GEOMETRY OF MEASURES IN RANDOM SYSTEMS WITH COMPLETE CONNECTIONS

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Abstract. We study new relations between countable iterated function systems, Smale endomorphisms and random systems with complete connections. We prove that the stationary measures of arbitrary countable conformal IFS with place-dependent probabilities, are exact dimensional, and we determine their Hausdorff dimension. Next, we construct a family of fractals in the limit set of a countable IFS with overlaps $\mathcal{S}$, and study the dimension for certain measures supported on these sub-fractals. In particular, we obtain families of measures on sub-fractals, related to the geometry of the system $\mathcal{S}$.

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1. Introduction

In this paper we study relations between countable conformal iterated function systems (IFS) with overlaps, Smale endomorphisms, and random systems with complete connections, from the point of view of their geometric and ergodic properties. We provide a surprising common framework for studying measures with certain invariance properties and their dimensions in these systems.

Finite and infinite iterated function systems were studied in many settings, and their invariant measures and limit sets have attracted a lot of interest in the literature, for eg [1], [2], [3], [6], [7], [8], [11], [16], [17], [18], to cite a few. Also there are several connections with the dimension theory for hyperbolic endomorphisms, for eg [20], [24], [12]-[15]. Iterated function systems with place-dependent probabilities (weights) were introduced and studied by Barnsley, Demko, Elton, Geronimo in [1]; see also [2], [3], [23]. Random systems with complete connections were introduced and studied by Iosifescu and Grigorescu in [10] (see also [8]), and are generalizations of the chains with complete connections introduced by Onicescu and Mihoc in [19]. Smale endomorphisms were introduced and studied by the authors in [17].

In the sequel, we first define/recall the notions of countable IFS with overlaps and place-dependent probabilities, the notion of random systems with complete connections, and the notion of Smale endomorphisms. We show that a countable IFS with place-dependent probabilities is a particular case of random system with complete connections. Then, in

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Theorem 1.5 we show that, to certain Smale endomorphisms, one can associate random systems with complete connections.

In Section 2, given a countable IFS $\mathcal{S}$ with place-dependent probabilities $\{p_i(\cdot), i \in I\}$ and arbitrary overlaps, we find its stationary measure. Then, in Theorem 2.1 we prove the exact dimensionality of such a stationary measure, and find its Hausdorff (and pointwise, box) dimension. The exact dimensionality largely characterizes the local and global metric properties of the respective measure.

Next, in Section 3, for an arbitrary countable IFS with overlaps $\mathcal{S}$ which satisfies a condition of pointwise non-accumulation, we associate a *maximal Smale endomorphism*. Using this method, we form a family of random fractals in the limit set Λ of $\mathcal{S}$, which correspond to sub-systems of iterations. In Theorem 3.2 we determine the pointwise dimension for a class of invariant measures supported on these random sub-fractals in Λ. In particular, we obtain families of such measures, which are related to the geometry of the system $\mathcal{S}$.

A finite measure μ on a metric space X is called *exact dimensional* (for eg [20], [24]) if there exists a number $\delta \geq 0$ such that for μ-a.e $x \in X$,

$$\lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = \delta.$$

In this case it follows that $HD(\mu) = \delta$. Exact dimensionality of a measure μ is a strong geometric property, and implies that the fractal dimensions of μ (Hausdorff, pointwise, box dimension) coincide. For finite conformal IFS with overlaps, the exact dimensionality of projections of ergodic measures was proved in [7]. For countable conformal IFS with overlaps, the exact dimensionality of projections of ergodic measures satisfying a finite entropy condition was proved in [16]. The property was studied for hyperbolic diffeomorphisms for eg in [20], [24], and for hyperbolic endomorphisms (non-invertible maps) for eg in [14].

Now, let us recall the three main notions that will be used in the sequel:

First, there is the notion of *finite iterated function systems with place-dependent probabilities*, introduced in [1]; see also the papers [2], [3], [23]. These iterated function systems are particular cases of the chains with complete connections introduced in [19]; see also [5], [9].

A *chain with complete connections* is a sequence of random variables $\xi_1, \xi_2, \ldots$ taking real values in a countable set Ω, where the probability that at step n, ξ_n takes value $\omega \in \Omega$, depends on the values taken by all previous random variables. Thus,

$$P(\xi_{n+1} = \omega | \xi_n = \omega_0, \xi_{n-1} = \omega_{-1}, \ldots) = P(c, \omega),$$

where $\omega \in \Omega$ and $c = (\ldots, \omega_{-1}, \omega_0)$ is a trajectory in $\Sigma_\Omega^- := \prod\limits_{j \in \mathbb{Z}, j \leq 0} \Omega$. One assumes in general that $P^1(c, \omega) = P(c, \omega)$ and for every $n \geq 1$,

$$P^{n+1}(c, \omega) = \sum_{\omega' \in \Omega} P(c, \omega') P^n(a(c, \omega'), \omega),$$

where $a(c, \omega') := (\ldots, \omega_{-1}, \omega_0, \omega')$, for $c = (\ldots, \omega_{-1}, \omega_0) \in \Sigma_\Omega^-$.

In the sequel, we extend the notion of finite IFS with place-dependent probabilities to *countable iterated function systems with overlaps and place-dependent probabilities*. Such a system $\mathcal{S}$ is given by continuous functions $\phi_i : V \longrightarrow V, \quad i \in I$, defined on a compact set $V \subset \mathbb{R}^D$ and indexed by a countable set I, and continuous probability functions (weights), $p_i : V \longrightarrow [0, 1],\ i \in I$, satisfying

$$\sum_{i \in I} p_i(x) = 1.$$

By IFS with overlaps we mean that the sets $\phi_i(V), i \in I$ may intersect in any way, thus no Open Set Condition is assumed (see for eg [6], [7], [16]). Assume also that there exists a number $s \in (0, 1)$ such that,

$$|\phi_i(x) - \phi_i(y)| \leq s|x - y|,\ \forall i \in I,\ x, y \in V. \tag{1.1}$$

The countable IFS case is different from the finite case, since the fractal limit set may be **non-compact**, and many methods from the finite IFS case do not work.

