A VARIATIONAL PRINCIPLE IN THE PARAMETRIC GEOMETRY OF NUMBERS

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Dedicated to Professor Wolfgang M. Schmidt on the occasion of his 85th birthday

ABSTRACT. We extend the parametric geometry of numbers (initiated by Schmidt and Summerer, and deepened by Roy) to Diophantine approximation for systems of m linear forms in n variables, and establish a new connection to the metric theory via a variational principle that computes fractal dimensions of a variety of sets of number-theoretic interest. The proof relies on two novel ingredients: a variant of Schmidt's game capable of computing the Hausdorff and packing dimensions of any set, and the notion of templates, which generalize Roy's rigid systems. In particular, we compute the Hausdorff and packing dimensions of the set of singular systems of linear forms and show they are equal, resolving a conjecture of Kadyrov, Kleinbock, Lindenstrauss and Margulis, as well as a question of Bugeaud, Cheung and Chevallier. As a corollary of Dani's correspondence principle, the divergent trajectories of a one-parameter diagonal action on the space of unimodular lattices with exactly two Lyapunov exponents with opposite signs has equal Hausdorff and packing dimensions. Other applications include quantitative strengthenings of theorems due to Cheung and Moshchevitin, which originally resolved conjectures due to Starkov and Schmidt respectively; as well as dimension formulas with respect to the uniform exponent of irrationality for simultaneous and dual approximation in two dimensions, completing partial results due to Baker, Bugeaud, Cheung, Chevallier, Dodson, Laurent and Rynne.

CONTENTS

Convention 1. In what follows, $A \lesssim B$ means that there exists a constant C (the *implied constant*) such that $A \leq CB$. $A \asymp B$ means $A \lesssim B \lesssim A$. Similarly, $A \lesssim_+ B$ means that $A \leq B + C$ for some constant C. When we write $A \lesssim_\beta B$ or $A \lesssim_{+,\beta} B$ this signifies that the implied constant depends on β. We use $A \asymp_+ B$ to mean $A \lesssim_+ B$ and $B \lesssim_+ A$. For instance, this allows us to write $A \asymp_+ B = C \asymp_+ D$ without having to write $O(1)$ everywhere, which would obscure some of the information and also be more cluttered.

Convention 2. Recall that $\Theta(x)$ denotes any number such that $x/C \leq \Theta(x) \leq Cx$ for some uniform constant C.

Convention 3. All measures and sets are assumed to be Borel, and measures are assumed to be locally finite. Sometimes we restate these hypotheses for emphasis.

Convention 4. Given a vector space V and some index set I we use the notation $\langle x_i \in V : i \in I\rangle$ to mean the set generated by $\{x_i \in V : i \in I\}$, or the smallest subspace containing $\{x_i \in V : i \in I\}$.

Glossary of Notation. For the reader's convenience we summarize a list of notations and terminology in the order that they appear in the sequel.

- $\mathrm{Sing}(m,n)$ .... The set of singular $m \times n$ matrices
- $\delta_{m,n}$ .... $\delta_{m,n} \stackrel{\text{def}}{=} mn\left(1 - \frac{1}{m+n}\right)$
- $\mathrm{BA}(m,n)$ .... The set of badly approximable $m \times n$ matrices
- $\mathrm{VWA}(m,n)$ .... The set of very well approximable $m \times n$ matrices
- $\dim_H(S)$ .... The Hausdorff dimension of a set S
- $\dim_P(S)$ .... The packing dimension of a set S
- I_k .... The k-dimensional identity matrix
- d .... $d \stackrel{\text{def}}{=} m + n$
- $\lambda_j(\Lambda)$ $(1 \leq j \leq d)$ .... The jth successive minimum of a lattice $\Lambda \subseteq \mathbb{R}^d$
- g_t .... For $t \in \mathbb{R}$, $g_t \stackrel{\text{def}}{=} \begin{bmatrix} e^{t/m}\mathrm{I}_m & \\ & e^{-t/n}\mathrm{I}_n \end{bmatrix} \in \mathrm{SL}_d(\mathbb{R})$
- u_A .... For an $m \times n$ matrix A, $u_A \stackrel{\text{def}}{=} \begin{bmatrix} \mathrm{I}_m & A \\ & \mathrm{I}_n \end{bmatrix} \in \mathrm{SL}_d(\mathbb{R})$
- $\widehat{\omega}(A)$ .... The uniform exponent of irrationality of an $m \times n$ matrix A
- $\mathrm{VSing}(m,n)$ .... The set of very singular $m \times n$ matrices, i.e. $\{A : \widehat{\omega}(A) > n/m\}$
- $\widehat{\tau}(A)$ .... $\widehat{\tau}(A) \stackrel{\text{def}}{=} \liminf_{t\to\infty} \frac{-1}{t} \log \lambda_1(g_t u_A \mathbb{Z}^d)$
- τ .... $\tau = \frac{1}{n}\frac{\omega - \frac{n}{m}}{\omega+1}$
- $\mathrm{Sing}_{m,n}(\omega)$ .... $\mathrm{Sing}_{m,n}(\omega) \stackrel{\text{def}}{=} \{A : \widehat{\omega}(A) = \omega\} = \{A : \widehat{\tau}(A) = \tau\}$

