12.4). The proof is finished.  $\square$

By making use Proposition 10.7, as a fairly easy consequence of Theorem 12.5, we get the following main result of this section.

**Theorem 12.6.** *Let  $G$  be a C–F balanced TNR rational semigroup of finite type generated by a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$ . If  $t \in \Delta_G^*$ , then for the dynamical system  $(\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f}), \mu_t)$  the following hold.*

- (a) *Fix  $\alpha \in (0, 1]$  and a bounded function  $g : J(\tilde{f}) \rightarrow \mathbb{R}$  which is Hölder continuous with the exponent  $\alpha$ . Then for every bounded measurable function  $\psi : J(\tilde{f}) \rightarrow \mathbb{R}$ , we have that*

$$\left| \int \psi \circ \tilde{f}^n \cdot g \, d\mu_t - \int g \, d\mu_t \int \psi \, d\mu_t \right| = O(\theta^n)$$

for some  $0 < \theta < 1$  depending on  $\alpha$ .

- (b) *The Central Limit Theorem holds for every Hölder continuous function  $g : J(\tilde{f}) \rightarrow \mathbb{R}$  that is not cohomologous to a constant in  $L^2(\mu_t)$ , i.e. for which there is no square integrable function  $\eta$  for which  $g = \text{const} + \eta \circ f - \eta$ . Precisely, we recall, this means that there exists  $\sigma > 0$  such that*

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g \circ \tilde{f}^j \longrightarrow \mathcal{N}(0, \sigma)$$

in distribution.

- (c) *The Law of Iterated Logarithm holds for every Hölder continuous function  $g : J(\tilde{f}) \rightarrow \mathbb{R}$  that is not cohomologous to a constant in  $L^2(\mu_t)$ . This, we recall, means that there exists a real positive constant  $A_g$  such that  $\mu_t$  almost everywhere*

$$\limsup_{n \rightarrow \infty} \frac{S_n g - n \int g \, d\mu_t}{\sqrt{n \log \log n}} = A_g.$$

*Proof.* Using in turn Proposition 10.7, formula (10.1) defining  $\hat{\mu}_t$ , the fact that  $H|_{\mathcal{D}_U^\infty} = \pi_U$ , and Theorem 12.2 (a), we get

$$(12.6) \quad (\mu_t)_{J_U} = \hat{\mu}_t = (\tilde{\mu}_t \circ \pi_U^{-1})_{J_U} = (\tilde{\mu}_t \circ H^{-1})_{J_U} = ((\nu_t)_{\mathcal{D}_U^\infty} \circ H^{-1})_{J_U} \prec\prec (\nu_t \circ H^{-1})_{J_U}.$$

But, by virtue of Theorem 12.2, the measure  $\nu_t$  is ergodic with respect to the map  $T : \hat{\mathcal{D}}_U^\infty \rightarrow \hat{\mathcal{D}}_U^\infty$ , defined by formula (12.3). So, the measure  $\nu_t \circ H^{-1}$  is ergodic with respect to the map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$ . As  $\mu_t$  is also ergodic with respect to this map, both measures  $(\mu_t)_{J_U}$  and  $(\nu_t \circ H^{-1})_{J_U}$  are ergodic with respect to the first return map of  $\tilde{f}$  from  $J_U$  to  $J_U$ . Therefore, by virtue of (12.6), they coincide. Hence,  $\mu_t$  and  $\nu_t \circ H^{-1}$  are not singular with respect to each other. Since both are ergodic, these must be thus equal, i.e.  $\mu_t = \nu_t \circ H^{-1}$ . Theorem 12.6 now directly follows from Theorem 12.5. The proof is complete.  $\square$

### Part 3. Geometry of \*Semi–hyperbolic and TNR Rational Semigroups

Part 3 of our paper is devoted to study finer fractal and geometrical properties of the fiber Julia sets  $J_\omega$  and the global Julia set  $J(G)$ .

#### 13. NICE OPEN SET CONDITION

The notion of the Open Set Condition was introduced by Hutchinson in [13] in the context of finite alphabet iterated function systems consisting of contracting similarities, was adopted in [18] to the setting of countable iterated function systems consisting of contracting conformal maps, and has commonly been used in the theory of iterated function systems since the work of Hutchinson. A version of the open set condition for expanding rational semigroups was introduced in [33]. In the paper [46] we defined the Open Set Condition in the context of (loosely meaning) non–recurrent rational semigroups. Unlike [33], in [46] we had to allow critical points of  $G$  to lie in the Julia set  $J(G)$ . This makes the whole situation more complex and demanding, and forced us in [46] to strengthen the Open Set Condition a little bit. We named such modified condition the Nice Open Set Condition.

We proved in [46] several of its crucial properties and used it heavily therein. In the present section we slightly weaken the Nice Open Set Condition of [46] to fit better to our present geometrical needs and we reprove, with essentially new proofs, its properties that we will need and use throughout the end of the manuscript. The Nice Open Set Condition defined below is however still stronger than the one of [46].

So, motivated by [46], we adopt the following definition.

**Definition 13.1.** *Let  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  be a multi–map and let  $G = \langle f_1, \dots, f_u \rangle$ . We say that  $G$  (or  $f$ ) satisfies the Open Set Condition if there exists a non–empty open subset  $U$  of  $\hat{\mathbb{C}}$  with the following two properties:*

(osc1)

$$f_1^{-1}(U) \cup f_2^{-1}(U) \cup \dots \cup f_u^{-1}(U) \subset U,$$

(osc2)

$$f_i^{-1}(U) \cap f_j^{-1}(U) = \emptyset$$

whenever  $i \neq j$ .

Moreover, we say that  $G$  (or  $f$ ) satisfies the Nice Open Set Condition if, in addition, the following condition is satisfied.

(osc3)  $\exists(\alpha \in (0, 1)) \forall(0 < r \leq 1) \forall(x \in \bar{U})$

$$l_2(U \cap B_s(x, r)) \geq \alpha l_2(B_s(x, r)),$$

where  $l_2$  denotes 2–dimensional Lebesgue measure on  $\hat{\mathbb{C}}$ .

**Remark 13.2.** *Condition (osc3) is not needed if our semigroup  $G$  is expanding (see [33] or note that our proofs would use only (osc1) and (osc2) under this assumption). Condition (osc3) is satisfied in the theory of conformal infinite iterated function systems (see [18], comp. [19]), where it follows from the open set condition and the cone condition. Moreover, condition (osc3) holds for example if the boundary of  $U$  is smooth enough; piecewise smooth with no exterior cusps suffices. Furthermore, (osc3) holds if  $U$  is a John domain (see [2]).*

As an immediate consequence of Definition 13.1, we get the following.

**Lemma 13.3.** *If a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  and  $G = \langle f_1, \dots, f_u \rangle$  satisfy the Open Set Condition, and if  $U \subset \hat{\mathbb{C}}$  is an open set which witnesses this condition, then*

$$J(G) \subset \bar{U}.$$

*Proof.* This is obvious if  $\bar{U} = \hat{\mathbb{C}}$ . So, suppose that

$$(13.1) \quad \bar{U} \neq \hat{\mathbb{C}}.$$

It follows immediately from Definition 13.1 that

$$G^*(\hat{\mathbb{C}} \setminus \bar{U}) \subset \hat{\mathbb{C}} \setminus \bar{U}.$$

Since, by (13.1),  $\hat{\mathbb{C}} \setminus \bar{U}$  contains at least three points, it follows from Montel's Theorem that  $G^*$  is normal on  $\hat{\mathbb{C}} \setminus \bar{U}$ . This means that  $\hat{\mathbb{C}} \setminus \bar{U} \subset F(G)$ , whence  $J(G) \subset \bar{U}$  and we are done.  $\square$

The following consequence of Definition 13.1 is immediate.

**Lemma 13.4.** *If a multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  and  $G = \langle f_1, \dots, f_u \rangle$  satisfy the Open Set Condition, and if  $\omega, \tau \in \Sigma_u^*$  are two incomparable words, then*

$$\phi_\omega^{-1}(U) \cap \phi_\tau^{-1}(U) = \emptyset.$$

Further auxiliary, but necessary for us, consequences of Definition 13.1, proved in [46], are these.

**Lemma 13.5.** *Let  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  be a multi-map and let  $G = \langle f_1, \dots, f_u \rangle$ . Assume that  $f$  and  $G$  satisfy the Nice Open Set Condition with some open set  $U \subset \hat{\mathbb{C}}$ . Then, there exist  $\beta_1 > 0$ ,  $\eta \in (0, 1/8]$ , and  $\kappa_1 > 0$  such that*

$$l_2(f_j^{-1}(U) \cap B_s(x, r)) \geq \kappa_1 r^2$$

for each  $j = 1, \dots, u$ , for each  $x \in f_j^{-1}(\bar{U}) \setminus \text{Crit}(f_j)$ , and for each

$$r \in (0, \min\{\beta_1, \eta \text{dist}(x, \text{Crit}(f_j))\}).$$

*Proof.* Let  $\delta > 0$  and  $\eta \in (0, 1/8]$  be so small that for every  $j \in \{1, 2, \dots, u\}$ , every  $c \in \text{Crit}(f_j)$ , and every  $z \in B(c, \delta) \setminus \{c\}$  the map

$$f_j|_{B(z, 8\eta|z-c|)} : B(z, 8\eta|z-c|) \longrightarrow \hat{\mathbb{C}}$$

is 1-to-1. Let

$$\gamma := \min\{\beta_1, \eta \cdot \text{dist}(x, \text{Crit}(f_j))\}.$$

Let  $\beta_1 \in (0, 1]$  be so small that for every  $j \in \{1, \dots, u\}$  and every  $z \in \hat{\mathbb{C}} \setminus B(\text{Crit}(G), \delta)$ , the map

$$f_j|_{B(z, 8\beta_1)} : B(z, 8\beta_1) \longrightarrow \hat{\mathbb{C}}$$

is 1-to-1 and

$$(13.2) \quad 4\beta_1 \max\{\|f'_j\|_\infty : j \in \{1, \dots, u\}\} \leq 1.$$

Suppose now that

$$(13.3) \quad x \in f^{-j}(\overline{U}) \setminus \text{Crit}(f_j).$$

Then, the inverse map

$$f_{j,x}^{-1} := (f_j|_{B(x,4\gamma)})^{-1} : f_j(B(x,4\gamma)) \longrightarrow B(x,4\gamma)$$

is well defined, holomorphic, and, because of the  $\frac{1}{4}$ -Koebe's Distortion Theorem,

$$f_j(B(x,8\gamma)) \supset B(f_j(x), 2|f'_j(x)|\gamma).$$

Hence, it follows from the standard Koebe's Distortion Theorem, applied this time to the map  $f_j^{-1}$ , from condition (osc3) of Definition 13.1, and from (13.2) that for every  $r \in (0, \gamma)$ , we have that

$$(13.4) \quad \begin{aligned} l_2(f_j^{-1}(U) \cap B(x,r)) &\geq l_2(f_{j,x}^{-1}(U \cap B(f_j(x), K^{-1}|f'_j(x)|r))) \\ &\geq K^{-2}|f'_{j,x}(x)|^{-2} l_2(U \cap B(f_j(x), K^{-1}|f'_j(x)|r)) \\ &\geq K^{-2}\alpha|f'_{j,x}(x)|^{-2} l_2(B(f_j(x), K^{-1}|f'_j(x)|r)) \\ &= \pi^2 K^{-2} \alpha r^2. \end{aligned}$$

So, the proof is completed by taking  $\kappa_1 := \pi^2 K^{-2} \alpha$ . □

**Lemma 13.6.** *Let  $f = (f_1, \dots, f_u) \in (\text{Rat})^u$  be a multi-map and let  $G = \langle f_1, \dots, f_u \rangle$ . Assume that  $f$  and  $G$  satisfy the Nice Open Set Condition with some open set  $U \subset \hat{\mathbb{C}}$ . Then there exist two constants  $\beta_2 > 0$  and  $\kappa_2 > 0$ , such that*

$$l_2(f_j^{-1}(U) \cap B_s(c,r)) \geq \kappa_2 r^2$$

for all  $j \in \{1, \dots, u\}$ , all points  $c \in \text{Crit}(f_j) \cap f_j^{-1}(\overline{U})$ , and all radii  $r \in (0, \beta_2)$ .

*Proof.* Denote the order of the critical point  $c$  of  $f_j$  by  $q$ . Let  $\beta_2 > 0$  be so small that

$$A^{-1}|z - c|^q \leq |f_j(z) - f_j(c)| \leq A|z - c|^q$$

and

$$A^{-1}|z - c|^{q-1} \leq |f'_j(z)| \leq A|z - c|^{q-1}$$

for some constant  $A \geq 1$  and all  $z \in B(c, \beta_2)$ . Then,

$$f_j(B(c,r)) \supset B(f_j(c), A^{-1}r^q)$$

for all  $r \in (0, \beta_2)$ . Since also

$$f_j(f_j^{-1}(U) \cap B(c,r)) = U \cap f_j(B(c,r)),$$

using (osc3), we thus get that

$$(13.5) \quad \begin{aligned} l_2(f_j(f_j^{-1}(U) \cap B(c,r))) &= l_2(U \cap f_j(B(c,r))) \geq l_2(U \cap B(f_j(c), A_c^{-1}r^q)) \\ &\geq \alpha l_2(B(f_j(c), A_c^{-1}r^q)) \\ &= \pi \alpha A_c^{-2} r^{2q}. \end{aligned}$$

But also,

$$\begin{aligned} l_2(f_j(f_j^{-1}(U) \cap B(c, r))) &\leq \int_{f_j^{-1}(U) \cap B(c, r)} |f_j'(z)|^2 dl_2(z) \\ &\leq A_{j,c}^2 \int_{f_j^{-1}(U) \cap B(c, r)} |z - c|^{2(q-1)} dl_2(z) \\ &\leq A_{j,c}^2 r^{2(q-1)} l_2(f_j^{-1}(U \cap B(c, r))). \end{aligned}$$

Combining this with (13.5), we get

$$l_2(f_j^{-1}(U) \cap B(c, r)) \geq \pi \alpha A_{j,c}^{-4} r^2.$$

Since there are only finitely many (precisely  $u$ ) maps  $f_j$  and since each map  $f_j$  has only finitely many critical points, we complete the proof by taking

$$\kappa_2 := \pi \alpha \min \{A_{j,c}^{-4} : j \in \{1, \dots, u\}, c \in \text{Crit}(f_j)\}.$$

□

Combining Lemma 13.5 and Lemma 13.6, we obtain the following lemma.

**Lemma 13.7.** *Let  $G = \langle f_1, \dots, f_u \rangle$  be a rational semigroup satisfying the Nice Open Set Condition witnessed by an open set  $U$ . Then, for every  $\beta > 0$  there exists  $\kappa_\beta > 0$  such that for each  $j = 1, \dots, u$ , for each  $x \in f_j^{-1}(\bar{U})$ , and for each radius  $r \in (0, \beta]$ , we have that*

$$l_2(f_j^{-1}(U) \cap B(x, r)) \geq \kappa_\beta r^2.$$

*Proof.* It suffices to deal with  $r \in (0, \min\{\beta_1, \beta_2\})$ . Fix  $j \in \{1, \dots, u\}$ . Because of Lemma 13.5, we need only consider the case when

$$r \in (\min\{\beta_1, \eta \cdot \text{dist}(x, \text{Crit}(f_j))\}, \min\{\beta_1, \beta_2\}),$$

with  $\eta \in (0, 1/8]$  coming from Lemma 13.5. Consider two cases. Assume first that

$$r \geq 2\text{dist}(x, \text{Crit}(f_j)).$$

Then there exists a point  $c \in \text{Crit}(f_j)$  such that  $|x - c| \leq r/2$ . Hence,

$$B(x, r) \supset B(c, r/2).$$

It therefore follows from Lemma 13.6 that

$$l_2(B(x, r) \cap f_j^{-1}(U)) \geq l_2(B(c, r/2) \cap f_j^{-1}(U)) \geq r\kappa_2/4.$$

Now assume that

$$r < 2\text{dist}(x, \text{Crit}(f_j)).$$

Then, by Lemma 13.5, we get that

$$\begin{aligned} l_2(B(x, r) \cap f_j^{-1}(U)) &\geq l_2(B(x, 1/2 \min\{\beta_1, \eta \cdot \text{dist}(x, \text{Crit}(f_j))\}) \cap f_j^{-1}(U)) \\ &\geq \kappa_1 (1/2 \min\{\beta_1, \eta \cdot \text{dist}(x, \text{Crit}(f_j))\})^2 \\ &\geq \frac{\kappa_1}{4} \left(\frac{\eta r}{2}\right)^2 = \frac{\kappa_1 \eta^2}{16} r^2. \end{aligned}$$

We are done. □

For every family  $\mathcal{F} \subset \Sigma_u^*$  let

$$\hat{\mathcal{F}} = \{\hat{\tau} : \tau \in \mathcal{F}\} \quad \text{and} \quad \mathcal{F}_* = \{\tau_* : \tau \in \mathcal{F}\}.$$

**Definition 13.8.** Let  $G = \langle f_1, \dots, f_u \rangle$  be a rational semigroup satisfying the Nie Open Set Condition. Suppose that  $J(G) \subset \mathbb{C}$ . Fix a number  $M > 0$ , a number  $a > 0$ , and  $V$ , an open subset of  $\Sigma_u$ . Suppose  $x \in J(G)$  and  $r \in (0, 1]$ . A family  $\mathcal{F} \subset \Sigma_u^*$  is called  $(M, a, V)$ -essential for the pair  $(x, r)$  provided that the following conditions are satisfied.

(ess0) For every  $\tau \in \mathcal{F}$ ,

$$\text{dist}(f_{\hat{\tau}}(x), f_{\tau_*}^{-1}(U)) \leq \frac{1}{2}R_{\tau}.$$

(ess1) For every  $\tau \in \mathcal{F}$  there exist a number  $R_{\tau}$  with  $0 < R_{\tau} < a$  and  $f_{\hat{\tau},x}^{-1} : B(f_{\hat{\tau}}(x), 2R_{\tau}) \rightarrow \mathbb{C}$ , an analytic branch of  $f_{\hat{\tau}}^{-1}$  sending  $f_{\hat{\tau}}(x)$  to  $x$ , such that

$$M^{-1}R_{\tau} \leq |f'_{\hat{\tau}}(x)|r \leq \frac{1}{4}R_{\tau}.$$

(ess2) The family  $\mathcal{F}$  consists of mutually incomparable words.

(ess3)  $\bigcup_{\tau \in \mathcal{F}} [\tau] = V$ .

If  $V = \Sigma_u$ , the family  $\mathcal{F}$  is simply called  $(M, a)$ -essential for the pair  $(x, r)$ .

We shall prove the following.

**Proposition 13.9.** Let  $G = \langle f_1, \dots, f_u \rangle$  be a rational semigroup satisfying the Nice Open Set Condition with  $U$ . Suppose that  $J(G) \subset \mathbb{C}$ . Then, for every number  $M > 0$  and for every  $a > 0$  there exists an integer  $\#_{(M,a)} \geq 1$  such that if  $V$  is an open subset of  $\Sigma_u$ ,  $x \in J(G)$ ,  $r \in (0, 1]$ , and  $\mathcal{F} \subset \Sigma_u^*$  is an  $(M, a, V)$ -essential family for  $(x, r)$ , then we have the following:

(a)

$$B(x, r) \subset f_{\hat{\tau},x}^{-1}(B(f_{\hat{\tau}}(x), R_{\tau})) \subset \bigcup_{\gamma \in \mathcal{F}} f_{\hat{\gamma},x}^{-1}(B(f_{\hat{\gamma}}(x), R_{\gamma})) \subset B(x, KM r)$$

for all  $\tau \in \mathcal{F}$ ,

(b)

$$J(\tilde{f}) \cap (V \times B(x, r)) \subset \bigcup_{\tau \in \mathcal{F}} \tilde{f}_{\hat{\tau},x}^{-|\hat{\tau}|} (p_2^{-1}(B(f_{\hat{\tau}}(x), R_{\tau}))) = \bigcup_{\tau \in \mathcal{F}} [\tau] \times f_{\hat{\tau},x}^{-1}(B(f_{\hat{\tau}}(x), R_{\tau})),$$

(c)  $\#\mathcal{F} \leq \#_{(M,a)}$ .

*Proof.* Item (a) follows immediately from Theorem 2.5 ( $\frac{1}{4}$ -Koebe's Distortion Theorem), and Theorem 2.6. The equality part in item (b) is obvious. In order to prove the inclusion take  $(\omega, z) \in J(\tilde{f}) \cap (V \times B(x, r))$ . By item (ess3) of Definition 13.8 there exists  $\tau \in \mathcal{F}$  such that  $\omega \in [\tau]$ . But then, by the first in item (a),  $(\omega, z) \in [\tau] \times f_{\hat{\tau},x}^{-1}(B(f_{\hat{\tau}}(x), R_{\tau}))$ , and item (b) is entirely proved. Let us deal with item (c). By item (osc2) of Definition 13.1,

$$\{f_{\hat{\tau},x}^{-1}((f_{\tau_*}|_{B(f_{\hat{\tau}}(x), R_{\tau})})^{-1}(U))\}_{\tau \in \mathcal{F}}$$

is a family of mutually disjoint sets. Hence, using also (a), we get

$$(13.6) \quad \sum_{\tau \in \mathcal{F}} l_2(f_{\hat{\tau},x}^{-1}((f_{\tau_*}|_{B(f_{\hat{\tau}}(x), R_{\tau})})^{-1}(U))) \leq l_2(B(x, KM r)) = C\pi(KM)^2 r^2,$$



where  $C > 0$  is a constant independent of  $r, M$ , and  $a$ . Let  $L_a := \xi \min\{(T/a)^2, 1\}$ , where  $\xi$  and  $T$  come from Lemma 13.7. By Lemma 13.7, we obtain that for each  $j = 1, \dots, u$ , for each  $y \in \overline{f_j^{-1}(U)}$ , and for each  $0 \leq b \leq a$ ,

$$(13.7) \quad l_2(B(y, b) \cap f_j^{-1}(U)) \geq \kappa_a b^2.$$

By (ess0) there exists a point  $z \in f_{\tau_*}^{-1}(\overline{U})$  such that  $|f_{\hat{\tau}}(x) - z| \leq R_{\tau}/2$ . Then

$$B(f_{\hat{\tau}}(x), R_{\tau}) \supset B(z, R_{\tau}/2).$$

It follows from this, Theorem 2.6, (13.7), and (ess1) that for all  $\tau \in \mathcal{F}$ , we have

$$\begin{aligned} l_2(f_{\hat{\tau}, x}^{-1}((f_{\tau_*}|_{B(f_{\hat{\tau}}(x), R_{\tau})})^{-1}(U))) &\geq K^{-2}|f'_{\hat{\tau}}(x)|^{-2}l_2((f_{\tau_*}|_{B(f_{\hat{\tau}}(x), R_{\tau})})^{-1}(U)) \\ &= K^{-2}|f'_{\tau}(x)|^{-2}l_2(B(f_{\hat{\tau}}(x), R_{\tau}) \cap f_{\tau_*}^{-1}(U)) \\ &\geq K^{-2}|f'_{\tau}(x)|^{-2}l_2(B(z, R_{\tau}/2) \cap f_{\tau_*}^{-1}(U)) \\ &\geq (2K)^{-2}|f_{\hat{\tau}}(x)|^{-2}\kappa_a R_{\tau}^2 \\ &\geq 4K^{-2}1\kappa_a r^2. \end{aligned}$$

Combining this with (13.6) we get that

$$\#\mathcal{F} \leq (4\kappa_a)^{-1}\pi CK^4 M^2.$$

We are done. □

**Remark 13.10.** *Because of Lemma 13.3, if*

$$\text{dist}(f_{\hat{\tau}}(x), f_{\tau_*}^{-1}(J(G))) \leq \frac{1}{2}R_{\tau},$$

*then condition (ess0) holds.*

The following lemma sheds additional light on the nature of the Nice Open Set Condition.

**Lemma 13.11.** *If  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  is a multi-map such that the rational semigroup  $G = \langle f_1, \dots, f_u \rangle$  is TNR and satisfies the Nice Open Set Condition, then the semigroup  $G$  is of finite type, i.e. the set  $\text{Crit}_*(\tilde{f})$  is also finite.*

*Proof.* Fix  $c \in \text{Crit}(f)$  and  $i \in \{1, \dots, u\}$  such that  $f_i(c) \in J(G)$  and  $f'_i(c) = 0$ . Since  $G$  is a TNR rational semigroup, there exists  $R \in (0, 1)$  such that

$$B(f_{\omega}(f_i(c)), 2R) \cap \text{PCV}(G) = \emptyset$$

for all  $\omega \in \Sigma_u^*$  such that  $f_{\omega}(f_i(c)) \in J(G)$ . In particular, there exists a unique holomorphic branch

$$f_{\omega, i}^{-1} : B(f_{\omega}(f_i(c)), 2R) \longrightarrow \mathbb{C}$$

of  $f_{\omega}^{-1}$  mapping  $f_{\omega}(f_i(c))$  to  $f_i(c)$ . Fix now an integer  $n \geq 1$  and consider the set

$$\Sigma_n(c, i) = \{\omega \in \{1, 2, \dots, u\}^n : f_{\omega}(f_i(c)) \in J(G)\}.$$

Let

$$\gamma_n := \min \{1, \min\{K^{-1}R|f'_{\omega}(f_i(c))|^{-1} : \omega \in \Sigma_n(c, i)\}\}.$$

Then for every  $\omega \in \Sigma_n(c, i)$ , we have that

$$Kr_n|f'_{\omega}(f_i(c))| \leq R.$$

So, using Koebe's Distortion Theorem, we get that

$$\begin{aligned} f_\omega^{-1} (U \cap B(f_\omega(f_i(c)), K^{-1}r_n|f'_\omega(f_i(c))|)) \\ \subset f_\omega^{-1} (B(f_\omega(f_i(c)), K^{-1}r_n|f'_\omega(f_i(c))|)) \subset B(f_i(c), \gamma_n), \end{aligned}$$

and, invoking also condition (osc3) of Definition 13.1, Lemma 13.3, and the fact that  $f_\omega(f_i(c)) \in J(G)$ , we also get that

$$\begin{aligned} l_2 (f_\omega^{-1} (U \cap B(f_\omega(f_i(c)), K^{-1}r_n|f'_\omega(f_i(c))|))) \\ \geq K^{-2}|f'_\omega(f_i(c))|^{-2}l_2 (U \cap B(f_\omega(f_i(c)), K^{-1}r_n|f'_\omega(f_i(c))|)) \\ \geq K^{-2}\alpha|f'_\omega(f_i(c))|^{-2}l_2 (B(f_\omega(f_i(c)), K^{-1}r_n|f'_\omega(f_i(c))|)) \\ = \pi K^{-4}\alpha r_n^2 = \alpha K^{-4}l_2(B(f_i(c), \gamma_n)). \end{aligned}$$

Since by condition (osc2) of Definition 13.1, all of the sets

$$f_\omega^{-1} (U \cap B(f_\omega(f_i(c)), K^{-1}r_n|f'_\omega(f_i(c))|)), \quad \omega \in \Sigma_n(c, i)$$

are mutually disjoint, we conclude that

$$\#\Sigma_n(c, i) \leq K^4\alpha^{-1}.$$

Hence, the set

$$\text{Crit}_*(f) = \bigcup_{(c,i) \in \text{CP}(t)} \left\{ (\omega, c) \in \text{Crit}_*(\tilde{f}) : \omega_1 = i \right\}$$

is finite as it is a finite union of finite sets. The proof is complete.  $\square$

#### 14. MULTIFRACTAL ANALYSIS OF INVARIANT MEASURES $\mu_t$

Throughout this section, we assume that  $G = (f_1, \dots, f_u) \in \text{Rat}^u$  is a C-F balanced TNR semigroup satisfying the Nice Open Set Condition. In particular, because of Lemma 13.11, the semigroup  $G$  is of finite type. We also keep  $t \in \Delta_G^*$  fixed. Our ultimate goal in this section is to provide a full account of multifractal analysis for the Gibbs/equilibrium measure  $\mu_t$  (and  $m_t$ ).

We first recall some notions and results from geometric measure theory which will be used in this and the following sections. Let  $\mu$  be a Borel probability measure on a metric space  $X$ . The Hausdorff dimension of  $\mu$ , denoted  $\text{HD}(\mu)$ , is defined to be

$$\text{HD}(\mu) := \inf\{\text{HD}(Y) : \mu(Y) = 1\} = \sup\{\text{HD}(Y) : \mu(X \setminus Y) = 0\}.$$

Furthermore, the lower and upper pointwise dimensions of  $\mu$  at a point  $x \in X$  are respectively defined by

$$\underline{d}_\mu(x) = \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \quad \text{and} \quad \bar{d}_\mu(x) = \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}.$$

The proof of the following theorem can be found for example in [26], [14], and [23].