For $x \in V$ and a Borel set $B \subset V$, the *probability of transfer* from x to B is equal to $P(x, B) = \sum_{i=1}^{\infty} p_i(x)\delta_{\phi_i(x)}(B)$. We have then the associated transfer operator:

$$\mathcal{L}g(x) = \int_V g(y)P(x, dy) = \sum_{i=1}^{m} p_i(x)g(\phi_i(x)),$$

where $g : V \to \mathbb{R}$ is measurable. If $M(V)$ denotes the space of finite signed Borel measures on V, then the operator $\mathcal{L}^*$ adjoint to $\mathcal{L}$, restricted to the space $M(V)$, is given by

$$\mathcal{L}^*\mu(B) = \int P(x, B)d\mu(x) = \sum_{i=1}^{\infty} \int_{\phi_i^{-1}B} p_i(x)d\mu(x) \tag{1.2}$$

A Borel probability measure μ on V is called *stationary* for the above system if

$$\mathcal{L}^*\mu = \mu, \tag{1.3}$$

and *attractive* if for all probabilities ν on V and all bounded measurable $g : V \to \mathbb{R}$,

$$\lim_{n\to\infty} \int_X g\, d(\mathcal{L}^{*n}\nu) = \int_V g\, d\mu.$$

One of the central problems in the theory of chains with complete connections and of IFS with place–dependent probabilities is to find stationary measures and to study their ergodic and metric properties. In many of these results in the finite case, the probability functions p_i, $i = 1, \ldots, m$, satisfy a Hölder type condition ([1], [2], [10]).

The second main notion is that of *random systems with complete connections*, which are generalizations of chains with complete connections (for eg [8], [10]).

Definition 1.1 ([10])**.** *A random system with complete connections (or RSCC) is a quadruple $((W, \mathcal{W}), (X, \mathcal{X}), u, P)$ where:*

i) $(W, \mathcal{W})$ and $(X, \mathcal{X})$ are measurable spaces,

ii) $u : W \times X \to W$ is a measurable map, with the product σ-algebra $\mathcal{W} \times \mathcal{X}$ on $W \times X$, *iii) P is a transition probability function from $(W, \mathcal{W})$ to $(X, \mathcal{X})$, i.e. $P(w, \cdot)$ is a probability on $\mathcal{X}$ for any $w \in W$ and $P(\cdot, A)$ is a random variable on Z for any set $A \in \mathcal{X}$.*

We call W the state space and X the index space. W is assumed to be a locally compact and σ-compact metric space. For $\mathcal{W}$ we take the σ-algebra generated by open sets of W.

An example of RSCC is an urn scheme with replacement. Consider an initial urn U_0, which contains $a_j = a_j^{(0)}$ balls of color $j, 1 \leq j \leq m$. If on trial $n \geq 1$ we extract a ball of color j, then this ball is replaced together with d_j balls of same color (the rest of the balls being left unchanged), hence $a_i^{(n)} = a_i^{(n-1)} + \delta_{ij} d_i, 1 \leq i \leq m$, with $d_1, \ldots, d_m$ being non-negative integers. Thus the probability of choosing a certain color at step n depends on all previous steps.

Remark 1.2. *If in Definition 1.1 the index space X is countable and $\mathcal{X}$ is the algebra of subsets of X, and we define the maps $\phi_i(w) := u(w, i)$, then we obtain a countable IFS on W. Denote $x^{(n)} := (x_1, \ldots, x_n) \in X^n$, $n \geq 1$. By induction define $u^{(n)} : W \times X^n \to W$,*

$$u^{(n+1)}(w, x^{(n+1)}) = u(w, x_1), n = 0, \ \text{and } u^{(n+1)}(w, x^{(n+1)}) = u(u^{(n)}(w, x^{(n)}), x_{n+1}), n \geq 1$$

We can write wx for $u(w, x)$, and wx^n for $u^{(n)}(w, x^{(n)})$. For $w \in W$ and A Borel set in X, let $P_1(w, A) = P(w, A)$, and if $w \in W, m > 1$ and A Borel set in X^m, let the m-transfer probability of w into A be,

$$P_m(w, A) = \int_X P(w, dx_1) \int_X P(wx_1, dx_2) \ldots \int_X P(wx^{(m-1)}, dx_m) d\chi_A(x^{(m)}$$

And for $w \in W, n, m \geq 1$ and $A \subset X^m$, define $P_m^n(w, A) = P_{n+m-1}(w, X^{n-1} \times A)$.

The following existence result was proved in [10].

Theorem 1.3. *Let a random system with complete connections $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P)$ and an arbitrary given point $w_0 \in W$. Then, there exist a probability space $(\Omega, \mathcal{K}, P_{w_0})$ and a sequence $(\xi_n)_{n \geq 1}$ of X-valued random variables defined on Ω such that, for all $m, n, q \geq 1$ and $A \in \mathcal{X}^m$, we have:*
i) $P_{w_0}([\xi_n, \ldots, \xi_{n+m-1}] \in A) = P_m^n(w_0, A)$,
ii) $P_{w_0}([\xi_{n+q}, \ldots, \xi_{n+q+m-1}] \in A | \xi^{(n)}) = P_m^q(w_0 \xi^{(n)}, A)$, P_{w_0}-a.e.

Finally, the third main notion we will use is that of *Smale skew product endomorphism*, introduced in [17]. Let I be a countable alphabet, and let Σ_I^+ be the associated 1-sided shift space, with shift map $\sigma : \Sigma_I^+ \to \Sigma_I^+$. Given $\beta > 0$, the metric d_β on Σ_I^+ is:

$$d_\beta\big((\omega_n)_0^\infty, (\tau_n)_0^\infty\big) = \exp\big(-\beta \max\{n \geq 0 : (0 \leq k \leq n) \Rightarrow \omega_k = \tau_k\}\big)$$

with the standard convention that $e^{-\infty} = 0$. All metrics d_β, $\beta > 0$, on Σ_I^+ are Hölder continuously equivalent and all induce the product topology on Σ_I^+. We define also the 2-sided shift space Σ_I with the same metric as above.

For every $\omega \in \Sigma_I$ and integers $m \leq n$, define the (m, n)-truncation $\omega|_m^n = \omega_m \omega_{m+1} \ldots \omega_n$. Let Σ_I^* be the set of finite words. For $\tau = \tau_m \tau_{m+1} \ldots \tau_n$, the cylinder from m to n is $[\tau]_m^n = \{\omega \in \Sigma_I : \omega|_m^n = \tau\}$. The family of cylinders from m to n is denoted by C_m^n. If $m = 0$, write $[\tau]$ for $[\tau]_m^n$.