- *trivially singular* .... See Section § 1.2.1
- $\mathrm{Sing}^*_{m,n}(\omega)$ .... $\mathrm{Sing}^*_{m,n}(\omega) \stackrel{\mathrm{def}}{=} \{A \in \mathrm{Sing}_{m,n}(\omega) : A$ is not trivially singular$\}$
- $\mathcal{P}(A)$ .... $\mathcal{P}(A) \stackrel{\mathrm{def}}{=} \lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{T}\lambda\big(\{t \in [0,T] : \lambda_1(g_t u_A \mathbb{Z}^d) \leq \varepsilon\}\big)$
- *singular on average* .... A is *singular on average* if $\mathcal{P}(A) = 1$
- *k-singular* .... See Definition 1.13
- $\{x\}$ .... $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$
- $f_{m,n}(k)$ .... $f_{m,n}(k) \stackrel{\mathrm{def}}{=} mn - \frac{k(m+n-k)mn}{(m+n)^2} - \left\{\frac{km}{m+n}\right\}\left\{\frac{kn}{m+n}\right\}$
- $\mathbf{h}, \mathbf{h}_A, h_i(t)$ .... $\mathbf{h} = \mathbf{h}_A = (h_1, \ldots, h_d) : [0,\infty) \to \mathbb{R}^d$, $h_i(t) \stackrel{\mathrm{def}}{=} \log \lambda_i(g_t u_A \mathbb{Z}^d)$
- $V_{j,t}$ .... $V_{j,t} \stackrel{\mathrm{def}}{=} \mathrm{span}_{\mathbb{R}}\{\mathbf{r} \in \mathbb{Z}^d : \|g_t u_A \mathbf{r}\| \leq \lambda_j(g_t u_A \mathbb{Z}^d)\}$
- $F_{j,I}(t)$ .... $F_{j,I}(t) \stackrel{\mathrm{def}}{=} \log \|g_t u_A(V_{j,t} \cap \mathbb{Z}^d)\|$
- $Z(j)$ .... $Z(j) \stackrel{\mathrm{def}}{=} \left\{\frac{L_+}{m} - \frac{L_-}{n} : L_\pm \in [0, d_\pm]_{\mathbb{Z}},\ \ L_+ + L_- = j\right\}$
- *template* .... See Def. 2.1
- *balanced template* .... See Def. 2.1
- *partial template* .... See Def. 2.1
- $\mathcal{T}_{m,n}$ .... The space of $m \times n$ templates
- F_j .... $F_j \stackrel{\mathrm{def}}{=} \sum_{i \leq j} f_i$ for a piecewise linear map $\mathbf{f} : [0,\infty) \to \mathbb{R}^d$
- *convexity condition* .... F_j is convex when $f_j < f_{j+1}$
- *quantized slope condition* .... Slopes of the pieces of F_j are in $Z(j)$ when $f_j < f_{j+1}$
- f_0, f_{d+1} .... $f_0 \stackrel{\mathrm{def}}{=} -\infty$ and $f_{d+1} \stackrel{\mathrm{def}}{=} +\infty$
- $\mathcal{M}(\mathbf{f})$ .... $\mathcal{M}(\mathbf{f}) \stackrel{\mathrm{def}}{=} \{A : \mathbf{h}_A \asymp_+ \mathbf{f}\}$
- $\mathcal{M}(\mathcal{F})$ .... $\mathcal{M}(\mathcal{F}) \stackrel{\mathrm{def}}{=} \bigcup_{\mathbf{f} \in \mathcal{F}} \mathcal{M}(\mathbf{f})$
- $\underline{\delta}(\mathbf{f}), \overline{\delta}(\mathbf{f})$ .... Lower and upper average contraction rates of a template $\mathbf{f}$
- $\widehat{\tau}(\mathbf{f})$ .... The uniform dynamical exponent of $\mathbf{f}$: $\widehat{\tau}(\mathbf{f}) \stackrel{\mathrm{def}}{=} \liminf_{t\to\infty} \frac{-1}{t} f_1(t)$
- $\widetilde{\mathrm{Sing}}^*_{m,n}(\omega)$ .... $\widetilde{\mathrm{Sing}}^*_{m,n}(\omega) \stackrel{\mathrm{def}}{=} \{A : \widehat{\omega}(A) \geq \omega,\ A$ not trivially singular$\}$
- $\mathcal{H}^s(A)$ .... The s-dimensional Hausdorff measure of a set $A \subseteq \mathbb{R}^d$
- $\mathcal{P}^s(A)$ .... The s-dimensional packing measure of a set $A \subseteq \mathbb{R}^d$
- $\dim_H(A)$ .... The Hausdorff dimension of a set $A \subseteq \mathbb{R}^d$
- $\dim_P(A)$ .... The packing dimension of a set $A \subseteq \mathbb{R}^d$
- $B(\mathbf{x}, \rho)$ .... The closed ball centered at $\mathbf{x} \in \mathbb{R}^d$ with radius $\rho > 0$
- $\underline{\dim}_{\mathbf{x}}(\mu)$ .... $\underline{\dim}_{\mathbf{x}}(\mu) \stackrel{\mathrm{def}}{=} \liminf_{\rho\to 0} \log \mu(B(\mathbf{x},\rho)) / \log \rho$
- $\overline{\dim}_{\mathbf{x}}(\mu)$ .... $\overline{\dim}_{\mathbf{x}}(\mu) \stackrel{\mathrm{def}}{=} \limsup_{\rho\to 0} \log \mu(B(\mathbf{x},\rho))\ \log \rho$
- $\underline{\delta}(\mathcal{A})$ .... $\underline{\delta}(\mathcal{A}) \stackrel{\mathrm{def}}{=} \liminf_{k\to\infty} \frac{1}{k} \sum_{i=0}^k -\log \#(A_i) / \log(\beta)$
- $\overline{\delta}(\mathcal{A})$ .... $\overline{\delta}(\mathcal{A}) \stackrel{\mathrm{def}}{=} \limsup_{k\to\infty} \frac{1}{k} \sum_{i=0}^k -\log \#(A_i) / \log(\beta)$