**Theorem 14.1.** *If  $\mu$  is a Borel probability measure on a metric space  $X$ , then*

$$\text{HD}(\mu) = \text{ess sup}(\underline{d}_\mu).$$

*In particular, if there exists some  $\theta \geq 0$  such that  $\underline{d}_\mu(x) = \theta$  for  $\mu$ -a.e.  $x \in X$ , then  $\text{HD}(\mu) = \theta$ .*

We first prove the following preparatory result which is also interesting on its own.

**Theorem 14.2.** *If  $G$  is a C–F balanced TNR semigroup satisfying the Nice Open Set Condition and  $t \in \Delta_G^*$ , then*

$$\text{HD}(\mu_t \circ p_2^{-1}) = \frac{h_{\mu_t}(\tilde{f})}{\chi_{\mu_t}} = t + \frac{P(t)}{\chi_{\mu_t}}.$$

*Proof.* By Birkhoff's Ergodic Theorem there exists a measurable set  $\Gamma \subset J(\tilde{f})$  such that  $\mu_t(\Gamma) = 1$  and

$$(14.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log |(\tilde{f}^n)'(w, z)| = \chi_{\mu_t} := \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu_t.$$

Fix  $(\omega, z) \in \Gamma \setminus \text{Sing}(\tilde{f})$ . Put  $\eta := \eta(\omega, z) > 0$  coming from Proposition 3.9. Let  $(n_j)_{j=1}^\infty$  be the increasing sequence of integers also coming from Proposition 3.9. Let

$$\tilde{f}_{\omega|n_j, z}^{-n_j} : p_2^{-1} \left( B \left( f_{\omega|n_j}(z), \eta \right) \right) \longrightarrow \Sigma_u \times \hat{\mathbb{C}}$$

be the holomorphic inverse branch of  $\tilde{f}^{n_j}$  resulting also from Proposition 3.9 which sends  $\tilde{f}^{n_j}(\omega, z)$  to  $(\omega, z)$ . By Koebe's Distortion Theorem we have

$$B \left( z, \frac{1}{2} K \left| f'_{\omega|n_j}(z) \right|^{-1} \eta \right) \supset f_{\omega|n_j, z}^{-1} \left( B \left( f_{\omega|n_j}(z), \eta/2 \right) \right).$$

So, by conformality of  $m_t$  and Koebe's Distortion Theorem again, putting

$$r_j = \frac{1}{2} K \left| f'_{\omega|n_j}(z) \right|^{-1} \eta,$$

we get that

$$\begin{aligned} m_t \circ p_2^{-1}(B(z, r_j)) &\geq m_t \left( \tilde{f}_{\omega|n_j, z}^{-n_j} \left( p_2^{-1} \left( B \left( f_{\omega|n_j}(z), \eta/2 \right) \right) \right) \right) \\ &= \int_{p_2^{-1}(B(f_{\omega|n_j}(z), \eta/2))} e^{-P(t)n_j} \left| \left( \tilde{f}_{\omega|n_j}^{-n_j} \right)' \right|^t dm_t \\ &\geq K^{-t} e^{-P(t)n_j} \left| \left( \tilde{f}_{\omega|n_j}^{n_j} \right)'(\omega, z) \right|^{-t} m_t \circ p_2^{-1} \left( B \left( f_{\omega|n_j}(z), \eta/2 \right) \right) \\ &\geq K^{-t} M_t(\eta/2) e^{-P(t)n_j} \left| \left( \tilde{f}_{\omega|n_j}^{n_j} \right)'(\omega, z) \right|^{-t} \\ &= 2(K^2 q)^{-t} (\eta/2) M_t(\eta/2) e^{-P(t)n_j} r_j^t, \end{aligned}$$

where

$$M_t(r) := \inf \{ m_t \circ p_2^{-1} (B(x, r)) : x \in J(G) \}$$

is positive since  $\text{supp}(m_t \circ p_2^{-1}) = J(G)$ . Therefore, using also (14.1), we get

$$\begin{aligned}
\liminf_{r \rightarrow 0} \frac{\log m_t \circ p_2^{-1}(B(z, r))}{\log r} &\leq \liminf_{j \rightarrow \infty} \frac{\log m_t \circ p_2^{-1}(B(z, r_j))}{\log r_j} \\
&\leq \liminf_{j \rightarrow \infty} \frac{\log(2(K^2\eta)^{-t} M_t(\eta/2)) - P(t)n_j + t \log r_j}{\log r_j} \\
&= t - P(t) \lim_{j \rightarrow \infty} \frac{n_j}{\log r_j} \\
(14.2) \qquad \qquad \qquad &= t + \frac{P(t)}{\chi_{\mu_t}}.
\end{aligned}$$

Hence, invoking Theorem 14.1, we see that

$$(14.3) \qquad \text{HD}(m_t \circ p_2^{-1}) \leq t + \frac{P(t)}{\chi_{\mu_t}}.$$

For the opposite inequality fix a point  $\xi \in J(\tilde{f}) \setminus \text{PCV}(\tilde{f})$ . Fix any number

$$(14.4) \qquad H \in \left(0, \frac{\text{dist}(p_2(\xi), \text{PCV}(G))}{2(2K+1)H}\right),$$

which is positive because of Lemma 2.4. For every  $(\omega, z) \in J(\tilde{f})$  let  $(n_k(\omega, z))_{k=1}^{\infty}$  be the increasing sequence of all visits of  $(\omega, z)$  to  $B(\xi, R)$ , i.e.

$$(14.5) \qquad \tilde{f}^{n_k(\omega, z)}(\omega, z) \in B(\xi, H).$$

Fix an arbitrary measurable set  $Z \subset J(\tilde{f})$  with  $\mu_t(Z) = 1$ . By Birkhoff's Ergodic Theorem there exists a measurable set  $Y \subset Z$  such that  $\mu_t(Y) = 1$ ,

$$(14.6) \qquad \lim_{k \rightarrow \infty} \frac{n_{k+1}(\omega, z)}{n_k(\omega, z)} = 1, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log |(\tilde{f}^n)'(\omega, z)| = \chi_{\mu_t}$$

for all  $(\omega, z) \in Y$ . Fix  $\varepsilon > 0$ . By Egorov's Theorem there exists an integer  $N_\varepsilon \geq 1$  and a measurable set  $Y_\varepsilon \subset Y$  such that  $m_t(Y_\varepsilon) \geq 1 - \varepsilon$ ,

$$(14.7) \qquad \frac{n_{k+1}(\tau, y)}{n_k(\tau, y)} \leq 1 + \varepsilon, \quad \text{and} \quad \chi_{\mu_t} - \varepsilon \leq \frac{1}{n} \log |(\tilde{f}^n)'(\tau, y)| \leq \chi_{\mu_t} + \varepsilon$$

for all  $(\tau, y) \in Y_\varepsilon$  and all  $n_k(\tau, y), n \geq N_\varepsilon$ . Let

$$m_{t, \varepsilon} := m_t|_{Y_\varepsilon}.$$

Fix  $y \in p_2^{-1}(Y_\varepsilon)$  and  $r > 0$ . Then, for every  $\omega \in p_1(Y_\varepsilon \cap p_2^{-1}(B(y, r)))$  fix an element  $x_\omega \in B(y, r)$  such that

$$(14.8) \qquad (\omega, x_\omega) \in Y_\varepsilon \cap p_2^{-1}(B(y, r)).$$

Since for every  $j \geq 0$

$$|f'_{\omega|n_j}(x_\omega)| \leq \|f'\|_\infty^{n_j},$$

if

$$r \leq \frac{1}{8} H \|f'\|_\infty^{-N_\varepsilon},$$

then there exists a largest integer  $k := k(\omega; r) \geq 1$  such that

$$(14.9) \quad \frac{1}{4}H |f'_{\omega|_{n_k}}(x_\omega)|^{-1} \geq 2r,$$

and

$$(14.10) \quad n_k \geq N_\varepsilon,$$

where we have abbreviated

$$n_k := n_{k(\omega; r)}(\omega, x_\omega).$$

Immediately we have

$$(14.11) \quad \frac{1}{4}H |f'_{\omega|_{n_{k+1}}}(x_\omega)|^{-1} < 2r.$$

Let

$$\hat{\mathcal{F}}(y; r) := \{ \omega|_{n_{k+1}} : \omega \in p_1(Y_\varepsilon \cap p_2^{-1}(B(y, r))) \},$$

and for every  $\omega \in p_1(Y_\varepsilon \cap p_2^{-1}(B(y, r)))$  let

$$(14.12) \quad R_{\omega|_{n_{k+1}}} := 16Kr \left| f'_{\omega|_{n_k}}(x_\omega) \right| \leq 2KH,$$

with the convention that if  $\omega|_{n_{k(\omega, r)+1}} = \tau|_{n_{k(\tau, r)+1}}$ , then we arbitrarily choose either  $x_\omega$  or  $x_\tau$  with the inequality part following from (14.9). Let  $\mathcal{F}(y, r)$  be the maximal subfamily of  $\hat{\mathcal{F}}(y; r)$  consisting of mutually incomparable words. We aim to check that this family satisfies the conditions of Definition 13.8 with appropriate numbers  $M$ ,  $a$ , and an open set  $V$ . Indeed, it satisfies (ess2) by its very definition. It also satisfies (ess3) if we set

$$(14.13) \quad V := \bigcup_{\varrho \in \mathcal{F}(y, r)} [\varrho] \supset p_1(Y_\varepsilon \cap p_2^{-1}(B(y, r))).$$

Now we aim to prove conditions (ess0) and (ess1). It follows from (14.4), (14.5), the definition of  $x_\omega$ , and (14.12) that

$$\begin{aligned} \text{dist}(f_{\omega|_{n_k}}(x_\omega), \text{PCV}(G)) &\geq \text{dist}(p_2(\xi), \text{PCV}(G)) - |f_{\omega|_{n_k}}(x_\omega) - p_2(\xi)| \\ &\geq 4H - H \\ &= 3H \geq 16r \left| f'_{\omega|_{n_k}}(x_\omega) \right|. \end{aligned}$$

Therefore, the unique holomorphic branch

$$f_{\omega|_{n_k}, x_\omega}^{-1} : B\left(f_{\omega|_{n_k}}(x_\omega), 16r \left| f'_{\omega|_{n_k}}(x_\omega) \right| \right) \longrightarrow \mathbb{C}$$

of  $f_{\omega|_{n_k}}^{-1}$  sending  $f_{\omega|_{n_k}}(x_\omega)$  to  $x_\omega$  is well defined. It therefore follows from Koebe's  $\frac{1}{4}$ -Distortion Theorem and (14.8) that

$$f_{\omega|_{n_k}, x_\omega}^{-1} : B\left(f_{\omega|_{n_k}}(x_\omega), 4r \left| f'_{\omega|_{n_k}}(x_\omega) \right| \right) \supset B(x_\omega, r) \ni y.$$

Hence,

$$(14.14) \quad f_{\omega|_{n_k}}(y) \in B\left(f_{\omega|_{n_k}}(x_\omega), 4r \left| f'_{\omega|_{n_k}}(x_\omega) \right| \right) \subset B\left(f_{\omega|_{n_k}}(x_\omega), \frac{1}{4K} R_{\omega|_{n_{k+1}}}\right).$$

Since also

$$f_{\omega|n_k}(x_\omega) \in f_{\omega|n_{k+1}}^{-1}(f_{\omega|n_{k+1}}(x_\omega)) \subset f_{\omega|n_{k+1}}^{-1}(J(G)),$$

using Remark 13.10, we conclude that condition (ess0) holds.

Now, passing to proving condition (ess1), note that because of (14.14), (14.5), (14.12), and the definition of  $x_\omega$ , for every  $z \in B(f_{\omega|n_k}(y), 2R_{\omega|n_{k+1}})$ , we get that

$$\begin{aligned} |z - p_2(\xi)| &< 2R_{\omega|n_{k+1}} + |f_{\omega|n_k}(y) - f_{\omega|n_k}(x_\omega)| + |f_{\omega|n_k}(x_\omega) - p_2(\xi)| + \frac{1}{4K}R_{\omega|n_{k+1}} \\ &\leq 2R_{\omega|n_{k+1}} + \frac{1}{4K}R_{\omega|n_{k+1}} + H \\ &\leq 4KH + H + H \\ &= 2(2K + 1)H. \end{aligned}$$

Hence, using also (14.4), we get that

$$\begin{aligned} \text{dist}(z, \text{PCV}(G)) &\geq \text{dist}(p_2(\xi), \text{PCV}(G)) - |z - p_2(\xi)| \\ &\geq \text{dist}(p_2(\xi), \text{PCV}(G)) - 2(2K + 1)H \\ &> 0. \end{aligned}$$

Therefore, the unique holomorphic branch

$$f_{\omega|n_k, y}^{-1} : B(f_{\omega|n_k}(y), 2R_{\omega|n_{k+1}}) \longrightarrow \mathbb{C}$$

of  $f_{\omega|n_k}^{-1}$  sending  $f_{\omega|n_k}(y)$  to  $y$  is well defined. Also, by invoking (14.12), we obtain

$$(16K)^{-1}R_{\omega|n_{k+1}} \leq |f'_{\omega|n_k}(x_\omega)|4 \leq \frac{1}{4}R_{\omega|n_{k+1}} \leq \frac{1}{2}KH \leq KH,$$

which yields condition (ess1). In conclusion, the conditions (ess0)–(ess3) of Definition 13.8 are satisfied with  $M = 16K$  and  $a = KH$ . For every  $\varrho \in \mathcal{F}(y, r)$  fix an element  $\tilde{\varrho} \in \Sigma_u$  such that  $x_\varrho = x_{\tilde{\varrho}}$  and

$$\tilde{\varrho}|_{n_k(\tilde{\varrho}, x_\varrho, r)(\tilde{\varrho}, x_\varrho)+1} = \varrho.$$

We claim that

$$(14.15) \quad \bigcup_{\varrho \in \mathcal{F}(y, r)} [\varrho] \times f_{\tilde{\varrho}, x_\varrho}^{-1}(B(f_{\tilde{\varrho}}(x_\varrho), R_\varrho)) \supset Y_\varepsilon \cap p_2^{-1}(B(y, r)).$$

Indeed, fix  $(\beta, x) \in Y_\varepsilon \cap p_2^{-1}(B(y, r))$ . It then follows from (14.13) that there exists  $\varrho \in \mathcal{F}(y, r)$  such that

$$(14.16) \quad \beta \in [\varrho].$$

Now,  $\varrho = \tilde{\varrho}|_{n_k(\tilde{\varrho}, x_\varrho, r)(\tilde{\varrho}, x_\varrho)+1}$ . It follows from the Koebe's  $\frac{1}{4}$ -Distortion Theorem that

$$\begin{aligned} f_{\tilde{\varrho}, x_\varrho}^{-1}(B(f_{\tilde{\varrho}}(x_\varrho), R_\varrho)) &= f_{\tilde{\varrho}|_{n_k(\tilde{\varrho}, x_\varrho, r)(\tilde{\varrho}, x_\varrho), x_\varrho}}^{-1} \left( B \left( f_{\tilde{\varrho}|_{n_k(\tilde{\varrho}, x_\varrho, r)(\tilde{\varrho}, x_\varrho)}}(x_\varrho), 16Kr \left| f'_{\tilde{\varrho}|_{\varrho}(\tilde{\varrho}, x_\varrho, r)}(x_\varrho) \right|^{-1} \right) \right) \\ &\supset B(x_\varrho, 4r) \\ &\supset B(x, r) \ni x. \end{aligned}$$

Along with (14.16), this implies that

$$(\beta, x) \in \bigcup_{\varrho \in \mathcal{F}(y, r)} [\varrho] \times f_{\tilde{\varrho}, x_{\varrho}}^{-1}(B(f_{\tilde{\varrho}}(x_{\varrho}), R_{\varrho})),$$

and formula (14.15) is proved.

It now follows from (14.15), conformality of the measure  $m_t$ , Koebe's Distortion Theorem again, and (14.11), that

(14.17)

$$\begin{aligned} m_{t, \varepsilon}(B(y, r)) &= m_t(Y_{\varepsilon} \cap p_2^{-1}(B(y, r))) \leq \\ &\leq m_t \left( \bigcup_{\varrho \in \mathcal{F}(y, r)} [\varrho] \times f_{\tilde{\varrho}, x_{\varrho}}^{-1}(B(f_{\tilde{\varrho}}(x_{\varrho}), R_{\varrho})) \right) \\ &= \sum_{\varrho \in \mathcal{F}(y, r)} m_t \left( [\varrho] \times f_{\tilde{\varrho}, x_{\varrho}}^{-1}(B(f_{\tilde{\varrho}}(x_{\varrho}), R_{\varrho})) \right) \\ &= \sum_{\varrho \in \mathcal{F}(y, r)} m_t(\Sigma_u \times (B(f_{\tilde{\varrho}}(x_{\varrho}), R_{\varrho}))) \int e^{-P(t)(|\varrho|-1)} |(f_{\tilde{\varrho}, x_{\varrho}}^{-1})'|^t dm_t \\ &\leq K^t \sum_{\varrho \in \mathcal{F}(y, r)} m_t(\Sigma_u \times (B(f_{\tilde{\varrho}}(x_{\varrho}), R_{\varrho}))) |f'_{\tilde{\varrho}|_{n_k(\tilde{\varrho}, x_{\varrho}, r)}(\tilde{\varrho}, x_{\varrho})}(x_{\varrho})|^{-t} \exp(-P(t)n_k(\tilde{\varrho}, x_{\varrho}, r)(\tilde{\varrho}, x_{\varrho})) \\ &\leq (8KH^{-1})^t r^t \sum_{\varrho \in \mathcal{F}(y, r)} \exp(-P(t)n_k(\tilde{\varrho}, x_{\varrho}, r)(\tilde{\varrho}, x_{\varrho})). \end{aligned}$$

Now, denoting  $\tilde{n}_k = n_k(\tilde{\varrho}, x_{\varrho}, r)(\tilde{\varrho}, x_{\varrho})$ , it follows from (14.7)–(14.11) that

$$(14.18) \quad \tilde{n}_k \leq \frac{\log |f'_{\tilde{\varrho}|_{\tilde{n}_k}}(x_{\varrho})|}{\chi_{\mu_t} - \varepsilon} \leq \frac{\log(R/8) - \log r}{\chi_{\mu_t} - \varepsilon}$$

and

$$(14.19) \quad \tilde{n}_k \geq (1 + \varepsilon)^{-1} \tilde{n}_{k+1} \geq (1 + \varepsilon)^{-1} \frac{\log |f'_{\tilde{\varrho}|_{\tilde{n}_{k+1}}}(x_{\varrho})|}{\chi_{\mu_t} + \varepsilon} \geq (1 + \varepsilon)^{-1} \frac{\log(R/8) - \log r}{\chi_{\mu_t} + \varepsilon}.$$

Making use of Proposition 13.9 (c) and inserting (14.18) or (14.19) into (14.17), respectively if  $P(t) \leq 0$  or if  $P(t) \geq 0$ , we get that

$$(14.20) \quad \liminf_{r \rightarrow 0} \frac{\log(m_{t, \varepsilon}(B(y, r)))}{\log r} \geq \begin{cases} t + \frac{P(t)}{\chi_{\mu_t} - \varepsilon} & \text{if } P(t) \leq 0 \\ t + \frac{P(t)}{\chi_{\mu_t} + \varepsilon} \frac{1}{1 + \varepsilon} & \text{if } P(t) \geq 0 \end{cases}.$$

Hence, invoking Theorem 14.1, we get that

$$(14.21) \quad \text{HD}(Z) \geq \text{HD}(Y) \geq \text{HD}(m_{t, \varepsilon}) \geq \begin{cases} t + \frac{P(t)}{\chi_{\mu_t} - \varepsilon} & \text{if } P(t) \leq 0 \\ t + \frac{P(t)}{\chi_{\mu_t} + \varepsilon} \frac{1}{1 + \varepsilon} & \text{if } P(t) \geq 0 \end{cases}.$$

In either case, letting  $\varepsilon \rightarrow 0$ , we get that

$$\text{HD}(Z) \geq t + \frac{P(t)}{\chi_{\mu_t}}.$$

Therefore,

$$\text{HD}(\mu_t \circ p_2^{-1}) = t + \frac{P(t)}{\chi_{\mu_t}}.$$

This proves the “right-hand” side of Theorem 14.2. By making use of Theorem 11.2, i.e. the Variational Principle, we get

$$t + \frac{P(t)}{\chi_{\mu_t}} = t + \frac{h_{\mu_t} - t\chi_{\mu_t}}{\chi_{\mu_t}} = \frac{h_{\mu_t}}{\chi_{\mu_t}},$$

and Theorem 14.2 is entirely proved.  $\square$

As an immediate consequence of this theorem and the Variational Principle (Theorem 11.2), we get the following.

**Corollary 14.3.** *If  $G$  is a C-F balanced TNR semigroup satisfying the Nice Open Set Condition and  $t \in \Delta_G^*$ , then*

$$\text{HD}(\mu_h \circ p_2^{-1}) = h = \text{HD}(J(G)) \text{ and } \text{HD}(\mu_t \circ p_2^{-1}) < h$$

if  $t \neq h$ .

*Proof.* The first part of this corollary immediately follows from Theorem 14.2 by substituting  $h$  for  $t$  and remembering that  $P(h) = 0$ . For the second part, suppose that

$$\text{HD}(\mu_t \circ p_2^{-1}) = h$$

for some  $t \in \Delta_G^*$ . It then follows from Theorem 14.2 that

$$h_{\mu_t}(\tilde{f}) - h\chi_{\mu_t} = 0 = P(h).$$

So, by virtue of Theorem 11.2, applied with  $t = h$ , we conclude that  $\mu_t = \mu_h$ . The proof is complete.  $\square$

Now, passing to actual multifractal analysis, fix  $t \geq 0$ . Our first result is the following.

**Lemma 14.4.** *If  $t \in \Delta_G^*$ , then for every  $q \in [0, 1]$  there exists a unique  $T_t(q) \geq 0$  such that*

$$P(T_t(q) + qt) = qP(t).$$

*Proof.* Because of Proposition 7.8, also  $qt \in \Delta_G^*$ . So, by Theorem 11.2, i.e. the Variational Principle, we have

$$P(qt) \geq h_{\mu_t} - qt\chi_{\mu_t} \geq q(h_{\mu_t} - t\chi_{\mu_t}) = qP(t).$$

So, employing Proposition 4.3 (b) and (d), the lemma immediately follows from the Intermediate Value Theorem.  $\square$



We denote

$$(14.22) \quad \mu_{t,q} := \mu_{T_t(q)+qt} \quad \text{and} \quad m_{t,q} := m_{T_t(q)+qt}.$$

We further denote

$$(14.23) \quad \alpha_t(q) := t + \frac{P(t)}{\chi_{\mu_{t,q}}}.$$

Now, for every point  $(\omega, z) \in J(\tilde{f})$ , we denote

$$\bar{\chi}(\omega, z) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log |(\tilde{f}^n)'(\omega, z)| \quad \text{and} \quad \underline{\chi}(\omega, z) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log |(\tilde{f}^n)'(\omega, z)|$$

and call them respectively the upper and lower Lyapunov exponents at the point  $(\omega, z)$ . If  $\underline{\chi}(\omega, z) = \bar{\chi}(\omega, z)$ , we denote the common value by  $\chi(\omega, z)$  and call it the Lyapunov exponent at  $(\omega, z)$ . Given  $\chi \geq 0$ , we denote

$$K(\chi) := \left\{ (\omega, z) \in J(\tilde{f}) : \underline{\chi}(\omega, z) = \bar{\chi}(\omega, z) = \chi \right\}.$$

We now shall prove the key statement of the main result of this section.

**Lemma 14.5.** *If  $t \in \Delta_G^*$ , then for every  $q \in [0, 1]$ ,*

$$\text{HD}(p_2(K(\chi_{\mu_{t,q}}))) = T_t(q) + q\alpha_t(q).$$

*Proof.* By Birkhoff's Ergodic Theorem there exists a measurable set  $X \subset J(\tilde{f})$  such that  $\mu_{t,q}(X) = 1$  and

$$\chi(\omega, z) = \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu_{t,q} = \chi_{\mu_{t,q}}$$

for all  $(\omega, z) \in X$ . Hence,  $X \subset K(\chi_{\mu_{t,q}})$ , and applying Theorem 14.2 and Lemma 14.4 along with (14.23), we get

$$\begin{aligned} \text{HD}(p_2(K(\chi_{\mu_{t,q}}))) &\geq \text{HD}(p_2(X)) \geq \text{HD}(\mu_{t,q} \circ p_2^{-1}) \\ (14.24) \quad &= T_t(q) + qt + \frac{P(T_t(q) + qt)}{\chi_{\mu_{t,q}}} = T_t(q) + qt + \frac{qP(t)}{\chi_{\mu_{t,q}}} \\ &= T_t(q) + q \left( t + \frac{P(t)}{\chi_{\mu_{t,q}}} \right) \\ (14.25) \quad &= T_t(q) + q\alpha_t(q). \end{aligned}$$

The proof of the opposite inequality is almost the same as the proof of the “ $\leq$ ” inequality of Theorem 14.2. However, since it does not formally follow from this theorem, we therefore present it here in full.

We start by fixing  $(\omega, z) \in K(\chi_{\mu_{t,q}}) \setminus \text{Sing}(\tilde{f})$ . Put  $\eta := \eta(\omega, z) > 0$  coming from Proposition 3.9. Let  $(n_j)_{j=1}^\infty$  be an increasing sequence of integers also coming from this Proposition. Let

$$\tilde{f}_{\omega|_{n_j}, z}^{-n_j} : p_2^{-1}(B(f_{\omega|_{n_j}}(z), \eta)) \longrightarrow \Sigma_u \times \hat{\mathbb{C}}$$

be the holomorphic inverse branch of  $\tilde{f}^{n_j}$  resulting also from Proposition 3.9 which sends  $\tilde{f}^{n_j}(\omega, z)$  to  $(\omega, z)$ . By Koebe's Distortion Theorem

$$B\left(z, \frac{1}{2}K \left|f'_{\omega|n_j}(z)\right|^{-1} \eta\right) \supset f_{\omega|n_j, z}^{-1}\left(B\left(f_{\omega|n_j}(z), \frac{\eta}{2}\right)\right).$$

So, by conformality of  $m_t$  and Koebe's Distortion Theorem again, we get with

$$r_j = \frac{1}{2}K \left|f'_{\omega|n_j}(z)\right|^{-1} \eta$$

that

$$\begin{aligned} m_{t,q} \circ p_2^{-1}(B(z, r_j)) &\geq \\ &\geq m_t\left(\tilde{f}_{\omega|n_j, z}^{-n_j}\left(p_2^{-1}\left(B\left(f_{\omega|n_j}(z), \frac{\eta}{2}\right)\right)\right)\right) \\ &= \int_{p_2^{-1}(B(f_{\omega|n_j}(z), \eta))} e^{-qP(t)n_j} \left|\left(\tilde{f}_{\omega|n_j, z}^{-n_j}\right)'\right|^{- (T_t(q)+qt)} dm_{t,q} \\ &\geq K^{-(T_t(q)+qt)} e^{-qP(t)n_j} \left|\left(\tilde{f}_{\omega|n_j}^{n_j}\right)'(\omega, z)\right|^{- (T_t(q)+qt)} m_{t,q} \circ p_2^{-1}\left(B\left(f_{\omega|n_j}(z), \eta/2\right)\right) \\ &\geq K^{-(T_t(q)+qt)} M_{t,q}(\eta/2) e^{-qP(t)n_j} \left|\left(\tilde{f}_{\omega|n_j}^{n_j}\right)'(\omega, z)\right|^{- (T_t(q)+qt)} \\ &= 2(K^2q)^{-(T_t(q)+qt)} M_{t,q}(\eta/2) e^{-qP(t)n_j} r_j^{T_t(q)+qt}, \end{aligned}$$

where

$$M_{t,q}(r) := \inf \{m_{t,q} \circ p_2^{-1}(B(x, r)) : x \in J(G)\}$$

is positive since  $\text{supp}(m_{t,q} \circ p_2^{-1}) = J(G)$ . Therefore, using also the fact that  $(\omega, z) \in K(\chi_{\mu_{t,q}})$ , we get

$$\begin{aligned} (14.26) \quad \liminf_{r \rightarrow 0} \frac{\log m_{t,q} \circ p_2^{-1}(B(z, r))}{\log r} &\leq \\ &\leq \liminf_{j \rightarrow \infty} \frac{\log m_{t,q} \circ p_2^{-1}(B(z, r_j))}{\log r_j} \\ &\leq \liminf_{j \rightarrow \infty} \frac{\log \left(2(K^2q)^{-(T_t(q)+qt)} M_{t,q}(\eta/2)\right) - qP(t)n_j + (T_t(q) + qt) \log r_j}{\log r_j} \end{aligned}$$

$$(14.27) \quad = T_t(q) + qt - qP(t) \lim_{j \rightarrow \infty} \frac{n_j}{\log r_j} = T_t(q) + qt + \frac{qP(t)}{\chi_{\mu_{t,q}}}$$

$$\begin{aligned} (14.28) \quad &= T_t(q) + q \left(t + \frac{P(t)}{\chi_{\mu_{t,q}}}\right) \\ &= T_t(q) + q\alpha_t(q). \end{aligned}$$

Therefore,

$$\text{HD}(p_2(K(\chi_{\mu_{t,q}}))) \leq T_t(q) + q\alpha_t(q).$$

Along with (14.25), this completes the proof of Lemma 14.5.  $\square$

Let us prove the following auxiliary result.