Let $\psi : \Sigma_I \to \mathbb{R}$ continuous. Topological pressure plays an important role in thermodynamic formalism and extends the notion of entropy (for eg [4], [22]). By extension, in our case the *topological pressure* of ψ is: $\mathrm{P}(\psi)$ is,

$$\mathrm{P}(\psi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in C_0^{n-1}} \exp\big(\sup\big(S_n\psi|_{[\omega]}\big)\big), \tag{1.4}$$

where the limit above exists by subadditivity. A shift-invariant Borel probability μ on the 2-sided shift space Σ_I with countable alphabet I, is called a *Gibbs measure* of ψ if there exist constants $C \geq 1$, $P \in \mathbb{R}$ such that

$$C^{-1} \leq \frac{\mu([\omega|_0^{n-1}])}{\exp(S_n\psi(\omega) - Pn)} \leq C \tag{1.5}$$

for all $n \geq 1, \omega \in \Sigma_I$. From (1.5), if ψ admits a Gibbs state, then $P = \mathrm{P}(\psi)$. The function $\psi : \Sigma_I \to \mathbb{R}$ is called *summable* if

$$\sum_{e\in E} \exp\big(\sup\big(\psi|_{[e]}\big)\big) < \infty$$

In [17] we proved that a Hölder continuous $\psi : \Sigma_I \to \mathbb{R}$ is summable if and only if $\mathrm{P}(\psi) < \infty$, and that for every Hölder continuous summable function $\psi : \Sigma_I \to \mathbb{R}$ there exists a unique Gibbs state μ_ψ on Σ_I, and the measure μ_ψ is ergodic. We also showed that if $\psi : \Sigma_I \to \mathbb{R}$ is a Hölder continuous summable function, then

$$\sup\left\{ \mathrm{h}_\mu(\sigma) + \int_{\Sigma_\mathrm{I}} \psi \mathrm{d}\mu : \mu\circ\sigma^{-1} = \mu \ \text{ and } \ \int \psi \mathrm{d}\mu > -\infty \right\} = \mathrm{P}(\psi) = \mathrm{h}_{\mu_\psi}(\sigma) + \int_{\Sigma_I} \psi d\mu_\psi,$$

and this supremum is attained only at μ_ψ. Let now the measurable partition of Σ_I,

$$\mathcal{P}_- = \{[\omega|_0^{+\infty}] : \omega \in \Sigma_I\} = \{[\omega] : \omega \in \Sigma_I^+\}. \tag{1.6}$$

If μ is a probability on Σ_I, let the canonical conditional measures associated to $\mathcal{P}_-$ ([21]),

$$\{\overline{\mu}^\tau : \tau \in \Sigma_I^+\}$$

Then $\overline{\mu}^\tau$ is a probability measure on the cylinder $[\tau|_0^{+\infty}]$ and we also write $\overline{\mu}^\omega$, $\omega \in \Sigma_I^+$, for the conditional measure on $[\omega]$. The canonical projection (truncation) is:

$$\pi_0 : \Sigma_I \to \Sigma_I^+, \ \ \pi_0(\tau) = \tau|_0^\infty, \tau \in \Sigma_I,$$

Recall that the system $\{\overline{\mu}^\omega : \omega \in \Sigma_I^+\}$ of conditional measures is uniquely determined (up to measure zero), by: $\int_{\Sigma_I} g\, d\mu = \int_{\Sigma_I^+} \int_{[\omega]} g\, d\overline{\mu}^\omega\, d(\mu\circ\pi_0^{-1})(\omega)$, $\forall g \in L^1(\mu)$ ([21]).

We introduced the notion of *Smale skew product endomorphisms.*

Definition 1.4. [17] *Let (Y, d) be a complete bounded metric space. For every $\omega \in \Sigma_I^+$ let $Y_\omega \subset Y$ be an arbitrary set and let $T_\omega : Y_\omega \longrightarrow Y_{\sigma(\omega)}$ be a continuous injective map. Let $\hat{Y} := \bigcup_{\omega\in\Sigma_I^+} \{\omega\} \times Y_\omega \subset \Sigma_I^+ \times Y$, and define the map $T : \hat{Y} \longrightarrow \hat{Y}, T(\omega, y) = (\sigma(\omega), T_\omega(y))$. The pair $(\hat{Y}, T : \hat{Y} \to \hat{Y})$ is called a model Smale endomorphism if there exists $\lambda > 1$ such that for all $\omega \in \Sigma_I^+$ and all $y_1, y_2 \in Y_\omega$, $d(T_\omega(y_2), T_\omega(y_1)) \leq \lambda^{-1} d(y_2, y_1)$.*

If $\tau = (\tau_{-n}, \ldots, \tau_0, \tau_1, \ldots)$ let $T^n_\tau = T_{\tau|^\infty_{-1}} \circ T_{\tau|^\infty_{-2}} \circ \ldots \circ T_{\tau|^\infty_{-n}} : Y_\tau \longrightarrow Y_{\tau|^\infty_0}$. If $\tau \in \Sigma_I$ let,

$$T^n_\tau := T^n_{\tau|^\infty_{-n}} := T_{\tau|^\infty_{-1}} \circ T_{\tau|^\infty_{-2}} \circ \ldots \circ T_{\tau|^\infty_{-n}} : Y_{\tau|^\infty_{-n}} \longrightarrow Y_{\tau|^\infty_0}$$

Then the sets $\big(T^n_\tau\big(Y_{\tau|^\infty_{-n}}\big)\big)^\infty_{n=0}$ form a descending sequence, and $\operatorname{diam}\big(\overline{T^n_\tau\big(Y_{\tau|^\infty_{-n}}\big)}\big) \leq \lambda^{-n}\operatorname{diam}(Y)$. As (Y, d) is complete, $\bigcap_{n=1}^\infty \overline{T^n_\tau\big(Y_{\tau|^\infty_{-n}}\big)}$ is a point denoted by $\hat{\pi}_2(\tau)$, which defines the map

$$\hat{\pi}_2 : \Sigma_I \longrightarrow Y, \tag{1.7}$$

and define also $\hat{\pi} : \Sigma_I \to \Sigma_I^+ \times Y$ by

$$\hat{\pi}(\tau) = \big(\tau|^\infty_0, \hat{\pi}_2(\tau)\big), \tag{1.8}$$

and the truncation to non-negative indices by $\pi_0 : \Sigma_I \longrightarrow \Sigma_I^+$, $\pi_0(\tau) = \tau|^\infty_0$.