- *Λ-rational* .... A subspace $V \subseteq \mathbb{R}^d$ is Λ-rational if $V \cap \Lambda$ is a lattice in V
- $\mathcal{V}_q(\Lambda)$ .... Set of all q-dimensional Λ-rational subspaces of $\mathbb{R}^d$
- $\|V\|$ .... Covolume of $V \cap \Lambda$ in V, where Λ is understood from context
- $\mathcal{L}$ .... $\mathcal{L} \stackrel{\text{def}}{=} \{\mathbf{0}\} \times \mathbb{R}^n$ is the subspace of $\mathbb{R}^d$ contracted by the (g_t) flow
- $\mathcal{C}(V, \varepsilon)$ .... Conical ε-neighborhood of a subspace $V \subseteq \mathbb{R}^d$
- *η-integral* .... see Def. 8.1
- *splits, mergers, transfers* .... See Def. 8.2
- *simple* .... See Def. 8.2
- *convex hull function* .... See Def. 8.6
- $\mathbf{b}$-*perturbation* of $\mathbf{f}$ at t_0 .... See Lemma 8.14
- $\mathcal{G} = \mathcal{G}(d, n)$ ...... The Grassmannian variety of n-dimensional subspaces of $\mathbb{R}^d$
- *standard template* .... See Def. 12.1
- $\mathbf{s}[(t_k, -\varepsilon_k), (t_{k+1}, -\varepsilon_{k+1})]$ .... See Def. 12.1

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Part 1. **Introduction**

1. Main results

The notion of singularity (in the sense of Diophantine approximation) was introduced by Khintchine, first in 1937 in the setting of simultaneous approximation [38], and later in 1948 in the more general setting of matrix approximation [39].[1] Since then

[1] Although Khintchine's 1926 paper [37] includes a proof of the existence of 2×1 and 1×2 matrices possessing a certain property which clearly implies that they are singular, it does not include a definition of singularity nor discuss any property equivalent to singularity.

this notion has been studied within Diophantine approximation and allied fields, see Moshchevitin's 2010 survey [47]. An $m \times n$ matrix A is called *singular* if for all $\varepsilon > 0$, there exists Q_ε such that for all $Q \geq Q_\varepsilon$, there exist integer vectors $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q} + \mathbf{p}\| \leq \varepsilon Q^{-n/m} \quad \text{and} \quad 0 < \|\mathbf{q}\| \leq Q.$$

Here $\|\cdot\|$ denotes an arbitrary norm on $\mathbb{R}^m$ or $\mathbb{R}^n$. We denote the set of singular $m \times n$ matrices by $\mathrm{Sing}(m,n)$. For 1×1 matrices (i.e. numbers), being singular is equivalent to being rational, and in general any matrix A which satisfies an equation of the form $A\mathbf{q} = \mathbf{p}$, with $\mathbf{p}, \mathbf{q}$ integral and $\mathbf{q}$ nonzero, is singular. However, Khintchine proved that there exist singular 2×1 matrices whose entries are linearly independent over $\mathbb{Q}$ [37, Satz II], and his argument generalizes to the setting of $m \times n$ matrices for all $(m,n) \neq (1,1)$. The name *singular* derives from the fact that $\mathrm{Sing}(m,n)$ is a Lebesgue nullset for all m, n, see e.g. [38, p.431] or [13, Chapter 5, §7]. Note that singularity is a strengthening of the property of *Dirichlet improvability* introduced by Davenport and Schmidt [21].

In contrast to the measure zero result mentioned above, the computation of the Hausdorff dimension of $\mathrm{Sing}(m,n)$ has been a challenge that so far only met with partial progress. The first breakthrough was made in 2011 by Cheung [16], who proved that the Hausdorff dimension of $\mathrm{Sing}(2,1)$ is $4/3$; this was extended in 2016 by Cheung and Chevallier [17], who proved that the Hausdorff dimension of $\mathrm{Sing}(m,1)$ is $m^2/(m+1)$ for all $m \geq 2$; while most recently Kadyrov, Kleinbock, Lindenstrauss, and Margulis (KKLM) [35] proved that the Hausdorff dimension of $\mathrm{Sing}(m,n)$ is at most $\delta_{m,n} \stackrel{\text{def}}{=} mn\big(1 - \frac{1}{m+n}\big)$, and went on to conjecture that their upper bound is sharp for all $(m,n) \neq (1,1)$ (see also [11, Problem 1]).

Cheung and Chevallier's result for singular vectors was an equality and they needed to develop separate tools to deal with upper and lower bounds. They developed the notion of *best approximation vectors* and a multidimensional extension of Legendre's theorem on convergents of real continued fraction expansions, as well as the notion of *self-similar coverings* that construct Cantor sets with "inhomogeneous" tree structures. On the other hand, though KKLM were only able to prove an upper bound rather than an equality, their methods, which were orthogonal to those of Cheung and Chevallier, leveraged the technology of integral inequalities developed by Eskin, Margulis and Mozes [25] and extend Cheung and Chevallier's upper bound to the matrix framework.