**Lemma 14.6.** *If  $t \in \Delta_G^*$  and  $q \in [0, 1]$ , then  $T_t(q) + qt \in \Delta_G^*$ .*

*Proof.* Consider two cases. First assume that  $t \leq h_f$ . Then  $P(t) \geq 0$ , whence  $qP(t) \geq 0$ . Since also  $P(T_t(q) + qt) = qP(t)$ , and, by Proposition 4.3 (b), the function  $s \mapsto P(s)$  is strictly decreasing, we have that  $T_t(q) + qt \leq h_f$ . Thus  $T_t(q) + qt \in \Delta_G^*$  by Proposition 7.8. Assume thus that  $t \geq h_f$ . Then  $P(t) \leq 0$ , and hence  $qP(t) \geq P(t)$ . So, applying Proposition 4.3 (b) again, we see that  $T_t(q) + qt \leq t$ . Thus,  $T_t(q) + qt \in \Delta_G^*$  by Proposition 7.8 again.  $\square$

Now, we will show that the function  $T_t(q)$  depends in a real-analytic way on both  $q$  and  $t$ .

**Lemma 14.7.** *The two functions*

$$\Delta_G^* \times [0, 1] \ni (t, q) \mapsto T_t(q) \in \mathbb{R} \quad \text{and} \quad \Delta_G^* \times [0, 1] \ni (t, q) \mapsto \chi_{\mu_t, q} \in \mathbb{R}$$

*are real-analytic with respect to both variables  $t$  and  $q$ .*

*Proof.* Consider the following function of three variables

$$[0, +\infty) \times \Delta_G^* \times [0, 1] \ni (s, t, q) \mapsto P(s + qt) \in \mathbb{R}.$$

Because of Lemma 14.6 the derivative  $\frac{\partial P}{\partial s}|_{(T_t(q), t, q)}$  is well defined and by Theorem 10.14,

$$(14.29) \quad \left. \frac{\partial P}{\partial s} \right|_{T_t(q), t, q} = -\chi_{\mu_t, q}.$$

Since  $\chi_{\mu_t, q} > 0$ , it therefore follows from Theorem 9.9 and the Implicit Function Theorem that both functions

$$\Delta_G^* \ni t \mapsto T_t(q) \quad \text{and} \quad [0, 1] \ni q \mapsto T_u(q), \quad (u \in \Delta_G^*),$$

are real-analytic. Of course the function  $(t, q) \mapsto \chi_{\mu_t, q}$  is real-analytic immediately from Theorem 9.9 and formula (14.29).  $\square$

Keep  $t \in \Delta_G^*$  fixed. Let

$$D_t(\tilde{f}) := \{\chi_{\mu_t, q} : q \in [0, 1]\}.$$

We call the parameter  $t$  exceptional if  $D_t(\tilde{f})$  is a singleton. Otherwise, we call it non-exceptional. Since the function  $[0, 1] \ni q \mapsto \chi_{\mu_t, q} \in (0, +\infty)$  is real-analytic,  $T_t(0) = h_f$  and  $T_t(1) = 0$ , it follows that  $D_t(\tilde{f})$  is a closed interval containing  $\chi_{\mu_{h_f}}$  and  $\chi_{\mu_t}$ . It is clear from Lemma 14.5 that if  $t$  is non-exceptional, then the multifractal analysis is non-trivial. We shall explore this issue now in greater detail. We shall prove the following.

**Proposition 14.8.** *If  $G$  is a C-F balanced TNR semigroup satisfying the Nice Open Set Condition, then the following are equivalent.*

- (a)  $\Delta_G^* \setminus \{h_f\}$  contains at least one exceptional parameter.
- (b) All elements of  $\Delta_G^* \setminus \{h_f\}$  are exceptional.
- (c) The set  $D(\tilde{f}) := \{\chi_{\mu_t} : t \in \Delta_G^*\}$  is a singleton.
- (d) There exist  $t, s \in \Delta_G^*$  with  $s \neq t$  such that  $\chi_{\mu_s} = \chi_{\mu_t}$ .

- (e) The function  $[0, +\infty) \ni t \mapsto P(t) \in \mathbb{R}$  is affine, i.e. there are  $\alpha, \beta \in \mathbb{R}$  such that  $P(t) = \alpha t + \beta$ .
- (f) The set  $\{\mu_t : t \in \Delta_G^*\}$  is a singleton.
- (g) There exist  $t, s \in \Delta_G^*$  with  $s \neq t$  such that  $\mu_s = \mu_t$ .

*Proof.* Of course (f)  $\implies$  (c)  $\implies$  (b) and (b)  $\implies$  (a). As for every  $t \in \Delta_G^* \setminus \{h_f\}$ ,

$$\Gamma_t := \{T_t(q) + qt : q \in [0, 1]\}$$

contains a (non-degenerate) interval between  $t$  and  $h_f$ , so we have that (a)  $\implies$  (d). In order to prove (d)  $\implies$  (e) assume without loss of generality that  $s < t$ . Then by (10.9) and (10.8), we have that

$$\chi_{\mu_s} \geq \chi_{\mu_u} \geq \chi_{\mu_t} = \chi_{\mu_s}$$

for every  $u \in [s, t]$ . So,  $\chi_{\mu_u} = \chi_{\mu_s}$ . Applying (10.8) again, it thus follows from Theorem 9.9 that the function  $\Delta_G^* \ni t \mapsto P'(t) \in \mathbb{R}$  is constant. This precisely means that the function  $\Delta_G^* \ni t \mapsto P(t) \in \mathbb{R}$  is affine and (e) is proved. Now, if (e) holds, then it follows from (10.8) again that  $D(\tilde{f})$  is a singleton, meaning that (c) holds. We have thus proved that conditions (a)–(e) are mutually equivalent and (f) entails all of them.

Now, we shall prove the next implication, namely that conditions (a)–(e) yield (f). Because of Theorem 11.2 and condition (e), we have for every  $t \in \Delta_G^*$  that

$$h_{\mu_t} - t\chi = \alpha t + \beta,$$

where  $\chi$  is the only element of  $D(\tilde{f})$ , which is a singleton by (c). Applying Theorem 14.2, we then get for every  $t \in \Delta_G^*$  that

$$(14.30) \quad \text{HD}(\mu_t \circ p_2^{-1}) = \frac{h_{\mu_t}}{\chi} = t + \frac{\alpha}{\chi}t + \frac{\beta}{\chi} = \left(1 + \frac{\alpha}{\chi}\right)t + \frac{\beta}{\chi}.$$

But, taking in (e),  $t = 0$ , we get by Theorem 11.2 that

$$(14.31) \quad \beta = P(0) = h_{\top}(\tilde{f}) > 0,$$

and, taking  $t = h$ , we get

$$\alpha = -\frac{\beta}{h}.$$

Substituting this to (14.30), we get

$$(14.32) \quad \text{HD}(\mu_t \circ p_2^{-1}) = \left(1 - \frac{\beta}{\chi h}\right)t + \frac{\beta}{\chi}.$$

Substituting here  $t = h$  or applying Corollary 14.3, we get that

$$(14.33) \quad \text{HD}(\mu_h \circ p_2^{-1}) = h.$$

Then from (14.30), (14.31), and from the ordinary Variational Principle for topological entropy, we obtain

$$(14.34) \quad \chi h = h_{\mu_h} \leq h_{\top}(\tilde{f}) = \beta.$$

Now, if  $\chi h < \beta$ , then it would follow from (14.32) that

$$\frac{d}{dt} (\text{HD}(\mu_t \circ p_2^{-1})) = 1 - \frac{\beta}{\chi h} < 0.$$

This would imply that for all  $t \in [0, h)$ ,

$$\text{HD}(\mu_t \circ p_2^{-1}) > \text{HD}(\mu_h \circ p_2^{-1}) = h = \text{HD}(J(G)).$$

Along with (14.34), this contradiction implies that

$$\chi h = \beta.$$

Substituting this to (14.32), we get that

$$\text{HD}(\mu_t \circ p_2^{-1}) = \frac{\beta}{\chi} = h$$

for every  $t \in \Delta_G^*$ . So, by applying Corollary 14.3, we get that  $\mu_t = \mu_h$  for every  $t \in \Delta_G^*$ , meaning that (f) holds. Thus all conditions (a)–(f) are equivalent. Of course (f) implies (g) and (g) implies (d). The proof of Proposition 14.8 is complete.  $\square$

If either one of the conditions (a)–(g) from Proposition 14.8 holds then we call the semigroup  $G$  and the skew product map  $\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \rightarrow \Sigma_u \times \hat{\mathbb{C}}$  exceptional. Otherwise, we call it non-exceptional. As an immediate consequence of Lemma 14.5, Lemma 14.7, Theorem 9.9, Proposition 14.8, and the fact that

$$(14.35) \quad \frac{\partial}{\partial q}(T_t(q) + qt) = -\frac{P(t)}{\chi_{\mu_{t,q}}} \begin{cases} < 0 & \text{if } t < h_f \\ > 0 & \text{if } t > h_f, \end{cases}$$

for all  $q \in [0, 1]$ , we get the following.

**Theorem 14.9.** *If  $G$  is a C–F balanced TNR non-exceptional semigroup satisfying the Nice Open Set Condition, then for every  $t \in \Delta_G^* \setminus \{h_f\}$ , the set  $D_t(\tilde{f})$  is a non-degenerate interval with endpoints  $\chi_{\mu_{h_f}}$  and  $\chi_{\mu_t}$ , and the function*

$$D_t(\tilde{f}) \ni \chi \mapsto \text{HD}(p_2(K(\chi))) \in [0, 2]$$

*is real-analytic.*

The class of non-exceptional semigroups is huge. This will be fully evidenced from Theorem 14.14 and Proposition 14.15. The first step in this direction is the following proposition which is also a complement of Proposition 14.8. Recall that the (co)homology of two functions was defined in formula (12.2). Denote by  $\tilde{N} : \mathcal{D}_U^\infty \rightarrow \mathbb{N}$  the function defined by

$$\tilde{N}(\tau) = \|\tau_1\|.$$

**Proposition 14.10.** *If  $G$  is a C–F balanced TNR semigroup satisfying the Nice Open Set Condition, then the following are equivalent.*

- (a)  $G$  is exceptional.
- (b) The set  $\{\tilde{\mu}_t : t \in \Delta_G\}$  is a singleton.
- (c) There exist  $s, t \in \Delta_G^*$  such that  $s \neq t$  and  $\tilde{\mu}_s = \tilde{\mu}_t$ .
- (d) All functions  $\zeta_{t, P(t)}$ ,  $t \in \Delta_G^*$ , are cohomologous to each other in the class of Hölder continuous real-valued bounded functions defined on  $\mathcal{D}_U^\infty$ .
- (e) All functions  $\zeta_{t, P(t)}$ ,  $t \in \Delta_G^*$ , are cohomologous to each other in the class of continuous real-valued bounded functions defined on  $\mathcal{D}_U^\infty$ .

- (f) *There are  $s, t \in \Delta_G^*$  such that  $s \neq t$  and  $\zeta_{s, P(s)}$  is cohomologous to  $\zeta_{t, P(t)}$  in the class of Hölder continuous real-valued bounded continuous functions defined on  $\mathcal{D}_U^\infty$ .*
- (g) *There are  $s, t \in \Delta_G^*$  such that  $s \neq t$  and  $\zeta_{s, P(s)}$  is cohomologous to  $\zeta_{t, P(t)}$  in the class of continuous real-valued bounded continuous functions defined on  $\mathcal{D}_U^\infty$ .*
- (h) *There exists  $\gamma \in \mathbb{R}$  such that the function  $\zeta := \zeta_{1,0} : \mathcal{D}_U^\infty \rightarrow \mathbb{R}$  and  $\gamma \tilde{N} : \mathcal{D}_U^\infty \rightarrow \mathbb{R}$  are cohomologous in the class of Hölder continuous real-valued bounded functions defined on  $\mathcal{D}_U^\infty$ .*
- (i) *There exists  $\gamma \in \mathbb{R}$  such that the function  $\zeta := \zeta_{1,0} : \mathcal{D}_U^\infty \rightarrow \mathbb{R}$  and  $\gamma \tilde{N} : \mathcal{D}_U^\infty \rightarrow \mathbb{R}$  are cohomologous in the class of continuous real-valued bounded functions defined on  $\mathcal{D}_U^\infty$ .*

*Proof.* The following implications are obvious:

$$(d) \implies (e) \implies (g) \implies (i), \quad (d) \implies (f) \implies (h) \implies (i), \quad \text{and} \quad (b) \implies (c).$$

In order to complete the proof we shall establish the following implications:

$$(i) \implies (a) \implies (b) \implies (d) \quad \text{and} \quad (c) \implies (a).$$

Having this we will be done. First, assume that (i) holds. This means that there exists a bounded continuous function  $u : \mathcal{D}_U^\infty \rightarrow \mathbb{R}$  such that

$$\zeta = \gamma \tilde{N} + u - u \circ \sigma.$$

Then, given any  $s, t \in \Delta_G^*$ , we have

$$\zeta_{t, P(t)} = (\gamma t - P(t)) \tilde{N} + tu - tu \circ \sigma \quad \text{and} \quad \zeta_{s, P(s)} = (\gamma s - P(s)) \tilde{N} + su - tu \circ \sigma.$$

Hence,

$$\zeta_{t, P(t)} - \zeta_{s, P(s)} = (\gamma(t-s) + P(s) - P(t)) \tilde{N} + (t-s)u - (t-s)u \circ \sigma.$$

Thus,

$$P \left( \sigma, \zeta_{t, P(t)} + (\gamma(t-s) + P(s) - P(t)) \tilde{N} + (t-s)u - (t-s)u \circ \sigma \right) = P \left( \sigma, \zeta_{t, P(t)} \right) = 0.$$

Therefore, as also  $P(\sigma, \zeta_{s, P(s)}) = 0$ , we get

$$\begin{aligned} & (\gamma(t-s) + P(s) - P(t)) \int_{\mathcal{D}_U^\infty} \tilde{N} d\tilde{\mu}_s = \\ & = h_{\tilde{\mu}_t}(\sigma) + \int_{\mathcal{D}_U^\infty} \zeta_{s, P(s)} d\tilde{\mu} + (\gamma(t-s) + P(s) - P(t)) \int_{\mathcal{D}_U^\infty} \tilde{N} d\tilde{\mu}_s \\ & \leq P(\sigma, \zeta_{s, P(s)} + (\gamma(t-s) + P(s) - P(t)) \tilde{N}) \\ & = 0. \end{aligned}$$

As

$$\int_{\mathcal{D}_U^\infty} \tilde{N} d\tilde{\mu}_s \geq 0,$$

this yields

$$\gamma(t-s) + P(s) - P(t) \leq 0.$$

Exchanging the roles of  $s$  and  $t$ , we also get

$$\gamma(s - t) + P(t) - P(s) \leq 0.$$

Hence,

$$P(t) - P(s) = \gamma(t - s).$$

This means that condition (e) of Proposition 14.8 holds, i.e. the semigroup  $G$  is exceptional. This in turn means that condition (a) of Proposition 14.10 holds, so the implication (i)  $\implies$  (a) is established.

Now, for proving the implication (a)  $\implies$  (b), assume that (a) holds. Then, because of Proposition 14.8 and Lemma 10.7, the set  $\{\mu_t : t \in \Delta_G^*\}$  is a singleton. By formula (10.1) this means that the set  $\{\tilde{\mu}_t \circ \pi_{\mathcal{U}}^{-1} : t \in \Delta_G^*\}$  is a singleton. Finally, because of Corollary 10.5, this implies that the set  $\{\tilde{\mu}_t : t \in \Delta_G\}$  is a singleton, meaning that (b) holds.

By Theorem 2.2.7 in [19], condition (b) of our present proposition is equivalent (keep also in mind that  $P(\sigma, \zeta_{s, P(s)}) = P(\sigma, \zeta_{t, P(t)}) = 0$ ) to (d), in particular, it entails (d).

For the second implication, assume that (c) holds. Then, by formula (10.1), we have that  $\hat{\mu}_s = \hat{\mu}_t$ . So, by Lemma 10.7, the two measures  $\mu_s$  and  $\mu_t$  are mutually singular. Since, by Theorem 4.20 both measures  $\mu_s$  and  $\mu_t$  are ergodic, they are equal. Because of Proposition 14.8 (see its item (g)), the item (a) of Proposition 14.10 is established, and thus we are done.  $\square$

We shall now prove that exceptional rational semigroups are exceptional indeed, i.e. we will almost fully classify all of them and will show that they form a very small sub-collection of all rational semigroups. We need some preparation. Keep  $G$ , a C-F balanced TNR semigroup satisfying the Nice Open Set Condition. Let

$$\mathcal{U} = \{U_s : s \in \text{Crit}_*(\tilde{f})\},$$

be a nice family produced in Theorem 6.6. Define the entrance time to the set  $J_{\mathcal{U}}^{\circ}$ ,

$$\hat{N} : J(\tilde{f}) \longrightarrow \mathbb{N}_0 \cup \{\infty\} = \{0, 1, 2, \dots, \infty\},$$

by declaring that  $\hat{N}(\xi)$  is the least element  $k$  of  $\mathbb{N}_0 \cup \{\infty\}$  such that

$$\tilde{f}^k(\xi) \in J_{\mathcal{U}}^{\circ}.$$

Further define

$$J_{\mathcal{U}}^{\circ,+} := \{z \in J(\tilde{f}) : \hat{N}(z) \in \mathbb{N}_0\}.$$

Now define the entrance map  $\tilde{f}_+ : J_{\mathcal{U}}^{\circ,+} \longrightarrow J_{\mathcal{U}}^{\circ}$  as

$$(14.36) \quad \tilde{f}_+(\xi) = \tilde{f}^{\hat{N}(\xi)}(\xi).$$

By its very definition the set  $J_{\mathcal{U}}^{\circ,+}$ , is backward invariant, i.e.

$$(14.37) \quad \bigcup_{k=0}^{\infty} \tilde{f}^{-k}(J_{\mathcal{U}}^{\circ,+}) \subset J_{\mathcal{U}}^{\circ,+}$$

and

$$(14.38) \quad \bigcup_{k=0}^{\infty} \tilde{f}^{-k}(\overline{J_{\mathcal{U}}^{\circ,+}}) \subset \overline{J_{\mathcal{U}}^{\circ,+}}.$$

We first shall prove the following.

**Lemma 14.11.** *If a C–F balanced TNR semigroup  $G$  satisfying the Nice Open Set Condition is exceptional, then there exist a constant  $\gamma \in \mathbb{R}$  and a Borel measurable function  $u^+ : J_{\mathcal{U}}^{\circ,+} \rightarrow \mathbb{R}$  such that*

- (a)  $-\log |\tilde{f}'| = \gamma + u^+ - u^+ \circ f$  everywhere on  $J_{\mathcal{U}}^{\circ,+}$ ,
- (b) If  $\xi \in \overline{J_{\mathcal{U}}^{\circ,+}} \setminus \text{PCV}(\tilde{f})$ , then there exists  $\Gamma \subset \Sigma_u \times \hat{\mathbb{C}}$ , an open neighborhood of  $\xi$  in  $\Sigma_u \times \hat{\mathbb{C}}$ , such that the function  $u^+|_{\Gamma \cap J_{\mathcal{U}}^{\circ,+}}$  is bounded.

*Proof.* Let

$$\mathcal{U} = \{U_s : s \in \text{Crit}_*(\tilde{f})\},$$

a nice family produced in Theorem 6.6. Since  $G$  is exceptional, by Proposition 14.10 (i) there exists a bounded continuous function  $u : \mathcal{D}_{\mathcal{U}}^{\infty} \rightarrow \mathbb{R}$  such that

$$(14.39) \quad \zeta = \gamma \tilde{N} + u - u \circ \sigma.$$

Since  $\zeta < 0$  everywhere in  $\mathcal{D}_{\mathcal{U}}^{\infty}$ , the integral of  $\zeta$  against any  $\tilde{f}$ -invariant measure is negative. Since also the integral of  $u - u \circ \sigma$  against any (at least one suffices) such measure vanishes, and since  $\tilde{N} \geq 1$  everywhere in  $\mathcal{D}_{\mathcal{U}}^{\infty}$ , we conclude that

$$(14.40) \quad \gamma < 0.$$

Now we define the function  $u^+ : J_{\mathcal{U}}^{\circ,+} \rightarrow \mathbb{R}$  by

$$(14.41) \quad u^+(\xi) = u\left(\pi_{\mathcal{U}}^{-1}(\tilde{f}_+(\xi))\right) - \log |\tilde{f}'_+(\xi)| - \gamma \hat{N}(\xi),$$

where  $\pi_{\mathcal{U}}^{-1}(\tilde{f}_+(\xi))$  is a singleton because of Corollary 10.5. By virtue of Lemma 10.2,

$$(14.42) \quad \tilde{f}(J_{\mathcal{U}}^{\circ,+}) \subset J_{\mathcal{U}}^{\circ,+}.$$

To prove (a) let  $\xi \in J_{\mathcal{U}}^{\circ,+}$ . We consider two cases. First assume that

$$\xi \notin J_{\mathcal{U}}^{\circ}.$$

Then  $\hat{N}(\xi) = 0$  and, because of item (b) of Corollary 10.6,

$$\hat{N}(\tilde{f}(\xi)) = \tilde{N}(\pi_{\mathcal{U}}^{-1}(\xi)) - 1 \text{ and } \tilde{f}_+(\tilde{f}(\xi)) = \tilde{f}|_{J_{\mathcal{U}}}(\xi).$$

Inserting this to (14.41) and using (14.39), along with item (b) of Corollary 10.6,

$$\begin{aligned} u^+(\tilde{f}(\xi)) &= u(\pi_{\mathcal{U}}^{-1}(\tilde{f}_+(\tilde{f}(\xi)))) - \log |\tilde{f}'_+(\tilde{f}(\xi))| - \gamma(\tilde{N}(\pi_{\mathcal{U}}^{-1}(\xi)) - 1) \\ &= u(\pi_{\mathcal{U}}^{-1}(\tilde{f}_{J_{\mathcal{U}}}(\xi))) - \log \left| \left( \tilde{f}^{\tilde{N}(\pi_{\mathcal{U}}^{-1}(\xi))} \right)'(\xi) \right| + \log |\tilde{f}'(\xi)| - \gamma \tilde{N}(\pi_{\mathcal{U}}^{-1}(\xi)) + \gamma \\ &= u(\pi_{\mathcal{U}}^{-1}(\tilde{f}_{J_{\mathcal{U}}}(\xi))) + u(\pi_{\mathcal{U}}^{-1}(\xi)) - u(\sigma(\pi_{\mathcal{U}}^{-1}(\xi))) + \log |\tilde{f}'(\xi)| + \gamma \\ &= u(\pi_{\mathcal{U}}^{-1}(\xi)) + \log |\tilde{f}'(\xi)| \\ &= u^+(\pi_{\mathcal{U}}^{-1}(\xi)) + \log |\tilde{f}'(\xi)| + \gamma, \end{aligned}$$

where the equality

$$(14.43) \quad u^+(\pi_{\mathcal{U}}^{-1}(\xi)) = u(\pi_{\mathcal{U}}^{-1}(\xi))$$



follows from the fact that  $\hat{N}(\xi) = 0$  and formula (14.41). The proof of (a) is thus complete.

Now to prove (b) we first note that if a point  $x \in U \cap J_{\mathcal{U}}^{\circ,+}$ , then  $x \in J_{\mathcal{U}}^{\circ}$ . It thus follows from (14.43) that

$$(14.44) \quad u^+|_{J_{\mathcal{U}}^{\circ,+} \cap U} = u|_{J_{\mathcal{U}}^{\circ,+} \cap U}$$

is bounded. Because of (14.37) and (14.38), and since the map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  is topologically exact, we see that for every  $\xi \in \overline{J_{\mathcal{U}}^{\circ,+}} \setminus \text{PCV}(\tilde{f})$  there exists  $j \geq 0$  and

$$x \in \left( U \cap \tilde{f}^{-j}(\xi) \cap \overline{J_{\mathcal{U}}^{\circ,+}} \right) \setminus \text{Crit}(\tilde{f}^j).$$

Since  $\text{Crit}(\tilde{f})$  is a closed set, there this exists an open set  $W \subset U$  such that

$$x \in W \subset \overline{W} \subset \hat{\mathbb{C}} \setminus \text{Crit}(\tilde{f}^j).$$

Then the function

$$W \ni z \mapsto \log |(\tilde{f}^j)'(z)| \in \mathbb{R}$$

is bounded. Hence, by iterating (a) (this is possible in view of (14.42)), which gives that

$$u^+ \circ f^j = \gamma j + u^+ + \log |(\tilde{f}^j)'|,$$

we get, with the use of (14.44), that the set  $u^+(\tilde{f}^j(W \cap J_{\mathcal{U}}^{\circ,+}))$  is bounded. But

$$\tilde{f}^j(W \cap J_{\mathcal{U}}^{\circ,+}) = \tilde{f}^j(W) \cap J_{\mathcal{U}}^{\circ,+}$$

since the set  $J_{\mathcal{U}}^{\circ,+}$  is, by (14.37), backward  $\tilde{f}$ -invariant, and, by (14.42), forward  $\tilde{f}$ -invariant. In addition, the set  $\tilde{f}^j(W)$  is open as  $W \subset \Sigma_u \times \hat{\mathbb{C}}$  is open and the map  $\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \rightarrow \Sigma_u \times \hat{\mathbb{C}}$  is open. Finally,

$$\xi = \tilde{f}^j(x) \in \tilde{f}^j(W).$$

So, setting  $\Gamma := \tilde{f}^j(W)$  finishes the proof of (b).  $\square$

As a fairly easy consequence of this lemma we get the following.