Now assume $Y_\omega = Y, \forall \omega \in \Sigma_I^+$, and for an arbitrary $\omega \in \Sigma_I^+$ denote the $\hat{\pi}_2$-projection of the cylinder $[\omega] \subset \Sigma_I$, $J_\omega := \hat{\pi}_2([\omega]) \subset Y$, and call these sets the *stable Smale fibers* of T. The global invariant set,

$$J := \hat{\pi}(\Sigma_I) = \bigcup_{\omega \in \Sigma_I^+} \{\omega\} \times J_\omega \subset \Sigma_I^+ \times Y,$$

is called the *Smale space* induced by T. Then the skew-product system

$$T : J \longrightarrow J, \tag{1.9}$$

is called the *Smale endomorphism* generated by $T : \hat{Y} \longrightarrow \hat{Y}$.

Now suppose more conditions about Y_ω, $\omega \in \Sigma_I^+$ and the maps $T_\omega : Y_\omega \to Y_{\sigma(\omega)}$, namely:

(a) Y_ω is a closed bounded subset of $\mathbb{R}^d$, with some $d \geq 1$ such that $\overline{\operatorname{Int}(Y_\omega)} = Y_\omega$.
(b) Each map $T_\omega : Y_\omega \to Y_{\sigma(\omega)}$ extends to a C^1 conformal embedding from Y^*_ω to $Y^*_{\sigma(\omega)}$, where Y^*_ω is a bounded connected open subset of $\mathbb{R}^d$ containing Y_ω. Then T_ω denotes also this extension and assume that the maps $T_\omega : Y^*_\omega \to Y^*_{\sigma(\omega)}$ satisfy:
(c) There is $\lambda > 1$ so that $d(T_\omega(y_1), T_\omega(y_2)) \leq \lambda^{-1} d(y_1, y_2), \forall \omega \in \Sigma_I^+, y_1, y_2 \in Y^*_\omega$.
(d) (Bounded Distortion Property 1) There are constants $\alpha > 0, H > 0$ s.t $\forall y, z \in Y^*_\omega$,
$$\big| \log |T'_\omega(y)| - \log |T'_\omega(z)| \big| \leq H ||y - z||^\alpha.$$
(e) The function $\Sigma_I \ni \tau \longmapsto \log |T_\omega(\hat{\pi}_2(\eta))| \in \mathbb{R}$ is Hölder continuous, where $\omega = \pi_0(\tau)$.
(f) (Open Set Condition) For every $\omega \in \Sigma_I^+$ and for all $a, b \in I$ with $a \neq b$, we have $T_{a\omega}(\operatorname{Int}(Y_{a\omega})) \cap T_{b\omega}(\operatorname{Int}(Y_{b\omega})) = \emptyset$.
(g) (Strong Open Set Condition) There exists a measurable function $\delta : \Sigma_I^+ \to (0, \infty)$ such that for every $\omega \in \Sigma_I^+$, $J_\omega \cap \big(Y_\omega \setminus \overline{B}(Y^c_\omega, \delta(\omega)\big) \neq \emptyset$.

A skew product Smale endomorphism satisfying conditions (a)–(g) will be called in the sequel a *conformal Smale endomorphism.*

We see that a countable IFS with place-dependent probabilities is a **particular case** of random system with complete connections with state space V, index space = countable alphabet I, and probability transition function determined by $p_i(\cdot), i \in I$, namely

$$P(x, B) = \sum_{i \in I} p_i(x) \delta_{\phi_{i(x)}}(B) \tag{1.10}$$

Also, to a conformal Smale endomorphism T with disjoint fibers $J_\omega, \omega \in \Sigma_I^+$, we will associate a random system with complete connections in the next result. Recall that $\pi_0 : \Sigma_I \to \Sigma_I^+, \pi_0(\eta) = \eta|_0^\infty, \eta \in \Sigma_I$.

Theorem 1.5. *Let the conformal Smale endomorphism $T : J \to J$ defined in (1.9), where we assume the spaces $Y_\omega = Y \subset \mathbb{R}^d, \forall \omega \in \Sigma_I^+$. Consider also a shift invariant measure μ on Σ_I. Define the quadruple $((W, \mathcal{W}), (X, \mathcal{X}), u, P)$ by:*

a) State space $W = \{(\omega, x),\ x \in J_\omega, \omega \in \Sigma_I^+\}$ with the Borel σ-algebra $\mathcal{W}$ induced from the Borel σ-algebra of the product space $\Sigma_I^+ \times Y \subset \Sigma_I^+ \times \mathbb{R}^d$, and index space $X = \Sigma_I^+$ with its Borel σ-algebra $\mathcal{X}$;

b) For $\tau \in \Sigma_I^+$ and $w \in W, w = (\omega, x), x \in J_\omega$, define the map $u(w, \tau) = (\sigma\omega, T_\omega(x))$;

c) For those $\omega \in \Sigma_I^+$ for which the conditional measure $\bar{\mu}^\omega$ is defined (their set has $\pi_{0*}\mu$-measure equal to 1), and for $w = (\omega, x), x \in J_\omega$, define the probability transition function $P_w(\cdot)$ by $P_w := \bar{\mu}^\omega$.

Then, $((W, \mathcal{W}), (X, \mathcal{X}), u, P)$ is a random system with complete connections.

Proof. On W we take the σ-algebra of Borel sets induced from $\Sigma_I^+ \times \mathbb{R}^d$. We defined the map $u(w, \tau) = (\sigma\omega, T_\omega(x))$, for $w = (\omega, x) \in W$, $\omega \in \Sigma_I^+$ and $x \in J_\omega$. On the other hand recall that J_ω is the set of points of the form $\hat{\pi}_2(\eta)$ for $\eta \in [\omega] \subset \Sigma_I$. So from (1.7) the map $u(\cdot, \cdot)$ is well-defined, since if $x = \hat{\pi}_2(\eta) = T_{\eta_{-1}\omega} \circ T_{\eta_{-2}\eta_{-1}\omega} \circ \ldots \in J_\omega$, then

$$T_\omega(x) = \hat{\pi}_2(\sigma\eta) \in J_{\sigma\omega},$$

as $\sigma\eta \in [\sigma\omega]$. Then, we use condition (e) from the definition of conformal Smale endomorphisms, to obtain that $u : W \times X \to W$ is measurable.