In this paper we prove that KKLM's conjecture is correct, as we announced in [20]. We will also show that the packing dimension of $\mathrm{Sing}(m,n)$ is the same as its Hausdorff dimension, thus answering a question of Bugeaud, Cheung, and Chevallier [11, Problem 7]. To summarize:

Theorem 1.1. *For all* $(m,n) \neq (1,1)$, *we have*

$$\dim_H(\mathrm{Sing}(m,n)) = \dim_P(\mathrm{Sing}(m,n)) = \delta_{m,n} \stackrel{\text{def}}{=} mn\left(1 - \tfrac{1}{m+n}\right),$$

where $\dim_H(S)$ *and* $\dim_P(S)$ *denote the Hausdorff and packing dimensions of a set* S, *respectively.*

1.1. **Dani correspondence.** The set of singular matrices is linked to homogeneous dynamics via the *Dani correspondence principle* [18, 40]. For each $t \in \mathbb{R}$ and for each matrix A, let

$$g_t \stackrel{\text{def}}{=} \begin{bmatrix} e^{t/m}\mathrm{I}_m & \\ & e^{-t/n}\mathrm{I}_n \end{bmatrix}, \qquad u_A \stackrel{\text{def}}{=} \begin{bmatrix} \mathrm{I}_m & A \\ & \mathrm{I}_n \end{bmatrix},$$

where I_k denotes the k-dimensional identity matrix. Finally, let $d = m + n$, and for each $j = 1, \ldots, d$, let $\lambda_j(\Lambda)$ denote the jth successive minimum of a lattice $\Lambda \subseteq \mathbb{R}^d$ (with respect to some fixed norm on $\mathbb{R}^d$), i.e. the infimum of λ such that the set $\{\mathbf{r} \in \Lambda : \|\mathbf{r}\| \leq \lambda\}$ contains j linearly independent vectors. Then the Dani correspondence principle is a dictionary between the Diophantine properties of a matrix A on the one hand, and the dynamical properties of the orbit $(g_t u_A \mathbb{Z}^d)_{t \geq 0}$ on the other.

Recall that an $m \times n$ matrix A is called *badly approximable* if there exists $c > 0$ such that for all integer vectors $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ we have $\|A\mathbf{q} + \mathbf{p}\| \geq c\|\mathbf{q}\|^{-\frac{n}{m}}$; and is called *very well approximable* if there exist $\varepsilon > 0$ and infinitely many integer vectors $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that $\|A\mathbf{q} + \mathbf{p}\| \leq \|\mathbf{q}\|^{-(\frac{n}{m}+\varepsilon)}$. Such classes have been intensively studied within the field of metric Diophantine approximation [5, 10, 23].

Diophantine properties of A	**Dynamical properties of $(g_t u_A x_0)_{t\geq 0}$**
A is *badly approximable*	$(g_t u_A x_0)_{t\geq 0}$ is bounded
A is *singular*	$(g_t u_A x_0)_{t\geq 0}$ is divergent
A is *very well approximable*	$\limsup_{t\to\infty} \frac{1}{t} d(x_0, g_t u_A x_0) > 0$

We denote the sets of badly approximable, singular, and very well approximable matrices by $\mathrm{BA}(m,n)$, $\mathrm{Sing}(m,n)$, and $\mathrm{VWA}(m,n)$, respectively. Using the Dani correspondence principle, the fact that they are all Lebesgue null sets can now be seen to follow from the ergodicity of the (g_t)-action (see [3, Corollary 2.2 in Chapter III]). Indeed, in each case it suffices to show that any trajectory that equidistributes is not in the respective set. An equidistributed trajectory is not bounded because the orbit must be dense, proving that $\mathrm{BA}(m,n)$ is Lebesgue null. An equidistributed trajectory is not divergent because that would imply escape of mass, proving that $\mathrm{Sing}(m,n)$ is Lebesgue null. Finally, an equidistributed trajectory does not escape to infinity at a linear rate because this would imply that it spends a proportionally long time near infinity infinitely often, which would imply escape of mass (along a subsequence); thereby proving that $\mathrm{VWA}(m,n)$ is Lebesgue null.

It follows from the Dani correspondence principle that Theorem 1.1 implies that the set of divergent trajectories of the one-parameter diagonal (g_t)-action (on the space of unimodular lattices that has exactly two Lyapunov exponents with opposite signs) has equal Hausdorff and packing dimensions. In the sequel, we focus on Diophantine statements and leave it to the interested reader to translate our results in the language of homogeneous dynamics.

Let us precisely state the result mentioned in the middle row of the table above as it is particularly germane to our theme.

