POROSITY IN CONFORMAL DYNAMICAL SYSTEMS

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ABSTRACT. In this paper we study various aspects of porosities for conformal fractals. We first explore porosity in the general context of infinite graph directed Markov systems (GDMS), and we show that their limit sets are porous in large (in the sense of category and dimension) subsets. We also provide natural geometric and dynamic conditions under which the limit set of a GDMS is upper porous or mean porous. On the other hand, we prove that if the limit set of a GDMS is not porous, then it is not porous almost everywhere. We also revisit porosity for finite graph directed Markov systems, and we provide checkable criteria which guarantee that limit sets have holes of relative size at every scale in a prescribed direction.

We then narrow our focus to systems associated to complex continued fractions with arbitrary alphabet and we provide a novel characterization of porosity for their limit sets. Moreover, we introduce the notions of upper density and upper box dimension for subsets of Gaussian integers and we explore their connections to porosity. As applications we show that limit sets of complex continued fractions system whose alphabet is co-finite, or even a co-finite subset of the Gaussian primes, are not porous almost everywhere, while they are uniformly upper porous and mean porous almost everywhere.

We finally turn our attention to complex dynamics and we delve into porosity for Julia sets of meromorphic functions. We show that if the Julia set of a tame meromorphic function is not the whole complex plane then it is porous at a dense set of its points and it is almost everywhere mean porous with respect to natural ergodic measures. On the other hand, if the Julia set is not porous then it is not porous almost everywhere. In particular, if the function is elliptic we show that its Julia set is not porous at a dense set of its points.

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1. Introduction

Let (X,d) be a metric space. A set $E \subset X$ is called *porous* if there exists a positive constant $c > 0$ such that every open ball centered at E with radius $r \in (0, \mathrm{diam}(E))$ contains an open ball of radius cr, which does not intersect E. If this condition is satisfied for balls centered at a fixed point x, then the set E is called *porous at* x. If (X,d) is a Euclidean space (or even a separable space equipped with a doubling Borel measure), then the Lebesgue density theorem easily implies that porous sets have zero Lebesgue measure. Therefore, one could obtain quantitative information about the size and structure of singular sets by investigating how “porous" they are. For example one could explore if holes of relative size appear at all or at just a fixed percentage of scales, or how the holes are locally spread around.

Indeed, various aspects of porosities have been introduced over the years in order to quantify the size and structure of exceptional sets in different contexts. A notion of porosity already appeared in the work of Denjoy [9] on trigonometric series in the 1920s. Since then, porosities have been studied widely, for example, in connection to geometric measure theory [1, 2, 13, 24, 25, 34, 52, 53, 59], geometric function theory [11, 28, 33, 41, 57], differentiability of Lipschitz maps [32, 43, 44, 55], harmonic analysis [4, 5, 17], fractal geometry and complex dynamics [5, 23, 47, 58].

In this paper we explore porosity for conformal dynamical systems. First, we perform a comprehensive study of various porosities in the context of *conformal graph directed Markov systems (GDMS)*, and we pay special attention to systems generated by complex continued fractions. Our results apply to a very broad family of fractals, as the general framework of conformal GDMS encompasses a wide selection of geometric objects, including limit sets of Kleinian and complex hyperbolic Schottky groups, Apollonian circle packings, self-conformal and self-similar sets. We then turn our attention to complex dynamics and we study various porosities for Julia sets of meromorphic functions.

Graph directed systems with a finite alphabet consisting of similarities were introduced by Mauldin and Williams in [37], see also [14]. Mauldin and the second named author developed an extensive theory of conformal GDMS with a countable alphabet in [36] stemming from [35]. In the recent monograph [7], the authors together with Tyson extended the theory of conformal GDMS in the setting of nilpotent stratified Lie groups (Carnot groups) equipped with a sub-Riemannian metric. We also refer to [6, 26, 27, 42, 51] for recent advances on various aspects of GDMS.

We defer the formal definition of a GDMS to Section 2, and we now only give a short heuristic description. A conformal GDMS $\mathscr{S}$ is modeled on a directed multigraph

(E,V), where E is a countable set of edges and V is a finite set of vertices. Each vertex $v \in V$ corresponds to a compact set X_v and each edge $e \in E$, which connects the vertices $t(e), i(e)$, corresponds to a contracting conformal map $\phi_e : X_{t(v)} \to X_{i(v)}$. An incidence matrix $A : E \times E \to \{0,1\}$ determines if a pair of these maps is allowed to be composed. The *limit set* of $\mathcal{S}$ is denoted by $J_{\mathcal{S}}$, and is defined as the image of a natural projection from the symbol space of admissible words to $X := \cup_{v\in V} X_v$.

If $\mathcal{S}$ is a finite and irreducible conformal GDMS (see Section 2 for the exact definitions) whose limit set $J_{\mathcal{S}}$ has zero Lebesgue measure then it is porous, see e.g. [36, Theorem 4.6.4], [58, Theorem 2.5] or [23, Theorem 2.6]. Nevertheless, if the system $\mathcal{S}$ is infinite the situation is very different. As we shall see in the following, there are many examples of conformal GDMS whose limit sets have Lebesgue measure zero but they are not porous; for example the limit set associated to complex continued fractions.

Although limit sets of finitely irreducible conformal GDMS are very often not porous, we will prove that they are *always* porous in large (in the sense of category and dimension) subsets. We record that (3.4) in the following theorem is a mild non-degeneracy condition which ensures that the limit set has Lebesgue measure zero.

Theorem 1.1. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS which satisfies* (3.4)*. Then the following hold:*

(i) The limit set $J_{\mathcal{S}}$ is porous at every fixed point of $\mathcal{S}$, in particular $J_{\mathcal{S}}$ is porous at a dense set of $J_{\mathcal{S}}$.

(ii) For every $\varepsilon > 0$ there exists some finite $F(\varepsilon) := F \subset E$ with

$$\dim_{\mathcal{H}}(J_{\mathcal{S}_F}) > \dim_{\mathcal{H}}(J_{\mathcal{S}}) - \varepsilon,$$

such that $J_{\mathcal{S}}$ is porous at every $x \in J_{\mathcal{S}_F}$ with porosity constant only depending on F and $\mathcal{S}$.

(iii) There exists some set $\tilde{J} \subset J_{\mathcal{S}}$ such that $\dim_{\mathcal{H}}(\tilde{J}) = \dim_{\mathcal{H}}(J_{\mathcal{S}})$ and $J_{\mathcal{S}}$ is porous at every $x \in \tilde{J}$.

It is not uncommon in the literature to call the notion of porosity discussed so far as lower porosity, whereas the weaker concept where the holes appear on arbitrarily small (but not necessarily all) scales is called *upper porosity*, see ex. [9, 11]. In Section 3.2, we provide a checkable pointwise criterion for the (pointwise) upper porosity of limit sets of conformal GDMSs.

Theorem 1.2. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS such that $\overline{J_{\mathcal{S}}} \neq X$. If $\omega \in E_A^{\mathbb{N}}$ and*

$$\overline{\lim}_{n\to\infty} \operatorname{dist}(\pi(\sigma^n(\omega)), X^c) > 0, \tag{1.1}$$

then $\overline{J_{\mathcal{S}}}$ is upper porous at $\pi(\omega)$.

We then employ Theorem 1.2 in order to show that under some additional separation assumptions the limit sets of conformal GDMSs are upper porous everywhere or almost everywhere with respect to natural measures.

Theorem 1.3. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS*

(i) If $\operatorname{dist}(J_{\mathcal{S}}, X^c) > 0$ then $\overline{J_{\mathcal{S}}}$ is upper porous at every $x \in J_{\mathcal{S}}$.

(ii) If $\mathcal{S}$ satisfies the strong open set condition, $\overline{J_{\mathcal{S}}} \neq X$, and $\tilde{\mu}$ is any σ-invariant ergodic measure on $E_A^{\mathbb{N}}$, then $\overline{J_{\mathcal{S}}}$ is upper porous at $\tilde{\mu} \circ \pi^{-1}$-a.e. $x \in J_{\mathcal{S}}$.

We record that there are always shift invariant ergodic measures on the limit set of a finitely irreducible conformal GDMS $\mathcal{S}$, see Theorem 2.1. In particular, there exists a shift invariant ergodic measure on $J_{\mathcal{S}}$ which is globally equivalent to the *h-conformal measure* m_h, where $h = \dim_{\mathcal{H}}(J_{\mathcal{S}})$. Hence, Theorem 1.3 (ii) holds for m_h as well.

It is well known that the h-conformal measure captures the right amount of information for the limit set of a conformal GDMS. If the system $\mathcal{S}$ is finite, m_h is, up to multiplicative constants, equal to the Hausdorff and packing measures restricted on $J_{\mathcal{S}}$, see e.g. [7, Theorem 7.18]. But if the system has infinitely many generators, then Hausdorff measure can vanish while packing measure can be infinite, and then (and only then) the above property may fail. See the end of Section 2 for a short introduction to conformal measures.

In Section 3.3, we show that under some natural assumptions limit sets of conformal GDMSs are porous on a fixed percentage of scales. This behavior is quantified by the notion of *mean porosity* whose formal definition is deferred to Section 3.3. Koskela and Rohde introduced mean porosity in [28] in connection with dimension estimates and quasiconformal mappings, and it was further investigated by several authors, ex. [1, 2, 41, 53]. We record that mean porosity is a stronger condition than upper porosity. We will show that limit sets of conformal GDMS are mean porous almost everywhere with respect to natural measures.

Theorem 1.4. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS such that (3.4) holds. Let $\tilde{\mu}$ be any σ-invariant ergodic measure on $E_A^{\mathbb{N}}$ with finite Lyapunov exponent $\chi_{\tilde{\mu}}(\sigma)$. Then $J_{\mathcal{S}}$ is mean porous at $\tilde{\mu} \circ \pi^{-1}$-a.e. $x \in J_{\mathcal{S}}$ with porosity constants depending only on $\mathcal{S}$ and $\tilde{\mu}$.*

Using Theorem 1.4 we show that if, moreover, $\mathcal{S}$ is strongly regular then $J_{\mathcal{S}}$ is mean porous at m_h-a.e. $x \in J_{\mathcal{S}}$, where, as before, $h := \dim_{\mathcal{H}}(J_{\mathcal{S}})$ and m_h is the h-conformal measure of $\mathcal{S}$, see Corollary 3.11.

As mentioned earlier, limit sets of finite and irreducible GDMSs with zero Lebesgue measure are porous; that is for any point of the limit set there are nearby holes of radius proportional to their distance from that point. However, if one is interested on the spatial distribution of the holes, *directed porosity* is the most appropriate kind of porosity to study. Given $v \in S^{n-1}$ we say that a set $E \subset \mathbb{R}^n$ is *v-directed porous* at x if the corresponding holes are centered in the line $\{x + tv, t \in \mathbb{R}\}$, see Section 3.4 for the exact definition.

As far as the authors know, directed (or directional) porosity first appears in the work of Preiss and Zajíček [43, 44] regarding differentiability of Lipschitz maps in Banach spaces. See also the book by Lindenstrauss, Preiss, and Tiser [32] and Speight's Ph.D thesis [55] for further advances and related references. Directed porosity was also introduced independently in [5] (as the first named author was not aware of [43] at the time) in the context of fractal geometry and harmonic analysis. Actually, more general notions of directed porosity were considered in [5], where lines could be replaced by any m-dimensional plane in $\mathbb{R}^n$. Among other things, it was shown in [5] that if the limit set of a finite conformal IFS has Hausdorff dimension less or equal than 1, then it is directed porous for all directions, see [5, Corollary 1.3]. In this paper we provide an easily checkable sufficient condition for a finite and irreducible GDMS to be directed porous at all directions.

Theorem 1.5. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finite and irreducible conformal GDMS such that*

$$\mathrm{dist}(\partial X, J_{\mathcal{S}}) > 0. \tag{1.2}$$

Then $J_{\mathcal{S}}$ is v-directed porous for every $v \in S^{n-1}$.

Using Theorem 1.5 we show that any finite and irreducible conformal GDMS which satisfies the well known Strong Separation Condition, is directed porous at all directions, see Theorem 3.17. We also consider systems where (1.2) does not necessarily hold. In Theorem 3.18 we show that if an IFS consists of rotation free similarities and there exist directions $v \in S^1$ such that the lines $\{x + tv, t \in \mathbb{R}\}$ miss the set of first iterations in the interior of the set X, then the limit set is v-directed porous.

Theorems 1.1, 1.4 and 1.5 all deal with various positive aspects of porosities for conformal GDMSs. On the opposite spectrum we also investigate non-porosity of infinite GDMSs. Among other things we prove that if the limit set of a finitely irreducible conformal GDMS is not porous at a single point of its closure then it is not porous in a set of full conformal measure.

Theorem 1.6. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS such that $J_{\mathcal{S}}$ is not porous, or $J_{\mathcal{S}}$ is not porous at some $\zeta \in \overline{J_{\mathcal{S}}}$. Then $J_{\mathcal{S}}$ is not porous at m_h-a.e. $x \in J_{\mathcal{S}}$.*

Moreover for every $h \in (0, n)$ we construct an infinite IFS $\mathcal{S}_h$ consisting of similarities in $\mathbb{R}^n$ such that $\dim_{\mathcal{H}}(J_{\mathcal{S}_h}) = h$ and $J_{\mathcal{S}_h}$ is not porous, and we use Theorem 1.6 to conclude that $J_{\mathcal{S}_h}$ is not porous m_h-a.e. See Theorem 3.21 for the construction.

In the last three sections we study porosity for some well-known families of dynamical systems. In Section 4 we perform a comprehensive study of porosity in the setting of *complex continued fractions*. The origins of complex continued fractions can be traced back to the works of the brothers Adolf Hurwitz [21] and Julius Hurwitz [22]. Since their pioneering contributions, complex continued fractions have been studied widely from different viewpoints, see e.g. [18, Chapter 5] and the references therein.

Recall that any irrational number in $[0,1]$ can be represented as a continued fraction

$$\cfrac{1}{e_1 + \cfrac{1}{e_2 + \cfrac{1}{e_3 + \dots}}},$$

where $e_i \in \mathbb{N}$ for all $i \in \mathbb{N}$. It is a remarkable fact that continued fractions can be described by the infinite conformal IFS

$$\mathcal{CF}_{\mathbb{N}} := \left\{\phi_n : [0,1] \to [0,1] : \phi_n(x) = \frac{1}{n+x} \text{ for } n \in I \subset \mathbb{N}\right\}.$$

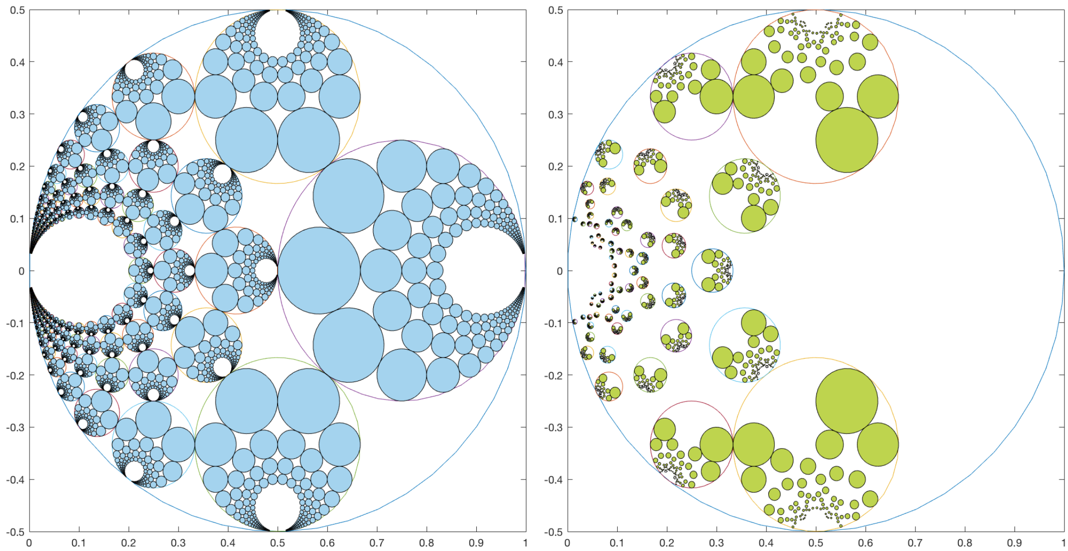

FIGURE 1. Approximation of the limit sets $\mathcal{CF}_E$ and of $\mathcal{CF}_{GP_{-\pi,\pi}}$ after two iterations.

As in the case of real continued fractions, complex continued fractions can be represented by the infinite conformal IFS

$$\mathcal{CF}_E = \{\phi_e : \bar{B}(1/2,1/2) \to \bar{B}(1/2,1/2)\}_{e\in E}$$

where

$$E = \{m + ni : (m,n) \in \mathbb{N}\times\mathbb{Z}\} \text{ and } \phi_e(z) = \frac{1}{e+z}.$$

This system was first considered in [35] and its geometric and dimensional properties were further studied by various authors, see e.g. [6, 15, 45, 58]. In particular it was shown in [58] that $\mathcal{CF}_E$ is not porous.

In this paper we consider porosity properties of $\mathcal{CF}_I$ when $I \subset E$ and we generalize the results from [58] in various ways. In the core of our approach lies Theorem 4.5, which provides a novel characterization of porosity for limit sets of complex continued fractions with arbitrary alphabet. One interesting aspect of our characterization is that given any alphabet $I \subset E$ one can check if J_I is porous by solely examining how I is distributed across E. The proof of Theorem 4.5, which is the most technical proof in our paper, is quite delicate and rather long as we have to consider several cases. See also Remark 4.10 on how Theorem 4.5 extends and streamlines earlier results from [58].

Of special interest to us, are complex continued fraction systems whose alphabet I is finite, co-finite, or even a radial subset of Gaussian primes of the form,

$$GP_{a,b} = \{w \in E : w \text{ is a Gaussian prime and } \arg w \in [a,b)\},$$

where $-\pi/2 \le a < b \le \pi/2$. We record that Gaussian primes are a very intriguing subset of $\mathbb{Z}[i]$ with many related important open questions, see e.g. [16].

Theorem 1.7. *Let $I \subset E$ and denote the limit set of $\mathcal{CF}_I$ by $J_I := J_{\mathcal{CF}_I}$. Moreover let m_{h_I} be the h_I-conformal measure of $\mathcal{CF}_I$, where $h_I = \dim_{\mathcal{H}}(J_I)$.*

(i) The limit set J_E is (uniformly) upper porous.
(ii) If I is finite then J_I is directed porous for all $v \in S^1$.
(iii) If I is co-finite then:
  (a) J_I is m_{h_I}-a.e. not porous,
  (b) J_I is m_{h_I}-a.e. mean porous with porosity constant only depending on I.
(iv) If I is a co-finite subset of $GP_{a,b}$ for some $-\pi/2 \le a < b \le \pi/2$ then:
  (a) J_I is m_{h_I}-a.e. not porous,
  (b) J_I is m_{h_I}-a.e. mean porous with porosity constant only depending on I.

Theorem 1.7 highlights the differences between finite and infinite continued fractions systems regarding their porosity properties. If the alphabet of a continued fractions system is finite, then it is porous in all directions. On the other hand, if the alphabet is infinite, even if it is relatively sparse as a radial subset of the Gaussian primes, then the limit set is not porous almost everywhere. However, continued fractions whose alphabets are co-finite subsets of E or co-finite subsets of radial sectors of Gaussian primes are a.e. porous on a fixed percentage of scales.

The proof of Theorem 1.7 employs several new ideas. In particular, we introduce *upper densities* and *upper box dimensions* (see Defintions 4.14 and 4.18 respectively) for subsets of Gaussian integers and we explore their connections to porosity. In Proposition 4.12 we show that if the limit set of a complex continued fractions system is porous, then its alphabet has upper density strictly less than 1. Since co-finite subsets of E have upper density equal to 1, we thus deduce that their limit sets are *not* porous. We then employ Theorem 1.6 in order to obtain Theorem 1.7 (iii)a. Although J_E and its co-finite subsystems are not porous, it is perhaps surprising that J_E is uniformly upper porous. This is the content of Theorem 4.1. In Theorem 4.19 we prove that if the limit set of a complex continued fractions system is porous, then its alphabet has upper box dimension strictly less than 2. Using Hecke's Prime Number theorem we then show that co-finite subsets of radial sectors of Gaussian primes have upper box dimension equal to 2, hence they define limit sets which are *not* porous. Thus, Theorem 1.7 (iv)a again follows from Theorem 1.6.

We finally turn our attention to complex dynamics and we conclude our paper with two fairly short sections which explore porosity for Julia sets of meromorphic, especially transcendental, functions and then, more specifically, of elliptic functions. We first investigate various porosity properties of tame meromorphic functions. We record that tameness is a mild hypothesis which is satisfied by many natural classes of maps, see Remark 5.2 for more details.

Theorem 1.8. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a tame meromorphic function such that $J(f) \neq \mathbb{C}$, where $J(f)$ denotes the Julia set of f. Moreover, let μ be a Borel probability f-invariant ergodic measure on $J(f)$ with full topological support.*

(i) The Julia set $J(f)$ is porous at a dense set of its points.

(ii) *If μ has finite Lyapunov exponent $\chi_\mu(f) := \int_{J(f)} \log|f'|\,d\mu$, then $J(f)$ is mean porous at μ-a.e. $x \in J(f)$ with porosity constants only depending on f.*

(iii) *If $J(f)$ is not porous in $\mathbb{C}$ then $J(f)$ is not porous at μ-a.e. $x \in J(f)$.*

The proof of Theorem 1.8 is based on the porosity results on conformal iterated function systems obtained in Section 3. Remarkably, such applications are possible due to a powerful tool of complex dynamics known as *nice set*. Roughly speaking, each sufficiently "good" meromorphic function admits a set, commonly named as a nice set, such that the holomorphic inverse branches of the first return map it induces, form a conformal iterated function system satisfying the open set condition. We carefully define nice sets in Section 5 and we show how they lead to conformal iterated function systems. More details about these sets, their properties, and their constructions can be found for example in [10, 31, 46, 50, 54].

Recall that a meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is called *elliptic* if it is doubly periodic. Iteration of elliptic functions have been studied widely in complex dynamics. A comprehensive and systematic account of iteration of elliptic functions can be found in the forthcoming book [31]; we also refer the reader to the following articles: [20, 29, 30, 40]. In addition to the results on general meromorphic function discussed above, the following theorem establishes several, more refined, porosity properties of Julia sets of elliptic functions. We would also like to emphasize that in this case we impose no tameness assumptions.

Theorem 1.9. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a non-constant elliptic function. Then:*

(i) *The Julia set $J(f)$ is not porous at a dense set of its points, in particular it is not porous at any point of the set*

$$P_f := \bigcup_{n=1}^{\infty} f^{-n}(\infty).$$

(ii) *For every $b \in P_f$ and for all $\kappa \in (0,1)$ there exists $R(b,\kappa) > 0$ such that*

$$\mathrm{por}(J(f), b, r) \le \kappa$$

for all $r \in (0, R(b,\kappa))$.

(iii) *If in addition $J(f) \neq \mathbb{C}$, then $J(f)$ is porous at a dense set of its points; the repelling periodic points of f.*

The paper is organized as follows. In Section 2 we introduce all the relevant concepts related to graph directed Markov systems and their thermodynamic formalism. Section 3 consists of four subsections, each of them devoted to a different aspect of porosity for general graph directed Markov systems. In Subsection 3.1 we explore (classical) porosity and we prove Theorem 1.1. Subsection 3.3 deals with mean porosity and contains, among other results, the proof of Theorem 1.4. In Subsection 3.4 we study directed porosity for finite systems and we prove Theorem 1.5. In Subsection 3.5 we investigate how non-porous are the limit sets of general graph directed Markov systems, and we prove Theorem 1.6. In Section 4 we narrow our focus to complex continued fractions and we prove Theorem 1.7. Sections 5 and 6 delve into porosity for Julia sets of meromorphic functions. In Section 5 we deal with tame meromorphic functions and we prove Theorem 1.8. Section 6 concerns elliptic functions and contains the proof of Theorem 1.9.

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2. Preliminaries

A *graph directed Markov system* (GDMS)

$$\mathcal{S} = \left\{V, E, A, t, i, \{X_v\}_{v\in V}, \{\phi_e\}_{e\in E}\right\}$$

consists of

- a directed multigraph (E, V) with a countable set of edges E, frequently referred to as the *alphabet* of $\mathcal{S}$, and a finite set of vertices V,
- an incidence matrix $A: E\times E \to \{0,1\}$,
- two functions $i, t: E \to V$ such that $t(a) = i(b)$ whenever $A_{ab} = 1$,
- a family of non-empty compact metric spaces $\{X_v\}_{v\in V}$,
- a family of injective contractions

$$\{\phi_e : X_{t(e)} \to X_{i(e)}\}_{e\in E}$$

  such that every ϕ_e, $e\in E$, has Lipschitz constant no larger than s for some $s \in (0,1)$.

We will always assume that the alphabet E contains at least two elements and for every $v \in V$ there exist $e, e' \in E$ such that $t(e) = v$ and $i(e') = v$. We will frequently use the simpler notation $\mathcal{S} = \{\phi_e\}_{e\in E}$ for a GDMS. If a GDMS has finite alphabet it will be called *finite*.

We now introduce some standard notation from symbolic dynamics. For every $\omega \in E^* := \bigcup_{n=0}^{\infty} E^n$, we denote by $|\omega|$ the unique integer $n \geq 0$ such that $\omega \in E^n$, and we call $|\omega|$ the *length* of ω. We also set $E^0 = \{\emptyset\}$. If $\omega \in E^{\mathbb{N}}$ and $n \geq 1$, we define

$$\omega|_n := \omega_1 \dots \omega_n \in E^n.$$

For $\tau \in E^*$ and $\omega \in E^* \cup E^{\mathbb{N}}$, we let

$$\tau\omega := (\tau_1, \dots, \tau_{|\tau|}, \omega_1, \dots).$$

Given $\omega, \tau \in E^{\mathbb{N}}$, we denote the longest initial block common to both ω and τ by $\omega \wedge \tau \in E^{\mathbb{N}} \cup E^*$. We also denote by

$$\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$$

the *shift map*, which is given by the formula

$$\sigma\left((\omega_n)_{n=1}^{\infty}\right) = \left((\omega_{n+1})_{n=1}^{\infty}\right).$$

Given a matrix $A: E\times E \to \{0,1\}$ we denote

$$E_A^{\mathbb{N}} := \{\omega \in E^{\mathbb{N}} :\ A_{\omega_i \omega_{i+1}} = 1 \text{ for all } i \in \mathbb{N}\}.$$

Elements of $E_A^{\mathbb{N}}$ are called A-*admissible* (infinite) words. We also set

$$E_A^n := \{w \in E^{\mathbb{N}} : \ A_{\omega_i \omega_{i+1}} = 1 \text{ for all } 1 \le i \le n-1\}, \quad n \in \mathbb{N},$$

and

$$E_A^* := \bigcup_{n=0}^{\infty} E_A^n.$$

The elements of E_A^* are called A-admissible (finite) words. A matrix $A : E \times E \to \{0,1\}$ is called *finitely irreducible* if there exists a finite set $\Lambda \subset E_A^*$ such that for all $i, j \in E$ there exists $\omega \in \Lambda$ for which $i\omega j \in E_A^*$. If the associated matrix of a GDMS is finitely irreducible, we will call the GDMS finitely irreducible as well. For every $\omega \in E_A^*$, we let

$$[\omega] := \{\tau \in E_A^{\mathbb{N}} : \ \tau_{|_{|\omega|}} = \omega\}.$$

Let $\mathcal{S} = \big\{V, E, A, t, i, \{X_v\}_{v \in V}, \{\phi_e\}_{e \in E}\big\}$ be a GDMS. For $\omega \in E_A^*$ we define the map coded by ω:

$$\phi_\omega = \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)} \qquad \text{if } \omega \in E_A^n. \tag{2.1}$$

Slightly abusing notation we will let $t(\omega) = t(\omega_n)$ and $i(\omega) = i(\omega_1)$ for ω as in (2.1).

For $\omega \in E_A^{\mathbb{N}}$, the sets $\{\phi_{\omega|_n}(X_{t(\omega_n)})\}_{n=1}^{\infty}$ form a decreasing (in the sense of inclusion) sequence of non-empty compact sets and therefore they have nonempty intersection. Moreover

$$\operatorname{diam}(\phi_{\omega|_n}(X_{t(\omega_n)})) \le s^n \operatorname{diam}(X_{t(\omega_n)}) \le s^n \max\{\operatorname{diam}(X_v) : v \in V\}$$

for every $n \in \mathbb{N}$, hence

$$\pi(\omega) := \bigcap_{n \in \mathbb{N}} \phi_{\omega|_n}(X_{t(\omega_n)})$$

is a singleton. Thus we can now define the coding map

$$\pi : E_A^{\mathbb{N}} \to \bigoplus_{v \in V} X_v := X, \tag{2.2}$$

the latter being a disjoint union of the sets X_v, $v \in V$. The set

$$J = J_{\mathcal{S}} := \pi(E_A^{\mathbb{N}})$$

will be called the *limit set* (or *attractor*) of the GDMS $\mathcal{S}$.

