DYNAMICS OF ENDOMORPHISMS AND COMPLEX CONTINUED FRACTIONS

EUGEN MIHAILESCU AND MARIUSZ URBAŃSKI

ABSTRACT. We study the dimension theory for invariant measures associated to a new class $\mathcal{E}$ of piecewise hyperbolic endomorphisms F_T on open sets in $\mathbb{C}^2$, associated to conformal Smale endomorphisms T (introduced in [17]). A significant example from this class is a transformation F which generates complex continued fractions with arbitrary digits from an open set in $\mathbb{C}$. The first coordinate map of F_T is only piecewise smooth on countably many pieces, and its second coordinate map is holomorphic and generates parametrized Julia sets $J_{T,\omega}$ in stable fibers. The conditional measures of equilibrium measures are shown to be exact dimensional on these fiber Julia sets. We prove then a global Volume Lemma for F_T which implies that projection measures are exact dimensional on the global basic set J_T in $\mathbb{C}^2$. We obtain a formula for the dimensions of these global measures, in terms of Lyapunov exponents and marginal entropies. The measures associated to geometric potentials $\psi_{T,s}$ and their dimensions are studied in the end.

MSC 2010: 37D35, 37A35, 37F40, 37C45, 37A45.
Kewords: Endomorphisms; Hausdorff dimension of measures; Julia sets; Lyapunov exponents; complex continued fractions; geometric potentials.

1. Introduction and Outline.

We study the dynamics and dimension theory for several types of invariant measures for a new class $\mathcal{E}$ of **endomorphisms** F_T on unbounded open sets in $\mathbb{C}^2$. In particular a special endomorphism F from $\mathcal{E}$ generates a new class of **complex continued fractions** with digits being complex numbers from an open set in $\mathbb{C}$. The non-smooth endomorphisms F_T of this class $\mathcal{E}$ are associated to Smale skew-product endomorphisms T over countable shifts (introduced in [17]); the first coordinate map of F_T is only piecewise differentiable on countably many pieces and non-conformal, while the second coordinate map consists of holomorphic or antiholomorphic parametrized functions. F_T is also piecewise hyperbolic on a non-compact set in $\mathbb{C}^2$. Using this method, for each endomorphism F_T from $\mathcal{E}$ we obtain a family of parametrized countably generated **fiber Julia sets** $J_{T,\omega}$, where the parameter ω belongs to a symbolic space Σ_I^+ with countable alphabet I.

We show first that the projections of conditional measures of arbitrary equilibrium measures are **exact dimensional** on the fiber Julia sets $J_{T,\omega}$, and find their dimensions.

Then, we study the geometric properties and Hausdorff dimension for the global invariant measures ν on the **global basic sets** J_T in $\mathbb{C}^2$, formed by taking the union of all fiber Julia sets of type $\{z\} \times J_{T,\omega}$, for $z = \tilde{\pi}(\omega)$, where $\tilde{\pi}$ is a coding map. We prove that the projections ν of equilibrium measures to these global basic sets J_T are **exact dimensional**.

Research of the first author supported in part by grant PN-III-P4-ID-PCE-2020-2693 from UEFISCDI Romania. Research of the second author supported in part by NSF Grant DMS 0400481.

Moreover, we find the **general formula** for the Hausdorff dimension of ν (and thus for its pointwise and box dimensions), which involves the Lyapunov exponents and marginal entropies. Global exact dimensionality is an important geometric property for these fractal measures.

We introduce also a special endomorphism $F \in \mathcal{E}$, which generates a new class of complex continued fractions with **digits from an open set** in $\mathbb{C}$. These are different from the complex continued fractions with integer digits of [9]. The second coordinate map of F is holomorphic of two complex variables. We study the dimensions of projections of equilibrium measures and the distribution of these complex continued fractions.

In general, for arbitrary endomorphisms $F_T \in \mathcal{E}$ and **geometric potentials** $\psi_{T,s}$ with associated measures $\nu_{T,s}^\omega$ on $J_{T,\omega}$ we show that $HD(\nu_{T,s}^\omega) = \delta_{T,s}$ for $\mu_{T,s}^+$-a.e $\omega \in \Sigma_I^+$, and that $\delta_{T,s}$ depends **real-analytically** on the parameter s. Also, we prove a **Variational Principle** for dimension on the fiber Julia sets $J_{T,\omega}$.

Dynamics of endomorphisms and the dimensions of their invariant sets and measures were studied for eg in [1], [5], [13], [12], [15], [17], [18], [21], [22], [23], to mention a few. Several differences appear for the dynamics of non-invertible transformations, as compared to the case of diffeomorphisms. In dimension theory, [20] Ruelle expressed first the Hausdorff dimension of an invariant repelling fractal set, as the zero of a pressure function. In [21], [22] he showed also that for endomorphisms (i.e non-invertible maps) the local unstable manifolds depend on the prehistories of points, and studied invariant measures on the spaces of prehistories. In [23] Young found a formula involving Lyapunov exponents, for the pointwise dimension of a hyperbolic measure μ (i.e μ has only non-zero Lyapunov exponents) invariant to a smooth diffeomorphism of a surface; this proves also the exact dimensionality of the measure μ. In [8] Manning found the formula for the Hausdorff dimension of an ergodic f-invariant measure on the Julia set of an analytic endomorphism f on the Riemann sphere $\mathrm{P}^1\mathbb{C}$ which is hyperbolic and has no critical points in the Julia set. Barreira, Pesin and Schmeling proved in [1] a property of local product structure for invariant hyperbolic measures under smooth diffeomorphisms, which allows to compute the pointwise dimension of the measure as the sum of stable and unstable pointwise dimensions. In [18] Pesin studied Carathéodory constructions as a way to present uniformly many analytic concepts such as dimension, entropy, etc, and investigated further applications of thermodynamic formalism in dimension theory. In [12] Mihailescu studied the local behavior and dimension for conditional measures on stable manifolds for hyperbolic endomorphisms. In [5] Fornaess and Mihailescu studied hyperbolic saddle sets of holomorphic endomorphisms on $\mathbb{P}^2\mathbb{C}$ and estimated the pointwise dimensions of equilibrium measures using the number of preimages. Also the dynamics of countable conformal iterated function systems and their applications were studied for eg in [6], [9], [10], [11], [16]. There are several aspects for countable iterated systems which are different from the case of finite iterated systems or from the dynamics of smooth transformations. In [9] Mauldin and Urbański studied the countable system of complex continued fractions with integer digits (introduced in [6]) from the point of view of Hausdorff and packing measures, and the Hausdorff dimension of their limit set.

In our current paper, we study a new class of endomorphisms on non-compact spaces, denoted by $\mathcal{E}$ and their invariant measures. In particular, a class of complex continued fractions with non-integer digits, different from the one in [9]. Our main results are Theorem 2.1 and its generalization Theorem 2.2, and Theorem 3.1 for global invariant measures of F_T. In Corollary 3.2 we prove exact dimensionality of global invariant measures for the map F which generates complex continued fractions with complex digits. In Corollary 3.3 we study geometric potentials $\psi_{T,s}$ for F_T and show that the dimensions of fiber measures depend real-analytically on s, and prove a Variational Principle for dimension.

In general, we take a conformal Smale skew-product endomorphism T as in Definition 1.6, with alphabet $E = I = \mathbb{N}^* \times \mathbb{N}^*$, and Y is the closure of a bounded open set in $\mathbb{C}$ and for every $\omega \in \Sigma_I^+$ the set Y_ω is equal to Y. So the conformal contractions in fibers are

$$T_\omega : Y \to Y, \quad \omega \in \Sigma_I^+$$

As $Y \subset \mathbb{C}$, the maps T_ω are injective and holomorphic or antiholomorphic, for all $\omega \in \Sigma_I^+$. Consider now the double representation in continued fractions for the irrational coordinates of points in $(1,\infty) \times (1,\infty)$ given by,

$$\tilde{\pi} : \Sigma_I^+ \to (1,\infty) \times (1,\infty),$$

$$\tilde{\pi}(\omega) = \tilde{\pi}\big(((m_0,n_0),(m_1,n_1),\ldots)\big) := \pi_1(\omega) + i\pi_2(\omega), \ \omega \in \Sigma_I^+,$$

where,

$$\pi_1(\omega) := m_0 + \frac{1}{m_1 + \frac{1}{m_2+\ldots}}, \text{ and } \ \pi_2(\omega) := n_0 + \frac{1}{n_1 + \frac{1}{n_2+\ldots}}$$

Then, given the conformal Smale endomorphism T as above, the new endomorphism F_T is defined on the unbounded open set $(1,\infty) \times (1,\infty) \times Y \ \subset \mathbb{C}^2$, such that if $z = \tilde{\pi}(\omega)$ (which happens if $Re(z), Im(z) \notin \mathbb{Q}$) and if $w \in Y$, then:

$$F_T : (1,\infty) \times (1,\infty) \times Y \longrightarrow (1,\infty) \times (1,\infty) \times Y,$$

$$(1.1) \qquad F_T(z,w) = \begin{cases} \Big(\frac{1}{\{Re(z)\}} + \frac{i}{\{Im(z)\}},\ T_\omega(w)\Big), & \text{if } Re(z) \notin \mathbb{Q}, Im(z) \notin \mathbb{Q}, z = \tilde{\pi}(\omega), \\ \\ (2,2,\frac{1}{2}) \qquad\qquad\qquad\qquad , & \text{if } Re(z) \in \mathbb{Q} \text{ or } Im(z) \in \mathbb{Q}, \end{cases}$$

where in general the fractional part of a positive number x is $\{x\} := x - [x]$.

By Definition 1.7, $\mathcal{E}$ is the **collection** of endomorphisms F_T for all conformal Smale skew-products T as above.

In case $Y = \overline{B(\frac{1}{2},\frac{1}{2})}$ is the closure of the open disk $B(\frac{1}{2},\frac{1}{2})$ in $\mathbb{C}$ centered at $(\frac{1}{2},0)$ and of radius $\frac{1}{2}$, then for any $\omega \in \Sigma_I^+$ we define the holomorphic map $T_\omega : Y \to Y$ by:

$$T_\omega(w) = \frac{1}{w + \tilde{\pi}(\omega)}$$

We will show in Section 2, that the maps T_ω form a conformal Smale skew-product T, and we denote the coresponding endomorphism F_T by F. This endomorphism F generates our class of *complex continued fractions with complex digits*, which are different from the complex continued fractions with integer digits of [9]. The fiber limit Julia set J_ω corresponds to a countable system of holomorphic maps depending on ω. In Theorem 2.1 we prove that the projections of conditional measures of arbitrary equilibrium measures for the endomorphism F, are *exact dimensional on* J_ω in stable fibers, and we find the *formula* for their Hausdorff dimension. Properties of conditional measures on stable fibers of endomorphisms were studied also in other settings (for eg in [12]). Then, in Theorem 2.2 we generalize this result to arbitrary endomorphisms F_T from $\mathcal{E}$.

In Section 3, in Theorem 3.1 we study the global endomorphism F_T associated to a conformal Smale skew-product T. If the complex coordinates of the map F_T are written as $F_T = (F_{T,1}, F_{T,2})$, then $F_{T,1}$ is a piecewise differentiable and piecewise hyperbolic map, on countably many pieces, and $F_{T,2}(z, w)$ is a conformal map. We prove a global Volume Lemma for the projection measures ν on the product space $(1, \infty) \times (1, \infty) \times B(\frac{1}{2}, \frac{1}{2})$. This is used to show that the measures ν are *exact dimensional on the global basic set* $J_T \subset (1, \infty) \times (1, \infty) \times B(\frac{1}{2}, \frac{1}{2})$. Moreover, in Theorem 3.1 we find the *general formula for the Hausdorff dimension* of the measures of type ν, which involves their Lyapunov exponents and the marginal entropies.

The proof of the global Theorem 3.1 is rather complicated by the fact that the first complex coordinate map $F_{T,1}$ is only piecewise differentiable on countably many pieces and it is not conformal on these pieces. Moreover, the measure ν can have non-equal Lyapunov exponents along the axes in the z-plane. The second coordinate map $F_{T,2}$ is conformal and the iterates coming from countably many n-preimages generate parametrized Julia sets $J_{T,\omega}$ in the w-fibers. All these facts require a proof with several steps of the global Volume Lemma, in which the generic points play an important role. Thus our setting and methods here are different from those in [17]. In particular, in Corollary 3.2 we obtain the exact dimensionality and formula for the dimension of global measures ν on the fractal set of *complex continued fractions* with complex digits.