**Theorem 14.12.** *If  $f = (f_1, \dots, f_u)$  is a multi-map and the corresponding rational semi-group  $G = \langle f_1, \dots, f_u \rangle$  is*

- C-F balanced,
- Totally Non-Recurrent (TNR),
- satisfies the Nice Open Set Condition,
- exceptional,

*then for every non-empty set  $D \subset \{1, 2, \dots, u\}$ , the corresponding rational semigroup  $F = \langle f_j : j \in D \rangle$  is exceptional.*

*Proof.* Since  $J(F)$  is closed and backward invariant under  $F$ . i.e.

$$\bigcup_{g \in F} g^{-1}(J(F)) \subset J(F),$$

there exists  $R \in (0, R_*(G))$  such that

$$\bigcup_{g \in F} g^{-1}(B(\text{Crit}_*(F), 4R)) \cap B(\text{Crit}_*(f) \setminus \text{Crit}_*(F), 4R) = \emptyset.$$

Let  $\mathcal{U}$  be a nice family for the semigroup  $G = \langle f_1, \dots, f_u \rangle$  induced by the aperiodic set  $S = \text{Crit}_*(f)$  with some arbitrary number  $\kappa \in (1, 2)$  and  $r \in (0, R]$  according to Theorem 6.6. Then

$$\mathcal{U}_F := \{U_s\}_{s \in \text{Crit}_*(F)}$$

is a nice family for the semigroup  $F = \langle f_j : j \in D \rangle$ . Furthermore,

$$\begin{aligned} \mathcal{D}_{\mathcal{U}}(F) &= \bigcup_{s \in \text{Crit}_*(F)} \bigcup_{n=1}^{\infty} \{\tau \in \mathcal{D}_n(G, s) : \tilde{\tau} \in D^n\} \\ &= \{\tau \in \mathcal{D}_{\mathcal{U}}(G) : \tilde{\tau} \in \Sigma_u^* \text{ and } t(\tau), i(\tau) \in \text{Crit}_*(F)\} \subset \mathcal{D}_{\mathcal{U}}(G) \end{aligned}$$

and

$$\mathcal{S}_{\mathcal{U}_F}(F) = \left\{ \tilde{f}_{\tau}^{-\|\tau\|} : X_{t(\tau)} \longrightarrow X_{i(\tau)} \right\}_{\tau \in \mathcal{D}_{\mathcal{U}}(F)} \subset \mathcal{S}_{\mathcal{U}}(G).$$

In addition,

$$A(F, \mathcal{U}_F) = A(G, \mathcal{U})|_{\mathcal{D}_{\mathcal{U}}(F) \times \tilde{\mathcal{D}}_{\mathcal{U}}(F)}$$

and

$$\mathcal{D}_{\mathcal{U}_F}^{\infty}(F) \subset \mathcal{D}_{\mathcal{U}}^{\infty}(G).$$

In particular, the subshift  $\sigma : \mathcal{D}_{\mathcal{U}_F}^{\infty}(F) \longrightarrow \mathcal{D}_{\mathcal{U}_F}^{\infty}(F)$  is a subsystem of the subshift  $\sigma : \mathcal{D}_{\mathcal{U}}^{\infty}(G) \longrightarrow \mathcal{D}_{\mathcal{U}}^{\infty}(G)$  and

$$(14.45) \quad \zeta_{1,0}^{(F)} = \zeta_{1,0}^{(G)}|_{\mathcal{D}_{\mathcal{U}_F}^{\infty}(F)},$$

where  $\zeta_{1,0}^{(F)}$  and  $\zeta_{1,0}^{(G)}$  respectively denote the  $\zeta_{1,0}$  functions associated to the semigroups  $F$  and  $G$ . Likewise,

$$(14.46) \quad \tilde{N}_F = \tilde{N}_G|_{\mathcal{D}_{\mathcal{U}_F}^{\infty}(F)}.$$

Since  $G$  is exceptional, it follows from item (i) of Proposition 14.10 that with some  $\gamma \in \mathbb{R}$  the functions  $\zeta_{1,0}^{(G)} : \mathcal{D}_{\mathcal{U}}^{\infty} \longrightarrow \mathbb{R}$  and  $\gamma \tilde{N}_G : \mathcal{D}_{\mathcal{U}}^{\infty}(G) \longrightarrow \mathbb{R}$  are cohomologous in the class of continuous real-valued bounded functions defined on  $\mathcal{D}_{\mathcal{U}}^{\infty}(G)$ . We therefore conclude from (14.45) and (14.46) that the functions  $\zeta_{1,0}^{(F)}$  and  $\gamma \tilde{N}_F$  are cohomologous in the class of continuous real-valued bounded functions defined on  $\mathcal{D}_{\mathcal{U}_F}^{\infty}(F)$ . The proof of Theorem 14.12 is thus complete by applying item (i) of Proposition 14.10 again.  $\square$

In order to formulate and to prove our main result about exceptional rational semigroups we need the concept of parabolic orbifolds. The notion of an orbifold which we will utilize was introduced by William Thurston in [48] and [49]. It is very useful to study the dynamics of some rational functions. It was in particular used by Anna Zdunik in [56] to classify all exceptional (in her sense) rational functions of the Riemann sphere  $\hat{\mathbb{C}}$ . Our brief introduction to (parabolic) orbifolds closely follows hers from [56] and we will substantially rely on some results from [56] to prove our Theorem 14.14. We consider only orbifolds homeomorphic to the Riemann sphere  $\hat{\mathbb{C}}$ . Such an orbifold is the sphere  $\hat{\mathbb{C}}$  with a collection of distinguished, mutually distinct, points  $x_1, x_2, \dots, x_k \in \hat{\mathbb{C}}$  and integers (including  $+\infty$ )  $\nu(x_1), \nu(x_2), \dots, \nu(x_k) \geq 2$  ascribed to them. It is denoted by

$$(\hat{\mathbb{C}}; x_1, x_2, \dots, x_k; \nu(x_1), \nu(x_2), \dots, \nu(x_k)).$$

Two orbifolds

$(\hat{\mathbb{C}}; x_1, x_2, \dots, x_k; \nu(x_1), \nu(x_2), \dots, \nu(x_k))$  and  $(\hat{\mathbb{C}}; y_1, y_2, \dots, y_l; \nu(y_1), \nu(y_2), \dots, \nu(y_l))$  are considered equivalent if and only if  $k = l$  and

$$\nu(x_1) = \nu(y_1), \nu(x_2) = \nu(y_2), \dots, \nu(x_k) = \nu(y_l).$$

We then refer to an orbifold simply by listing some numbers

$$\nu_1, \nu_2, \dots, \nu_k \in \{2, 3, \dots\} \cup \{+\infty\}.$$

William Thurston introduced in [48] and [49] the notion of Euler characteristic of a (general) orbifold. In our context of the Riemann sphere  $\hat{\mathbb{C}}$ , it is given by the following formula

$$\chi(\hat{\mathbb{C}}; \nu_1, \nu_2, \dots, \nu_k) = 2 - \sum_{j=1}^k \left(1 - \frac{1}{\nu_j}\right).$$

An orbifold  $(\hat{\mathbb{C}}; \nu_1, \nu_2, \dots, \nu_k)$  is called parabolic if and only if its Euler characteristic is equal to 0. It is easy to list all parabolic orbifolds associated to the Riemann sphere  $\hat{\mathbb{C}}$ . These are

$$(\hat{\mathbb{C}}; 2, 2, 2, 2), (\hat{\mathbb{C}}; 3, 3, 3), (\hat{\mathbb{C}}; 2, 4, 4), (\hat{\mathbb{C}}; 2, 3, 6), (\hat{\mathbb{C}}; 2, 2, +\infty), (\hat{\mathbb{C}}; +\infty, +\infty).$$

A rational function  $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  is called critically finite if and only if the forward trajectory of each of its critical points is finite, in other words, if and only if each critical point of  $f$  is either periodic or eventually periodic. There is a natural way of ascribing an orbifold to such a critically finite map  $f$ . The (finite) set of distinguished points is given by the direct postcritical set of  $f$ , i.e. the set

$$\{f^k(c) : c \in \text{Crit}(f) \text{ and } k \geq 1\}.$$

The numbers  $\nu(f^k(c))$  are required to satisfy the relation that  $\nu(f^{k+1}(c))$  is an integral multiple of  $\nu(f^k(c))$ . There is exactly one minimal (in an obvious sense) way of choosing these numbers. We would like to bring up (see [56]) the following fact.

**Fact 14.13.** *We have the following.*

- (1) *The orbifold  $(\hat{\mathbb{C}}; +\infty, +\infty)$  corresponds, up to a conjugacy by a Möbius map, to rational functions of the form*

$$\hat{\mathbb{C}} \ni z \mapsto z^d \in \hat{\mathbb{C}}$$

*where  $d$  is an integer with  $|d| \geq 2$ .*

- (2) *The orbifold  $(\hat{\mathbb{C}}; 2, 2, +\infty)$  corresponds, up to a conjugacy by a Möbius map, to  $\pm$ Tchebyshev's polynomials.*
- (3) *These two above are the only classes of critically finite rational functions on the Riemann sphere  $\hat{\mathbb{C}}$  yielding parabolic orbifolds and whose Julia sets are different (so nowhere dense) from  $\hat{\mathbb{C}}$ .*

Now, we shall prove the following main result about exceptional rational semigroups.

**Theorem 14.14.** *If  $f = (f_1, \dots, f_u)$  is a multi-map and the corresponding rational semigroup  $G = \langle f_1, \dots, f_u \rangle$  is*

- C–F balanced,
- Totally Non–Recurrent (TNR),
- satisfies the Nice Open Set Condition,
- exceptional,

then each element of  $G$  is a critically finite rational function with parabolic orbifold.

*Proof.* Fix an element of  $G$ . It is of the form

$$f_\tau : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}, \quad \tau \in \Sigma_u^*.$$

By our hypotheses and Theorem 14.14, the rational semigroup  $\langle g \rangle$  generated by  $g$  is exceptional. Since  $\langle g \rangle$  has exactly one generator, the corresponding skew product map  $\tilde{g}$  is canonically identified with the map  $g : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  itself. So, Lemma 14.11, applied to the semigroup  $\langle g \rangle$ , tells us that there exist a constant  $\gamma \in \mathbb{R}$  and a Borel measurable function  $u^+ : J_{\mathcal{U}}^{\circ,+} \rightarrow \mathbb{R}$  such that

$$(14.47) \quad -\log |g'| = \gamma + u^+ - u^+ \circ g$$

everywhere on  $J_{\mathcal{U}}^{\circ,+}(\langle g \rangle)$ . In addition, the function  $u^+$  is given by the formula (14.41). Let  $\mu_0$  be the measure of maximal entropy for  $g : J(g) \rightarrow J(g)$ . By Poincaré’s Recurrence Theorem and ergodicity of  $\mu_0$ , we have that

$$\mu_0(J_{\mathcal{U}}^{\circ,+}(\langle g \rangle)) = 1.$$

Since the function  $u : \mathcal{D}_{\mathcal{U}}^\infty(\langle g \rangle) \rightarrow \mathbb{R}$  is measurable and bounded, the first summand

$$J_{\mathcal{U}}^{\circ,+} \ni \xi \mapsto u(\pi_{\mathcal{U}}^{-1}(\tilde{f}_+(\xi))) \in \mathbb{R}$$

in the formula (14.41) defining  $u^+$ , is bounded, thus belongs to  $L^2(\mu_0)$ . The other two terms  $\xi \mapsto -\gamma \tilde{N}(\xi)$  and  $\xi \mapsto -\log |\tilde{f}'_+(\xi)|$  and belong to  $L^2(\mu_0)$  respectively because of Corollary 10.11 and Lemma 10.12, both applied with  $p = 2$ . In conclusion,

$$(14.48) \quad u^+ \in L^2(\mu_0).$$

Now we can integrate (14.47) against  $\mu_0$  to get that

$$\gamma = - \int \log |g'| d\mu_0 = -\chi_{\mu_0}(g).$$

Therefore,

$$\begin{aligned} \frac{h_{\mu_0}(g)}{\chi_{\mu_0}(g)} \log |g'| &= \frac{h_{\mu_0}(g)}{\chi_{\mu_0}(g)} = \frac{\log \deg(g)}{\chi_{\mu_0}(g)} (\chi_{\mu_0}(g) + u^+ \circ g - u^+) \\ &= \log \deg(g) + \left( \frac{\log \deg(g)}{\chi_{\mu_0}(g)} u^+ \right) \circ g - \frac{\log \deg(g)}{\chi_{\mu_0}(g)} u^+. \end{aligned}$$

Equivalently,

$$\frac{\log \deg(g)}{\chi_{\mu_0}(g)} \log |g'| - \log \deg(g) = \left( \frac{\log \deg(g)}{\chi_{\mu_0}(g)} u^+ \right) \circ g - \frac{\log \deg(g)}{\chi_{\mu_0}(g)} u^+,$$

and

$$\frac{\log \deg(g)}{\chi_{\mu_0}(g)} u^+ \in L^2(\mu_0)$$

by virtue of (14.48). Since the number  $\alpha$  of [56] is equal to

$$\frac{\log \deg(g)}{\chi_{\mu_0}(g)},$$

the equation (H) from page 634 of [56] is satisfied by our function  $g$ . It therefore follows from Corollary on page 637 (in Section 5) of [56], the (last) Corollary of Section 7 on page 644 in [56], and from Proposition 8 on page 645 (in Section 8) of [56] that  $g$  is a critically finite rational function with parabolic orbifold. The proof is complete.  $\square$

It is evident from Theorem 14.14 that critical points of exceptional rational semigroups are very special. We shall now prove it without invoking the theory of orbifolds and quite deep results of [56]. We call a critical point  $(\omega, c) \in J(\tilde{f})$  exceptional if for every integer  $n \geq 1$

$$\tilde{f}^{-n}(\tilde{f}^n(\omega, c)) \subset \text{Crit}(\tilde{f}^n) \cup \text{PCV}(\tilde{f}).$$

We shall prove the following.

**Proposition 14.15.** *If a C–F balanced TNR semigroup  $G$  satisfying the Nice Open Set Condition contains some non-exceptional critical points in  $J(\tilde{f})$ , then  $G$  is non-exceptional.*

*Proof.* Seeking contradiction, suppose that  $G$  is exceptional. Keep

$$\mathcal{U} = \{U_s : s \in \text{Crit}_*(\tilde{f})\},$$

a nice family produced in Theorem 6.6. Now, passing to the last step of the proof of Proposition 14.15, let  $c \in J(\tilde{f})$  be a non-exceptional point of  $\tilde{f}$ . This means that there exists an integer  $n \geq 1$  such that

$$\tilde{f}^{-n}(\tilde{f}^n(\omega, c)) \not\subset \text{Crit}(\tilde{f}^n) \cup \text{PCV}(\tilde{f}).$$

This in turn means that there exists a point

$$(14.49) \quad \xi \in J(\tilde{f}) \setminus \left( \text{Crit}(\tilde{f}^n) \cup \text{PCV}(\tilde{f}) \right)$$

such that

$$(14.50) \quad \tilde{f}^n(\xi) = c.$$

Since  $c \in J(\tilde{f}) \cap U$ , since  $\text{Trans}(f)$ , the set of transitive points of  $\tilde{f}$ , is dense in  $J(\tilde{f})$ , and since  $\text{Trans}(f) \cap U \subset J_U^{\circ,+}$ , we conclude that  $c \in \overline{J_U^{\circ,+}}$ . Since  $\hat{f}$  is a TNR map, we thus have that

$$(14.51) \quad c \in \overline{J_U^{\circ,+}} \setminus \text{PCV}(\tilde{f}).$$

So, by (14.38), (14.49), and (14.50), also

$$(14.52) \quad \xi \in \overline{J_U^{\circ,+}} \setminus \text{PCV}(\tilde{f}).$$

Let then  $\Gamma_c$  and  $\Gamma_\xi$  be the open neighborhoods respectively of  $c$  and  $\xi$  produced in Lemma 14.11. By (14.49), (14.51), and (14.38)

$$w := \tilde{f}^n(\xi) = \tilde{f}^n(c) \in J_U^{\circ,+}.$$

So, there exists  $(z_k)_{k=1}^\infty$ , a sequence in  $\tilde{f}^n(\Gamma_c) \cap \tilde{f}^n(\Gamma_\xi) \cap J_{\mathcal{U}}^{\circ,+}$  such that

$$\lim_{k \rightarrow \infty} z_k = w.$$

Then, there exist  $(x_k)_{k=1}^\infty$  and  $(y_k)_{k=1}^\infty$ , two sequences respectively of points in  $\Gamma_c$  and  $\Gamma_\xi$ , such that

$$\tilde{f}^n(x_k) = z_k = \tilde{f}^n(y_k)$$

for all  $k \geq 1$ ,

$$(14.53) \quad \lim_{k \rightarrow \infty} x_k = c, \quad \text{and} \quad \lim_{k \rightarrow \infty} y_k = \xi.$$

Then by (14.38),  $x_k, y_k \in J_{\mathcal{U}}^{\circ,+}$  for all  $k \geq 1$  and equation (b) of Lemma 14.11, iterated  $n$  times, gives

$$\log |(\tilde{f}^n)'(x_k)| = u^+(z_k) - u^+(x_k) - \gamma n$$

and

$$\log |(\tilde{f}^n)'(y_k)| = u^+(z_k) - u^+(y_k) - \gamma n$$

for every  $k \geq 1$ . So,

$$\log |(\tilde{f}^n)'(x_k)| = \log |(\tilde{f}^n)'(y_k)| + u^+(y_k) - u^+(x_k)$$

for every  $k \geq 1$ . But, the sequences  $(u^+(y_k))_{k=1}^\infty$  and  $(u^+(x_k))_{k=1}^\infty$  are both bounded because of item (b) of Lemma 14.11 while, because of (14.49) and the right-hand side of (14.53), there exists  $q \geq 1$  such that the sequence

$$\left( \log |(\tilde{f}^n)'(y_k)| \right)_{k=q}^\infty$$

is bounded. Thus, the sequence

$$\left( \log |(\tilde{f}^n)'(x_k)| \right)_{k=q}^\infty$$

is also bounded. This however is a contradiction since by the left-hand side of (14.53),

$$\lim_{k \rightarrow \infty} \log |(\tilde{f}^n)'(x_k)| = -\infty.$$

We are done. □

## 15. MEASURES $m_t$ AND $\mu_t$

VERSUS

HAUSDORFF MEASURES  $H_{t^\kappa}$  AND  $H_{t^\kappa \exp(c\sqrt{\log(1/t)\log^3(1/t)})}$

In this section we will establish relationships between the Gibbs/equilibrium measures  $\mu_t$  and generalized Hausdorff measures, for example corresponding to gauge functions of the form

$$(0, \varepsilon) \ni u \mapsto u^{\text{HD}(\mu_t)} \exp \left( s \sqrt{\log \frac{1}{u} \log^3 \frac{1}{u}} \right), \quad s > 0,$$

where  $\log^3(x) = \log(\log(\log(x)))$  and, more generally, for every integer  $q \geq 1$ ,  $\log^q(x)$  is the  $q$ th iterate of the logarithm applied to  $x$ . The general strategy is to work with the GDS  $\mathcal{S}_{\mathcal{U}}$  of Theorem 6.9 rather than the skew product map  $\tilde{f} : \Sigma_u \times \hat{\mathbb{C}} \rightarrow \Sigma_u \times \hat{\mathbb{C}}$  itself and it is

based on [52], comp. also Section 4.8 of [19]. We start with an appropriate stochastic law which is an extension of the Law of the Iterated Logarithm.

A monotone increasing function  $\psi : [1, +\infty) \rightarrow [0, +\infty)$  is said to belong to the lower class if

$$\int_1^\infty \frac{\psi(u)}{u} \exp\left(-\frac{1}{2}\psi^2(u)\right) d\mu < +\infty$$

and to the upper class if

$$\int_1^\infty \frac{\psi(u)}{u} \exp\left(-\frac{1}{2}\psi^2(u)\right) d\mu = +\infty.$$

Recall that given an integer  $q \geq 1$  and  $u > 0$  large enough, we mean by  $\log^q(u)$  the  $q$ th iteration of the logarithm applied to  $u$ ; for example:

$$\log^1(u) = \log(u), \quad \log^2(u) = \log(\log(u)), \quad \log^3(u) = \log(\log(\log(u))).$$

In what follows, in the proofs, we will need the following lemma providing suitable improvements of lower and upper functions. This lemma has been proved in [5] as Lemma 4.3 and was repeated, with proof, in [52] as Lemma 6.1. We provide here its formulation and a short proof for the sake of completeness and convenience of the reader.

**Lemma 15.1.** *Let  $\eta, \chi > 0$  and let  $\rho : [(\chi + \eta)^{-1}, \infty) \rightarrow \mathbb{R}_+$  belong to the upper (lower) class. Let*

$$\theta : [(\chi + \eta)^{-1}, \infty) \rightarrow \mathbb{R}_+$$

*be a function such that*

$$\lim_{t \rightarrow \infty} \rho(t)\theta(t) = 0.$$

*Then there exists respective upper and lower class functions*

$$\rho_+ : [1, \infty) \rightarrow \mathbb{R}_+ \quad \text{and} \quad \rho_- : [1, \infty) \rightarrow \mathbb{R}_+$$

*such that*

- (a)  $\rho(t(\chi + \eta)) + \theta(t(\chi + \eta)) \leq \rho_+(t)$ ,
- and
- (b)  $\rho(t(\chi - \eta)) - \theta(t(\chi - \eta)) \geq \rho_-(t)$

*for all  $t \geq 1$ .*

*Proof.* Since  $\lim_{t \rightarrow \infty} \rho(t)\theta(t) = 0$ , there exists a constant  $M$  such that

$$(\rho(t) + \theta(t))^2 \leq \rho(t)^2 + M.$$

Let  $\rho$  belong to the upper class. Then the function  $t \mapsto \rho(t/(\chi + \eta))$  also belongs to the upper class. Hence, we may assume that  $\chi + \eta = 1$ . Define

$$\rho_+(t)^2 := \inf \{u(t)^2 : u \text{ is non-decreasing and } u(t) \geq \rho(t) + \theta(t)\}.$$

Then  $\rho_+(t) \geq \rho(t) + \theta(t)$  for  $t \geq 1$  and  $\rho_+$  is non-decreasing. Since  $\rho_+(t)^2 \leq \rho(t)^2 + M$ , we also get

$$\int_1^\infty \frac{\rho_+(t)}{t} \exp(-(1/2)\rho_+^2(t)) dt \geq \exp(-M/2) \int_1^\infty \frac{\rho_+(t)}{t} \exp(-(1/2)\rho^2(t)) dt = +\infty.$$

The proof in the case of a function of the lower class is similar. □

As an immediate consequence of Theorem 5.2 and Lemma 5.3 of [52] (comp. Theorem 2.5.5 and Lemma 2.5.6 in [19]) along with Proposition 9.2, Corollary 10.11, and the inequality

$$\left( t \log \left| \left( \tilde{f}^{\|\tau_1\|} \right)' (\pi_{\mathcal{U}}(\sigma(\tau))) \right| \right)^\alpha \left( \left( \tilde{f}^{\|\tau_1\|} \right)' (\pi_{\mathcal{U}}(\sigma(\tau))) \right)^{-t} \leq \left| \left( \tilde{f}^{\|\tau_1\|} \right)' (\pi_{\mathcal{U}}(\sigma(\tau))) \right|^{-t'}$$

holding for all  $t' < t$ ,  $\alpha \geq 0$ , and  $\|\tau_1\|$  (depending on  $t'$ ) large enough (the last two properties yielding finite moments required in Lemma 5.3 of [52] or Lemma 2.5.6 of [19]), we get the following.

**Theorem 15.2.** *If  $G$  is a C–F balanced TNR rational semigroup of finite type and  $\mathcal{U}$  is a nice family of sets, then for all  $t \in \Delta_G^*$  and all  $\alpha, \beta \in \mathbb{R}$*

$$\begin{aligned} & \tilde{\mu}_t \left( \left\{ \tau \in \mathcal{D}_{\mathcal{U}}^\infty : \sum_{j=0}^{n-1} (\alpha\zeta + \beta N_\zeta) \circ \sigma^j(\tau) - \tilde{\mu}_t(\alpha\zeta + \beta N_\zeta) \right. \right. \\ & \qquad \qquad \qquad \left. \left. > \sigma_t(\alpha\zeta + \beta N_\zeta) \psi(n) \sqrt{n} \text{ for infinitely many } n \right\} \right) \\ &= \begin{cases} 0 & \text{if } \psi : [1, +\infty) \rightarrow (0, +\infty) \text{ belongs to the lower class} \\ 1 & \text{if } \psi : [1, +\infty) \rightarrow (0, +\infty) \text{ belongs to the upper class,} \end{cases} \end{aligned}$$

where  $\zeta : \mathcal{D}_{\mathcal{U}}^\infty \rightarrow \mathbb{R}$  is given by

$$\zeta(\tau) := \log \left| \left( \tilde{f}^{\|\tau_1\|} \right)' (\pi_{\mathcal{U}}(\sigma(\tau))) \right|,$$

$N_\zeta := \|\tau_1\|$ , and, with  $k := \alpha\zeta + \beta N_\zeta$ , the non-negative number

$$\sigma_t^2(k) := \int_{\mathcal{D}_{\mathcal{U}}^\infty} (k - \tilde{\mu}_t(k))^2 d\tilde{\mu}_t + 2 \sum_{n=1}^{\infty} \int_{\mathcal{D}_{\mathcal{U}}^\infty} (k - \tilde{\mu}_t(k))(k \circ \sigma^n - \tilde{\mu}_t(k)) d\tilde{\mu}_t$$

is assumed to be positive.

Define  $k^* : J(\tilde{f}) \rightarrow \mathbb{R}$  by the formula,

$$k^*(\omega, z) := \alpha \log |\tilde{f}'(\omega, z)| + \beta N_\zeta(\omega, z).$$

Further define:

$$\sigma_t^2(k^*) := \int_{J(\tilde{f})} (k^* - \mu_t(k^*))^2 d\mu_t + 2 \sum_{n=1}^{\infty} \int_{J(\tilde{f})} (k^* - \tilde{\mu}_t(k^*)) (k^* \circ \tilde{f}^n - \tilde{\mu}_t(k^*)) d\mu_t$$

and

$$(15.1) \quad \chi_{\mu_t} := \int_{J(\tilde{f})} \log |\tilde{f}'| d\mu_t \quad \text{and} \quad \chi_{\tilde{\mu}_t} := \int_{\mathcal{D}_{\mathcal{U}}^\infty} \zeta d\tilde{\mu}_t.$$

Then it follows from Lemma 10.7 and (a generalization of) Kac's Lemma that

$$(15.2) \quad \frac{\sigma_t^2(k)}{\chi_{\tilde{\mu}_t}} = \frac{\sigma_t^2(k^*)}{\chi_{\mu_t}}.$$



Put

$$\tilde{\chi}_t := \chi_{\tilde{\mu}_t} \quad \text{and} \quad \tilde{\sigma}_t^2 := \sigma_t^2((\kappa_t - t)\zeta - P(t)N_\zeta(\xi)).$$

The main technical result of this section is the following.

**Lemma 15.3.** *Let  $G$  be a C–F balanced TNR rational semigroup satisfying the Nice Open Set Condition. Assume that  $t \in \Delta_G^*$  and  $\tilde{\sigma}_t^2 > 0$ . If a slowly growing function  $\psi$  belongs to the upper class, then*

$$(15.3) \quad \limsup_{r \rightarrow 0} \frac{m_t \circ p_2^{-1}(B(x, r))}{r^{\text{HD}(\mu_t \circ p_2^{-1})} \exp\left(\tilde{\sigma}_t \chi_{\mu_t}^{-1/2} \psi\left(\log \frac{1}{r}\right) \sqrt{\log \frac{1}{r}}\right)} = +\infty$$

for  $\mu_t \circ p_2^{-1}$ -a.e.  $x \in J(G)$ .

If, on the other hand,  $\psi$  belongs to the lower class, then for every  $\varepsilon > 0$  there exists a measurable set  $\tilde{J}_\varepsilon \subset J(\tilde{f})$  with  $\mu_t(\tilde{J}_\varepsilon) \geq 1 - \varepsilon$  and a constant  $r_\varepsilon \in (0, 1]$  such that

$$(15.4) \quad \frac{m_t\left(\tilde{J}_\varepsilon \cap p_2^{-1}(B(x, r))\right)}{r^{\text{HD}(\mu_t \circ p_2^{-1})} \exp\left(\tilde{\sigma}_t \chi_{\mu_t}^{-1/2} \psi\left(\log \frac{1}{r}\right) \sqrt{\log \frac{1}{r}}\right)} \leq \varepsilon$$

for all  $x \in p_2(\tilde{J}_\varepsilon)$  and all  $r \in (0, r_\varepsilon]$ .