For $\alpha > 0$, we define the metrics d_α on $E_A^{\mathbb{N}}$ by setting

$$d_\alpha(\omega, \tau) = \mathrm{e}^{-\alpha|\omega \wedge \tau|}. \tag{2.3}$$

It is easy to see that all the metrics d_α induce the same topology. It is also well known, see [7, Proposition 4.2], that the coding map $\pi : E_A^{\mathbb{N}} \to \bigoplus_{v \in V} X_v$ is Hölder continuous, when $E_A^{\mathbb{N}}$ is equipped with any of the metrics d_α as in (2.3) and $\bigoplus_{v \in V} X_v$ is equipped with the direct sum metric.

In this paper we will focus on conformal GDMS. Let $U \subset \mathbb{R}^n$ be open and connected. Recall that a C^1 diffeomorphism $\phi : U \to \mathbb{R}^n$ is *conformal* if its derivative at every point of U is a similarity map. By $D\phi(z) : \mathbb{R}^n \to \mathbb{R}^n$ we denote the derivative of ϕ evaluated at the point z and by we denote by $\|D\phi(z)\|$ its norm, which in the conformal case coincides with the similarity ratio. Note that for

- $n = 1$ the map ϕ is conformal if and only if it is a C^1-diffeomorphism,

- $n = 2$ the map ϕ is conformal if and only if it is either holomorphic or antiholomorphic,
- $n \geq 3$ the map ϕ is conformal if and only if it is a Möbius transformation.

Definition 2.1. A graph directed Markov system $\mathcal{S}$ is called *conformal* if the following conditions are satisfied.

(i) For every vertex $v \in V$, X_v is a compact subset of a fixed Euclidean space $\mathbb{R}^n$ and $X_v = \overline{\mathrm{Int}(X_v)}$.

(ii) (*Open Set Condition* or *OSC*). For all $a, b \in E$, $a \neq b$,

$$\phi_a(\mathrm{Int}(X_{t(a)})) \cap \phi_b(\mathrm{Int}(X_{t(b)})) = \emptyset.$$

(iii) For every vertex $v \in V$ there exist open and connected sets $W_v \supset X_v$ such that for every $\omega \in E^*$, the map ϕ_ω extends to a C^1 conformal diffeomorphism of $W_{t(\omega)}$ into $W_{i(\omega)}$.

(iv) (*Bounded Distortion Property* or *BDP*) For each $v \in V$ there exist compact and connected sets S_v such that $X_v \subset \mathrm{Int}(S_v) \subset S_v \subset W_v$ so that

$$\left| \frac{\|D\phi_\omega(p)\|}{\|D\phi_\omega(q)\|} - 1 \right| \leq L|p-q|^\alpha \text{ for all } \omega \in E_A^* \text{ and } p, q \in S_{t(\omega)},$$

where $\alpha > 0$ and $L \geq 1$ are two constants depending only on $\mathcal{S}, S_v$ and W_v.

Remark 2.2. If $n \geq 2$ the definition of a conformal GDMS can be significantly simplified. First, condition (iii) can be replaced by the following weaker condition:

(iii)' For every vertex $v \in V$ there exists an open connected set $W_v \supset X_v$ such that for every $e \in E$, the map ϕ_e extends to a C^1 conformal diffeomorphism of $W_{t(e)}$ into $W_{i(e)}$.

Moreover, Condition (iv) is superfluous since Condition (iii)' $\Longrightarrow$ Condition (iv)(with $\alpha = 1$), see e.g. [36] and [31].

Remark 2.3. When V is a singleton and for every $e_1, e_2 \in E$, $A_{e_1e_2} = 1$ if and only if $t(e_1) = i(e_2)$, the GDMS is called an *iterated function system* (IFS).

Remark 2.4. In several instances we will use some stronger separation conditions than OSC. We will say that a conformal GDMS $\mathcal{S} = \{\phi_e\}_{e\in E}$ satisfies the

- *Strong Open Set Condition*, or *SOSC*, if

$$\mathrm{Int}(X) \cap J_{\mathcal{S}} \neq \emptyset,$$

- *Strong Separation Condition*, or *SSC*, if

$$\phi_a(X_{t(a)}) \cap \phi_b(X_{t(b)}) = \emptyset$$

for all $a, b \in E$ such that $a \neq b$.

Note that the Bounded Distortion Property implies that there exists some constant depending only on $\mathcal{S}, S_v, W_v$ such that

$$K^{-1} \leq \frac{\|D\phi_\omega(p)\|}{\|D\phi_\omega(q)\|} \leq K \tag{2.4}$$

for every $\omega \in E_A^*$ and every pair of points $p,q \in S_{t(\omega)}$.

For $\omega \in E_A^*$ we set

$$\|D\phi_\omega\|_\infty := \|D\phi_\omega\|_{X_{t(\omega)}}.$$

Note that (2.4) and the Leibniz rule easily imply that if $\omega \in E_A^*$ and $\omega = \tau\upsilon$ for some $\tau,\upsilon \in E_A^*$, then

$$K^{-1}\|D\phi_\tau\|_\infty \|D\phi_\upsilon\|_\infty \leq \|D\phi_\omega\|_\infty \leq \|D\phi_\tau\|_\infty \|D\phi_\upsilon\|_\infty. \tag{2.5}$$

Moreover, there exists a constant M, depending only on $\mathcal{S}$, such thatfor every $\omega \in E_A^*$, and every $p,q \in S_{t(\omega)}$,

$$d(\phi_\omega(p),\phi_\omega(q)) \leq MK\|D\phi_\omega\|_\infty d(p,q), \tag{2.6}$$

where d is the Euclidean metric on $\mathbb{R}^n$. In particular for every $\omega \in E_A^*$

$$\operatorname{diam}(\phi_\omega(X_{t(\omega)})) \leq MK\|D\phi_\omega\|_\infty \operatorname{diam}(X_{t(\omega)}). \tag{2.7}$$

The following lemma, which we'll use repeatedly, provides information on the distortion of the iterations ϕ_ω. It's proof can be found in [36, Section 4.1]. For each $v \in V$, we denote

$$\eta_{\mathcal{S}} = \min\{\operatorname{dist}(X_v, \partial S_v) : v \in V\}. \tag{2.8}$$

Lemma 2.5 (Egg Yolk Principle)**.** *Let $S = \{\phi_e\}_{e\in E}$ be a conformal GDMS. Then for all finite words $\omega \in E_A^*$, all $p \in X_{t(\omega)}$ and all $0 < r < \eta_{\mathcal{S}}$,*

$$B(\phi_\omega(p), K^{-1}\|D\phi_\omega\|_\infty r) \subset \phi_\omega(B(p,r)) \subset B(\phi_\omega(p), K\|D\phi_\omega\|_\infty r). \tag{2.9}$$

We also need to recall some well known facts from thermodynamic formalism. Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS. For $t \geq 0$ and $n \in \mathbb{N}$ let

$$Z_n(\mathcal{S},t) := Z_n(t) := \sum_{\omega\in E_A^n} \|D\phi_\omega\|_\infty^t. \tag{2.10}$$

By (2.5) we easily see that

$$Z_{m+n}(t) \leq Z_m(t)Z_n(t), \tag{2.11}$$

and consequently the sequence $(\log Z_n(t))_{n=1}^\infty$ is subadditive. Thus, the limit

$$\lim_{n\to\infty} \frac{\log Z_n(t)}{n}$$

exists and equals $\inf_{n\in\mathbb{N}}(\log Z_n(t)/n)$. The value of the limit is denoted by $P(t)$ (or if we want to be more precise by $P_{\mathcal{S}}(t)$) and it is called the *topological pressure* of the system $\mathcal{S}$ evaluated at the parameter t. We also define two special parameters related to topological pressure; we let

$$\theta(\mathcal{S}) := \theta = \inf\{t \geq 0 : P(t) < \infty\} \quad \text{and} \quad h(\mathcal{S}) := h = \inf\{t \geq 0 : P(t) \leq 0\}.$$

The parameter $h(\mathcal{S})$ is known as *Bowen's parameter*.

It is well known that $t \mapsto P(t)$ is decreasing on $[0,+\infty)$ with $\lim_{t\to+\infty} P(t) = -\infty$, and it is convex and continuous on $\overline{\{t \geq 0 : P(t) < \infty\}}$. Moreover

$$\theta(\mathcal{S}) := \theta = \inf\{t \geq 0 : P(t) < \infty\} = \inf\{t \geq 0 : Z_1(t) < \infty\}, \tag{2.12}$$

and for $t \geq 0$

$$P(t) < +\infty \text{ if and only if } Z_1(t) < +\infty. \tag{2.13}$$

For proofs of these facts see e.g. [7, Proposition 7.5] and [6, Lemma 3.10].

Definition 2.6. A finitely irreducible conformal GDMS $\mathcal{S}$ is:

(i) *regular* if $P(h) = 0$,
(ii) *strongly regular* if there exists $t \geq 0$ such that $0 < P(t) < +\infty$.
(iii) *co-finitely regular* if $P(\theta) = +\infty$.

It is well known, see e.g. [7, Proposition 7.8] that

$$\text{co-finitely regular} \Longrightarrow \text{strongly regular} \Longrightarrow \text{regular}. \tag{2.14}$$

Topological pressure plays a key role in the dimension theory of conformal dynamical systems:

Theorem 2.7. *If $\mathcal{S}$ is a finitely irreducible conformal GDMS, then*

$$h(S) = \dim_{\mathcal{H}}(J_{\mathcal{S}}) = \sup\{\dim_{\mathcal{H}}(J_F) : F \subset E \textit{ finite}\}.$$

For the proof see [7, Theorem 7.19] or [36, Theorem 4.2.13].

We close this section with a discussion regarding conformal measures. If $\mathcal{S} = \{\phi_e\}_{e\in E}$ is a finitely irreducible conformal GDMS we define

$$\mathrm{Fin}(\mathcal{S}) := \left\{ t : \sum_{e\in E} ||D\phi_e||_\infty^t < +\infty \right\}.$$

Moreover for $t \in \mathrm{Fin}(\mathcal{S})$ we define the *Perron-Frobenius operator* with respect to $\mathcal{S}$ and t as

$$\mathcal{L}_t g(\omega) = \sum_{i:\, A_{i\omega_1}=1} g(i\omega)\|D\phi_i(\pi(\omega))\|^t \quad \text{for } g \in C_b(E_A^{\mathbb{N}}) \text{ and } \omega \in E_A^{\mathbb{N}}, \tag{2.15}$$

where $C_b(E_A^{\mathbb{N}})$ is the Banach space of all real-valued bounded continuous functions on $E_A^{\mathbb{N}}$ endowed with the supremum norm $||\cdot||_\infty$. Note that $\mathcal{L}_t : C_b(E_A^{\mathbb{N}}) \to C_b(E_A^{\mathbb{N}})$. As usual we denote by $\mathcal{L}_t^* : C_b^*(E_A^{\mathbb{N}}) \to C_b^*(E_A^{\mathbb{N}})$ the dual operator of $\mathcal{L}_t$. We will use the following theorem repeatedly. Its proof can be found in [7, Theorem 7.4].

Theorem 2.1. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS and let $t \in \mathrm{Fin}(\mathcal{S})$.*

(i) There exists a unique eigenmeasure $\tilde{m}_t$ of the conjugate Perron-Frobenius operator $\mathcal{L}_t^*$ and the corresponding eigenvalue is $e^{P(t)}$.
(ii) The eigenmeasure $\tilde{m}_t$ is a Gibbs state for the potential

$$\omega \longmapsto t \log \|D\phi_{\omega_1}(\pi(\sigma(\omega))\| := t\zeta(\omega).$$

(iii) The potential $t\zeta : E_A^{\mathbb{N}} \to \mathbb{R}$ has a unique shift-invariant Gibbs state $\tilde{\mu}_t$ which is ergodic and globally equivalent to $\tilde{m}_t$.

For all $t \in \text{Fin}(\mathcal{S})$ we will denote

$$m_t := \tilde{m}_t \circ \pi^{-1} \text{ and } \mu_t := \tilde{\mu}_t \circ \pi^{-1}. \tag{2.16}$$

We record that the measures $\tilde{\mu}_t, \tilde{m}_t$ are probability measures, since they are Gibbs states, hence the measures m_t, μ_t are probability measures as well. Note also that $h = h(\mathcal{S}) \in \text{Fin}(\mathcal{S})$, and as it turns out the measure m_h, which we will call the h-*conformal measure* of $\mathcal{S}$, is particularly important for the geometry of $J_{\mathcal{S}}$. This is evidenced by the following theorem, which is straightforward to prove. We record that $\mathcal{H}^h|_{J_{\mathcal{S}}}$ and $\mathcal{P}^h|_{J_{\mathcal{S}}}$ are, respectively, the h-dimensional Hausdorff measure restricted to $J_{\mathcal{S}}$ and the h-dimensional packing measure restricted to $J_{\mathcal{S}}$.

Theorem 2.2. *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a finitely irreducible conformal GDMS. If either $\mathcal{H}^h(J_{\mathcal{S}}) > 0$ or $\mathcal{P}^h(J_{\mathcal{S}}) < \infty$, then the system $\mathcal{S}$ is regular, and in the former case*

$$m_h = \mathcal{H}^h(J_{\mathcal{S}})^{-1} \mathcal{H}^h|_{J_{\mathcal{S}}},$$

while in the latter case

$$m_h = \mathcal{P}^h(J_{\mathcal{S}})^{-1} \mathcal{P}^h|_{J_{\mathcal{S}}}.$$

Moreover, If the alphabet E is finite, then both $\mathcal{H}^h|_{J_{\mathcal{S}}}$ and $\mathcal{P}^h|_{J_{\mathcal{S}}}$ are positive and finite and one is a constant multiple of the other. In addition, both are Ahlfors h-regular measures.

See [7, Chapters 6,7,8] or [36, Chapter 4] for more information on Gibbs states and conformal measures.

3. Porosities and conformal graph directed Markov systems

In this section we study various aspects of porosities for general graph directed Markov systems. We note that the results obtained in Subsections 3.1, 3.3 and 3.5 hold for Carnot conformal GDMS as well, see [7] for more information on Carnot GDMS.

3.1. **Porosity.**

We start with the formal definition of porosity. If (X, d) is a metric space and $E \subset X$, $x \in X$ and $r \in (0, \text{diam}(E))$ we let

$$\text{por}(E, x, r) = \sup\{c \geq 0 : B(y, cr) \subset B(x, r) \setminus E \text{ for some } y \in X\}. \tag{3.1}$$

Definition 3.1. Let (X, d) be a metric space and let $E \subset X$ be a bounded set. Given $c \in (0, 1)$ and $x \in X$, we say that $E \subset X$ is:

(i) *c-porous at x* if there exists some $r_0 > 0$ such that $\text{por}(E, x, r) \geq c$ for every $r \in (0, r_0)$,

(ii) *porous at x* if there exists some $c \in (0, 1)$ such that E is *c-porous at x*,

(iii) *c-porous* if there exists some $r_0 > 0$ such that $\text{por}(E, x, r) \geq c$ for every $x \in E$ and $r \in (0, r_0)$,

(iv) *porous* if it is c-porous for some $c \in (0, 1)$.

Before proving our first auxiliary lemma we need to introduce some extra notation. Recall first that $X = \cup_{v\in V} X_v$. For $n \in \mathbb{N}$ and $v \in V$ let

$$X^n := \bigcup_{\omega\in E_A^n} \phi_\omega(X_t(\omega)), \tag{3.2}$$

and

$$X_v^n := X^n \cap X_v = \bigcup_{\omega\in E_A^n : i(\omega)=v} \phi_\omega(X_t(\omega)). \tag{3.3}$$

Lemma 3.2. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS such that*

$$X \setminus \overline{\cup_{e\in E}\phi_e(X_{t(e)})} \neq \emptyset. \tag{3.4}$$

Then

(i) For every $v \in V$ there exists some $m_v \in \mathbb{N}$ such that

$$\operatorname{Int} X_v \setminus \overline{X_v^{m_v}} \neq \emptyset. \tag{3.5}$$

(ii) There exists a family of open balls $\{B_v\}_{v\in V}, B_v \subset \operatorname{Int} X_v$, such that for every $\omega \in E_A^*$,

$$\phi_\omega(B_{t(\omega)}) \cap J_{\mathcal{S}} = \emptyset. \tag{3.6}$$

Proof. Throughout the proof we are going to use repeatedly the fact that the maps

$$\phi_e : W_{t(e)} \to \phi_e(W_{t(e)}) \subset W_{i(e)}$$

are homeomorphisms for every $e \in E$. The proof of (i) is similar to the proof of [7, Theorem 8.26], nevertheless we include the details. First note that (3.4) implies that

$$\operatorname{Int} X \setminus \overline{X^1} \neq \emptyset.$$

Since $\operatorname{Int} X = \cup_{v\in V} \operatorname{Int} X_v$, it follows that there exists some $v_0 \in V$ such that

$$\operatorname{Int} X_{v_0} \setminus \overline{X_{v_0}^1} \neq \emptyset. \tag{3.7}$$

Now let $v \in V$ and $e, e_0 \in E$ such that $i(e) = v$ and $t(e_0) = v_0$. Since $\mathcal{S}$ is finitely irreducible there exists some $\omega \in E_A^*$, with $|\omega| \le m_0$ for some $m_0 \in \mathbb{N}$ depending only on $\mathcal{S}$, such that $\omega' := e\omega e_0 \in E_A^*$. Therefore,

$$\phi_{\omega'}(\operatorname{Int} X_{v_0}) \setminus \phi_{\omega'}(\overline{X_{v_0}^1}) \neq \emptyset. \tag{3.8}$$

Now if $|\omega'| = k$ notice that

$$\overline{X_v^{k+1}} \cap \phi_{\omega'}(\operatorname{Int} X_{v_0}) \subset \phi_{\omega'}(\overline{X_{v_0}^1}). \tag{3.9}$$

To prove (3.9), first observe that

$$\begin{aligned} X_v^{k+1} &\subset \phi_{\omega'}\left(\bigcup_{a\in E: i(a)=v_0} \phi_a(X_{t(a)})\right) \cup \bigcup_{\tau\in I_v} \phi_\tau(X_{t(\tau)}) \\ &= \phi_{\omega'}(X_{v_0}^1) \cup \bigcup_{\tau\in I_v} \phi_\tau(X_{t(\tau)}), \end{aligned}$$

where $I_v = \{\tau \in E_A^{k+1} : i(\tau) = v \text{ and } \tau|_k \neq \omega'\}$. Therefore

$$\overline{X_v^{k+1}} \subset \phi_{\omega'}(\overline{X_{v_0}^1}) \cup \overline{\bigcup_{\tau \in I_v} \phi_\tau(X_{t(\tau)})}. \tag{3.10}$$

Note that for every $\tau \in I_v$ by the open set condition

$$\phi_\tau(X_{t(\tau)}) \cap \phi_{\omega'}(\operatorname{Int} X_{v_0}) \subset \phi_{\tau|_k}(X_{t(\tau_k)}) \cap \phi_{\omega'}(\operatorname{Int} X_{v_0}) = \emptyset,$$

hence

$$\bigcup_{\tau \in I_v} \phi_\tau(X_{t(\tau)}) \cap \phi_{\omega'}(\operatorname{Int} X_{v_0}) = \emptyset.$$

Since $\phi_{\omega'}(\operatorname{Int} X_{v_0})$ is open, we deduce that

$$\overline{\bigcup_{\tau \in I_v} \phi_\tau(X_{t(\tau)})} \cap \phi_{\omega'}(\operatorname{Int} X_{v_0}) = \emptyset. \tag{3.11}$$

Therefore (3.9) follows by (3.10) and (3.11). Recall that

$$\phi_{\omega'}(\operatorname{Int}(X_{v_0})) = \phi_{\omega'}(\operatorname{Int}(X_{t(e_0)})) = \phi_{\omega'}(\operatorname{Int}(X_{t(\omega')})) = \phi_{\omega'}(\operatorname{Int}(X_{t(e_0)})) \subset \operatorname{Int} X_{i(e)} = \operatorname{Int} X_v.$$

Hence,

$$\begin{aligned}\operatorname{Int} X_v \setminus \overline{X_v^{k+1}} &\supset \phi_{\omega'}(\operatorname{Int} X_{v_0}) \setminus \overline{X_v^{k+1}} = \phi_{\omega'}(\operatorname{Int} X_{v_0}) \cap (\phi_{\omega'}(\operatorname{Int} X_{v_0}) \setminus \overline{X_v^{k+1}}) \\ &\overset{(3.9)}{\supset} \phi_{\omega'}(\operatorname{Int} X_{v_0}) \setminus \phi_{\omega'}(\overline{X_{v_0}^1}) \overset{(3.8)}{\neq} \emptyset,\end{aligned}$$

and this implies (3.5) because $k = |\omega'| \leq m_0 + 2$. The proof of (i) is complete

We will now prove (ii). By (3.5), for all $v \in V$, there exist $z_v \in X_v$ and $r_v > 0$ such that

$$B_v := B(z_v, r_v) \subset \operatorname{Int} X_v \setminus \overline{X_v^{m_v}}. \tag{3.12}$$

Observe first that

$$B_v \cap J_{\mathcal{S}} = \emptyset \tag{3.13}$$

for all $v \in V$. Because if not, there exists some $y \in B_v \cap J_{\mathcal{S}}$ and since $y \in J_{\mathcal{S}}$ there exists some $\tau \in E_A^{\mathbb{N}}$ such that

$$y = \pi(\tau) = \bigcap_{n=1}^{\infty} \phi_{\tau|_n}(X_{t(\tau_n)}).$$

In that case $y \in \phi_{\tau|_{m_v}}(X_{t(\tau_{m_v})}) \subset X_v^{m_v}$, which contradicts (3.12). Hence (3.13) follows. Suppose now that there exists some $\omega \in E_A^*$ and some $y \in B_{t(\omega)}$ such that $\phi_\omega(y) \in J_{\mathcal{S}}$. So there exists some $\tau \in E_A^{\mathbb{N}}$, such that

$$\phi_\omega(y) = \pi(\tau) = \bigcap_{n=1}^{\infty} \phi_{\tau|_n}(X_{t(\tau_n)}). \tag{3.14}$$

We will show that $\omega = \tau|_\omega$. By contradiction assume that this is not the case. Let $n_0 < |\omega|$ be the smallest natural number such that $\omega_{n_0} \neq \tau_{n_0}$. Then by the open set condition,

$$\phi_{\omega_{n_0}}(\operatorname{Int}(X_{t(\omega_{n_0})})) \cap \phi_{\tau_{n_0}}(X_{t(\tau_{n_0})}) = \emptyset,$$

and consequently

$$\phi_{\omega|_{n_0}}(\operatorname{Int}(X_{t(\omega_{n_0})})) \cap \phi_{\tau|_{n_0}}(X_{t(\tau_{n_0})}) = \emptyset.$$

Note also that

$$\phi_{\omega_{n_0+1,\dots,|w|}}(\mathrm{Int}(X_{t(\omega_{|\omega|})})) = \mathrm{Int}\,\phi_{\omega_{n_0+1,\dots,|w|}}(X_{t(\omega_{|\omega|})}) \subset \mathrm{Int}(X_{t(\omega_{n_0})}),$$

and

$$\phi_{\tau_{n_0+1,\dots,|w|}}(X_{t(\tau_{|\omega|})}) \subset X_{t(\tau_{n_0})}.$$

Therefore

$$\phi_\omega(\mathrm{Int}(X_{t(\omega_{|\omega|})})) \cap \phi_{\tau|_{|\omega|}}(X_{t(\tau_{|\omega|})}) = \emptyset. \tag{3.15}$$

Recalling (3.14) and the fact that $y \in B_{t(\omega)} \subset \mathrm{Int}\,X_{t(\omega)}$ we deduce that

$$\phi_\omega(y) \in \phi_\omega(\mathrm{Int}(X_{t(\omega_{|\omega|})})) \cap \phi_{\tau|_{|\omega|}}(X_{t(\tau_{|\omega|})}),$$

which contradicts (3.15).

Therefore $\omega = \tau|_{|\omega|}$, and

$$\phi_\omega(y) = \phi_{\tau|_{|\omega|}}(\pi(\sigma^{|\omega|}(\tau))) = \phi_\omega(\pi(\sigma^{|\omega|}(\tau))).$$

Hence

$$y = \pi(\sigma^{|\omega|}(\tau)) \in J_{\mathcal{S}},$$

and this contradicts (3.13). So we have established (3.6) and the proof of (ii) is complete. □

Under the mild assumption (3.4), limit sets of finitely irreducible graph directed Markov systems are porous in large (in the sense of category and dimension) subsets. This is the content of Theorem 1.1, which we restate and prove in the following.

Theorem 3.3. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS which satisfies (3.4). Then the following hold:*

(i) The limit set $J_{\mathcal{S}}$ is porous at every fixed point of $\mathcal{S}$, in particular $J_{\mathcal{S}}$ is porous at a dense set of $J_{\mathcal{S}}$.

(ii) For every $\varepsilon > 0$ there exists some finite $F(\varepsilon) := F \subset E$ with $\dim_{\mathcal{H}}(J_{\mathcal{S}_F}) > \dim_{\mathcal{H}}(J_{\mathcal{S}}) - \varepsilon$, such that $J_{\mathcal{S}}$ is porous at every $x \in J_{\mathcal{S}_F}$ with porosity constant only depending on F and $\mathcal{S}$.

(iii) There exists some set $\tilde{J} \subset J_{\mathcal{S}}$ such that $\dim_{\mathcal{H}}(\tilde{J}) = \dim_{\mathcal{H}}(J_{\mathcal{S}})$ and $J_{\mathcal{S}}$ is porous at every $x \in \tilde{J}$.

Proof. Let $\{B(z_v, r_v)\}_{v\in V}$ as in Lemma 3.2. Throughout the proof, without loss of generality, we will assume that $r_0 := \min_{v\in V} r_v < \eta_{\mathcal{S}}$.

Let x_ω be a fixed point of $\mathcal{S}$ corresponding to some $\omega \in E_A^*$, that is

$$\{x_\omega\} = \bigcap_{n=1}^{\infty} \phi_{\omega^n}(X_{t(\omega)}).$$

Note that if $\omega \in E_A^*$ then there exists at least one $\omega' \in E_A^*$ such that $\omega'|_{|\omega|} = \omega$ and $\omega'\omega' \in E_A^*$. Indeed by the finite irreducibility of $\mathcal{S}$ there exists some $\rho \in E_A^*$ such that $\omega\rho\omega \in E_A^*$. Hence, if $\omega' = \omega\rho$ then $\omega'\omega' \in E_A^*$. Therefore the fixed points of $\mathcal{S}$ form a dense subset of $J_{\mathcal{S}}$.

By the Leibniz rule for every $n \in \mathbb{N}$,

$$\|D\phi_{\omega^n}(x_\omega)\| = \|D\phi_\omega(x_\omega)\|^n, \tag{3.16}$$

and

$$\|D\phi_{\omega^n}\|_\infty \overset{(3.16)\wedge(2.4)}{\leq} K\,\|D\phi_\omega(x_\omega)\|^n. \tag{3.17}$$

Let $r < \min\{\text{diam}(X_v) : v \in V\}$ and let $n \in \mathbb{N}$ be the largest natural number such that,

$$\phi_{\omega^n}(X_{t(\omega)}) \subset B(x_\omega, r). \tag{3.18}$$

Therefore, there exists some $z \in \phi_{\omega^{n-1}}(X_{t(\omega)}) \setminus B(x_\omega, r)$. Hence, $d(x_\omega, z) \geq r$ and since $x_\omega \in \phi_{\omega^{n-1}}(X_{t(\omega)})$ we also have that

$$\text{diam}(\phi_{\omega^{n-1}}(X_{t(\omega)})) \geq r. \tag{3.19}$$

Moreover, assuming without loss of generality that $\text{diam}(X) \leq 1$,

$$\begin{aligned}\text{diam}(\phi_{\omega^{n-1}}(X_{t(\omega)})) &\overset{(2.7)}{\leq} MK\|D\phi_{\omega^{n-1}}\|_\infty \overset{(3.17)}{\leq} MK^2\|D\phi_\omega(x_\omega)\|^{n-1}\\ &= \frac{MK^2}{\|D\phi_\omega(x_\omega)\|}\|D\phi_\omega(x_\omega)\|^n.\end{aligned} \tag{3.20}$$

By Lemma 2.5 (ii),

$$\begin{aligned}\phi_{\omega^n}(B_{t(\omega)}) &\overset{(2.9)}{\supset} B(\phi_{\omega^n}(z_{t(\omega)}), K^{-1}\|D\phi_{\omega^n}\|_\infty r_{t(\omega)})\\ &\supset B(\phi_{\omega^n}(z_{t(\omega)}), K^{-1}\|D\phi_{\omega^n}(x_\omega)\| r_{t(\omega)})\\ &\overset{(3.16)}{=} B(\phi_{\omega^n}(z_{t(\omega)}), K^{-1}\|D\phi_\omega(x_\omega)\|^n r_{t(\omega)})\\ &:= B^r.\end{aligned} \tag{3.21}$$

Therefore

$$B^r \subset \phi_{\omega^n}(B_{t(\omega)}) \subset \phi_{\omega^n}(X_{t(\omega)}) \overset{(3.18)}{\subset} B(x_\omega, r), \tag{3.22}$$

and,

$$J_{\mathcal{S}} \cap B^r \overset{(3.6)\wedge(3.21)}{=} \emptyset. \tag{3.23}$$

Moreover,

$$\frac{\text{radius}\,(B^r)}{r} \overset{(3.19)\wedge(3.20)\wedge(3.21)}{\geq} \frac{K^{-1}\|D\phi_\omega(x_\omega)\|^n r_{t(\omega)}}{\frac{MK^2}{\|D\phi_\omega(x_\omega)\|}\|D\phi_\omega(x_\omega)\|^n} \geq \frac{r_0\|D\phi_\omega(x_\omega)\|}{MK^3}. \tag{3.24}$$

Therefore, (3.22), (3.23) and (3.24) imply that $J_{\mathcal{S}}$ is $\frac{r_0\|D\phi_\omega(x_\omega)\|}{MK^3}$-porous at x_ω. The proof of (i) is complete.