Then, in Corollary 3.3 we study measures $\mu_{T,s}$ associated to *geometric potentials* $\psi_{T,s}$, and the measures $\nu^\omega_{T,s}$ on Julia sets $J_{T,\omega}$ associated to the endomorphism F_T. We prove that for $\mu^+_{T,s}$-a.e $\omega \in \Sigma_I^+$, $\nu^\omega_{T,s}$ is exact dimensional on $J_{T,\omega}$ and $HD(\nu^\omega_{T,s}) = \delta_{T,s}$. We show that $\delta_{T,s}$ varies *real-analytically* with the parameter s. In the end, we prove a *Variational Principle* for dimension on the fiber Julia sets $J_{T,\omega}$.

Exact dimensionality of a measure is an important property, and if a measure is exact dimensional, then its various dimensions (pointwise, Hausdorff, box; for eg [4] for definitions) do coincide ([23]). Notice that, differently from the setting of smooth diffeomorphisms on compact manifolds of [1] and [23], in our setting the non-invertible transformation F_T acts on a non-compact manifold, and it is only piecewise differentiable on countably many pieces.

Let us now recall the notion of *pointwise dimension* for a measure, and that of *exact dimensional measure* (for eg [18]).

Definition 1.1. *Let a measure μ on a metric space X, and for a point $x \in X$ define the upper pointwise dimension, respectively the lower pointwise dimension of μ at x by*

$$\overline{\delta}(\mu)(x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}, \text{ and } \underline{\delta}(\mu)(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

If $\overline{\delta}(\mu)(x) = \underline{\delta}(\mu)(x)$, then we say that the pointwise dimension of μ exists at x, and we denote it by $\delta(\mu)(x)$.
A measure μ is called exact dimensional on X, if the pointwise dimension of μ exists for μ-a.e $x \in X$, and $\delta(\mu)(\cdot)$ is constant μ-a.e.; this common value is denoted by $\delta(\mu)$.

In the sequel, we will use the notion of *Smale skew product endomorphism*, introduced and studied in [17]. Below are briefly recalled some notions, notations, and results. Let E be a countable alphabet. Given $\beta > 0$, the metric d_β on $E^{\mathbb{N}}$ is:

$$d_\beta\big((\omega_n)_0^\infty, (\tau_n)_0^\infty\big) = \exp\big(-\beta \max\{n \geq 0 : (0 \leq k \leq n) \Rightarrow \omega_k = \tau_k\}\big)$$

with the standard convention that $e^{-\infty} = 0$. All metrics d_β, $\beta > 0$, on $E^{\mathbb{N}}$ are Hölder continuously equivalent and induce the product topology on $E^{\mathbb{N}}$. Let

$$\Sigma_E^+ = \{(\omega_n)_0^\infty : \forall_{n \in \mathbb{N}}, \omega_n \in E\}$$

The shift map $\sigma : \Sigma_E^+ \to \Sigma_E^+$ is defined by $\sigma\big((\omega_n)_0^\infty\big) = \big((\omega_{n+1})_{n=0}^\infty\big)$, For every finite word $\omega = \omega_0\omega_1 \ldots \omega_{n-1}$, put $|\omega| = n$ the length of ω, and

$$[\omega] = \{\tau \in \Sigma_E^+, \forall_{(0 \leq j \leq n-1)} : \tau_j = \omega_j\}$$

is the *cylinder* determined by ω. If $\psi : \Sigma_E^+ \to \mathbb{R}$ is continuous, define the pressure $\mathrm{P}(\psi)$ by

$$\mathrm{P}(\psi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\big(\sup\big(S_n\psi|_{[\omega]}\big)\big)$$

and the above limit does exist, since the sequence $\log \sum_{|\omega|=n} \exp\big(\sup\big(S_n\psi|_{[\omega]}\big), n \in \mathbb{N}$, is subadditive. A function $\psi : \Sigma_E^+ \longrightarrow \mathbb{R}$ is called *summable* if

$$\sum_{e \in E} \exp\big(\sup\big(\psi|_{[e]}\big)\big) < \infty$$

A shift-invariant probability Borel measure μ on Σ_E^+ is called a *Gibbs state* of ψ, if there exist constants $C \geq 1$ and $\mathrm{P} \in \mathbb{R}$ such that

$$C^{-1} \leq \frac{\mu([\omega])}{\exp(S_n\psi(\tau) - \mathrm{P}n)} \leq C \tag{1.2}$$

for all $n \geq 1$, all words ω of length n and all $\tau \in [\omega]$. From (1.2) is easy to see that, if ψ admits a Gibbs state, then $P = \mathrm{P}(\psi)$. A measure which attains the supremum in the Variational Principle is called an *equilibrium measure* for ψ (compare also with [2], [7]).

Recall from [17] also some results from thermodynamic formalism of 2-sided shifts (Σ_E, σ) on countable alphabets. Again E is a countable set. For $\beta > 0$ the metric d_β on $E^{\mathbb{Z}}$ is

$$d_\beta\big((\omega_n)_{-\infty}^{+\infty}, (\tau_n)_{-\infty}^{+\infty}\big) = \exp\big(-\beta \max\{n \geq 0 : \forall_{k \in \mathbb{Z}} |k| \leq n \Rightarrow \omega_k = \tau_k\}\big)$$

with $e^{-\infty} = 0$. All metrics d_β, $\beta > 0$, on $E^{\mathbb{Z}}$ induce the product topology on $E^{\mathbb{Z}}$. We set

$$\Sigma_E = \big\{(\omega_n)_{-\infty}^{+\infty}, \omega_n \in E, \forall n \in \mathbb{Z}\big\}$$

Hölder continuity is defined similarly as before, for potentials $\psi : \Sigma_E \to \mathbb{R}$. For any $\omega \in \Sigma_E$ and $-\infty \le m \le n \le +\infty$, define the truncation between the m and n positions as:

$$\omega|_m^n = \omega_m \omega_{m+1} \dots \omega_n.$$

If $\tau \in \Sigma_E$, let the cylinder from m to n positions,

$$[\tau]_m^n = \{\omega \in \Sigma_E, \omega|_m^n = \tau|_m^n\}$$

The family of cylinders from m to n is denoted by $\mathcal{C}_m^n$. Let $\psi : \Sigma_E \to \mathbb{R}$ be a continuous function. Then the *topological pressure* $\mathrm{P}(\psi)$ is,

$$\mathrm{P}(\psi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{[\omega]\in\mathcal{C}_0^{n-1}} \exp\big(\sup\big(S_n\psi|_{[\omega]}\big)\big), \tag{1.3}$$

where the limit above exists by subadditivity. A shift-invariant Borel probability μ on Σ_E is called a *Gibbs measure* of ψ if there are constants $C \ge 1$, $P \in \mathbb{R}$ such that

$$C^{-1} \le \frac{\mu([\omega|_0^{n-1}])}{\exp(S_n\psi(\omega) - Pn)} \le C \tag{1.4}$$

for all $n \ge 1, \omega \in \Sigma_E$. From (1.4), it follows that if ψ has a Gibbs state, then automatically $P = \mathrm{P}(\psi)$. As before, $\psi : \Sigma_E \to \mathbb{R}$ is called *summable* if

$$\sum_{e\in E} \exp\big(\sup\big(\psi|_{[e]}\big)\big) < \infty$$

In [17] we proved the following results:

Proposition 1.2. *A Hölder continuous function $\psi : \Sigma_E \to \mathbb{R}$ is summable if and only if $P(\psi) < \infty$.*

Theorem 1.3. [17] *For every Hölder continuous summable potential $\psi : \Sigma_E \to \mathbb{R}$ there exists a unique Gibbs state μ_ψ on Σ_E, and the measure μ_ψ is ergodic.*

Theorem 1.4 (Variational Principle for Two-Sided Shifts, [17])**.** *Suppose that $\psi : \Sigma_E \to \mathbb{R}$ is a Hölder continuous summable potential. Then*

$$\sup\left\{\mathrm{h}_\mu(\sigma) + \int_{\Sigma_\mathrm{E}} \psi \mathrm{d}\mu : \mu\circ\sigma^{-1} = \mu \quad and \quad \int \psi \mathrm{d}\mu > -\infty\right\} = \mathrm{P}(\psi) = \mathrm{h}_{\mu_\psi}(\sigma) + \int_{\Sigma_E} \psi d\mu_\psi,$$

and μ_ψ is the only measure at which this supremum is attained.

Equilibrium states for ψ are defined as before. Many examples of Hölder continuous summable potentials and their equilibrium measures for EMR maps (like the Gauss transformation for continued fractions) were given in [17].

Consider now the partition of Σ_E into cylinders determined by the nonnegative indices:

$$\mathcal{P}_- = \{[\eta|_0^\infty] : \eta \in \Sigma_E\} = \{[\omega] : \omega \in \Sigma_E^+\}.$$

$\mathcal{P}_-$ is a measurable partition of Σ_E. If μ is a Borel probability on Σ_E, consider the canonical system of conditional measures associated to the partition $\mathcal{P}_-$ (see [19]), denoted by

$$\{\overline{\mu}^\tau : \tau \in \Sigma_E\}$$

Then $\overline{\mu}^\tau$ is a probability measure on the cylinder $[\tau|_0^\infty]$ and we denote by $\overline{\mu}^\omega$, $\omega \in \Sigma_E^+$, the conditional measure on $[\omega]$. The truncation map to non-negative indices is:

$$\pi_0 : \Sigma_E \to \Sigma_E^+, \ \pi_0(\tau) = \tau|_0^\infty, \tau \in \Sigma_E,$$

The system above $\{\overline{\mu}^\omega, \omega \in \Sigma_E^+\}$ of conditional measures is uniquely determined up to measure zero by the property (see [19]) that, for all $g \in L^1(\mu)$,

$$\int_{\Sigma_E} g \, d\mu = \int_{\Sigma_E^+} \int_{[\omega]} g \, d\overline{\mu}^\omega \, d(\mu \circ \pi_0^{-1})(\omega).$$

Recall now the notion of *Smale skew product endomorphisms* from [17].

Definition 1.5. *Let (Y, d) be a complete bounded metric space. For any $\omega \in \Sigma_E^+$ let $Y_\omega \subset Y$ be an arbitrary set and $T_\omega : Y_\omega \longrightarrow Y_{\sigma(\omega)}$ a continuous injective map. Define*

$$\hat{Y} := \bigcup_{\omega \in \Sigma_E^+} \{\omega\} \times Y_\omega \subset \Sigma_E^+ \times Y.$$

Define the map $T : \hat{Y} \longrightarrow \hat{Y}$ by $T(\omega, y) = (\sigma(\omega), T_\omega(y))$. The pair $(\hat{Y}, T : \hat{Y} \to \hat{Y})$ is called a skew product Smale endomorphism if there exists $\lambda > 1$ such that T is uniformly contracting on fibers, i.e for all $\omega \in \Sigma_E^+$ and all $y_1, y_2 \in Y_\omega$,

$$d(T_\omega(y_2), T_\omega(y_1)) \leq \lambda^{-1} d(y_2, y_1) \tag{1.5}$$

For an arbitrary $\tau \in \Sigma_E$ let,

$$T_\tau^n := T^n_{\tau|_{-n}^\infty} := T_{\tau|_{-1}^\infty} \circ T_{\tau|_{-2}^\infty} \circ \ldots \circ T_{\tau|_{-n}^\infty} : Y_{\tau|_{-n}^\infty} \longrightarrow Y_{\tau|_0^\infty}$$

Then the sets $\big(T_\tau^n\big(Y_{\tau|_{-n}^\infty}\big)\big)_{n=0}^\infty$ form a descending sequence, and $\mathrm{diam}\big(\overline{T_\tau^n\big(Y_{\tau|_{-n}^\infty}\big)}\big) \leq \lambda^{-n}\mathrm{diam}(Y)$. But the space (Y, d) is complete, hence the set

$$\bigcap_{n=1}^\infty \overline{T_\tau^n\big(Y_{\tau|_{-n}^\infty}\big)}$$

is a point denoted by $\hat{\pi}_2(\tau)$. In this way we define the map

$$\hat{\pi}_2 : \Sigma_E \longrightarrow Y \tag{1.6}$$

Define also the map $\hat{\pi} : \Sigma_E \to \Sigma_E^+ \times Y$ by

$$\hat{\pi}(\tau) = \big(\tau|_0^\infty, \hat{\pi}_2(\tau)\big), \tag{1.7}$$

and the truncation to non-negative indices by $\pi_0 : \Sigma_E \longrightarrow \Sigma_E^+$, $\pi_0(\tau) = \tau|_0^\infty$. Now for arbitrary $\omega \in \Sigma_E^+$ denote the $\hat{\pi}_2$-projection in Y of the cylinder $[\omega] \subset \Sigma_E$, by

$$J_{T,\omega} := \hat{\pi}_2([\omega]), \tag{1.8}$$

and call these sets the *stable Smale fibers* of T, or the *parametrized Julia sets.*

The global basic set associated to T is:

$$\Lambda_T := \hat{\pi}(\Sigma_E) = \bigcup_{\omega \in \Sigma_E^+} \{\omega\} \times J_\omega \subset \Sigma_E^+ \times Y, \tag{1.9}$$

and is called the *Smale space* induced by T. The dynamical system on this global limit set,

$$T : \Lambda_T \longrightarrow \Lambda_T,$$

is called the *skew product Smale endomorphism* generated by $T : \hat{Y} \longrightarrow \hat{Y}$.