*Proof.* We shall prove the first formula (15.3). Fix  $x = p_2(\pi_{\mathcal{U}}(\xi))$ , where  $\xi \in \mathcal{D}_{\mathcal{U}}^\infty$  (then  $\xi \in J(G)$ ) and for every integer  $n \geq 1$  set

$$r_n := \text{diam}\left(p_2(\phi_{\xi|n}(X_{t(\xi_n)}))\right).$$

Then

$$B(x, r_n) \supset p_2\left(\left|\phi_{\xi|n}(X_{t(\xi_n)})\right|\right),$$

and so,

$$(15.5) \quad \begin{aligned} m_t \circ p_2^{-1}(B(x, r_n)) &\geq m_t(\phi_{\xi|n}(X_{t(\xi_n)})) = \int_{X_{t(\xi_n)}} e^{-P(t)N_\zeta(\xi)} |\phi'_{\xi|n}|^t dm_t \\ &\geq Q^{-1} K^{-t} \exp((-tS_n\zeta - P(n)N_\zeta)(\xi)) \end{aligned}$$

where  $K \geq 1$  is the Koebe's Distortion Constant and

$$Q := \min\{m_t(X_s) : s \in S\} > 0.$$

Put

$$\kappa_t := \text{HD}(\mu_t \circ p_2^{-1}).$$

Using (15.5) and denoting by  $S_n$ ,  $n \geq 1$ , the Birkhoff's sum corresponding to the dynamical system  $\sigma : \mathcal{D}_U^\infty \rightarrow \mathcal{D}_U^\infty$ , we get

$$\begin{aligned}
(15.6) \quad & \frac{m_t \circ p_2^{-1}(B(x, r_n))}{r_n^{\kappa_t} \exp\left(\tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi\left(\log \frac{1}{r_n}\right) \sqrt{\log \frac{1}{r_n}}\right)} \geq \frac{Q^{-1} K^{-t} \exp((-t S_n \zeta - P(t) N_\zeta)(\xi))}{r_n^{\kappa_t} \exp\left(\tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi\left(\log \frac{1}{r_n}\right) \sqrt{\log \frac{1}{r_n}}\right)} \\
& \geq \frac{Q^{-1} K^{-t} \exp((-t S_n \zeta - P(t) N_\zeta)(\xi))}{K^{\kappa_t} \exp(-\kappa_t S_n \zeta(\xi)) \exp\left(\tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi\left(\log \frac{1}{r_n}\right) \sqrt{\log \frac{1}{r_n}}\right)} \\
& = (Q K^{t+\kappa_t})^{-1} \exp\left(S_n((\kappa_t - t)\zeta - P(t) N_\zeta)(\xi)\right) \exp\left(-\tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi\left(\log \frac{1}{r_n}\right) \sqrt{\log \frac{1}{r_n}}\right) \\
& = (Q K^{t+\kappa_t})^{-1} \exp\left(S_n((\kappa_t - t)\zeta - P(t) N_\zeta)(\xi) - \tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi\left(\log \frac{1}{r_n}\right) \sqrt{\log \frac{1}{r_n}}\right).
\end{aligned}$$

Now, we estimate  $\log(1/r_n)$  from above. By definition, we have

$$(15.7) \quad \log \frac{1}{r_n} \leq \log K + \log D + S_n \zeta(\xi).$$

Now, Birkhoff's Ergodic Theorem asserts that for  $\tilde{\mu}_t$ -a.e.  $\xi \in \mathcal{D}_U^\infty$ ,

$$(15.8) \quad \lim_{n \rightarrow \infty} \frac{1}{n} S_n \zeta(\xi) = \int_{\mathcal{D}_U^\infty} \zeta d\tilde{\mu}_t = \tilde{\chi}_t.$$

Hence, there exists a measurable set  $Y_1 \subset \mathcal{D}_U^\infty$  such that  $\tilde{\mu}_t(Y_1) = 1$  and for every  $\eta > 0$  and every  $\xi \in Y_1$  there exists  $n'_1(\xi, \eta) \geq 1$  such that

$$S_n \zeta(\xi) \leq \left(\tilde{\chi}_t + \frac{1}{2}\eta\right) n$$

for all  $n \geq n'_1(\xi, \eta)$ . Combining this with (15.7), we see that there exists  $n_1(\xi, \eta) \geq n'_1(\xi, \eta)$  such that

$$(15.9) \quad \log \frac{1}{r_n} \leq (\tilde{\chi}_t + \eta) n$$

for all  $n \geq n_1(\xi, \eta)$ . In fact, in what follows, we will need a better estimate on  $\log(1/r_n)$ . Indeed, since the function  $u \mapsto 3\sqrt{u \log^2 u}$  belongs to the lower class, by (15.7) and Theorem 15.2, there exists a measurable set  $Y_2 \subset Y_1$  such that  $\tilde{\mu}_t(Y_2) = 1$  and

$$(15.10) \quad \log \frac{1}{r_n} \leq \tilde{\chi}_t n + 3(\sigma_t^2(\zeta) + 1) \sqrt{n \log^2 n}$$

for every integer  $n \geq 1$  large enough, say  $n \geq n_2(\xi) \geq n_1(\xi, \eta)$ . As a matter of fact, we just derived (15.10) assuming that  $\sigma_t^2(\xi) > 0$ . If however  $\sigma_t^2(\xi) = 0$ , then by an appropriate theorem from [19] (or [27])  $\xi$  is cohomologous to a constant in the class of bounded Hölder continuous functions on  $\mathcal{D}_U^\infty$ , and then the factor  $3(\sigma_t^2(\zeta) + 1) \sqrt{n \log^2 n}$  can be replaced by a constant (so, much better) in (15.10). One would like to apply Theorem 15.2 to inequality (15.6) to conclude the proof. However, neither (15.9) nor even (15.10) are sufficiently

strong for such an argument. What we need is to make use of Lemma 15.1. So, consider the function

$$\theta(u) := \psi(u) \left( \sqrt{1 + \frac{3(\sigma_t^2(\xi) + 1)}{\tilde{\chi}_t} \sqrt{\frac{\log^2((\tilde{\chi}_t + \eta)^{-1}u)}{(\tilde{\chi}_t + \eta)^{-1}u}}} - 1 \right) + (\tilde{\chi}_t + \eta)^{\frac{1}{4}} u^{-\frac{1}{4}}.$$

Since  $\theta > 0$  and since  $\psi$  is a slowly growing function, we have that

$$\lim_{u \rightarrow \infty} \psi(u)\theta(u) = 0.$$

Therefore, by Lemma 15.1, there exists  $\psi_+$ , a function in the upper class, such that

$$(15.11) \quad \psi_+(u) \geq \psi((\tilde{\chi}_t + \eta)u) + \theta((\tilde{\chi}_t + \eta)u).$$

By Theorem 11.2, i.e. the Variational Principle, we have that

$$\begin{aligned} \int_{\mathcal{D}_{\tilde{\mathcal{U}}}^\infty} ((\kappa_t - t)\zeta - P(t)N_\zeta) d\tilde{\mu}_t &= (\kappa_t - t)\tilde{\chi}_t - P(t) \frac{1}{\mu_t(\tilde{J}_{\mathcal{U}})} = \frac{1}{\mu_t(\tilde{J}_{\mathcal{U}})} ((\kappa_t - t)\chi_{\mu_t} - P(t)) \\ &= \frac{1}{\mu_t(\tilde{J}_{\mathcal{U}})} \left( \left( \frac{h_{\mu_t}}{\chi_{\mu_t}} - t \right) \chi_{\mu_t} - P(t) \right) \\ &= \frac{1}{\mu_t(\tilde{J}_{\mathcal{U}})} (h_{\mu_t} - t\chi_{\mu_t} - P(t)) \\ (15.12) \quad &= 0. \end{aligned}$$

Hence, inserting this and (15.11) to Theorem 15.2, and using (15.9), (15.10), and the fact that  $\psi$  is monotone increasing, we obtain for infinitely many  $u$  that

$$\begin{aligned} S_n((\kappa_t - t)\zeta - P(t))(\xi) - \tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi(\log 1/r_n) \sqrt{1/r_n} &\geq \\ &\geq \tilde{\sigma}_t \sqrt{n} \psi_+(n) - \tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi((\tilde{\chi}_t + \eta)n) \tilde{\chi}_t^{1/2} \sqrt{n} \sqrt{1 + \frac{3(\sigma_t^2(\zeta) + 1)}{\tilde{\chi}_t} \sqrt{\frac{\log^2 n}{n}}} \\ &\geq \tilde{\sigma}_t \sqrt{nn^{-1/4}} = \tilde{\sigma}_t n^{1/4}. \end{aligned}$$

Inserting this to (15.6), we get

$$\begin{aligned} \limsup_{r \rightarrow 0} \frac{m_t \circ p_2^{-1}(B(x, r))}{r^{\kappa_t} \exp\left(\tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi(\log(1/r)) \sqrt{\log(1/r)}\right)} \\ (15.13) \quad &\geq \limsup_{n \rightarrow \infty} \frac{m_t \circ p_2^{-1}(B(x, r_n))}{r_n^{\kappa_t} \exp\left(\tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi(\log(1/r_n)) \sqrt{\log(1/r_n)}\right)} \\ &\geq (QK^{t+\kappa_t})^{-1} \limsup_{n \rightarrow \infty} \exp(\tilde{\sigma}_t n^{1/4}) \\ &= +\infty. \end{aligned}$$

Since, by Theorem 4.20, the measures  $\mu_t \circ p_2^{-1}$  and  $m_t \circ p_2^{-1}$  are equivalent, formula (15.3) of Lemma 15.3 results from (15.13) for  $\mu_t \circ p_2^{-1}$ -a.e.  $x \in J(G)$ .

We now pass to the proof of (15.4). For ever  $\xi \in \mathcal{D}_U^\infty$  and every radius  $r > 0$  let  $n = n(\xi, r) \geq 0$  be the first integer such that

$$(15.14) \quad \text{diam} \left( p_2 \left( \phi_{\xi|_{n+1}} \left( X_{t(\xi_{n+1})} \right) \right) \right) < r.$$

Of course

$$(15.15) \quad \lim_{r \rightarrow \infty} n(\xi, r) = +\infty,$$

and therefore there exists  $r_1(\omega) > 0$  such that for all  $r \in (0, r_1(\omega))$ , we have  $n(\xi, r) \geq 2$ ; so

$$(15.16) \quad \text{diam} \left( p_2 \left( \phi_{\xi|_n} \left( X_{t(\xi_n)} \right) \right) \right) \geq r.$$

Hence,

$$r \left| f'_{\xi|_n} \left( p_2(\pi_U(\xi)) \right) \right| \leq K \text{diam} \left( X_{t(\xi_n)} \right) \leq 2KR,$$

where  $R > 0$  comes from Definition 6.5. So, fix  $r \in (0, r_1(\omega))$ . It then follows from the definition 6.5 of nice families that

$$(15.17) \quad \text{dist} \left( f_{\xi|_n} \left( p_2(\pi_U(\sigma^n(\xi))) \right), \text{PCV}(G) \right) \geq 64Kr \left| f'_{\xi|_n} \left( p_2(\pi_U(\xi)) \right) \right|$$

and, consequently, the holomorphic inverse branch

$$f_{\xi|_n}^{-1} : B \left( f_{\xi|_n} \left( p_2(\pi_U(\sigma^n(\xi))) \right), 16Kr \left| f'_{\xi|_n} \left( p_2(\pi_U(\xi)) \right) \right| \right) \rightarrow \tilde{\mathcal{C}}$$

sending  $f_{\xi|_n} \left( p_2(\pi_U(\sigma^n(\xi))) \right)$  to  $p_2(\pi_U(\sigma^n(\xi)))$  is well defined, whence

$$\begin{aligned} m_t & \left( [\tilde{\xi}|_n] \times f_{\xi|_n}^{-1} \left( B \left( f_{\xi|_n} \left( p_2(\pi_U(\sigma^n(\xi))) \right), 8r \left| f'_{\xi|_n} \left( p_2(\pi_U(\xi)) \right) \right| \right) \right) \right) \\ & \leq |\phi'_{\xi|_n}(\pi_U(\sigma^n(\xi)))|^t e^{-P(t)n} m_t \left( \Sigma_u \times f_{\xi|_n}^{-1} \left( B \left( f_{\xi|_n} \left( p_2(\pi_U(\sigma^n(\xi))) \right), 8r \left| f'_{\xi|_n} \left( p_2(\pi_U(\xi)) \right) \right| \right) \right) \right) \\ & \leq \exp(-tS_n\zeta(\xi) - P(t)n). \end{aligned}$$

Hence

$$(15.18) \quad \begin{aligned} & m_t \left( [\tilde{\xi}|_n] \times f_{\xi|_n}^{-1} \left( B \left( f_{\xi|_n} \left( p_2(\pi_U(\sigma^n(\xi))) \right), 8r \left| f'_{\xi|_n} \left( p_2(\pi_U(\xi)) \right) \right| \right) \right) \right) \\ & \frac{r^{\kappa_t} \exp \left( \tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi(\log(1/r_n)) \sqrt{\log(1/r_n)} \right)}{\exp(-tS_n\zeta(\xi) - P(t)n)} \\ & \leq \frac{\text{diam}^{\kappa_t} \left( p_2 \left( \phi_{\xi|_n} \left( \chi_{t(\xi_n)} \right) \right) \right) \exp \left( \tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi(\log(1/r)) \sqrt{\log(1/r)} \right)}{\exp(-tS_n\zeta(\xi) - P(t)n)} \\ & \leq \frac{\left| \phi'_{\xi|_{n+1}}(\pi_U(\sigma^{n+1}(\xi))) \right|^{\kappa_t} \exp \left( \tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi(\log(1/r)) \sqrt{\log(1/r)} \right)}{\exp \left( \left( (\kappa_t - t)S_n\zeta - P(t)N_\zeta \right) (\xi) - \tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi(\log(1/r)) \sqrt{\log(1/r)} + \kappa_t \zeta(\sigma^{n+1}(\xi)) \right)}. \end{aligned}$$

Now, because (15.8) there exists a measurable set  $Y_2 \subset \mathcal{D}_U^\infty$  such that  $\tilde{\mu}_t(Y_2) = 1$  and for every  $\eta > 0$  and every  $\xi \in Y_2$  there exists  $n'_2(\xi, \eta) \geq 1$  such that

$$S_k \zeta(\xi) \geq \left( \tilde{\chi}_t - \frac{1}{2} \eta \right) k$$

for all  $k \geq n'_2(\xi, \eta)$ . Combining this with (15.16), we see that

$$\log(1/r) \geq -\log D + S_n \zeta(\xi) \geq -\log D + \left( \tilde{\chi}_t - \frac{1}{2} \eta \right) n,$$

whenever  $\xi \in Y_2$  and  $n = n(\xi, r) \geq n'_2(\xi, \eta)$ . Hence, there exists  $n_2(\xi, \eta) \geq n'_2(\xi, \eta)$  such that

$$(15.19) \quad \log(1/r) \geq (\tilde{\chi}_t - \eta) n$$

whenever  $\xi \in Y_2$  and  $n = n(\xi, r) \geq n_2(\xi, \eta)$ . As in the proof of (15.3), we will need a better estimate on  $\log(1/r_n)$ . Indeed, since the function  $u \mapsto 3\sqrt{u \log^2 u}$  belongs to the lower class, by (15.7) and Theorem 15.2, there exists a measurable set  $Y_3 \subset Y_2$  such that  $\tilde{\mu}_t(Y_3) = 1$  and

$$(15.20) \quad \log \frac{1}{r} \geq \tilde{\chi}_t n - 3\sigma_t(\xi) \sqrt{n \log^2 n}$$

if the integer  $n \geq 1$  is large enough, say  $n \geq n_3(\xi) \geq n_2(\xi, \eta)$ . As a matter of fact we just derived (15.20) assuming that  $\sigma_t^2(\xi) > 0$ . If however  $\sigma_t^2 = 0$ , then by an appropriate theorem from [19] (or [27])  $\xi$  is cohomologous to a constant in the class of bounded Hölder continuous functions on  $\mathcal{D}_U^\infty$ , and then the  $2(\sigma_t^2(\xi) + 1)\sqrt{n \log^2 n}$  term can be replaced by a constant (so, much better) in (15.10). Now again, as in the proof of formula (15.3), we consider an appropriate function  $\theta$ . It is now defined by

$$\theta(u) := \psi(u) \left( 1 - \sqrt{1 - \frac{3\sigma_t(\zeta)}{\tilde{\chi}_t} \sqrt{\frac{\log^2((\tilde{\chi}_t - \eta)^{-1}u)}{(\tilde{\chi}_t - \eta)^{-1}u}}} \right) + (\tilde{\chi}_t - \eta)^{\frac{1}{4}} u^{-\frac{1}{4}} + (\tilde{\chi}_t - \eta)^{\frac{1}{8}} u^{-\frac{1}{8}} \psi^{\frac{3}{4}}(u).$$

Since  $\theta > 0$  and since  $\psi$  is a slowly growing function, we have that

$$\lim_{u \rightarrow \infty} \psi(u) \theta(u) = 0.$$

Therefore, by Lemma 15.1, there exists  $\psi_-$ , a function in the lower class such that

$$(15.21) \quad \psi_-(u) \leq \psi((\tilde{\chi}_t - \eta)u) - \theta((\tilde{\chi}_t - \eta)u).$$

Using (15.12), (15.19), and (15.20), it follows from Theorem 15.2 that

$$(15.22) \quad \begin{aligned} & (\kappa_t - t) S_n(\zeta - P(t)N_\zeta)(\xi) - \tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi(\log 1/r) \sqrt{\log 1/r} \leq \\ & \leq \tilde{\sigma}_t \sqrt{n} \psi_-(n) - \tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi((\tilde{\chi}_t - \eta)n) \tilde{\chi}_t^{1/2} \sqrt{n} \sqrt{1 - \frac{3\sigma_t(\xi)}{\tilde{\chi}_t} \sqrt{\frac{\log^2 n}{n}}} \end{aligned}$$

for  $\tilde{\mu}_t$ -a.e.  $\xi \in \mathcal{D}_U^\infty$  and all  $n \geq 1$  large enough, say for all  $\xi \in Y_4 \subset Y_3$  with  $\tilde{\mu}_t(Y_4) = 1$  and all  $n \geq n_4(\xi) \geq n_3(\xi)$ . We also need to take care of the term  $\kappa_t \zeta(\sigma^{n+1}(\xi))$  appearing at the end of (15.18). Since the function  $|\zeta|$  has all moments, in particular since

$$\int |\zeta|^3 d\tilde{\mu}_t < +\infty,$$

it follows from Tchebyshev's Inequality that

$$\tilde{\mu}_t\left(\left\{\xi \in \mathcal{D}_U^\infty : \kappa_t |\zeta(\xi)| \geq \tilde{\sigma}_t \psi^{3/4}((\tilde{\chi}_t - \eta)k) k^{3/8}\right\}\right) \leq \tilde{\sigma}_t^{-3} \psi^{-1/4}((\tilde{\chi}_t - \eta)k) k^{-9/8} \int_{\mathcal{D}_U^\infty} |\zeta|^3 d\tilde{\mu}_t.$$

Since the measure  $\tilde{\mu}_t$  is  $\sigma$ -invariant, we therefore obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \tilde{\mu}_t\left(\left\{\xi \in \mathcal{D}_U^\infty : \kappa_t |\zeta(\sigma^{k+1}(\xi))| \geq \tilde{\sigma}_t \psi^{3/4}((\tilde{\chi}_t - \eta)k) k^{3/8}\right\}\right) &\leq \\ &\leq \tilde{\sigma}_t^{-3} \sum_{k=1}^{\infty} \psi^{-1/4}((\tilde{\chi}_t - \eta)k) \tilde{\mu}_t(|\zeta|^3) k^{-9/8} \\ &\leq \tilde{\sigma}_t^{-3} \psi^{-1/4}((\tilde{\chi}_t - \eta)) \tilde{\mu}_t(|\zeta|^3) \sum_{k=1}^{\infty} k^{-9/8} \\ &< +\infty. \end{aligned}$$

Therefore, in view of the Borel–Cantelli Lemma there exists a measurable set  $Y_5 \subset Y_4$  such that  $\tilde{\mu}_t(Y_5) = 1$  and for all  $\xi \in Y_5$  there exists an integer  $n_5(\xi) \geq n_4(\xi)$  such that

$$\kappa_t |\zeta(\sigma^{k+1}(\xi))| \leq \tilde{\sigma}_t \psi^{3/4}((\tilde{\chi}_t - \eta)k) k^{3/8}$$

for all integers  $k \geq n_5(\xi)$ . Along with (15.22) and (15.21), this yields

$$\begin{aligned} &((\kappa_t - t)S_n \zeta - P(t)N_\zeta)(\xi) - \tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi(\log(1/r)) \sqrt{\log(1/r)} + \kappa_t \zeta(\sigma^{n+1}(\xi)) \\ &\leq \tilde{\sigma}_t \sqrt{n} \psi_-(n) - \tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi((\tilde{\chi}_t - \eta)n) \tilde{\chi}_t^{1/2} \sqrt{n} \sqrt{1 - \frac{3\sigma_t(\xi)}{\tilde{\chi}_t} \sqrt{\frac{\log^2 n}{n}}} + \\ &\quad + \tilde{\sigma}_t \psi^{3/4}((\tilde{\chi}_t - \eta)n) n^{3/8} \\ &\leq \tilde{\sigma}_t \sqrt{n} \left( \psi((\tilde{\chi}_t - \eta)n) - \theta((\tilde{\chi}_t - \eta)n) - \psi((\tilde{\chi}_t - \eta)n) \sqrt{1 - \frac{3\sigma_t(\xi)}{\tilde{\chi}_t} \sqrt{\frac{\log^2 n}{n}}} \right) \\ &\quad + \tilde{\sigma}_t \psi^{3/4}((\tilde{\chi}_t - \eta)n) n^{3/8} \\ &= \tilde{\sigma}_t \psi^{3/4}((\tilde{\chi}_t - \eta)n) n^{3/8} - \tilde{\sigma}_t \sqrt{n} n^{-1/4} - \tilde{\sigma}_t \sqrt{n} \psi^{3/4}((\tilde{\chi}_t - \eta)n) n^{1/8} \\ &= -\tilde{\sigma}_t n^{1/4} \end{aligned}$$

for all  $\xi \in Y_5$  and  $n = n(\xi, r) \geq n_5(\xi)$ . Inserting this to (15.18), we get that

$$(15.23) \quad \frac{m_t \left( [\tilde{\xi}|_n] \times f_{\tilde{\xi}|_n}^{-1} \left( B \left( f_{\tilde{\xi}|_n} (p_2(\pi_{\mathcal{U}}(\sigma^n(\xi)))) \right), 8r \left| f'_{\tilde{\xi}|_n} (p_2(\pi_{\mathcal{U}}(\xi))) \right| \right) \right)}{r^{\kappa_t} \exp \left( \tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi(\log(1/r_n)) \sqrt{\log(1/r_n)} \right)} \leq \exp \left( -\tilde{\sigma}_t n^{1/4}(\xi, r) \right)$$

for every  $\xi \in Y_5$  and every  $r > 0$  so small that  $n(\xi, r) \geq n_5(\xi)$  (see also (15.15)). Now, fix  $\varepsilon \in (0, 1)$  and take  $k \geq 1$  so large that

$$\tilde{\mu}_t \left( Z'_\varepsilon := Y_5 \cap n_5^{-1}(\{0, 1, \dots, k\}) \right) \geq 1 - \frac{\varepsilon}{2}$$

and

$$(15.24) \quad \exp \left( -\tilde{\sigma}_t k^{1/4} \right) \leq \frac{\varepsilon}{\#(16K^2, a)},$$

where  $a$  is given by the formula

$$(15.25) \quad a := 32K^2 R,$$

with, we recall,  $R > 0$  comes from Definition 6.5 and  $\#(16K^2, a)$  comes from Proposition 13.9. By (15.15) there exists  $r(\varepsilon) > 0$  so small that

$$(15.26) \quad \tilde{\mu}_t \left( Z_\varepsilon := \{\xi \in Z'_\varepsilon : n(\xi, r(\varepsilon)) \geq k\} \right) \geq 1 - \varepsilon.$$

Fix  $r \in (0, r(\varepsilon)]$ . Our goal now is to apply Proposition 13.9. For every  $x \in p_2(\pi_{\mathcal{U}}(Z_\varepsilon))$ , we define

$$\mathcal{F}_0(x, r) := \{ \tilde{\xi}|_{n(\xi, r)+1} : \xi \in Z_\varepsilon \cap (p_2 \circ \pi_{\mathcal{U}})^{-1}(B(x, r)) \}.$$

Furthermore, we take  $\mathcal{F}(x, r)$  to be a family of mutually incomparable elements of  $\mathcal{F}_0(x, r)$  such that

$$(15.27) \quad \bigcup_{\tau \in \mathcal{F}(x, r)} [\tau] = \bigcup_{\tau \in \mathcal{F}_0(x, r)} [\tau].$$

This is the largest family of mutually incomparable words in  $\mathcal{F}_0(x, r)$ . For every  $\tau \in \mathcal{F}(x, r)$  fix  $\tau^+ \in Z_\varepsilon \cap (p_2 \circ \pi_{\mathcal{U}})^{-1}(B(x, r))$  such that

$$(15.28) \quad \tilde{\tau}^+|_{|\tau|} = \tau.$$

Define

$$R_\tau := 16Kr \left| f'_{\tilde{\tau}^+} (p_2(\pi_{\mathcal{U}}(\tau^+))) \right|.$$

The condition (ess2) is obviously satisfied by the definition of  $\mathcal{F}(x, r)$ . We now deal with condition (ess0). Keeping  $\tau \in \mathcal{F}(x, r)$ , it follows from  $\frac{1}{4}$ -Koebe's Distortion Theorem that

$$f_{\tilde{\tau}^+}^{-1} \left( B \left( f_{\tilde{\tau}^+} (p_2 \circ \pi_{\mathcal{U}}(\tau^+)), 4r \left| f'_{\tilde{\tau}^+} (p_2(\pi_{\mathcal{U}}(\tau^+))) \right| \right) \right) \supset B(p_2 \circ \pi_{\mathcal{U}}(\tau^+), r).$$

Hence,

$$(15.29) \quad x \in f_{\tilde{\tau}^+}^{-1} \left( B \left( f_{\tilde{\tau}^+} (p_2 \circ \pi_{\mathcal{U}}(\tau^+)), 4r \left| f'_{\tilde{\tau}^+} (p_2(\pi_{\mathcal{U}}(\tau^+))) \right| \right) \right).$$

Consequently,

$$f_{\tilde{\tau}^+}(x) \in B \left( f_{\tilde{\tau}^+} (p_2 \circ \pi_{\mathcal{U}}(\tau^+)), 4r \left| f'_{\tilde{\tau}^+} (p_2(\pi_{\mathcal{U}}(\tau^+))) \right| \right) = B \left( f_{\tilde{\tau}^+} (p_2 \circ \pi_{\mathcal{U}}(\tau^+)), (4K)^{-1} R_\tau \right).$$

But

$$f_{\hat{\tau}}(p_2 \circ \pi_{\mathcal{U}}(\tau^+)) \in f_{\tau_*}^{-1}(p_2 \circ \pi_{\mathcal{U}}(\tau^+)) \in f_{\tau_*}^{-1}(J(G)).$$

Thus,

$$\text{dist}(f_{\hat{\tau}}(x), f_{\tau_*}^{-1}(J(G))) \leq \frac{1}{2}R_{\tau},$$

and, consequently, (ess0) holds because of Remark 13.10. It follows from (15.17) that

$$\begin{aligned} \text{dist}(f_{\hat{\tau}}(x), \text{PCV}(G)) &\geq \text{dist}(f_{\hat{\tau}}(p_2 \circ \pi_{\mathcal{U}}(\tau^+)), \text{PCV}(G)) - |f_{\hat{\tau}}(x) - f_{\hat{\tau}}(p_2 \circ \pi_{\mathcal{U}}(\tau^+))| \\ &\geq 64Kr |f'_{\hat{\tau}}(p_2(\pi_{\mathcal{U}}(\tau^+)))| - 4r |f'_{\hat{\tau}}(p_2(\pi_{\mathcal{U}}(\tau^+)))| \\ &= 60Kr |f'_{\hat{\tau}}(p_2(\pi_{\mathcal{U}}(\tau^+)))| \\ &> 2R_{\tau}. \end{aligned}$$

Hence, a unique analytic branch  $f_{\hat{\tau},x}^{-1} : B(f_{\hat{\tau}}(x), 2R_{\tau}) \rightarrow \mathbb{C}$  of  $f_{\hat{\tau}}^{-1}$  sending  $f_{\hat{\tau}}(x)$  to  $x$  exists. It also follows from (15.29) that

$$K^{-1} \leq \frac{|f'_{\hat{\tau}}(x)|}{|f'_{\hat{\tau}}(p_2(\pi_{\mathcal{U}}(\tau^+)))|} \leq K.$$

Therefore,

$$\frac{1}{16K^2} \leq \frac{|f'_{\hat{\tau}}(x)|r}{R_{\tau}} \leq \frac{1}{16}.$$

In conclusion, condition (ess1) of Definition 13.8 is satisfied with  $M = 16K^2$  and, by (15.27), with  $a$  given by (15.25). So,  $\mathcal{F}(x, r)$  is  $(16K^2, a, V)$ -essential for  $(x, r)$  with

$$V := \bigcup_{\tau \in \mathcal{F}(x, r)} [\tau].$$

In consequence, Proposition 13.9 applies, and, in particular, its item (c) yields

$$(15.30) \quad \#\mathcal{F}(x, r) \leq \#(16K^2, a).$$

We want to show that

$$(15.31) \quad \begin{aligned} &\pi_{\mathcal{U}}(Z_{\varepsilon} \cap (p_2 \circ \pi_{\mathcal{U}})^{-1}(B(x, r))) \subset \\ &\subset \bigcup_{\tau \in \mathcal{F}(x, r)} [\hat{\tau}] \times f_{\hat{\tau}}^{-1}\left(B(p_2(\pi_{\mathcal{U}}(\sigma^{|\hat{\tau}|}(\tau^+))), 8r |f'_{\hat{\tau}}(p_2(\pi_{\mathcal{U}}(\tau^+)))|\right). \end{aligned}$$