We now move to the proof of (ii). Let $\Lambda \subset E$ be a set witnessing finite irreducibility for E. Let F be a finite set such that $\Lambda \subset F \subset E$, and let

$$m_F = \min\{\|D\phi_e\|_\infty : e \in F\}.$$

Let $\omega \in F_A^{\mathbb{N}}$ and $x = \pi(\omega)$. Let $r < \min\{\text{diam}(X_v) : v \in V\}$ and let $n \in \mathbb{N}$ be the largest natural number such that

$$\phi_{\omega|_n}(X_{t(\omega_n)}) \subset B(x, r). \tag{3.25}$$

By Lemma 2.5 we have that

$$\phi_{\omega|_n}(B_{t(\omega_n)}) \supset B(\phi_{\omega|_n}(z_{t(\omega_n)}), K^{-1}\|D\phi_{\omega|_n}\|_\infty r_{t_{\omega_n}}) := B_F^r. \tag{3.26}$$

By (3.6),

$$J_{\mathcal{S}} \cap B_F^r = \emptyset.$$

Arguing as in (3.19) and (3.20) we also get that

$$r \leq \operatorname{diam}(\phi_{\omega|_{n-1}}(X_{t(\omega_{n-1})})) \leq \frac{K}{m_F}\|D\phi_{\omega|_n}\|_\infty.$$

Therefore,

$$\frac{\operatorname{radius}(B_F^r)}{r} \geq \frac{K^{-1}\|D\phi_{\omega|_n}\|_\infty r_{t(\omega)}}{\frac{K}{m_F}\|D\phi_{\omega|_n}\|_\infty} \geq \frac{r_0 m_F}{K^2}. \tag{3.27}$$

Hence (3.25), (3.26) and (3.27) imply that $J_{\mathcal{S}}$ is $\frac{r_0 m_F}{K^2}$-porous at every $x \in \pi(F^{\mathbb{N}}) = J_{\mathcal{S}_F}$. Now (ii) follows by Theorem 2.7.

Finally (iii) follows easily from (ii). Take

$$\tilde{J} = \cup\{J_F : F \text{ is finite and irreducible}\}.$$

We then have that $\tilde{J} \subset J_{\mathcal{S}}$ and by (ii) $J_{\mathcal{S}}$ is porous at every point of $\tilde{J}$. Moreover by Theorem 2.7, $\dim_{\mathcal{H}}(J_{\mathcal{S}}) = \dim_{\mathcal{H}}(\tilde{J})$. The proof is complete. □

3.2. **Upper Porosity.** We will now prove some general results regarding *upper porosity* for graph directed Markov systems. Recall (3.1) for the formal definition of upper porosity which we state below.

Definition 3.4. Let (X,d) be a metric space. The set $E \subset X$ is *upper porous at x* if

$$\limsup_{r\to 0} \operatorname{por}(E,x,r) > 0.$$

Moreover the set E is c-*upper porous* if

$$\limsup_{r\to 0} \operatorname{por}(E,x,r) > c.$$

for all $x \in E$. The set E is called (uniformly) upper porous if it is c-upper porous for some $c > 0$.

We are now ready to restate and prove Theorem 1.2 which provides a useful criterion for pointwise upper porosity.

Theorem 3.1. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS such that $\overline{J_{\mathcal{S}}} \neq X$. If $\omega \in E_A^{\mathbb{N}}$ and*

$$\overline{\lim}_{n\to\infty} \operatorname{dist}(\pi(\sigma^n(\omega)), X^c) > 0, \tag{3.28}$$

then $\overline{J_{\mathcal{S}}}$ is upper porous at $\pi(\omega)$.

Proof. Let $\omega \in E_A^{\mathbb{N}}$ which satisfies (3.28). Recalling (2.8) we pick some $\theta > 0$ such that

$$\theta < \frac{1}{2}\min\{\eta_{\mathcal{S}}, \overline{\lim}_{n\to\infty} \operatorname{dist}(\pi(\sigma^n(\omega)), X^c)\}.$$

We then choose a sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that

$$\operatorname{dist}(\pi(\sigma^{n_k}(\omega)), X^c) > \theta \text{ for all } k \in \mathbb{N}. \tag{3.29}$$

Let ξ be an accumulation point of $(\pi(\sigma^{n_k}(\omega))_{k\in\mathbb{N}}$. After possibly passing to a subsequence, we can assume that

$$\lim_{k\to\infty} \pi(\sigma^{n_k}(\omega)) = \xi.$$

Note that

$$\xi \in \mathrm{Int}(X) \setminus B(X^c, \theta/2). \tag{3.30}$$

Indeed, (3.29) implies that $\xi \notin B(X^c, \theta/2)$. So $\xi \notin \partial X$ and since X is compact, $\xi \in X \setminus \partial X = \mathrm{Int}(X)$. Thus, (3.30) holds.

Since $\overline{J_{\mathcal{S}}} \neq X$, is is easy to see that $\mathrm{Int}(X) \setminus \overline{J_{\mathcal{S}}} \neq \emptyset$. Hence, [36, Theorem 4.6.1] implies that $\overline{J_{\mathcal{S}}}$ is nowhere dense. Therefore, there exists some $R < \eta_{\mathcal{S}}$ and $z \in B(\xi, \theta/8)$ such that

$$B(z, R) \subset B(\xi, \theta/8) \setminus \overline{J_{\mathcal{S}}}. \tag{3.31}$$

Let $k_0 \in \mathbb{N}$ big enough such that

$$d(\pi(\sigma^{n_k}(\omega)), \xi) < \theta/8 \text{ for all } k \geq k_0. \tag{3.32}$$

Hence,

$$B(\pi(\sigma^{n_k}(\omega)), \theta/4) \overset{(3.32)}{\subset} B(\xi, \theta/2) \overset{(3.30)}{\subset} \mathrm{Int}(X). \tag{3.33}$$

Moreover, for all $k \geq k_0$,

$$B(z, R) \overset{(3.31)\wedge(3.32)}{\subset} B(\pi(\sigma^{n_k}(\omega)), \theta/4) \setminus \overline{J_{\mathcal{S}}},$$

and consequently

$$\phi_{\omega|_{n_k}}(B(z, R)) \subset \phi_{\omega|_{n_k}}(B(\pi(\sigma^{n_k}(\omega)), \theta/4)) \setminus \phi_{\omega|_{n_k}}(\overline{J_{\mathcal{S}}}). \tag{3.34}$$

Note that since the sets X_v, $v \in V$, are pairwise disjoint and

$$\pi(\sigma^{n_k}(\omega)) \in \phi_{\omega_{n_k+1}}(X_{t(\omega_{n_k+1})}) \subset X_{i(\omega_{n_k+1})} = X_{t(\omega_{n_k})},$$

we have that

$$B(\pi(\sigma^{n_k}(\omega)), \theta/4) \overset{(3.33)}{\subset} \mathrm{Int}(X_{t(\omega_{n_k})}). \tag{3.35}$$

We will now show that,

$$\phi_{\omega|_{n_k}}(\mathrm{Int}(X_{t(\omega_{n_k})})) \cap (\overline{J_{\mathcal{S}}} \setminus \phi_{\omega|_{n_k}}(\overline{J_{\mathcal{S}}})) = \emptyset. \tag{3.36}$$

Since

$$J_{\mathcal{S}} \setminus \phi_{\omega|_{n_k}}(J_{\mathcal{S}}) = \cup_{\tau\in E_A^{n_k}:\tau\neq\omega|_{n_k}} \phi_\tau(J_{\mathcal{S}}) \subset \cup_{\tau\in E_A^{n_k}:\tau\neq\omega|_{n_k}} \phi_\tau(X_{t(\tau)}),$$

and the open set condition implies that

$$\phi_{\omega|_{n_k}}(\mathrm{Int}(X_{t(\omega_{n_k})})) \cap \phi_\tau(X_{t(\tau)}) = \emptyset \text{ for all } \tau \in E_A^{n_k} \setminus \{\omega|_{n_k}\},$$

we deduce that

$$\phi_{\omega|_{n_k}}(\mathrm{Int}(X_{t(\omega_{n_k})})) \cap (J_{\mathcal{S}} \setminus \phi_{\omega|_{n_k}}(J_{\mathcal{S}})) = \emptyset. \tag{3.37}$$

Moreover, since

$$\overline{J_{\mathcal{S}}} \setminus \phi_{\omega|_{n_k}}(\overline{J_{\mathcal{S}}}) = \overline{J_{\mathcal{S}}} \setminus \overline{\phi_{\omega|_{n_k}}(J_{\mathcal{S}})} \subset \overline{J_{\mathcal{S}} \setminus \phi_{\omega|_{n_k}}(J_{\mathcal{S}})},$$

and $\phi_{\omega|_{n_k}}(\mathrm{Int}(X_{t(\omega_{n_k})}))$ is an open set, we see that (3.37) implies (3.36).

Now that (3.36) has been established, we obtain

$$\phi_{\omega|_{n_k}}(B(z,R)) \overset{(3.34)\wedge(3.35)\wedge(3.36)}{\subset} \phi_{\omega|_{n_k}}(B(\pi(\sigma^{n_k}(\omega)),\theta/4)) \setminus \overline{J_{\mathscr{S}}}, \tag{3.38}$$

for all $k \geq k_0$. By Lemma 2.5 we have that for all $k \in \mathbb{N}$,

$$\phi_{\omega|_{n_k}}(B(z,R)) \supset B(\phi_{\omega|_{n_k}}(z), K^{-1}R\|D\phi_{\omega|_{n_k}}\|_\infty). \tag{3.39}$$

and for all $k \geq k_0$,

$$\phi_{\omega|_{n_k}}(B(z,R)) \overset{(3.38)}{\subset} B(\pi(\omega), 4^{-1}\theta\|D\phi_{\omega|_{n_k}}\|_\infty) \setminus \overline{J_{\mathscr{S}}}. \tag{3.40}$$

Setting $r_k = 4^{-1}\theta\|D\phi_{\omega|_{n_k}}\|_\infty$ we see that

$$\mathrm{por}(\overline{J_{\mathscr{S}}}, \pi(\omega), r_k) \overset{(3.39)\wedge(3.40)}{\geq} \frac{4R}{K\theta},$$

for all $k \geq k_0$, and consequently,

$$\limsup_{r\to 0} \mathrm{por}(\overline{J_{\mathscr{S}}}, \pi(\omega), r) \geq \frac{4R}{K\theta}.$$

The proof is complete. □

Remark 3.5. Note that the hypothesis $\overline{J_{\mathscr{S}}} \neq X$ in Theorem 3.1 is necessary. Indeed, if $\mathscr{S}$ is the conformal IFS of real continued fractions, i.e. $\mathscr{S} = \{\phi_n : [0,1] \to [0,1]\}_{n\in\mathbb{N}}$ where $\phi_n(x) = (n+x)^{-1}$, then $\overline{J_{\mathscr{S}}} = [0,1]$ and $\overline{J_{\mathscr{S}}}$ is not porous at any of its points.

We will now discuss several rather straightforward corollaries of the previous theorem. They attest that under some (additional) natural separation conditions, limit sets of finitely irreducible conformal GDMSs are upper porous everywhere or almost everywhere. The first one is just a restatement of Theorem 1.3 (i).

Corollary 3.6. *If $\mathscr{S} = \{\phi_e\}_{e\in E}$ is a finitely irreducible conformal GDMS such that*

$$\mathrm{dist}(J_{\mathscr{S}}, X^c) > 0, \tag{3.41}$$

then $\overline{J_{\mathscr{S}}}$ is upper porous at every $x \in J_{\mathscr{S}}$.

Proof. Let $\omega \in E_A^{\mathbb{N}}$ and note that the set $P_\omega = \{\pi(\sigma^n(\omega)) : n \in \mathbb{N}\}$ is contained in $J_{\mathscr{S}}$. Therefore, $\mathrm{dist}(P_\omega, X^c) > 0$ for all $\omega \in E_A^{\mathbb{N}}$, and in particular (3.28) is satisfied for all $\omega \in E_A^{\mathbb{N}}$. Note also that $\mathrm{dist}(J_{\mathscr{S}}, X^c) > 0$ implies that $\mathrm{Int}(X) \setminus J_{\mathscr{S}} \neq \emptyset$. Hence Theorem 3.1 implies that $\overline{J_{\mathscr{S}}}$ is upper porous at $\pi(\omega)$ for all $\omega \in E_A^{\mathbb{N}}$. The proof is complete. □

Since $J_{\mathscr{S}} \subset \cup_{e\in E} X_{t(e)}$, Corollary (3.7) immediately implies the following.

Corollary 3.7. *If $\mathscr{S} = \{\phi_e\}_{e\in E}$ is a finitely irreducible conformal GDMS such that*

$$\overline{\cup_{e\in E} X_{t(e)}} \cap \partial X = \emptyset, \tag{3.42}$$

then $\overline{J_{\mathscr{S}}}$ is upper porous at every $x \in J_{\mathscr{S}}$.

Even without assuming (3.41) we can show that the limit set of a finitely irreducible conformal GDMS which satisfies the strong open set condition is upper porous μ-a.e. with respect to any shift invariant ergodic measure μ on $J_{\mathcal{S}}$. In particular, $J_{\mathcal{S}}$ is m_h-a.e upper porous, where m_h is the h-conformal measure of $J_{\mathcal{S}}$ and $h = \dim_{\mathcal{H}}(J_{\mathcal{S}})$. The following corollary is a restatement of Theorem 1.3 (i).

Corollary 3.8. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS which satisfies $\overline{J_{\mathcal{S}}} \neq X$ and the strong open set condition. If $\tilde{\mu}$ is a shift invariant ergodic probability measure on $E_A^{\mathbb{N}}$ with full topological support, then $J_{\mathcal{S}}$ is upper porous at $\tilde{\mu}\circ\pi^{-1}$-a.e. $x \in J_{\mathcal{S}}$.*

Proof. Recalling Remark 2.4, the SOSC means that $J_{\mathcal{S}} \cap \mathrm{Int}(X) \neq \emptyset$. Hence, there exists some $x \in J_{\mathcal{S}}$ and $r > 0$ such that

$$B(x,2r) \subset \mathrm{Int}(X). \tag{3.43}$$

Let $\tilde{\mu}$ be a shift invariant ergodic probability measure on $E_A^{\mathbb{N}}$ with full topological support. Birkhoff's Ergodic Theorem implies that

$$\lim_{n\to\infty} \frac{\sharp\{k \in \{0,1,\dots,n-1\} : \sigma^k(\omega) \in \pi^{-1}(B(x,r))\}}{n} = \tilde{\mu}(\pi^{-1}(B(x,r))), \tag{3.44}$$

for $\tilde{\mu}$-a.e. $\omega \in E_A^{\mathbb{N}}$. Since $\tilde{\mu}$ has full topological support and $\pi : E_A^{\mathbb{N}} \to X$ is continuous, we deduce that $\tilde{\mu}(\pi^{-1}(B(x,r))) > 0$. Hence, if $\omega \in E_A^{\mathbb{N}}$ satisfies (3.44), then there exists a strictly increasing sequence of natural numbers $(n_k(\omega))_{k\in\mathbb{N}}$ such that

$$\pi(\sigma^{n_k(\omega)}(\omega)) \in B(x,r) \text{ for all } k \in \mathbb{N}.$$

Thus, if $\omega \in E_A^{\mathbb{N}}$ satisfies (3.44), then

$$\mathrm{dist}(\pi(\sigma^{n_k(\omega)}(\omega)), X^c) \geq r \text{ for all } k \in \mathbb{N}.$$

Therefore, Theorem 3.1 implies that $\overline{J_{\mathcal{S}}}$ is upper porous for $\tilde{\mu}$-a.e. $\omega \in E_A^{\mathbb{N}}$. Consequently, we deduce that $\overline{J_{\mathcal{S}}}$ is upper porous for $\tilde{\mu}\circ\pi^{-1}$-a.e. $x \in J_{\mathcal{S}}$. The proof is complete.

3.3. **Mean porosity.** In this section we will investigate *mean porosity* for graph directed Markov systems. Recall (3.1) for the formal definition of mean porosity which we state below.

Definition 3.9. Let (X,d) be a metric space. The set $E \subset X$ is (α,p)-*mean porous at* x if

$$\begin{aligned} S\underline{A}(E,x;\alpha) :&= \sup_{s\in(0,1/2]} \sup_{\beta>\alpha} \left\{ \liminf_{i\to+\infty} \frac{\sharp\{j \in [1,i]\cap\mathbb{N} : \mathrm{por}(E,x,s2^{-j}) > \beta\}}{i} \right\} \\ &= \sup_{s\in(0,+\infty)} \sup_{\beta>\alpha} \left\{ \liminf_{i\to+\infty} \frac{\sharp\{j \in [1,i]\cap\mathbb{N} : \mathrm{por}(E,x,s2^{-j}) > \beta\}}{i} \right\} > p. \end{aligned}$$

Denote also, for future reference,

$$\underline{A}(E,x;\beta,s) := \liminf_{i\to+\infty} \frac{\sharp\{j \in [1,i]\cap\mathbb{N} : \mathrm{por}(E,x,s2^{-j}) > \beta\}}{i}.$$

We will now prove Theorem 1.4 which we restate in a more precise manner. We also record that if $\mathcal{S} = \{\phi_e\}_{e\in E}$ is a finitely irreducible conformal GDMS and μ is a Borel probability shift-invariant ergodic measure on $E_A^{\mathbb{N}}$, the *characteristic Lyapunov exponent* with respect to μ and σ is defined as

$$\chi_\mu(\sigma) = -\int_{E_A^{\mathbb{N}}} \log \|D\phi_{\omega_1}(\pi(\sigma(\omega))\| \, d\mu(\omega).$$

Note that $\chi_\mu(\sigma) > 0$.

Theorem 3.10. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS which satisfies (3.4). Let $\tilde{\mu}$ be any Borel probability σ-invariant ergodic measure on $E_A^{\mathbb{N}}$ with finite Lyapunov exponent $\chi_{\tilde{\mu}}(\sigma)$. Then there exists some $\alpha_{\mathcal{S}} \in (0,1/2)$ such that*

$$\underline{A}\big(J_{\mathcal{S}}, x; 2\alpha_{\mathcal{S}}, 1\big) \geq \frac{\log 2}{\chi_{\tilde{\mu}}(\sigma)}.$$

for $\tilde{\mu}\circ\pi^{-1}$-a.e. $x \in J_{\mathcal{S}}$. In consequence,

$$S\underline{A}\big(J_{\mathcal{S}}, x; \alpha_{\mathcal{S}}\big) \geq \frac{\log 2}{\chi_{\tilde{\mu}}(\sigma)}$$

for $\tilde{\mu}\circ\pi^{-1}$-a.e. $x \in J_{\mathcal{S}}$ and the set $J_{\mathcal{S}}$ is $(\alpha_{\mathcal{S}}, p)$–mean porous for every

$$p < \frac{\log 2}{\chi_{\tilde{\mu}}(\sigma)}$$

at $\tilde{\mu}\circ\pi^{-1}$-a.e. $x \in J_{\mathcal{S}}$.

Proof. Let $\{B(z_v, r_v)\}_{v\in V}$ as in Lemma 3.2 and recall (3.12). Let $\omega \in E_A^{\mathbb{N}}$ and set

$$w_j = \phi_{\omega|_j}(z_{t(\omega_j)}).$$

By Lemma 3.2 (ii) we know that

$$\phi_{\omega|_j}(B_{t(\omega_j)}) \cap J_{\mathcal{S}} = \emptyset \tag{3.45}$$

for all $j \in \mathbb{N}$. Without loss of generality assume again that $r_0 = \min_{v\in V} r_v < \eta_{\mathcal{S}}$. By Lemma 2.5,

$$\begin{aligned} \phi_{\omega|_j}(B_{t(\omega_j)}) &\supset B(\phi_{\omega|_j}(z_{t(\omega_j)}), K^{-1}\|D\phi_{\omega|_j}\|_\infty r_{t(\omega_j)}) \\ &\supset B(w_j, K^{-1}\|D\phi_{\omega|_j}\|_\infty r_0). \end{aligned} \tag{3.46}$$

Hence, for all $j \in \mathbb{N}$,

$$\operatorname{dist}(w_j, J_{\mathcal{S}}) \overset{(3.45)\wedge(3.46)}{\geq} K^{-1}\|D\phi_{\omega|_j}\|_\infty r_0. \tag{3.47}$$

Moreover, since without loss of generality we can assume that $\operatorname{diam}(\cup_{v\in V} X_v) \leq 1$, we have that

$$d(w_j, \pi(\omega)) = d(\phi_{\omega|_j}(z_{t(\omega_j)}), \phi_{\omega|_j}(\pi(\sigma^j(\omega)))) \overset{(2.6)}{\leq} MK\|D\phi_{\omega|_j}\|_\infty. \tag{3.48}$$

Let

$$\alpha' := \min\left\{\frac{r_0}{2K}, \frac{1}{2}\right\}.$$

Then

$$B(w_j, 2\alpha' \|D\phi_{\omega|_j}\|_\infty) \cap J_{\mathcal{S}} \overset{(3.45)\wedge(3.46)}{=} \emptyset$$

and

$$B(w_j, 2\alpha' \|D\phi_{\omega|_j}\|_\infty) \overset{(3.48)}{\subset} B(\pi(\omega), 2MK\|D\phi_{\omega|_j}\|_\infty).$$

Therefore, for all $j \in \mathbb{N}$,

$$B(w_j, 2\alpha' \|D\phi_{\omega|_j}\|_\infty) \subset B(\pi(\omega), 2MK\|D\phi_{\omega|_j}\|_\infty) \setminus J_{\mathcal{S}}. \tag{3.49}$$

Let $(n_j)_{j\in\mathbb{N}}, n_j \in \mathbb{R}$, such that

$$2MK\|D\phi_{\omega|_j}\|_\infty = 2^{-n_j},$$

that is

$$n_j = -\frac{\log(2MK\|D\phi_{\omega|_j}\|_\infty)}{\log 2}.$$

Since $\lim_{j\to\infty} \|D\phi_{\omega|_j}\|_\infty = 0$, see e.g. [7, Lemma 4.18], there exists some some $j_0 \in \mathbb{N}$ such that $n_j > 0$ for all $j \geq j_0$, and $n_j \to \infty$. Moreover, n_j is increasing by (2.5). Notice that for $j \geq j_0$, (3.49) implies that

$$\operatorname{por}(J_{\mathcal{S}}, \pi(\omega), 2^{-n_j}) \geq \alpha,$$

where $\alpha = \frac{a'}{MK}$. It then follows that for all $j \geq j_0$

$$\operatorname{por}(J_{\mathcal{S}}, \pi(\omega), 2^{-\lfloor n_j \rfloor}) \geq \alpha/2,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Therefore

$$\begin{aligned} \underline{A}\big(J_{\mathcal{S}}, \pi(\omega); \alpha/3, 1\big) &\geq \liminf_{i\to+\infty} \frac{\sharp\{j \in [1,i] \cap \mathbb{N} : \operatorname{por}(J_{\mathcal{S}}, \pi(\omega), 2^{-j}) \geq \alpha/2\}}{i} \\ &\geq \liminf_{j\to+\infty} \frac{j}{\lfloor n_j \rfloor}. \end{aligned} \tag{3.50}$$

Since $\tilde{\mu}$ is σ-invariant and ergodic, Birkhoff's ergodic theorem implies that for $\tilde{\mu}$-a.e. $\omega \in E_A^{\mathbb{N}}$,

$$-\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \zeta \circ \sigma^k(\omega) = -\int_{E_A^{\mathbb{N}}} \zeta \, d\tilde{\mu} = \chi_{\tilde{\mu}}(\sigma), \tag{3.51}$$

where $\zeta = \log \|D\phi_{\omega_1}(\pi(\sigma(\omega)))\|$. Notice that for $\omega \in E_A^{\mathbb{N}}$,

$$\begin{aligned} \sum_{k=0}^{n-1} -\zeta \circ \sigma^k(\omega) &= -\sum_{k=0}^{n-1} \log \|D\phi_{\omega_{k+1}}(\pi(\sigma^{k+1}(\omega)))\| \\ &= -\log\big(\|D\phi_{\omega_1}(\pi(\sigma^1(\omega)))\| \, \|D\phi_{\omega_2}(\pi(\sigma^2(\omega)))\| \cdots \|D\phi_{\omega_n}(\pi(\sigma^n(\omega)))\|\big). \end{aligned}$$

By the Leibniz rule and the fact that

$$\pi(\sigma^m(\omega)) = \phi_{\omega_{m+1}} \circ \cdots \circ \phi_{\omega_n}(\pi(\sigma^n(\omega))) \text{ for } 1 \leq m \leq n,$$

we deduce that

$$\sum_{k=0}^{n-1} -\zeta \circ \sigma^k(\omega) = -\log \|D\phi_{\omega|_n}(\pi(\sigma^n(\omega)))\|. \tag{3.52}$$

By (2.4) we have that for every $j \in \mathbb{N}$,

$$K^{-1}\|D\phi_{\omega|_j}\|_\infty \le \|D\phi_{\omega|_j}(\pi(\sigma^j(\omega)))\| \le \|D\phi_{\omega|_j}\|_\infty. \tag{3.53}$$

Therefore for $\tilde{\mu}$-a.e. $\omega \in E_A^{\mathbb{N}}$,

$$\chi_{\tilde{\mu}}(\sigma) \overset{(3.51)\wedge(3.52)}{=} \lim_{j\to+\infty} -\frac{\log\|D\phi_{\omega|_j}(\pi(\sigma^j(\omega)))\|}{j} \overset{(3.53)}{=} \lim_{j\to+\infty} -\frac{\log\|D\phi_{\omega|_j}\|_\infty}{j}. \tag{3.54}$$

Hence

$$\begin{aligned}\lim_{j\to+\infty}\frac{n_j}{j} &= \lim_{j\to+\infty}\frac{-\log(2\|D\phi_{\omega|_j}\|_\infty)}{j\log 2}\\ &= \frac{1}{\log 2}\lim_{j\to+\infty}\frac{-\log\|D\phi_{\omega|_j}\|_\infty}{j} \overset{(3.54)}{=} \frac{\chi_{\tilde{\mu}}(\sigma)}{\log 2}.\end{aligned}$$

Moreover since by our assumption $\chi_{\tilde{\mu}}(\sigma) \in (0,+\infty)$,

$$\liminf_{j\to+\infty}\frac{j}{\lfloor n_j\rfloor} \ge \lim_{j\to+\infty}\frac{j}{n_j} = \frac{\log 2}{\chi_{\tilde{\mu}}(\sigma)}. \tag{3.55}$$

Combining (3.50) and (3.55) we deduce that

$$\underline{A}(J_{\mathcal{S}},\pi(\omega);\alpha/3,1) \ge \frac{\log 2}{\chi_{\tilde{\mu}}(\sigma)}.$$

The proof is complete by choosing $\alpha_{\mathcal{S}} := \alpha/6$. □

We will now present a corollary of Theorem 3.10 which asserts that the limit set of a strongly regular finitely irreducible GDMS is almost everywhere mean porous with respect to its conformal measure.

Corollary 3.11. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a strongly regular finitely irreducible conformal GDMS which satisfies (3.4). Then*

(i)

$$S\underline{A}(J_{\mathcal{S}},x;\alpha_{\mathcal{S}}) \ge \underline{A}(J_{\mathcal{S}},x;2\alpha_{\mathcal{S}},1) \ge \frac{\log 2}{\chi_{\tilde{\mu}_h}(\sigma)}$$

for m_h-a.e. $x \in J_{\mathcal{S}}$, and the set $J_{\mathcal{S}}$ is $(\alpha_{\mathcal{S}}, p)$–mean porous for every

$$p < \frac{\log 2}{\chi_{\tilde{\mu}_h}(\sigma)}$$

at m_h-a.e. $x \in J_{\mathcal{S}}$.