We will work in the sequel with a certain type of Smale skew-product endomorphism:

Definition 1.6 ([17]). *Assume now more about the spaces Y_ω, $\omega \in \Sigma_E^+$, and the fiber maps $T_\omega : Y_\omega \to Y_{\sigma(\omega)}$ from Definition 1.5, namely:*

(a) *There is a closed bounded set $Y \subset \mathbb{C}$ s.t $\overline{\text{Int}(Y)} = Y$ and $Y_\omega = Y$, for any $\omega \in \Sigma_I^+$.*
(b) *Each map $T_\omega : Y \to Y$ extends to a C^1 conformal embedding from Y^* to Y^*, where Y^* is a bounded connected open subset of $\mathbb{C}$ containing Y. Then T_ω denotes also this extension and assume that the maps $T_\omega : Y^* \to Y^*$ satisfy:*
(c) *Formula (1.5) holds for all $y_1, y_2 \in Y^*$, perhaps with some smaller constant $\lambda > 1$.*
(d) *(Bounded Distortion Property 1) There are constants $\alpha > 0, H > 0$ s.t $\forall y, z \in Y^*$,*

$$\big| \log |T'_\omega(y)| - \log |T'_\omega(z)| \big| \leq H ||y - z||^\alpha.$$

(e) *The function $\Sigma_E \ni \tau \longmapsto \log |T'_\tau(\hat{\pi}_2(\tau))| \in \mathbb{R}$ is Hölder continuous.*
(f) *(Open Set Condition) For every $\omega \in \Sigma_E^+$ and for all $a, b \in E$ with $A_{a\omega_0} = A_{b\omega_0} = 1$ and $a \neq b$, we have $T_{a\omega}(\text{Int}(Y)) \cap T_{b\omega}(\text{Int}(Y)) = \emptyset$.*
(g) *(Strong Open Set Condition) There exists a measurable function $\delta : \Sigma_E^+ \to (0, \infty)$, so that for every $\omega \in \Sigma_E^+$, $J_\omega \cap \big(Y \setminus \overline{B}(Y^c, \delta(\omega)\big) \neq \emptyset$.*

*A skew product Smale endomorphism satisfying conditions (a)–(g) will be called in the sequel a **conformal skew-product Smale endomorphism**.*

Also without loss of generality we give the same name to the generalized conformal skew product Smale endomorphisms of [17], namely endomorphisms which are coded by endomorphisms as above on $\Sigma_I^+ \times Y$.

We can now define formally the collection $\mathcal{E}$ of endomorphisms F_T,

Definition 1.7. *Denote by $\mathcal{E}$ the collection of all endomorphisms F_T, defined by 1.1 for all conformal skew-products Smale endomorphisms T from Definition 1.6.*

2. Conditional measures on parametrized fibers.

We start our investigation with the complex continued fractions generated by maps of type $z \to \frac{1}{z+w}$, for arbitrary parameters $w \in \mathbb{C}$ with both real and imaginary parts irrational and larger than 1. Let $X := (1, \infty) \times (1, \infty) \subset \mathbb{C}$, and the countable alphabet

$$I = \mathbb{N}^* \times \mathbb{N}^*$$

Define the coding map for continued fractions in X, for $\omega = ((m_0, n_0), (m_1, n_1), \ldots) \in \Sigma_I^+$,

$$\tilde{\pi} : \Sigma_I^+ \to X, \tag{2.1}$$

$$\tilde{\pi}(\omega) = \tilde{\pi}\big(((m_0, n_0), (m_1, n_1), \ldots)\big) := \pi_1(\omega) + i\pi_2(\omega),\ \omega \in \Sigma_I^+,$$

where,

$$\pi_1(\omega) := m_0 + \frac{1}{m_1 + \frac{1}{m_2 + \ldots}}, \quad \text{and} \quad \pi_2(\omega) := n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \ldots}}$$

Now consider the disk in $\mathbb{C}$ centered at $(\frac{1}{2}, 0)$ of radius 0.55, denoted by $B(\frac{1}{2}, 0.55)$, and let the holomorphic contraction $T_\omega : B(\frac{1}{2}, 0.55) \to B(\frac{1}{2}, 0.55)$ defined by

$$T_\omega(z) = \frac{1}{z + \tilde{\pi}(\omega)} \tag{2.2}$$

We see from above that T_ω is well defined and holomorphic and that it is a contraction, with contraction factor independent of $\omega \in \Sigma_I^+$. Indeed, if we denote $V := B(\frac{1}{2}, 0.55)$, and $z = x + iy \in V$, then $z + \tilde{\pi}(\omega) = x + \pi_1(\omega) + i(y + \pi_2(\omega)) = x' + iy'$, and

$$\begin{aligned} |\frac{1}{z + \tilde{\pi}(\omega)} - \frac{1}{2}|^2 = |\frac{x' - iy'}{x'^2 + y'^2} - \frac{1}{2}|^2 = |\frac{x'}{x'^2 + y'^2} - \frac{1}{2} - i\frac{y'}{x'^2 + y'^2}|^2 \\ = (\frac{x'}{x'^2 + y'^2} - \frac{1}{2})^2 + \frac{y'^2}{(x'^2 + y'^2)^2} = \frac{1}{x'^2 + y'^2} - \frac{x'}{x'^2 + y'^2} + \frac{1}{4} = \frac{1 - x'}{x'^2 + y'^2} + \frac{1}{4} \end{aligned}$$

But now since $z \in V$ and $\pi_1(\omega) > 1, \pi_2(\omega) > 1$, it follows that $x' \in (0.95, \infty)$ and $y' \in (0.45, \infty)$. Hence $x'^2 + y'^2 > 1.1$ and $1 - x' < 0.05$, so

$$\frac{1 - x'}{x'^2 + y'^2} < \frac{0.05}{1.1} < 0.55^2 - \frac{1}{4} = 0.0525$$

Thus $T_\omega(z) \in V = B(\frac{1}{2}, 0.55)$, for any $z \in V, \omega \in \Sigma_I^+$. Recall that the disk in $\mathbb{C}$ of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$ is denoted by $B(\frac{1}{2}, \frac{1}{2})$, and denote

$$Y := \overline{B(\frac{1}{2}, \frac{1}{2})}$$

Let the contractions $T_\omega,\ \omega \in \Sigma_I^+$ only on Y. Since for any $z \in Y$, $Re(z + \tilde{\pi}(\omega)) \geq 1$, it follows that $T_\omega(z) \in \overline{B(\frac{1}{2}, \frac{1}{2})}$. We define now for every $\omega \in \Sigma_I^+$ and $\tau \in \Sigma_I$ with $\tau|_0^\infty = \omega$,

$$T_\tau^n = T_{\tau|_{-1}^\infty} \circ T_{\tau|_{-2}^\infty} \circ \ldots \circ T_{\tau|_{-n}^\infty}$$

Then define the **parametrized Julia set** J_ω corresponding to T from (2.2) and to any $\omega \in \Sigma_I^+$, as the set of points in $\mathbb{C}$ of type

$$J_\omega := \{ \bigcap_{n \geq 1} \overline{T_{\tau|_n^\infty}(Y)},\ \tau \in [\omega] \} \tag{2.3}$$

This is similar to the case of expanding maps from one-dimensional complex dynamics ([3]).

The union in $\mathbb{C}^2$ of the family of fiber Julia sets $\{\tilde{\pi}(\omega)\} \times J_\omega$ for $\omega \in \Sigma_I^+$, is called the **global basic set** of F,

$$J := \bigcup_{\omega \in \Sigma_I^+} \{\tilde{\pi}(\omega)\} \times J_\omega \ \subset \mathbb{C}^2 \tag{2.4}$$

Recall now that $\pi_0 : \Sigma_I \to \Sigma_I^+$ is the truncation of $\tau \in \Sigma_I$ to $\tau|_0^\infty$; also the projection $\hat{\pi}_2$ was defined in (1.6).

Theorem 2.1. *Consider the above system of complex continued fractions given for every $\omega \in \Sigma_I^+$ by the holomorphic map $T_\omega : \overline{B(\frac{1}{2}, \frac{1}{2})} \to \overline{B(\frac{1}{2}, \frac{1}{2})}$, $T_\omega(z) = \frac{1}{z + \tilde{\pi}(\omega)}$. Let also $\psi : \Sigma_I \to \mathbb{R}$ be a Hölder continuous summable potential with equilibrium measure μ_ψ.*

Then, for $\pi_{0*}\mu_\psi$-a.e $\omega \in \Sigma_I^+$, the projection $\hat{\pi}_{2*}\bar{\mu}_\psi^\omega$ of the conditional measure $\bar{\mu}_\psi^\omega$ is exact dimensional on J_ω, and its dimension is:

$$HD(\hat{\pi}_{2*}\bar{\mu}_\psi^\omega) = \frac{h_{\mu_\psi}(\sigma)}{\chi_{\mu_\psi}}$$

Proof. First we prove the uniform contraction for all the maps $T_\omega, \omega \in \Sigma_I^+$. Indeed, $T'_\omega(z) = -\frac{1}{(z+\tilde{\pi}(\omega))^2}$, and write $z+\tilde{\pi}(\omega) = x+\pi_1(\omega)+i(y+\pi_2(\omega))$. But $x > -0.05, y > -0.55$, and $\pi_1(\omega) > 1, \pi_2(\omega) > 1$ since $m_0, n_0 \geq 1$, so

$$|z+\tilde{\pi}(\omega)|^2 = (x+\pi_1(\omega))^2 + (y+\pi_2(\omega))^2 > 0.95^2 + 0.45^2 > 1.1$$

Therefore for all $z \in V, \omega \in \Sigma_I^+$, we have

$$|T'_\omega(z)| > \sqrt{1.1} > 1$$

The advantage of letting T_ω on Y is that we have also Open Set Condition. Indeed, if $(m_0, n_0) \neq (m'_0, n'_0)$, then $[B(\frac{1}{2}, \frac{1}{2}) + m_0 + in_0] \cap [B(\frac{1}{2}, \frac{1}{2}) + m'_0 + in'_0] = \emptyset$. As the alphabet I is $\mathbb{N}^* \times \mathbb{N}^*$, the Julia limit set J_ω is not contained in the boundary of Y. Thus the Strong Open Set Condition from the definition of conformal Smale skew products ([17]) is satisfied. Now, we show that the Bounded Distortion Property (BDP) is satisfied. Let us mention first that this follows from Koebe Distortion Theorem (for eg [3]). However we give also a direct proof, in which the constants of distortion are obtained. Clearly $T'_\omega(z) = -\frac{1}{(z+\tilde{\pi}(\omega))^2}$ for any $\omega \in \Sigma_I^+$. For a fixed $\omega \in \Sigma_I^+$, denote by $\gamma_1 + i\gamma_2 := \tilde{\pi}(\omega)$ with $\gamma_1, \gamma_2 \in \mathbb{R}$, and by

$$f(x,y) = \log((x+\gamma_1)^2 + (y+\gamma_2)^2),\ z = x+iy \in \overline{B(\frac{1}{2}, \frac{1}{2})}$$