Indeed, let  $\xi \in Z_{\varepsilon} \cap (p_2 \circ \pi_{\mathcal{U}})^{-1}(B(x, r))$ . By (15.27) there exists  $\xi' \in Z_{\varepsilon} \cap (p_2 \circ \pi_{\mathcal{U}})^{-1}(B(x, r))$  such that

$$(15.32) \quad \begin{aligned} &\xi' |_{n(\xi', r)+1} \in \mathcal{F}(x, r), \\ &n(\xi, r) \geq n(\xi', r), \end{aligned}$$

and

$$(15.33) \quad \xi |_{n(\xi, r)+1} = \xi' |_{n(\xi', r)+1}.$$

Denoting  $n(\xi', r)$  by  $n$  and  $\xi' |_{n(\xi', r)+1}$  by  $\gamma$ , it follows from the  $\frac{1}{4}$ -Koebe's Distortion Theorem that

$$f_{\hat{\gamma}}^{-1}\left(B(p_2(\pi_{\mathcal{U}}(\sigma^{|\hat{\gamma}|}(\gamma^+))), 8r |f'_{\hat{\gamma}}(p_2(\pi_{\mathcal{U}}(\gamma^+)))|\right) \supset B(p_2(\pi_{\mathcal{U}}(\gamma^+)), 2r) \ni p_2(\pi_{\mathcal{U}}(\xi)).$$



Since, also by (15.33) and (15.32),

$$p_1(\pi_{\mathcal{U}}(\xi)) = \tilde{\xi} \in [\tilde{\xi}|_{n(\xi,r)}] \subset [\xi'|_{n(\xi',r)}] = [\hat{\gamma}],$$

we conclude that (15.31) holds. This formula, along with (15.23), (15.24), (15.26), and (15.30), gives

$$\frac{m_t(\pi_{\mathcal{U}}(Z_\varepsilon \cap (p_2 \circ \pi_{\mathcal{U}})^{-1}(B(x,r))))}{r^{\kappa_t} \exp\left(\tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi(\log(1/r)) \sqrt{\log(1/r)}\right)} \leq \varepsilon.$$

But

$$\pi_{\mathcal{U}}(Z_\varepsilon \cap (p_2 \circ \pi_{\mathcal{U}})^{-1}(B(x,r))) = \pi_{\mathcal{U}}(Z_\varepsilon) \cap p_2^{-1}(B(x,r)),$$

so

$$(15.34) \quad \frac{m_t(\pi_{\mathcal{U}}(Z_\varepsilon) \cap p_2^{-1}(B(x,r)))}{r^{\kappa_t} \exp\left(\tilde{\sigma}_t \tilde{\chi}_t^{-1/2} \psi(\log(1/r)) \sqrt{\log(1/r)}\right)} \leq \varepsilon.$$

Since, also by (15.26),

$$\mu_t(\pi_{\mathcal{U}}(Z_\varepsilon)) \geq \tilde{\mu}_t(Z_\varepsilon) \geq 1 - \varepsilon,$$

formula (15.4) of Lemma 15.3 follows by setting  $\tilde{J}_\varepsilon := \pi_{\mathcal{U}}(Z_\varepsilon)$ . The proof is now complete.  $\square$

Given  $\psi : [0, +\infty) \rightarrow (0, +\infty)$  and  $t \in \Delta_G^*$ , let

$$\tilde{\psi}_t : (1, +\infty) \rightarrow (0, +\infty)$$

be given by the formula

$$\tilde{\psi}_t(u) := u^{\text{HD}(\mu_t \circ p_2^{-1})} \exp\left(\tilde{\sigma}_t \chi_{\mu_t}^{-1/2} \psi(\log(1/u)) \sqrt{\log(1/u)}\right),$$

and let  $H_{\tilde{\psi}_t}$  be the corresponding (generalized) Hausdorff measure. If  $(X, \varrho)$  is a metric space and  $\delta > 0$ , we denote by  $H_{\tilde{\psi}_t}(X, \varrho)$  the infimum of countable sums

$$\sum_{k=1}^{\infty} \tilde{\psi}(\text{diam}(X_k)),$$

where  $\{X_k\}_{k=1}^{\infty}$  are countable covers of  $X$  with  $\text{diam}(X_k) \leq \delta$  for each  $k \geq 1$ . Now, the proof of the following, main result of this section, is fairly standard.

**Theorem 15.4.** *Let  $G$  be a C–F balanced TNR rational semigroup satisfying the Nice Open Set Condition. Assume that  $t \in \Delta_G^*$  and  $\tilde{\sigma}_t > 0$ . If  $\psi : [0, +\infty) \rightarrow (0, +\infty)$  is a slowly growing function, then*

- (a) *If  $\psi$  is in the upper class, then the measures  $\mu_t \circ p_2^{-1}$  and  $H_{\tilde{\psi}_t}$  on  $J(G)$  are mutually singular,*
- (b) *If  $\psi$  is in the lower class, then  $\mu_t \circ p_2^{-1}$  is absolutely continuous with respect to  $H_{\tilde{\psi}_t}$  on  $\tilde{C}$ . Moreover,  $H_{\tilde{\psi}_t}(E) = +\infty$  whenever  $E \subset J(G)$  is a Borel set such that  $\mu_t \circ p_2^{-1}(E) > 0$ .*

*Proof.* To prove item (a), first fix  $\varepsilon \in (0, 1)$ . By formula (15.3) of Lemma 15.3 and by Egorov's Theorem for every integer  $n \geq 1$  there exists a Borel set  $E_n(\varepsilon) \subset J(G)$  such that

$$(15.35) \quad \mu_t \circ p_2^{-1}(E_n(\varepsilon)) > 1 - \varepsilon 2^{-n}$$

and for every  $x \in E_n(\varepsilon)$  there exists a closed ball  $B_n(x)$  contained in  $X$  with  $\text{diam}(B_n(x)) < 1/n$  and

$$m_t \circ p_2^{-1}(B_n(x)) \geq n\tilde{\psi}(\text{diam}(B_n(x))).$$

Let  $b(2)$  be the constant from Besicovič's Theorem corresponding to dimension 2. Applying this theorem to the cover  $\{B_n(x) : x \in E_n(\varepsilon)\}$  of  $E_n(\varepsilon)$ , we get a countable set  $S \subset E_n(\varepsilon)$  such that

$$\bigcup_{x \in S} B_n(x) \supset E_n(\varepsilon)$$

and  $S$  can be represented as a disjoint union  $S_1 \cup S_2 \cup \dots \cup S_{b(2)}$  such that for every  $1 \leq k \leq b(2)$ , the family  $\{B_n(x) : x \in S_k\}$  consists of mutually disjoint sets. Then

$$\begin{aligned} H_{\tilde{\psi}_t}(E_n(\varepsilon), 1/n) &\leq \sum_{x \in S} \tilde{\psi}_t(\text{diam}(B_n(x))) \leq \frac{1}{n} \sum_{x \in S} m_t \circ p_2^{-1}(B_n(x)) \\ &= \frac{1}{n} \sum_{k=1}^{b(2)} \sum_{x \in S_k} m_t \circ p_2^{-1}(B_n(x)) = \frac{1}{n} \sum_{k=1}^{b(2)} m_t \circ p_2^{-1} \left( \bigcup_{x \in S_k} B_n(x) \right) \\ &\leq \frac{b(2)}{n}. \end{aligned}$$

So, if

$$E_\varepsilon := \bigcap_{n=1}^{\infty} E_n(\varepsilon),$$

then

$$H_{\tilde{\psi}_t}(E_\varepsilon) = 0 \quad \text{and} \quad \mu_t \circ p_2^{-1}(E_\varepsilon) \geq 1 - \varepsilon.$$

Hence, setting

$$E := \bigcup_{\ell=1}^{\infty} E_{1/\ell},$$

we have  $H_{\tilde{\psi}_t}(E) = 0$  and  $\mu_t \circ p_2^{-1}(E) = 1$ . Thus, item (a) is proved.

Now to prove item (b) put

$$\eta := \frac{1}{2} m_t \circ p_2^{-1}(E) > 0,$$

as by Theorem 4.20, the measures  $m_t \circ p_2^{-1}$  and  $\mu_t \circ p_2^{-1}$  are equivalent. Fix  $\kappa > 0$  arbitrarily. Using Theorem 4.20 again, it follows from Lemma 15.3 that there exists  $\varepsilon \in (0, \kappa)$  such that  $m_t(\tilde{J}_\varepsilon) \geq 1 - \eta$ . Then

$$(15.36) \quad m_t(\tilde{J}_\varepsilon \cap p_2^{-1}(E)) \geq m_t \circ p_2^{-1}(E) - \eta = \eta.$$

Fix  $\delta \in (0, r(\varepsilon))$ , where  $r(\varepsilon) > 0$  also comes from Lemma 15.3. Consider an arbitrary set  $S \subset E \cap p_2(\tilde{J}_\varepsilon)$  and for every  $x \in S$  an arbitrary radius  $r(x) \in (0, \delta]$  such that

$$\bigcup_{x \in S} B(x, r(x)) \supset E \cap p_2(\tilde{J}_\varepsilon).$$

We then get from formula (15.4) of Lemma 15.3, the choice of  $\varepsilon$ , and (15.3) that

$$\begin{aligned} \sum_{x \in S} \tilde{\psi}_t(r(x)) &\geq \frac{1}{\varepsilon} \sum_{x \in S} m_t \left( \tilde{J}_\varepsilon \cap p_2^{-1}(B(x, r(x))) \right) \geq \frac{1}{\varepsilon} m_t \left( \tilde{J}_\varepsilon \cap \left( \bigcup_{x \in S} p_2^{-1}(B(x, r(x))) \right) \right) \\ &\geq \frac{1}{\varepsilon} m_t \left( \tilde{J}_\varepsilon \cap p_2^{-1} \left( \bigcup_{x \in S} B(x, r(x)) \right) \right) \geq \frac{1}{\varepsilon} m_t \left( \tilde{J}_\varepsilon \cap p_2^{-1} (E \cap p_2(\tilde{J}_\varepsilon)) \right) \\ &\geq \frac{1}{\varepsilon} m_t \left( \tilde{J}_\varepsilon \cap p_2^{-1}(E) \right) \geq \frac{\eta}{\kappa}. \end{aligned}$$

Hence,

$$H_{\tilde{\psi}_t}(E \cap p_2(\tilde{J}_\varepsilon), \delta) \geq A \frac{\eta}{\kappa},$$

where  $A \in (0, +\infty)$  is some universal constant, see [16] or [27] for example. Therefore,

$$H_{\tilde{\psi}_t}(E) \geq H_{\tilde{\psi}_t}(E \cap p_2(\tilde{J}_\varepsilon)) \geq A \frac{\eta}{\kappa}.$$

Since  $\kappa > 0$  is arbitrary, this gives that  $H_{\tilde{\psi}_t}(E) = +\infty$ , finishing the proof of item (b) and of the whole Theorem 15.4.  $\square$

Good examples of lower and upper class functions are ones of the form

$$\ell_c(u) = c \sqrt{\log^2 u}, \quad c > 0.$$

Indeed, if  $0 \leq c \leq 2$ , then  $\ell_c$  belongs to the upper class, and if  $c > 2$ , then  $\ell_c$  belongs to the lower class. Therefore, as an immediate consequence of Theorem 15.4, we get the following.

**Corollary 15.5.** *Let  $G$  be a C–F balanced TNR rational semigroup type satisfying the Nice Open Set Condition. Assume that  $t \in \Delta_G^*$  and  $\tilde{\sigma}_t > 0$ . We have the following.*

- (a) *If  $0 \leq c \leq 2$ , then  $\mu_t \circ p_2^{-1}$  and  $H_{(\tilde{\ell}_c)_t}$  on  $J(G)$  are mutually singular. In particular, the measures  $\mu_t \circ p_2^{-1}$  and  $H_{\text{HD}(\mu_t \circ p_2^{-1})}$  are mutually singular.*
- (b) *If  $c > 2$ , then  $\mu_t \circ p_2^{-1}$  is absolutely continuous with respect to  $H_{(\tilde{\ell}_c)_t}$  on  $J(G)$ . Moreover,  $H_{(\tilde{\ell}_c)_t}(E) = +\infty$  whenever  $E \subset J(G)$  is a Borel set such that  $\mu_t \circ p_2^{-1}(E) > 0$ .*

## 16. HD( $J(G)$ ) VERSUS HAUSDORFF DIMENSION OF FIBER JULIA SETS $J_\omega$ , $\omega \in \Sigma_u$

In this section our goal is to relate the global Hausdorff dimension  $\text{HD}(J(G))$  with the Hausdorff dimension of the fibers  $\text{HD}(J_\omega)$ . We will show that for our systems, i.e. C–F balanced TNR rational semigroups of finite type satisfying the Nice Open Set Condition,

$$\text{HD}(J(G)) > \text{HD}(J_\omega)$$

for every  $\omega \in \Sigma_u$ . If in addition our semigroup is expanding, then

$$\sup\{\text{HD}(J_\omega) : \omega \in \Sigma_u\} < \text{HD}(J(G)).$$

First we shall prove the following auxiliary result.

**Lemma 16.1.** *Let  $G$  be a C–F balanced TNR rational semigroup generated by a rational multi-map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  and satisfying the Nice Open Set Condition. If  $V \subset \Sigma_u \times \hat{\mathbb{C}}$  is a non-empty neighborhood of  $\text{Crit}_*(\tilde{f})$ , then*

$$\text{HD}(p_2(K(V))) < h = \text{HD}(J(G)).$$

*Proof.* Have  $R_2 > 0$  coming from Lemma 7.1. Since the set  $K(V)$  is compact there exists a finite set  $\Xi \subset K(V)$  such that

$$\bigcup_{\xi \in \Xi} B(\xi, R_2) \supset K(V).$$

Let

$$(16.1) \quad \underline{P}_V^\Xi(t) := \liminf_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\xi \in \Xi} \sum_{x \in K(V) \cap \tilde{f}^{-n}(\xi)} \left| (\tilde{f}^n)'(x) \right|^{-t}.$$

Then by convexity, the function  $\mathbb{R} \ni t \mapsto \underline{P}_V^\Xi(t) \in \mathbb{R}$  is continuous, and it follows from Lemma 8.1 that there exists  $t \in (0, h)$  such that

$$(16.2) \quad \underline{P}_V^\Xi(t) < 0.$$

By Lemma 8.1, for every  $\xi \in \Xi$ , every integer  $n \geq 1$ , and every  $x \in \tilde{f}^{-n}(\xi) \cap K(V)$  there exists a unique holomorphic branch  $\tilde{f}_x^{-n} : B(\xi, 2R_2) \rightarrow \Sigma_u \times \hat{\mathbb{C}}$  which sends  $\xi$  to  $x$ . It now follows from (16.1) and (16.2) that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\xi \in \Xi} \sum_{x \in K(V) \cap \tilde{f}^{-n}(\xi)} \text{diam}^t(p_2(\tilde{f}_x^{-n}(B(\xi, R_2)))) = \underline{P}_V^\Xi(t) < 0.$$

Therefore,

$$\liminf_{n \rightarrow \infty} \sum_{\xi \in \Xi} \sum_{x \in K(V) \cap \tilde{f}^{-n}(\xi)} \text{diam}^t(p_2(\tilde{f}_x^{-n}(B(\xi, R_2)))) = 0.$$

Since also

$$\bigcup_{\xi \in \Xi} \bigcup_{x \in K(V) \cap \tilde{f}^{-n}(\xi)} p_2(\tilde{f}_x^{-n}(B(\xi, R_2))) \supset K(V),$$

we thus conclude that  $H_t(p_2(K(V))) = 0$ . Hence,

$$\text{HD}(p_2(K(V))) \leq t < h,$$

and the proof is complete.  $\square$

So, now we shall prove the following.

**Theorem 16.2.** *If  $G$  is a C–F balanced TNR rational semigroup type generated by a rational multi–map  $f = (f_1, \dots, f_u) \in \text{Rat}^u$  and satisfying the Nice Open Set Condition, then*

$$\text{HD}(J_\omega) < h = \text{HD}(J(G))$$

for every  $\omega \in \Sigma_u$ . If in addition,  $G$  is expanding, then

$$\sup \{ \text{HD}(J_\omega) : \omega \in \Sigma_u \} < h = \text{HD}(J(G)).$$

*Proof.* Fix  $\mathcal{U} = \{U_s\}_{s \in S}$ , an arbitrary nice family for  $f$ . For every  $\rho \in \Sigma_u^* \cup \Sigma_u$ , let

$$[\rho]^\sim := \{ \omega \in \mathcal{D}_U^\infty : \tilde{\omega}|_{[\rho]} = \rho \}.$$

Still for every integer  $n \geq 0$  and also for every  $\tau \in \Sigma_u$ , we have

$$(16.3) \quad J_\tau(\mathcal{U}) := p_2(\pi_U([\tau]^\sim)) \subset p_2 \circ \pi_U \left( \bigcup \{ [\omega|_n] : \omega \in [\tau|_n]^\sim \} \right)$$

$$= p_2 \circ \pi_U \left( \bigcup \{ [\omega] : \omega \in \mathcal{D}_U^n \text{ and } \tilde{\omega}|_n = \tau|_n \} \right)$$

$$(16.4) \quad \subset \bigcup \{ p_2(\phi_\omega(X_{t(\omega)})) : \omega \in [\tau|_n]^\sim|_n \}.$$

Now, fixing for every  $s \in X$ ,  $\xi_s \in \mathcal{D}_U^\infty$  such that  $\pi_U(\xi_s) \in U_s \subset X_s$ , we have

$$(16.5) \quad \sum_{\omega \in [\tau|_n]^\sim|_n} \text{diam}^h(p_2(\phi_\omega(X_{t(\omega)}))) \asymp \sum_{\omega \in [\tau|_n]^\sim|_n} |\phi'_\omega(\pi_U(\xi_{t(\omega)}))|^h = \sum_{s \in S} \mathcal{L}_h^n(\mathbb{1}_{[\tau|_n]^\sim})(\xi_s),$$

where  $\mathcal{L}_h : C_b(\mathcal{D}_U^\infty) \rightarrow C_b(\mathcal{D}_U^\infty)$  is the Perron–Frobenius (transfer) operator corresponding to the potential

$$\zeta_{h, \text{P}(h)} = \zeta_{h, 0} : \mathcal{D}_U^\infty \longrightarrow \mathbb{R}.$$

This operator is given by the formula

$$(16.6) \quad \mathcal{L}_h(g)(\xi) = \sum_{\substack{e \in \mathcal{D}_U \\ A_{e, \xi_1}(\mathcal{U})=1}} e^{\zeta_{h, 0}(e\xi)} g(e\xi) = \sum_{\substack{e \in \mathcal{D}_U \\ A_{e, \xi_1}(\mathcal{U})=1}} |\phi'_e(\pi_U(\xi))|^h g(e\xi).$$

We want to show that the functions  $\mathcal{L}_h(\mathbb{1}_{[\tau|_n]^\sim})$ ,  $n \geq 1$ , converge to zero uniformly exponentially fast. For this we want to use the existence of a spectral gap for the operator  $\mathcal{L}_h$ . For the operator  $\mathcal{L}_h$  acting on the Banach space of bounded Hölder continuous functions from  $\mathcal{D}_U^\infty$  to  $\mathbb{R}$ , this was proved in [19]. This is however not sufficient for us as the Hölder norms of the characteristic functions of cylinders increase to infinity (even exponentially fast) with the length of the cylinder. We therefore turn our attention to a more sophisticated Banach space and apply the results of [24]. We proceed slightly more generally than is really needed for the sake of arguments of this section. Namely except restricting ourselves to the parameter  $h$ , we fix, and we deal with, any parameter

$$t \in \Delta_G^*.$$

Given a function  $g \in L^1(\tilde{\mu}_t)$  and an integer  $m \geq 0$ , we define the *oscillation function*  $\text{osc}_m(g) : \mathcal{D}_U^\infty \rightarrow [0, \infty)$  by the following formula:

$$(16.7) \quad \text{osc}_m(g)(\omega) := \text{ess sup} \{ |g(\alpha) - g(\beta)| : \alpha, \beta \in [\omega|_m] \}$$

and

$$\text{osc}_0(g) := \text{ess sup}(g) - \text{ess inf}(g).$$

We further define:

$$(16.8) \quad |g|_{t,\theta} := \sup_{m \geq 0} \{\theta^{-m} \|\text{osc}_m(g)\|_1\},$$

where  $\|\cdot\|$  denotes the  $L^1$ -norm with respect to the measure  $\tilde{\mu}_t$ . The announced, non-standard (it even depends on the dynamics – via  $\tilde{\mu}_t$ ) Banach space is defined as follows:

$$\mathcal{B}_\theta(t) := \{g \in L^1(\tilde{\mu}_t) : |g|_\theta < +\infty\}$$

and we denote

$$(16.9) \quad \|g\|_{t,\theta} := \|g\|_1 + |g|_{t,\theta}.$$

Of course  $\mathcal{B}_\theta(t)$  is a vector space and the function

$$(16.10) \quad \mathcal{B}_\theta(t) \ni g \longmapsto \|g\|_{t,\theta}$$

is a norm on  $\mathcal{B}_\theta(t)$ . This is the non-standard Banach space we will be working with now.

A direct observation shows that for every  $\tau \in \Sigma_u$  and every integer  $n \geq 0$ , we have:

$$(16.11) \quad \text{osc}_k(\mathbb{1}_{[\tau|_n]^\sim})(\omega) = \begin{cases} 0 & \text{if } k \geq n \\ 0 & \text{if } k < n \text{ and } \tilde{\omega}|_k \neq \tau|_k \\ 1 & \text{if } k < n \text{ and } \tilde{\omega}|_k = \tau|_k. \end{cases}$$

Therefore,

$$(16.12) \quad \int_{\mathcal{D}_u^\infty} \text{osc}_k(\mathbb{1}_{[\tau|_n]^\sim})(\omega) d\tilde{\mu}_t(\omega) = \begin{cases} 0 & \text{if } k \geq n \\ \tilde{\mu}_t([\tau|_k]^\sim) & \text{if } k < n. \end{cases}$$

Thus

$$(16.13) \quad |\mathbb{1}_{[\tau|_n]^\sim}|_{t,\theta} = \max \{\theta^{-k} \tilde{\mu}_t([\tau|_k]^\sim) : 0 \leq k < n\}.$$

We want to show that this number is uniformly bounded above. For this, we need a good upper estimate on  $\tilde{\mu}_t([\tau|_k]^\sim)$ . We will prove it now.

**Claim 1:** We have that

$$\lim_{k \rightarrow \infty} \max \{\omega \in \Sigma_u^k : \tilde{\mu}_t([\omega]^\sim)\} = 0.$$

*Proof.* For every  $\omega \in \Sigma_u$  we have  $[\omega]^\sim \subset \pi_{\mathcal{U}}^{-1}(\{\omega\} \times J_\omega)$ , whence, by Lemma 10.7, we get that

$$(16.14) \quad \tilde{\mu}_t([\omega]^\sim) \leq \tilde{\mu}_t(\pi_{\mathcal{U}}^{-1}(\{\omega\} \times J_\omega(\mathcal{U}))) = \mu_t(\{\omega\} \times J_\omega(\mathcal{U})) \leq \mu_t(\{\omega\} \times J_\omega).$$

Now, if  $\omega$  is not eventually periodic, then the sets  $\{\tilde{f}^n(\{\omega\} \times J_\omega)\}_{n=0}^\infty$  are mutually disjoint. Since also

$$\mu_t(\tilde{f}^{n+1}(\{\omega\} \times J_\omega)) \geq \mu_t(\tilde{f}^n(\{\omega\} \times J_\omega))$$

and since  $\mu_t(J(\tilde{f})) = 1 < +\infty$ , we conclude that

$$(16.15) \quad \mu_t(\{\omega\} \times J_\omega) = 0.$$

If  $\omega$  is eventually periodic, then  $\sigma^q(\omega)$  with some  $q \geq 0$  is periodic, and if  $\mu_t(\{\omega\} \times J_\omega) > 0$ , then by (16.15),

$$\mu_t(\{\sigma^q(\omega)\} \times J_{\sigma^q(\omega)}) = \mu_t(\tilde{f}^q(\{\omega\} \times J_\omega)) > 0.$$

So, if  $p \geq 1$  is a period of  $\sigma^q(\omega)$ , then

$$\mu_t\left(\bigcup_{j=0}^{p-1} \tilde{f}^j(\{\sigma^q(\omega)\} \times J_{\sigma^q(\omega)})\right) > 0,$$

and

$$\tilde{f}\left(\bigcup_{j=0}^{p-1} \tilde{f}^j(\{\sigma^q(\omega)\} \times J_{\sigma^q(\omega)})\right) = \bigcup_{j=0}^{p-1} \tilde{f}^{j+1}(\{\sigma^q(\omega)\} \times J_{\sigma^q(\omega)}).$$

Hence, the ergodicity of the measure  $\mu_t$  yields

$$(16.16) \quad \mu_t\left(\bigcup_{j=0}^{p-1} \tilde{f}^j(\{\sigma^q(\omega)\} \times J_{\sigma^q(\omega)})\right) = 1.$$

But,

$$\bigcup_{j=0}^{p-1} \tilde{f}^j(\{\sigma^q(\omega)\} \times J_{\sigma^q(\omega)})$$

is a closed proper subset of  $J(\tilde{f})$ . So, (16.16) contradicts the fact that  $\text{supp}(\mu_t) = J(\tilde{f})$ , i.e. that  $\tilde{\mu}_t$  is a positive on non-empty open subsets of  $J(\tilde{f})$ . In consequence,

$$\mu_t(\{\omega\} \times J_\omega) = 0.$$

Along with (16.15) and (16.14) this implies that

$$(16.17) \quad \tilde{\mu}_t([\omega]^\sim) = 0.$$

Now seeking a contradiction, suppose that Claim 1 fails, i.e. that

$$\eta := \frac{1}{2} \limsup_{k \rightarrow \infty} \max \{ \tilde{\mu}_t([\omega]^\sim) : \omega \in \Sigma_u^k \} \in (0, 1/2).$$

Then for every  $k \geq 0$ , the family

$$\mathcal{F}_k := \{ \omega \in \Sigma_u^k : \tilde{\mu}_t([\omega]^\sim) > \eta \}$$

is not empty and for every  $\omega \in \Sigma_u^{k+1}$ ,  $\omega|_k \in \Sigma_u^k$ . This means that the families  $(\mathcal{F}_k)_{k \geq 0}$  form a tree rooted at  $\mathcal{F}_0 = \{\emptyset\}$ . Since this tree is finitely branched, the number of branches outgoing from each vertex being (uniformly) bounded above by  $u$ , it follows from König's Lemma that there exists  $\omega \in \Sigma_u$  such that  $\omega|_k \in \mathcal{F}_k$  for all  $k \geq 0$ . Hence,

$$\tilde{\mu}_t([\omega|_k]^\sim) > \eta$$

for all  $k \geq 0$ . Thus,

$$\tilde{\mu}_t([\omega]^\sim) = \lim_{k \rightarrow \infty} \tilde{\mu}_t([\omega|_k]^\sim) \geq \eta,$$

contrary to (16.17). Thus, Claim 1 is proved.  $\square$

Now we shall prove a substantial strengthening of Claim 1.

