

**THE EXACT VALUE OF HAUSDORFF DIMENSION
OF
ESCAPING SETS
OF
CLASS \mathcal{B} MEROMORPHIC FUNCTIONS**

VOLKER MAYER AND MARIUSZ URBAŃSKI

ABSTRACT. We consider the subclass of class \mathcal{B} consisting of meromorphic functions $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ for which infinity is not an asymptotic value and whose all poles have orders uniformly bounded from above. This class was introduced in [7] and the Hausdorff dimension $\text{HD}(\mathcal{I}(f))$ of the set $\mathcal{I}(f)$ of all points escaping to infinity under forward iteration of f was estimated therein. In this paper we provide a closed formula for the exact value of $\text{HD}(\mathcal{I}(f))$ identifying it with the critical exponent of the natural series introduced in [7]. This exponent is very easy to calculate for many concrete functions. In particular, we construct a function from this class which is of infinite order and for which $\text{HD}(\mathcal{I}(f)) = 0$.

1. INTRODUCTION

We deal with the dynamics of meromorphic functions from class \mathcal{B} . This class consists in all meromorphic functions $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ for which the set $S(f)$ of finite asymptotical and critical values is bounded. For entire functions, class \mathcal{B} was introduced by Eremenko and Lyubich in [9].

Recalling standard definitions, the *Fatou* set $\mathcal{F}(f)$ of a function $f \in \mathcal{B}$ consists in all points in \mathbb{C} that admit a neighborhood on which all iterates of f are defined and normal in the sense of Montel. Then, the *Julia* set $\mathcal{J}(f)$ of f is the complement of the *Fatou* set $\mathcal{F}(f)$.

In this paper we focus on another dynamically significant set, namely the *escaping* set of f

$$(1.1) \quad \mathcal{I}(f) = \{z \in \mathbb{C} : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}$$

which is also the intersection over all $R > 0$ of the sets

$$(1.2) \quad \mathcal{I}_R(f) = \{z \in \mathbb{C} : \liminf_{n \rightarrow \infty} |f^n(z)| \geq R\}.$$

Date: March 3, 2021.

1991 *Mathematics Subject Classification.* Primary 37F10; Secondary 30D05, 37F45, 28A80.

If f is a polynomial then $\mathcal{I}(f)$ is simply the basin of (immediate) attraction to infinity which is a central object in the study of the dynamics of polynomials. If $f \in \mathcal{B}$ is transcendental then $\mathcal{I}(f) \neq \emptyset$ and $\mathcal{I}(f) \subset \mathcal{J}(f)$ (see [9] and [21]) and we are interested in

$$\text{HD}(\mathcal{I}(f)),$$

where $\text{HD}(E)$ denotes the *Hausdorff dimension* of a set E contained in a metric space.

McMullen showed in [15] that the Julia set of any exponential function $z \mapsto \lambda e^z$, $\lambda \neq 0$, has Hausdorff dimension equal to 2 and that the Julia set of any sine function $z \mapsto \sin(\alpha z + \beta)$, $\alpha \neq 0$, has even positive Lebesgue measure. His result has been extended by Barański [3] and Schubert [22] to all entire functions in class \mathcal{B} that have finite order. As it is mentioned in [7], their proof actually shows that

$$\text{HD}(\mathcal{I}(f)) = 2$$

for all such functions and in fact for all meromorphic functions in the Eremenko-Lyubich class \mathcal{B} that have finite order and for which ∞ is an asymptotic value. Bergweiler, Karpińska, and Stallard showed in [6] that this result even also holds for some functions $f \in \mathcal{B}$ having infinite order. The situation is totally different when ∞ is not an asymptotical value as was shown by Bergweiler and Kotus in [7] who, improving former results for elliptic functions by Kotus and Urbański (see [12], comp. [13]), obtained the following upper estimate.

Theorem 1.1 (Bergweiler and Kotus). *Let $f \in \mathcal{B}$ be a transcendental meromorphic function having finite order $\rho < \infty$, for which ∞ is not an asymptotic value and such that there exists $M \in \mathbb{N}$ such that the multiplicity of all poles, except possibly finitely many, is at most M . Then*

$$(1.3) \quad \text{HD}(\mathcal{I}(f)) \leq \frac{2M\rho}{2 + M\rho}$$

and

$$(1.4) \quad \lim_{R \rightarrow \infty} \text{HD}(\mathcal{I}_R(f)) \leq \frac{2M\rho}{2 + M\rho}.$$

In this article we determine the exact values of these Hausdorff dimensions and we do this for the whole class of functions in Theorem 1.1, and beyond. The exact value of the Hausdorff dimension of the escaping set has been so far known only in some special cases, mainly for elliptic functions (see [13]), for the functions of the form $R(e^z)$, where R is a rational function, obtained in the paper [11] by Galazka and Kotus, and for Nevanlinna functions obtained in the paper [8] by Cui, where the special case $z \mapsto \lambda \tan z$ has been treated

in [17]. Notice that for this later case one has strict inequality

$$\text{HD}(\mathcal{I}(f)) < \frac{2M\rho}{2 + M\rho}.$$

Equality holds for all elliptic functions and for some interesting examples considered in [7]. Finally, there are also original new examples provided by Aspenberg and Cui in [2] which consist in some functions that have finite order and a finite singular set, that belong to Speiser class \mathcal{S} , and whose escaping set may have any Hausdorff dimension in $[0, 2]$.

Let f be like in Theorem 1.1, let $(a_j)_{j=1}^{\infty}$ be the sequence of all its poles, and let m_j be the multiplicity of a_j , $j \geq 1$. Then

$$(1.5) \quad \limsup_{j \rightarrow \infty} m_j = M < \infty \quad \text{and} \quad f(z) \sim \left(\frac{b_j}{z - a_j} \right)^{m_j} \quad \text{as} \quad z \rightarrow a_j$$

for every $j \geq 1$ with some $b_j \in \mathbb{C} \setminus \{0\}$. Bergweiler and Kotus considered the series

$$(1.6) \quad \sum_{j \text{ s.t. } a_j \neq 0} \left(\frac{|b_j|}{|a_j|^{1+1/M}} \right)^t,$$

and an essential step in their proof of Theorem 1.1 was to show the convergence of it for all $t > \frac{2M\rho}{2+M\rho}$.

Definition 1.2. *The critical exponent δ of the series (1.6) is the infimum over all $t \in \mathbb{R}$ for which the series (1.6) converges.*

Notice that δ does depend on M which itself does depend on f . Also, as explained above, this critical exponent is at most $\frac{2M\rho}{2+M\rho}$ but in general it is strictly smaller. On the other hand, the functions under consideration have infinitely many poles and thus $\delta \geq 0$.

The main result of this article is the following result. Not only that it holds for all functions of Theorem 1.1 but it also applies to functions of infinite order.

Theorem 1.3. *Let f be a meromorphic function in class \mathcal{B} for which ∞ is not an asymptotic value and whose poles have uniformly bounded multiplicities. Then*

$$\text{HD}(\mathcal{I}(f)) = \lim_{R \rightarrow \infty} \text{HD}(\mathcal{I}_R(f)) = \delta,$$

where δ is the critical exponent of the series (1.6).

Rempe-Gillen and Stallard called in [19] the number $\lim_{R \rightarrow \infty} \text{HD}(\mathcal{I}_R(f))$ the *eventual dimension* of f and asked if it can differ from $\text{HD}(\mathcal{I}(f))$. Theorem 1.3 shows that this is not the case for any function considered therein.

Combining our result with the aforementioned results by Barański and Schubert, one obtains the following full description for the Hausdorff dimension of escaping sets.

Corollary 1.4. *Let $f \in \mathcal{B}$ be a meromorphic function of finite order with poles having bounded multiplicity. Then*

- (1) $\text{HD}(\mathcal{I}(f)) = 2$ if ∞ is an asymptotic value of f and
- (2) $\text{HD}(\mathcal{I}(f)) = \delta$ otherwise.

The upper bound in Theorem 1.1 does depend on the order ρ of f and converges to 2 when $\rho \rightarrow \infty$. This suggests that $\text{HD}(\mathcal{I}(f)) = 2$ whenever f has infinite order, and this is the reason why in this case the behavior of $\mathcal{I}(f)$ with respect to Lebesgue measure has been studied rather than the Hausdorff dimension of $\mathcal{I}(f)$ (see [7, Theorem 1.3] and the discussion preceding it). However, our result, i.e. Theorem 1.3, shows that $\text{HD}(\mathcal{I}(f))$ does not really depend on the order. It certainly depends on the distribution of the poles but is also influenced by the coefficients b_j .

It is quite easy to determine the critical exponent δ as soon as the poles a_j and the coefficients b_j are known. We exploit this in providing some quite simple explicit examples in Section 7. Among them are functions having the following properties which, up to our best knowledge, have not appeared in the literature.