Then

$$Df(x,y) = \Big(\frac{2(x+\gamma_1)}{(x+\gamma_1)^2+(y+\gamma_2)^2}, \frac{2(y+\gamma_2)}{(x+\gamma_1)^2+(y+\gamma_2)^2}\Big)$$

Now if $z = x+iy \in \overline{B(\frac{1}{2}, \frac{1}{2})}$, it follows that $0 \leq x \leq 1$ and $-\frac{1}{2} \leq y \leq \frac{1}{2}$. Since $\gamma_1, \gamma_2 \geq 1$, $(x+\gamma_1)^2 + (y+\gamma_2)^2 > (x+\gamma_1)^2$, hence $\frac{2(x+\gamma_1)}{(x+\gamma_1)^2+(y+\gamma_2)^2} < \frac{2}{|x+\gamma_1|} < 2$. But since $y+\gamma_2 \geq \frac{1}{2}$,

$$\frac{2(y+\gamma_2)}{(x+\gamma_1)^2+(y+\gamma_2)^2} \leq \frac{2}{|y+\gamma_2|} \leq 4$$

Hence we obtain the BDP, namely for any $z, z' \in \overline{B(\frac{1}{2}, \frac{1}{2})}$,

$$| \log |T'_\omega(z)| - \log |T_\omega(z')| \leq 8|z-z'| \tag{2.5}$$

Also,

$$|T_\omega(z) - T_\eta(z)| = |\frac{1}{z+\tilde{\pi}(\omega)} - \frac{1}{z+\tilde{\pi}(\eta)}| = |\frac{\tilde{\pi}(\eta)-\tilde{\pi}(\omega)}{(z+\tilde{\pi}(\omega))(z+\tilde{\pi}(\eta))}|$$

Therefore the system $T_\omega : \overline{B(\frac{1}{2}, \frac{1}{2})} \to \overline{B(\frac{1}{2}, \frac{1}{2})}$ given by $T_\omega(z) = \frac{1}{z+\tilde{\pi}(\omega)}$, $\omega \in \Sigma_I^+$, forms a *conformal Smale skew product*. We want to show now that T is a Smale skew product of global character (for definition see [17]). Indeed for $z \in Y$ and $\omega, \eta \in \Sigma_I^+$,

$$|T_\omega(z) - T_\eta(z)| = |\frac{1}{z+\tilde{\pi}(\omega)} - \frac{1}{z+\tilde{\pi}(\eta)} = \frac{|\tilde{\pi}(\omega)-\tilde{\pi}(\eta)|}{(z+\tilde{\pi}(\omega))(z+\tilde{\pi}(\eta)}$$

But since, $\omega = (\omega_0, \omega_1, \dots) = ((m_0, n_0), (m_1, n_1), \dots)$, and $\eta = (\eta_0, \eta_1, \dots) = ((m'_0, n'_0), (m'_1, n'_1), \dots)$, with all $m_i, m'_i, n_i, n'_i, i \in \mathbb{N}$ larger than or equal to 1, it follows that $|z + \tilde{\pi}(\omega)| \geq 1, |z + \tilde{\pi}(\eta)| \geq 1$. Thus from the inequalities above,

$$|T_\omega(z) - T_\eta(z)| \leq |\tilde{\pi}(\omega) - \tilde{\pi}(\eta)| \tag{2.6}$$

Assume now that $\omega_0 = \eta_0, \dots, \omega_k = \eta_k$ but $\omega_{k+1} \neq \eta_{k+1}$, for some $k \geq 1$. Then,

$$\pi_1(\omega) - \pi_1(\eta) = \frac{\frac{1}{m_{k+1} + \frac{1}{\dots}} - \frac{1}{m'_{k+1} + \frac{1}{\dots}}}{(m_1 + \frac{1}{m_2 + \dots}) \dots (m_k + \frac{1}{m_{k+1} + \dots}) \cdot (m'_1 + \frac{1}{m'_2 + \dots}) \dots (m'_k + \frac{1}{m'_{k+1} + \dots})}$$

Notice now that, since $m_{j+1} + \frac{1}{m_{j+2} + \dots} < m_{j+1} + 1 < 2m_{j+1}$ for any $j \geq 1$, we have

$$\begin{aligned}(m_j + \frac{1}{m_{j+1} + \dots})(m_{j+1} + \frac{1}{m_{j+2} + \dots}) &> m_{j+1}(m_j + \frac{1}{m_{j+1} + \dots}) \\ &= m_j m_{j+1} + \frac{m_{j+1}}{m_{j+1} + \dots} \geq m_j m_{j+1} + \frac{1}{2}\end{aligned}$$

Hence as all m_j are nonzero natural numbers, we obtain

$$(m_j + \frac{1}{m_{j+1} + \dots})(m_{j+1} + \frac{1}{m_{j+2} + \dots}) \geq \frac{3}{2} \tag{2.7}$$

Therefore from above,

$$|\pi_1(\omega) - \pi_1(\eta)| \leq 2 \cdot (\sqrt{\frac{2}{3}})^k \tag{2.8}$$

We obtain a similar estimate for $|\pi_2(\omega) - \pi_2(\eta)|$. But if $\omega|_0^k = \eta|_0^k$ and $\omega_{k+1} \neq \eta_{k+1}$, then $d_\beta(\omega, \eta) = \beta^k$, for the metric d_β on Σ_I^+, for $\beta \in (0,1)$ fixed. Therefore from (2.8) and (2.6), it follows that there exist constants $C > 0, \beta \in (0,1)$ such that

$$|T_\omega(z) - T_\eta(z)| \leq C d_\beta(\omega, \eta) \tag{2.9}$$

Hence T is of global character, so from [17], T is a Hölder conformal Smale skew product. So the conclusion follows from Theorem 6.2 in [17].

□

In general, if F_T is associated to a conformal Smale skew-product T, then for every $\omega \in \Sigma_I^+$, let $J_{T,\omega}$ be the associated fiber Julia set of F_T given by (1.8). Namely we have,

$$J_{T,\omega} := \{ \bigcap_{n \geq 1} \overline{T_{\tau|_n^\infty}(Y)}, \ \tau \in [\omega] \} \tag{2.10}$$

Then the following generalization of Theorem 2.1 to endomorphisms F_T from $\mathcal{E}$ follows similarly as above.

Theorem 2.2. *Consider an endomorphism F_T associated to an arbitrary conformal Smale skew-product T by formula (1.1). Let $\psi : \Sigma_I \to \mathbb{R}$ be a Hölder continuous summable potential, which has an equilibrium measure μ_ψ on Σ_I^+.*

Then, for $\pi_{0*}\mu_\psi$-a.e $\omega \in \Sigma_I^+$, the projection $\hat{\pi}_{2*}\bar{\mu}_\psi^\omega$ of the conditional measure $\bar{\mu}_\psi^\omega$ of μ_ψ on $[\omega]$, is exact dimensional on the Julia set $J_{T,\omega} \subset \mathbb{C}$. Moreover, its dimension satisfies:

$$HD(\hat{\pi}_{2*}\bar{\mu}_\psi^\omega) = \frac{h_{\mu_\psi}(\sigma)}{\chi_{\mu_\psi}}.$$

3. Global exact dimensionality and dimension formulas.

We now prove that the global projections ν of equilibrium measures, are exact dimensional for the endomorphism $F_T \in \mathcal{E}$, associated to a conformal Smale skew-product T (as in Definition 1.6), and we find their dimensions. Consider thus the spaces

$$X = (1, \infty) \times (1, \infty) \subset \mathbb{C}, \text{ and } Y \subset \mathbb{C},$$

and given T, let the skew-product transformation $F_T : X \times Y \to X \times Y$,

(3.1)
$$F_T(z, w) = \Big(\frac{1}{\{Re(z)\}} + \frac{i}{\{Im(z)\}}, \ T_\omega(w)\Big), \ z \in X, Re(z) \notin \mathbb{Q}, Im(z) \notin \mathbb{Q}, \ z = \tilde{\pi}(\omega), \ w \in Y,$$

and

$$F(z, w) = (2, 2, \frac{1}{2}), \text{ if } Re(z) \in \mathbb{Q} \text{ or } Im(z) \in \mathbb{Q}.$$

Recall that for $x \in \mathbb{R}$, $\{x\} := x - [x]$. Using $\tilde{\pi}$ from (2.1), we define the coding

$$\tilde{\pi}_Y : \Sigma_I^+ \times Y \to X \times Y,$$

$$\tilde{\pi}_Y(\omega, y) = (\tilde{\pi}(\omega), y), \ (\omega, y) \in \Sigma_I^+ \times Y \tag{3.2}$$

Recall also that we denoted the truncation to non-negative indices by

$$\pi_0 : \Sigma_I \to \Sigma_I^+, \ \pi_0(\eta) = \eta|_0^\infty, \ \eta \in \Sigma_I,$$

and given the conformal Smale skew-product T, there exists from (1.7) a coding map,

$$\hat{\pi} : \Sigma_I \to \Sigma_I^+ \times Y, \ \ \hat{\pi}(\eta) = (\pi_0(\eta), \hat{\pi}_2(\eta)), \ \eta \in \Sigma_I \tag{3.3}$$

Introduce also the projection

$$\pi : \Sigma_I \to X \times Y, \ \pi := \tilde{\pi}_Y \circ \hat{\pi}, \ \pi(\eta) = (\tilde{\pi}(\eta|_0^\infty), \hat{\pi}_2(\eta)), \eta \in \Sigma_I \tag{3.4}$$

As $I = \mathbb{N}^* \times \mathbb{N}^*$, denote the canonical projections on "coordinates" of points in Σ_I^+ by,

$$p_1 : \Sigma_I^+ \to \Sigma_{\mathbb{N}^*}^+, \text{ and } p_2 : \Sigma_I^+ \to \Sigma_{\mathbb{N}^*}^+,$$

where for any $((m_0, n_0), (m_1, n_1), \ldots) \in \Sigma_I^+$, with $m_i, n_i \in \mathbb{N}^*, \ i \in \mathbb{N}$,

$$p_1((m_0, n_0), (m_1, n_1), \ldots) = (m_0, m_1, \ldots), \ \ p_2((m_0, n_0), (m_1, n_1), \ldots) = (n_0, n_1, \ldots) \tag{3.5}$$

If μ^+ is a σ-invariant measure on Σ_I^+, denote the projection measures of μ^+ on $\Sigma_{\mathbb{N}^*}^+$ by

$$\mu_1 := p_{1*}\mu^+, \text{ and } \mu_2 := p_{2*}\mu^+, \tag{3.6}$$

and call μ_1, μ_2 the **marginal mesures** of μ^+. The entropies of μ_1, μ_2 with respect to the shift on $\Sigma_{\mathbb{N}^*}^+$ are called the **marginal entropies** of μ^+.

Since we work with continued fractions, define for $n \in \mathbb{N}^*$, the contraction map,

$$\varphi_n(x) = \frac{1}{x+n}, \ x \in [0,1) \tag{3.7}$$

Denote the coding $\rho_0 : \Sigma^+_{\mathbb{N}^*} \to (0,1)$, as the representation in continued fractions,

$$\rho_0(\omega) = \frac{1}{\omega_0 + \frac{1}{\omega_1 + \dots}}, \text{ for } \omega = (\omega_0, \omega_1, \dots) \in \Sigma^+_{\mathbb{N}^*} \tag{3.8}$$

For an arbitrary σ-invariant measure μ on Σ_I, denote by $\mu^+ := \pi_{0*}\mu$ on Σ_I^+, and let μ_1, μ_2 be the measures from (3.6) associated to μ^+.

Using (3.6) denote the **marginal Lyapunov exponents of** μ^+ by:

$$\chi_1(\mu^+) := -\int_{\Sigma^+_{\mathbb{N}^*}} \log|\varphi'_{\zeta_0}(\rho_0\sigma\zeta)| \ d\mu_1(\zeta) \tag{3.9}$$

$$\chi_2(\mu^+) := -\int_{\Sigma^+_{\mathbb{N}^*}} \log|\varphi'_{\zeta_0}(\rho_0\sigma\zeta)| \ d\mu_2(\zeta)$$

If $\mu^+ = \pi_{0*}\mu$ is the projection of a measure μ on Σ_I (as in the sequel), write also $\chi_1(\mu), \chi_2(\mu)$ for $\chi_1(\mu^+), \chi_2(\mu^+)$ respectively.