(ii)

$$S\underline{A}(J_{\mathcal{S}},x;\alpha_{\mathcal{S}}) \ge \underline{A}(J_{\mathcal{S}},x;2\alpha_{\mathcal{S}},1) \ge \frac{\log 2}{\chi_{\tilde{\mu}_h}(\sigma)}$$

for $\mathcal{H}^h$-a.e. $x \in J_{\mathcal{S}}$, and the set $J_{\mathcal{S}}$ is $(\alpha_{\mathcal{S}}, p)$–mean porous for every

$$p < \frac{\log 2}{\chi_{\tilde{\mu}_h}(\sigma)}$$

at $\mathcal{H}^h$-a.e. $x \in J_{\mathcal{S}}$, *where $\alpha_{\mathcal{S}}$ is as in Theorem 3.10, $h = \dim_{\mathcal{H}}(J_{\mathcal{S}})$, m_h is the h-conformal measure of $\mathcal{S}$, and $\tilde{\mu}_h$ is the unique shift invariant ergodic Gibbs state globally equivalent to $\tilde{m}_h$.*

Proof. We will first show that if $\mathcal{S}$ is strongly regular then $\chi_{\tilde{\mu}_h}(\sigma)$, the Lyapunov exponent of the unique ergodic shift invariant Gibbs state $\tilde{\mu}_h$, is finite. Since $\mathcal{S}$ is strongly regular there exists some $t > 0$ such that $P(t) \in (0,\infty)$. It follows then, for example by [7, Proposition 7.5], that P is continuous and decreasing on $[t,+\infty)$. Hence, there exists some $\eta > 0$ such that $P(h-\eta) < \infty$. Thus, (2.13) implies that $Z_1(h-\eta) < \infty$. Equivalently,

$$\sum_{e\in E} \|D\phi_e\|_\infty^{h-\eta} < \infty. \tag{3.56}$$

Observe that for all but finitely many $e \in E$,

$$\|D\phi_e\|_\infty^{-\eta} \geq -\log(\|D\phi_e\|_\infty). \tag{3.57}$$

Indeed, if (3.57) was false we could find infinitely many $(e_n)_{n\in\mathbb{N}}, e_n \in E$, such that

$$\|D\phi_{e_n}\|_\infty^{\eta} \log(\|D\phi_{e_n}\|_\infty^{-1}) > 1.$$

Nevertheless, this is impossible because by [7, Lemma 4.18] we know that $\|D\phi_{e_n}\|_\infty \to 0$ as $n\to\infty$, while $\lim_{x\to 0^+} x^\eta \log(1/x) = 0$. Therefore,

$$\sum_{e\in E} -\log(\|D\phi_e\|_\infty)\|D\phi_e\|_\infty^h \overset{(3.56)\wedge(3.57)}{<} \infty. \tag{3.58}$$

We can now estimate $\chi_{\tilde{\mu}_h}(\sigma)$:

$$\begin{aligned}
\chi_{\tilde{\mu}_h}(\sigma) &= -\int_{E_A^{\mathbb{N}}} \log\|D\phi_{\omega_1}(\pi(\sigma(\omega))\| \, d\tilde{\mu}_h(\omega) \\
&= \sum_{e\in E} \int_{[e]} -\log\|D\phi_{\omega_1}(\pi(\sigma(\omega))\| \, d\tilde{\mu}_h(\omega) \\
&\overset{(2.4)}{\leq} \log K \sum_{e\in E} \tilde{\mu}_h([e]) + \sum_{e\in E}(-\log(\|D\phi_e\|_\infty)\tilde{\mu}_h([e]) \\
&= \log K\, \tilde{\mu}_h(E_A^{\mathbb{N}}) + \sum_{e\in E}(-\log(\|D\phi_e\|_\infty)\tilde{\mu}_h([e]).
\end{aligned}$$

Recall that $\tilde{\mu}_h$ is a probability measure on $E_A^{\mathbb{N}}$ (as a Gibbs state), hence in order to show that $\chi_{\tilde{\mu}_h}(\sigma) < \infty$ it suffices to show that

$$\sum_{e\in E}(-\log(\|D\phi_e\|_\infty)\tilde{\mu}_h([e]) < \infty. \tag{3.59}$$

Since $\mathcal{S}$ is strongly regular, and thus regular by (2.14), it is well known, see e.g. [7, Equation 7.18], that there exists some constant $c_h \geq 1$ such that for all $\omega \in E_A^*$,

$$c_h^{-1}\|D\phi_\omega\|_\infty \leq \tilde{\mu}_h([\omega]) \leq c_h\|D\phi_\omega\|_\infty. \tag{3.60}$$

Thus, (3.59) follows by (3.58) and (3.60).

As we showed that $\chi_{\tilde{\mu}_h}(\sigma) < \infty$, (i) follows from Theorem 3.10 and the fact that $\tilde{\mu}_h$ is globally equivalent to the conformal measure $\tilde{m}_h$. Finally (ii) follows from (i) and [7, Theorem 10.1]. □

Note that all the quantities and concepts introduced in Definition 3.9 are invariant under isometries of the ambient metric space (X,d). It is also fairly obvious that these are invariant under all similarity self-maps of X. In the following lemma, which will be useful in Section 5, we will show that they are also conformal invarant.

Lemma 3.12. *Let $X \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$. Let $\xi \in X$ and suppose that W is an open connected neighborhood of ξ and $\phi : W \to \mathbb{R}^n$ is a C^1 conformal diffeomorphism from W onto $\phi(W)$. If X is (α,p)-mean porous at ξ in $\mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ is any set such that $\phi(\xi) \in Y$ and $Y \cap \phi(W) \subset \phi(X \cap W)$ then Y is (α,p)-mean porous at $\phi(\xi)$.*

Proof. Since W is an open set containing ξ and since $\phi(W)$ is an open set containing $\phi(\xi)$, there exists $i_1 \geq 0$ such that

$$B(\xi,2^{-j}) \subset W \text{ and } B(\phi(\xi),2^{-j}) \subset \phi(W)$$

for all integers $j \geq i_1$. Now suppose that $s \in (0,1/2]$ and $\beta > \alpha$ are such that $\underline{A}(X,\xi;\beta,s) > p$. Fix $\varepsilon > 0$ so small that $(1+\varepsilon)^{-2}\beta > \alpha$. Since $\phi : W \to \phi(W)$ is a C^1 conformal diffeomorphism from W onto $\phi(W)$ (in particular $|\phi'(\xi)| \neq 0$) then there exists $i_2 \geq i_1$ such that

$$B(\phi(a),(1+\varepsilon)^{-1}|\phi'(\xi)|r) \subset \phi(B(a,r)) \subset B(\phi(a),(1+\varepsilon)|\phi'(\xi)|r) \tag{3.61}$$

and

$$B(\phi^{-1}(b),(1+\varepsilon)^{-1}|\phi'(\xi)|^{-1}r) \subset \phi^{-1}(B(b,r)) \subset B(\phi^{-1}(b),(1+\varepsilon)|\phi'(\xi)|^{-1}r) \tag{3.62}$$

for every $a \in B(\xi,2^{-i_2}), b \in B(\phi(\xi),2^{-i_2})$ and $r \in (0,2^{-i_2})$. Take then $j \geq i_2$ such that $\text{por}(X,\xi,s2^{-j}) > \beta$. Then there exists $z \in \mathbb{R}^n$ such that $B(z,\beta s2^{-j}) \subset B(\xi,s2^{-j}) \setminus X$. Since $Y \cap \phi(W) \subset \phi(X \cap W)$ we have that

$$\begin{aligned} \phi(B(z,\beta s2^{-j})) &\subset \phi(B(\xi,s2^{-j}) \setminus X) \subset \phi(B(\xi,s2^{-j})) \setminus Y \\ &\overset{(3.61)}{\subset} B(\phi(\xi),(1+\varepsilon)|\phi'(\xi)|s2^{-j}) \setminus Y. \end{aligned} \tag{3.63}$$

Moreover,

$$\phi(B(z,\beta s2^{-j})) \overset{(3.61)}{\supset} B(\phi(z),(1+\varepsilon)^{-1}|\phi'(\xi)|\beta s2^{-j}). \tag{3.64}$$

Therefore,

$$B(\phi(z),(1+\varepsilon)^{-1}|\phi'(\xi)|\beta s2^{-j}) \overset{(3.63)\wedge(3.64)}{\subset} B(\phi(\xi),(1+\varepsilon)|\phi'(\xi)|s2^{-j}) \setminus Y, \tag{3.65}$$

and consequently

$$\text{por}(Y,\phi(\xi),(1+\varepsilon)|\phi'(\xi)|s2^{-j}) \geq (1+\varepsilon)^{-2}\beta.$$

Thus,

$$\underline{A}(Y,\phi(\xi);(1+\varepsilon)^{-2}\beta,(1+\varepsilon)|\phi'(\xi)|s) \geq \underline{A}(X,\xi;\beta,s) > p$$

Therefore,

$$S\underline{A}(Y,\phi(\xi);\alpha) \geq \underline{A}(Y,\phi(\xi);(1+\varepsilon)^{-2}\beta,(1+\varepsilon)|\phi'(\xi)|s) > p,$$

and the proof is complete. □

□

3.4. **Directed porosity.** We now turn our attention to directed porosity, whose formal definition we provide below. Given $v \in S^{n-1}$ we will denote by l_v the line in $\mathbb{R}^n$ containing the origin and v. For $E \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $r \in (0, \mathrm{diam}(E))$ we let

$$\mathrm{por}_v(E,x,r) = \sup\{c \geq 0 : B(y,cr) \subset B(x,r) \setminus E \text{ for some } y \in x + l_v\}. \tag{3.66}$$

Definition 3.13. Let $E \subset \mathbb{R}^n$ be a bounded set. Given $v \in S^{n-1}, c \in (0,1)$ and $x \in \mathbb{R}^n$, we say that E is:

(i) *v-directed c-porous at x* if there exists some $r_0 > 0$ such that $\mathrm{por}_v(E,x,r) \geq c$ for every $r \in (0, r_0)$,
(ii) *v-directed porous at x* if there exists some $c \in (0,1)$ such that E is v-directed c-porous at x,
(iii) E is *v-directed c-porous* if there exists some $r_0 > 0$ such that $\mathrm{por}_v(E,x,r) \geq c$ for every $x \in E$ and $r \in (0, r_0)$,
(iv) *v-directed porous* if it is v-directed c-porous for some $c \in (0,1)$.

Our first theorem in this section provides a sufficient condition for a finite and irreducible conformal GDMS to be directed porous at every direction.

Theorem 3.14. *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a finite and irreducible conformal GDMS in $\mathbb{R}^n$ such that*

$$\mathrm{dist}(\partial X, J_{\mathcal{S}}) := R > 0. \tag{3.67}$$

Then $J_{\mathcal{S}}$ is v-directed porous for every $v \in S^{n-1}$.

Proof. Let $v \in S^{n-1}$, $\omega \in E_A^{\mathbb{N}}$ and $r > 0$. Let k be the smallest integer such that

$$\phi_{\omega|_k}(X_{t(\omega_k)}) \subset B(\pi(\omega), r). \tag{3.68}$$

Now let $I_v \subset \phi_{\omega|_k}(X_{t(\omega_k)})$ be a line segment at direction v, joining $\pi(\omega)$ to $\partial \phi_{\omega|_k}(X_{t(\omega_k)})$. Then there exists some $y \in \partial X_{t(\omega_k)}$ such that

$$\mathrm{length}(I_v) \geq \mathrm{dist}(\pi(\omega), \partial \phi_{\omega|_k}(X_{t(\omega_k)})) = |\pi(\omega) - \phi_{\omega|_k}(y)|.$$

Hence, by [7, Lemma 4.14] there exists some constant $c > 0$ such that

$$\begin{aligned}\mathrm{length}(I_v) &\geq |\phi_{\omega|_k}(\pi(\sigma^k(\omega)) - \phi_{\omega|_k}(y)| \\ &\geq c\|D\phi_{\omega|_k}\|_\infty |\pi(\sigma^k(\omega)) - y| \geq c\|D\phi_{\omega|_k}\|_\infty R > 0.\end{aligned}$$

Note that $\gamma_v := \phi^{-1}_{\omega|_k}(I_v) \subset X_{t(\omega_k)}$ is a smooth curve joining $\pi(\sigma^k(\omega))$ to ∂X. In particular γ_v joins $\pi(\sigma^k(\omega))$ to $\partial X_{t(\omega_k)}$. Therefore there exists $z \in \gamma_v$ such that

$$B(z, R/4) \subset X_{t(\omega_k)} \subset X \text{ and } B(z, R/4) \cap J_{\mathcal{S}} = \emptyset, \tag{3.69}$$

and consequently

$$\phi_{\omega|_k}(B(z, R/4)) \cap J_{\mathcal{S}} = \emptyset. \tag{3.70}$$

Let $R' := \min\{R/4, \eta_{\mathcal{S}}/2\}$, where $\eta_{\mathcal{S}}$ was defined in (2.8). Thus

$$\begin{aligned}B(\phi_{\omega|_k}(z), K^{-1}\|D\phi_{\omega|_k}\|_\infty R') &\overset{(2.9)}{\subset} \phi_{\omega|_k}(B(z,R')) \\ &\overset{(3.69)}{\subset} \phi_{\omega|_k}(X_{t(\omega_k)}) \overset{(3.68)}{\subset} B(\pi(\omega), r).\end{aligned} \tag{3.71}$$

Therefore,

$$J_{\mathcal{S}} \cap B(\phi_{\omega|_k}(z), K^{-1}\|D\phi_{\omega|_k}\|_\infty R') \overset{(3.70)\wedge(3.71)}{=} \emptyset. \tag{3.72}$$

We also record that

$$\phi_{\omega|_k}(z) \in I_v \subset \phi_{\omega|_k}(X_{t(\omega_k)}) \subset B(\pi(\omega), r). \tag{3.73}$$

Without loss of generality we can assume that $\text{diam}(X) = 1$. So if $m_0 = \min\{\|D\phi_e\|_\infty : e \in E\}$ we have that

$$r \leq \text{diam}(\phi_{\omega|_{k-1}}(X_{t(\omega_{k-1})})) \overset{(2.7)}{\leq} \frac{KM}{m_0}\|D\phi_{\omega|_{k-1}}\|_\infty\|D\phi_{\omega_k}\|_\infty \overset{(2.5)}{\leq} \frac{K^2M}{m_0}\|D\phi_{\omega|_k}\|_\infty. \tag{3.74}$$

Hence

$$J_{\mathcal{S}} \cap B(\phi_{\omega|_k}(z), m_0(MK^3)^{-1}R'r) \overset{(3.72)\wedge(3.74)}{=} \emptyset. \tag{3.75}$$

Combining (3.75) and (3.73) we deduce that $J_{\mathcal{S}}$ is v-directed $m_0(MK^3)^{-1}R'$-porous at $\pi(\omega)$. The proof is complete. □

We will now see how we can use Theorem 3.14 to show that if a finite and irreducible conformal GDMS satisfies the strong separation condition (recall Remark 2.4) then it is directed porous at every direction. Before doing so, we recall the notion of *equivalent* graph directed Markov systems which was introduced in [7].

Definition 3.15. Two conformal GDMS $\mathcal{S}$ and $\mathcal{S}'$ are called *equivalent* if:

(i) they share the same associated directed multigraph (E, V),
(ii) they have the same incidence matrix A and the same functions $i, t : E \to V$,
(iii) they are defined by the same set of conformal maps $\{\phi_e : W_{t(e)} \to W_{i(e)}\}$, where W_v are open connected sets, and for every $v \in V$, $X_v \cup X'_v \subset W_v$.

Lemma 3.16. *Let $\mathcal{S} = \{V, E, A, t, i, \{X_v\}_{v\in V}, \{\phi_e\}_{e\in E}\}$ be a finite conformal GDMS which satisfies the strong separation condition. Then there exists a finite conformal $\mathcal{S}' = \{V, E, A, t, i, \{X'_v\}_{v\in V}, \{\phi_e\}_{e\in E}\}$ which is equivalent to $\mathcal{S}$ and*

$$\text{dist}(\partial X', \cup_{e\in E}\phi_e(X'_{t(e)})) := R > 0,$$

where $X' = \cup_{v\in V} X'_v$.

Proof. Since $\mathcal{S}$ satisfies the strong separation condition there exists $\varepsilon < \eta_{\mathcal{S}}$ such that

$$\phi_a(X_{t(a)}(\varepsilon)) \cap \phi_b(X_{t(b)}(\varepsilon)) = \emptyset \tag{3.76}$$

for all distinct $a, b \in E$, where $A(\varepsilon) = \{x : d(x, A) < \varepsilon\}$ for $A \subset \mathbb{R}^n$. We can also assume that

$$\max\{\|D\phi_e\|_\infty : e \in E\} = s' < 1.$$

We now let

$$X'_v = \overline{X_v(\varepsilon)}.$$

We will show that for all $e \in E$,

$$\phi_e(X'_{t(e)}) \subset \overline{X_{i(e)}(s'\varepsilon)}. \tag{3.77}$$

For all $p \in X'_{t(e)}$ there exists some $q \in X_{t(e)}$ such that $d(p,q) \le \varepsilon$. Hence, by the mean value theorem, if $\phi_e(p) \notin X_{i(e)}$,

$$d(X_{i(e)}, \phi_e(p)) \le d(\phi_e(X_{t(e)}), \phi_e(p)) \le |\phi_e(q) - \phi_e(p)| \le \|D\phi_e\|_\infty |p-q| \le s'\varepsilon.$$

Therefore, (3.77) follows.

Hence, (3.76) and (3.77) imply that $\mathcal{S}' = \{V, E, A, t, i, \{X'_v\}_{v\in V}, \{\phi_e\}_{e\in E}\}$ is a conformal GDMS and moreover (3.77) implies that

$$\operatorname{dist}(\partial X', \cup_{e\in E}\phi_e(X'_{t(e)})) \ge (1-s')\varepsilon > 0.$$

The proof is complete. □

Theorem 3.17. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finite and irreducible conformal GDMS in $\mathbb{R}^n$ which satisfies the strong separation condition. Then $J_{\mathcal{S}}$ is v-directed porous for every $v \in S^{n-1}$.*

Proof. Note that if two conformal GDMS are equivalent then they generate the same limit set. Therefore Theorem 3.17 follows by Theorem 3.14 and Lemma 3.16. □

In the next theorem we consider systems where (3.67) does not necessarily hold. We prove that if an IFS consists of rotation free similarities and there exist directions $v \in S^{n-1}$ such that the lines $l_v + x$ miss all the first iterations in the interior of the set X, then the limit set is v-directed porous.

Theorem 3.18. *Let $\mathcal{S} = \{\phi_e : X \to X\}_{e\in E}$ be a finite CIFS consisting of rotation free similarities. Suppose that there exists a set $V \subset S^{n-1}$ such that*

$$(\operatorname{Int}(X) \cap (l_v + x)) \setminus \bigcup_{e\in E} \phi_e(X) \ne \emptyset \tag{3.78}$$

for every $x \in \bigcup_{e\in E}\phi_e(X)$ and $v \in S^{n-1}\setminus V$. Then $J_{\mathcal{S}}$ is v-directed porous for every $v \in S^{n-1}\setminus V$ and every $x \in J_{\mathcal{S}}$.

Proof. Let $v \in S^{n-1}\setminus V$. We will first show that there exists some $\eta > 0$ such that for all $x \in \cup_{e\in E}\phi_e(X)$ there exists some $z_x \in l_v + x$ such that

$$B(z_x, \eta) \subset \operatorname{Int}(X) \setminus J_{\mathcal{S}}. \tag{3.79}$$

If $x \in \cup_{e\in E}\phi_e(X)$ let

$$\eta_x = \sup\{\theta \ge 0 : \exists z \in l_v + x \text{ such that } B(z,\theta) \subset \operatorname{Int}(X)\setminus J_{\mathcal{S}}\}.$$

In order to establish (3.79) it suffices to show that

$$\inf\{\eta_x : x \in \cup_{e\in E}\phi_e(X)\} > 0.$$

By way of contradiction assume that $\inf\{\eta_x : x \in \cup_{e\in E}\phi_e(X)\} = 0$. So there exists a positive sequence $(\eta_n)_{n\in\mathbb{N}}$ such that $\eta_n \to 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in $\cup_{e\in E}\phi_e(X)$ such that if $z \in (l_v + x_n) \cap \operatorname{Int}(X)$ satisfies $B(z,\theta) \subset \operatorname{Int}(X)\setminus J_{\mathcal{S}}$, then $\theta \le \eta_n$. By compactness, passing to subsequences if necessary, there exists $x \in \cup_{e\in E}\phi_e(X)$ such that $x_n \to x$. By (3.78) there exists $y_x \in (\operatorname{Int}(X) \cap (l_v + x)) \setminus \bigcup_{e\in E}\phi_e(X)$ and $R > 0$ such that

$$B(y_x, R) \subset \operatorname{Int}(X) \setminus \bigcup_{e\in E} \phi_e(X). \tag{3.80}$$

For every $n \in \mathbb{N}$ let y_n be the unique point in $(l_v^\perp + y_x) \cap (l_v + x_n)$. Let $n_0 \in \mathbb{N}$ big enough such that $|x_n - x| < R/4$ for all $n \geq n_0$. Notice that

$$|y_n - y_x| \leq |x_n - x| < R/4$$

for all $n \geq n_0$. Hence $B(y_n, R/2) \subset B(y_x, R)$ for all $n \geq n_0$. Therefore by (3.80),

$$B(y_n, R/2) \subset \mathrm{Int}(X) \setminus \bigcup_{e \in E} \phi_e(X), \text{ for all } n \geq n_0.$$

Hence, we have found $y_n \in X \cap (l_v + x_n)$ such that

$$B(y_n, R/2) \subset \mathrm{Int}(X) \setminus J_{\mathcal{S}}.$$

But this is a contradiction because $\eta_n \to 0$ and (3.79) has been proven.

Now let $x = \pi(\omega)$ for some $\omega \in E^{\mathbb{N}}$ and fix some $r > 0$. Let n be the smallest integer such that

$$\phi_{\omega|_n}(X) \subset B(\pi(\omega), r).$$

Denote $l_\omega := l_v + x$ and let $l_0 := \phi_{\omega|_n}^{-1}(l_\omega)$. Since $\mathcal{S}$ consists of rotation free similarities, l_0 is a line parallel to l_ω. Observe that

$$\pi(\sigma^n(\omega)) \in l_0 \cap \phi_{\omega_{n+1}}(X). \tag{3.81}$$

This follows because $\pi(\omega) = \phi_{\omega|_n}(\pi(\sigma^n(\omega)))$, hence $\pi(\sigma^n(\omega)) \in \phi_{\omega|_n}^{-1}(l_\omega) = l_0$. Also by definition of the projection π,

$$\pi(\sigma^n(\omega)) = \bigcap_{m=1}^{\infty} \phi_{\sigma^n(\omega)|_m}(X) \subset \phi_{\omega_{n+1}}(X).$$

Hence $l_0 = \pi(\sigma^n(\omega)) + l_v$ and by (3.79) there exists some $z \in l_0$ such that

$$B(z, \eta) \subset \mathrm{Int}(X) \setminus J_{\mathcal{S}}. \tag{3.82}$$

Without loss of generality we can assume that $\eta < \eta_{\mathcal{S}}$. Hence

$$B(\phi_{\omega|_n}(z), K^{-1}\eta \|D\phi_{\omega|_n}\|_\infty) \overset{(2.9)}{\subset} \phi_{\omega|_n}(B(z, \eta)). \tag{3.83}$$

We will now show that

$$\phi_{\omega|_n}(B(z, \eta)) \cap J_{\mathcal{S}} = \emptyset. \tag{3.84}$$

By way of contradiction assume that there exists some $y \in B(z, \eta)$ such that $\phi_{\omega|_n}(y) \in J_{\mathcal{S}}$. Hence there exists some $\tau \in E_A^{\mathbb{N}}$ such that

$$\phi_{\omega|_n}(y) = \pi(\tau) = \phi_{\tau|_n}(\pi(\sigma^n(\tau))). \tag{3.85}$$

If $\omega|_n = \tau|_n$ then $y = \pi(\sigma^n(\tau))$. Hence, $y \in J_{\mathcal{S}}$ and this contradicts (3.82). If $\omega|_n \neq \tau|_n$ let $m \leq n$ be the smallest integer such that $\omega_m \neq \tau_m$. By the open set condition and the fact that the maps ϕ_e are homeomorphisms for every $e \in E$, we deduce that

$$\phi_{\omega_m}(\mathrm{Int}(X)) \cap \phi_{\tau_m}(X) = \emptyset,$$

and consequently

$$\phi_{\omega|_m}(\mathrm{Int}(X)) \cap \phi_{\tau|_m}(X) = \emptyset.$$

But this contradicts (3.85) because $\phi_{\omega|_n}(y) \in \phi_{\omega|_m}(\mathrm{Int}(X))$ and $\pi(\tau) \in \phi_{\tau|_m}(X)$. Therefore we have established (3.84).

We then have

$$B(\phi_{\omega|_n}(z), K^{-1}\eta\|D\phi_{\omega|_n}\|_\infty) \cap J_{\mathcal{S}} \overset{(3.83)\wedge(3.84)}{=} \emptyset. \tag{3.86}$$

Since $z \in l_0 \cap X$ it follows that $\phi_{\omega|_n}(z) \in l_\omega \cap \phi_{\omega|_n}(X) \subset B(\pi(\omega), r)$. Without loss of generality we can assume that $\mathrm{diam}(X) \leq 1$. So, by the choice of n, we have that

$$r \leq \mathrm{diam}(\phi_{\omega|_{n-1}}(X)) \overset{(2.7)}{\leq} \frac{K}{m_0}\|D\phi_{\omega|_{n-1}}\|_\infty\|D\phi_{\omega_n}\|_\infty \overset{(2.5)}{\leq} \frac{K^2}{m_0}\|D\phi_{\omega|_n}\|_\infty, \tag{3.87}$$

where $m_0 = \min\{\|D\phi_e\|_\infty : e \in E\}$. Therefore

$$B(\phi_{\omega|_n}(z), \frac{m_0\eta}{K^3} r) \cap J_{\mathcal{S}} \overset{(3.86)\wedge(3.87)}{=} \emptyset.$$

Thus, $J_{\mathcal{S}}$ is v-directed $\frac{m_0\eta}{K^3}$-porous at x. The proof is complete. □

3.5. **Non-porosity.** So far, we have been investigating porosity properties of conformal GDMS. Nevertheless, limit sets of conformal GDMS very frequently are non-porous. In the following, we will prove that if the limit set of a finitely irreducible conformal GDMS is not porous at a single point, then it is not porous in a set of full measure.

Theorem 3.19. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS such that $J_{\mathcal{S}}$ is not porous, or $J_{\mathcal{S}}$ is not porous at some $\zeta \in \overline{J_{\mathcal{S}}}$. If $\tilde{\mu}$ is a shift invariant ergodic probability measure on $E_A^{\mathbb{N}}$ with full topological support, then $J_{\mathcal{S}}$ is not porous at $\tilde{\mu}\circ\pi^{-1}$ a.e. $x \in J_{\mathcal{S}}$.*

Proof. We will first assume that $J_{\mathcal{S}}$ is not porous. Let $\varepsilon \in (0, 2K^{-2})$. Since $J_{\mathcal{S}}$ is not ε-porous then there exists $r := r_\varepsilon \in (0, \eta_{\mathcal{S}}/2)$ (recall (2.8)) and $\xi := \xi_\varepsilon \in J_{\mathcal{S}}$ such that $\mathrm{por}(J_{\mathcal{S}}, \xi, r) < \varepsilon$. Hence, for all $x \in B(\xi, r)$

$$B(x, \varepsilon r) \cap J_{\mathcal{S}} \neq \emptyset. \tag{3.88}$$

The measure $\tilde{\mu}$ has full topological support hence $\tilde{\mu}(\pi^{-1}(B(\xi, s))) > 0$, for every $s > 0$. Moreover, since $\tilde{\mu}$ is ergodic and Birkhoff's Ergodic Theorem implies that if

$$Y_\varepsilon := \{\omega \in E_A^{\mathbb{N}} : \sigma^n(\omega) \in \pi^{-1}(B(\xi, (4MK^2)^{-1}r)) \text{ for infinitely many } n \in \mathbb{N}\},$$

then $\tilde{\mu}(Y_\varepsilon) = 1$.