Theorem 1.5. *There exist meromorphic functions $f \in \mathcal{B}$ of positive finite and also of infinite order for which ∞ is not an asymptotic value, whose all poles have bounded multiplicities, and*

$$\text{HD}(\mathcal{I}(f)) = 0.$$

Our paper is organized as follows. First we present some background in Section 2. In the next section, Section 3, we collect some properties of the behavior of the functions from Theorem 1.3, near poles. The first step of the proof of Theorem 1.3 is also done therein. This part provides an upper estimate of $\text{HD}(\mathcal{I}(f))$ which is a straightforward adaption of the proof from [7]. Indeed, Bergweiler and Kotus used the most natural dynamical covers of the escaping set, in fact of $\mathcal{I}_R(f)$, from which the upper bound in Theorem 1.1 follows. The starting point of the present paper is the observation that their argument gives in fact that

$$\text{HD}(\mathcal{I}(f)) \leq \lim_{R \rightarrow \infty} \text{HD}(\mathcal{I}_R(f)) \leq \delta.$$

Establishing a precise lower bound is the new and delicate step. For all specific functions for which $\text{HD}(\mathcal{I}(f))$ was known, such as elliptic and Nevanlinna functions mentioned above, the precise lower bound was obtained using McMullen's method in [15]. However, for this method to give sharp estimates

the poles must be very uniformly distributed in addition with a uniform behavior of coefficients b_j , $j \geq 1$. This is clearly the case for elliptic and Nevanlinna functions as well as for the examples in [7]. In order to treat our general case we had to proceed totally differently. Our idea was to use the concept of non-autonomous iterated function systems and some results proved for them. Indeed, in Section 4, we briefly summarize some facts from this theory. Then, in Section 5, we prove the inequality

$$\text{HD}(\mathcal{I}(f)) \geq \delta,$$

by associating to the escaping set $\mathcal{I}(f)$ some non-autonomous conformal iterated function system. This allows us to apply then some results of Rempe-Gillen and the second named author obtained in [20]. We construct our non-autonomous iterated function system by carefully choosing appropriate inverse branches of the third iterate of the function f , and by doing some desingularizing change of variables which moves infinity to zero. Theorem 1.3 is then proved.

In Section 6 we slightly generalize Theorem 1.3 by going beyond the class of functions whose poles have orders uniformly bounded from above.

Finally in Section 7 we prove Theorem 1.5 by providing simple concrete constructions.

2. PRELIMINARIES

The disk centered at $a \in \mathbb{C}$ and of radius $R > 0$ will be denoted by $\mathbb{D}(a, R)$ and we define

$$\mathbb{D}_R^* := \{z \in \mathbb{C} : |z| > R\}.$$

If for some constant $C \geq 1$ we have $A/C \leq B \leq CA$ then this will be denoted by $A \asymp B$. If we need quantitative estimates, we write more precisely

$$A \asymp^C B.$$

The inverse of a transcendental meromorphic function f can have two types of singularities: asymptotic and critical values. A point $b \in \widehat{\mathbb{C}}$ is an asymptotic value of f if there exists a curve $\gamma \subset \mathbb{C}$ tending to infinity such that $f(z) \rightarrow b$ as $z \rightarrow \infty$ along γ . A critical value of f is any point of the form $f(c)$ where

$$c \in \mathcal{C}_f = \{z \in \mathbb{C} : f \text{ is defined at } z \text{ and } f'(z) \neq 0\},$$

the set of critical points of f . The set of all finite singular values of f is denoted by $S(f)$ and is commonly referred to as the singular set of f . A detailed description of singular values of meromorphic functions can be found for example in [5].

We repeat from the introduction that class \mathcal{B} consists in all those meromorphic functions f for which the singular set $S(f)$ is bounded. This class goes

back to Eremenko and Lyubich [9] who introduced it for entire functions. The functions in \mathcal{B} are also frequently called of *bounded type*.

Unless stated otherwise, we consider in this paper transcendental meromorphic functions f having the following properties:

- (i) $f \in \mathcal{B}$.
- (ii) ∞ is not asymptotic value of f .
- (iii) All poles a_j of f have uniformly bounded multiplicities denoted by m_j , so that the number M in (1.5) is well defined, and consequently so is the critical exponent δ . Also, we can and we will assume without loss of generality that 0 is not a pole of f .

One should always have in mind that a transcendental meromorphic function f for which ∞ is not an asymptotic value always has infinitely many poles. Moreover, we can use them to characterize the order $\rho = \rho(f)$ of the function f as the the critical exponent of the series

$$(2.1) \quad \sum_{j=1}^{\infty} \frac{1}{|a_j|^t},$$

meaning that this series converges if $t > \rho$ and diverges if $t < \rho$; all these facts are in [7, Section 2] and they go back to Iversen and Teichmüller.

We fix a number $R_0 > 0$ such that

$$(2.2) \quad S(f) \subset \mathbb{D}(0, R_0/2) \quad \text{and} \quad f(0) \subset \mathbb{D}(0, R_0/2).$$

Let $R \geq R_0$. Since ∞ is not an asymptotic value, Lemma 2.1 in [21] asserts that $f^{-1}(\mathbb{D}_R^*)$ is a countable union of simply connected bounded components $U_j(R)$, $j \geq 1$, each of which contains exactly one pole $a_j \in U_j(R)$. When $R \geq R_0$ is fixed then we often denote these components simply U_j .

All relevant information on the dynamics of transcendental functions, in particular the definition of the Julia set, can be found in the survey [4] by Bergweiler, see also [13]. We deal here only with the escaping set $\mathcal{I}(f)$ and with the sets $\mathcal{I}_R(f)$ which have been introduced in (1.1) and (1.2).

Falconer's book [10], [18], and [13] contain all relevant information on fractal dimensions we need; especially the two latter of them are designed for dynamical needs. One can find the definition of Hausdorff dimension in Section 3 and we recall that the Hausdorff dimension of a set E will be denoted by $\text{HD}(E)$. Many results on fractal dimensions for meromorphic functions have been obtained by several authors, the interested reader can find an overview of them in the survey [23] by Stallard.

The restriction of every conformal map $g : \mathbb{D}(0, 2r) \rightarrow \mathbb{C}$ to $\mathbb{D}(0, r)$ has *bounded distortion* in the sense that there exists an absolute constant $K \geq 1$

such that

$$(2.3) \quad |g'(z_1)| \leq K|g'(z_2)| \quad \text{for all } z_1, z_2 \in \mathbb{D}(0, r).$$

This is the well known Koebe's distortion theorem.

We will need its more sophisticated version. It relies on the concept of modulus of ring domains that we briefly recall now, see for example [1] for more details.

The modulus of the round annulus $A(r, R) = \{r < |z| < R\}$ is

$$\text{mod}(A(r, R)) = \frac{1}{2\pi} \log \frac{R}{r}, \quad 0 < r < R$$

and the modulus is a conformal invariant which can be used to define the modulus of general annuli. These are domains $\mathcal{A} \subset \mathbb{C}$ such that, for some radii $0 < r_1 < r_2$, \mathcal{A} is the conformal image of $A(r_1, r_2)$ and thus in fact the conformal image of $A(1, R)$ where $R = \frac{r_2}{r_1}$. Set then

$$\text{mod}(\mathcal{A}) = \frac{1}{2\pi} \log R.$$

We need the following important fact which results from Theorem 2 in [1, p.10].

Fact 2.1. *If $\mathcal{A} \subset \mathbb{C}$ is an annuli containing a round annuli $A(r, R) \subset \mathcal{A}$ for some $0 < r < R < \infty$ then*

$$\text{mod}(\mathcal{A}) \geq \text{mod} A(r, R) = \frac{1}{2\pi} \log R/r.$$

The following generalization of (2.3) is contained in Theorem 2.9 in [16].

Theorem 2.2. *For every $m > 0$ there exists $K(m) \geq 1$ such that the following holds. Let U and V be open domains in \mathbb{C} both conformally equivalent to \mathbb{D} such that $U \subset \bar{U} \subset V$ and*

$$\text{mod}(V \setminus \bar{U}) \geq m.$$

If $F : V \rightarrow \mathbb{C}$ is holomorphic univalent map. Then

$$|F'(z_2)| \leq K(m)|F'(z_1)| \quad \text{for all } z_1, z_2 \in U.$$

3. ESTIMATES AND UPPER BOUND

This part is a straightforward adaption of [7]. Throughout the whole section f is a meromorphic function satisfying all the hypotheses of Theorem 1.3. We first give a fairly complete description of the behavior of the function f near its poles and then we give the required upper estimate of the Hausdorff dimension of $\mathcal{I}_R(f)$ for large R .

3.1. Estimates near poles. The behavior of a function f near a pole a_j follows from (1.1). So, let us recall it:

$$f(z) \sim \left(\frac{b_j}{z - a_j} \right)^{m_j}$$

as $z \rightarrow a_j$ for some complex number $b_j \neq 0$ and with m_j being the multiplicity of f at a_j that satisfies $1 \leq m_j \leq M$ except for at most finitely many poles. Since we will only consider poles that lie outside some large disk we can assume that

$$(3.1) \quad 1 \leq m_i \leq M = \limsup_{j \rightarrow \infty} m_j.$$

for every $i \geq 1$. We recall that $R_0 > 0$ has been fixed large enough so that (2.2) holds. We shall prove the following.

Lemma 3.1. *For every $j \geq 1$ there exists a conformal homeomorphism $\Psi_j : U_j(R_0) \rightarrow \mathbb{D}(0, R_0^{-1/m_j})$ such that $\Psi_j(a_j) = 0$, $\Psi'_j(a_j) = 1/b_j$, and*

$$f|_{U_j(R_0)} = 1/\Psi_j^{m_j}$$

Proof. Let $\Psi_j : U_j(R_0) \rightarrow \mathbb{D}(0, R_0^{-1/m_j})$ be any conformal homeomorphism with $\Psi_j(a_j) = 0$. It follows from the asymptotic behavior of f near a_j that the function $f \Psi_j^{m_j}$ is well defined, holomorphic, and non-zero on $U_j(R_0)$ and in addition

$$\lim_{U_j(R_0) \ni z \rightarrow \partial U_j(R_0)} |f(z) \Psi_j^{m_j}(z)| = 1.$$

Thus, the maximum principle implies that $f \Psi_j^{m_j}$ is constant and of modulus one. The required conformal map is now just this Ψ_j normalized by a rotation and then clearly $\Psi'_j(a_j) = 1/b_j$. \square

As the first application of this lemma, we can now describe the geometry of the domains U_j . In the next lemma, the constant K is the absolute constant from Koebe's distortion property (2.3).