Also define the **Lyapunov exponent of the endomorphism** T of (2.2) with respect to a shift-invariant probability measure μ on Σ_I by,

$$\chi_T(\mu) := -\int_{\Sigma_I} \log|T'_{\eta|_0^\infty}(\hat{\pi}_2(\eta))| \ d\mu(\eta) \tag{3.10}$$

If T is fixed, we will denote also $\chi(\mu)$ for $\chi_T(\mu)$. We notice that $\chi_1(\mu^+), \chi_2(\mu^+), \chi(\mu)$ are all positive numbers. Also let

$$\lambda_1(\mu^+) := \exp(-\chi_1(\mu^+)), \ \lambda_2(\mu^+) := \exp(-\chi_2(\mu^+)) \tag{3.11}$$

So $\lambda_1(\mu^+), \lambda_2(\mu^+) \in (0,1)$. If $\mu^+ = \pi_{0*}\mu$, write also $\lambda_1(\mu), \lambda_2(\mu)$ for $\lambda_1(\mu^+), \lambda_2(\mu^+)$.

We are now ready to prove the *global* exact dimensionality of the projection measure ν of an equilibrium measure μ, for endomorphisms from the class $\mathcal{E}$. If the endomorphism F_T is associated to a conformal Smale skew-product T, then for any $\omega \in \Sigma_I^+$ the fiber Julia set was denoted by $J_{T,\omega}$. Let J_T be the **global basic set** of F_T,

$$J_T := \bigcup_{\omega \in \Sigma_I^+} \{\tilde{\pi}(\omega)\} \times J_{T,\omega} \ \subset \mathbb{C}^2$$

The following Theorem establishes **exact dimensionality** for ν on the global Julia (basic) set J_T, and gives the **formula for its dimension**.

Theorem 3.1. *In the setting of Theorem 2.2 let the endomorphism $F_T : X \times Y \to X \times Y$ of $\mathcal{E}$ associated to a conformal Smale skew-product T by (1.1). Let $\psi : \Sigma_I \to \mathbb{R}$ be a Hölder continuous summable potential with μ_ψ its equilibrium measure on Σ_I. Denote the measures*

$$\mu^+_\psi := (\pi_0)_*\mu_\psi \ \textit{on} \ \Sigma_I^+, \ \textit{and} \ \nu_\psi := \pi_*\mu_\psi \ \textit{on} \ X \times Y$$

Let $\mu_{1,\psi}, \mu_{2,\psi}$ be the marginal measures μ_1, μ_2 from (3.6) associated to μ^+_ψ. Then:

a) The projection measure ν_ψ is exact dimensional on the global basic set $J_T \subset \mathbb{C}^2$.

b) If $\lambda_1(\mu_\psi) < \lambda_2(\mu_\psi)$, the pointwise (and Hausdorff) dimension of ν_ψ is,

$$\delta(\nu_\psi) = \frac{h_{\mu_\psi} - h_{\mu_{1,\psi}}(1 - \frac{\chi_2(\mu_\psi)}{\chi_1(\mu_\psi)})}{\chi_2(\mu_\psi)} + \frac{h_{\mu_\psi}}{\chi(\mu_\psi)}$$

c) If $\lambda_1(\mu_\psi) \geq \lambda_2(\mu_\psi)$, then

$$\delta(\nu_\psi) = \frac{h_{\mu_\psi} - h_{\mu_{2,\psi}}(1 - \frac{\chi_1(\mu_\psi)}{\chi_2(\mu_\psi)})}{\chi_1(\mu_\psi)} + \frac{h_{\mu_\psi}}{\chi(\mu_\psi)}.$$

Proof. The general idea is to take first projections of the conditional measures of the measure μ onto the fibers $J_{T,\omega}, \omega \in \Sigma_I^+$ (defined in (2.3), then to look at the projections of ν on the first complex coordinate z. At the same time, the first complex coordinate map $F_1(z,w)$, of $F(z,w)$, is only piecewise differentiable on countably many pieces and it is not conformal on these pieces, and the measure μ can have different Lyapunov exponents $\chi_1(\mu), \chi_2(\mu)$ in the two real directions in the z-plane. We will cover an arbitrarily small ball $B(z,r)$ with thin rectangles obtained as projections of n-cylinders $[\omega_1 \ldots \omega_n] \times [\eta_0 \ldots \eta_n]$ from Σ_I^+, and will estimate their number by looking at their interval projections onto the first real coordinate. The number of such intervals which are obtained by projection, will be estimated using the exact dimensionality and the formula for the Hausdorff dimension of an invariant measure for a random countable iterated system from [16].
Then, we will prove a Volume Lemma in the z-direction, namely we will add the z-projection measures of all the above rectangles in the z-direction, in order to estimate the measure of $B(z,r)$. This will give the exact dimensionality and the Hausdorff dimension of the projection of the measure ν in the z-direction. Next we will prove a global Volume Lemma for ν itself, by adding the obtained dimension in z-direction with the dimension of projections of conditional measures in w-fibers given by Theorem 2.1. This implies exact dimensionality for ν on $X \times Y$ and will give the dimension of ν.

We now provide the complete proof, emphasizing below its **main steps**:

1. Codings, notations, and the measure Υ.

It will be more convenient to work on $(0,1)\times(0,1)$ instead of X, so denote

$$Z := (0,1)\times(0,1),$$

and consider the bijective transformation

$$\theta : X \to Z,\ \theta(x,y) = (\frac{1}{x}, \frac{1}{y}),\ (x,y) \in X$$

Recalling the notation in (3.8), introduce then the coding map $\rho : \Sigma_I^+ \to Z$,

$$\rho((m_0,n_0),(m_1,n_1),\ldots) = (\frac{1}{m_0 + \frac{1}{m_1+\ldots}}, \frac{1}{n_0 + \frac{1}{n_1+\ldots}}) = (\rho_0(m_0,m_1,\ldots), \rho_0(n_0,n_1,\ldots)),$$

for every $((m_0,n_0),(m_1,n_1),\ldots) \in \Sigma_I^+$. Let us denote also

$$\rho_Y : \Sigma_I^+ \times Y \to Z \times Y,\ \rho_Y(\omega,y) = (\rho(\omega),y)$$

For the Hölder continuous summable potential $\psi : \Sigma_I \to \mathbb{R}$ from above, define the measures:

$$\vartheta_\psi := (\rho_Y)_*\mu_\psi^+ \text{ on } Z\times Y, \text{ and } \Upsilon_\psi := \rho_*\mu_\psi^+ \text{ on } Z \tag{3.12}$$

Notice that $(\theta \times id_Y)_* \nu_\psi = \vartheta_\psi$.

In the sequel, we assume the potential ψ **is fixed**, and drop the index ψ from the notations above. Thus we will denote μ for μ_ψ, ν for ν_ψ, μ^+ for μ^+_ψ, as well. Also write Υ for Υ_ψ, μ_1 for $\mu_{1,\psi}$, μ_2 for $\mu_{2,\psi}$, and ϑ for ϑ_ψ.
Recalling(3.6) denote by μ_1, μ_2 respectively the measures $\mu_{1,\psi}$ and $\mu_{2,\psi}$ (associated to μ^+_ψ). Also recalling (3.9) and (3.10) we denote χ_1 for $\chi_1(\mu^+_\psi)$, χ_2 for $\chi_2(\mu^+_\psi)$, χ for $\chi(\mu_\psi)$. And from (3.11) denote similarly λ_1 for $\lambda_1(\mu^+_\psi)$, and λ_2 for $\lambda_2(\mu^+_\psi)$.

2. Past-independent potentials and the projection measure Υ_1.

We observe now that from [2] and [17], there exists a past-independent Hölder continuous summable function ψ^+, which is cohomologous to ψ in the class of bounded Hölder continuous functions, where by *past-independent* we mean that

$$\psi^+(\eta) = \psi^+(\eta'),$$

for any $\eta, \eta' \in \Sigma_I$ with $\eta_j = \eta'_j$, for $j \geq 0$. Denote the restriction of ψ^+ to Σ_I^+ by $\tilde{\psi}^+$. Let us now look at little closer at these two probability measures μ on Σ_I and μ^+ on Σ_I^+. Since $\mu = \mu_\psi = \mu_{\psi^+}$, it follows from the Gibbs property that, for any integer $n > 1$ and for any point $\zeta \in \Sigma_I$,

$$\mu([\zeta_0 \ldots \zeta_n]) \approx \exp(S_n \psi^+(\zeta)) - nP(\psi^+)),$$

where the pressure $P(\psi^+)$ is taken with respect to the shift on Σ_I, and where the comparability constants do not depend on n, ζ.

But, on the other hand, if $\mu_{\tilde{\psi}^+}$ denotes the equilibrium measure of $\tilde{\psi}^+$ on Σ_I^+, we have from the Gibbs property that, for any $\xi \in \Sigma_I^+$, any $n > 1$ and any $\zeta \in \Sigma_I$ with $\zeta|_0^\infty = \xi$, the following estimate holds:

$$\mu_{\tilde{\psi}^+}([\xi_0 \ldots \xi_n]) \approx \exp(S_n \tilde{\psi}^+(\xi) - nP(\tilde{\psi}^+)) = \exp(S_n \psi^+(\zeta) - nP(\tilde{\psi}^+)),$$

where now the pressure $P(\tilde{\psi}^+)$ is taken with respect to the shift on Σ_I^+, and where the comparability constants do not depend on n, ξ. But clearly $P(\psi^+) = P(\tilde{\psi}^+)$. Hence from the uniqueness of Gibbs measures for given Hölder continuous summable potentials on Σ_I^+,

$$\pi_{0*}\mu_\psi = \mu^+_\psi = \pi_{0*}\mu_{\psi^+} = \mu_{\tilde{\psi}^+} \tag{3.13}$$

Also, since (Σ_I, σ, μ) is the natural extension of the system $(\Sigma_I^+, \sigma, \mu^+)$ (or by using the Brin-Katok formula and the estimates for the measure on Bowen balls)), it follows that

$$h_\mu = h_{\mu^+} \tag{3.14}$$

Now, consider the canonical projection on the X-coordinate

$$p_X : X \times Y \to X$$

Since by (3.12), $\Upsilon = (\theta \circ p_X)_* \nu = \rho_* \mu^+$ on Z, we see immediately from the definition of the pointwise dimension δ that, if $z = \theta(x, y)$ with $(x, y) \in X$, then

$$\delta(\Upsilon)(z) = \delta((p_X)_* \nu)(x, y) \tag{3.15}$$

Recall that Υ is a measure on $Z = (0,1) \times (0,1)$, and denote its canonical projection in the first coordinate on $(0,1)$ by Υ_1. Hence, in the notations of (3.5), (3.6) and (3.8),

$$\Upsilon_1 = (\rho_0)_* \mu_1 \tag{3.16}$$

Since μ^+ is σ-invariant on Σ_I^+, it follows that μ_1 is σ-invariant on $\Sigma_{\mathbb{N}^*}^+$.