Next, for $\pi_{0*}\mu$-a.e $\omega \in \Sigma_I^+$, the conditional measure $\bar{\mu}^\omega$ is defined on the cylinder $[\omega] \subset \Sigma_I$, and this cylinder can be identified with Σ_I^+. If A is a Borel set in Σ_I^+ and $w = (\omega, x) \in W, x \in J_\omega$, define $P(w, A) = \bar{\mu}^\omega(A)$, thus P_w can be viewed as a probability measure on Σ_I^+. From the uniqueness of the system of conditional measures associated to the partition $\mathcal{P}_-$ (see [21]) with the property that

$$\int_{\Sigma_I} g(\xi) d\mu = \int_{\Sigma_I^+} \int_{[\omega]} g(\xi) d\bar{\mu}^\omega(\xi) d\pi_{0*}\mu(\omega),$$

for any integrable function $g : \Sigma_I \to \mathbb{R}$, we obtain that for any set A as above, $P_w(A)$ depends measurably on $\omega \in \Sigma_I^+$, where $w = (\omega, x), x \in J_\omega$. Hence the function

$$P(\cdot, A) : W \to \mathbb{R},\ w \to P(w, A)$$

is also measurable with respect to the σ-algebra $\mathcal{W}$ induced on W from $\Sigma_I^+ \times \mathbb{R}^d$.

□

2. Stationary measures of countable systems with overlaps and place-dependent probabilities

In this section we study the case of a countable IFS with place–dependent probabilities and overlaps, and prove the exact dimensionality of stationary measures, and compute the pointwise (and Hausdorff) dimension.

Consider a system of smooth contractions defined on a compact set $V \subset \mathbb{R}^D$ indexed by a countable alphabet I, $\mathcal{S} = \big\{\phi_i : V \longrightarrow V\big\}_{i\in I}$ with limit set Λ, and the weights $p_i : V \to \mathbb{R}$, $i \in I$, satisfying for any $x \in V$,

$$\sum_{i\in I} p_i(x) = 1. \tag{2.1}$$

Σ_I^+ denotes the 1-sided shift space with alphabet I. If $i_1, \ldots, i_n \in I, n \geq 1$, denote

$$\phi_{i_1\ldots i_n} := \phi_{i_1} \circ \ldots \circ \phi_{i_n}.$$

Let $\pi : \Sigma_I^+ \to \Lambda, \pi(\omega) = \lim\limits_{n\to\infty} \phi_{i_1 i_2\ldots i_n}$ if $\omega = (i_1, i_2, \ldots) \in \Sigma_I^+$, be the canonical coding map for the limit set Λ.

Assume also that $p_i(\cdot)$ depend uniformly Hölder continuously on $x \in V$, for $i \in I$, i.e. there exist constants $\alpha, C > 0$ such that for all $i \in I$ and all $x, y \in V$,

$$|p_i(x) - p_i(y)| \leq C|x - y|^\alpha. \tag{2.2}$$

The *transfer probability* in this case is $P(x, B) := \sum\limits_{i\in I} p_i(x)\delta_{\phi_i(x)}(B)$, and the *transfer operator* $\mathcal{L} : \mathcal{C}(V) \to \mathcal{C}(V)$ is given by:

$$\mathcal{L}(f)(x) = \int_X f(y)P(x, dy).$$

A measure μ on V is called *stationary* if it is a fixed point of the dual operator of $\mathcal{L}$,

$$\mathcal{L}^*(\nu)(B) = \int P(x, B)d\nu(x) = \sum_{i\in I} \int_{\phi_i^{-1}(B)} p_i(x)d\nu(x).$$

Define also the *Lyapunov exponent* of a shift-invariant measure μ on Σ_I^+ by:

$$\chi_\mu := -\int_{\Sigma_I^+} \log|\phi'_{\omega_1}(\pi(\sigma\omega))| \, d\mu(\omega).$$

Let us now recall the notion of *projection entropy* for an invariant measure of a countable iterated function system from [16]; this is a generalization of the notion of projection entropy from the finite case of [7]. As a matter of fact, in [16] we defined the projection entropy in the more general case of random countable iterated function systems, but here we need it only for countable deterministic systems.

So let $\mathcal{S} = \{\phi_i, i \in I\}$ be a countable system and μ be a σ-invariant probability measure on Σ_I^+. Denote by ξ the partition of Σ_I^+ into initial 1-cylinders, and by $\epsilon_{\mathbb{R}^D}$ the point partition of $\mathbb{R}^D$, and by $\pi : \Sigma_I^+ \to \Lambda$ the canonical coding map for the limit set $\Lambda \subset \mathbb{R}^D$ of the function system $\mathcal{S}$. Then $\pi^{-1}\epsilon_{\mathbb{R}^D}$ and $\sigma^{-1}(\pi^{-1}\epsilon_{\mathbb{R}^D})$ are measurable partitions of Σ_I^+. The *projection entropy* of μ with respect to $\mathcal{S}$ is defined then by,

$$h_\mu(\mathcal{S}) := H_\mu(\xi|\sigma^{-1}(\pi^{-1}\epsilon_{\mathbb{R}^D})) - H_\mu(\xi|\pi^{-1}\epsilon_{\mathbb{R}^D}). \tag{2.3}$$

We prove next that the stationary measure from Theorem 1.3 is exact dimensional.

Theorem 2.1. *In the above setting if the system $\mathcal{S} = \{\phi_i, i \in I\}$ is countable and conformal and if the probabilities $\{p_i(\cdot), i \in I\}$ satisfy (2.1)-(2.2), then the stationary measure $\tilde{\mu}_P$ for the system $\mathcal{S}$ with place-dependent probabilities $P = \{p_i(\cdot), i \in I\}$ is exact dimensional, and*

$$HD(\tilde{\mu}_P) \leq \frac{h_{\mu_\psi}(\sigma)}{\chi_{\mu_\psi}},$$

where $\psi : \Sigma_I^+ \to \mathbb{R}, \psi(\omega) := \log p_{\omega_0}(\pi(\sigma\omega)), \omega \in \Sigma_I^+$, and μ_ψ is the equilibrium measure of ψ on Σ_I^+.