Lemma 3.2. *For every $R \geq 2^M R_0$,*

$$(3.2) \quad \mathbb{D}\left(a_j, \frac{1}{K} R^{-1/m_j} |b_j|\right) \subset U_j(R) \subset \mathbb{D}\left(a_j, K R^{-1/m_j} |b_j|\right).$$

Moreover, there exists $R_1 \geq 2^M R_0$ such that for every $R \geq R_1$ there exists $\kappa_R > 1$ such that $\kappa_R \leq 2$ for every $R \geq R_1$, $\lim_{r \rightarrow \infty} \kappa_R = 1$, and

$$(3.3) \quad \frac{1}{\kappa_R} \leq \frac{|z|}{|a_j|} \leq \kappa_R$$

for every $z \in U_j(R)$.

Proof. We have $\Psi_j(U_j(R)) = \mathbb{D}(0, R^{-1/m_j})$ where Ψ_j is the conformal map of Lemma 3.1. Since $m_j \leq M$, we have that $R^{-1/m_j} \leq \frac{1}{2}R_0^{-1/m_j}$. This allows us to apply the bounded distortion property (2.3) to Ψ_j^{-1} from which (3.2) follows.

First let $R := 2^M R_0$. By (2.2), $0 \notin U_j(R)$ and, using the first inclusion of (3.2), we see that

$$(3.4) \quad |a_j| - \frac{|b_j|}{KR^{1/m_j}} > 0, \quad \text{thus} \quad |b_j| \leq KR|a_j| = K2^M R_0|a_j|.$$

Now, take an arbitrary $R \geq 2^M R_0$ and then $z \in U_j(R)$. Then,

$$|z - a_j| \leq \frac{K}{R^{1/m_j}}|b_j| \leq \frac{K}{R^{1/M}}K2^M R_0|a_j|,$$

and thus

$$\left| \frac{z}{a_j} - 1 \right| \leq 2^M K^2 \frac{R_0}{R^{1/M}}$$

which implies (3.3) if R is large enough, which determines R_1 . \square

The next technical result provides uniform estimates for the derivative f' in the domains $U_j(R_1)$.

Lemma 3.3. *For every $j \geq 1$ we have that*

$$|f'| \asymp^{KM} \frac{1}{|b_j|} |f|^{1+\frac{1}{m_j}} \quad \text{in} \quad U_j(R_1).$$

Proof. By Lemma 3.1 we have $1/f = \Psi_j^{m_j}$ in U_j . Differentiating this relation yields

$$|f'| = m_j |\Psi_j^{m_j-1} \Psi_j' f^2| = m_j |\Psi_j'| |f|^{1+\frac{1}{m_j}} \asymp m_j \frac{1}{|b_j|} |f|^{1+\frac{1}{m_j}} \quad \text{in} \quad U_j$$

by (2.3) and since $\Psi'(a_j) = 1/b_j$. This proves the lemma since $m_j \leq M$. \square

3.2. Upper estimate. Let us recall that this part is taken from [7] except that we alter the final conclusion. We include it for the sake of completeness and as a preparation for the proof of the lower bound. Here is the precise statement.

Proposition 3.4. *Let f be a meromorphic function of class \mathcal{B} for which ∞ is not an asymptotic value and with poles having bounded multiplicity. Then*

$$\text{HD}(\mathcal{I}(f)) \leq \lim_{R \rightarrow \infty} \text{HD}(\mathcal{I}_R(f)) \leq \delta,$$

where δ is the critical exponent of the series (1.6).

Proof. For every $R \geq R_1$ define the set

$$\mathcal{I}_R^0(f) := \{z \in \mathbb{C} : f^n(z) \geq R \text{ for all } n \geq 0\}$$

and, given an integer $l \geq 0$,

$$\mathcal{I}_{R,l}^0(f) := \{z \in \mathbb{D}_R^* : f^k(z) \in \mathbb{D}_R^* \text{ for all } 1 \leq k < l\} \quad , \quad l \geq 0.$$

Of course

$$\mathcal{I}_R^0(f) \subset \mathcal{I}_{R,l}^0(f)$$

for every $l \geq 0$. We first will estimate the Hausdorff dimension of the $\mathcal{I}_R^0(f)$; for the ease of notation we will do this for the sets $\mathcal{I}_{4R}^0(f)$

The pole neighborhoods U_j and their successive pullbacks by inverse branches of f give natural covers of the sets $\mathcal{I}_{4R,l}^0(f)$, $l \geq 2$. We now explain it. By Lemma 3.2 we have $U_j \subset \mathbb{D}_R^*$ whenever $U_j \cap \mathbb{D}_{4R}^* \neq \emptyset$ since then $|a_j| > 2R$, whence

$$|z| \geq \frac{1}{2}|a_j| > R \text{ for every } z \in U_j.$$

Since $S(f) \cap \mathbb{D}_{R_0}^* = \emptyset$, all holomorphic inverse branches of f are well defined on every simply connected subdomain of $\mathbb{D}_{R_0}^*$. In particular, given any j, k such that $U_j, U_k \subset \mathbb{D}_{R_0}^*$, there exist precisely m_j such branches that map U_k into U_j . Let $g_{j,k} : U_k \rightarrow U_j$ be any of them. Lemma 3.3 along with (3.3) give precise estimate for its derivative.

$$(3.5) \quad |g'_{j,k}(z)| \asymp^{KM} |b_j|/|z|^{1+\frac{1}{m_j}} \asymp^2 |b_j|/|a_k|^{1+\frac{1}{m_j}} \leq |b_j|/|a_k|^{1+\frac{1}{M}} \quad , \quad z \in U_k.$$

Given $l \geq 2$ denote by $Z_l(R)$ the set of all l -tuples $\omega = (j_1, \dots, j_l)$ of positive integers such that $U_{j_1}, \dots, U_{j_l} \subset \mathbb{D}_R^*$. Then set

$$(3.6) \quad V_\omega := g_{j_1, j_2} \circ \dots \circ g_{j_{l-1}, j_l}(U_{j_l}),$$

and we have

$$\bigcup_{\omega \in Z_l} V_\omega \supset \mathcal{I}_{4R,l}^0(f) \supset \mathcal{I}_{4R}^0(f).$$

We have to estimate from above spherical diameters of the sets V_ω , $\omega \in Z_l(R)$, in order to get an upper bound of $\text{HD}(\mathcal{I}_{4R}^0(f))$. Notice that

$$\text{diam}_{sph}(V) \asymp \text{diam}(V)/|a_{j_1}|^2 \leq \text{diam}(V)/|a_{j_1}|^{1+\frac{1}{M}}.$$

We are thus left to estimate the Euclidean diameter $\text{diam}(V_\omega)$. It follows from (3.2) that

$$\text{diam}(U_{j_i}) \leq KR^{-1/M}|b_{j_i}|.$$

Employing (3.5) shows that there exists a constant C such that

$$\text{diam}(V_\omega) \leq C^{l-1} \frac{|b_{j_1}|}{|a_{j_2}|^{1+\frac{1}{M}}} \dots \frac{|b_{j_{l-1}}|}{|a_{j_l}|^{1+\frac{1}{M}}} \text{diam}(U_{j_l})$$

and thus for the spherical derivative one gets

$$\text{diam}_{sph}(V_\omega) \leq \frac{KC^{l-1}}{R^{1/M}} \prod_{k=1}^l \frac{|b_{j_k}|}{|a_{j_k}|^{1+\frac{1}{M}}}.$$

Summing now over all sets V_ω , $\omega \in Z_l(R)$, and taking into account that for each pair j, k there are at most M inverse branches of f mapping U_j into U_k , gives us for every $t > 0$ the following estimate

$$\sum_{\omega \in Z_l(R)} \text{diam}_{sph}(V_\omega)^t \leq \frac{K^t}{R^{t/M}} \left[CM \sum_{j \in Z_0} \left(\frac{|b_j|}{|a_j|^{1+\frac{1}{M}}} \right)^t \right]^l$$

If $t > \delta$, the critical exponent of the series (1.6), then for all $R > R_1$ large enough,

$$\sum_{\omega \in Z_l(R)} \text{diam}_{sph}(V_\omega)^t \leq \frac{K^t}{R^{t/M}} \frac{1}{2^l} \longrightarrow 0 \quad \text{as } l \rightarrow \infty.$$

Therefore,

$$\text{HD}(\mathcal{I}_{4R}^0(f)) \leq t.$$

Hence

$$\text{HD}(f^{-k}(\mathcal{I}_{4R}^0(f))) \leq t$$

for every integer $k \geq 0$. Since also

$$\mathcal{I}_{5R}(f) \subset \bigcup_{k=0}^{\infty} f^{-k}(\mathcal{I}_{4R}^0(f)),$$

we thus get that

$$\text{HD}(\mathcal{I}_{5R}(f)) \leq t$$

for all $R > R_1$ large enough. Thus, $\lim_{R \rightarrow \infty} \text{HD}(\mathcal{I}_R(f)) \leq t$. Since $t > \delta$ was arbitrary, Proposition 3.4 follows. \square

Remark 3.5. By (3.5), in fact by Lemma 3.3 along with (3.3), we have for all $j, k \geq 1$ that

$$|g'_{j,k}(z)| \leq L|b_j|/|a_k|^{1+\frac{1}{M}}, \quad z \in U_k, \quad \text{where } L = 2MK.$$

On the other hand $|b_j| \leq K2^M R_0 |a_j|$ (see (3.4)). Therefore,

$$|g'_{j,k}| \leq LK2^M R_0 \frac{1}{|a_k|^{1/M}} \leq LK2^M R_0 \frac{1}{R_1^{1/M}} \quad \text{on } U_k.$$

In the following we assume R_1 to be so large such that Lemma 3.2 holds and that

$$(3.7) \quad R_1^{1/M} \geq LK2^{M+1} R_0$$

so that then

$$|g'_{j,k}(z)| \leq 1/2$$

for all $j, k \geq 1$, meaning that all these maps are uniformly contracting.