3. Pointwise dimension for Υ_1.
From the definition (3.8) of the map ρ_0 as the representation of irrational numbers as continued fractions, we obtain that the measure Υ_1 is the projection on $(0,1)$ of an invariant probability measure μ_1 on $\Sigma_{\mathbb{N}^*}^+$, with respect to a countable iterated function system.
Therefore, by the main result in [16] (restricted to the case when the parameter space Λ in that paper consists of only one point), it follows that the projection measure Υ_1 is exact dimensional on $(0,1)$. Hence the pointwise dimension of Υ_1 is the same at Υ_1-a.e. point, and this common value is equal to the Hausdorff dimension of Υ_1.
Let us then denote the pointwise dimension of Υ_1 by δ_1. But in our case there are no overlaps in the countable iterated function system (open set condition for countable iterated function system is satisfied). Thus, the projectional entropy of μ_1 is the same as its usual entropy h_{μ_1}. Hence from [16], it follows that for Υ_1-a.e. point $x \in (0,1)$, we have:

$$\delta_1 = \delta(\Upsilon_1)(x) = \lim_{r \to 0} \frac{\log \Upsilon_1(B(x,r))}{\log r} = \frac{h_{\mu_1}}{\chi_1} \tag{3.17}$$

4. Geometry in the z-direction, and generic points.
Let us assume first, as in part b) of the statement, that

$$\lambda_1 < \lambda_2 \tag{3.18}$$

Consider arbitrary numbers $n > 1$ and $\varepsilon > 0$, and define the Borel set $C(n,\varepsilon) \subset [0,1)$ by:

$$C(n,\varepsilon) := \{x \in (0,1),\ \frac{\log \Upsilon_1(B(x,r))}{\log r} \in (\delta_1 - \varepsilon, \delta_1 + \varepsilon), \text{for } 0 < r \le \lambda_2^{n(1-\varepsilon)}\}$$

From the exact dimensionality of Υ_1, it follows that for every $\varepsilon > 0$ there exists an integer $n(\varepsilon) > 1$ and a positive function $\kappa(\cdot)$ with $\lim_{\varepsilon \to 0} \kappa(\varepsilon) = 0$, such that for every $n > n(\varepsilon)$,

$$\Upsilon_1(C(n,\varepsilon)) > 1 - \kappa(\varepsilon) \tag{3.19}$$

We identify in the sequel the space Σ_I^+ with $\Sigma_{\mathbb{N}^*}^+ \times \Sigma_{\mathbb{N}^*}^+$ by the map $\Psi : \Sigma_I^+ \to \Sigma_{\mathbb{N}^*}^+ \times \Sigma_{\mathbb{N}^*}^+$,

$$\Psi(\zeta) = (\omega, \eta), \tag{3.20}$$

where for an arbitrary $\zeta = ((m_0,n_0),(m_1,n_1),\ldots) \in \Sigma_I^+$, write $\omega = (m_0, m_1, \ldots) \in \Sigma_{\mathbb{N}^*}^+$, and $\eta = (n_0, n_1, \ldots) \in \Sigma_{\mathbb{N}^*}^+$.
Given the contractions $\phi_n, n \ge 1$ from (3.7), and an arbitrary sequence of integers $\omega \in \Sigma_{\mathbb{N}^*}^+$, denote by

$$\phi_\omega^n := \phi_{\omega_0} \circ \phi_{\omega_1} \circ \ldots \circ \phi_{n-1}$$

Now for arbitrary $n > 1$, $\varepsilon > 0$, define the following Borel measurable set
(3.21)

$$A(n,\varepsilon) := \{(\omega,\eta) \in \Sigma^+_{\mathbb{N}^*} \times \Sigma^+_{\mathbb{N}^*},\ |\frac{S_j\psi^+(\omega,\eta)}{j} - \int \psi^+ d\mu^+| < \varepsilon, \text{ and}$$
$$|(\phi^j_\omega)'(\rho_0\sigma^j\omega)| \in (e^{j(-\chi_1-\varepsilon)}, e^{j(-\chi_1+\varepsilon)}) \text{ and } |(\phi^j_\eta)'(\rho_0\sigma^j\eta)| \in (e^{(j(-\chi_2-\varepsilon)}, e^{j(-\chi_2+\varepsilon)}),\ \forall j \geq n\}$$

Since μ^+ and μ_1, μ_2 are ergodic with respect to the respective shift maps, and using Birkhoff Ergodic Theorem and (3.9), it follows that $\mu^+(A(n,\varepsilon))$ is close to 1 for large n. Without loss of generality we can assume that, for the functions $n(\varepsilon)$ and $\kappa(\varepsilon)$ above, we have for any $n > n(\varepsilon)$ the lower bound,

$$\mu^+(A(n,\varepsilon)) > 1 - \kappa(\varepsilon). \tag{3.22}$$

Therefore if $n > n(\varepsilon)$ then,

$$\mu_1(p_1(A(n,\varepsilon))) > 1 - \kappa(\varepsilon)$$

Let us denote now by

$$\tilde{C}(n,\varepsilon) := C(n,\varepsilon) \cap \rho_0(p_1(A(n,\varepsilon)))\ \subset (0,1)$$

Since by (3.16), $\Upsilon_1 = \rho_{0*}\mu_1$, it follows from (3.19) and (3.22) that if $n > n(\varepsilon)$,

$$\Upsilon_1(\tilde{C}(n,\varepsilon)) > 1 - 2\kappa(\varepsilon) \tag{3.23}$$

Now consider $(\omega,\eta) \in A(n,\varepsilon)$ and its ρ-projection,

$$z = \rho(\omega,\eta) = (\rho_0(\omega), \rho_0(\eta)) \in (0,1) \times (0,1)$$

We want to estimate the measure $\Upsilon(B(z, \lambda_2^{(1+\varepsilon)n}))$, by covering a sufficiently large portion of $B(z, \lambda_2^{(1+\varepsilon)n})$ with an optimal cover, consisting of ρ-projections of cylinders of type

$$[\omega'_0 \ldots \omega'_n] \times [\eta'_0 \ldots \eta'_n]$$

The Υ-measure of the projection of such a cylinder is equal to the μ^+-measure of the respective cylinder $[(\omega'_0,\eta'_0)\ldots(\omega'_n,\eta'_n)]$. Thus it can be estimated by using the Gibbs property of μ^+. Then, we shall estimate the number of such cylinders in the optimal cover, by looking at their projections on the first real coordinate, and we will take into account also the bounded multiplicity of this cover.

From (3.20) we identify Σ^+_I with $\Sigma^+_{\mathbb{N}^*} \times \Sigma^+_{\mathbb{N}^*}$, and can consider without loss of generality that ψ^+ is defined on $\Sigma^+_{\mathbb{N}^*} \times \Sigma^+_{\mathbb{N}^*}$. Since the IFS with contractions $\varphi_n, n \geq 1$ satisfies the Open Set Condition, and by using the Gibbs property of μ^+, it follows that there exists a constant $C > 0$ so that for any $n \geq 1$ and any $(\omega',\eta') \in \Sigma^+_I$,

$$\begin{aligned}\Upsilon(\rho([(\omega'_0,\eta'_0)\ldots(\omega'_n,\eta'_n)])) = &\mu^+[(\omega'_0,\eta'_0)\ldots(\omega'_n,\eta'_n)] \in \\ &\in \Big(\frac{e^{S_n\psi^+(\omega',\eta') - nP(\psi^+)}}{C},\ Ce^{S_n\psi^+(\omega',\eta') - nP(\psi^+)}\Big)\end{aligned} \tag{3.24}$$

Introduce now for arbitrary $n > 1$, $\varepsilon > 0$, the following Borel subset of $(0,1) \times (0,1)$,
(3.25)

$$\hat{\Omega}(n,\varepsilon) := \big\{z \in Z,\ \Upsilon\big(B(z,r) \cap \rho(A(n,\varepsilon)) \cap (C(n,\varepsilon) \times (0,1))\big) > \frac{1}{2}\Upsilon(B(z,r)),\ \forall\, 0 < r \leq \lambda_2^{(1+\varepsilon)n}\big\}$$

Denote the complement of $\hat{\Omega}(n, \varepsilon)$ in $(0,1) \times (0,1)$ by $\hat{\Omega}^c(n, \varepsilon))$. From definition, if $z \in \hat{\Omega}^c(n, \varepsilon)$, then $\exists 0 < r = r(z) < \lambda_2^{(1+\varepsilon)n}$ so that:

$$\Upsilon(B(z,r) \setminus (\rho(A(n,\varepsilon)) \cap C(n,\varepsilon) \times (0,1))) > \frac{1}{2}\Upsilon(B(z,r)) \tag{3.26}$$

In this case we can cover the set $\hat{\Omega}^c(n, \varepsilon)$ with balls of type $B(z, r(z))$ for all $z \in \hat{\Omega}^c(n, \varepsilon)$. Then, from Besicovitch Covering Theorem applied to a bounded subset of $Z = (0,1) \times (0,1)$, there exists a subcover with such balls for $z \in \mathcal{F}$, such that

$$\hat{\Omega}^c(n, \varepsilon) \subset \bigcup_{z \in \mathcal{F}} B(z, r(z)),$$

and the multiplicity of this subcover is finite and bounded above by a constant M which does not depend on n, ε. Thus from property (3.26) and the bounded multiplicity of $\mathcal{F}$, and using (3.19) and (3.22), it follows that,

$$\begin{aligned}
\Upsilon(\hat{\Omega}^c(n,\varepsilon)) &\leq \sum_{z \in \mathcal{F}} \Upsilon(B(z, r(z)) \leq 2 \sum_{z \in \mathcal{F}} \Upsilon(B(z,r) \setminus (\rho(A(n,\varepsilon)) \cap C(n,\varepsilon) \times (0,1))) \\
&\leq 2M\Upsilon\big(\bigcup_{z \in \mathcal{F}} B(z, r(z)) \setminus (\rho(A(n,\varepsilon)) \cap C(n,\varepsilon) \times (0,1))\big) \\
&\leq 2M\Upsilon\big(Z \setminus (\rho(A(n,\varepsilon)) \cap C(n,\varepsilon) \times (0,1))\big) \\
&= 2M\big(1 - \Upsilon(\rho(A(n,\varepsilon)) \cap C(n,\varepsilon) \times (0,1))\big) < 4M\kappa(\varepsilon)
\end{aligned}$$

Hence,

$$\Upsilon(\hat{\Omega}(n,\varepsilon)) \geq 1 - 4M\kappa(\varepsilon) \tag{3.27}$$

Now we can do the same argument as before for the projection measure Υ_1 of Υ on the first coordinate, i.e. $\Upsilon_1 = p_{1*}\Upsilon$, where p_1 denotes the projection on the first real coordinate. Hence we obtain a Borel set $\Omega_1(n, \varepsilon) \subset (0,1)$ with

$$\Upsilon_1(\Omega_1(n,\varepsilon)) > 1 - 4M\kappa(\varepsilon),$$

and such that for any point $x \in \Omega_1(n, \varepsilon)$ and any $0 < r < \lambda_2^{n(1+\varepsilon)}$, we have:

$$\Upsilon_1\big(B(x,r) \cap C(n,\varepsilon) \cap p_1(\rho(A(n,\varepsilon)))\big) > \frac{1}{2}\Upsilon_1(B(x,r)) \tag{3.28}$$

Denote in the sequel

$$\Omega(n,\varepsilon) := \hat{\Omega}(n,\varepsilon) \cap p_1^{-1}\Omega_1(n,\varepsilon) \tag{3.29}$$

From above, since $\Upsilon(p_1^{-1}\Omega_1(n,\varepsilon)) = \Upsilon_1(\Omega_1(n,\varepsilon))$, it follows that

$$\Upsilon(\Omega(n,\varepsilon)) > 1 - 8M\kappa(\varepsilon) \tag{3.30}$$

Let us then take a point $z \in \Omega(n, \varepsilon)$ and assume $z = \rho(\omega, \eta)$. This means that for any integer $n' \geq n$,

$$\Upsilon(B(z, \lambda_2^{(1+\varepsilon)n'})) \leq 2\Upsilon\big(B(z, \lambda_2^{(1+\varepsilon)n'}) \cap \rho(A(n,\varepsilon)) \cap C(n,\varepsilon) \times (0,1))\big) \tag{3.31}$$

So for any $n' \geq n$, in order to estimate the Υ-measure of the ball

$$B(z, \lambda_2^{(1+\varepsilon)n'}) \subset (0,1) \times (0,1),$$

it is enough to consider only its generic points in the sense of (3.31). Let us take then a point $w \in B(z, \lambda_2^{(1+\varepsilon)n'}) \cap \rho(A(n,\varepsilon)) \cap C(n,\varepsilon) \times (0,1)$, with $w = \rho(\omega', \eta')$. Then, from the definition of Υ as the projection measure $\rho_* \mu^+$,

$$\Upsilon(\rho([\omega'_0 \dots \omega'_{n'}] \times [\eta'_0 \dots \eta'_{n'}]) = \mu^+([\omega'_0 \dots \omega'_{n'}] \times [\eta'_0 \dots \eta'_{n'}])$$

For any $j \geq 1$ and any $(\omega', \eta') \in \Sigma_I^+$ (recall that Σ_I^+ is identified by Ψ with $\Sigma_{\mathbb{N}^*}^+ \times \Sigma_{\mathbb{N}^*}^+$), denote the projection of the associated j-cylinder $[\omega'_0 \dots \omega'_j] \times [\eta'_0 \dots \eta'_j]$ by:

$$R_j(\omega', \eta') := \rho([\omega'_0 \dots \omega'_j] \times [\eta'_0 \dots \eta'_j]) \ \subset (0,1) \times (0,1) \tag{3.32}$$

Now let $(\omega', \eta') \in \Sigma_I^+$ with $\rho(\omega', \eta') \in \rho(A(n,\varepsilon) \cap C(n,\varepsilon) \times (0,1))$. Hence for any $n' \geq n$,

$$|\frac{S_{n'}\psi^+(\omega', \eta')}{n'} - \int \psi^+ d\mu^+| < \varepsilon$$

Next we use the definition of $\Upsilon = \rho_* \mu^+$ and the Gibbs property of μ^+ from (3.24), together with the above genericity of (ω', η') and the fact that

$$P(\psi^+) = h_{\mu^+} + \int \psi^+ d\mu^+,$$

to obtain that for any $(\omega', \eta') \in A(n,\varepsilon)) \cap \rho^{-1}(C(n,\varepsilon) \times (0,1))$ and any $n' \geq n$,

$$\frac{1}{C} e^{-n'(h_{\mu^+} + \varepsilon)} < \Upsilon(R_{n'}(\omega', \eta')) < C e^{-n'(h_{\mu^+} - \varepsilon)}, \tag{3.33}$$

where the constant $C > 0$ does not depend on n, n', ω', η'.