Let $\omega \in Y_\varepsilon$ and $n \in \mathbb{N}$ such that $\sigma^n(\omega) \in \pi^{-1}(B(\xi, (4K)^{-1}r))$. Since $\xi \in J_{\mathcal{S}}$ there exists some $v \in V$ such that $\xi \in X_v$. On the other hand $\pi(\sigma^n(\omega)) \in X_{t(\omega_n)}$ and

$$d(\pi(\sigma^n(\omega)), \xi) \leq \frac{r}{4MK^2} < \frac{\eta_{\mathcal{S}}}{8K^2}. \tag{3.89}$$

By the definition of $\eta_{\mathcal{S}}$ and the fact that the sets $X_v, v \in V$, are pairwise disjoint we deduce that $\xi \in X_{t(\omega_n)}$. Hence, Lemma 2.5 implies that

$$B(\phi_{\omega|_n}(\xi), K^{-1}\|D\phi_{\omega|_n}\|_\infty r) \subset \phi_{\omega|_n}(B(\xi, r)) \subset B(\phi_{\omega|_n}(\xi), K\|D\phi_{\omega|_n}\|_\infty r). \tag{3.90}$$

Moreover

$$\begin{aligned} d(\pi(\omega), \phi_{\omega|_n}(\xi)) &= d(\phi_{\omega|_n}(\pi(\sigma^n(\omega))), \phi_{\omega|_n}(\xi)) \\ &\overset{(2.6)}{\leq} KM\|D\phi_{\omega|_n}\|_\infty d(\pi(\sigma^n(\omega)), \xi) \overset{(3.89)}{\leq} \frac{\|D\phi_{\omega|_n}\|_\infty r}{4K}. \end{aligned} \tag{3.91}$$

Hence

$$B(\pi(\omega),(2K)^{-1}\|D\phi_{\omega|_n}\|_\infty r) \overset{(3.91)}{\subset} B(\phi_{\omega|_n}(\xi),K^{-1}\|D\phi_{\omega|_n}\|_\infty r) \overset{(3.90)}{\subset} \phi_{\omega|_n}(B(\xi,r)). \tag{3.92}$$

Let $\omega\in Y_\varepsilon$. We will show that $J_\mathscr{S}$ is not ε'-porous at $\pi(\omega)$ where $\varepsilon'=2K^2M\varepsilon$. Let $\delta'>0$ and choose $n\in\mathbb{N}$ big enough such that $(2K)^{-1}\|D\phi_{\omega|_n}\|_\infty r<\delta'$; this is possible for example by [7, Lemma 4.18]. Now take $z\in B(\pi(\omega),(2K)^{-1}\|D\phi_{\omega|_n}\|_\infty r)$ and note that by (3.92), $z=\phi_{\omega|_n}(b)$ for some $b\in B(\xi,r)$. Therefore, (3.88) implies that $B(b,\varepsilon r)\cap J_\mathscr{S}\neq\emptyset$. Let $y\in B(b,\varepsilon r)\cap J_\mathscr{S}$. Then, we have that $d(y,b)<\varepsilon r<\eta_\mathscr{S}/2$ and

$$d(\phi_{\omega|_n}(y),\phi_{\omega|_n}(b)) \overset{(2.6)}{\leq} KM\|D\phi_{\omega|_n}\|_\infty d(y,b)<KM\|D\phi_{\omega|_n}\|_\infty \varepsilon r.$$

Therefore

$$J_\mathscr{S}\cap B(z,KM\|D\phi_{\omega|_n}\|_\infty\varepsilon r)\neq\emptyset.$$

Hence $J_\mathscr{S}$ is not ε'-porous at $\pi(\omega)$.

Let $\mu=\tilde\mu\circ\pi^{-1}$. We have shown that for every $\varepsilon\in(0,1)$ there exists $A_\varepsilon:=\pi(Y_\varepsilon)\subset J_\mathscr{S}$ such that $\mu(A_\varepsilon)=1$ and $J_\mathscr{S}$ is not $2K^2M\varepsilon$-porous for all $z\in A_\varepsilon$. Therefore there exist sets $A_n\subset J_\mathscr{S}$ such that $\mu(A_n)=1$ and $J_\mathscr{S}$ is not $2K^2Mn^{-1}$-porous for all $z\in A_n$. Let $A:=\cap_{n\in\mathbb{N}}A_n$. Then $\mu(A)=1$ and we will show that $J_\mathscr{S}$ is not porous at every $z\in A$. Let $z\in A$ and suppose by contradiction that $J_\mathscr{S}$ is d-porous at z for some $d>0$. Choose $n\in\mathbb{N}$ such that $2K^2Mn^{-1}<d$. Since $z\in A_n$ for every $\delta>0$ there exists some $r_\delta\in(0,\delta)$ such that if $y\in B(z,r_\delta)$ then $J_\mathscr{S}\cap B(y,2K^2Mn^{-1}r_\delta)\neq\emptyset$. Hence, $J_\mathscr{S}\cap B(y,dr_\delta)\neq\emptyset$ for all $y\in B(z,r_\delta)$ and we have reached a contradiction since we had assumed that $J_\mathscr{S}$ is d-porous at z. So the proof is complete when $J_\mathscr{S}$ is not porous.

If $J_\mathscr{S}$ is not porous at some $\zeta\in\overline{J_\mathscr{S}}$ the proof is almost identical. In that case for all $\varepsilon>0$ there exists $r:=r_\varepsilon\in(0,\frac{\eta_\mathscr{S}}{2})$ such that $\mathrm{por}(J_\mathscr{S},\zeta,r)<\varepsilon$. Hence, for all $x\in B(\zeta,r)$

$$B(x,\varepsilon r)\cap J_\mathscr{S}\neq\emptyset.$$

The rest of the proof follows exactly in the same manner as in the previous case. The proof is complete. □

We are now ready to see how Theorem 1.6 follows from Theorem 3.19. For the concepts appearing in the following corollary, recall Theorem 2.1 and the related discussion in the end of Section 2.

Corollary 3.20. *Let $\mathscr{S}=\{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS such that $J_\mathscr{S}$ is not porous, or $J_\mathscr{S}$ is not porous at some $\zeta\in\overline{J_\mathscr{S}}$. If $t\in\mathrm{Fin}(\mathscr{S})$ then $J_\mathscr{S}$ is not porous at m_t-a.e. $x\in J_\mathscr{S}$. In particular $J_\mathscr{S}$ is not porous at m_h-a.e. $x\in J_\mathscr{S}$ where $h=\dim_\mathscr{H}(J_\mathscr{S})$.*

Proof. Let $\tilde\mu_t$ be the unique shift invariant ergodic measure on $E_A^\mathbb{N}$ which is a Gibbs state for the potential $t\zeta$, see Theorem 2.1. It follows by the definition of Gibbs states, see e.g. [7, Section 6.2], that $\tilde\mu_t([\omega])>0$ for all $\omega\in E_A^*$. Since the collection of cylinders forms a countable base for the topology of $E_A^\mathbb{N}$ (recall (2.3)) it follows that $\tilde\mu_t$ has full topological support. Hence the conclusion follows by Theorem 3.19 and Theorem 2.1

because the unique eigenmeasure $\tilde{m}_t$ of the conjugate Perron-Frobenius operator $\mathscr{L}_t^*$ is globally equivalent to $\tilde{\mu}_t$. □

We record that since $\mathscr{H}^h\lfloor_{J_\mathscr{S}} \ll m_h$ (see e.g. [7, Theorem 10.1]), Corollary 3.20 implies that if $J_\mathscr{S}$ is not porous, or $J_\mathscr{S}$ is not porous at some $\zeta \in \overline{J_\mathscr{S}}$, then $J_\mathscr{S}$ is not porous at $\mathscr{H}^h$-a.e. $x \in J_\mathscr{S}$. Nevertheless this statement is vacuous when $\mathscr{H}^h(J_\mathscr{S}) = 0$.

It is known that if $\mathscr{S}$ is a finitely irreducible GDMS which consists of finitely many conformal maps and $\text{Int}(X) \setminus J_\mathscr{S} \neq \emptyset$ then $J_\mathscr{S}$ is porous, see e.g. [36, Theorem 4.6.4] and [58, Theorem 2.5]. According to our next Theorem the situation is very different when the alphabet is infinite. For every h less than the dimension of the ambient space we construct an infinite IFS consisting of similarities whose limit set is not porous almost everywhere.

Theorem 3.21. *For every $h \in (0,n)$ there exists a CIFS $\mathscr{S}_h$ consisting of similarities in $\mathbb{R}^n$ such that $\dim_\mathscr{H}(J_{\mathscr{S}_h}) = h$ and $J_{\mathscr{S}_h}$ is not porous at m_h-a.e. $x \in J_{\mathscr{S}_h}$, where m_h is the h-conformal measure of $\mathscr{S}$.*

Proof. We start by fixing some $\xi \in \mathbb{R}^n$ and $h \in (0,n)$. By Corollary 3.20 it suffices to construct a CIFS $\mathscr{S}_h$ consisting of similarities such that $\dim_\mathscr{H}(J_{\mathscr{S}_h}) = h$, $\xi \in \overline{J_{\mathscr{S}_h}}$ and $J_{\mathscr{S}_h}$ is not porous at ξ.

Let $A = (a_k)_{k\in\mathbb{N}}$ such that,

(i) $A \subset \overline{B}(0,1/2)$,
(ii) every $p \in A$ is an isolated point of A,
(iii) $\xi \in \bar{A} \setminus A$,
(iv) there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, with $\varepsilon_k \in (0,10^{-1})$ such that if $z_k \in \overline{B}(a_k,\varepsilon_k)$ then $(z_k)_{k\in\mathbb{N}}$ is not porous at ξ.

It is not difficult to produce countable sets with the above properties. For example, start with $A = \cup_{j=4}^\infty \Sigma_j$ where Σ_j is a maximal family in $\partial B(0,1/j)$ such that for all $p,q \in \Sigma_j$, $p \neq q$, $d(p,q) \geq j^{-2}$.

Pick some $r_1 \in (0,1)$ such that $r_1^h < 1$ and $r_1 < \varepsilon_1$. Proceeding inductively we obtain a sequence $(r_k)_{k\in\mathbb{N}}$ such that $0 < r_k < \varepsilon_k$ and

(i) $\sum_{k=1}^l r_k^h < 1$ for all $l \in \mathbb{N}$,
(ii) $\overline{B}(a_k,r_k) \cap (\bar{A} \setminus \{a_k\}) = \emptyset$,
(iii) the balls $\overline{B}(a_k,r_k)$ are pairwise disjoint.

Of course in that case

$$\sum_{k=1}^\infty r_k^h \leq 1. \tag{3.93}$$

For $k \in \mathbb{N}$ let $\phi_k : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\phi_k = \tau_{a_k} \circ \delta_{r_k},$$

where $\tau_q(p) = q + p$ and $\delta_r(p) = rp$. Note that $\phi_k(\overline{B}(0,1)) = \overline{B}(a_k, r_k)$, hence the sequence $(\phi_k)_{k\in\mathbb{N}}$ defines a system of similitudes $\mathcal{S}$ which satisfies the SSC, recall Remark 2.4. Note also that since for all $k \in \mathbb{N}$,

$$J_{\mathcal{S}} \cap \overline{B}(a_k, r_k) \neq \emptyset,$$

we deduce that $\xi \in \overline{J_{\mathcal{S}}}$. By the choices of a_k and r_k we also have that if $x_k \in J_{\mathcal{S}} \cap B(a_k, r_k)$ for $k \in \mathbb{N}$, then the set $\{x_k\}_{k\in\mathbb{N}}$ is not porous at ξ. Therefore $J_{\mathcal{S}}$ is not porous at ξ.

If $\sum_{k\in\mathbb{N}} r_k^h = 1$, then by [7, Corollary 7.22] we have that $\dim_{\mathcal{H}}(J_{\mathcal{S}}) = h$ and we are done. So we only have to check the case when $\sum_{k\in\mathbb{N}} r_k^h < 1$. Since

$$J_{\mathcal{S}} \subset \bigcup_{k=1}^{\infty} \overline{B}(a_k, r_k),$$

we deduce that $B(0,1) \setminus \overline{J_{\mathcal{S}}} \neq \emptyset$. Let

$$\psi_m : B(0,1) \to B_m,\ m = 1, \dots, n_0,$$

be a finite set of similarities such that the open balls B_m are pairwise disjoint, $B_m \subset B(0,1) \setminus \overline{J_{\mathcal{S}}}$, and

$$\sum_{n\in\mathbb{N}} r_n^h + \sum_{m=1}^{n_0} \|D\psi_m\|_\infty^h = 1.$$

Note that this is possible because the radius of each B_m is $\|D\psi_m\|_\infty$ and we can choose a_k and r_k such that $\cup_{k=1}^{\infty} \overline{B}(a_k, r_k) \subset B(0, 1/2)$. Hence if $\mathcal{S}_h = \mathcal{S} \cup \{\psi_m\}_{m=1}^{n_0}$, $J_{\mathcal{S}_h}$ is not porous at ξ and $\dim_{\mathcal{H}}(J_{\mathcal{S}_h}) = h$. Hence, Corollary 3.20 implies that $J_{\mathcal{S}_h}$ is not porous at m_h-a.e. $x \in J_{\mathcal{S}_h}$.The proof is complete. □

4. Porosity for complex continued fractions

In this section we explore porosity in the setting of *complex continued fractions*. We denote by

$$E := \{m + ni : (m,n) \in \mathbb{N} \times \mathbb{Z}\}$$

the Gaussian integers with positive real part. For $e \in E$ we consider the maps

$$\phi_e(z) := \frac{1}{e+z},$$

which are all holomorphic when $\mathrm{Re}(z) > -1$. Moreover, it is easy to check that

$$\mathrm{Re}(z) \geq 1 \text{ if and only if } z \in \overline{B}(1/2, 1/2). \tag{4.1}$$

Hence, if we set

$$X := \overline{B}(1/2, 1/2),$$

we see that

$$\phi_e : X \to X \text{ for all } e \in E. \tag{4.2}$$

The following proposition, whose proof can be found in [35], gathers some properties of the maps ϕ_e.

Proposition 4.1. For every $e \in E$

(i) $4^{-1}|e|^{-2} \leq |\phi_e'(z)| \leq 4|e|^{-2}$ for all $z \in X$,
(ii) $4^{-1}|e|^{-2} \leq \operatorname{diam}(\phi_e(X)) \leq 4|e|^{-2}$.

It is easy to check that $\phi_e(\operatorname{Int}(X)) \cap \phi_a(\operatorname{Int}(X)) = \emptyset$ for $e, a \in E, e \neq a$. Hence, $\{\phi_e\}_{e\in E}$ satisfies the OSC but formally $\{\phi_e\}_{e\in E}$ is *not* a conformal IFS because $\phi_1'(0) = 1$. For this reason we need the following lemma.

Lemma 4.2. *There exist $\delta > 0$ and $s \in (0,1)$ such that for all $\omega \in E^* \setminus E$,*

$$\phi_\omega(B(1/2, 1/2+\delta)) \subset B(1/2, 1/2 + s\delta) \tag{4.3}$$

and

$$\|\phi_\omega'\|_{B(1/2,1/2+\delta)} \leq s. \tag{4.4}$$

Proof. It suffices to prove (4.4) for $\omega \in E^2 \cup E^3$. Indeed, once (4.4) is established for $\omega \in E^2 \cup E^3$, then the Mean Value Theorem implies (4.3) for $\omega \in E^2 \cup E^3$. Since for every odd natural number $n \geq 3$ there exists some $k \in \mathbb{N}$ such that $n = 2k+3$, if (4.3) holds for $\omega \in E^2 \cup E^3$, then it holds for all $\omega \in E^*$ with $|\omega| \geq 2$. Finally, (4.3) for all $\omega \in E^* \setminus E$ combined with (4.4) for $\omega \in E^2 \cup E^3$ and the Leibniz rule, implies (4.4) for all $\omega \in E^*$ with $|\omega| \geq 2$.

We will first consider the case when $\omega = ab \in E^2$. Note that if $a, b \in E$, then

$$\phi_{ab}(z) = \frac{b+z}{1+a(b+z)},$$

and if $z \in B(1/2, 3/4)$,

$$|1 + a(b+z)| = |a|\left|\frac{1}{a} + b + z\right| \geq \left|\frac{1}{a} + b + z\right| \geq \operatorname{Re}(a^{-1}) + \operatorname{Re}(b) + \operatorname{Re}(z) > 1 - 1/4. \tag{4.5}$$

Hence the maps ϕ_{ab} are conformal on $B(1/2, 3/4)$ for all $a, b \in E$. From now on we assume that $z \in B(1/2, 3/4)$ and we record that for such z,

$$\phi_{ab}'(z) = \frac{1}{(1+a(z+b))^2}. \tag{4.6}$$

We will prove (4.4) (for $\omega = ab \in E^2$) by considering two cases. We first assume that $|a| \leq 2$, that is $a \in \{1, 1+i, 1-i, 2\}$. Note that in this case $\operatorname{Re}(a^{-1}) \geq 1/2$. Therefore, arguing as in (4.5),

$$|1 + a(z+b)| \geq \operatorname{Re}(a^{-1}) + \operatorname{Re}(b) + \operatorname{Re}(z) \geq 3/2 + \operatorname{Re}(z). \tag{4.7}$$

Hence, if $z \in B(1/2, 1/2+\delta)$ then $\operatorname{Re}(z) > -\delta$, and if we let $\delta < 1/2$ (whose exact value will be determined later) we have that

$$\|\phi_{ab}'\|_{B(1/2,1/2+\delta)} \overset{(4.6)\wedge(4.7)}{\leq} \frac{1}{(3/2-\delta)^2} < 1. \tag{4.8}$$

We now move to the case when $a \in E, |a| > 2$. Observe that if $a \in E$ and $|a| > 2$ then $|a| \geq \sqrt{5}$. Hence, for such $a \in E$ (and any $b \in E$), if we choose $\delta = \frac{\sqrt{5}-2}{2\sqrt{5}}$ we see that for any $z \in B(1/2, 1/2+\delta)$,

$$|1 + a(z+b)| \geq |a||z+b| - 1 \geq \sqrt{5}\operatorname{Re}(z+b) - 1 > \sqrt{5}(1-\delta) - 1 = \frac{\sqrt{5}}{2}. \tag{4.9}$$

Thus,

$$\|\phi'_{ab}\|_{B(1/2,1/2+\delta)} \overset{(4.6)\wedge(4.9)}{\leq} 4/5. \tag{4.10}$$

If we plug in $\delta = \frac{\sqrt{5}-2}{2\sqrt{5}}$ in (4.8) we deduce that for $a \in E$ with $|a| \leq 2$

$$\|\phi'_{ab}\|_{B(1/2,1/2+\delta)} < (5/4)^{-2} = 16/25 < 4/5,$$

where we used that $\frac{\sqrt{5}-2}{2\sqrt{5}} < 1/4$. Hence, we have shown that if $\omega \in E^2$,

$$\|\phi'_{ab}\|_{B(1/2,1/2+\delta)} \leq 4/5, \tag{4.11}$$

and we have established (4.4) for $\omega \in E^2$ with $\delta = \frac{\sqrt{5}-2}{2\sqrt{5}}$ and $s_0 = 4/5$.

Now, as mentioned earlier, (4.3) for $\omega \in E^2$ follows from (4.4) after a straightforward application of the Mean Value Theorem. Indeed, let $\omega \in E^2$ and $p \in B(1/2, 1/2+\delta)$. Then, there exists some $q \in \partial B(1/2,1/2)$ such that $|p-q| < \delta$. By the Mean Value Theorem,

$$|\phi_\omega(p) - \phi_\omega(q)| \leq \|\phi'_\omega\|_{B(1/2,1/2+\delta)} |p-q| < s_0\delta.$$

By (4.2) we deduce that $\phi_\omega(\overline{B}(1/2,1/2)) \subset \overline{B}(1/2,1/2)$, hence

$$|\phi_\omega(p) - 1/2| \leq |\phi_\omega(p) - \phi_\omega(q)| + |\phi_\omega(q) - 1/2| < s_0\delta + 1/2,$$

and (4.3) for $\omega \in E^2$ has been proven.

It remains to show (4.4) for $\omega \in E^3$. Let $\delta = \frac{\sqrt{5}-2}{2\sqrt{5}}$ and $s_0 = 4/5$ as in the previous case. From now on we will assume that $z \in B(1/2, 1/2+\delta)$. Note that for $\omega \in E^3$,

$$\phi_\omega(z) = \phi_{\omega_1}(\phi_{\omega_2\omega_3}(z)) = \frac{1}{\omega_1 + \phi_{\omega_2\omega_3}(z)},$$

and by (4.3) for E^2 we know that $\phi_{\omega_2\omega_3}(z) \in B(1/2, 1/2 + s_0\delta)$. Hence,

$$|\omega_1 + \phi_{\omega_2\omega_3}(z)| \geq \mathrm{Re}(\omega_1) + \mathrm{Re}(\phi_{\omega_2\omega_3}(z)) > 1 - s_0\delta > 0. \tag{4.12}$$

So the maps $\phi_\omega, \omega \in E^3$, are conformal in $B(1/2, 1/2+\delta)$. By (4.4) for words of length 2 and the chain rule, we obtain that

$$\begin{aligned} |\phi'_\omega(z)| &= |\phi'_{\omega_1}(\phi_{\omega_2\omega_3}(z))||\phi'_{\omega_2\omega_3}(z)| \\ &\overset{(4.11)}{\leq} s_0|\phi'_{\omega_1}(\phi_{\omega_2\omega_3}(z))| = \frac{s_0}{|\omega_1 + \phi_{\omega_2\omega_3}(z)|^2} \overset{(4.12)}{<} \frac{s_0}{(1-s_0\delta)^2}. \end{aligned} \tag{4.13}$$

However,

$$s := \frac{s_0}{(1-s_0\delta)^2} = \frac{400}{(10\sqrt{5} - 4(\sqrt{5}-2))^2} < 1. \tag{4.14}$$

Hence, we have proven (4.4) for $\omega \in E^3$ with $\delta = \frac{\sqrt{5}-2}{2\sqrt{5}}$ and s as in (4.14). Arguing exactly as in the case for $\omega \in E^2$, we also obtain (4.3) for $\omega \in E^3$. The proof is complete. □

Lemma 4.2 combined with [31, Remarks 10.2.2 and 10.3.3] allows us to treat $\{f_e : X \to X\}_{e\in E}$ as a conformal IFS. Alternatively, Lemma 4.2 shows that the family $\{\phi_e \circ \phi_j : (e,j) \in E \times E\}$ is a conformal IFS. From now on, perhaps slightly abusing notation, given $I \subset E$ we will call $\mathcal{CF}_I := \{\phi_e\}_{e\in I}$, a *complex continued fractions* IFS. Moreover we will denote its corresponding limit set by J_I. With regard to the bounded distortion properties of $\mathcal{CF}_I$ we record that the best distortion constant is $K = 4$, see [35, Remark 6.7].

We will repeatedly use the following version of Koebe's distortion theorem. For a relevant discussion of Koebe-type distortion theorems, see [31].

Theorem 4.3. *There exists a nonnegative, continuous, and increasing function $t \to K_t$, $t \in [0,1)$ with $K_0 = 1$, such that if $p \in \mathbb{C}$, $r > 0$ and $f : B(p,r) \to \mathbb{C}$ is a holomorphic function, then for every $t \in [0,1)$ and for all $w, z \in B(p, tr)$,*

$$\frac{|f'(w)|}{|f'(z)|} \leq K_t.$$

Consequently, if $w, z \in B(p, tr)$ and $B(w,s) \subset B(p, tr)$ for some $s > 0$, then

$$B(f(w), K_t^{-1}|f'(z)|s) \subset f(B(w,s)) \subset B(f(w), K_t|f'(z)|s).$$

In addition,

$$f(B(p,r)) \supset B(f(p), 4^{-1}|f'(p)|r).$$

It follows by [58, Theorem 2.5], or [23, Theorem 2.6], that if $F \subset E$ is finite then J_F is porous. Nevertheless, using Theorem 3.14 we can show something much stronger; as it turns out J_F is directed porous at all directions. We thus establish Theorem 1.7 (ii), which we restate and prove below.

Theorem 4.4. *Let $F \subset E$ be finite and let J_F be the limit set associated to the complex continued fractions system $\mathcal{CF}_F$. Then J_F is v-directed porous for all $v \in S^1$.*

Proof. We will first show that $\phi_e(z) \in \partial X$ if and only if $\text{Re}(e) = 1$ and $z = 0$. Note that $\phi_e(z) \in \partial X$ means that

$$|\phi_e(z) - 1/2| = \left| \frac{1}{z+e} - \frac{1}{2} \right| = 1/2,$$

or equivalently

$$|2 - (z+e)| = |z+e|. \tag{4.15}$$

Developing (4.15) we see that $\phi_e(z) \in \partial X$ if and only if $\text{Re}(z+e) = 1$, which holds exactly when $\text{Re}(e) = 1$ and $z = 0$, since $e \in E$ and $z \in \bar{B}(1/2, 1/2)$.

For $e \in E$ with $\text{Re}(e) = 1$ let $\{b_e\} = \partial X \cap \phi_e(X)$ and note that $\phi_e^{-1}(b_e) = 0$. We want to apply Theorem 3.14 hence we need to verify that

$$\text{dist}(J_F, \partial X) > 0. \tag{4.16}$$

Since J_F is a compact subset of X, it suffices to show that $J_F \cap \partial X = \emptyset$. Suppose by way of contradiction that there exists some $x \in J_F \cap \partial X$. Then $x = \pi(\omega)$ for some $\omega \in E^{\mathbb{N}}$. Then $x = \phi_{\omega_1}(\pi(\sigma(\omega)))$ and we let $e = \omega_1$. Therefore $x \in J_F \cap \phi_e(X) \cap \partial X$. By the previous discussion we then deduce that $x = b_e \in J_F$. In that case we would have that

$$0 = \phi_e^{-1}(b_e) = \phi_e^{-1}(x) = \pi(\sigma(\omega)) \in J_F,$$

and this is contradiction, since $0 \notin J_F$. The proof is complete. $\square$

Our first main theorem in this section, with many implications in the following, provides a characterization of porous limit sets of complex continued fractions with alphabet $I \subset E$. One interesting aspect of our characterization is that everything reduces to properties of the alphabet I. Hence, given $I \subset E$ one can check if J_I is porous by solely examining a certain property of the alphabet I.

Theorem 4.5. *Let I be an infinite subset of E. Then $\overline{J_I}$ is porous if and only if there exist $\theta \in (0,1), \kappa \in (0,1)$ and $\rho > 0$, such that for every $i \in I$ and every $R \in [\rho, \kappa|i|]$, there exists some $y_{i,R} \in B(i,R)$ such that*

$$E \cap B(y_{i,R}, \theta R) \subset E \setminus I.$$

Before giving the proof of Theorem 4.5 we state a general characterization of porous limit sets of infinite CIFS which we will employ in our proof.

Theorem 4.6 ([58])**.** *Let $\mathcal{S} = \{\phi_i\}_{i \in I}$ be an infinite CIFS. Then $\overline{J_I}$ is porous if there exists a cofinite set $F \subset I$ and parameters $\eta \geq 1, c > 0, \xi > 0, \beta \geq 1$, such that for all $i \in I \setminus F$ and every $r \in [\beta \operatorname{diam}(\phi_i(X)), \xi)$, there exist $x_i \in B(\phi_i(X), \eta r) \cap X$ such that*

$$B(x_i, cr) \cap J_I = \emptyset.$$

Proof of Theorem 4.5. Throughout the proof we are going to denote $M(z) = z^{-1}$ for $z \in \mathbb{C} \setminus \{0\}$. We will first show that if $\overline{J_I}$ is porous then there exist $\theta \in (0,1), \kappa \in (0,1)$ and $\rho > 0$, such that for every $i \in I$ and every $R \in [\rho, \kappa|i|]$, there exists some $y_{i,R} \in B(i,R)$ such that $E \cap B(y_{i,R}, \theta R) \subset E \setminus I$.

Since $\overline{J_I}$ is porous there exists $\alpha \in (0,1)$ such that for all $z \in \overline{J_I}$ and $r \in (0,1)$ there exists some $w := w_{z,r} \in \mathbb{C}$ such that

$$B(w, \alpha r) \subset B(z,r) \setminus \overline{J_I}.$$

Fix κ so small so that $K_\kappa \kappa \leq 1$, where K_κ is as in Theorem 4.3. Note that this is possible because by Theorem 4.3 the function $t \to K_t, t \in [0,1)$ is continuous and $\lim_{t \to 0} K_t = 1$. Theorem 4.3 implies that for every $i \in I$ and every $R \in (0, \kappa|i|]$

$$\begin{aligned} B(i^{-1}, K_\kappa^{-1} R|i|^{-2}) \subset M(B(i,R)) &\subset B(i^{-1}, K_\kappa R|i|^{-2}) \\ &\subset B(i^{-1}, K_\kappa \kappa |i|^{-1}) \subset B(i^{-1}, |i|^{-1}). \end{aligned} \tag{4.17}$$

Note that if $i \in I$ then $i^{-1} \in \overline{J_I}$. Indeed, it is easy to see that, since I is infinite, for every $\varepsilon > 0$ there exists some $\omega_\varepsilon \in I^{\mathbb{N}}$ such that $|\pi(\omega_\varepsilon)| < \varepsilon$. Therefore by the Mean Value Theorem,

$$|\pi(i\omega) - i^{-1}| = |\phi_i(\pi(\omega_\varepsilon)) - \phi_i(0)| \leq \|\phi_i'\|_\infty \varepsilon,$$

and our claim follows by choosing ε appropriately small.