4. NON AUTONOMOUS IFS

A conformal iterated function system, IFS for short, in the complex plane is given by a family $\Phi = \{\Phi_i\}_{i \in I}$ of conformal contractions $\Phi_i : \Omega \rightarrow \Omega$, where I is an arbitrary countable set and where Ω is a bounded simply connected domain in \mathbb{C} having some geometric properties that we will describe below. We assume that the functions Φ_i are in fact defined and continuous on $\overline{\Omega}$ and that $\Phi_i(\overline{\Omega}) \subset \Omega$.

One is interested by the dynamics of all possible compositions

$$\Phi_{i_1} \circ \dots \circ \Phi_{i_n} : \Omega \longrightarrow \Omega, \quad i_k \in I \text{ for every } 1 \leq k \leq n.$$

Non-autonomous dynamical systems vary in time. In the setting of IFS this means that for each k the functions Φ_{i_k} in the above composition belong to a different family of functions. For example, consider the family $\Phi = \{\Phi_i\}_{i \in I}$ and define for each $n \geq 1$ a set $I^{(n)} \subset I$ hence a family $\Phi^{(n)} = \{\Phi_i\}_{i \in I^{(n)}}$. The corresponding non autonomous dynamical system is given by the compositions

$$\Phi_{i_1} \circ \dots \circ \Phi_{i_n} : \Omega \longrightarrow \Omega, \quad (i_1, \dots, i_n) \in \mathcal{I}_n = I^{(1)} \times \dots \times I^{(n)}, \quad n \geq 1,$$

and the limit set of this non-autonomous IFS is $\Phi^{(n)} = \{\Phi_i\}_{i \in I^{(n)}}$, $n \geq 1$, is defined to be

$$\mathcal{J} := \bigcap_{n \geq 1} \bigcup_{(i_1, \dots, i_n) \in \mathcal{I}_n} \Phi_{i_1} \circ \dots \circ \Phi_{i_n}(\Omega).$$

We only consider the case where all the sets $I^{(n)}$ are finite but their cardinalities are allowed to vary and grow with n , however only subexponentially, meaning that

$$(4.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \#I^{(n)} = 0.$$

We then say that the non-autonomous IFS $\Phi^{(n)} = \{\Phi_i\}_{i \in I^{(n)}}$, $n \geq 1$, is of subexponential growth. We are interested in the Hausdorff dimension $\text{HD}(\mathcal{J})$ of the limit set \mathcal{J} of this system, and it is shown in [20] that this dimension is given by a Bowen's Formula provided the system Φ has the following properties.

- (1) Ω is bounded and convex.
- (2) Open Set Condition, meaning that

$$\Phi_i(\Omega) \cap \Phi_j(\Omega) = \emptyset$$

for all $n \geq 1$ and all $i, j \in I^{(n)}$.

- (3) Bounded Distortion: all the compositions $\Phi_{i_1} \circ \dots \circ \Phi_{i_n} : \Omega \rightarrow \Omega$, $(i_1, \dots, i_n) \in \mathcal{I}_n$ have the property (2.3) on Ω .
- (4) Uniform contraction: there exists $s \in (0, 1)$ such that

$$|\Phi'(z)| \leq s \text{ for every } z \in \Omega \text{ and } i \in I.$$

Also Bowen's Formula is not true in general but we have from Theorem 1.1 in [20]:

Theorem 4.1 (Bowen's Formula). *If a non-autonomous IFS $\Phi^{(n)} = \{\Phi_i\}_{i \in I^{(n)}}$, $n \geq 1$, has all the properties listed above and is of subexponential growth, then*

$$\text{HD}(\mathcal{J}) = \inf\{t \geq 0 : \underline{P}(t) < 0\},$$

where

$$\underline{P}(t) := \liminf_{n \rightarrow \infty} \frac{1}{n} \log \sum_{(i_1, \dots, i_n) \in \mathcal{I}_n} \|(\Phi_{i_1} \circ \dots \circ \Phi_{i_n})'\|_\infty^t$$

is the, commonly called, lower pressure at t of the non-autonomous system $\Phi^{(n)} = \{\Phi_i\}_{i \in I^{(n)}}$, $n \geq 1$.

5. LOWER BOUND

Given Proposition 3.4, Theorem 1.3 will be proved if we can show that

$$\text{HD}(\mathcal{I}(f)) \geq \delta.$$

This will be done here and the strategy is to associate to f a non-autonomous IFS consisting of compositions of some selected holomorphic inverse branches of f , for which Theorem 4.1 (Bowen's Formula) will be applicable. First of all we have to determine a good domain Ω on which our non-autonomous IFS is to act.

5.1. The Domain for the IFS. For $0 < \alpha_1 < \alpha_2 < 2\pi$ let

$$\mathcal{C}(\alpha_1, \alpha_2) := \exp\left(\{z \in \mathbb{C} : \alpha_1 < \Im z < \alpha_2\}\right)$$

be the cone delimited by the directions α_1, α_2 . We recall that δ is the critical exponent of (1.6).

Lemma 5.1. *There exists a cone $\mathcal{C} = \mathcal{C}(\alpha, \alpha + 2\pi \cdot 3/4)$ such that*

$$\sum_{j \text{ s.t. } a_j \in \mathcal{C}} \left(\frac{|b_j|}{|a_j|^{1+1/M}} \right)^t = \infty$$

for all $0 \leq t < \delta$ and such that \mathcal{C} contains infinitely many poles a_j of maximal multiplicity $m_j = M$.

Proof. Since $\mathbb{C} \setminus \{0\}$ can be covered by five cones $\mathcal{C}(\alpha_1, \alpha_2)$ with $(\alpha_2 - \alpha_1)/2\pi = 1/4$, at least one of them must contain infinitely many poles of maximal multiplicity. Denote it by \mathcal{C}_0 .

The plane minus the origin can also be covered by two cones $\mathcal{C}_1, \mathcal{C}_2$ of the form $\mathcal{C}(\alpha, \alpha + 2\pi 3/4)$ each of them containing \mathcal{C}_0 so that both \mathcal{C}_1 and \mathcal{C}_2 contain infinitely many poles of maximal multiplicity. Now, let $t_0 < \delta$. Then the series in Lemma 5.1 diverges if $\mathcal{C} = \mathcal{C}_1$ or if $\mathcal{C} = \mathcal{C}_2$. Furthermore, if this is the case, say for $\mathcal{C} = \mathcal{C}_1$, then the series with $\mathcal{C} = \mathcal{C}_1$ diverges for all $t \leq t_0$. The conclusion now comes by considering a sequence $t_n \rightarrow \delta$ with all $t_n < \delta$. \square

For $R \geq R_1$ let

$$\mathcal{C}_R := \mathcal{C} \cap \mathbb{D}_R^*$$

where $\mathcal{C} = \mathcal{C}(\alpha, \alpha + 2\pi 3/4)$ is the cone of Lemma 5.1.

Lemma 5.2. *There exists $R_2 \geq 2K^{2M}R_1$ such that for every pole $a_j \in \mathcal{C}_{2R_2}$ we have that*

$$U_j(R_2) \subset U_j\left(\frac{R_2}{2K^{2M}}\right) \subset \hat{\Omega} = \mathcal{C}_{R_2}(\alpha - \pi/8, \alpha + 2\pi 13/16).$$

Proof. This follows immediately from the inequality (3.3) in Lemma 3.2. \square

In the following, R_2 will be the number from this lemma and let

I be the set of all j such that the pole $a_j \in \mathcal{C}_{2R_2}$.

Since $\hat{\Omega}$ is a simply connected subdomain of $\mathbb{D}_{R_2}^*$ and $\mathbb{D}_{R_2}^*$ is disjoint from $S(f)$, all holomorphic inverse branches of f are well defined on $\hat{\Omega}$ and, for every $j \in I$, there exists m_j such branches mapping $\hat{\Omega}$ into $U_j = U_j(R_2) \subset \hat{\Omega}$. Denote by

$$f_j^{-1} : \hat{\Omega} \rightarrow U_j$$

one, arbitrarily chosen, of these inverse branches. We already gave a precise expression for the derivative of this function in the formula (3.5). Here we repeat it.

$$(5.1) \quad |(f_j^{-1})'(z)| \asymp^{KM} \frac{|b_j|}{|z|^{1+\frac{1}{m_j}}}, \quad z \in \hat{\Omega}.$$

Since, by Lemma 5.2, $f_j^{-1}(\hat{\Omega}) \subset \hat{\Omega}$ for all $j \in I$, we could build an IFS out of this family of functions. However it would fail to have some necessary properties. Especially, given the above expression for the derivative of f_j^{-1} , these functions have no bounded distortion property on $\hat{\Omega}$. Nevertheless, there is a somewhat miraculous solution to this problem by applying, what we would call, a singular change of variables.

5.2. Desingularization and Derivative Estimates. Consider the map $u : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ given by the formula

$$u(\xi) = \xi^{-M}.$$

It maps $\mathbb{D}\left(0, R_2^{-\frac{1}{M}}\right) \setminus \{0\}$ onto $\mathbb{D}_{R_2}^*$ and admits all holomorphic inverse branches on $\widehat{\Omega}$. Choose again arbitrarily one of them, denote it by u_*^{-1} , and define

$$\Omega := u_*^{-1}(\widehat{\Omega}).$$

Notice that Ω is bounded and convex.