Theorem 1.2 ([18, Theorem 2.14])**.** *An $m \times n$ matrix A is singular if and only if the trajectory $(g_t u_A \mathbb{Z}^d)_{t\geq 0}$ is divergent in the space of unimodular lattices in $\mathbb{R}^d$, or equivalently (via Mahler's compactness criterion* [24, Theorem 11.33]*) if*

$$\lim_{t\to\infty} \lambda_1(g_t u_A \mathbb{Z}^d) = 0.$$

It is natural to ask about the set of matrices such that the above limit occurs at a prescribed rate, such as the set of matrices such that $-\log \lambda_1(g_t u_A \mathbb{Z}^d)$ grows linearly with respect to t. This question is closely linked with the concept of uniform exponents of irrationality. The *uniform exponent of irrationality* of an $m \times n$ matrix A, denoted $\widehat{\omega}(A)$, is the supremum of ω such that for all Q sufficiently large, there exist integer vectors $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q} + \mathbf{p}\| \leq Q^{-\omega} \text{ and } 0 < \|\mathbf{q}\| \leq Q.$$

By Dirichlet's theorem ([22] or [55, Theorem 1E in §II]), every $m \times n$ matrix A satisfies $\widehat{\omega}(A) \geq \frac{n}{m}$. Moreover, it is immediate from the definitions that any matrix A satisfying $\widehat{\omega}(A) > \frac{n}{m}$ is singular. We call a matrix *very singular* if it satisfies the inequality

$\widehat{\omega}(A) > \frac{n}{m}$, in analogy with the set of *very well approximable* matrices, which satisfy a similar inequality for the regular (non-uniform) exponent of irrationality. We denote the set of very singular $m \times n$ matrices by $\mathrm{VSing}(m,n)$. The relationship between uniform exponents of irrationality and very singular matrices on the one hand, and homogeneous dynamics on the other, is given as follows:

Theorem 1.3. *A matrix A is very singular if and only $\widehat{\tau}(A) > 0$, where*

$$\widehat{\tau}(A) \stackrel{\text{def}}{=} \liminf_{t\to\infty} \frac{-1}{t} \log \lambda_1(g_t u_A \mathbb{Z}^d).$$

Moreover, the quantities $\tau = \widehat{\tau}(A)$ and $\omega = \widehat{\omega}(A)$ are related by the formula

$$\tau = \frac{1}{n}\frac{\omega - \frac{n}{m}}{\omega + 1}. \tag{1.1}$$

This theorem is a straightforward example of the Dani correspondence principle and is probably well-known, but we have not been able to find a reference.

Proof. The first assertion follows from (1.1), so it suffices to prove (1.1). Let $\omega = \widehat{\omega}(A)$, and let τ be given by (1.1); then we need to prove that $\widehat{\tau}(A) = \tau$. We prove the $\geq$ direction; the $\leq$ direction is similar. Fix $\varepsilon > 0$ and $t \geq 0$, and let $Q = e^{(1/n-\tau)t}$. By the definition of ω, if t (and thus Q) is sufficiently large then there exist $\mathbf{p}, \mathbf{q}$ such that $\|A\mathbf{q} + \mathbf{p}\| \leq Q^{-\omega+\varepsilon}$ and $0 < \|\mathbf{q}\| \leq Q$. Now let

$$\mathbf{r} = g_t u_A(\mathbf{p}, \mathbf{q}) = (e^{t/m}(A\mathbf{q} + \mathbf{p}), e^{-t/n}\mathbf{q}).$$

Then

$$\begin{aligned}
\lambda_1(g_t u_A \mathbb{Z}^d) \leq \|\mathbf{r}\| &\asymp \max(e^{t/m}\|A\mathbf{q} + \mathbf{p}\|, e^{-t/n}\|\mathbf{q}\|) \\
&\leq \max(e^{t/m} Q^{-\omega+\varepsilon}, e^{-t/n} Q) \\
&= \max(e^{t/m} e^{(1/n-\tau)(-\omega+\varepsilon)}, e^{-\tau t}) \\
&= \exp\left(-t \min\left(\tau, \left(\tfrac{1}{n} - \tau\right)(\omega - \varepsilon) - \tfrac{1}{m}\right)\right).
\end{aligned}$$

Since t was arbitrary, it follows that

$$\widehat{\tau}(A) = \liminf_{t\to\infty} \frac{-1}{t} \log \lambda_1(g_t u_A \mathbb{Z}^d) \geq \min\left(\tau, \left(\tfrac{1}{n} - \tau\right)(\omega - \varepsilon) - \tfrac{1}{m}\right).$$

Taking the limit as $\varepsilon \to 0$ we get

$$\widehat{\tau}(A) \geq \min\left(\tau, \left(\tfrac{1}{n} - \tau\right)\omega - \tfrac{1}{m}\right) = \min(\tau, \tau) = \tau. \qquad \square$$

1.2. **Dimensions of very singular matrices.** Perhaps unsurprisingly, the set of very singular matrices has the same dimension properties as the set of singular matrices.

Theorem 1.4. *For all $(m,n) \neq (1,1)$, we have*

$$\dim_H(\mathrm{VSing}(m,n)) = \dim_P(\mathrm{VSing}(m,n)) = \delta_{m,n}.$$

One can also ask for more precise results regarding the function $\widehat{\omega}$. Specifically, for each $\omega > \frac{n}{m}$ we can consider the levelset[2]

$$\mathrm{Sing}_{m,n}(\omega) \stackrel{\mathrm{def}}{=} \{A : \widehat{\omega}(A) = \omega\} = \{A : \widehat{\tau}(A) = \tau\} \stackrel{\mathrm{def}}{=} \mathrm{Sing}_{m,n}(\tau),$$

where τ is given by (1.1). It would be desirable to obtain precise formulas for the Hausdorff and packing dimensions of $\mathrm{Sing}_{m,n}(\omega)$ in terms of ω, m, and n, see e.g. [11, Problem 2]. However, this appears to be extremely challenging at the present juncture. We have made significant progress towards this question: solving it completely in the cases $(m,n) = (1,2)$ and $(m,n) = (2,1)$, and for packing dimension in the case where $n \geq 2$. See Theorems 1.8 and 1.10 for details.