Proof. First let us define the potential $\psi : \Sigma_I^+ \to \mathbb{R}$ by:

$$\psi(\omega) := \log p_{\omega_0}(\pi(\sigma\omega)),\ \omega \in \Sigma_I^+$$

From (2.1) and (2.2) it follows that ψ is summable and Hölder continuous on Σ_I^+. Then there exists an equilibrium measure μ_ψ on Σ_I^+, which projects to the probability measure ν_p on Λ, associated to the system of weights $p := (p_i, i \in I)$. The transfer operator $\mathcal{L} : \mathcal{C}(\Sigma_I^+) \to C(\Sigma_I^+)$ is,

$$\mathcal{L}(\phi)(\omega) := \sum_{i=1}^{\infty} p_i(\pi\omega)\phi(i\omega),$$

where $\mathcal{C}(\Sigma_I^+)$ is the space of bounded continuous real-valued functions on Σ_I^+. We see from above that

$$\mathcal{L}(\phi)(\omega) = \sum_{i=1}^{\infty} e^{\psi(i\omega)}\phi(i\omega)$$

For such transfer operators, it was proved (see [11]) that if ψ is Hölder continuous and summable, then there exists a unique probability measure $\tilde{\nu}_\psi$ on Σ_I^+, such that

$$\mathcal{L}^*(\tilde{\nu}_\psi) = \tilde{\nu}_\psi$$

The projection, denoted by $\tilde{\mu}_P$, of the measure $\tilde{\nu}_\psi$ onto the limit set Λ of $\mathcal{S}$, is the stationary measure of the system $\mathcal{S}$ with the place-dependent probabilities P. Hence,

$$\tilde{\mu}_P = \pi_*\tilde{\nu}_\psi$$

On the other hand, for the expanding map $\sigma : \Sigma_I^+ \to \Sigma_I^+$ and the Hölder continuous potential ψ, let us notice that there exists also a shift-invariant equilibrium measure μ_ψ of ψ on Σ_I^+, and moreover there exists a function θ such that

$$\theta\tilde{\mu}_\psi = \mu_\psi$$

Moreover this function θ satisfies

$$\theta \geq M > 0,$$

for some constant M depending on ψ (see [11]). On the other hand, the projection of the shift-invariant equilibrium measure μ_ψ onto the limit set Λ is denoted by μ_P and we have

$$\mu_P = \pi_*\mu_\psi.$$

In [16] we proved that the projection μ_P of the invariant measure μ_ψ is exact dimensional even when the system $\mathcal{S}$ has overlaps. And we found a formula for its pointwise dimension, involving the *projection entropy* (recalled above). Indeed in the main Theorem of [16], it is enough to consider a random system where the parameter space consists of only one point, and to take the identity as the evolution map on the space of parameters. This means that for μ_P-a.e $x \in \Lambda$,

$$\lim_{r\to 0} \frac{\log \mu_P(B(x,r))}{\log r} = \delta = \frac{h_{\mu_\psi}(\mathcal{S})}{\chi_{\mu_\psi}}, \tag{2.4}$$

and δ does not depend on $x \in \Lambda$. But $\mu_P(B(x,r)) = \mu_\psi(\pi^{-1}(B(x,r)))$, and we know that $\mu_\psi = \theta\tilde{\mu}_\psi$, hence

$$\tilde{\mu}_\psi(\pi^{-1}(B(x,r))) = \int_{\pi^{-1}(B(x,r))} \theta d\mu_\psi.$$

On the other hand let us recall that $\theta > M$ and that θ is a continuous bounded function on Σ_I^+. Hence using (2.4) and the fact that $\tilde{\mu}_P = \pi_*\tilde{\mu}_\psi$, we see that for μ_P-a.e $x \in \Lambda$,

$$\lim_{r\to 0} \frac{\log \tilde{\mu}_P(B(x,r))}{\log r} = \delta. \tag{2.5}$$

Therefore, the stationary measure $\tilde{\mu}_p$ of the system $\mathcal{S}$ with the place-dependent probabilities P is exact dimensional, and from (2.5) its Hausdorff (and pointwise) dimension is given by:

$$HD(\tilde{\mu}_p) = \frac{h_{\mu_\psi}(\mathcal{S})}{\chi_{\mu_\psi}}, \tag{2.6}$$

where $h_{\mu_\psi}(\mathcal{S})$ is the projection entropy of μ_ψ with respect to $\mathcal{S}$ and χ_{μ_ψ} is its Lyapunov exponent. From the definition (2.3) of $h_{\mu_\psi}(\mathcal{S})$ it follows that $h_{\mu_\psi}(\mathcal{S}) \le h_{\mu_\psi}(\sigma)$. Therefore, we obtain the desired dimension formula.

□

3. Families of fractals and unfoldings of countable IFS.

We want now to associate a Smale endomorphism to a countable iterated function systems with overlaps, by unfolding, in such a way as to control the structure of overlappings. We will consider in general equilibrium measures of real-valued summable functions ψ on Σ_I.

Let us consider a countable conformal IFS with overlaps $\mathcal{S} = \{\phi_i, i \in I\}$, where the maps $\phi_i : V \to V$ are contractions defined on a neighbourhood of a compact set $V \subset \mathbb{R}^D$, and $|\phi_i'| < \alpha < 1, i \in I$ on V. Denote by Λ the limit set of $\mathcal{S}$, and assume that Λ is not contained in the boundary of V. Since we work with a countable system, the limit set Λ may be non-compact. Assume that $I = \mathbb{N}^*$ and that $\mathcal{S}$ satisfies the Bounded Distortion Property, i.e there exist constants $H > 0, \beta > 0$ such that for all $i \in I$,

$$|\log|\phi_i'(y)| - \log|\phi_i'(z)|| \le H|y-z|^\beta, \ \forall y, z \in V \tag{3.1}$$

We will associate to $\mathcal{S}$, a Smale skew product T and a random system with complete connections, with the goal to separate the images of the compositions of maps ϕ_i along any given sequence $\omega = (\omega_1, \omega_2, \dots) \in \Sigma_I^+$. This is realised by an inductive process of *unfolding the overlaps* of $\mathcal{S}$. Recall that Λ is the limit set of $\mathcal{S}$.