Recall now that $z = \rho(\omega, \eta) = (x, y) \in \Omega(n, \varepsilon)$, and we want to cover the set

$$B(z, \lambda_2^{n'(1+\varepsilon)}) \cap \rho(A(n,\varepsilon)) \cap C(n,\varepsilon) \times (0,1),$$

with rectangles in $(0,1) \times (0,1)$ of type $R_{n'}(\omega', \eta')$, with $(\omega', \eta') \in A(n,\varepsilon) \cap \rho^{-1}(C(n,\varepsilon) \times (0,1))$. But by definition,

$$R_{n'}(\omega', \eta') = \rho_0([\omega'_0 \dots \omega'_{n'}]) \times \rho_0([\eta'_0 \dots \eta'_{n'}])$$

If $(\omega', \eta') \in A(n,\varepsilon) \cap \rho^{-1}(C(n,\varepsilon) \times (0,1))$, $n' \geq n$, and $(z'_1, z'_2) := \rho(\omega', \eta')$, then by (3.21),

$$B(z'_1, e^{n'(\chi_1 - \varepsilon)}) \times B(z'_2, e^{n'(\chi_2 - \varepsilon)}) \subset \ R_{n'}(\omega', \eta') \ \subset B(z'_1, e^{n'(\chi_1 + \varepsilon)}) \times B(z'_2, e^{n'(\chi_2 + \varepsilon)}). \tag{3.34}$$

5. Estimates for the number of covering rectangles and Υ.

Let us now cover the set $B(z, \lambda_2^{n'(1+\varepsilon)}) \cap \rho(A(n,\varepsilon)) \cap C(n,\varepsilon) \times (0,1))$ with rectangles $R_{n'}(\omega', \eta')$, with (ω', η') from a finite family $\mathcal{G}$, such that the projections of these rectangles on the first real coordinate intersect with multiplicity bounded by M; this is possible by using the Besicovitch Covering Theorem. Denote the number of rectangles in $\mathcal{G}$ by $N(n', \varepsilon)$; clearly $N(n', \varepsilon)$ depends also on z. We want to estimate in the sequel this number of rectangles $N(n', \varepsilon)$.

In order to do this, notice from (3.34) that the projection on first coordinate of an arbitrary rectangle from $\mathcal{G}$ is a ball of some radius r, with all these radii r satisfying

$$r \in (\lambda_1^{n'(1+\varepsilon)}, \lambda_1^{n'(1-\varepsilon)})$$

Denote the projection on first coordinate of $R_{n'}(\omega', \eta')$ by $D_{n'}(\omega', \eta')$, and denote by $\mathcal{G}_1$ the set of projections $D_{n'}(\omega', \eta')$ for the rectangles from $\mathcal{G}$. From construction, $\mathcal{G}_1$ has multiplicity bounded above by M. Since in any set $D_{n'}(\omega', \eta')$ from $\mathcal{G}_1$ there are points from $C(n, \varepsilon)$ (by construction), and since $\lambda_1 < \lambda_2$, it follows that there exists a constant (denoted also by C) so that for every $n' \ge n$,

$$\frac{1}{C}\lambda_1^{n'(\delta_1+\varepsilon)} < \Upsilon_1(D_{n'}(\omega', \eta')) < C\lambda_1^{n'(\delta_1-\varepsilon)} \tag{3.35}$$

From the definition of $\mathcal{G}_1$ and (3.29), it follows that the sets $D_{n'}(\omega', \eta')$ from $\mathcal{G}_1$ cover $B(x, \lambda_2^{n'(1+\varepsilon)}) \cap C(n, \varepsilon) \cap \rho_0(p_1(A(n, \varepsilon)))$. But from (3.28),

$$\Upsilon_1(B(x, \lambda_2^{n'(1+\varepsilon)}) \cap C(n, \varepsilon) \cap \rho_0(p_1(A(n, \varepsilon)))) \ge \frac{1}{2}\Upsilon_1(B(x, \lambda_2^{n'(1+\varepsilon)})$$

Therefore from (3.35) and as $x \in C(n, \varepsilon)$ and $\mathcal{G}_1$ has multiplicity bounded by M, it follows that

$$\frac{1}{MC}\lambda_1^{n'(\delta_1+\varepsilon)}N(n', \varepsilon) < \lambda_2^{n'(\delta_1-\varepsilon)} \text{ and } 2C\lambda_1^{n'(\delta_1-\varepsilon)}N(n', \varepsilon) > \lambda_2^{n'(\delta_1+\varepsilon)}$$

So there exists a constant $C_1 > 0$ such that for all $z \in \Omega(n, \varepsilon)$ and any $n' \ge n$,

$$\frac{1}{C_1}\lambda_2^{n'(\delta_1+\varepsilon)}\lambda_1^{n'(-\delta_1+\varepsilon)} \le N(n', \varepsilon) \le C_1\lambda_2^{n'(\delta_1-\varepsilon)}\lambda_1^{-n'(\delta_1+\varepsilon)} \tag{3.36}$$

We now use this estimate on the number of rectangles $N(n', \varepsilon)$, in order to estimate the Υ-measure of the ball $B(z, \lambda_2^{n'(1+\varepsilon)})$. From (3.31) it follows however that it is enough to estimate the measure

$$\Upsilon\big(B(z, \lambda_2^{(1+\varepsilon)n'} \cap \rho(A(n, \varepsilon)) \cap C(n, \varepsilon) \times (0, 1))\big),$$

and to notice that the set $B(z, \lambda_2^{(1+\varepsilon)n'} \cap \rho(A(n, \varepsilon)) \cap C(n, \varepsilon) \times (0, 1))$ is covered with the $N(n', \varepsilon)$ rectangles above from $\mathcal{G}$. But the Υ-measure of every such rectangle from the cover $\mathcal{G}$ was estimated in (3.33). Hence there exists a constant $C_2 > 0$ such that for any point $z \in \Omega(n, \varepsilon)$, and any integer $n' > n$,

$$\frac{1}{C_2}\lambda_2^{n'(\delta_1+\varepsilon)}\lambda_1^{n'(-\delta_1+\varepsilon)}e^{-n'(h_{\mu^+}+\varepsilon)} < \Upsilon(B(z, \lambda_2^{n'(1+\varepsilon)})) < C_2\lambda_2^{n'(\delta_1-\varepsilon)}\lambda_1^{n'(-\delta_1-\varepsilon)}e^{-n'(h_{\mu^+}-\varepsilon)} \tag{3.37}$$

6. Volume Lemmas for the measures Υ and ν.

Now, if $r > 0$ is arbitrarily small, there must exist some large integer $n' > n$ such that

$$\lambda_2^{(n'+1)(1+\varepsilon)} < r < \lambda_2^{n'(1+\varepsilon)}$$

This implies that,

$$B(z, \lambda_2^{(n'+1)(1+\varepsilon)}) \subset B(z, r) \subset B(z, \lambda_2^{n'(1+\varepsilon)})$$

But when $n' \to \infty$, then we have $r \to 0$, and viceversa. Hence if $z \in \Omega(n, \varepsilon)$, then from (3.37) and the last two displayed estimates above, one obtains the following:

(3.38)
$$\frac{\log \lambda_2(\delta_1-\varepsilon)-\log \lambda_1(\delta_1+\varepsilon)-h_{\mu^+}+\varepsilon}{\log \lambda_2(1+\varepsilon)} \le \lim_{r\to 0}\frac{\log \Upsilon(B(z,r))}{\log r} \le$$
$$\le \frac{\log \lambda_2(\delta_1+\varepsilon)-\log \lambda_1(\delta_1-\varepsilon)-h_{\mu^+}-\varepsilon}{\log \lambda_2(1+\varepsilon)}$$

But on the other hand, $\Omega(n,\varepsilon)\subset \Omega(n+1,\varepsilon), n\ge 1$, and thus from (3.30),
$$\Upsilon\big(\bigcup_{n\ge 1}\Omega(n,\varepsilon)\big)\ge 1-8M\kappa(\varepsilon)$$
Therefore, if we define the Borel set
$$\Omega := \bigcap_{m\ge 1}\bigcup_{n\ge 1}\Omega(n,\frac{1}{m}),$$
then from the last displayed inequality, and since $\lim\limits_{\varepsilon\to 0}\kappa(\varepsilon)=0$, it follows that $\Upsilon(\Omega)=1$. In conclusion, from (3.38) and (3.17), the measure Υ is exact dimensional and for all $z\in\Omega$,
$$\delta(\Upsilon)(z)=\lim_{r\to 0}\frac{\log\Upsilon(B(z,r))}{\log r}=\frac{h_{\mu^+}-h_{\mu_1}(1-\frac{\chi_2}{\chi_1})}{\chi_2}.$$

Now we use the exact dimensionality of the projection measure $\Upsilon=\rho_*\mu^+$ proved above, and the exact dimensionality of the conditional measures on fibers from Theorem 2.2, together with Theorem 8.7 of [17] applied to ν. Recall that $\mu=\mu_\psi$. Thus the measure $\nu=\nu_\psi$ is exact dimensional on $X\times Y$. Moreover, from the last displayed formula and (3.14), it follows that the Hausdorff (and pointwise) dimension of ν is given by,
$$HD(\nu)=\delta(\nu)=\delta(\Upsilon)+\frac{h_\mu}{\chi(\mu)}=\frac{h_\mu-h_{\mu_1}(1-\frac{\chi_2(\mu)}{\chi_1(\mu)})}{\chi_2(\mu)}+\frac{h_\mu}{\chi(\mu)}.$$

The other case $\lambda_1\ge\lambda_2$ is proved similarly. Hence we showed exact dimensionality of ν on $X\times Y$, and the formulas for $HD(\nu)$, which concludes the proof of the Theorem.

□

In particular, Theorem 3.1 applies to the endomorphism F determined by T of (2.2) which generates complex continued fractions with complex digits. The first coordinate map of F is only piecewise smooth on countably many pieces, and its second coordinate map is holomorphic of two complex variables. We obtain:

Corollary 3.2. *Let $X=(1,\infty)\times(1,\infty)$, $B(\frac12,\frac12)\subset\mathbb{C}$, and $F: X\times B(\frac12,\frac12)\to X\times B(\frac12,\frac12)$,*
$$F(z,w)=\Big(\frac{1}{\{Re(z)\}}+\frac{i}{\{Im(z)\}},\ \frac{1}{w+z}\Big),\ z\in X, Re(z)\notin\mathbb{Q}, Im(z)\notin\mathbb{Q},\ w\in Y,$$
and $F(z,w)=(2,2,\frac12)$ otherwise. Let $\psi:\Sigma_I\to\mathbb{R}$ be a Hölder continuous summable potential, and μ_ψ be the equilibrium measure of ψ on Σ_I. Denote the measures
$$\mu_\psi^+ := (\pi_0)_*\mu_\psi \text{ on } \Sigma_I^+, \text{ and } \nu_\psi := \pi_*\mu_\psi \text{ on } X\times Y$$

Then, ν_ψ is exact dimensional on $J\subset\mathbb{C}^2$, and its dimension is given by the formula in Theorem 3.1.