In general, we have obtained asymptotic formulas of two types: estimates valid when ω is small and estimates valid when ω is large. Note that while the minimum value of $\widehat{\omega}$ is always $\frac{n}{m}$ (corresponding to $\widehat{\tau} = 0$), the maximum value depends on whether or not n is at least 2. If $n \geq 2$, then the maximum value of $\widehat{\omega}$ is ∞ (corresponding to $\widehat{\tau} = \frac{1}{n}$), while if $n = 1$, then the maximum value of $\widehat{\omega}$ (excluding rational points) is 1 (corresponding to $\widehat{\tau} = \frac{m-1}{2m}$).[3] Consequently, we have two different asymptotic estimates of the dimensions of $\mathrm{Sing}_{m,n}(\omega)$ when ω is large corresponding to these two cases. In all of the formulas below, τ is related to ω by the formula (1.1).

Theorem 1.5. *Suppose that $(m,n) \neq (1,1)$. Then for all $\omega > \frac{n}{m}$ sufficiently close to $\frac{n}{m}$, we have*

$$\begin{aligned} \dim_H(\mathrm{Sing}_{m,n}(\omega)) &= \delta_{m,n} - \Theta\left(\sqrt{\omega - \tfrac{n}{m}}\right) & \dim_P(\mathrm{Sing}_{m,n}(\omega)) &= \delta_{m,n} - \Theta\left(\omega - \tfrac{n}{m}\right) \\ &= \delta_{m,n} - \Theta\left(\sqrt{\tau}\right) & &= \delta_{m,n} - \Theta\left(\tau\right) \end{aligned}$$

unless $(m,n) = (2,2)$, in which case

$$\begin{aligned} \dim_H(\mathrm{Sing}_{m,n}(\omega)) &= \delta_{m,n} - \Theta\left(\omega - \tfrac{n}{m}\right) & \dim_P(\mathrm{Sing}_{m,n}(\omega)) &= \delta_{m,n} - \Theta\left(\omega - \tfrac{n}{m}\right) \\ &= \delta_{m,n} - \Theta\left(\tau\right) & &= \delta_{m,n} - \Theta\left(\tau\right). \end{aligned}$$

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[2]For results considering the superlevelset, see Theorem 2.9.

[3]The reason for this is that if $n = 1$, then for trivial reasons the value of $\widehat{\omega}$ at a point $\mathbf{x} \in \mathbb{R}^m$ is at most the minimum value of $\widehat{\omega}$ over the coordinates $x_1, \ldots, x_m$, and if $\mathbf{x}$ is irrational, then for some $i = 1, \ldots, m$, x_i is irrational and therefore (since we are in one dimension) satisfies $\widehat{\omega}(x_i) = 1$.

In the sequel, we refer to the dimension formulas in the case $(m,n) \notin \{(1,1),(2,2)\}$ *as "the first case of Theorem 1.5", and to the dimension formulas in the case* $(m,n) = (2,2)$ *as "the second case of Theorem 1.5".*

Theorem 1.6. *Suppose that* $n \geq 2$*. Then for all* $\omega < \infty$ *sufficiently large, we have*

$$\begin{aligned} \dim_H(\mathrm{Sing}_{m,n}(\omega)) &= mn - 2m + \Theta\left(\tfrac{1}{\omega}\right) \qquad \dim_P(\mathrm{Sing}_{m,n}(\omega)) = mn - m. \\ &= mn - 2m + \Theta\left(\tfrac{1}{n} - \tau\right) \end{aligned}$$

Theorem 1.7. *Suppose that* $n = 1$ *and* $m \geq 2$*. Then for all* $\omega < 1$ *sufficiently close to* 1*, we have*

$$\begin{aligned} \dim_H(\mathrm{Sing}_{m,n}(\omega)) &= \Theta\left(1 - \omega\right) \qquad \dim_P(\mathrm{Sing}_{m,n}(\omega)) = 1. \\ &= \Theta\left(\tfrac{m-1}{2m} - \tau\right) \end{aligned}$$

Beyond the results above, we have a precise formula for the packing dimension when $n \geq 2$, which remains a lower bound when $n = 1$.

Theorem 1.8. *Define the function*

$$\overline{\delta}_{m,n}(\tau) \stackrel{\text{def}}{=} \max\left(mn - m,\ \delta_{m,n} - \frac{mn}{m+n}(d+m)\tau,\ mn - \frac{mn}{m+n}\frac{1+m\tau}{1 - \frac{mn}{m-1}\tau}\right).$$

Then we have

$$\dim_P(\mathrm{Sing}_{m,n}(\tau)) \geq \overline{\delta}_{m,n}(\tau), \tag{1.2}$$

with the understanding that the last piece of $\overline{\delta}_{m,n}(\tau)$ *is ignored if* $m = 1$*. If* $n \geq 2$*, then equality holds in* (1.2).