Assume that for any point $x \in \Lambda$, the $\mathcal{S}$-images $\phi_i(x), i \in I$ of x, satisfy the following **Non-accumulation Condition**:

$$\phi_i(x) \notin \overline{\{\phi_j(x), j \in I \setminus \{i\}\}}, \ \forall i \in I \tag{3.2}$$

This condition is quite general, and it can be checked on many systems (see for instance the examples in [18]). Recall that $\pi(\omega) = \phi_{\omega_1\omega_2\ldots}$ is the canonical projection from Σ_I^+ to the limit set Λ. Then, for an arbitrary $\omega = (\omega_1, \omega_2, \ldots) \in \Sigma_I^+$, we define inductively the contractions $T_{i\omega}$ for $i \in I$. Let us start by defining

$$T_{1\omega} := \phi_{1\omega_1\ldots\omega_{n_1(\omega)}} = \phi_1 \circ \phi_{\omega_1} \circ \ldots \circ \phi_{\omega_{n_1(\omega)}},$$

where $n_1(\omega)$ is defined as the smallest integer $n_1 \geq 1$ such that

$$\phi_j(\pi(\omega)) \notin \phi_{1\omega_1\ldots\omega_{n_1}}(V), \text{ for all } j \neq 1$$

Next, since from above $\phi_2(\pi(\omega)) \notin \phi_{1\omega_1\ldots\omega_{n_1}}(V)$, take $n_2(\omega)$ to be the smallest integer $n_2 > n_1$ such that

$$\phi_{2\omega_1\ldots\omega_{n_2(\omega)}}(V) \cap \phi_{1\omega_1\ldots\omega_{n_1(\omega)}}(V) = \emptyset, \text{ and } \phi_j(\pi\omega) \notin \phi_{2\omega_1\ldots\omega_{n_2(\omega)}}(V), \text{ for } j \neq 2$$

Then we define

$$T_{2\omega} := \phi_{2\omega_1\ldots\omega_{n_2(\omega)}}$$

Inductively, if we defined $n_k(\omega)$ up to some $k \geq 1$, we now define $n_{k+1}(\omega) \geq 1$ as the smallest integer $n_{k+1} > n_k$ with the property that

(3.3)

$$\phi_{(k+1)\omega_1\ldots\omega_{n_{k+1}(\omega)}}(V) \cap \phi_{j\omega_1\ldots\omega_{n_j(\omega)}}(V) = \emptyset, \ 1 \leq j \leq k, \text{ and } \phi_\ell(\pi\omega) \notin \phi_{(k+1)\omega_1\ldots\omega_{n_{k+1}(\omega)}}(V), \ell \neq k+1,$$

and then the fiber map

$$T_{k+1\omega} := \phi_{(k+1)\omega_1\ldots\omega_{n_{k+1}(\omega)}} \tag{3.4}$$

Thus we constructed for any $\eta \in \Sigma_I^+$, a contraction $T_\eta = T_{\eta_1\sigma(\eta)}$ as above. Consider now a sequence $\tau \in \Sigma_I$, then for any $n \geq 1$ define the map

$$T_\tau^n = T_{\tau_{-1}^\infty} \circ T_{\tau|_{-2}^\infty} \circ \ldots \circ T_{\tau|_{-n}^\infty}$$

Then $\hat{\pi}_2(\tau) = \bigcap_{n=1}^\infty \overline{T_\tau^n(V)}$, and the fractal $J_\omega = \hat{\pi}_2([\omega])$ is contained in Λ, where $[\omega]$ is the cylinder in Σ_I determined by $\omega \in \Sigma_I^+$.

Definition 3.1. *Let us define the space $\hat{Y} := \Sigma_I^+ \times V$, and the skew product $T : \hat{Y} \to \hat{Y}$,*

$$T(\omega, x) = (\sigma(\omega), T_\omega(x)), \ (\omega, x) \in \hat{Y}$$

*We will call $T : \Sigma_I^+ \times V \to \Sigma_I^+ \times V$ the **maximal Smale system** associated to the countable IFS with overlaps $\mathcal{S}$.*

From definition we see that the map T_ω depends on the whole sequence $\omega \in \Sigma_I^+$, and not just on the projection point $\pi(\omega)$. Thus the dynamical system $(\hat{Y}, T)$ describes *how* the overlappings are formed through iterations.

If μ is a probability measure on Σ_I, define its *Lyapunov exponent with respect to T* as

$$\chi_\mu(\sigma) = -\int_{\Sigma_I} \log |T'_{\tau|_0^\infty}(\hat{\pi}_2(\tau))| \, d\mu(\tau)$$

Denote by $\pi_0 : \Sigma_I \to \Sigma_I^+$, $\pi_0(\tau) = (\tau_0, \tau_1, \ldots)$, the canonical truncation map.

We constructed above the family of random fractals $J_\omega \subset \Lambda, \omega \in \Sigma_I^+$, and now we prove that certain invariant probability measures supported on J_ω are exact dimensional.

Theorem 3.2. *Let a countable IFS with overlaps $\mathcal{S}$ as above satisfying (3.2), and let T be its associated maximal Smale system. Consider $\psi : \Sigma_I \to \mathbb{R}$ summable Hölder continuous function with equilibrium measure μ_ψ, and let ν_ψ be the canonical projection $\pi_{0,*}\mu_\psi$ of μ_ψ on Σ_I^+. Take the conditional measure μ_ψ^ω of μ_ψ on the cylinder $[\omega]$ and let $\nu_\psi^\omega := \hat{\pi}_{2*}\mu_\psi^\omega$ be its projection on J_ω, with $\hat{\pi}_2$ defined in 1.8. Then, for ν_ψ-a.e $\omega \in \Sigma_I^+$, the measure ν_ψ^ω is exact dimensional on the sub-fractal $J_\omega \subset \Lambda$, and*

$$HD(\nu_\psi^\omega) = \frac{h_{\mu_\psi}(\sigma)}{\chi_{\mu_\psi}(\sigma)}.$$

Proof. First let us notice that, from our construction, for any $i \neq j$,

$$T_{i\omega}(V) \cap T_{j\omega}(V) = \emptyset$$

Therefore the open set condition in fibers from the definition of the Smale skew-product, is satisfied. Moreover if $||\phi_i'|| < \alpha < 1, i \in I$, it follows that the same uniform contractivity condition is satisfied by all the maps $T_\omega, \omega \in \Sigma_I^+$. Hence the uniform contractivity of the maps T_ω is satisfied.