Transfer then the maps f_j^{-1} to Ω by conjugation to get

$$\varphi_j := u_*^{-1} \circ f_j^{-1} \circ u : \Omega \longrightarrow V_j := u_*^{-1}(U_j) \subset \Omega \quad \text{for all } j \in I.$$

After having done this change of variables, not all new maps will have bounded distortion yet. However, the maps φ_j associated to poles with maximal multiplicity will do have this property. The following lemma along with Lemma 5.4 are for us the two main ingredients to get, in Lemma 5.5, the required bounded distortion.

Lemma 5.3. *For every $j \in I$, we have that*

$$|\varphi_j'(\xi)| \asymp^{2KM} \frac{|b_j|}{|a_j|^{1+\frac{1}{M}}} |z|^{\frac{1}{M}-\frac{1}{m_j}} = \frac{|b_j|}{|a_j|^{1+\frac{1}{M}}} |\xi|^{\frac{M}{m_j}-1}, \quad \xi \in \Omega \text{ and } z = u(\xi).$$

In particular, when a_j is a pole with maximal multiplicity $m_j = M$ then

$$|\varphi_j'(\xi)| \asymp^{2KM} \frac{|b_j|}{|a_j|^{1+\frac{1}{M}}}, \quad \xi \in \Omega.$$

Proof. Let $\xi \in \Omega$ and set $z := u(\xi) = \xi^{-M}$. since $u \circ \varphi_j = f_j^{-1} \circ u$, we get

$$(\varphi_j(\xi))^{-M} = f_j^{-1}(\xi^{-M}).$$

So, taking derivatives, gives

$$|\varphi_j'(\xi)| = |(f_j^{-1})'(z)| \frac{|\varphi_j(\xi)|^{M+1}}{|\xi|^{M+1}} = |(f_j^{-1})'(z)| \frac{|z|^{1+\frac{1}{M}}}{|f_j^{-1}(z)|^{1+\frac{1}{M}}}$$

Since $f_j^{-1}(z) \in U_j$, Lemma 3.2 implies that $|f_j^{-1}(z)| \asymp^2 |a_j|$. The affirmations in Lemma 5.3 now result from (5.1). \square

5.3. Non-autonomous IFS. Our goal now is to build an appropriate non-autonomous IFS out of the functions φ_j . In order to do this, we first define an appropriate family of holomorphic maps on Ω and verify that they have all necessary properties listed in Section 4. Then we explain how to choose for each $n \geq 1$ its subfamily, depending on n , so that all these subfamilies taken together define a conformal non-autonomous IFS that fulfills all the requirements of Bowen's Formula in Theorem 4.1.

There are several constraints for a construction of such IFS. First of all, it must consist in maps that are uniformly contracting and have bounded distortion. Secondly, its limit set must be, up to the change of variables u , a subset of the escaping set of f , and it must have a sufficiently large Hausdorff dimension. It turns out that the multiplicity of the poles play a crucial role and we get limit sets of large dimension only if the dynamics of the IFS pass sufficiently often through poles of maximal multiplicity. Since these poles are so important we introduce a second notation for them. Let $(j_l)_{l=1}^\infty$ be a strictly increasing sequence of elements in I such that $(a_{j_l})_{l=1}^\infty$ are all the poles with multiplicity $m_{j_l} = M$ in the set $\{a_i\}_{i \in I}$. Define then

$$\mathbf{a}_{l,M} := a_{j_l}, \mathbf{b}_{l,M} := b_{j_l}, \mathbf{V}_{l,M} := u^{-1}(\mathbf{U}_{l,M}) = u^{-1}(U_{j_l}) \text{ and } \varphi_{l,M} := \varphi_{j_l}.$$

Also, since I is infinite, we can assume without loss of generality that $I = \mathbb{N}$.

The maps φ_j cannot be used directly but it turns out that there is a way to compose these maps carefully in order to obtain a good family of generators for an IFS we want. So, consider the following maps that correspond to inverse branches of the third iterate of f :

$$(5.2) \quad \Phi_{l,j} := \varphi_{l,M} \circ \varphi_j \circ \varphi_{l,M} : \Omega \longrightarrow \Omega \quad , \quad l \geq 1, j \in I.$$

These have indeed bounded distortion on Ω and are uniformly contracting:

Lemma 5.4. *For every $j \in I$, every $l \geq 1$, and every $z \in \Omega$, we have*

$$|\Phi'_{l,j}(z)| \asymp \frac{|\mathbf{b}_{l,M}|^2}{|\mathbf{a}_{l,M}|^{2+\frac{1}{m_j}+\frac{1}{M}}} \frac{|b_j|}{|a_j|^{1+\frac{1}{M}}}$$

and

$$|\Phi'_{l,j}(z)| \leq \frac{1}{4}.$$

Proof. First of all

$$\Phi'_{l,j} = \left(\varphi_{l,M} \circ \varphi_j \right)' \circ \varphi_{l,M} \varphi'_{l,M}.$$

Lemma 5.3 and the fact that $\mathbf{a}_{l,M}$ is a pole of maximal multiplicity yield

$$(5.3) \quad |\varphi'_{l,M}| \asymp^{2KM} \frac{|\mathbf{b}_{l,M}|}{|\mathbf{a}_{l,M}|^{1+\frac{1}{M}}} \quad \text{on } \Omega.$$

Let $\xi \in \Omega$ and $\eta = \varphi_{l,M}(\xi) \in \mathbf{V}_{l,M}$. We have to estimate the other factor of $\Phi'_{l,j}$ at η . Since $u(\eta) \in \mathbf{U}_{l,M}$, we have that

$$|\eta|^{-M} = |u(\eta)| \asymp^2 |\mathbf{a}_{l,M}|.$$

Using now Lemma 5.3 once more and remembering that $\mathbf{a}_{l,M}$ has maximal multiplicity, we get

$$\begin{aligned} \left| \left(\varphi_{l,M} \circ \varphi_j \right)'(\eta) \right| &\asymp (2KM)^2 \frac{|\mathbf{b}_{l,M}|}{|\mathbf{a}_{l,M}|^{1+\frac{1}{M}}} \frac{|b_j|}{|a_j|^{1+\frac{1}{M}}} |\eta|^{\frac{M}{m_j}-1} \\ &\asymp^2 \frac{|\mathbf{b}_{l,M}|}{|\mathbf{a}_{l,M}|^{1+\frac{1}{M}}} \frac{|b_j|}{|a_j|^{1+\frac{1}{M}}} |\mathbf{a}_{l,M}|^{\frac{1}{M}-\frac{1}{m_j}} = \frac{|\mathbf{b}_{l,M}|}{|\mathbf{a}_{l,M}|^{1+\frac{1}{m_j}}} \frac{|b_j|}{|a_j|^{1+\frac{1}{M}}}. \end{aligned}$$

Multiplying this with (5.3) gives the required estimate for $|\Phi'_{l,j}|$.

It remains to verify the contraction property. The, already proved, first formula of our lemma gives

$$|\Phi'_{l,j}| \leq 2L^3 \frac{|\mathbf{b}_{l,M}|^2}{|\mathbf{a}_{l,M}|^{2+\frac{1}{m_j}+\frac{1}{M}}} \frac{|b_j|}{|a_j|^{1+\frac{1}{M}}}$$

where $L = 2KM$. By (3.4) we have $|b_i| \leq K2^M R_0 |a_i|$ for all i . So, since $|a_i| > R_1$ and $m_i \leq M$,

$$|\Phi'_{l,j}| \leq 2L^3 (K2^M R_0)^3 \frac{1}{R_1^{3/M}}.$$

Along with (3.7), this implies that $|\Phi'_{l,j}| \leq 1/4$. The proof is complete. \square

We also need bounded distortion for arbitrary compositions of the maps $\Phi_{l,j}$.

Lemma 5.5. *There exists $\hat{K} \geq 1$ such that for all $n \geq 2$ and all $(l_k, i_k) \in I$, $1 \leq k \leq n$, the map*

$$F = \Phi_{l_1, i_1} \circ \Phi_{l_2, i_2} \circ \dots \circ \Phi_{l_n, i_n} : \Omega \longrightarrow \Omega$$

satisfies

$$(5.4) \quad |F'(z_2)| \leq \hat{K} |F'(z_1)| \quad \text{for all } z_1, z_2 \in \Omega.$$

Proof. Write $F = F_{n-1} \circ \Phi_{l_n, i_n}$ where $F_{n-1} = \Phi_{l_1, i_1} \circ \dots \circ \Phi_{l_{n-1}, i_{n-1}}$. The map Φ_{l_n, i_n} satisfies (5.4) with some constant \hat{K} , say $\hat{K}_1 \geq 1$, by Lemma 5.4. Therefore, Lemma 5.5 will hold if we can show that there exists \hat{K}_2 such that

$$|F'_{n-1}(z_1)| \leq \hat{K}_2 |F'_{n-1}(z_2)|$$

for all $n \geq 2$, all $(l_k, i_k) \in I$ with $1 \leq k \leq n$, and all $z_1, z_2 \in \Phi_{l_n, i_n}(\Omega)$.

But this follows from the general Koebe distortion theorem, i.e. Theorem 2.2, if we can find an open disk D such that $\Phi_{l_n, i_n}(\Omega) \subset D$, the map F_{n-1} has a holomorphic univalent extension to D , and

$$\text{mod}(D \setminus \overline{\Phi_{l_n, i_n}(\Omega)}) \geq m$$

where $m > 0$ is a constant independent of all $n \geq 1$, $(l_k, i_k) \in I$, and $1 \leq k \leq n$.