Thus, since $\overline{J_I}$ is porous, there exists some $w \in \mathbb{C}$ such that

$$B := B(w, \alpha r') \subset B(i^{-1}, r') \setminus \overline{J_I}, \tag{4.18}$$

where $r' = \frac{\kappa R}{K_\kappa |i|^2}$. Note that $0 \notin B(i^{-1}, r')$ because $r' < |i|^{-1}$ and observe that

$$I \cap M^{-1}(B) = \emptyset. \tag{4.19}$$

Suppose that (4.19) is false, then there exists some $a \in I \cap M^{-1}(B)$. So $a^{-1} = M(a) \in B$, and since $a^{-1} \in \overline{J_I}$, we deduce that $B \cap \overline{J_I} \neq \emptyset$. This contradicts (4.18).

We will now show that

$$M^{-1}(B) \supset B\left(w^{-1}, \frac{\alpha\kappa R}{4K_\kappa^2}\right). \tag{4.20}$$

By Theorem 4.3,

$$M^{-1}(B) \supset B(w^{-1}, 4^{-1}\alpha r'|w|^{-2}) = B\left(w^{-1}, \frac{\alpha\kappa R}{4K_\kappa |i|^2|w|^2}\right). \tag{4.21}$$

Since $w \in B\left(i^{-1}, \kappa\frac{R}{K_\kappa|i|^2}\right)$, Theorem 4.3 implies that

$$|i|^2 = |M'(i^{-1})| \leq K_\kappa |M'(w)| = K_\kappa |w|^{-2}. \tag{4.22}$$

Therefore (4.20) follows by (4.21) and (4.22).

Hence,

$$I \cap B\left(w^{-1}, \frac{\alpha\kappa R}{4K_\kappa^2}\right) \overset{(4.19)\wedge(4.20)}{=} \emptyset. \tag{4.23}$$

Since $w \in B\left(i^{-1}, \kappa\frac{R}{K_\kappa|i|^2}\right)$, Theorem 4.3 implies that

$$|w^{-1} - i| = |M(w) - M(i^{-1})| \leq K_\kappa |i|^2 \frac{\kappa R}{K_\kappa |i|^2} = \kappa R < R. \tag{4.24}$$

The proof of this implication follows by (4.23) and (4.24) after choosing

$$y_{i,R} = w^{-1}, \theta = 4^{-1}\alpha\kappa K_\kappa^{-2}, \text{ and } \rho = \kappa/2.$$

Actually, it follows by the proof that we could even choose $\rho = 0$, see Theorem 4.7.

We will now prove the other direction. Assume that there exist $\theta \in (0,1), \kappa \in (0,1)$ and $\rho > 0$, such that for every $i \in I$ and every $R \in [\rho, \kappa|i|]$, there exists some $y_{i,R} \in B(i,R)$ such that

$$E \cap B(y_{i,R}, \theta R) \subset E \setminus I.$$

Note that we can assume that

$$\kappa \leq (4K_{1/2})^{-1}, \tag{4.25}$$

$$\theta^{-1} \leq \frac{\rho}{2}. \tag{4.26}$$

Under this assumption we will show that $\overline{J_I}$ is porous.

We will start by proving the following claim.

Claim 1. Let

$$\eta_0 := \frac{1}{4} K_{1/2}\kappa(1+\theta),$$
$$c_0 := (8K_{1/2})^{-1}\theta\kappa(1+\kappa)^{-2},$$
$$\beta_0 := 20\rho\kappa^{-1},$$

Then for all $i \in I$ and $r \in [\beta_0 \operatorname{diam}(\phi_i(X)), 4/|i|]$, there exists $x_i \in B(\phi_i(X), \eta_0 r) \cap X$ such that

$$B(x_i, c_0 r) \cap \overline{J_I} = \emptyset.$$

Proof. First notice that for $i \in I$ and $r > 0$ as in the claim, by Proposition 4.1 (ii) we have that,

$$r \geq \beta_0 \operatorname{diam} \phi_i(X) = \frac{20\rho}{\kappa} \operatorname{diam} \phi_i(X) \geq \frac{20\rho}{4\kappa} |i|^{-2}.$$

Hence, recalling the assumption for i,

$$\frac{\rho}{\kappa}\frac{1}{|i|^2} < \frac{r}{4} \leq \frac{1}{|i|}. \tag{4.27}$$

Then $0 \notin \overline{B}(\frac{1}{i}, \frac{r}{4})$ and Theorem 4.3 implies that

$$M\left(B\left(\frac{1}{i}, \frac{r}{8}\right)\right) \subset B\left(i, K_{1/2}^{-1}|i|^2\frac{r}{8}\right).$$

Let $R := \frac{\kappa}{4} r|i|^2$ and note that by (4.27),

$$\frac{\kappa}{4} r|i|^2 \leq \kappa \frac{1}{|i|}|i|^2 = \kappa|i| \quad \text{and} \quad \frac{\kappa}{4} r|i|^2 > \rho.$$

Hence, $R \in [\rho, \kappa|i|]$ and by our assumption there exists some $y_{i,R} \in B(i,R)$ such that

$$I \cap B(y_{i,R}, \theta R) = \emptyset. \tag{4.28}$$

Note that

$$\frac{1}{4}\theta\kappa r|i|^2 + R \leq \frac{|i|}{2}. \tag{4.29}$$

This is easy to check:

$$\frac{1}{4}\theta\kappa r|i|^2 + R = \frac{1}{4}\theta\kappa r|i|^2 + \frac{1}{4}\kappa r|i|^2 = (1+\theta)\kappa\frac{r}{4}|i|^2 \overset{(4.27)}{\leq} (1+\theta)\kappa|i| \overset{(4.25)}{\leq} \frac{|i|}{2}.$$

Since $y_{i,R} \in B(i,R)$,

$$B(y_{i,R}, 4^{-1}\theta\kappa r|i|^2) \subset B(i, 4^{-1}\theta\kappa r|i|^2 + R) \overset{(4.29)}{\subset} B\left(i, \frac{|i|}{2}\right), \tag{4.30}$$

hence Theorem 4.3 implies that

$$B\left(y_{i,R}^{-1}, (4K_{1/2})^{-1}\theta\kappa r\right) \subset M(B(y_{i,R}, 4^{-1}\theta\kappa r|i|^2)). \tag{4.31}$$

Again by Theorem 4.3 and (4.30)

$$\begin{aligned} M(B(y_{i,R}, 4^{-1}\theta\kappa r|i|^2)) &\subset M(B(i, 4^{-1}\theta\kappa r|i|^2 + R)) \\ &\subset B(i^{-1}, 4^{-1}K_{1/2}\theta\kappa r + K_{1/2}R|i|^{-2}) \\ &= B(i^{-1}, 4^{-1}K_{1/2}\kappa(1+\theta)r). \end{aligned} \tag{4.32}$$

Therefore

$$M(B(y_{i,R}, 4^{-1}\theta\kappa r|i|^2)) \subset B(\phi_i(X), 4^{-1}K_{1/2}\kappa(1+\theta)r). \tag{4.33}$$

Now assume that Claim 1 is false. Then there exists some $i \in I$ and some

$$\beta_0 \operatorname{diam}(\phi_i(X)) \leq r \leq 4/|i|$$

such that for all $y \in B(\phi_i(X), \eta_0 r) \cap X$ it holds that

$$B(y, c_0 r) \cap \overline{J_I} \neq \emptyset.$$

By (4.33) we know that $y_{i,R}^{-1} \in B(\phi_i(X), \eta_0 r)$, where as before $R = \frac{\kappa}{4} r|i|^2$. Hence, there exists some $j \in I$ such that

$$B(y_{i,R}^{-1}, c_0 r) \cap \phi_j(X) \neq \emptyset.$$

Note that

$$M(\phi_j(X)) = M \circ M \circ \tau_j(\overline{B}(1/2, 1/2)) = \tau_j(\overline{B}(1/2, 1/2)) = \overline{B}(j + 1/2, 1/2) \subset \overline{B}(j, 1),$$

where $t_j(x) = j + x$. So, we conclude that

$$M(B(y_{i,R}^{-1}, c_0 r)) \cap \overline{B}(j, 1) \neq \emptyset. \tag{4.34}$$

Moreover,

$$B\left(y_{i,R}^{-1}, (4K_{1/2})^{-1}\theta\kappa r\right) \overset{(4.31)\wedge(4.32)}{\subset} B(i^{-1}, 4^{-1}K_{1/2}\kappa(1+\theta)r)$$
$$\overset{(4.27)}{\subset} B(i^{-1}, K_{1/2}\kappa|i|^{-1}) \overset{(4.25)}{\subset} B(i^{-1}, (2|i|)^{-1}).$$

Therefore since $c_0 < (4K_{1/2})^{-1}\theta\kappa$, $y_{i,R} \in B(i, R)$, and $R \leq \kappa|i|$, Theorem 4.3 implies that

$$M(B(y_{i,R}^{-1}, c_0 r)) \subset B(y_{i,R}, K_{1/2}\, c_0\, |y_{i,R}|^2 r) \subset B(y_{i,R}, K_{1/2}\, c_0\, (1+\kappa)^2|i|^2 r). \tag{4.35}$$

Consequently by (4.34) and (4.35) we conclude that

$$B(y_{i,R}, K_{1/2}\, c_0\, (1+\kappa)^2|i|^2 r) \cap \overline{B}(j, 1) \neq \emptyset,$$

and

$$|y_{i,R} - j| \leq 1 + K_{1/2}\, c_0\, (1+\kappa)^2|i|^2 r. \tag{4.36}$$

Note also that

$$2\theta^{-1} \overset{(4.26)}{\leq} \rho \overset{(4.27)}{<} \frac{\kappa}{4} r|i|^2. \tag{4.37}$$

So, recalling the definition of c_0,

$$|y_{i,R} - j| \overset{(4.36)\wedge(4.37)}{<} \frac{r}{8}\theta\kappa|i|^2 r + K_{1/2}\, c_0\, (1+\kappa)^2|i|^2 r \leq \frac{r}{4}\theta\kappa|i|^2 = \theta R. \tag{4.38}$$

Now notice that (4.38) contradicts (4.28). The proof of Claim 1 is complete. □

We now continue with the proof of the theorem. Note that since $0 \notin \phi_e(X)$ for all $e \in E$, we have that

$$r_0 := \operatorname{dist}\left(\bigcup_{i \in I : |i| \leq 8} \phi_i(X), 0\right) > 0. \tag{4.39}$$

We will prove the following claim.

Claim 2. Let $F = \{i \in I : |i| > 4/r_0\}$, and

$$\eta := \max\left\{5, \frac{1}{4}K_{1/2}\kappa(1+\theta)+6+\frac{\kappa}{25\rho}\right\}, \tag{4.40}$$

$$c := c_0 = (8K_{1/2})^{-1}\theta\kappa(1+\kappa)^{-2}, \tag{4.41}$$

$$\beta := \beta_0 = \frac{20\rho}{\kappa}, \tag{4.42}$$

$$\xi := \min\left\{\frac{1}{8}, \frac{r_0}{24}, \frac{\kappa}{10^5\rho}\right\}. \tag{4.43}$$

Then for all $i \in I \setminus F$ and $r \in [\beta \operatorname{diam}(\phi_i(X)), \xi]$, there exists $x_i \in B(\phi_i(X), \eta r) \cap X$ such that

$$B(x_i, cr) \cap \overline{J_I} = \emptyset.$$

Proof. Let $i \in I \setminus F$ and $r \in [\beta \operatorname{diam}(\phi_i(X)), 4/|i|]$. In that case, since $\eta \geq \eta_0$, $c = c_0$ and $\beta = \beta_0$, Claim 2 follows by Claim 1. Therefore we can assume that $i \in I \setminus F$ and

$$\max\{\beta \operatorname{diam}\phi_i(X), 4/|i|\} \leq r \leq \xi. \tag{4.44}$$

We are going to distinguish two cases. First assume that

$$(B(i^{-1}, 6r) \setminus B(i^{-1}, 2r)) \cap \bigcup_{a \in I} \phi_a(X) = \emptyset. \tag{4.45}$$

It is easy to see that for every $e \in E$,

$$e^{-1} \in X = \overline{B}(1/2, 1/2).$$

Moreover by the choice of ξ we have that $4r \leq 1/2$, hence $X \cap \partial B(i^{-1}, 4r) \neq \emptyset$. Let

$$x_i \in X \cap \partial B(i^{-1}, 4r).$$

Since $x_i \in \partial B(i^{-1}, 4r)$ we have that $B(x_i, r) \subset B(i^{-1}, 6r) \setminus B(i^{-1}, 2r)$, therefore (4.45) implies that $B(x_i, r) \cap \overline{J_I} = \emptyset$. Since $x_i \in B(\phi_i(X), 5r)$ the claim has been proven in that case.

We are left with the case when $(B(i^{-1}, 6r) \setminus B(i^{-1}, 2r)) \cap \bigcup_{a \in I} \phi_a(X) \neq \emptyset$. Hence there exists some $a \in I$ such that

$$(B(i^{-1}, 6r) \setminus B(i^{-1}, 2r)) \cap \phi_a(X) \neq \emptyset. \tag{4.46}$$

Therefore,

$$\begin{aligned}|a|^{-1} &\geq |a^{-1} - i^{-1}| - |i|^{-1} \overset{(4.44)\wedge(4.46)}{\geq} 2r - \operatorname{diam}(\phi_a(X)) - \frac{r}{4} \\ &\geq \frac{7r}{4} - 4|a|^{-2} \geq \frac{7r}{4} - 4|a|^{-1},\end{aligned}$$

where in the third inequality we used Proposition 4.1 (ii). Hence,

$$|a|^{-1} \geq \frac{7r}{20} > \frac{r}{4}. \tag{4.47}$$

Observe that

$$|a| > 8. \tag{4.48}$$

To see this, assume by contradiction that $|a| \leq 8$. Then (4.39) implies that

$$\operatorname{dist}(\phi_a(X), 0) \geq r_0. \tag{4.49}$$

Moreover, by (4.46) there exists $z \in (B(i^{-1}, 6r) \setminus B(i^{-1}, 2r)) \cap \phi_a(X)$. Since $i \in F$ and $r \leq \xi$ we deduce that

$$|z| \leq |i|^{-1} + 6r < \frac{r_0}{4} + \frac{r_0}{4} = \frac{r_0}{2}.$$

And this contradicts (4.49), establishing (4.48).

We record again that (4.46) implies that

$$|i^{-1} - a^{-1}| \leq 6r + \operatorname{diam}(\phi_a(X)). \tag{4.50}$$

We now have,

$$\begin{aligned} |a|^{-1} &\leq |i|^{-1} + |i^{-1} - a^{-1}| \overset{(4.44)\wedge(4.50)}{\leq} 7r + \operatorname{diam}(\phi_a(X)) \\ &\leq 7r + 4|a|^{-2} \overset{(4.48)}{\leq} 7r + \frac{1}{2}|a|^{-1}, \end{aligned} \tag{4.51}$$

where in the third inequality we used Proposition 4.1 (ii). Hence

$$|a|^{-2} \overset{(4.51)}{\leq} 14^2 r \leq 14^2 r^2 \xi \leq \overset{(4.43)}{\leq} r \frac{\kappa}{10^2 \rho}. \tag{4.52}$$

Consequently, Proposition 4.1 (ii) implies that

$$\frac{25\rho}{\kappa} \operatorname{diam}(\phi_a(X)) \leq \frac{100\rho}{\kappa} |a|^{-2} \overset{(4.52)}{\leq} r. \tag{4.53}$$

Since $\frac{25\rho}{\kappa} > \beta_0$, (4.47) and (4.53) allows us to use Claim 1 and obtain some $x_a \in B(a^{-1}, \eta_0 r)$ such that

$$B(x_a, c_0 r) \cap \overline{J_I} = \emptyset. \tag{4.54}$$

Notice then, that

$$\begin{aligned} |x_a - i^{-1}| &\leq |x_a - a^{-1}| + |a^{-1} - i^{-1}| \overset{(4.50)}{\leq} \eta_0 r + 6r + \operatorname{diam}(\phi_a(X)) \\ &\overset{(4.53)}{\leq} \left(\eta_0 + 6 + \frac{\kappa}{25\rho}\right) r \overset{(4.40)}{\leq} \eta r. \end{aligned} \tag{4.55}$$

Hence Claim 2 follows by (4.54) and (4.55) after choosing $x_i := x_a$. □

Notice that Claim 2 and Theorem 4.6 imply that $\overline{J_I}$ is porous. The proof is complete. □

Observe that the proof of the first implication of Theorem 4.5 implies that if $\overline{J_I}$ is porous we can choose the parameter $\rho = 0$. While, what we prove in the second implication of Theorem 4.5 is stronger than assuming that $\rho = 0$. Therefore we have also shown the following.

Theorem 4.7. *Let I be an infinite subset of E. Then $\overline{J_I}$ is porous if and only if there exist $\theta \in (0,1)$ and $\kappa \in (0,1)$ such that for every $i \in I$ and every $R \in [0, \kappa|i|]$, there exists some $y_{i,R} \in B(i, R)$ such that*

$$E \cap B(y_{i,R}, \theta R) \subset E \setminus I.$$

Note that if $|i|, i \in E$, is large enough there exist $R \in [0, \kappa|i|]$ such that $E \cap B(y_{i,R}, \theta R) \neq e\emptyset$ Hence, as an immediate consequence of Theorem 4.7 the limit set of the (full) complex continued fractions IFS $\mathcal{CF}_E$ is not porous. However, since $\mathcal{CF}_E$ satisfies condition (3.4), Theorem 3.3 (i), implies that the limit set J_E is porous at a dense set of its points. We gather these consequences in the following theorem.

Theorem 4.8. *If $\mathcal{CF}_E$ is a (full) complex continued fractions IFS, then its limit set J_E is not porous but it is porous at a dense set of its points.*

Remark 4.9. The balls appearing in Theorems 4.7 and 4.5 are taken with respect to the Euclidean norm. Nevertheless, Theorems 4.7 and 4.5 hold true (with different constants θ, κ and ρ) if the balls are taken with respect to any norm in $\mathbb{C}$, since all norms in finite dimensional spaces are bi-Lipschitz equivalent.

Remark 4.10. It is interesting to compare Theorem 4.5 with [58, Theorem 3.3]. The latter concerns real continued fractions and, strictly speaking, it is entirely independent of Theorem 4.5. Indeed, porosity is not an intrinsic property of a subset of a metric space but does depend on the (ambient) metric space as well. For example, the interval $[0,1]$ is trivially porous in $\mathbb{C}$, hence every limit set of real continued fractions, as considered in [58], is porous in $\mathbb{C}$. However, as it was proven in [58], there exist many such limit sets which are not porous as subsets of $\mathbb{R}$.

Nevertheless, if we carry on the proof of Theorem 4.5 with $E \subset \mathbb{N}$ and $\mathbb{C}$ replaced by $\mathbb{R}$, we will obtain a characterization of limit sets J_E that are porous in $\mathbb{R}$, analogous to Theorem 4.5 (in that case $y_{i,R} \in \mathbb{R}$ and the balls $B(i,R)$ and $B(y_{i,R}, \theta R)$ are taken with respect to the usual topology of $\mathbb{R}$). It is not hard to see that such a characterization is equivalent to that of [58, Theorem 3.3].

Although the limit set of $\mathcal{CF}_E$ is not porous, it is (uniformly) upper porous. This is proved in our next theorem.

Theorem 4.1. *There exists a positive constant c_u such that J_E, the limit set of $\mathcal{CF}_E$, is c_u-upper porous.*

Proof. For $a \in \mathbb{R}$ let $H_a^+ = \{z \in \mathbb{C} : \text{Re}(z) > a\}$. Let $\delta > 0$ as in Lemma 4.2. It is not difficult to check that there exists some $\eta \in (0, 1/10)$ such that

$$M(H^+_{1-4\eta}) \subset B(1/2, 1/2 + \delta/2), \tag{4.56}$$

where $M(z) = z^{-1}$. It is also geometrically evident that there exists some $r \in (0, \eta)$ such that for all $z \in H_1^+$ there exists some $\xi(z) \in B(z, 1/2 + \eta) \cap H_1^+$ such that

$$B(\xi(z), 2r) \subset H_1^+ \text{ and } B(\xi(z), r) \cap \cup_{e \in E} \overline{B}(e + 1/2, 1/2) = \emptyset. \tag{4.57}$$

Hence,

$$B(\xi(z), r) \subset B(z, 1/2 + \eta + r) \cap H_1^+ \subset B(z, 1/2 + 2\eta) \cap H_1^+ \tag{4.58}$$

For $r > 0$ and $z \in H_1^+$ we denote

$$C(z, 1/2 + r) := B(z, 1/2 + r) \cap H^+_{1-r}.$$

Note that by (4.56)

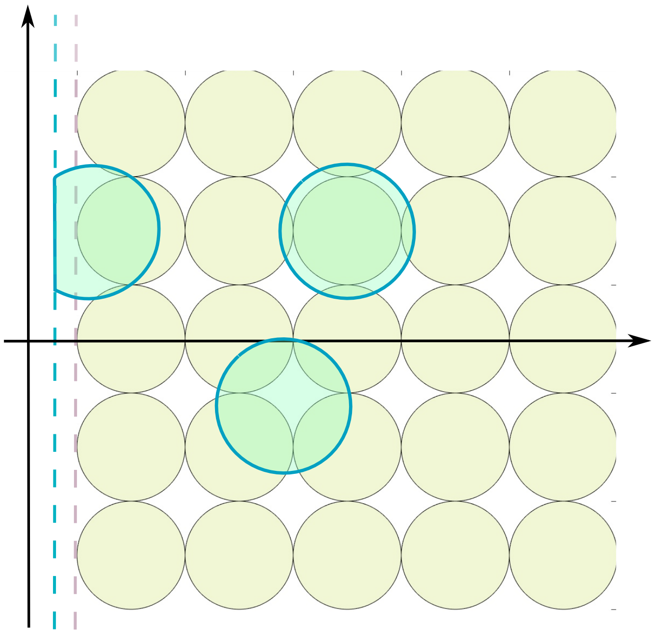

FIGURE 2. The grid of balls $\cup_{e\in E}\overline{B}(e+1/2,1/2)$ and examples of the sets $C(z,1/2+r)$ for $z\in H_1^+$ and $r\le 4\eta$.

$$M(C(z,1/2+4\eta))\subset B(1/2,1/2+\delta/2)\,. \tag{4.59}$$

Let $\omega\in E^{\mathbb{N}}$ and $n>1$. Let

$$z_\omega=\omega_{n+1}+\pi(\sigma^{n+2})$$

and note that $z_\omega\in H_1^+$. Hence, Lemma 4.2 implies that

$$\phi_{\omega|_n}\circ M(C(z_\omega,1/2+2\eta))\subset\phi_{\omega|_n}\circ M(C(z_\omega,1/2+4\eta))\overset{(4.59)\wedge(4.3)}{\subset}B(1/2,1/2+\delta).$$

In particular $\phi_{\omega|_n}\circ M$ is conformal on $C(z_\omega,1/2+4\eta)$. Since $C(z_\omega,1/2+4\eta)$ is convex (as an intersection of two convex sets), the Mean Value Theorem implies that for all $p\in C(z_\omega,1/2+2\eta)$,

$$\begin{aligned}|\phi_{\omega|_n}(M(z_\omega))-\phi_{\omega|_n}(M(p))|&\le\|(\phi_{\omega|_n}\circ M)'\|_{C(z_\omega,1/2+2\eta)}|p-z_\omega|\\&<\|(\phi_{\omega|_n}\circ M)'\|_{C(z_\omega,1/2+2\eta)}(1/2+2\eta).\end{aligned}\tag{4.60}$$

Moreover, if $q\in C(z_\omega,1/2+2\eta)$ then

$$\begin{aligned}|(\phi_{\omega|_n}\circ M)'(q)|=|\phi'_{\omega|_n}(M(q))||M'(q)|&\le\|\phi'_{\omega|_n}\|_{M(C(z_\omega,1/2+2\eta))}\|M'\|_{C(z_\omega,1/2+2\eta)}\\&\overset{(4.59)}{\le}\|\phi'_{\omega|_n}\|_{B(1/2,1/2+\delta/2)}\|M'\|_{B(z_\omega,1/2+2\eta)}\end{aligned}\tag{4.61}$$

Since $\phi_{\omega|_n}$ is conformal in $B(1/2, 1/2+\delta)$, recall also (2.4), we deduce that there exists a constant $K_1 \geq 1$ (depending only on δ) such that

$$\frac{|\phi'_{\omega|_n}(p)|}{|\phi'_{\omega|_n}(q)|} \leq K_1 \text{ for all } p, q \in B(1/2, 1/2+\delta/2). \tag{4.62}$$

In particular,

$$\|\phi'_{\omega|_n}\|_{B(1/2,1/2+\delta/2)} \leq K_1 \|\phi'_{\omega|_n}\|_\infty. \tag{4.63}$$

Since $\eta < 1/10$, Theorem 4.3 implies that there exists some constant K_2 (depending only on η) such that

$$\frac{|M'(p)|}{|M'(q)|} \leq K_1 \text{ for all } p, q \in B(z_\omega, 1/2+2\eta). \tag{4.64}$$

In particular

$$\|M'\|_{B(z_\omega,1/2+2\eta)} \leq K_2 |M'(z_\omega)|. \tag{4.65}$$

Observe also that

$$M(z_\omega) = \frac{1}{\omega_{n+1} + \pi(\sigma^{n+2}(\omega))} = \phi_{\omega_{n+1}}(\pi(\sigma^{n+2}(\omega))). \tag{4.66}$$

Therefore,

$$\begin{aligned}
&\phi_{\omega|_n} \circ M(C(z_\omega, 1/2+2\eta)) \\
&\overset{(4.60)\wedge(4.61)\wedge(4.63)\wedge(4.65)}{\subset} B(\phi_{\omega|_n}(M(z_\omega)), K_1K_2\|\phi'_{\omega|_n}\|_\infty |M'(z_\omega)|(1/2+2\eta)) \\
&= B(\phi_{\omega|_n}(M(\omega_{n+1} + \pi(\sigma^{n+2}(\omega)))), K_1K_2\|\phi'_{\omega|_n}\|_\infty |M'(z_\omega)|(1/2+2\eta)) \\
&\overset{(4.66)}{=} B(\phi_{\omega|_n}(\phi_{\omega_{n+1}}(\pi(\sigma^{n+2}(\omega)))), K_1K_2\|\phi'_{\omega|_n}\|_\infty |M'(z_\omega)|(1/2+2\eta)) \\
&= B(\pi(\omega), K_1K_2\|\phi'_{\omega|_n}\|_\infty |M'(z_\omega)|(1/2+2\eta)).
\end{aligned} \tag{4.67}$$

Note that,

$$B(\xi(z_\omega), r) \overset{(4.58)}{\subset} B(z_\omega, 1/2+2\eta) \cap H_1^+ \subset C(z_\omega, 1/2+2\eta),$$

hence,

$$\phi_{\omega|_n} \circ M(B(\xi(z_\omega), r)) \overset{(4.67)\wedge(4.69)}{\subset} B(\pi(\omega), K_1K_2\|\phi'_{\omega|_n}\|_\infty |M'(z_\omega)|(1/2+2\eta)). \tag{4.68}$$

Recalling (4.57) we see that $\phi_{\omega|_n} \circ M$ is conformal on $B(\xi(z_\omega), 2r)$. Thus, Theorem 4.3 implies that there exists some constant K_3 depending only on r such that

$$\phi_{\omega|_n} \circ M(B(\xi(z_\omega), r)) \supset B(\phi_{\omega|_n} \circ M(\xi(z_\omega)), K_3^{-1} |(\phi_{\omega|_n} \circ M)'(\xi(z_\omega))| r). \tag{4.69}$$

By the chain rule,

$$|(\phi_{\omega|_n} \circ M)'(\xi(z_\omega))| = |\phi'_{\omega|_n}(M(\xi(z_\omega)))||M'(\xi(z_\omega))|. \tag{4.70}$$

Since $\xi(z_\omega) \in H_1^+$, we have that $M(\xi(z_\omega)) \in B(1/2, 1/2)$ and consequently, (4.62) implies that

$$|\phi'_{\omega|_n}(M(\xi(z_\omega)))| \geq K_1^{-1} \|\phi'_{\omega|_n}\|_\infty. \tag{4.71}$$

In addition, since $\xi(z_\omega) \in B(z_\omega, 1/2+\eta)$ we also have that

$$|M'(\xi(z_\omega))| \overset{(4.64)}{\geq} K_2^{-1}|M'(z_\omega))|. \tag{4.72}$$

Therefore, combining (4.69), (4.70), (4.71) and (4.72) we obtain that

$$\phi_{\omega|_n} \circ M(B(\xi(z_\omega),r)) \supset B(\phi_{\omega|_n} \circ M(\xi(z_\omega)), (K_1K_2K_3)^{-1}\|\phi'_{\omega|_n}\|_\infty |M'(z_\omega))|r). \tag{4.73}$$

Observe that,

$$\phi_{\omega|_n} \circ M(B(\xi(z_\omega),r)) \subset \phi_{\omega|_n}(B(1/2,1/2)) \setminus J \subset B(1/2,1/2) \setminus J. \tag{4.74}$$

To verify (4.74) pick any $t \in B(\xi(z_\omega),r)$ and note that by (4.57),

$$M(t) \notin \cup_{e\in E}\phi_e(\overline{B}(1/2,1/2)).$$

As a consequence, $M(t) \notin J$. Since $\phi_{\omega|_n} \circ M(B(\xi(z_\omega),r)) \subset \phi_{\omega|_n}(B(1/2,1/2))$ it suffices to show that $\phi_{\omega|_n}(M(t)) \in J$. Assume by contradiction that $\phi_{\omega|_n}(M(t)) \in J$. In that case, there exists some $\tau \in E^{\mathbb{N}}$ such that

$$\phi_{\omega|_n}(M(t)) = \pi(\tau) = \phi_{\tau|_n}(\pi(\sigma^{n+1}(\tau))).$$

Since $M(t) \in B(1/2,1/2)$ and $\pi(\sigma^{n+1}(\tau)) \in \overline{B}(1/2,1/2)$, the open set condition implies that $\tau|_n = \omega|_n$. But, then, the injectivity of $\phi_{\omega|_n}$ implies that $M(t) = \pi(\sigma^{n+1}(\tau)) \in J$ which is a contradiction.