Remark. The cases of the maximum correspond to $\tau \in [\tau_2, \frac{1}{n}]$, $\tau \in [\tau_1, \tau_2]$, and $\tau \in [0, \tau_1]$, respectively, where $\tau_1 = \frac{m^2-d}{mn(d+m)}$ and $\tau_2 = \frac{m}{n(m+d)}$. Note that $\tau_1 > 0$ if and only if $m^2 > d$. When $\tau_1 \leq 0$, then the second case of the maximum holds for all $\tau \in [0, \tau_2]$.

When $n = 1$, the inequality (1.2) is strict for some values of τ, as shown by the following theorem:

Theorem 1.9. *We have*

$$\begin{aligned} &\dim_P(\mathrm{Sing}_{m,1}(\tau)) \geq 1 && \text{for all } 0 < \tau \leq \tfrac{m-1}{2m}, \text{ and} \\ &\dim_P(\mathrm{Sing}_{m,1}(\tau)) \geq m - 1 && \text{for all } 0 < \tau \leq \tfrac{1}{m^2}. \end{aligned}$$

Remark. To see that Theorem 1.9 implies that the inequality (1.2) in Theorem 1.8 is strict for some values of τ, note that $\overline{\delta}_{m,1}(\frac{m-1}{2m}) = \frac{1}{2} < 1$. For $m \geq 3$, we have

$$\overline{\delta}_{m,1}\left(\frac{1}{m^2}\right) = m - 1 - \frac{1}{m^2 - m - 1} < m - 1.$$

When $m = 2$, we instead have

$$\overline{\delta}_{m,1}\left(\frac{1}{m^2}\right) = \overline{\delta}_{m,1}\left(\frac{m-1}{2m}\right) = \frac{1}{2} < 1 = m - 1.$$

1.2.1. *Trivially singular matrices.* Call a matrix A *trivially singular* if there exists $j = 1, \ldots, d - 1$ such that

$$\log \lambda_{j+1}(g_t u_A \mathbb{Z}^d) - \log \lambda_j(g_t u_A \mathbb{Z}^d) \to \infty \text{ as } t \to \infty.$$

Then all of the formulas above in Theorems 1.5-1.9 remain true if $\mathrm{Sing}_{m,n}(\omega)$ is replaced by the set

$$\mathrm{Sing}^*_{m,n}(\omega) \stackrel{\text{def}}{=} \{A \in \mathrm{Sing}_{m,n}(\omega) : A \text{ is not trivially singular}\}.$$

Similarly, the formulas in Theorems 1.1 and 1.4 above and in Theorems 1.10-1.14 below remain true if we restrict to the respective sets of matrices that are not trivially singular. The reason for this is since while proving lower bounds none of the templates (see Definition 2.1) we construct are trivially singular.

Moreover, for $n \geq 2$ we have

$$\dim_H(\mathrm{Sing}^*_{m,n}(\infty)) = mn - 2m \qquad \dim_P(\mathrm{Sing}^*_{m,n}(\infty)) = mn - m$$

and for $n = 1$, $m \geq 2$ we have

$$\dim_H(\mathrm{Sing}^*_{m,n}(1)) = 0 \qquad \dim_P(\mathrm{Sing}^*_{m,n}(1)) = 1.$$

Note that the class of trivially singular matrices is smaller than the class of matrices with degenerate trajectories in the sense of [18, Definition 2.8], but larger than the class considered in [11, p.2] consisting of matrices A such that the group $A\mathbb{Z}^n + \mathbb{Z}^m$ does not have full rank. A $d \times 1$ or $1 \times d$ matrix is trivially singular if and only if it is contained in a rational hyperplane of $\mathbb{R}^d$.

1.3. 1×2 **and** 2×1 **matrices.** Beyond our asympototic formulas stated in the previous section, we obtain precise formulas for the Hausdorff and packing dimensions of $\mathrm{Sing}_{m,n}(\omega)$ for the cases $(m, n) = (1, 2)$ and $(m, n) = (2, 1)$. Our dimension formulas complete a cornucopia of bounds due to Baker, Bugeaud–Laurent, Laurent, Dodson,

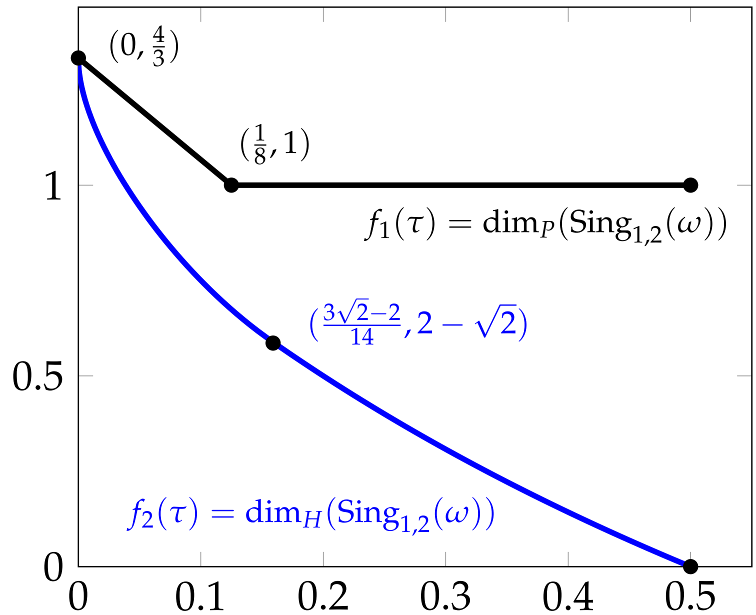

FIGURE 1. Graphs of the dimension functions

$$f_1(\tau) \overset{\text{def}}{=} \dim_P(\mathrm{Sing}_{1,2}(\omega)) \text{ and } f_2(\tau) \overset{\text{def}}{=} \dim_H(\mathrm{Sing}_{1,2}(\omega)).$$

The packing dimension function f_1 is linear on the intervals $[0, 1/8]$ and $[1/8, 1/2]$, while the Hausdorff dimension function f_2 is real-analytic on the intervals $[0, \tau_0]$ and $[\tau_0, 1/2]$, where $\tau_0 = (3\sqrt{2}-2)/14 \sim 0.1602$.