We have that $\Phi_{l_n, i_n}(\Omega) \subset \mathbf{V}_{l_n, M} = u_*^{-1}(\mathbf{U}_{l_n, M}) = u_*^{-1}(U_{j_{l_n}})$ and, by Lemma 5.2,

$$U_{j_{l_n}}(R_2) \subset U_{j_{l_n}}(R_2/Q) \subset \hat{\Omega} \quad \text{where} \quad Q = 2K^{2M}.$$

The map F_{n-1} is well defined, holomorphic, and univalent on the disk

$$u_*^{-1}(U_{j_{l_n}}(R_2/Q)).$$

It will be our disk D , and as

$$\text{mod}(u_*^{-1}(U_{j_{l_n}}(R_2/Q) \setminus \overline{\mathbf{V}_{l_n, M}})) = \text{mod}(U_{j_{l_n}}(R_2/Q) \setminus \overline{U_{j_{l_n}}(R_2)}),$$

it remains to estimate from below the modulus of the annulus

$$\mathcal{A} := U_{j_{l_n}}(R_2/Q) \setminus \overline{U_{j_{l_n}}(R_2)}.$$

This can be done by applying Lemma 3.2. Indeed, it follows from this lemma that

$$U_{j_{l_n}}(R_2) \subset \mathbb{D}(a_{j_{l_n}}, KR_2^{-1/m_{j_{l_n}}} |b_{j_{l_n}}|)$$

and

$$\mathbb{D}(a_{j_{l_n}}, \frac{1}{K}(R_2/Q)^{-1/m_{j_{l_n}}} |b_{j_{l_n}}|) \subset U_{j_{l_n}}(R_2/Q).$$

Thus, \mathcal{A} contains the round annulus

$$A\left(KR_2^{-1/m_{j_{l_n}}} |b_{j_{l_n}}|, \frac{1}{K}(R_2/Q)^{-1/m_{j_{l_n}}} |b_{j_{l_n}}|\right),$$

which implies, by Fact 2.1, that

$$\text{mod}(\mathcal{A}) \geq \frac{1}{2\pi} \log \frac{Q^{1/m_{j_{l_n}}}}{K^2} \geq \frac{1}{2\pi M} \log 2.$$

So, the proof is complete by putting $m := \frac{1}{2\pi M} \log 2 > 0$. \square

Having now the right family of contractions defined on an appropriate domain Ω we are ready to define the non-autonomous system. In order to do so, we will choose for each $n \geq 1$ a set $I^{(n)} \subset \mathbb{N} \times I$ and then consider compositions

$$\Phi_{l_1, i_1} \circ \Phi_{l_2, i_2} \circ \dots \circ \Phi_{l_n, i_n} : \Omega \longrightarrow \Omega$$

with $(l_k, i_k) \subset I^{(k)}$ for all $1 \leq k \leq n$. The choice of these index sets does depend on the parameter t . So, fix $t < \delta$ arbitrary. Then, by the definition of δ , the series (1.6) is divergent for this t . Lemma 5.4 then implies that

$$\sum_{j \geq j_l} \inf |\Phi'_{l,j}|^t = +\infty$$

for every $l \geq 1$, where $(j_l)_{l=1}^\infty$ is again the sequence corresponding to the poles of maximal multiplicity $\mathbf{a}_{l,M} = a_{j_l}$. Therefore, for every $l \geq 1$, there exists $k_l > j_l$ with

$$\sum_{j \geq j_l}^{k_l} \inf |\Phi'_{l,j}|^t \geq 2$$

and, adding some terms in these sums if necessary, we may assume that

$$k_{l+1} - j_{l+1} \geq k_l - j_l$$

for all $l \geq 1$. Define now

$$\hat{I}^{(l)} := \{(l, j) : j_l \leq j \leq k_l\}, \quad l \geq 1.$$

$\#\hat{I}^{(l)}$, the number of elements of $I^{(l)}$, increases with l but may vary a lot. This is the reason why we define the sets $I^{(n)}$ as follows. They interpolate the sets $\hat{I}^{(l)}$ but in such a way that

$$\#I^{(n+1)} \leq \#I^{(n)} + 1$$

for all $n \geq 1$. One way of doing this is the following. Let

$$I^{(1)} := \hat{I}^{(1)}.$$

If $\#\hat{I}^{(2)} = \#I^{(1)}$, then put

$$I^{(2)} := \hat{I}^{(2)}.$$

Otherwise, we have $\#\hat{I}^{(2)} = \#I^{(1)} + N_1$ for some $N_1 \geq 1$. We then define inductively

$$\begin{aligned} I^{(2)} &:= I^{(1)} \cup \{(1, k_1 + 1)\}, \quad I^{(3)} := I^{(2)} \cup \{(1, k_1 + 2)\}, \dots, \\ I^{(N_1+1)} &:= I^{(N_1)} \cup \{(1, k_1 + N_1)\}, \end{aligned}$$

and then

$$I^{(N_1+2)} := \hat{I}^{(2)}.$$

By continuing inductively this definition, we obtain a sequence of sets $(I^{(n)})_{n=1}^\infty$ having the following properties.

Lemma 5.6. *For every $t < \delta$ there exist $(I^{(n)})_{n=1}^\infty$ a sequence of sets, and $(l_n)_{n=1}^\infty$, a diverging sequence of positive integers, such that for every $n \geq 1$ the following hold:*

$$(1) \quad I^{(n)} = \{(l_n, j) : j_{l_n} \leq j \leq h_n\} \quad \text{with some } h_n \geq k_{l_n}.$$

$$(2) \quad \sum_{(l_n, j) \in I^{(n)}} \inf |\Phi'_{l_n, j}|^t \geq 2.$$

$$(3) \quad \#I^{(n+1)} \leq \#I^{(n)} + 1.$$

The last property immediately implies that the system has subexponential growth, i.e. it satisfies (4.1), meaning that

$$(5.5) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \#I^{(n)} = 0.$$

whereas the second property yields positivity of the lower pressure for the parameter t :

$$(5.6) \quad \underline{P}(t) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log \sum_{(i_1, \dots, i_n) \in \mathcal{I}_n} \|(\Phi_{(l_1, i_1)} \circ \dots \circ \Phi_{(l_n, i_n)})'\|_\infty^t \geq \log 2 > 0.$$

5.4. Conclusion of the Proof of Theorem 1.3. Theorem 1.3 results from Proposition 3.4 and the next result.

Proposition 5.7. *If \mathcal{J} is the limit set of the non-autonomous conformal IFS $\Phi^{(n)} = \{\Phi_i\}_{i \in I^{(n)}}$, $n \geq 1$, constructed in the previous subsection, then*

$$u(\mathcal{J}) \subset \mathcal{I}(f)$$

and

$$\text{HD}(\mathcal{J}) \geq \delta.$$

Proof. The first assertion of this proposition follows immediately from the construction of the non-autonomous IFS $\Phi^{(n)} = \{\Phi_i\}_{i \in I^{(n)}}$ done in the previous subsection.

The second assertion follows immediately from Lemma 5.4, Lemma 5.5, Formula (5.5), Formula (5.6), and Bowen's Formula, i.e. Theorem 4.1. \square

6. GENERAL FUNCTIONS OF CLASS \mathcal{B}

As mentioned in the Introduction, McMullen and various other authors who extended his work [15] estimated the Hausdorff dimension of the Julia set by considering the escaping set and, more precisely, the set of points whose orbit escapes through a tract of logarithmic singularity. Here we consider, for general functions of class \mathcal{B} , orbits that escape to infinity “through poles”

To make this more precise, let $f \in \mathcal{B}$ be an arbitrary transcendental meromorphic function and let again $R_0 > 0$ be such that (2.2) holds so that in particular $S(f) \subset \mathbb{D}(0, R_0/2)$. For $R \geq R_0$ let again $U_j(R)$ be the connected component of $f^{-1}(\mathbb{D}_R^*)$ that contains a pole a_j . Consider then the set

$$\mathcal{U}_{M,R} := \bigcup_{j: m_j \leq M} U_j(R)$$

where, as always, m_j is the multiplicity of a_j and where $M \geq 1$. Consider now the subsets

$$\mathcal{I}(f, \mathcal{U}_{M,R})$$

of the escaping set of f consisting of all those points $z \in \mathcal{I}(f)$ for which there exists $N \geq 1$ such that

$$f^n(z) \in \mathcal{U}_{M,R}$$

for all $n \geq N$. Clearly, these subsets of the escaping set are only relevant if f has infinitely many poles. If f is entire or has only finitely many poles then the sets $\mathcal{I}(f, \mathcal{U}_{M,R})$ are empty.

Theorem 6.1. *Let $f \in \mathcal{B}$ be a transcendental meromorphic function and assume that there exists M_f such that f has infinitely many poles a_j , $j \geq 1$, with multiplicities $m_j \leq M_f$. Then, for every $M \geq M_f$, there exists $R_M > R_0$ such that for all $R \geq R_M$*

$$\text{HD}(\mathcal{I}(f, \mathcal{U}_{M,R})) = \delta_M,$$

where δ_M is the critical exponent of the series

$$\sum_{j \text{ s.t. } m_j \leq M} \left(\frac{|b_j|}{|a_j|^{1+1/\mu}} \right)^t$$

and where μ is the upper limit of all the numbers m_j for which $m_j \leq M$.

Proof. The proofs of the Propositions 3.4 and 5.7 are both local. Indeed, let $M \geq M_f$ and consider the restriction of f to the set \mathcal{U}_{M,R_0} . All inverse branches $f_j^{-1} : \mathbb{D}_{R_0}^* \rightarrow U_j(R_0)$, $m_j \leq M$, are well defined and we can proceed exactly as in these proofs using this family of inverse branches in place of the full family of inverse branches considered in the proofs of the Propositions 3.4 and 5.7. \square

Theorem 6.1 contains Theorem 1.3 as its particular case. Also, it is clear that it gives the following estimate valid for all functions in class \mathcal{B} having infinitely many poles. The critical exponents δ_M are increasing with M so that $\sup_M \delta_M = \lim_{M \rightarrow \infty} \delta_M$. Denote this limit by δ_∞ .