**Claim 2:** There exist two constants  $A_t > 0$  and  $\alpha_t \in (0, 1)$  such that for every  $k \geq 0$

$$\max \{ \tilde{\mu}_t([\omega]^\sim) : \omega \in \Sigma_u^k \} \leq A_t \alpha_t^k.$$

*Proof.* It directly follows from Theorem 9.4 (a) that there exists a constant  $B \geq 1$  such that

$$(16.18) \quad \tilde{\mu}_t([\gamma\xi]) \leq B \tilde{\mu}_t([\gamma]) \tilde{\mu}_t([\xi])$$

for all  $\gamma, \xi \in \mathcal{D}_u^\infty$ . By Claim 1 there exists  $q \geq 1$  so large that

$$\beta := \max \{ \tilde{\mu}_t([\omega]^\sim) : \omega \in \Sigma_u^q \} < B^{-1}.$$

Put

$$s := B\beta < 1.$$

We shall show by induction that

$$(16.19) \quad \beta_k := \max \{ \tilde{\mu}_t([\omega]^\sim) : \omega \in \Sigma_u^{qk} \} \leq s^k$$

for every  $k \geq 0$ . Indeed for  $k = 0$ , we have  $1=1$ . So, suppose that (16.19) holds for some  $k \geq 0$ . Using (16.18), we then get for every  $\omega \in \Sigma_u^{q(k+1)}$  that

$$\begin{aligned} \tilde{\mu}_t([\omega]^\sim) &= \tilde{\mu}_t \left( \bigcup \left\{ \gamma\xi : \gamma \in [\omega|_{qk}]^\sim, \xi \in [\sigma^{qk}(\omega)]^\sim, \gamma\xi \in \mathcal{D}_u^{q(k+1)} \right\} \right) \\ &= \sum_* \tilde{\mu}_t([\gamma\xi]) \leq B \sum_* \tilde{\mu}_t([\gamma]) \mu_t([\xi]) \\ &\leq B \tilde{\mu}_t([\xi]) \sum_{\gamma \in [\omega|_{qk}]^\sim} \tilde{\mu}_t([\gamma]) \sum_{\xi \in [\sigma^{qk}(\omega)]^\sim} = B \tilde{\mu}_t([\omega|_{qk}]^\sim) \tilde{\mu}_t([\sigma^{qk}(\omega)]^\sim) \\ &\leq B s^k \beta = s^{k+1}, \end{aligned}$$

where  $\sum_*$  indicates the summation over the set

$$\left\{ \gamma\xi : \gamma \in [\omega|_{qk}]^\sim, \xi \in [\sigma^{qk}(\omega)]^\sim, \gamma\xi \in \mathcal{D}_u^{q(k+1)} \right\}.$$

So,  $\beta_{k+1} \leq s^{k+1}$ , and formula (16.19) is proved. This formula directly entails Claim 2.  $\square$

As indicated in [24], for the results of this paper one can take  $\theta \in (0, 1)$  as close to 1 as one wishes. We take an arbitrary

$$\theta \in (\alpha_t, 1).$$

It then follows from 16.13 and Claim 2 that

$$(16.20) \quad |\mathbb{1}_{[\tau|_n]^\sim}|_{t,\theta} \leq A_t$$

for every  $\tau \in \Sigma_u$  and every integer  $n \geq 0$ . Now we want to apply some results from [24]; for this, strictly speaking, we need to consider the normalized operator

$$(16.21) \quad \mathcal{L}_{h,0} := \frac{1}{\rho_h} \mathcal{L}_h \circ \rho_h.$$



It formally follows from Proposition 2.4.2 (Fundamental Perturbative Result) of [24], with the sequence  $U_n = \emptyset$  for all  $n \geq 1$ , (in fact it is much easier than the full fledged Proposition 2.4.2) that

$$(16.22) \quad \mathcal{L}_{h,0}^n g := \tilde{\mu}_h(g) + \Delta_h^n g,$$

for all  $n \geq 0$  and all  $g \in \mathcal{B}_\theta(h)$ , where

$$\|\Delta_h^n g\|_\infty \leq \|\Delta_h^n g\|_{h,\theta} \leq C_h \kappa_h^n \|g\|_{h,\theta}$$

for all  $n \geq 0$  with some constants  $C_h \in (0, +\infty)$  and  $\kappa_h \in (0, 1)$ . Fix  $\tau \in \Sigma_u$  arbitrarily. It directly follows from (16.20)–(16.22), Theorem 9.4 (e), and Claim 2 that with some constant  $D \in [1, +\infty)$ , we have for every integer  $n \geq 1$  that

$$(16.23) \quad \begin{aligned} \|\mathcal{L}_h^n(\mathbb{1}_{[\tau|_n]^\sim})\|_\infty &\leq \tilde{\mu}_h([\tau|_n]^\sim) + \|\Delta_h^n \mathbb{1}_{[\tau|_n]^\sim}\|_\infty \leq A_h \alpha_h^n + C_h \kappa_h^n A_h \\ &\leq A_h(1 + C_h) \max(\alpha_h^n, \kappa_h^n). \end{aligned}$$

We now want some generalizations of formulas (16.5) and (16.6).

Keep  $\tau \in \Sigma_u$  fixed. For every  $t \in \Delta_G^*$ , let

$$\underline{P}_\tau(t) := \liminf_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{L}_t^n(\mathbb{1}_{[\tau|_n]^\sim})\|_\infty.$$

Invoking the last assertion of Corollary 9.8 and making use of the convexity argument, we conclude that the function

$$(h - \delta, h) \ni t \longmapsto \underline{P}_\tau(t) \in \mathbb{R}$$

is convex, thus continuous. Since also by (16.23),

$$(16.24) \quad \underline{P}_\tau(h) \leq \log \max(\alpha_h, \kappa_h) < 0,$$

it thus follows that there exists  $\alpha \in (h - \delta, h)$  such that

$$(16.25) \quad \underline{P}_\tau(\alpha) < 0.$$

Since formula (16.5) holds with  $h$  replaced by any  $t \geq 0$ , in particular by  $\alpha$ , and  $\mathcal{L}_h$  replaced by  $\mathcal{L}_t$ , the Perron–Frobenius operator corresponding to the potential  $\zeta_{t,0} : \mathcal{D}_U^\infty \rightarrow \mathbb{R}$ , we conclude, using (16.25), that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega \in [\tau|_n]^\sim|_n} \text{diam}^\alpha(p_2(\phi_\omega(X_{t(\omega)}))) \leq \underline{P}_\tau(\alpha) < 0.$$

Therefore,

$$\liminf_{n \rightarrow \infty} \sum_{\omega \in [\tau|_n]^\sim|_n} \text{diam}^\alpha(p_2(\phi_\omega(X_{t(\omega)}))) = 0.$$

Hence, invoking also (16.3), we conclude that  $H_\alpha(\phi_\omega(X_{t(\omega)})) = 0$ . Thus,

$$(16.26) \quad \text{HD}(J_\tau(\mathcal{U})) \leq \alpha < h.$$

Now we pass to the second step of the proof. Since the map  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  is topologically exact, there exists an integer  $q \geq 1$  such that

$$(16.27) \quad \tilde{f}^q(U_s) \supset J(\tilde{f})$$

for all  $s \in S$ . The key technical part of the second step is the following.

**Claim 3:** Keeping  $\tau \in \Sigma_u$ , we have for every integer  $n \geq q$  that

$$J_{\sigma^n(\tau)}(\mathcal{U}) \subset \bigcup_{\omega \in \Sigma_u^q} f_{\omega\tau|_{q+1}}^n(J_{\omega\sigma^q(\tau)}(\mathcal{U})).$$

*Proof.* Fix  $s \in S$  and  $n \geq q$ . Take an arbitrary  $\xi \in [\sigma^n(\tau)]^\sim$ . Then, by the definition of the system  $\mathcal{S}_u$ , there exists  $\rho = \tilde{f}_\rho^{-(n-q)} : B(X_{i(\tau_{n+1})}, R) \longrightarrow \Sigma_u \times \hat{\mathbb{C}}$ , a holomorphic inverse branch of  $\tilde{f}^{n-q}$  such that

$$(16.28) \quad \tilde{\rho} = \tau|_{q+1}^n.$$

By (16.27) there exists  $z \in U_s$  such that

$$\tilde{f}^q(z) = \tilde{f}_\rho^{-(n-q)}(\pi_{\mathcal{U}}(\xi)).$$

By the definition of the system  $\mathcal{S}_u$  again, there exists

$$\gamma = \tilde{f}_\gamma^{-q} : \tilde{f}_\rho^{-(n-q)}(B(X_{i(\tau_{n+1})}, R)) \longrightarrow \Sigma_u \times \hat{\mathbb{C}},$$

a holomorphic inverse branch of  $\tilde{f}^q$  such that  $\tilde{f}_\gamma^{-q}(\tilde{f}^q(z)) = z$ . But, as  $z \in U_s$ , we then have that

$$\tilde{f}_\gamma^{-q} \circ \tilde{f}_\rho^{-(n-q)}(X_{i(\tau_{n+1})}) \cap U_s \neq \emptyset.$$

But then, by the definition of nice sets and the system  $\zeta$  for the final time,  $\gamma\rho \in \mathcal{D}_u^*$ . Hence,

$$\gamma\rho\xi \in \mathcal{D}_u^\infty.$$

In addition,

$$(16.29) \quad \tilde{\gamma} \in \Sigma_u^q.$$

Therefore, using (16.28) and the choice of  $\xi$ , we obtain

$$\begin{aligned} p_2 \circ \pi_{\mathcal{U}}(\xi) &= p_2 \circ \pi_{\mathcal{U}}(\sigma^{|\gamma\rho|}(\gamma\rho\xi)) \\ &= f_{\tilde{\gamma}\tilde{\rho}}(p_2 \circ \pi_{\mathcal{U}}(\gamma\rho\xi)) \in f_{\tilde{\gamma}\tilde{\rho}}(p_2 \circ \pi_{\mathcal{U}}([\tilde{\gamma}\sigma^q(\tau)]^\sim)) = f_{\tilde{\gamma}\tilde{\rho}}(J_{\tilde{\gamma}\sigma^q(\tau)}(\mathcal{U})). \end{aligned}$$

Invoking (16.28) again, and also (16.29), we conclude that

$$p_2 \circ \pi_{\mathcal{U}}(\xi) \in \bigcup_{\omega \in \Sigma_u^q} f_{\omega\tau|_{q+1}}^n(J_{\omega\sigma^q(\tau)}(\mathcal{U})).$$

Claim 3 is thus proved.  $\square$

Since the collection  $\{\omega\tau|_{q+1}^n : \omega \in \Sigma_u^q\}$  is finite, as an immediate consequence of Claim 3 and (16.26), we get that

$$(16.30) \quad \sup \{\text{HD}(J_{\sigma^n(\tau)}(\mathcal{U})) : n \geq 0\} < h.$$

Now observe that each point  $z \in U_s$  such that  $\tilde{f}^n(z) \in U_s$  for infinitely many  $n \geq 0$ , belongs in fact to  $J_{\mathcal{S}_u}$ . So, for each point  $x$  of

$$J_\tau \setminus \bigcup_{n=0}^{\infty} f_{\tau|_n}^{-1}(J_{\sigma^n(\tau)}(\mathcal{U}))$$

there are only finitely many integers  $n \geq 0$  such that  $\tilde{f}^n(\tau, x) \in U_s$ . Therefore,

$$J_\tau \setminus \bigcup_{n=0}^{\infty} f_{\tau|n}^{-1}(J_{\sigma^n(\tau)}(\mathcal{U})) \subset \bigcup_{k=0}^{\infty} f_{\tau|k}^{-1}(p_2(K(U_s))).$$

Hence, combining Lemma 16.1 and (16.30), we conclude that

$$\text{HD}(J_\tau) < h.$$

So, the first part of Theorem 16.2 is proved.

Now, we pass to the expanding case. Then the dynamical system  $\tilde{f} : J(\tilde{f}) \rightarrow J(\tilde{f})$  admits (finite!) Markov partitions with arbitrarily small diameters. In the language of the present paper it means that there exists a nice family (in fact nice sets with arbitrarily small diameters)  $\mathcal{U} = \{U_s\}_{s \in S}$  such that

$$\bigcup_{s \in S} U_s \supset J(\tilde{f}),$$

the alphabet  $\mathcal{D}_\mathcal{U}$  of the corresponding GDMS  $\mathcal{S}_\mathcal{U}$  is finite, and all elements  $\phi_e$  of  $\mathcal{S}_\mathcal{U}$  are holomorphic inverse branches of generating maps  $f_j$ ,  $j = 1, \dots, u$ . Moreover, for every  $\tau \in \Sigma_u$ ,

$$J_\tau = J_\tau(\mathcal{U}).$$

Having  $t \geq 0$ ,  $\tau \in \Sigma_u$ , and  $n \geq 1$ , we put

$$P_\tau(t, n) := \frac{1}{n} \log \|\mathcal{L}_h^n(\mathbb{1}_{[\tau|n]^\sim})\|_\infty.$$

But in the current expanding case,

$$\|\mathcal{L}_h^n(\mathbb{1}_{[\tau|n]^\sim})\|_\infty = \sum_{\xi \in [\tau|n]^\sim} |\phi'_\xi(x_\tau(n))|^t$$

for some  $x_\tau(n) \in J(\tilde{f})$ . So, then

$$P'_\tau(t, n) = \frac{1}{n} \cdot \frac{\sum_{\xi \in [\tau|n]^\sim} |\phi'_\xi(x_\tau(n))|^t \log |\phi'_\xi(x_\tau(n))|^t}{\sum_{\xi \in [\tau|n]^\sim} |\phi'_\xi(x_\tau(n))|^t},$$

and in our special case there exists  $\chi > 0$  such that

$$|\phi'_\xi(x)| \geq \exp(-\chi|\xi|)$$

for every  $\xi \in \mathcal{D}_\mathcal{U}^*$  and every  $x \in X_{t(\xi)}$ . This entails

$$P'_\tau(t, n) \geq -\chi t.$$

So, denoting  $\text{HD}(J_\tau)$  by  $h_\tau$ , we get for every  $n \geq 1$  that

$$P_\tau(h, n) - P_\tau(h_\tau, n) \geq -\chi(h - h_\tau).$$

Therefore,

$$h - h_\tau \geq \frac{P_\tau(h_\tau, n) - P_\tau(h, n)}{\chi}.$$

Hence, noting also that  $\underline{P}_\tau(h_\tau) \geq 0$ , using (16.24), and taking the limit as  $n \rightarrow \infty$ , we obtain

$$h - h_\tau \geq \frac{1}{\chi} (\underline{P}_\tau(h_\tau) - P_\tau(h)) \geq -\frac{P_\tau(h)}{\chi} \geq -\frac{1}{\chi} \log \max\{\alpha_h, \kappa_h\} > 0.$$

The proof of the second part of Theorem 16.2 is complete, and we are done.  $\square$

## 17. EXAMPLES

In this section, we describe some examples of semi-hyperbolic rational semigroups with the Nice Open Set Condition.

**Example 17.1** ([46], comp. [32, 34]). *Let*

$$f_1(z) := z^2 + 2 \quad \text{and} \quad f_2(z) := z^2 - 2.$$

*Let*  $f = (f_1, f_2)$  *and*

$$G := \langle f_1, f_2 \rangle.$$

*In addition, let*

$$U := \{z \in \mathbb{C} : |z| < 2\}.$$

*Then,  $G$  is semi-hyperbolic but not hyperbolic (so, not expanding) ([32, Example 5.8]). Since  $\text{Crit}_*(f) = \{0\}$  is a singleton, it follows from Observation 5.4 that  $G$  is a TNR semigroup. Moreover,  $G$  satisfies the Nice Open Set Condition with  $U$ . Since*

$$J(G) \subset f_1^{-1}(\overline{U}) \cup f_2^{-1}(\overline{U}) \subsetneq \overline{U},$$

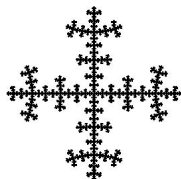
[34, Theorem 1.25] *implies that  $J(G)$  is porous and  $\text{HD}(J(G)) < 2$ . It is also easy to see that the multi-map  $f$  is C-F balanced. It follows from Lemma 13.11 that  $G$  is of finite type. So,  $G$  satisfies all our requirements imposed on a rational semigroup throughout the manuscript. In addition, by Theorem 1.1 we have that*

$$h_f = \text{HD}(J(G)) = \text{PD}(J(G)) = \text{BD}(J(G)).$$

*Furthermore (see Figure 1),*

$$f_1^{-1}(\overline{U}) \cap f_2^{-1}(\overline{U}) \neq \emptyset.$$

FIGURE 1. The Julia set of  $\langle f_1, f_2 \rangle$ , where  $f_1(z) = z^2 + 2$ ,  $f_2(z) = z^2 - 2$ .



**Example 17.2.** *A large class of examples is provided by the following.*

**Proposition 17.3** ([46], comp. [37, 40]). *Assume the following.*

- Let  $f_1$  be a semi-hyperbolic polynomial with  $\deg(f_1) \geq 2$  such that  $J(f_1)$  is connected.
- Let  $K(f_1)$  be the filled-in Julia set of  $f_1$  and suppose that  $\text{Int}(K(f_1)) \neq \emptyset$ .
- Let  $b \in \text{Int}(K(f_1))$  be a point.
- Let  $d$  be a positive integer such that  $d \geq 2$ .
- Suppose that  $(\deg(f_1), d) \neq (2, 2)$ .

Then, there exists a real number  $c > 0$  such that for each  $\lambda \in \{\lambda \in \mathbb{C} : 0 < |\lambda| < c\}$ , setting

$$f_\lambda := (f_{\lambda,1}, f_{\lambda,2}) = (f_1, \lambda(z - b)^d + b) \quad \text{and} \quad G_\lambda := \langle f_1, f_{\lambda,2} \rangle,$$

we have the following.

- (1)  $G_\lambda$  is \*semi-hyperbolic.
- (2)  $f_\lambda$  satisfies the Nice Open Set Condition with an open set  $U_\lambda$ .
- (3)  $J(G_\lambda)$  is porous.
- (4)  $\text{HD}(J(G_\lambda)) = h_{f_\lambda} < 2$ .
- (5)  $\text{PCV}(G_\lambda) \setminus \{\infty\}$  is bounded in  $\mathbb{C}$ .
- (6)  $G_\lambda$  is C-F balanced.
- (7) If, in addition,  $f_1$  is totally non-recurrent TNR, then so is  $G_\lambda$ , and it is of finite type.

*Proof.* We will closely follow the exposition from [46]. Conjugating  $f_1$  by a Möbius transformation, we may assume that  $b = 0$  and that the coefficient of the highest degree term of  $f_1$  is equal to 1.

Let  $r > 0$  be a real number such that  $\overline{B(0, r)} \subset \text{Int}K(f_1)$ . We set  $d_1 := \deg(f_1)$ . Let  $\alpha > 0$  be a number. Since  $d \geq 2$  and  $(d, d_1) \neq (2, 2)$ , it is easy to see that

$$\left(\frac{r}{\alpha}\right)^{\frac{1}{d}} > 2 \left(2 \left(\frac{1}{\alpha}\right)^{\frac{1}{d-1}}\right)^{\frac{1}{d_1}}$$

if and only if

$$(17.1) \quad \log \alpha < \frac{d(d-1)d_1}{d+d_1-d_1d} \left( \log 2 - \frac{1}{d_1} \log \frac{1}{2} - \frac{1}{d} \log r \right).$$

We set

$$(17.2) \quad c_0 := \exp \left( \frac{d(d-1)d_1}{d+d_1-d_1d} \left( \log 2 - \frac{1}{d_1} \log \frac{1}{2} - \frac{1}{d} \log r \right) \right) \in (0, \infty).$$

Let  $0 < c < c_0$  be a number required to be sufficiently small later in the proof and let  $\lambda \in \mathbb{C}$  be a number with  $0 < |\lambda| < c$ . Put

$$f_{\lambda,2}(z) := \lambda z^d.$$

Then, we have

$$K(f_{\lambda,2}) = \left\{ z \in \mathbb{C} : |z| \leq \left( \frac{1}{|\lambda|} \right)^{\frac{1}{d-1}} \right\}$$

and

$$f_{\lambda,2}^{-1}(\{z \in \mathbb{C} : |z| = r\}) = \left\{ z \in \mathbb{C} : |z| = \left( \frac{r}{|\lambda|} \right)^{\frac{1}{d}} \right\}.$$

Let

$$D_\lambda := \overline{B \left( 0, 2 \left( \frac{1}{|\lambda|} \right)^{\frac{1}{d-1}} \right)}.$$

Since

$$f_1(z) = z^{d_1}(1 + o(1)) \quad \text{as } z \rightarrow \infty,$$

it follows that if  $0 < c < c_0$  is small enough, then for every  $\lambda \in \mathbb{C}$  with  $0 < |\lambda| < c$ , we have that

$$f_1^{-1}(D_\lambda) \subset \left\{ z \in \mathbb{C} : |z| \leq 2 \left( 2 \left( \frac{1}{|\lambda|} \right)^{\frac{1}{d-1}} \right)^{\frac{1}{d_1}} \right\}.$$

This implies that

$$(17.3) \quad f_1^{-1}(D_\lambda) \subset f_{\lambda,2}^{-1}(\{z \in \mathbb{C} : |z| < r\}).$$

Hence, setting

$$U_\lambda := \text{Int}(K(f_{\lambda,2})) \setminus K(f_1),$$

we will have

$$f_1^{-1}(U_\lambda) \cup f_{\lambda,2}^{-1}(U_\lambda) \subset U_\lambda \quad \text{and} \quad f_1^{-1}(U_\lambda) \cap f_{\lambda,2}^{-1}(U_\lambda) = \emptyset.$$

Furthermore, since  $f_1$  is semi-hyperbolic, we get that  $\hat{\mathbb{C}} \setminus K(f_1)$  is a John domain by [2]. Hence,  $U_\lambda$  satisfies (osc3). Therefore,  $G_\lambda$  satisfies the Nice Open Set Condition witnessed by  $U_\lambda$ .

We have

$$J(G_\lambda) \subset \overline{U_\lambda} \subset K(f_{\lambda,2}) \setminus \text{Int}(K(f_1)).$$

In particular,

$$\text{Int}(K(f_1)) \cup (\hat{\mathbb{C}} \setminus K(f_{\lambda,2})) \subset F(G_\lambda).$$

Furthermore, (17.3) implies that  $f_{\lambda,2}(K(f_1)) \subset \text{Int}(K(f_1))$ . Thus, we have

$$\text{PCV}(G_\lambda) \setminus \{\infty\} = \bigcup_{g \in G_\lambda^*} g(\text{CritV}^*(f_1) \cup \text{CritV}^*(f_{\lambda,2})) \subset K(f_1),$$

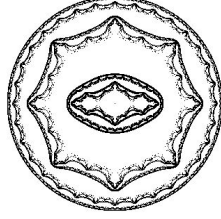
where  $\text{CritV}^*(\cdot)$  denotes the set of all critical values in  $\mathbb{C}$ . Hence,  $\text{PCV}(G_\lambda) \setminus \{\infty\}$  is bounded in  $\mathbb{C}$ .

Since  $f_1$  is semi-hyperbolic, there exist an  $N \in \mathbb{N}$  and a  $\delta_1 > 0$  such that for each  $x \in J(f_1)$  and for each  $n \in \mathbb{N}$ ,

$$\deg(f_1^n : V \rightarrow B(x, \delta_1)) \leq N$$

for each connected component  $V$  of  $f_1^{-n}(B(x, \delta_1))$ . Also,  $f_{\lambda,2}^{-1}(J(f_1)) \cap K(f_1) = \emptyset$ , and so

$$f_{\lambda,2}^{-1}(J(f_1)) \subset \hat{\mathbb{C}} \setminus \text{PCV}(G_\lambda).$$

FIGURE 2. The Julia set of  $\langle f_1^2, f_2^2 \rangle$ , where  $f_1(z) = z^2 - 1$ ,  $f_2(z) = z^2/4$ .

It follows from the above that there exists a  $0 < \delta_2 < \delta_1$  such that for each  $x \in J(f_1)$  and each  $g \in G_\lambda$ ,

$$\deg(g : V \longrightarrow B(x, \delta_2)) \leq N$$

for each connected component  $V$  of  $g^{-1}(B(x, \delta_2))$ . Since  $\text{PCV}(G_\lambda) \setminus \{\infty\} \subset K(f_1)$  again, we obtain that there exists a number  $0 < \delta_3 < \delta_2$  such that for each  $x \in J(G_\lambda)$  and each  $g \in G_\lambda$ ,

$$\deg(g : V \longrightarrow B(x, \delta_3)) \leq N$$

for each connected component  $V$  of  $g^{-1}(B(x, \delta_3))$ . Thus,  $G_\lambda$  is semi-hyperbolic.

Since  $J(G_\lambda) \subset f_1^{-1}(\overline{U_\lambda}) \cup f_{\lambda,2}^{-1}(\overline{U_\lambda}) \subsetneq \overline{U_\lambda}$ , [34] implies that  $J(G_\lambda)$  is porous and

$$\text{HD}(J(G_\lambda)) < 2.$$

Also, by Theorem 1.1, we have that

$$h_{f_\lambda} = \text{HD}(J(G_\lambda)).$$

Finally, it is easy to see that item (7) holds, and we are done.  $\square$

## 18. APPENDIX 1; ABSOLUTELY CONTINUOUS $\sigma$ -FINITE INVARIANT MEASURES: MARCO-MARTENS APPROACH

In this appendix we state again Theorem 4.19 and provide its full proof. As we have already explained in the paragraph preceding Definition 4.17, we do this (provide the proof) for the sake of completeness, the importance of Theorem 4.19 for our overall approach, and since there is no published proof of this theorem in so big generality (see [14] and [23]).

**Theorem 18.1.** *If  $(X, \mathcal{B}, m)$  is a probability space and  $T : X \longrightarrow X$  is a Marco-Martens map with a Marco-Martens cover  $\{X_j\}_{j=0}^\infty$ , then we have the following.*

- *There exists a  $\sigma$ -finite  $T$ -invariant measure  $\mu$  on  $X$  equivalent to  $m$ .*
- *In addition,  $0 < \mu(X_j) < +\infty$  for each  $j \geq 0$ .*

*The measure  $\mu$  is constructed in the following way: Let  $l_B : l^\infty \longrightarrow \mathbb{R}$  be a Banach limit. For each  $A \in \mathcal{B}$ , set*

$$m_n(A) := \frac{\sum_{k=0}^n m(T^{-k}(A))}{\sum_{k=0}^n m(T^{-k}(X_0))}.$$

*If  $A \in \mathcal{B}$  and  $A \subset Y_j$  with some  $j \geq 0$ , then we obtain  $(m_n(A))_{n=1}^\infty \in l^\infty$ . We set*

$$\mu(A) := l_B((m_n(A))_{n=1}^\infty).$$

For a general measurable subset  $A \subset X$ , set

$$\mu(A) := \sum_{j=0}^{\infty} \mu(A \cap Y_j).$$

In addition, if for a measurable subset  $A \subset X$ , the sequence  $(m_n(A))_{n=1}^{\infty}$  is bounded, then we have the following formula:

$$(18.1) \quad \mu(A) = l_B((m_n(A))_{n=1}^{\infty}) - \lim_{l \rightarrow \infty} l_B \left( \left( m_n \left( A \cap \bigcup_{j=l}^{\infty} Y_j \right) \right)_{n=0}^{\infty} \right).$$

In particular, if  $A \in \mathcal{B}$  is contained in a finite union of sets  $X_j$ ,  $j \geq 0$ , then

$$\mu(A) = l_B((m_n(A))_{n=1}^{\infty}).$$

Furthermore, if the measure-preserving transformation  $T : X \rightarrow X$  is ergodic (equivalently with respect to the measure  $m$  or  $\mu$ ), then the  $T$ -invariant measure  $\mu$  is unique up to a multiplicative constant.

The proof of this theorem will consist of several lemmas. We start with the following.

**Lemma 18.2.** *Let  $(Z, \mathcal{F})$  be a measurable space such that:*

$$(a) \quad Z = \bigcup_{j=0}^{\infty} Z_j \text{ for some mutually disjoint sets } Z_j \in \mathcal{F},$$

and

$$(b) \quad \nu_j \text{ is a finite measure on } Z_j \text{ for each } j \geq 0.$$

Then the set function  $\nu : \mathcal{F} \rightarrow [0, \infty]$  defined by

$$\nu(F) := \sum_{j=0}^{\infty} \nu_j(F \cap Z_j)$$

is a  $\sigma$ -finite measure on  $Z$ .

*Proof.* Clearly,  $\nu(\emptyset) = 0$ . Let  $F \in \mathcal{F}$  and let  $\{F_n\}_{n=1}^{\infty}$  be a partition of  $F$  into sets in  $\mathcal{F}$ . Then

$$\begin{aligned} \nu(F) &= \sum_{j=0}^{\infty} \nu_j(F \cap Z_j) = \sum_{j=0}^{\infty} \nu_j \left( \bigcup_{n=1}^{\infty} (F_n \cap Z_j) \right) \\ &= \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} \nu_j(F_n \cap Z_j) = \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \nu_j(F_n \cap Z_j) \\ &= \sum_{n=1}^{\infty} \nu(F_n), \end{aligned}$$

where the order of summation could be changed since all terms involved are non-negative. Thus,  $\nu$  is a measure. Moreover, by definition,  $Z = \bigcup_{j=0}^{\infty} Z_j$  and  $\nu(Z_j) = \nu_j(Z_j) < \infty$  for all  $j \geq 0$ . Therefore  $\nu$  is  $\sigma$ -finite.  $\square$

From this point on, all lemmas rely on the same main hypotheses as Theorem 4.19.