Let us see now if the maps T_ω satisfy Bounded Distortion Property (BDP). For this consider an arbitrary $\omega \in \Sigma_I^+$. Then $T_\omega = \phi_{\omega_1\omega_2\ldots\omega_n}$ for some integer n which depends on ω. Thus, there exists $L \in (0,1)$ and $H' > 0$ such that for any $y, z \in V$, we have

$$\begin{aligned}&|\log|T_\omega'(y)| - \log|T_\omega'(z)|| \leq \\ &\leq |\log|\phi_{\omega_1}(\phi_{\omega_2\ldots\omega_n}(y))| - \log|\phi_{\omega_1}(\phi_{\omega_2\ldots\omega_n}(z))|| + \ldots + |\log|\phi_{\omega_n}'(y)| - \log|\phi_{\omega_n}'(z)|| \leq \\ &\leq H|y-z|^\beta(1 + L + \ldots + L^n) = H'|y-z|^\beta\end{aligned}$$

Another condition in the definition of a conformal Smale skew product is the Holder continuity of the real-valued map on Σ_I given by:

$$\tau \longrightarrow \log|T_\tau'(\hat{\pi}_2(\tau))| \tag{3.5}$$

Let us take $\omega \in \Sigma_I^+$ and $\tau \in \Sigma_I$ such that $\tau \in [\omega]$. Then $T_\tau = T_{\tau_{-1}\omega}$, and consider the integer $n_{\tau_{-1}}(\omega)$. Then from definition, we have

$$T_{\tau_{-1}\omega} = \phi_{\tau_{-1}} \circ \phi_{\omega_1} \circ \ldots \circ \phi_{\omega_{n_{\tau_{-1}}(\omega)}}$$

But if $\eta \in [\tau_{-m} \ldots \tau_m]$ for $m \geq n_{\tau_{-1}(\omega)}$, we have that $T_\tau = T_\eta$. On the other hand, due to the form of $\hat{\pi}_2(\tau)$ given in (1.7), we have

$$d(\hat{\pi}_2(\tau), \hat{\pi}_2(\eta)) \leq C\frac{1}{2^m},$$

for some constant C independent of τ, η, m. Hence the map from (3.5) is indeed Hölder continuous on Σ_I.

Therefore, the maximal Smale system $T : \hat{Y} \to \hat{Y}$ defined above, where we defined

$$T_{\tau|_0^\infty} = \phi_{\tau_0\tau_1\ldots\tau_{n(\sigma\tau|_0^\infty)}},$$

satisfies the properties of a conformal Smale skew product endomorphism. We then apply our result from [17], to obtain the exact dimensionality and the formula for the dimension of the projection measures ν_ψ^ω on $J_\omega \subset \Lambda$, for ν_ψ-a.e. $\omega \in \Sigma_I^+$. In particular, it follows that

$$HD(\nu_\psi^\omega) = \frac{h_{\mu_\psi}(\sigma)}{\chi_{\mu_\psi}(\sigma)},$$

where $h_{\mu_\psi}(\sigma)$ is the entropy of the measure μ_ψ, and $\chi_{\mu_\psi}(\sigma)$ is its Lyapunov exponent.

□

Examples of functions ψ, based on the maximal Smale system T associated to a countable IFS with overlaps $\mathcal{S}$ as above, are given next. In Theorem 3.3 we also construct a family of measures on sub-fractals in the limit set Λ, which are related to the intricate geometry of S.

Theorem 3.3. *In the setting of Theorem 3.2 let the countable IFS $\mathcal{S}$ with limit set Λ, and for any $s > 0$ define the function $\psi_s : \Sigma_I \to \mathbb{R}$,*

$$\psi_s(\eta) = s \log |T'_{i\omega}(\hat{\pi}_2(\eta))|, \;\; \eta \in \Sigma_I, \eta_0 = i, \omega = \eta|_0^\infty.$$

Then for any $s > 0$,

a) ψ_s is summable and Hölder continuous on Σ_I.

b) If $\mu_s := \mu_{\psi_s}$ is the equilibrium measure of ψ_s on Σ_I, and $\nu_s := \pi_{0*}\mu_s$ on Σ_I^+, then for ν_s-a.e. $\omega \in \Sigma_I^+$, the measure $\nu_s^\omega := \nu_{\psi_s}^\omega$ is exact dimensional on the sub-fractal $J_\omega \subset \Lambda$ and,

$$HD(\nu_s^\omega) = \frac{h_{\mu_s}(\sigma)}{\left| \int_{\Sigma_I} \log |\phi'_{\tau_0\tau_1\dots\tau_{n_{\tau_0}(\sigma\tau)}}(\hat{\pi}_2(\tau))| \; d\mu_s(\tau) \right|}.$$

Proof. a) Since the maximal system T associated to $\mathcal{S}$ by Definition 3.1 was shown in Theorem 3.2 to be a conformal Smale endomorphism, it follows that ψ_s is Hölder continuous on Σ_I.

Also, recall the definition of $T_{i\omega}$ as a composition of maps given by the inductive relation (3.3). Also, the initial contractions $\phi_i : V \to V$ from $\mathcal{S}$ are defined on a compact set $V \subset \mathbb{R}^D$, such that $|\phi'_i| \le \alpha < 1, i \in I$ on V. From the definition (3.3), we have the increasing sequence of integers, $n_1(\omega) < n_2(\omega) < \dots$, and thus for any $\omega \in \Sigma_I^+$, the positive integers $n_k(\omega)$ satisfy the inequality,

$$n_k(\omega) \ge k, \;\; k \ge 1.$$

Now from (3.3) we have $T_{i\omega} = \phi_{i\omega_1\dots\omega_{n_i(\omega)}}$, and thus it follows from above that for any $i \in I$,

$$|T'_{i\omega}| \le \alpha^{n_i(\omega)} \le \alpha^i.$$

Therefore, since we assumed $s > 0$, it follows that,

$$\sum_{i \in I} e^{\sup \psi_s|_{[i]}} \le \sum_{k \ge 1} \alpha^{sk} < \infty.$$

Hence the real-valued function ψ_s is also summable in this case.

b) For any $s > 0$, if $\nu_s := \nu_{\psi_s}$ is the canonical projection of the equilibrium measure μ_{ψ_s} onto Σ_I^+, then in the notation of Theorem 3.2 we have for ν_s-a.e $\omega \in \Sigma_I^+$,

$$\nu_s^\omega := \nu_{\psi_s}^\omega = \hat{\pi}_{2*}\mu_{\psi_s}^\omega,$$

which is a measure supported on $J_\omega \subset \Lambda$. Now due to the properties of ψ_s proved in a), one can apply Theorem 3.2 for the measure ν_s^ω on the fiber set $J_\omega \subset \Lambda$, for ν_s-a.e. $\omega \in \Sigma_I^+$. Thus we obtain the exact dimensionality of ν_s^ω on J_ω. Then, by using the specific expression of $T_{i\omega}$ in this case, we compute the Hausdorff dimension of ν_s^ω by the above formula.

□

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