Corollary 6.2. *If $f \in \mathcal{B}$ is a transcendental meromorphic function having infinitely many poles, then*

$$\text{HD}(\mathcal{I}(f)) \geq \delta_\infty.$$

7. EXAMPLES

Using our main result, i.e. Theorem 1.3, it is particularly easy to determine the value of the Hausdorff dimension of the escaping set as soon as one has a sufficiently good information about the poles of the function. We illustrate this here by providing some simple examples. The first class of examples

consists in functions with finite order, in fact order two (but this can be easily generalized to any finite order), such that $\text{HD}(\mathcal{I}(f))$ realizes any value in $[0, 2)$. Then we show, by explicit construction, that functions of infinite order fulfilling the hypotheses of Theorem 1.3 can have $\text{HD}(\mathcal{I}(f)) < 2$ and even $\text{HD}(\mathcal{I}(f)) = 0$.

Theorem 1.5 follows from the Theorems 7.1 and 7.6 proven below.

7.1. Examples of Finite Order. We consider functions of the form

$$f(z) = \sum_{|j|, |k| \geq N} \left(\frac{b_{j,k}}{z - a_{j,k}} \right)^M$$

with poles $a_{j,k} = j + ik$, $j, k \in \mathbb{Z}$, all of same multiplicity $M \geq 1$. Remember that the neighborhood $U_{j,k}$ of a pole $a_{j,k}$ is approximately a disk of size $|b_{j,k}|$; see Lemma 3.2. Hence, these numbers can not be too large and we take

$$0 < b_{j,k} \leq |a_{j,k}|^{-\alpha} \quad \text{for some fixed } \alpha > 2/M.$$

Since $\alpha M > 2$, we can fix the integer $N \geq 1$ sufficiently large so that

$$(7.1) \quad \sum_{|j|, |k| \geq N} |a_{j,k}|^{-\alpha M} \leq \frac{1}{2^{M+1}}.$$

Denote

$$\mathbb{Z}_N^2 := \{(j, k) \in \mathbb{Z}^2 : |j|, |k| \geq N\}$$

Theorem 7.1. *Under these assumptions, f is a well defined meromorphic function in class \mathcal{B} and of order two for which ∞ is not an asymptotic value and all of its poles have multiplicity M . furthermore:*

(1) *If $b_{j,k} = |a_{j,k}|^{-\alpha}$ for all $(j, k) \in \mathbb{Z}_N^2$, then*

$$\text{HD}(\mathcal{I}(f)) = \frac{2}{\alpha + 1 + \frac{1}{M}} \in \left(0, \frac{2}{1 + \frac{3}{M}} \right).$$

(2) *If $b_{j,k} = \exp(-|a_{j,k}|)$ for all $(j, k) \in \mathbb{Z}_N^2$, then $\text{HD}(\mathcal{I}(f)) = 0$.*

Proof. Fix $z \in \mathbb{C}$ distinct from all the poles. For all those $(j, k) \in \mathbb{Z}_N^2$ for which $|a_{j,k}| \geq 2|z|$ (keep in mind that the set of j, k failing to satisfy this inequality is finite), we have

$$\left| \frac{b_{j,k}}{z - a_{j,k}} \right|^M \leq \left| \frac{|a_{j,k}|^{-\alpha}}{|a_{j,k}| - |z|} \right|^M \leq 2^M |a_{j,k}|^{-(1+\alpha)M}.$$

So, since $\alpha M \geq 2$, the series defining f converges absolutely and uniformly on some neighborhood of each such point z and, moreover, $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ is a well defined meromorphic function with poles being equal to the numbers $a_{j,k}$, $(j, k) \in \mathbb{Z}_N^2$. Furthermore, f is of order two since the series (2.1) converges if and only if $t > 2$.

Consider the disjoint disks $\mathbb{D}(a_{j,k}, b_{j,k})$, $(j, k) \in \mathbb{Z}_N^2$, and let

$$\mathcal{U} := \bigcap_{(j,k) \in \mathbb{Z}_N^2} \mathbb{D}^c(a_{j,k}, b_{j,k}).$$

Claim 7.2. *The set \mathcal{C}_f of critical points of f satisfies $\mathcal{C}_f \subset \mathcal{U}$.*

Proof. Indeed, since

$$f'(z) = -M \sum_{(j,k) \in \mathbb{Z}_N^2} \frac{b_{j,k}^M}{(z - a_{j,k})^{M+1}}$$

we have for all $(j, k) \in \mathbb{Z}_N^2$, and all $z \in \mathbb{D}(a_{j,k}, b_{j,k}) \setminus \{a_{j,k}\}$ that

$$\begin{aligned} |f'(z)| &\geq M \left(\frac{b_{j,k}^M}{|z - a_{j,k}|^{M+1}} - \sum_{(u,v) \in \mathbb{Z}_N^2 \setminus \{(j,k)\}} \frac{b_{u,v}^M}{|z - a_{u,v}|^{M+1}} \right) \\ &\geq M \left(b_{j,k}^{-1} - \sum_{(u,v) \in \mathbb{Z}_N^2 \setminus \{(j,k)\}} 2^{M+1} b_{u,v}^M \right) \end{aligned}$$

since $|z - a_{u,v}| \geq 1/2$ if $(u, v) \neq (j, k)$. Thus,

$$|f'(z)| \geq M \left(|a_{j,k}|^\alpha - 2^{M+1} \sum_{(u,v) \neq (j,k)} |a_{u,v}|^{-\alpha M} \right) \geq M \left((\sqrt{2}N)^\alpha - 1 \right) > 0.$$

□

Claim 7.3. *$|f(z)| \leq 2$ for every $z \in \mathcal{U}$.*

Proof. Let $z \in \mathcal{U}$. If there exists $(j, k) \in \mathbb{Z}_N^2$ such that $|z - a_{j,k}| \leq 1/2$, then $|z - a_{u,v}| \geq 1/2$ for all $(u, v) \in \mathbb{Z}_N^2 \setminus \{(j, k)\}$ and

$$\begin{aligned} |f(z)| &\leq \left(\frac{b_{j,k}}{|z - a_{j,k}|} \right)^M + \sum_{(u,v) \in \mathbb{Z}_N^2 \setminus \{(j,k)\}} \left(\frac{b_{u,v}}{|z - a_{u,v}|} \right)^M \\ &\leq \left(\frac{b_{j,k}}{b_{j,k}} \right)^M + \sum_{(u,v) \in \mathbb{Z}_N^2 \setminus \{(j,k)\}} \left(\frac{b_{u,v}}{1/2} \right)^M \\ &\leq 1 + 2^M 2^{-(M+1)} \leq 2 \end{aligned}$$

since $b_{u,v} \leq |a_{u,v}|^{-\alpha}$ and since we have (7.1).

In the other case, i.e. if $|z - a_{j,k}| > 1/2$ for all $(j, k) \in \mathbb{Z}_N^2$, then

$$|f(z)| \leq \sum_{(j,k) \in \mathbb{Z}_N^2} \left(\frac{b_{j,k}}{1/2} \right)^M \leq 2^M 2^{-(M+1)} \leq 2$$

also since $b_{j,k} \leq |a_{j,k}|^{-\alpha}$ and since we have (7.1). Thus the claim holds. □

Claim 7.4.

$$S(f) \subset \overline{\mathbb{D}}(0, 2),$$

and consequently $f \in \mathcal{B}$ and ∞ is not asymptotic value.

Proof. It directly follows from Claims 7.2 and 7.3 that

$$f(\mathcal{C}_f) \subset \overline{\mathbb{D}}(0, 2).$$

Concerning the asymptotic values, if $\gamma \subset \mathbb{C}$ is a curve tending to infinity whose image under f converges to an asymptotic value $b \in \hat{\mathbb{C}}$, then there are arbitrary large values $z \in \gamma$ that are in \mathcal{U} . Claim 7.3 implies that for such values z we have $|f(z)| \leq 2$, consequently $|b| \leq 2$. All of this shows that $S(f) \subset \overline{\mathbb{D}}(0, 2)$ and that ∞ is not an asymptotic value. We are done. \square

Thus the function f satisfies all hypotheses of our Theorem 1.3, and thus it suffices to determine the critical exponent δ of (1.6) in order to determine $\text{HD}(\mathcal{I}(f))$.

Case (1): $b_{j,k} = |a_{j,k}|^{-\alpha}$ for all $(j, k) \in \mathbb{Z}_N^2$. Then the series

$$\sum_{(j,k) \in \mathbb{Z}_N^2} \left(\frac{b_{j,k}}{|a_{j,k}|^{1+\frac{1}{M}}} \right)^t = \sum_{(j,k) \in \mathbb{Z}_N^2} |a_{j,k}|^{-(\alpha+1+\frac{1}{M})t}$$

converges if and only if $t(\alpha + 1 + \frac{1}{M}) > 2$. This, with the help of Theorem 1.3, shows that

$$\text{HD}(\mathcal{I}(f)) = \delta = \frac{2}{\alpha + 1 + \frac{1}{M}}.$$

Case (2): $b_{j,k} = \exp(-|a_{j,k}|)$ for all $(j, k) \in \mathbb{Z}_N^2$. Then the series

$$\sum_{(j,k) \in \mathbb{Z}_N^2} \left(\frac{b_{j,k}}{|a_{j,k}|^{1+\frac{1}{M}}} \right)^t = \sum_{(j,k) \in \mathbb{Z}_N^2} \left(\frac{\exp(-|a_{j,k}|)}{|a_{j,k}|^{1+\frac{1}{M}}} \right)^t$$

converges for all $t > 0$ which, in the same way as in Case 1 shows that

$$\text{HD}(\mathcal{I}(f)) = \delta = 0.$$

\square

Remark 7.5. *In both cases, i.e. if $b_{j,k} = |a_{j,k}|^{-\alpha}$ or if $b_{j,k} = \exp(-|a_{j,k}|)$, we see that the series (1.6) diverges at the critical exponent $t = \delta$. This in fact gives further information on the dimension $\dim \mathcal{I}_R(f)$ since it was observed in [7] that one can then employ [14] in order to get*

$$\text{HD}(\mathcal{I}_R(f)) > \delta$$

for every $R > 0$ although, by Theorem 1.3, $\lim_{R \rightarrow \infty} \dim \mathcal{I}_R(f) = \delta$.