**Lemma 18.3.** *For all  $n, j \geq 0$  and all  $A, B \in \mathcal{A}$  with  $A \cup B \subset X_j$ , we have*

$$m_n(A) m(B) \leq K_j m(A) m_n(B).$$

*Proof.* This follows directly from the definition of  $m_n$  and condition (4) of Definition 4.17.  $\square$

**Lemma 18.4.** *For every  $j \geq 0$ , we have  $(m_n(X_j))_{n=1}^\infty \in l^\infty$  and  $\mu(Y_j) \leq \mu(X_j) < \infty$ .*

*Proof.* Fix  $j \geq 0$ . By virtue of condition (3) of Definition 4.17, there exists  $q \geq 0$  such that  $m(X_j \cap T^{-q}(X_0)) > 0$ . By Lemma 18.3 and the definition of  $m_n$ , for all  $n \geq 0$  we have that

$$\begin{aligned} m_n(Y_j) \leq m_n(X_j) &\leq K_j \frac{m(X_j)}{m(X_j \cap T^{-q}(X_0))} m_n(X_j \cap T^{-q}(X_0)) \\ &\leq K_j \frac{m(X_j)}{m(X_j \cap T^{-q}(X_0))} m_n(T^{-q}(X_0)) \\ &= K_j \frac{m(X_j)}{m(X_j \cap T^{-q}(X_0))} \frac{\sum_{k=0}^{n+q} m(T^{-k}(X_0))}{\sum_{k=0}^n m(T^{-k}(X_0))} \\ &= K_j \frac{m(X_j)}{m(X_j \cap T^{-q}(X_0))} \left[ 1 + \frac{\sum_{k=n+1}^{n+q} m(T^{-k}(X_0))}{\sum_{k=0}^n m(T^{-k}(X_0))} \right] \\ (18.2) \quad &\leq K_j \frac{m(X_j)}{m(X_j \cap T^{-q}(X_0))} \left[ 1 + \frac{q}{m(X_0)} \right]. \end{aligned}$$

Consequently,  $(m_n(X_j))_{n=1}^\infty \in l^\infty$ , and standard properties of a Banach limit yield that

$$\mu(Y_j) \leq K_j \frac{m(X_j)}{m(X_j \cap T^{-q}(X_0))} \left[ 1 + \frac{q}{m(X_0)} \right] < \infty.$$

Since  $X_j = \bigcup_{i=0}^j Y_i$  and the  $Y$ s are mutually disjoint, we deduce that

$$\mu(Y_j) \leq \sum_{i=0}^j \mu(X_j \cap Y_i) = \sum_{i=0}^\infty \mu(X_j \cap Y_i) =: \mu(X_j) \leq \sum_{i=0}^j \mu(Y_i) < \infty.$$

$\square$

Now, for every  $j \geq 0$ , set

$$\mu_j := \mu|_{Y_j}.$$

**Lemma 18.5.** *For every  $j \geq 0$  such that  $\mu(Y_j) > 0$  and for every measurable set  $A \subset Y_j$ , we have*

$$K_j^{-1} \frac{\mu(Y_j)}{m(Y_j)} m(A) \leq \mu_j(A) \leq K_j \frac{\mu(Y_j)}{m(Y_j)} m(A).$$

*Proof.* This follows from the definition of the measure  $\mu$  and by setting  $B = Y_j$  in Lemma 18.3 and using the standard properties of a Banach limit.  $\square$

**Lemma 18.6.** *For each  $j \geq 0$ ,  $\mu_j$  is a finite measure on  $Y_j$ .*

*Proof.* Let  $j \geq 0$ . Assume without loss of generality that  $\mu_j(Y_j) > 0$ . Let  $A \subset Y_j$  be a measurable set and  $(A_k)_{k=1}^\infty$  a countable measurable partition of  $A$ . Using termwise operations on sequences, for every  $l \in \mathbb{N}$  we have

$$\begin{aligned} \left( \sum_{k=1}^{\infty} m_n(A_k) \right)_{n=1}^{\infty} - \sum_{k=1}^l (m_n(A_k))_{n=1}^{\infty} &= \left( \sum_{k=1}^{\infty} m_n(A_k) \right)_{n=1}^{\infty} - \left( \sum_{k=1}^l m_n(A_k) \right)_{n=1}^{\infty} \\ &= \left( \sum_{k=l+1}^{\infty} m_n(A_k) \right)_{n=1}^{\infty}. \end{aligned}$$

It therefore follows from Lemma 18.3 (with  $A = A_k$  and  $B = Y_j$ ) that

$$\begin{aligned} \left\| \left( \sum_{k=1}^{\infty} m_n(A_k) \right)_{n=1}^{\infty} - \sum_{k=1}^l (m_n(A_k))_{n=1}^{\infty} \right\|_{\infty} &= \left\| \left( \sum_{k=l+1}^{\infty} m_n(A_k) \right)_{n=1}^{\infty} \right\|_{\infty} \\ &\leq \left\| \frac{K_j}{m(Y_j)} \left( m_n(Y_j) \sum_{k=l+1}^{\infty} m(A_k) \right)_{n=1}^{\infty} \right\|_{\infty} \\ &= \frac{K_j}{m(Y_j)} \left\| \left( m_n(Y_j) \sum_{k=l+1}^{\infty} m(A_k) \right)_{n=1}^{\infty} \right\|_{\infty}. \end{aligned}$$

Since  $(m_n(Y_j))_{n=1}^{\infty} \in l^\infty$  by Lemma 18.4 and since  $\lim_{l \rightarrow \infty} \sum_{k=l+1}^{\infty} m(A_k) = 0$ , we conclude that

$$\lim_{l \rightarrow \infty} \left\| \left( \sum_{k=1}^{\infty} m_n(A_k) \right)_{n=1}^{\infty} - \sum_{k=1}^l (m_n(A_k))_{n=1}^{\infty} \right\|_{\infty} = 0.$$

This means that

$$\left( \sum_{k=1}^{\infty} m_n(A_k) \right)_{n=1}^{\infty} = \sum_{k=1}^{\infty} (m_n(A_k))_{n=1}^{\infty}$$

in  $l^\infty$ . Hence, using the continuity of the Banach limit  $l_B : l^\infty \rightarrow \mathbb{R}$ , we get

$$\begin{aligned} \mu(A) &= l_B((m_n(A))_{n=1}^{\infty}) = l_B \left( \left( m_n \left( \bigcup_{k=1}^{\infty} A_k \right) \right)_{n=1}^{\infty} \right) = l_B \left( \left( \sum_{k=1}^{\infty} m_n(A_k) \right)_{n=1}^{\infty} \right) \\ &= \sum_{k=1}^{\infty} l_B((m_n(A_k))_{n=1}^{\infty}) = \sum_{k=1}^{\infty} \mu(A_k). \end{aligned}$$

So  $\mu_j$  is countably additive. Also,  $\mu_j(\emptyset) = 0$ . Thus  $\mu_j$  is a measure. By Lemma 18.4,  $\mu_j$  is finite.  $\square$

Combining Lemmas 18.2, 18.4, 18.5, and 18.6, and condition (2) of Definition 4.17, we get the following.

**Lemma 18.7.**  $\mu$  is a  $\sigma$ -finite measure on  $X$  equivalent to  $m$ . Moreover,  $\mu(Y_j) \leq \mu(X_j) < \infty$  and  $\mu(X_j) > 0$  for all  $j \geq 0$ .

**Lemma 18.8.** Formula (4.17) holds.

*Proof.* Fix  $A \in \mathcal{A}$  such that  $(m_n(A))_{n=1}^\infty \in l^\infty$ . Then for every  $l \in \mathbb{N}$  we have

$$\begin{aligned} l_B((m_n(A))_{n=1}^\infty) &= l_B\left(\sum_{j=0}^l (m_n(A \cap Y_j))_{n=1}^\infty\right) + l_B\left(\left(m_n\left(\bigcup_{j=l+1}^\infty A \cap Y_j\right)\right)_{n=1}^\infty\right) \\ &= \sum_{j=0}^l l_B\left(\left(m_n(A \cap Y_j)\right)_{n=1}^\infty\right) + l_B\left(\left(m_n\left(A \cap \bigcup_{j=l+1}^\infty Y_j\right)\right)_{n=1}^\infty\right). \end{aligned}$$

Letting  $l \rightarrow \infty$ , we obtain that

$$\begin{aligned} l_B((m_n(A))_{n=1}^\infty) &= \sum_{j=0}^\infty l_B\left(\left(m_n(A \cap Y_j)\right)_{n=1}^\infty\right) + \lim_{l \rightarrow \infty} l_B\left(\left(m_n\left(A \cap \bigcup_{j=l+1}^\infty Y_j\right)\right)_{n=1}^\infty\right) \\ &= \sum_{j=0}^\infty \mu(A \cap Y_j) + \lim_{l \rightarrow \infty} l_B\left(\left(m_n\left(A \cap \bigcup_{j=l}^\infty Y_j\right)\right)_{n=1}^\infty\right) \\ &= \mu(A) + \lim_{l \rightarrow \infty} l_B\left(\left(m_n\left(A \cap \bigcup_{j=l}^\infty Y_j\right)\right)_{n=1}^\infty\right). \end{aligned}$$

This establishes formula (4.17). In particular, if  $A \subset \bigcup_{j=0}^k X_j$  for some  $k \in \mathbb{N}$ , then  $A \cap \bigcup_{j=l}^\infty Y_j \subset \left(\bigcup_{j=0}^k X_j\right) \cap \left(X \setminus \bigcup_{i < l} X_i\right) = \emptyset$  for all  $l > k$ . In that case, the equation above reduces to

$$l_B((m_n(A))_{n=1}^\infty) = \mu(A).$$

□

**Lemma 18.9.** *The  $\sigma$ -finite measure  $\mu$  is  $T$ -invariant.*

*Proof.* Let  $i \geq 0$  be such that  $m(Y_i) > 0$ . Fix a measurable set  $A \subset Y_i$ . By definition,  $\mu(A) = l_B((m_n(A))_{n=1}^\infty)$ . Furthermore, for all  $n \geq 0$  notice that

$$|m_n(T^{-1}(A)) - m_n(A)| = \frac{|m(T^{-(n+1)}(A)) - m(A)|}{\sum_{k=0}^n m(T^{-k}(X_0))} \leq \frac{1}{\sum_{k=0}^n m(T^{-k}(X_0))}.$$

Thus,  $(m_n(T^{-1}(A)))_{n=1}^\infty \in l^\infty$  because  $(m_n(A))_{n=1}^\infty \in l^\infty$ . Moreover, by condition (5) of Definition 4.17, it follows from the above and the standard properties of a Banach limit that

$$l_B((m_n(T^{-1}(A)))_{n=1}^\infty) = l_B((m_n(A))_{n=1}^\infty) = \mu(A).$$

Keep  $A$  a measurable subset of  $Y_i$ . Fix  $l \in \mathbb{N}$ . We then have

$$\begin{aligned} m_n \left( T^{-1}(A) \cap \bigcup_{j=l}^{\infty} Y_j \right) &= \frac{\sum_{k=0}^n m(T^{-k}(T^{-1}(A) \cap \bigcup_{j=l}^{\infty} Y_j))}{\sum_{k=0}^n m(T^{-k}(X_0))} \\ &\leq \frac{\sum_{k=0}^n m(T^{-(k+1)}(A \cap T(\bigcup_{j=l}^{\infty} Y_j)))}{\sum_{k=0}^n m(T^{-k}(X_0))} \\ &\leq m_{n+1} \left( A \cap T \left( \bigcup_{j=l}^{\infty} Y_j \right) \right) \cdot \frac{\sum_{k=0}^{n+1} m(T^{-k}(X_0))}{\sum_{k=0}^n m(T^{-k}(X_0))} \\ &\leq K_i \frac{m_{n+1}(Y_i)}{m(Y_i)} \cdot m \left( A \cap T \left( \bigcup_{j=l}^{\infty} Y_j \right) \right) \cdot \frac{\sum_{k=0}^{n+1} m(T^{-k}(X_0))}{\sum_{k=0}^n m(T^{-k}(X_0))}, \end{aligned}$$

where the last inequality sign holds by Lemma 18.3 since  $A \subset Y_i$ . When  $n \rightarrow \infty$ , the last quotient on the right-hand side approaches 1. Therefore

$$0 \leq l_B \left( \left( m_n \left( T^{-1}(A) \cap \bigcup_{j=l}^{\infty} Y_j \right) \right)_{n=1}^{\infty} \right) \leq K_i \frac{\mu(Y_i)}{m(Y_i)} m \left( T \left( \bigcup_{j=l}^{\infty} Y_j \right) \right).$$

Hence, by virtue of condition (6) of Definition 4.17,

$$0 \leq \lim_{l \rightarrow \infty} l_B \left( \left( m_n \left( T^{-1}(A) \cap \bigcup_{j=l}^{\infty} Y_j \right) \right)_{n=1}^{\infty} \right) \leq K_i \frac{\mu(Y_i)}{m(Y_i)} \lim_{l \rightarrow \infty} m \left( T \left( \bigcup_{j=l}^{\infty} Y_j \right) \right) = 0.$$

So

$$\lim_{l \rightarrow \infty} l_B \left( \left( m_n \left( T^{-1}(A) \cap \bigcup_{j=l}^{\infty} Y_j \right) \right)_{n=1}^{\infty} \right) = 0.$$

It thus follows from Lemma 18.8 that

$$\mu(T^{-1}(A)) = l_B((m_n(T^{-1}(A)))_{n=1}^{\infty}) = l_B((m_n(A))_{n=1}^{\infty}) = \mu(A).$$

For an arbitrary  $A \in \mathcal{A}$ , write  $A = \bigcup_{j=0}^{\infty} A \cap Y_j$  and observe that

$$\mu(T^{-1}(A)) = \mu \left( \bigcup_{j=0}^{\infty} T^{-1}(A \cap Y_j) \right) = \sum_{j=0}^{\infty} \mu(T^{-1}(A \cap Y_j)) = \sum_{j=0}^{\infty} \mu(A \cap Y_j) = \mu(A).$$

We are done.  $\square$

**Proof of Theorem 4.19:** Combining Lemmas 18.4, 18.7, 18.8, and 18.9, we obtain the full statement of Theorem 4.19 except its last assertion. This last assertion however holds because of the following well known theorem:

**Theorem 18.10.** *Let  $T : (X, \mathcal{A}) \rightarrow (X, \mathcal{A})$  be a measurable transformation and  $m$  a  $\sigma$ -finite quasi- $T$ -invariant measure. If  $T$  is ergodic and conservative with respect to  $m$  then, up to a positive multiplicative constant, there exists at most one non-zero  $\sigma$ -finite  $T$ -invariant measure  $\mu$  which is absolutely continuous with respect to  $m$ .*

$\square$

**Remark 18.11.** *In the course of the proof of Theorem 4.19 we have shown that*

$$0 < \inf\{m_n(A) : n \in \mathbb{N}\} \leq \sup\{m_n(A) : n \in \mathbb{N}\} < +\infty$$

for all  $j \geq 0$  and all measurable sets  $A \subset X_j$  such that  $m(A) > 0$ .

#### APPENDIX A. CORRECTED PROOFS OF LEMMA 7.9 AND LEMMA 7.10 FROM [46]

The formulations and proofs of Lemma 7.9 and Lemma 7.10 from [46] were not entirely correct. Although we do not rely on them in our current manuscript, we take now the opportunity to provide their corrected formulations and proofs based on the progress we have made in the current manuscript. For the convenience of the reader we formulate first Proposition 5.3 from [46], which is one of the main ingredients in the proof of Lemma A.2 but has not been formulated in our current manuscript yet. We also use other results and notation from [46], but since [46] is published and easily reachable we do not explain or copy them here.

**Proposition A.1** (Proposition 5.3 in [46]). *Fix  $\theta \in (0, \min\{1, \gamma\})$ . For all  $(\tau, z) \in J(\tilde{f})$  and  $r > 0$  there exists a minimal integer  $s = s(\theta, (\tau, z), r) \geq 0$  with the following properties (a) and (b).*

- (a)  $|(\tilde{f}^s)'(\tau, z)| \neq 0$ .
- (b) *Either  $r|(\tilde{f}^s)'(\tau, z)| > \|\tilde{f}'\|_\infty^{-1}$  or there exists  $c \in \text{Crit}(f_{\tau_{s+1}})$  such that*

$$f_{\tau_{s+1}}(c) \in J(G) \quad \text{and} \quad |f_{\tau|s}(z) - c| \leq \theta r |f'_{\tau|s}(z)|.$$

*In addition, for this  $s$ , we have*

- (c)  $\theta r |f'_{\tau|s}(z)| \leq \theta < \gamma$  and

$$\text{Comp}(z, f_{\tau|s}, (KA_f^2)^{-1} 2^{-\#\text{Crit}(f)} \theta r |f'_{\tau|s}(z)|) \cap \text{Crit}(f_{\tau|s}) = \emptyset.$$

**Lemma A.2** (Lemma 7.9 in [46]). *Suppose that  $\Gamma$  is a closed subset of  $J(G)$  such that  $g(\Gamma) \cap J(G) \subset \Gamma$  for each  $g \in G$ , and that  $\tilde{m}$  is a Borel probability nearly upper  $t$ -conformal measure on  $J(\tilde{f})$  respective to  $\Gamma$ .*

*Fix  $i \in \{0, 1, \dots, p\}$  and suppose that for every critical point  $c \in S_i(f) \cap \Gamma$  the measure  $\tilde{m}|_{\Sigma(c) \times \hat{\mathbb{C}}} \circ p_2^{-1}$  is upper  $t$ -estimable at  $c$ .*

*Then the measure  $m$  is uniformly upper  $t$ -estimable at all points  $z \in J_i(G) \cap \Gamma$ .*

*Proof.* Since  $\Gamma$  is a closed set and  $\text{Crit}(f)$  is finite, the number  $\Delta = \text{dist}(\Gamma, \text{Crit}(f) \setminus \Gamma)$  is positive (if  $\text{Crit}(f) \setminus \Gamma = \emptyset$  then we put  $\Delta = \infty$ ). Fix  $\theta \in (0, \min\{1, \gamma\})$  so small that

$$(A.1) \quad \theta \|\tilde{f}'\|_\infty^{-1} < \min\{\Delta, \rho\}.$$

Put

$$\alpha = \theta (KA_f^2)^{-1} 2^{-\#\text{Crit}(f)}.$$

Let  $z \in J_i(G) \cap \Gamma$ . Fix  $\tau \in \Sigma_u$  such that  $(\tau, z) \in J(\tilde{f})$ , i.e.  $\tau \in p_1(J(\tilde{f}) \cap p_2^{-1}(z))$ . Assume  $r \in (0, R_f]$  to be sufficiently small. Let

$$s(\tau, r) := s(\theta, (\tau, z), 8\alpha^{-1}r) \geq 0$$

be the integer produced in Proposition A.1. Set

$$R_{\tau|_{s(\tau,r)+1}} := 4r|f'_{\tau|_{s(\tau,r)}}(z)|.$$

It then follows from Proposition A.1 that the family

$$\mathcal{F}(z, r) = \{\tau|_{s(\tau,r)+1} : \tau \in p_1(J(\tilde{f}) \cap p_2^{-1}(z))\}$$

is  $(4, \gamma, V)$ -essential for the pair  $(z, r)$ , where

$$V := \bigcup \{[\tau|_{s(\tau,r)+1}] : \tau \in p_1(J(\tilde{f}) \cap p_2^{-1}(z))\}.$$

Keep

$$\tau \in p_1(J(\tilde{f}) \cap p_2^{-1}(z)) \quad \text{and} \quad s = s(\tau, r).$$

Suppose that the first alternative of (b) in Proposition A.1 holds. Then  $8\alpha^{-1}r|f'_{\tau|_s}(z)| > \|\tilde{f}'\|_\infty^{-1}$ . So, using Koebe's Distortion Theorem, and assuming that  $\theta$  is small enough, we get from the nearly upper  $t$ -conformality of  $\tilde{m}$  respective to  $\Gamma$  that

$$\begin{aligned} \tilde{m}(\tilde{f}_{\tau|_s,z}^{-s}([\tau_{s+1}] \times B(f_{\tau|_s}(z), R_{\tau|_{s+1}}))) &\leq \tilde{m}(\tilde{f}_{\tau|_s,z}^{-s}(p_2^{-1}(B(f_{\tau|_s}(z), R_{\tau|_{s+1}})))) \\ (A.2) \quad &\leq K^t|f'_{\tau|_s}(z)|^{-t}\tilde{m}p_2^{-1}(B(f_{\tau|_s}(z), R_{\tau|_{s+1}}))) \\ &\leq K^t|f'_{\tau|_s}(z)|^{-t} \\ &\leq (8K\alpha^{-1}\|\tilde{f}'\|_\infty)^t r^t. \end{aligned}$$

Now suppose that

$$8\alpha^{-1}r|f'_{\tau|_s}(z)| \leq \|\tilde{f}'\|_\infty^{-1}.$$

This implies that the second alternative of (b) in Proposition A.1 holds. Let  $c \in \text{Crit}(f_{\tau_{s+1}})$  come from item (b) of this proposition. In particular,

$$f_{\tau_{s+1}}(c) \in J(G).$$

Since  $z \in J_i(G)$  (and  $\theta\|\tilde{f}'\|_\infty^{-1} < \rho$ , where  $\rho$  was defined in [46]), it follows from the formula (16) in [46] and from Proposition A.1 that  $c \in S_i(f)$ . Since  $8\alpha^{-1}r|f'_{\tau|_s}(z)| \leq \|\tilde{f}'\|_\infty^{-1}$ , it follows from Proposition A.1(b) and (A.1) that

$$|f_{\tau|_s}(z) - c| \leq \theta\|\tilde{f}'\|_\infty^{-1} < \Delta.$$

Because of the definition of  $z$  and  $\tau$ , we have that  $f_{\tau|_s}(z) \in J(G)$ . Since also  $f_{\tau|_s} \in G$ , we thus obtain that  $f_{\tau|_s}(z) \in \Gamma$ . In conclusion,  $c \in \Gamma$ . Hence, making use of Proposition A.1(b), (c), as well as Koebe's Distortion Theorem, nearly upper  $t$ -conformality of  $\tilde{m}$ , and our  $t$ -upper estimability assumption, and assuming  $\theta$  is small enough, we get with some universal

constant  $C_1$  that

$$\begin{aligned}
& \tilde{m}\left(\tilde{f}_{\tau|s,z}^{-s}([\tau_{s+1}] \times B(f_{\tau|s}(z), R_{\tau|s+1}))\right) \\
& \leq K^t |f'_{\tau|s}(z)|^{-t} \tilde{m}([\tau_{s+1}] \times B(f_{\tau|s}(z), R_{\tau|s+1})) \\
& \leq K^t |f'_{\tau|s}(z)|^{-t} \tilde{m}|_{\Sigma(c) \times \hat{c}} \circ p_2^{-1}(B(f_{\tau|s}(z), R_{\tau|s+1})) \\
& \leq K^t |f'_{\tau|s}(z)|^{-t} \tilde{m}|_{\Sigma(c) \times \hat{c}} \circ p_2^{-1}(B(c, R_{\tau|s+1} + 8\theta\alpha^{-1}r|f'_{\tau|s}(z)|)) \\
& \leq K^t |f'_{\tau|s}(z)|^{-t} \tilde{m}|_{\Sigma(c) \times \hat{c}} \circ p_2^{-1}(B(c, 4(1 + 2\theta\alpha^{-1})r|f'_{\tau|s}(z)|)) \\
& \leq K^t |f'_{\tau|s}(z)|^{-t} C_1 (4(1 + 2\theta\alpha^{-1})r|f'_{\tau|s}(z)|)^t \\
& = C_1 (4K(1 + 2\theta\alpha^{-1}))^t r^t.
\end{aligned}$$

Combining this with (A.2) and applying Proposition 13.9, we get that

$$(A.3) \quad m(B(z, r)) \leq \#_{4,4} C_1 \max\{8K\alpha^{-1} \|\tilde{f}'\|_\infty, 4K(1 + 2\theta\alpha^{-1})\}^t r^t.$$

We are done.  $\square$

**Lemma A.3** (Lemma 7.10 in [46]). *There are two functions  $(R, S) \mapsto R^*$  and  $L \mapsto \hat{L}$  with the following property.*

- Suppose that  $\Gamma$  is a closed subset of  $J(G)$  such that

$$g(\Gamma) \cap J(G) \subset \Gamma$$

for each  $g \in G$ , and that  $\tilde{m}$  is a Borel probability nearly upper  $t$ -conformal measure on  $J(f)$  relative to  $\Gamma$  with nearly upper conformality radius  $S$ .

- Fix  $i \in \{0, 1, \dots, p\}$  and suppose that the measure  $m|_{J_i(G)}$  is uniformly upper  $t$ -estimable at all points  $z \in J_i(G) \cap \Gamma$  with the corresponding estimability constant  $L$  and estimability radius  $R$ .
- Then the measure  $\tilde{m}|_{\Sigma(c) \times \hat{c}} \circ p_2^{-1}|_{J_i(G)}$  is  $t$ -upper estimable, with upper estimability constant  $\hat{L}$  and radius  $R^*$  at every point  $c \in Cr_{i+1}(f)$  such that

$$\bigcup_{|\omega|=l} f_\omega(c_+) \cap J(G) \subset \Gamma.$$

*Proof.* Fix  $c \in Cr_{i+1}(f)$  such that  $\bigcup_{|\omega|=l} f_\omega(c_+) \subset \Gamma$  and also  $j \in \{0, 1, \dots, u\}$  such that  $f'_j(c) = 0$ . Consider an arbitrary  $\tau \in \Sigma_u$  such that  $\tau_1 = j$  and  $(\tau, c) \in J(\tilde{f})$ . In view of Lemma 4.8 in [46], we get that

$$f_{\tau|l+1}(c) \in J_i(G) \cap \Gamma.$$

Let  $R > 0$  (sufficiently small) be the radius resulting from uniform  $t$ -upper estimability at all points of  $J_i(G) \cap \Gamma$ . Let  $D_{\tau|l+1}(c)$  be the connected component of  $f_{\tau|l+1}^{-1}(B(f_{\tau|l+1}(c), R))$  containing  $c$ . Set

$$\nu_{\tau|l+1} = \tilde{m}|_{[\tau|l+1] \times \hat{c}} \circ p_2^{-1}|_{D_{\tau|l+1}(c) \cap J_i(G) \cap f_{\tau|l+1}^{-1}(J(G))}.$$

Applying nearly upper  $t$ -conformality of  $\tilde{m}$  for every Borel set  $A \subset D_{\tau|_{l+1}}(c) \setminus \{c\}$  such that  $f_{\tau|_{l+1}}|_A$  is injective we get that

$$m(f_{\tau|_{l+1}}(A)) = \tilde{m}(\Sigma_u \times f_{\tau|_{l+1}}(A)) = \tilde{m}(\tilde{f}^{l+1}([\tau|_{l+1}] \times A)) \geq \int_A |f'_{\tau|_{l+1}}(x)|^t d\nu_{\tau|_{l+1}}(x).$$

It therefore follows from Lemma 2.10 and item (c) of Definition 7.8 in [46] that the measure  $\nu_{\tau|_{l+1}}$  is upper  $t$ -estimable at  $c$  with upper estimability constant  $L_0$  and radius  $R_0$  independent of  $\tilde{m}$  (but possibly  $R_0$  depends on  $(R, S)$  and  $L_0$  depends on  $L$ ). Let

$$\mathcal{F} = \{\tau|_{l+1} : (\tau, c) \in J(\tilde{f}) \text{ and } f'_{\tau_1}(c) = 0\}.$$

Let

$$D_c := \bigcap_{\omega \in \mathcal{F}} D_\omega(c) \cap J_i(G) \cap f_{\tau|_{l+1}}^{-1}(J(G)).$$

Since  $\#\mathcal{F} \leq u^{l+1}$  and since

$$\tilde{m}|_{\Sigma(c) \times \hat{\mathbb{C}}} \circ p_2^{-1}|_{D_c} = \sum_{\omega \in \mathcal{F}} \nu_\omega|_{D_c},$$

we conclude that the measure  $\tilde{m}|_{\Sigma(c) \times \hat{\mathbb{C}}} \circ p_2^{-1}|_{J_i(G)}$  is  $t$ -upper estimable at the point  $c$  with upper estimability constant  $\hat{L}$  and radius  $R^*$  independent of  $\tilde{m}$ . We are done.  $\square$

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