In the end, we study a class of examples given by **geometric potentials**. Consider a fixed conformal Smale skew-product T and the associated endomorphism $F_T \in \mathcal{E}$ given by (1.1). Then we define the *s-geometric potentials*, for $s > 0$, by:

$$\psi_{T,s} : \Sigma_I \to \mathbb{R}, \quad \psi_{T,s}(\eta) := s \log |T'_{\eta|_0^\infty}(\hat{\pi}_2\eta)|, \text{ for } \eta \in \Sigma_I \tag{3.39}$$

We show below that $\psi_{T,s}$ is Hölder continuous. By Proposition 1.2, $\psi_{T,s}$ is summable if and only if its pressure $P(\psi_{T,s})$ is finite. In the sequel, we shall always assume that $\psi_{T,s}$ is summable. If $\psi_{T,s}$ is summable on Σ_I and Hölder continuous, then by Theorem 1.3, $\psi_{T,s}$ has a unique equilibrium measure $\mu_{T,s}$ on Σ_I. If T is fixed, we may drop T from the notation of ψ_s, μ_s.

Denote also the fiber Julia sets by $J_{T,\omega}$, for $\omega \in \Sigma_I^+$. Recall the definition of the Lyapunov exponent $\chi(\mu_{T,s})$ from (3.10). Let the measure $\mu_{T,s}^+ := \pi_{0*}\mu_{T,s}$ be the canonical projection of $\mu_{T,s}$ on Σ_I^+. For $\mu_{T,s}^+$-a.e. $\omega \in \Sigma_I^+$, let us also denote by $\nu_{T,s}^\omega$ the $\hat{\pi}_2$-projection on $J_{T,\omega}$ of the conditional measure $\mu_{T,s}^\omega$ of $\mu_{T,s}$ on $[\omega]$, namely,

$$\nu_{T,s}^\omega = \hat{\pi}_{2*}\mu_{T,s}^\omega$$

The next Corollary describes geometrically the fiber measures $\nu_{T,s}^\omega$, and shows that their dimensions $\delta_{T,s}$ depend **real-analytically** on s. Moreover, it gives a **Variational Principle** for dimension on the Julia sets $J_{T,\omega}$.

Corollary 3.3. *Let a conformal Smale skew-product T and the associated endomorphism F_T defined in (1.1), and let $\psi_{T,s}$ be as in (3.39). Then:*

a) If $\psi_{T,s}$ is summable, then $\nu_{T,s}^\omega$ is exact dimensional on $J_{T,\omega}$ and for $\mu_{T,s}^+$-a.e $\omega \in \Sigma_I^+$,

$$HD(\nu_{T,s}^\omega) = \delta_{T,s} = \frac{h(\mu_{T,s})}{\chi(\mu_{T,s})}$$

b) The dimension value $\delta_{T,s}$ depends real-analytically on the parameter s.
c) For any $\omega \in \Sigma_I^+$,

$$HD(J_{T,\omega}) = \sup_s \delta_{T,s}$$

Proof. For part a), recall first from (1.6) that for T fixed as above and for any $\eta \in \Sigma_I$,

$$\hat{\pi}_2(\eta) = T_{\eta|_{-1}^\infty} \circ T_{\eta|_{-2}^\infty} \circ \dots ,$$

where $T_\omega(w)$ is the fiber map of T, and $\omega = ((m_0, n_0), (m_1, n_1), \dots) \in \Sigma_I^+$. Also, $\pi(\eta) = (\pi_0(\eta|_0^\infty), \hat{\pi}_2(\eta))$, where $\tilde{\pi}(\omega) = m_0 + \frac{1}{m_1+\dots} + i(n_0 + \frac{1}{n_1+\dots})$. The Julia set $J_{T,\omega}$ is equal to $\hat{\pi}_2([\omega])$, $\omega \in \Sigma_I^+$. By (3.39) and Definition 1.6 it follows that $\psi_{T,s}$ is Hölder continuous, since the map $\eta \to \hat{\pi}_2(\eta)$ is Hölder continuous on Σ_I. Denote by

$$\mathcal{F}(T) := \{t \geq 0, \sum_{i \in I} \sup_{\omega \in \Sigma_I^+} \sup_{\xi \in \hat{\pi}_2[i\omega]} |T'_{i\omega}(\xi)|^t < \infty\},$$

the set of s for which $\psi_{T,s}$ is summable. From Proposition 1.2, $\mathcal{F}(T)$ is exactly the set of s for which $P(\psi_{T,s}) < \infty$, and one can see that $\mathcal{F}(T)$ is an interval. For $s \in \mathcal{F}(T)$, since

$\psi_{T,s}$ is summable and Hölder continuous on Σ_I, then it follows by Theorem 2.2 that, for $\mu^+_{T,s}$-a.e. $\omega \in \Sigma_I^+$, $HD(\nu^\omega_{T,s}) = \delta_{T,s} := \frac{h(\mu_{T,s})}{\chi(\mu_{T,s})}$.

For part b), since $\mu_{T,s}$ is the equilibrium measure of $\psi_{T,s}$, we have $P(\psi_{T,s}) = h(\mu_{T,s}) + \int \psi_{T,s} d\mu_{T,s}$. Thus $h(\mu_{T,s}) = P(\psi_{T,s}) - \int \psi_{T,s} d\mu_{T,s}$. But the pressure $P(\psi_{T,s})$ depends real-analytically on s ([20], [21]). From the Ruelle formula for the derivative of the pressure ([21]), we have:

$$\frac{\partial P(\psi_{T,s} + v\psi_{T,s})}{\partial v}|_{v=0} = \int \psi_{T,s}\ d\mu_{T,s}$$

Hence the integral $\int \psi_{T,s} d\mu_{T,s}$, and from above also the entropy $h(\mu_{T,s})$, depend real-analytically on the parameter s. Thus from a), the dimension $\delta_{T,s} = \frac{h(\mu_{T,s})}{\chi(\mu_{T,s})}$ also depends real-analytically on s.

For part c), we apply Theorem 7.2 of [17], since the uniform geometry condition is satisfied. Hence for any $\omega \in \Sigma_I^+$, we have

$$HD(J_{T,\omega}) = \inf\{s > 0, P(\psi_{T,s}) \le 0\}$$

Let now $\varepsilon > 0$ arbitrary, and from above there exists $s = s(\varepsilon) > 0$ so that $P(\psi_{T,s}) \le 0$ and

$$s - \varepsilon < HD(J_{T,\omega}) \le s \tag{3.40}$$

But $P(\psi_{T,s'}) > P(\psi_{T,s})$ for any $s' < s$ from $\mathcal{F}(T)$, and the pressure function $s \to P(\psi_{T,s})$ is Lipschitz continuous on $\mathcal{F}(T)$. Hence there exists $\beta(\varepsilon) < 0$ with $\beta(\varepsilon) \underset{\varepsilon\to 0}{\to} 0$, so that for $s = s(\varepsilon)$ found above, $P(\psi_{T,s}) > \beta(\varepsilon)$. Thus $P(\psi_{T,s}) = h(\mu_{T,s}) - s\chi(\mu_{T,s}) > \beta(\varepsilon)$, and

$$\delta_{T,s} = \frac{h(\mu_{T,s})}{\chi(\mu_{T,s})} > s + \frac{\beta(\varepsilon)}{\chi(\mu_{T,s})}$$

However from (3.10) and using the properties of T from Definition 1.6, it follows that there exists $\chi_0 > 0$, such that $\chi(\mu_{T,s}) > \chi_0$ for all $s \in \mathcal{F}(T)$. Hence from (3.40),

$$HD(J_{T,\omega}) \le s \le \delta_{T,s} + |\frac{\beta(\varepsilon)}{\chi_0}|$$

But $\beta(\varepsilon) \underset{\varepsilon\to 0}{\to} 0$, hence $HD(J_{T,\omega}) \le \sup\limits_{s\in\mathcal{F}(T)} \delta_{T,s}$. Since $P(\psi_{T,s}) = h(\mu_{T,s}) - s\chi(\mu_{T,s}) \le 0$ and by using (3.40), we obtain that,

$$\delta_{T,s} \le s \le HD(J_{T,\omega}) + \varepsilon$$

Thus from the last two displayed inequalities, $HD(J_{T,\omega}) = \sup\{\delta_{T,s},\ s \in \mathcal{F}(T)\}$.

□

Remark 3.4.

1) If T is the special endomorphism in (2.2), giving the complex continued fractions with complex digits, then the geometric potentials are

$$\psi_s(\eta) = -2s \log |\tilde\pi(\eta|_0^\infty) + \hat\pi_2(\eta)|,\ \eta \in \Sigma_I$$

This implies that in this case, ψ_s is summable on Σ_I for any $s > 1$.

2) For T arbitrary, Theorem 3.1 shows that the measure $\nu_{T,s} = \pi_*\mu_{T,s}$ is exact dimensional on $J_T \subset \mathbb{C}^2$, and it gives the Hausdorff (and pointwise) dimension of $\nu_{T,s}$.

□

References

[1] L. Barreira, Y. Pesin, J. Schmeling, Dimension and product structure of hyperbolic measures, Ann. of Math. 149 (1999), 755-783.
[2] R. Bowen, Equilibrium States and Ergodic Theory for Anosov Diffeomorphisms. Lecture Notes, 1975.
[3] L. Carleson, T. Gamelin, Complex Dynamics, Springer Verlag, 1993.
[4] K. Falconer, Techniques in Fractal Geometry, J. Wiley & Sons, Chichester, 1997.
[5] J.E. Fornaess, E. Mihailescu, Equilibrium measures on saddle sets of holomorphic maps on $\mathbb{P}^2\mathbb{C}$, Math. Annalen, 356, 1471-1491, 2013.
[6] R. J. Gardner, R. D. Mauldin, On the Hausdorff dimension of a set of complex continued fractions, Illinois J. Math, 27 (1983) 334-344.
[7] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems. Cambridge University Press, 1995.
[8] A. Manning, The dimension of maximal measure for a polynomial map, Ann Math 119, 425-30, 1984.
[9] D. Mauldin, M. Urbański, Dimensions and measures in infinite iterated function systems, Proc. London Math. Soc. (3) 73 (1996), 105-154.
[10] D. Mauldin, M. Urbański, Gibbs states on the symbolic space over an infinite alphabet, Israel J. Math., 125 (2001), 93-130.
[11] D. Mauldin, M. Urbański, Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets, Cambridge University Press, 2003.
[12] E. Mihailescu, On a class of stable conditional measures, Ergod Th Dynam Syst, 31 (2011), 1499-1515.
[13] E. Mihailescu, Unstable directions and fractal dimension for skew products with overlaps in fibers, Math Zeit, 269, 2011, 733-750.
[14] E. Mihailescu, B. Stratmann, Upper estimates for stable dimensions on fractal sets with variable numbers of foldings, International Math. Res. Notices, 2014(23), (2014), 6474-6496.
[15] E. Mihailescu, M. Urbański, Estimates for the stable dimension for holomorphic maps, Houston J Math, vol 31, 2, 2005, 367-389.
[16] E. Mihailescu, M. Urbański, Random countable iterated function systems with overlaps and applications, Advances Math, 298, 2016, 726-758.
[17] E. Mihailescu, M. Urbański, Skew product Smale endomorphisms over countable shifts of finite type, Ergod Th Dynam Syst, vol 40, 11, 2020, 3105-3149.
[18] Y. Pesin, Dimension Theory in Dynamical Systems, Chicago Lectures in Mathematics, 1997.
[19] V. A Rokhlin, Lectures on the theory of entropy of transformations with invariant measures, Russian Math Surveys, 22, 1967, 1-54.
[20] D. Ruelle, Repellers for real-analytic maps, Ergod Th Dynam Syst, 2, 1982, 99-107.
[21] D. Ruelle, Thermodynamic formalism. The mathematical structures of equilibrium statistical mechanics, Second edition, Cambridge University Press, Cambridge, 2004.
[22] D. Ruelle, Elements of Differentiable Dynamics and Bifurcation Theory, Academic Press, 1989.
[23] L. S. Young, Dimension, entropy and Lyapunov exponents, Ergod Th Dynam Syst, 2, 1982, 109-124.

Eugen Mihailescu, Institute of Mathematics of the Romanian Academy, P.O Box 1-764, RO 014700, Bucharest, Romania

Email address: `Eugen.Mihailescu@imar.ro` Web: `www.imar.ro/∼mihailes`

Mariusz Urbański, Department of Mathematics, University of North Texas, Denton, TX 76203-1430, USA

Email address: `urbanski@unt.edu` Web: `www.math.unt.edu/∼urbanski`