7.2. Examples of Infinite Order. There are many ways to modify the preceding example in order to get such functions having infinite order. Here is one way to do it. Let

$$a_j := \log j, \quad b_j := e^{-j},$$

and define

$$(7.2) \quad f(z) := \sum_{j \geq 8} \frac{b_j}{z - a_j}, \quad z \in \mathbb{C}.$$

Theorem 7.6. *The function f defined above in the formula (7.2) is a well defined meromorphic function of class \mathcal{B} of infinite order infinity for which ∞ is not an asymptotic value. Furthermore, all poles of f have multiplicity one and the escaping set of f satisfies*

$$\text{HD}(\mathcal{I}(f)) = 0.$$

Proof. Since $\frac{b_j}{a_j} = \frac{e^{-j}}{\log j}$, $j \geq 8$, the same argument, even easier, as in the proof of Theorem 7.1, shows that the series of (7.2) defines a meromorphic function from \mathbb{C} to $\widehat{\mathbb{C}}$ with poles being equal to the numbers a_j , $j \geq 8$. Furthermore, since $a_j = \log j$, $j \geq 8$, the series (2.1) associated with our present function f diverges for all $t > 0$. This shows that the order ρ of f is equal to ∞ .

The other estimates needed to prove Theorem 7.6 are (also) similar and even simpler than in the proof of Theorem 7.1. Consider again the sets

$$\mathbb{D}(a_j, b_j), \quad j \geq 8, \quad \text{and} \quad \mathcal{U} := \bigcap_{i \geq 8} \mathbb{D}^c(a_i, b_i).$$

Let $z \in \mathcal{U}$. If $z \in \mathbb{D}\left(a_j, \frac{1}{3j}\right)$ for some $j \geq 8$ then, by an elementary calculation,

$$(7.3) \quad |z - a_k| \geq \frac{1}{3k} \quad \text{for all } k \neq j, k \geq 8.$$

This implies

$$|f(z)| \leq \frac{b_j}{|z - a_j|} + \sum_{k \neq j, k \geq 8} \frac{e^{-k}}{|z - a_k|} \leq 1 + 3 \sum_{k \geq 8} k e^{-k} \leq 2.$$

If $z \notin \mathbb{D}\left(a_j, \frac{1}{3j}\right)$ for all $j \geq 8$, then

$$|f(z)| \leq \sum_{j \geq 8} \frac{b_j}{|z - a_j|} \leq 3 \sum_{j \geq 8} j e^{-j} \leq 1.$$

This shows that $|f(z)| \leq 2$ for every $z \in \mathcal{U}$.

The next step is to show that $\mathcal{C}_f \subset \mathcal{U}$. We have

$$f'(z) = - \sum_{j \geq 8} \frac{b_j}{(z - a_j)^2}.$$

Let $z \in \mathcal{U}$. Then $z \in \mathbb{D}(a_j, b_j) \setminus \{a_j\}$ for some $j \geq 8$. So, using (7.3) we get

$$|f'(z)| \geq \frac{b_j}{|z - a_j|^2} - \sum_{k \neq j, k \geq 8} \frac{b_k}{|z - a_k|^2} \geq \frac{b_j}{b_j^2} - 9 \sum_{k \geq 8} k^2 e^{-k} \geq e^8 - 1 > 0.$$

Therefore we have again $\mathcal{C}_f \subset \mathcal{U}$ and $f(\mathcal{C}_f) \subset \overline{\mathbb{D}}(0, 2)$. Arguing exactly as in the proof of Theorem 7.1, it also follows that all possible asymptotic values of f belong to $\overline{\mathbb{D}}(0, 2)$. This shows that $f \in \mathcal{B}$ and that ∞ is not an asymptotic value of f .

Thus, the function f is in the class of functions considered in Theorem 1.3, and this theorem implies that $\text{HD}(\mathcal{I}(f)) = \delta$. It remains to determine the value of δ . Since all the poles of f are simple, the series determining this value is

$$\sum_{j \geq 8} \left(\frac{b_j}{a_j^2} \right)^t = \sum_{j \geq 8} \left(\frac{e^{-j}}{(\log j)^2} \right)^t.$$

It converges for every $t > 0$. In consequence (using Theorem 1.3), $\text{HD}(\mathcal{I}(f)) = \delta = 0$. \square

REFERENCES

- [1] Lars V. Ahlfors. *Lectures on quasiconformal mappings*, volume 38 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 2006. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. 2
- [2] Magnus Aspenberg and Weiwei Cui. Hausdorff dimension of escaping sets of meromorphic functions. *Trans. Amer. Math. Soc.*, to appear. 1
- [3] Krzysztof Barański. Hausdorff dimension of hairs and ends for entire maps of finite order. *Math. Proc. Camb. Phil. Soc.*, 145:719–737, 2008. 1
- [4] Walter Bergweiler. Iteration of meromorphic functions. *Bull. Amer. Math. Soc.*, 29:151–188, 1993. 2
- [5] Walter Bergweiler and Alexandre Eremenko. On the singularities of the inverse to a meromorphic function of finite order. *Revista Matemática Iberoamericana*, 11(2):355–373, 1995. 2
- [6] Walter Bergweiler, Bogusława Karpińska, and Gwyneth M. Stallard. The growth rate of an entire function and the Hausdorff dimension of its Julia set. *J. Lond. Math. Soc. (2)*, 80(3):680–698, 2009. 1
- [7] Walter Bergweiler and Janina Kotus. On the Hausdorff dimension of the escaping set of certain meromorphic functions. *Transactions of the American Mathematical Society*, 364(10):5369–5394, Oct 2012. (document), 1, 1, 1, 1, 2, 3, 3.2, 7.5
- [8] W. Cui. Hausdorff dimension of escaping sets of nevanlinna functions. *Int. Math. Res. Not. IMRN*, to appear, 2019. 1

- [9] A. Eremenko and M. Yu Lyubich. Dynamical properties of some classes of entire functions. *Annales de l'institut Fourier*, 42(4):989–1020, 1992. 1, 1, 2
- [10] Kenneth Falconer. *Fractal geometry: Mathematical Foundations and Applications*. John Wiley & Sons Inc., third edition, 2014. 2
- [11] Piotr Galazka and Janina Kotus. Hausdorff dimension of sets of escaping points and escaping parameters for elliptic functions. *Proc. Edinb. Math. Soc. (2)*, 59(3):671–690, 2016. 1
- [12] Janina Kotus and Mariusz Urbański. Hausdorff dimension and hausdorff measures of julia sets of elliptic functions. *Bulletin of the London Mathematical Society*, 35(2):269–275, 2003. 1
- [13] Janina Kotus and Mariusz Urbanski. Ergodic theory, geometric measure theory, conformal measures and the dynamics of elliptic functions. *arXiv:2007.13235*, 2020. 1, 1, 2
- [14] Volker Mayer. The sizes of the Julia set of meromorphic functions. *Math. Nachr.*, 282(8):1189–1194, 2009. 7.5
- [15] Curt McMullen. Area and Hausdorff dimension of Julia sets of entire functions. *Trans. Amer. Math. Soc.*, 300(1):329–342, 1987. 1, 1, 6
- [16] Curtis T. McMullen. *Complex dynamics and renormalization*, volume 135 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1994. 2
- [17] G. Piotr and J.Kotus. Escaping points and escaping parameters for singly periodic meromorphic maps: Hausdorff dimensions outlook. *Complex Var. Elliptic Equ.*, 63(4):547–568, 2018. 1
- [18] Feliks Przytycki and Mariusz Urbański. *Conformal fractals: ergodic theory methods*, volume 371 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2010. 2
- [19] Lasse Rempe and Gwyneth M. Stallard. Hausdorff dimensions of escaping sets of transcendental entire functions. *Proc. Amer. Math. Soc.*, 138(5):1657–1665, 2010. 1
- [20] Lasse Rempe-Gillen and Mariusz Urbański. Non-autonomous conformal iterated function systems and Moran-set constructions. *Trans. Amer. Math. Soc.*, 368(3):1979–2017, 2016. 1, 4, 4
- [21] P. J. Rippon and G. M. Stallard. Iteration of a class of hyperbolic meromorphic functions. *Proc. Amer. Math. Soc.*, 127(11):3251–3258, 1999. 1, 2
- [22] H. Schubert. Über die hausdorff-dimension der julia menge von funktionen endlicher ordnung. *Dissertation, University of Kiel*, 2007. 1
- [23] Gwyneth M. Stallard. Dimensions of Julia sets of transcendental meromorphic functions. In *Transcendental dynamics and complex analysis*, volume 348 of *London Math. Soc. Lecture Note Ser.*, pages 425–446. Cambridge Univ. Press, Cambridge, 2008. 2

VOLKER MAYER, UNIVERSITÉ DE LILLE, DÉPARTEMENT DE MATHÉMATIQUES, UMR 8524 DU CNRS, 59655 VILLENEUVE D'ASCQ CEDEX, FRANCE

Email address: volker.mayer@univ-lille.fr

Web: math.univ-lille1.fr/~mayer

MARIUSZ URBAŃSKI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON, TX 76203-1430, USA

Email address: urbanski@unt.edu

Web: www.math.unt.edu/~urbanski