We can now finish the proof of our theorem. For all $\omega \in E^{\mathbb{N}}$ and $n > 1$ let

$$R_n = K_1K_2\|\phi'_{\omega|_n}\|_\infty |M'(z_\omega)|(1/2+2\eta).$$

We have proven that

$$\begin{aligned}\operatorname{por}(J,\pi(\omega),R_n) &\overset{(4.68)\wedge(4.73)\wedge(4.74)}{\geq} \frac{(K_1K_2K_3)^{-1}\|\phi'_{\omega|_n}\|_\infty |M'(z_\omega))|r}{K_1K_2\|\phi'_{\omega|_n}\|_\infty |M'(z_\omega)|(1/2+2\eta)} \\ &= \frac{r}{(K_1^2K_2^2K_3)(1/2+2\eta)} := c_u.\end{aligned}$$

Since $R_n \to 0$, the proof is complete. □

Definition 4.11. For any $I \subset E = \{m+ni : m \in \mathbb{N}, n \in \mathbb{Z}\}$ we define the *upper and lower densities of I in E* as

$$\overline{\rho}_E(I) := \limsup_{R\to+\infty} \frac{\sharp(I \cap \overline{B}_\infty(1,R))}{\sharp(E \cap \overline{B}_\infty(1,R))} \quad \text{and} \quad \underline{\rho}_E(I) := \liminf_{R\to+\infty} \frac{\sharp(I \cap \overline{B}_\infty(1,R))}{\sharp(E \cap \overline{B}_\infty(1,R))},$$

where $B_\infty(z,r)$ denotes the ball centered at z with radius r, with respect to the norm $\|w\|_\infty = \max\{|\mathrm{Re}(w)|, |\mathrm{Im}(w)|\}$.

Slightly abusing notation, we will call a set $I \subset E$ *porous* in E if $\overline{J_I}$ is porous in $\mathbb{C}$. The following proposition relates porosity to the notion of upper density which we just defined. The proof is based on Theorem 4.7.

Proposition 4.12. If $I \subset E$ is porous then $\overline{\rho}_E(I) < 1$.

Proof. According to Theorem 4.7 and Remark 4.9 there exist constants $\theta \in (0,1)$ and $\kappa \in (0,1)$ such that for every $i \in I$ and every $R \in [0, \kappa \|i\|_\infty]$, there exists some $y_{i,R} \in B_\infty(i,R)$ such that
$$E \cap B_\infty(y_{i,R}, \theta R) \subset E \setminus I.$$
Let $R > 100\theta^{-1}\kappa^{-1}$. We will estimate the quantity
$$\frac{\sharp(I \cap \overline{B}_\infty(1,R))}{\sharp(E \cap \overline{B}_\infty(1,R))}.$$
Notice that since $\sharp(S \cap \overline{B}_\infty(1,R)) = \sharp(S \cap B_\infty(1,\lfloor R \rfloor))$ for every $S \subset E$ and $R > 0$ we can assume that $R \in \mathbb{N}$.

We distinguish two cases. First assume that
$$I \cap A_\infty(1, R/4, R/2) = \emptyset, \tag{4.75}$$
where $A_\infty(z,r,s) = \{w \in \mathbb{C} : r \le \|w - z\|_\infty \le s\}$. Notice that (4.75) implies that
$$\sharp(I \cap \overline{B}_\infty(1,R)) < \sharp(E \cap \overline{B}_\infty(1,R)) - \lfloor R/4 \rfloor^2 < \sharp(E \cap \overline{B}_\infty(1,R)) - (R/8)^2. \tag{4.76}$$
Now assume that (4.75) fails. Then, there exists some $b \in I \cap A_\infty(1, R/4, R/2)$. Let $R' = \kappa R/4$. Then $R' \le \kappa \|b\|_\infty$. Hence there exists some $y_{b,R'} \in B_\infty(b, R')$ such that
$$I \cap B_\infty(y_{b,R'}, \theta R') = \emptyset.$$
Note that
$$\begin{aligned} \sharp(I \cap \overline{B}_\infty(1,R)) &\le \sharp(E \cap \overline{B}_\infty(1,R)) - \sharp(E \cap B_\infty(y_{b,R'}, \theta R')) \\ &\le \sharp(E \cap \overline{B}_\infty(1,R)) - (\theta R')^2 \\ &= \sharp(E \cap \overline{B}_\infty(1,R)) - (4^{-1}\theta\kappa R)^2. \end{aligned} \tag{4.77}$$
Let
$$c = \min\left\{\frac{1}{64}, \frac{\kappa^2\theta^2}{16}\right\}.$$
Then, by (4.76) and (4.77) we deduce that for every $R > 100\theta^{-1}\kappa^{-1}$, we have that
$$\sharp(I \cap \overline{B}_\infty(1,R)) \le \sharp(E \cap \overline{B}_\infty(1,R)) - cR^2.$$
Observe that $\sharp(E \cap \overline{B}_\infty(1,R)) = (R+1)\cdot(2R+1)$, hence
$$\frac{\sharp(I \cap \overline{B}_\infty(1,R))}{\sharp(E \cap \overline{B}_\infty(1,R))} \le 1 - \frac{cR^2}{2R^2 + 3R + 1},$$
and consequently
$$\limsup_{R \to \infty} \frac{\sharp(I \cap \overline{B}_\infty(1,R))}{\sharp(E \cap \overline{B}_\infty(1,R))} \le 1 - c/2 < 1.$$
The proof is complete. □

We are now ready to prove Theorem 1.7 (iii), which improves and extends significantly [58, Theorem 4.2]. As the reader can check, the proof will follow easily from Proposition 4.12, Corollary 3.20 and Corollary 3.11.

Theorem 4.13. *Let I be any co-finite subset of E, let J_I be the limit set associated to the complex continued fractions system $\mathcal{CF}_I$ and let $h_I = \dim_{\mathcal{H}}(J_I)$. Let also m_{h_I} be the h_I-conformal measure of $\mathcal{CF}_I$. Then:*

(i) The limit set J_I is not porous at m_{h_I}-a.e. $x \in J_I$.
(ii) There exists a constant c_I such that J_I is c_I-mean porous at m_{h_I}-a.e. $x \in J_I$.

Proof. As an immediate corollary of Proposition 4.12 we deduce that if $I \subset E$ is co-finite then the limit set J_I is not porous. Hence (i) follows by Corollary 3.20.

Moreover, it follows by [35, Proposition 6.1] that $\mathcal{CF}_E$ is co-finitely regular. Since I is co-finite [6, Lemma 3.10] implies that $\mathcal{CF}_I$ is co-finitely regular. Hence, $\mathcal{CF}_I$ is strongly regular by (2.14). Therefore (ii) follows by Corollary 3.11. □

Definition 4.14. For any $I \subset \mathbb{N}$ we define the *upper and lower densities of I in $\mathbb{N}$* as

$$\overline{\rho}_{\mathbb{N}}(I) := \limsup_{n\to+\infty} \frac{\sharp(I \cap [1,n])}{n} \quad \text{and} \quad \underline{\rho}_{\mathbb{N}}(I) := \liminf_{n\to+\infty} \frac{\sharp(I \cap [1,n])}{n}.$$

In a similar manner if $I \subset \mathbb{Z}$ we define the *upper and lower densities of I in $\mathbb{Z}$* by

$$\overline{\rho}_{\mathbb{Z}}(I) := \limsup_{n\to+\infty} \frac{\sharp(I \cap [-n,n])}{2n+1} \quad \text{and} \quad \underline{\rho}_{\mathbb{Z}}(I) := \liminf_{n\to+\infty} \frac{\sharp(I \cap [-n,n])}{2n+1}.$$

Definition 4.15. We say that $I \subset \mathbb{N}$ is $\mathbb{N}$-*porous* if the second condition of Theorem 4.5 holds with

$$y_{i,R} \in (\mathbb{N} \times \{0\}) \cap B(i,R),\ i \in I \times \{0\}.$$

In the same manner, we say that $I \subset \mathbb{Z}$ is $\mathbb{Z}$-*porous* if the second condition of Theorem 4.5 holds with

$$y_{i,R} \in (1 \times \mathbb{Z}) \cap B(i,R),\ i \in \{1\} \times I.$$

The proof of the following proposition is straightforward and we leave it to the reader.

Proposition 4.16. Let $I_1 \subset \mathbb{N}$ and $I_2 \subset \mathbb{Z}$.

(i) If $\overline{\rho}_{\mathbb{N}}(I_1) < 1$ and $\overline{\rho}_{\mathbb{N}}(I_2) < 1$ then $\overline{\rho}_E(I_1 \times I_2) < 1$.
(ii) If I_1 is $\mathbb{N}$-porous and I_2 is $\mathbb{Z}$-porous then $I_1 \times I_2$ is porous.

It follows by [58, Theorem 3.15] that if $a \geq 2$ then the set $I_a := \{\alpha^n\}_{n\in\mathbb{N}}$ is $\mathbb{N}$-porous. Hence, the following corollary follows by Proposition 4.16 (ii).

Corollary 4.17. *Let $a, b, c \geq 2$. If $I_1 \subset I_a$ and $I_2 \subset I_b \cup (-I_c)$ then the set $I_1 \times I_2$ is porous.*

For $z \in \mathbb{C}$ and $r > 0$ we are going to denote by $Q(z,r)$ the closed filled square centered at z with sides parallel to the axis and sidelength $\ell(Q) = r$, i.e. $Q(z,r) = \overline{B}_\infty(z, r/2)$. Moreover for $r > 0$ we will denote

$$\Delta(r) = \{Q(z,r) : z \in \mathbb{C}\}.$$

We will also use the notation

$$\mathcal{D} = \bigcup_{r>0} \Delta(r)$$

for the collection of all closed squares with sides parallel to the axis. Being motivated by the concept of upper density dimension for subsets of positive integers, introduced in Section 3 of [58], we propose the following analogous but improved definition for subsets of $\mathbb{Z}^2$.

Definition 4.18. If $I \subset \mathbb{Z}^2$ the *upper box dimension* of I is defined as

$$\overline{\mathrm{BD}}(I) = \overline{\lim}_{R\to+\infty} \sup \left\{ \frac{\log \sharp(I\cap Q)}{\log R} : Q \in \Delta(R) \right\}.$$

Note that if $I \subset \mathbb{Z}^2$ then $\sharp(I\cap Q) \le (R+1)^2$ for every $R > 0$ and every $Q \in \Delta(R)$. Therefore

$$\overline{\mathrm{BD}}(I) \le 2 \quad \text{for all } \; I \subset \mathbb{Z}^2. \tag{4.78}$$

In our next theorem we show that if $I \subset \mathbb{N}\times\mathbb{Z} = E$ is porous then the inequality in (4.78) is strict.

Theorem 4.19. *If $I \subset E$ is porous then* $\overline{\mathrm{BD}}(I) < 2$.

Proof. Fix some $R \ge 2$ and consider a square $Q_0 \in \Delta(R)$ such that

$$\mathrm{Re}(Q) := \{\mathrm{Re}(z) : z \in Q\} \subset [0,+\infty).$$

We are going to construct inductively a finite number of families of squares from $\mathscr{D}$ with mutually disjoint interiors, whose union contains $Q_0 \cap I$.

The first family contains only Q_0 and we denote it by $\Sigma_1 = \{Q_0\}$. Now suppose that Σ_n has been defined. Passing to the inductive step we start with any

$$Q := Q(w,\ell(Q)) \in \Sigma_n.$$

We then decompose Q into 81 squares from $\Delta(\frac{1}{9}\ell(Q))$ with mutually disjoint interiors. We call Q_c the square from $\Delta(\frac{1}{9}\ell(Q))$ which shares the same center with Q.

If $Q_c \cap I = \emptyset$, we remove Q_c and we denote by $\Sigma^1_{n+1}(Q)$ the family of the remaining 80 squares from $\Delta(\frac{1}{9}\ell(Q))$ whose union is $Q \setminus \mathrm{Int}(Q_c)$. Note also that

$$\mathrm{Area}\left(\bigcup_{Q' \in \Sigma^1_{n+1}(Q)} Q \right) = \ell(Q)^2 - \frac{1}{81}\ell(Q)^2 = \frac{80}{81}\ell(Q)^2 = \frac{80}{81}\mathrm{Area}(Q). \tag{4.79}$$

If $Q_c \cap I \neq \emptyset$ we pick some $\xi \in Q_c \cap I$. According to Theorem 4.7 and Remark 4.9, since I is porous there exist constants $\theta \in (0,1)$ and $\kappa \in (0,1)$ such that for every $i \in I$ and every $R \in [0,\kappa\|i\|_\infty]$, there exists some $y_{i,R} \in B_\infty(i,R)$ such that $E \cap B_\infty(y_{i,R},\theta R) \subset E \setminus I$. Notice that $\ell(Q) \le 3\|\xi\|_\infty$, since $\mathrm{Re}(Q) \subset [0,+\infty)$. In particular, if we choose

$$L = \frac{\kappa}{9}\ell(Q), \tag{4.80}$$

we have that $L < \kappa\|\xi\|_\infty$. Therefore there exists a point $z \in Q(\xi, L)$ such that

$$I \cap Q(z,\theta L) = \emptyset.$$

Since we can assume that $\theta < 1/9$, we have that

$$\begin{aligned} Q(z,\theta L) &\subset Q\left(w, \theta L + L + \frac{1}{9}\ell(Q)\right) \overset{(4.80)}{\subset} Q\left(w, (1+\theta)\frac{1}{9}\ell(Q) + \frac{1}{9}\ell(Q)\right) \\ &\subset Q(w,\ell(Q)) = Q. \end{aligned} \tag{4.81}$$

Let k be the smallest natural number such that $2^{-k}\ell(Q) \leq \frac{\theta}{9}L$, or equivalently the smallest natural number such that $2^{-k} \leq \frac{\theta\kappa}{81}$, and decompose Q into elements of $\Delta(2^{-k}\ell(Q))$. We record that by the definition of k,

$$2^{-k}\ell(Q) > \frac{\theta}{18}L. \tag{4.82}$$

Let $P \in \Delta(2^{-k}\ell(Q))$ such that

$$P \cap Q\left(z, \frac{\theta}{9}L\right) \neq \emptyset.$$

Then

$$P \subset Q\left(z, \frac{\theta}{9}L + 2 2^{-k}\ell(Q)\right) = Q\left(z, \frac{\theta}{3}L\right).$$

We remove P and we denote by $\Sigma^2_{n+1}(Q)$ the family of the remaining $2^{2k}-1$ squares from $\Delta(2^{-k}\ell(Q))$ whose union is $Q \setminus \mathrm{Int}(P)$. We also have that

$$\begin{aligned}\mathrm{Area}\left(\bigcup_{Q' \in \Sigma^2_{n+1}(Q)} Q\right) &= \ell(Q)^2 - 2^{-2k}\ell(Q)^2 \overset{(4.82)}{<} \left(1 - \left(\frac{\theta L}{18\ell(Q)}\right)^2\right)\ell(Q)^2 \\ &\overset{(4.80)}{=} \left(1 - \left(\frac{\theta\kappa}{162}\right)^2\right)\ell(Q)^2 := \eta\,\ell(Q)^2 = \eta\,\mathrm{Area}(Q),\end{aligned} \tag{4.83}$$

and we record that $\eta \in (0,1)$.

So we can now complete the inductive step. For any $Q \in \Sigma_n$ we let

$$\Sigma_{n+1}(Q) = \begin{cases} \Sigma^1_{n+1}(Q) : \text{ if } Q_c \cap I = \emptyset \\ \Sigma^2_{n+1}(Q) : \text{ if } Q_c \cap I \neq \emptyset, \end{cases}$$

and we define

$$\Sigma_{n+1} := \{\Sigma_{n+1}(Q) : Q \in \Sigma_n\}.$$

The process terminates at level $N \in \mathbb{N}$ if at least one element of Σ_N contains a square with sidelength less than 1.

Let $C_n = \bigcup_{Q \in \Sigma_n} Q$ and notice that for all $n = 1, \dots, N-1$,

$$\mathrm{Area}(C_{n+1}) = \sum_{Q \in \Sigma_n} \mathrm{Area}\left(\bigcup_{Q' \in \Sigma_{n+1}(Q)} Q'\right) \overset{(4.79)\wedge(4.83)}{\leq} \eta \sum_{Q \in \Sigma_n} \mathrm{Area}(Q) = \eta\,\mathrm{Area}(C_n),$$

and

$$\mathrm{Area}(C_n) \leq \eta^{n-1}\mathrm{Area}(C_1) = \eta^{n-1}\mathrm{Area}(Q_0) = \eta^{n-1}R^2. \tag{4.84}$$

Moreover, notice that for all $Q \in \Sigma_N$,

$$\ell(Q) \overset{(4.82)}{\geq} \left(\frac{\theta\kappa}{162}\right)^{N-1} \ell(Q_0) := \alpha^{N-1}R, \tag{4.85}$$

and $\alpha \in (0,1)$. Observe also that

$$I \cap Q_0 = I \cap C_n$$

for all $n = 1, \dots, N$.

Since the process terminates at level N there exists at least one $Q \in \Sigma_N$ such that $\ell(Q) < 1$. Therefore (4.85) implies that $(N-1)\log\alpha + \log R < 0$, or equivallently

$$(N-1)\frac{\log\alpha}{\log R} < -1. \tag{4.86}$$

Let $Q = Q(w, \ell(Q)) \in \Sigma_{N-1}$ such that $I \cap Q \neq \emptyset$. Note that since $I \subset \mathbb{N} + \mathbb{Z}i$, if $i, j \in I \cap Q, i \neq j$, then

$$\operatorname{Int}(Q(i,1)) \cap \operatorname{Int}(Q(j,1)) = \emptyset.$$

Therefore, since $\ell(Q) \geq 1$,

$$\sharp(I \cap Q) = \operatorname{Area}\left(\bigcup_{i \in I \cap Q} Q(i,1)\right) \subset \operatorname{Area}(Q(w, 2\ell(Q))) = 4\operatorname{Area}(Q), \tag{4.87}$$

and

$$\sharp(I \cap Q_0) \leq \sum_{Q \in \Sigma_{N-1}} \sharp(I \cap Q) \overset{(4.87)}{\leq} 4 \sum_{Q \in \Sigma_{N-1}} \operatorname{Area}(Q) = 4\operatorname{Area}(C_{N-1}). \tag{4.88}$$

From now on we will assume that $R \geq \alpha^{-2}$. Notice that this implies that $N \geq 3$. Hence, using that $\eta, \alpha \in (0,1)$, we get the following estimate

$$\begin{aligned}
\frac{\log \sharp(I \cap Q_0)}{\log R} &\overset{(4.88)}{\leq} \frac{\log\left(\operatorname{Area}(C_{N-1})\right)}{\log R} + \frac{\log 4}{\log R} \\
&\overset{(4.84)}{\leq} \frac{(N-2)\log\eta + 2\log R}{\log R} + \frac{\log 4}{\log R} \\
&= 2 + \frac{N-2}{N-1}(N-1)\frac{\log\alpha}{\log R} \cdot \frac{\log\eta}{\log\alpha} + \frac{\log 4}{\log R} \\
&\overset{(4.86)}{<} 2 - \frac{N-2}{N-1} \cdot \frac{\log\eta}{\log\alpha} + \frac{\log 4}{\log R} \\
&\overset{N\geq 3}{\leq} 2 - \frac{1}{2}\frac{\log\eta}{\log\alpha} + \frac{\log 4}{\log R}.
\end{aligned} \tag{4.89}$$

Finally notice that since $I \subset \mathbb{N} \times \mathbb{Z}$,

$$\overline{\operatorname{BD}}(I) = \overline{\lim}_{R\to+\infty} \sup\left\{\frac{\log \sharp(I \cap Q)}{\log R} : Q \in \Delta(R) \text{ and } \operatorname{Re}(Q) \subset [0, +\infty)\right\}$$

Therefore, (4.89) implies that $\overline{\operatorname{BD}}(I) \leq 2 - \frac{1}{2}\frac{\log\eta}{\log\alpha} < 2$. The proof is complete. □

Remark 4.20. Theorem 4.19 corresponds to Theorem 3.5 in [58]. Although, logically speaking, both theorems are independent as one of them concerns subsets of $\mathbb{N}$ while the other concerns Gaussian integers, their assertions are analogous and the proofs are related. However, the proof provided in the current paper is clearer, simpler, and better describes the key ideas. One could easily adopt it to give a better proof of Theorem 3.5 in [58].

We will apply Theorem 4.19 to the set of *Gaussian primes* which has been studied extensively in number theory. Recall that $\mathbb{Z}[i]$, the set of Gaussian integers, has exactly four *units* : $1,-1,i$ and $-i$. These are the only elements of $\mathbb{Z}[i]$ whose Euclidean norm is equal to 1. Multiplying any $z \in \mathbb{Z}[i]$ by the units of $\mathbb{Z}[i]$ we obtain its associates, i.e. the *associates* of z are $z,-z,iz$ and $-iz$. A Gaussian integer $z \in \mathbb{Z}[i]$ is called *prime* if $|z|>1$ and it is divisible only by the units and its associates. There are many good sources of information for the basic divisibility properties of Gaussian integers and in particular about Gaussian primes, see e.g. [12, 56]. See also [19] for a treatment of several analytic topics related to Gaussian primes.

If $-\pi \le a < b \le \pi$ we denote

$$GP_{a,b} = \{w \in E : w \text{ is a Gaussian prime and } \arg w \in [a,b)\},$$

and recall that E is the set of all Gaussian integers with positive real part.

Lemma 4.21. *If $-\pi/2 \le a < b \le \pi/2$ then the complex continued fractions system $\mathcal{CF}_{GP_{a,b}}$ is co-finitely regular.*

Proof. By Hecke's Prime Number Theorem, see e.g. [19, Theorem 4, pages 134-135], if

$$\pi_{a,b}(x) = \sharp\{w : w \text{ is a Gaussian prime and } a \le \arg w < b, |w|^2 \le x\}$$

then

$$\pi_{a,b}(x) \sim \frac{2}{\pi}(b-a)\frac{x}{\log x}. \tag{4.90}$$

Note that if $a \in (-\pi/2, \pi/2)$ then for $R>0$

$$\sharp(GP_{a,b} \cap \bar{B}(0,R)) = \pi_{a,b}(R^2).$$

While, if $a=-\pi/2$ then

$$\sharp(GP_{a,b} \cap \bar{B}(0,R)) + \sharp(N \cap \bar{B}(0,R)) = \pi_{a,b}(R^2),$$

where

$$N := \{w \text{ is a Gaussian prime such that} \operatorname{Re} w = 0 \text{ and } \operatorname{Im} w \le 0\}.$$

Note that $\sharp(N \cap \bar{B}(0,R)) \le R$ therefore, if $a=-\pi/2$ then

$$\sharp(GP_{a,b} \cap \bar{B}(0,R)) \ge \pi_{a,b}(R^2) - R.$$

Hence, using (4.90) it is not difficult to show that if $-\pi/2 \le a < b \le \pi/2$ then there exists some $R_0>0$ such that for all $R \ge R_0$,

$$\frac{1}{16\pi}(b-a)\frac{R^2}{\log R} \le \sharp(GP_{a,b} \cap \bar{B}(0,2R) \setminus \bar{B}(0,R)) \le \frac{8}{\pi}(b-a)\frac{R^2}{\log R}. \tag{4.91}$$

By (2.10) and Proposition 4.1 (i), for $t \ge 0$

$$Z_1(\mathcal{CF}_{GP_{a,b}}, t) := Z_1(t) \approx \sum_{e \in GP_{a,b}} |e|^{-2t}. \tag{4.92}$$

Now let $A_n = \bar{B}(0, 2^{n+1}R_0) \setminus \bar{B}(0, 2^n R_0) \cap GP_{a,b}$, $n \in \mathbb{N}$, and note that

$$\begin{aligned}\sum_{e \in GP_{a,b}, |e| \geq 2R_0} |e|^{-2t} &= \sum_{n=1}^{\infty} \sum_{e \in A_n} |e|^{-2t} \approx \sum_{n=1}^{\infty} \sharp A_n \cdot 2^{-2nt} \\ &\overset{(4.91)}{\approx} \sum_{n=1}^{\infty} \frac{(2^n R_0)^2}{\log(2^n \cdot R_0)} 2^{-2nt} \approx \sum_{n=1}^{\infty} \frac{2^{2n(1-t)}}{n}.\end{aligned} \tag{4.93}$$

Since

$$Z_1(t) \overset{(4.93)}{\approx} \sum_{e \in GP_{a,b}, |e| < 2R_0} |e|^{-2t} + \sum_{n=1}^{\infty} \frac{2^{2n(1-t)}}{n},$$

(2.12) implies that $\theta(\mathcal{CF}_{GP_{a,b}}) = 1$ and $Z_1(1) = +\infty$. Therefore, recalling Definition 2.6 and (2.13), we deduce that $\mathcal{CF}_{GP_{a,b}}$ is co-finitely regular. The proof is complete. $\square$

We will conclude this section with the proof of Theorem 1.7 (iv).

Theorem 4.22. *Let $-\pi/2 \leq a < b \leq \pi/2$ and let I be any co-finite subset of $GP_{a,b}$. Let J_I be the limit set associated to the complex continued fractions system $\mathcal{CF}_I$ and let $h_I = \dim_{\mathcal{H}}(J_I)$. Then:*

(i) The limit set J_I is not porous at m_{h_I}-a.e. $x \in J_I$.
(ii) There exists a constant c_I such that J_I is c_I-mean porous at m_{h_I}-a.e. $x \in J_I$.

Proof. Using (4.91), we deduce that there exits some $c \in (0,1)$ such that if R is large enough

$$\sharp(I \cap Q(0,R)) \geq \frac{cR^2}{\log R}. \tag{4.94}$$

Hence, for R large enough

$$\frac{\log \sharp(I \cap Q(0,R))}{\log R} \overset{(4.94)}{\geq} 2 + \frac{\log c}{\log R} - \frac{\log\log R}{\log R}, \tag{4.95}$$

and consequently

$$\sup\left\{\frac{\log \sharp(I \cap Q)}{\log R} : Q \in \Delta(R)\right\} \overset{(4.95)}{\geq} 2 + \frac{\log c}{\log R} - \frac{\log\log R}{\log R}. \tag{4.96}$$

Recalling Definition 4.18 we see that (4.96) implies that $\overline{\text{BD}}(I) \geq 2$. Hence by (4.78) we deduce that $\overline{\text{BD}}(I) = 2$, and Theorem 4.19 implies that I is not porous, i.e. the set J_I is not porous. Therefore Corollary 3.20 implies that J_I is not porous at m_{h_I}-a.e. $x \in J_I$.

By Lemma 4.21 the system $\mathcal{CF}_{GP_{a,b}}$ is co-finitely regular. Now the proof of (ii) follows exactly as in the proof of Theorem 4.13 (i). The proof is complete. $\square$

Taking $a = -\pi/2$ and $b = \pi/2$ in Theorem 4.22 we obtain the following corollary involving the complex continued fractions system whose alphabet is the set of Gaussian primes with positive real part, see also Figure 2.

Corollary 4.23. *Let GP^+ be the set of Gaussian primes with positive real part. Let J_{GP^+} be the limit set associated to the complex continued fractions system $\mathcal{CF}_{GP^+}$ and let $h_{GP^+} = \dim_{\mathcal{H}}(J_{GP^+})$. Then:*

(i) The limit set J_{GP^+} is not porous at $m_{h_{GP^+}}$-a.e. $x \in J_{GP^+}$.
(ii) There exists a constant c_{GP^+} such that J_{GP^+} is c_{GP^+}-mean porous at $m_{h_{GP^+}}$-a.e. $x \in J_{GP^+}$.

5. POROSITY FOR MEROMORPHIC FUNCTIONS

In this short section we deal with some quite general classes of meromorphic (either rational functions or transcendental) functions from $\mathbb{C}$ to $\hat{\mathbb{C}}$. A very powerful tool of meromorphic dynamics is the concept of a *nice set*. Roughly speaking a nice set of a meromorphic function is a set such that the holomorphic inverse branches of the first return map to this set form a conformal IFS. By means of nice sets we will apply our results on mean porosity of conformal IFSs to the realm of some large classes of meromorphic functions from $\mathbb{C}$ to $\hat{\mathbb{C}}$.

More precisely, let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. Let $\mathrm{Sing}(f^{-1})$ be the set of all *singular points* of f^{-1}, i. e. the set of all points $w \in \hat{\mathbb{C}}$ such that if W is any open connected neighborhood of w, then there exists a connected component U of $f^{-1}(W)$ such that the map $f : U \to W$ is not bijective. Of course, if f is a rational function, then $\mathrm{Sing}(f^{-1}) = f(\mathrm{Crit}(f))$, where

$$\mathrm{Crit}(f) := \{w \in \mathbb{C} : f'(w) = 0\}.$$

We also define

$$\mathrm{PS}(f) := \bigcup_{n=0}^{\infty} f^n(\mathrm{Sing}(f^{-1})).$$

We are now going to recall the definitions of Fatou and Julia sets of meromorphic functions.

Definition 5.1. Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. The *Fatou set* $F(f)$ of the function f is the set of all points $z \in \mathbb{C}$ for which there exists an open neighborhood U_z of z such that all iterates $f^n|_{U_z}$, $n \in \mathbb{N}$, are well defined and form a normal family in the sense of Montel.

We also define the *Julia set* of f, as

$$J(f) := \hat{\mathbb{C}} \setminus F(f).$$

Following [48] and [54] a meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is called *tame* if and only if

$$J(f) \setminus \overline{\mathrm{PS}(f)} \neq \emptyset.$$

Remark 5.2. Tameness is a very mild hypothesis which is satisfied by many natural classes of maps. These include:

(1) Quadratic maps $\hat{\mathbb{C}} \ni z \mapsto z^2 + c \in \hat{\mathbb{C}}$ for which $c \in \mathbb{R}$ and the Julia set is not contained in the real line;
(2) Rational maps for which the restriction to the Julia set is expansive which includes the case of expanding rational functions; and
(3) Misiurewicz maps, where the critical point is not recurrent.

(4) Dynamically regular meromorphic functions introduced and considered in [38] and [39].

In this paper the main advantage of dealing with tame functions is that these admit *nice sets*. Before giving their formal definition (included in the next theorem) we record that Rivera-Letelier [50] introduced the concept of nice sets in the realm of dynamics of rational maps of the Riemann sphere. In [10] Dobbs proved their existence for tame meromorphic functions from $\mathbb{C}$ to $\hat{\mathbb{C}}$; see also [31] for an extended treatment of nice sets. Before quoting Dobbs' theorem, given a tame meromorphic function $f:\mathbb{C}\to\hat{\mathbb{C}}$ a set $F\subset\hat{\mathbb{C}}$ and an integer $n\geq 0$, we denote by $\mathscr{C}_{F,f}(n):=\mathscr{C}_F(n)$ the collection of all connected components of $f^{-n}(F)$.

Theorem 5.3. *Let $f:\mathbb{C}\to\hat{\mathbb{C}}$ be a tame meromorphic function. Fix a non-periodic point $z\in J(f)\setminus\overline{\mathrm{PS}(f)}$, and two parameters $\kappa>1$, and $K>1$. Then for all $L>1$ and for all $r>0$ sufficiently small there exists an open connected simply connected set $V=V(z,r)\subset\mathbb{C}\setminus\overline{\mathrm{PS}(f)}$, called a nice set, such that*

(i) If $U\in\mathscr{C}_V(n)$ and $U\cap V\neq\emptyset$, then $U\subseteq V$.
(ii) If $U\in\mathscr{C}_V(n)$ and $U\cap V\neq\emptyset$, then, for all $w,w'\in U$,

$$|(f^n)'(w)|\geq L \quad \textit{and} \quad \frac{|(f^n)'(w)|}{|(f^n)'(w')|}\leq K.$$

(iii) $\overline{B(z,r)}\subset V\subset B(z,\kappa r)\subset B(z,2\kappa r)\subset\mathbb{C}\setminus\overline{\mathrm{PS}(f)}$.

Each nice set of a tame meromorphic function canonically gives rise to a countable alphabet conformal iterated function system in the sense considered in the previous sections of the present paper. Namely, let V be a nice set of a tame meromorphic function $f:\mathbb{C}\to\hat{\mathbb{C}}$, and put

$$\mathscr{C}_V^*=\bigcup_{n=1}^{\infty}\mathscr{C}_V(n).$$

=== It is easy to see (comp. [31] or [54] for more details) that for every $U\in\mathscr{C}_V^*$ let $\tau_V(U)\geq 1$ the unique integer $n\geq 1$ such that $U\in\mathscr{C}_V(n)$. Since $V\subset\mathbb{C}$ is open, connected, simply connected and disjoint from $\overline{\mathrm{PS}(f)}$, using the Inverse Function Theorem and the Monodromy Theorem in the standard way (see [8]), we see that there exists

$$f_U^{-\tau_V(U)}:B(z,2\kappa r)\to\mathbb{C}$$

a unique holomorphic branch of $f^{-\tau_V(U)}$ such that

$$f_U^{-\tau_V(U)}(V)=U.$$

Denote

$$\phi_U:=f_U^{-\tau_V(U)}$$

and keep in mind that

$$\phi_U(V)=U.$$

Denote by E_V the collection of all elements U of $\mathscr{C}_V^*$ such that

(a) $\phi_U(V)\subset V$,

(b) $f^k(U)\cap V=\emptyset$ for all $k=1,2,\dots,\tau_V(U)-1$.

Of course (a) yields

(a') $\phi_U(\overline{V})\subset\overline{V}$.

We note that the collection E_V is not empty and the details can be found in the proof of Lemma 5.4.

We now form a conformal iterated function system as follows. Let

$$X:=\overline{V} \quad\text{and}\quad W:=B(z,2\kappa r).$$

Assume now that the parameter L from Theorem 5.3 is so large that $2L^{-1}<1$. Take any $w\in W=B(z,2\kappa r)$ and let w' be the point in $[z,w]\cap\overline{B(z,r)}$ closest to w, where $[z,w]$ denotes the line segment with endpoints z and w. Then for every $U\in E_V$ we have that $\phi_U(w')\in\overline{V}$, and

$$|\phi_U(w)-\phi_U(w')|\le L^{-1}|w-w'|\le L^{-1}2\kappa r.$$

Hence,

$$|\phi_U(w)-z|\le|\phi_U(w)-\phi_U(w')|+|\phi_U(w')-z|\le L^{-1}2\kappa r+\kappa r<2\kappa r.$$

Therefore, $\phi_U(w)\in B(z,2\kappa r)$ and consequently $\phi_U(W)\subset W$.

The collection of maps

$$\mathcal{S}_V:=\{\phi_U:W\to W\}_{U\in E_V} \tag{5.1}$$

is therefore well defined and, by (a'),

$$\phi_U(X)\subset X$$

for all $U\in E_V$. We claim that $\mathcal{S}_V$ is a conformal IFS satisfying the Open Set Condition. We only mention that all the sets $\{U=\phi_U(V):U\in E_V\}$ are mutually disjoint by their very definition which is given by means of continuous inverse branches of positive iterates of a single meromorphic map. Therefore, the Open Set Condition follows. Furthermore, uniform contraction of the elements of the system $\mathcal{S}_V$ follows immediately from item (b) of Theorem 5.3.

In other words the elements of $\mathcal{S}_V$ are determined by all holomorphic inverse branches of the first return map $F_V:V\to V$. In particular, $\tau_V(U)$ is the first return time of all points in $U=\phi_U(V)$ to V.

It is easily seen from this construction, and we provide a proof, that the following holds.

Lemma 5.4. *Let $f:\mathbb{C}\to\hat{\mathbb{C}}$ be a tame meromorphic function. If V is a nice set produced in Theorem 5.3 and $\mathcal{S}_V$ is the corresponding conformal IFS produced in formula (5.1), then the limit set J_V of the system $\mathcal{S}_V$ contains all transitive points of the map $f:J(f)\to J(f)$ lying in V, i.e. the set of points in $V\cap J(f)$ whose orbit under iterates of f is dense in $J(f)$. In particular:*

$$\overline{J_V}=\overline{V}\cap J(f).$$

In addition, if $J(f)\ne\mathbb{C}$ (this in fact means that $J(f)$ is nowhere dense in $\mathbb{C}$), then condition (3.4) holds for the IFS $\mathcal{S}_V$.

Proof. The inclusion $\overline{J_V} \subset \overline{V}$ is obvious. The inclusion $\overline{J_V} \subset J(f)$ follows since $\overline{J_V}$ is the closure of all fixed points of all elements ϕ_ω, $\omega \in E_V^*$, and these are repelling periodic points of f which are all in the closed set $J(f)$. Thus,

$$\overline{J_V} \subset \overline{V} \cap J(f).$$

In order to see the opposite inclusion, recall first that the set $V \cap J(f)$ contains transitive points, see e.g. [3, 31]. Take any transitive point $w \in V \cap J(f)$ and let et $(n_j)_{j=1}^\infty$ be the unbounded increasing sequence of all consecutive visits of w in V under the action of f. In other words

$$f^{n_j}(w) \in V$$

for all $j \geq 1$ and

$$f^k(w) \notin V$$

if $k \neq n_j$ for all $j \geq 1$. Let $m_j = n_j - n_{j-1}$, $j \in \mathbb{N}$ and let $n_0 = 0$. Then, using Theorem 5.3, we see that for every $j \geq 1$ there exists a unique holomorphic branch of f^{-m_j} defined on W and mapping V into V, and furthermore, that this branch belongs to $\mathscr{S}_V$. Hence, this branch is equal to ϕ_{U_j} for some $U_j \in E_V$. Note that for all $k \in \mathbb{N}$,

$$\phi_{U_1} \circ \phi_{U_2} \circ \cdots \circ \phi_{U_k} \circ f^{m_1} \circ f^{m_2} \circ \ldots f^{m_k}(w) = w. \tag{5.2}$$

Since $f^{m_1} \circ f^{m_2} \circ \ldots f^{m_k}(w) = f^{n_k}(w) \in V$, (5.2) implies that $w \in \phi_{U_1} \circ \phi_{U_2} \circ \cdots \circ \phi_{U_k}(V)$. Therefore,

$$w = \pi(U_1 U_2 U_3 \ldots) \in J_V$$

and we are done. □

As an almost immediate consequence of this lemma and Theorem 3.3 (i), we get the following.

Theorem 5.5. *If $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a tame meromorphic function such that $J(f) \neq \mathbb{C}$, then the Julia set $J(f)$ is porous at a dense set of its points.*

Proof. Since $J(f) \neq \mathbb{C}$, it is nowhere dense in $\mathbb{C}$, and therefore the conformal IFS $\mathscr{S}_V$ produced in formula (5.1), satisfies condition (3.4). Hence, the proof concludes by a direct application of Theorem 3.3 (i) and Lemma 5.4. □

As a fairly direct consequence of Lemma 5.4, Theorem 5.3, and Theorem 3.10, we shall prove the following.

Theorem 5.6. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a tame meromorphic function such that $J(f) \neq \mathbb{C}$. If μ is a Borel probability f-invariant ergodic measure on $J(f)$ with full topological support and with finite Lyapunov exponent $\chi_\mu(f) := \int_{J(f)} \log|f'|\,d\mu$, then there exist some $\alpha_f, p_f \in (0,1]$ such that set $J(f)$ is (α_f, p_f)–mean porous at μ-a.e. $x \in J(f)$.*

Proof. Let V be a nice set as in Lemma 5.4. Notice then, that

$$V \cap \bigcup_{n=1}^{\infty} f^n(\partial V) = \emptyset. \tag{5.3}$$

To see this, suppose by contradiction that there exists $x \in \partial V$ such that $f^n(x) \in V$ for some $n \in \mathbb{N}$. Then there exists a holomorphic branch $\phi : V \to \mathbb{C}$ of f^{-n} such that

$\phi(f^n(x)) = x$. Hence, $\phi(V) \in \mathscr{C}_V(n)$ and since non-constant meromorphic functions are open, $\phi(V)$ is open. Therefore, $\phi(V) \cap V^c \neq \emptyset$, because $\phi(V) \cap \partial V \neq \emptyset$. But this contradicts Theorem 5.3 (i), and (5.3) has been proven.

Since $\mathrm{spt}(\mu) = J(f)$, we have that $\mu(V) > 0$. Hence, $\mu(\cup_{n=1}^{\infty} f^n(\partial V)) < 1$. Since the set

$$\bigcup_{n=1}^{\infty} f^n(\partial V)$$

is f-forward invariant, that is

$$f\left(\bigcup_{n=1}^{\infty} f^n(\partial V)\right) \subset \bigcup_{n=1}^{\infty} f^n(\partial V),$$

its complement is f-backwards invariant, that is

$$f^{-1}\left(\hat{\mathbb{C}} \setminus \bigcup_{n=1}^{\infty} f^n(\partial V)\right) \subset \hat{\mathbb{C}} \setminus \bigcup_{n=1}^{\infty} f^n(\partial V).$$

Hence, the ergodicity of the measure μ implies that

$$\mu\left(\bigcup_{n=1}^{\infty} f^n(\partial V)\right) = 0. \tag{5.4}$$

Since $\partial V \subset f^{-1}(f(\partial V))$ and the measure μ is f-invariant, this implies that $\mu(\partial V) = 0$. So, again by the f-invariance of μ, we get that

$$\mu\left(\bigcup_{n=0}^{\infty} f^{-n}(\partial V)\right) = 0. \tag{5.5}$$

In particular, if μ_{J_V} is the conditional measure on J_V, i.e.

$$\mu_{J_V}(F) = \frac{\mu(F)}{\mu(J_V)}, \quad F \subset J_V \text{ Borel},$$

then

$$\mu_{J_V}\left(J_V \cap \bigcup_{n=0}^{\infty} f^{-n}(\partial V)\right) = 0. \tag{5.6}$$

We will now show that if $\pi : E_V^{\mathbb{N}} \to J_V$ is the projection map generated by the IFS $\mathscr{S}_V$, then

$$\mu\big(\{z \in J_V : \pi^{-1}(z) \text{ is a singleton}\}\big) = 1. \tag{5.7}$$

Note first that (5.7) will follow from (5.8) if we prove that every point

$$z \in J_V \setminus \bigcup_{n=0}^{\infty} f^{-n}(\partial V)$$

has a unique code. Suppose that this is not the case, then there exists

$$z \in J_V \setminus \bigcup_{n=0}^{\infty} f^{-n}(\partial V)$$

and two distinct words $\omega, \tau \in E_V^{\mathbb{N}}$ such that $\pi(\omega) = \pi(\tau) = z$. Let $k \in \mathbb{N}$ be the first instance such that $\omega_k \neq \tau_k$. Then

$$z \in \phi_{\omega|_k}(\overline{V}) \cap \phi_{\tau|_k}(\overline{V}).$$

Hence, the formula

$$\tilde{\mu}(A) := \mu_{J_V}(\pi_V(A)) \quad A \subset E_V^{\mathbb{N}} \text{ Borel}, \tag{5.8}$$

defines a Borel probability measure on $E^{\mathbb{N}}$. Also then, $\mu_{J_V} = \tilde{\mu} \circ \pi_V^{-1}$ and the following diagram commutes

$$\begin{array}{ccc} E^{\mathbb{N}} & \xrightarrow{\sigma} & E^{\mathbb{N}} \\ \downarrow \pi_V & & \downarrow \pi_V \\ J_V & \xrightarrow{f_V} & J_V \end{array},$$

where $f_V : J_V \to J_V$ is the first return map from J_V to J_V defined (by the Poincaré Recurrence Theorem) μ_{J_V}-a.e. Furthermore, the projection map $\pi_V : (E^{\mathbb{N}}, \tilde{\mu}) \to (J_V, \mu_V)$ is a metric isomorphism. Since, the measure μ is f-invariant and ergodic, the measure μ_{J_V} is f_V-invariant and ergodic. Consequently, the measure $\tilde{\mu}$ is $\sigma-$ invariant and ergodic. Moreover, by Kac's Lemma,

$$\chi_{\tilde{\mu}}(\sigma) = \int_{J_V} \log|f_V'| \, d\mu_{J_V} = \frac{\chi_\mu(f)}{\mu(J_V)} < +\infty.$$

Therefore, an application of Theorem 3.10 gives that the set J_V is

$$\left(\alpha_{\mathcal{S}_V}, \frac{\mu(J_V)\log 2}{2\chi_{\tilde{\mu}}(f)}\right) - \text{mean porous}$$

at μ_{J_V}-a.e. $x \in J_V$. Since, by Lemma 5.4, $\overline{J_V} \supset J(f) \cap V$, the set $J(f)$ is

$$\left(\alpha_{\mathcal{S}_V}, \frac{\mu(J_V)\log 2}{2\chi_{\tilde{\mu}}(f)}\right) - \text{mean porous}$$

at μ-a.e. $x \in J(f) \cap V$. Note that apart from critical points and poles of f (which is a countable set and so of μ measure 0), the property of being (α, p)-mean porous is f-invariant because of Lemma 3.12. Indeed, if p is not a critical point or a pole of f and W is a neighborhood of p such that f is injective on W then $J(f) \cap f(W) = f(J(f) \cap W)$. Thus, we can apply Lemma 3.12 with W as above and $X = Y = J(f)$. Hence, the ergodicity of μ implies that the set $J(f)$ is

$$\left(\alpha_{\mathcal{S}_V}, \frac{\mu(J_V)\log 2}{2\chi_{\tilde{\mu}}(f)}\right) - \text{mean porous}$$

at μ-a.e. point of $J(f)$. The proof is complete. □

Remark 5.7. We would like to note that using Pesin's theory, as developed in [49], we could remove the tameness hypothesis from Theorem 5.6 if the Lyapunov exponent $\chi_\mu(f)$ is positive.

In essentially the same way (only simpler) as Theorem 3.19, we can prove the following. Alternatively, we could deduce it from Theorem 3.19, Theorem 5.3, and Lemma 5.4, arguing in a similar way as in the proof of Theorem 5.6.

Theorem 5.8. *Let $f:\mathbb{C}\to\hat{\mathbb{C}}$ be a tame meromorphic function such that $J(f)\neq\mathbb{C}$ and $J(f)$ is not porous in $\mathbb{C}$. If μ is a Borel probability f-invariant ergodic measure on $J(f)$ with full topological support, then $J(f)$ is not porous at μ-a.e. $x\in J(f)$.*

6. Porosity for elliptic functions

In the last section of our paper we prove that the Julia sets of non-constant elliptic functions are *not* porous. We first recall the definition of elliptic functions.

Definition 6.1. A meromorphic function $f:\mathbb{C}\to\hat{\mathbb{C}}$ is called *elliptic* if it is doubly periodic; i.e. if there exist two complex numbers $w_1,w_2\in\mathbb{C}\setminus\{0\}$ such that $\mathrm{Im}(w_2/w_1)\neq 0$ and

$$f(z+w_1)=f(z+w_2)=f(z)$$

for all $z\in\mathbb{C}$. We also define the *basic fundamental parallelogram* of f by

$$\mathcal{R}_f:=\{t_1w_1+t_2w_2:t_1,t_2\in[0,1]\}.$$

and we denote by Λ_f the lattice generated by w_1 and w_2, i.e.

$$\Lambda_f:=\{n_1w_1+n_2w_2:n_1,n_2\in\mathbb{Z}\}$$

In the following we collect some basic well-known facts about elliptic functions which are going to be used in the following, see [31] for more information. If G is an open subset of $\mathbb{C}$ and b is a pole of a meromorphic function $g:G\to\hat{\mathbb{C}}$, we denote by $q_b\geq 1$ the order of g at b. The following proposition presents some elementary facts about poles in the context we will make use of them.

Proposition 6.2. If $f:\mathbb{C}\to\hat{\mathbb{C}}$ is a non-constant meromorphic function, then for all $n\in\mathbb{N}$ and for every $b\in f^{-n}(\infty)$ there exists some $R_1:=R_1(n,b)>0$ and some $A:=A(n,b)\geq 1$ such that for all $z\in B(b,R_1)$

$$A^{-1}|z-b|^{-q_b}\leq|f^n(z)|\leq A|z-b|^{-q_b}, \tag{6.1}$$

and

$$A^{-1}|z-b|^{-(q_b+1)}\leq|(f^n)'(z)|\leq A|z-b|^{-(q_b+1)}. \tag{6.2}$$

where, we recall, $q_b\geq 1$ is the order of b as the pole of f^n.

Proposition 6.3. If $f:\mathbb{C}\to\hat{\mathbb{C}}$ is a non-constant elliptic function, then there exists some $R_2>0$ such that for all $w\in\mathbb{C}$,

$$f(B(w,R_2))=\hat{\mathbb{C}}. \tag{6.3}$$

Proof. Just take $R_2>0$ so large that each ball $B(w,R_2)$ contains a congruent, mod Λ_f, copy of the fundamental parallelogram $\mathcal{R}_f$. □

Remark 6.4. It is immediate from Montel's Theorem that if f is a non-constant elliptic function and U is an open set such that all iterates $f^n|_U$, $n\in\mathbb{N}$, are well defined (in fact $f^{-1}(\infty)\cap\bigcup_{n\geq 0}f^n(U)=\emptyset$ and remember that $f^{-1}(\infty)$ is an infinite set, so in particular it contains three different points), then they form a normal family in the sense of Montel.

The following proposition gathers some properties of the Julia set of elliptic functions.

Proposition 6.5. Let $f:\mathbb{C}\to\hat{\mathbb{C}}$ be a non-constant elliptic function. Then

$$J(f)=\overline{\cup_{n=1}^{\infty} f^{-n}(\infty)}\neq\emptyset. \tag{6.4}$$

and

$$f(J(f)\setminus\{\infty\})=J(f)=f^{-1}(J(f))\cup\{\infty\}. \tag{6.5}$$

Proof. The formula (6.4) is an immediate consequence of Remark 6.4 while (6.5) is one of the most basic facts in the theory of iteration of meromorphic functions and for example it is formulated in [3, Lemma 2]. □

Notice that if f is a non-constant elliptic function then for all $w\in\mathbb{C}$

$$J(f)\cap B(w,R_2)\neq\emptyset. \tag{6.6}$$

This follows because

$$J(f)\cap f(B(w,R_2))\overset{(6.3)\wedge(6.4)}{\neq}\emptyset.$$

Hence there exists some

$$z\in f^{-1}(J(f))\cap B(w,R_2)\overset{(6.5)}{\subset}J(f)\cap B(w,R_2),$$

and (6.6) follows.

We are now ready to prove Theorem 1.9 which we restate for the convenience of the reader.

Theorem 6.6. *Let $f:\mathbb{C}\to\hat{\mathbb{C}}$ be a non-constant elliptic function. Then:*

(i) The Julia set $J(f)$ is not porous at a dense set of its points, in particular it is not porous at any point of the set

$$P_f:=\bigcup_{n=1}^{\infty} f^{-n}(\infty).$$

(ii) For every $b\in P_f$ and for all $\kappa\in(0,1)$ there exists $R(b,\kappa)>0$ such that

$$\mathrm{por}(J(f),b,r)\leq\kappa$$

for all $r\in(0,R(b,\kappa))$.

(iii) If in addition $J(f)\neq\mathbb{C}$, then $J(f)$ is porous at a dense set of its points; the repelling periodic points of f.

Proof. Without loss of generality we can assume that $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ and $f(\infty)=\infty$. of course (i) follows from (ii). So for the proof of (ii) let

$$b\in\bigcup_{n=1}^{\infty} f^{-n}(\infty),$$

and fix some $n\in\mathbb{N}$ such that

$$b\in f^{-n}(\infty).$$

We will show that for all $\kappa\in(0,1)$ and all $r\in(0,1)$ small enough, we have that

$$J(f)\cap B(z,\kappa r)\neq\emptyset, \tag{6.7}$$

for all $z\in B(b,r)\setminus\{b\}$.

Note that since f' is a non-constant elliptic function with the same basic fundamental parallelogram as f, the set $\text{Crit}(f)\cap\mathcal{R}_f$ is finite. Hence $f(\text{Crit}(f)) = f(\text{Crit}(f)\cap\mathcal{R}_f)$ is finite as well. As a consequence the set $\bigcup_{k=1}^n f^k(\text{Crit}(f))$ is also finite. Therefore there exists some $R_3 > 0$ (depending on b) such that

$$\left(\bigcup_{k=1}^{n} f^k(\text{Crit}(f))\right)\cap B(0,R_3)^c = \emptyset. \tag{6.8}$$

Observe that (6.8) implies that

$$f^n(\text{Crit}(f^n))\cap B(0,R_3)^c = \emptyset. \tag{6.9}$$

To verify (6.9) let $\xi \in \text{Crit}(f^n)$. Then there exists some $l = 0,\dots,n-1$ such that $f^l(\xi) \in \text{Crit}(f)$, hence

$$f^n(\xi) \in f^{n-l}(\text{Crit}(f)).$$

Thus $f^n(\text{Crit}(f^n)) \subset \bigcup_{k=1}^n f^k(\text{Crit}(f))$, and (6.9) follows from (6.8). Arguing in the same way it actually follows that $f^n(\text{Crit}(f^n)) = \bigcup_{k=1}^n f^k(\text{Crit}(f)$.

Now let $\kappa \in (0,1)$ and let

$$0 < r < \min\{R_1, R_2, \kappa(K_{1/2}AR_2)^{-1/q_b}, (A(2R_2+R_3))^{-1/q_b}\} := R(b,\kappa), \tag{6.10}$$

where $q_b \geq 1$ is the order of b as pole of f^n. In the following, $z \in B(b,r)\setminus\{b\}$. Note then that

$$|f^n(z)| \overset{(6.1)}{\geq} A^{-1}|z-b|^{-q_b} \geq A^{-1}r^{-q_b} \overset{(6.10)}{\geq} 2R_2+R_3. \tag{6.11}$$

Therefore (6.9), (6.11), and the Monodromy Theorem, imply that there exists a unique holomorphic inverse branch of f^n,

$$f_z^{-n}(B(f_n(z), 2R_2)) \to \mathbb{C},$$

such that $f_z^{-n}(f_n(z)) = z$. By Theorem 4.3 we have that,

$$\begin{aligned} f_z^{-n}(B(f^n(z),R_2)) &\subset B(z, K_{1/2}|(f^n)'(z)|^{-1}R_2) \overset{(6.2)}{\subset} B(z, K_{1/2}R_2A|z-b|^{q_b+1}) \\ &\subset B(z, K_{1/2}R_2Ar^{q_b+1}) \overset{(6.10)}{\subset} B(z,\kappa r). \end{aligned} \tag{6.12}$$

Recalling (6.6) we have that $J(f)\cap B(f^n(z),R_2) \neq \emptyset$. Hence, there exists some $x \in f_z^{-n}(J(f))\cap f_z^{-n}(B(f^n(z),R_2))$. Since (6.5) implies that $f_z^{-n}(J(f)) \subset J(f)$ we deduce that

$$J(f)\cap f_z^{-n}(B(f^n(z),R_2)) \neq \emptyset. \tag{6.13}$$

Combing (6.12) and (6.13) we deduce that

$$J(f)\cap B(z,\kappa r) \neq \emptyset.$$

The proof of (iii) is essentially the same as the proof of Theorem 3.3 (i).The proof of Theorem 6.6 is complete. □

Remark 6.7. Theorem 5.6 implies that if f is a tame elliptic function such that $J(f) \neq \mathbb{C}$ and μ is a Borel probability f-invariant ergodic measure on $J(f)$ with full topological support and with finite Lyapunov exponent, then there exist some $\alpha_f, p_f \in (0,1]$ such that $J(f)$ is (α_f, p_f)-mean porous at μ-a.e. $x \in J(f)$. As in Remark 5.7, we note that tameness of the elliptic function f is in fact not needed if the Lyapunov exponent $\chi_\mu(f)$ is positive.

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