Yavid, Rynne, and Bugeaud–Cheung–Chevallier (1977–2016). We refer to [11] for a detailed history of the prior results.

Theorem 1.10. *For all $\omega \in (2, \infty)$ (corresponding to $\tau \in (0, 1/2)$) we have*

$$\dim_H(\mathrm{Sing}_{1,2}(\omega)) = \begin{cases} \frac{4}{3} - \frac{4}{3}\sqrt{\tau - 6\tau^3 + 4\tau^4} - 2\tau + \frac{8}{3}\tau^2 & \text{if } \tau \le \tau_0 \overset{\text{def}}{=} \frac{3\sqrt{2}-2}{14} \\ \frac{1-2\tau}{1+\tau} & \text{if } \tau \ge \tau_0 \end{cases}$$

$$\dim_P(\mathrm{Sing}_{1,2}(\omega)) = \begin{cases} \frac{4-8\tau}{3} & \text{if } \tau \le \tau_1 \overset{\text{def}}{=} \frac{1}{8} \\ 1 & \text{if } \tau \ge \tau_1 \end{cases}$$

(cf. Figure 1).

Remark. There had been a lot of partial progress towards the Hausdorff dimension part of Theorem 1.10. In particular, the $\ge$ direction follows from [11, Corollary 2 and Theorem 3]. For $\tau \ge \tau_0$ the upper bound follows from [11, Corollary 2] and for $\tau < \tau_0$, a non-optimal upper bound is given in [11, Theorem 1].

Remark. By Jarník's identity [34] (see also [29, Theorem A]), for all $\omega \in [2,\infty)$ we have

$$\mathrm{Sing}_{1,2}(\omega) = \mathrm{Sing}_{2,1}(\omega')$$

where $\omega' = 1 - \frac{1}{\omega}$, and

$$\mathrm{Sing}_{1,2}(\infty) = \mathrm{Sing}_{2,1}(1) \cup \mathrm{Sing}_{2,1}(\infty).$$

Thus by applying an appropriate substitution to the above formulas and using the fact that $\mathrm{Sing}_{2,1}(\infty)$ is countable (it is the set of rational points), it is possible to get explicit formulas for $\dim_H(\mathrm{Sing}_{2,1}(\omega'))$ and $\dim_P(\mathrm{Sing}_{2,1}(\omega'))$, either in terms of ω' or in terms of

$$\tau' = \frac{\omega' - \frac{1}{2}}{\omega' + 1} = \frac{\tau}{1+2\tau}.$$

However, the resulting formulas are not very elegant so we omit them.

Remark. The transition point $\tau_0 = (3\sqrt{2} - 2)/14$ in the above formula for Hausdorff dimension corresponds to

$$\omega_0 = 2 + \sqrt{2}\,,\ \omega_0' = \sqrt{2}/2\,,\ \tau_0' = (4 - 3\sqrt{2})/2\,,\text{ and } \dim_H(\mathrm{Sing}_{1,2}(\omega_0)) = 2 - \sqrt{2}.$$

The transition point $\tau_1 = 1/8$ for packing dimension corresponds to

$$\omega_1 = 3\,,\ \omega_1' = 2/3\,,\ \tau_1' = 1/10\,,\text{ and } \dim_P(\mathrm{Sing}_{1,2}(\omega_1)) = 1.$$

Remark. Theorem 1.10 implies that $\dim_H(\mathrm{Sing}_{1,2}(\omega)) < \dim_P(\mathrm{Sing}_{1,2}(\omega))$ for all $\omega \in (2,\infty)$. This answers the first part of [11, Problem 7] in the affirmative.

1.4. **Singularity on average.** A different way of quantifying the notion of singularity is the notion of *singularity on average* introduced in [35]. Given a matrix A, we define the *proportion of time spent near infinity* to be the number

$$\mathcal{P}(A) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{T}\lambda\big(\{t \in [0,T] : \lambda_1(g_t u_A \mathbb{Z}^d) \leq \varepsilon\}\big) \in [0,1],$$

where λ denotes Lebesgue measure. The matrix A is said to be *singular on average* if $\mathcal{P}(A) = 1$. Clearly, every singular matrix is singular on average.

Theorem 1.11. *For all $p \in [0,1]$, we have*

$$\dim_H(\{A : \mathcal{P}(A) = p\}) = \dim_P(\{A : \mathcal{P}(A) = p\}) = p\delta_{m,n} + (1-p)mn.$$

In particular, the dimension of the set of matrices singular on average is $\delta_{m,n}$.

Note that the Hausdorff dimension part of this theorem proves the conjecture stated in [35, Remark 2.1], where the upper bound was proven. However, we give an independent proof of the upper bound. Also note that when $p = 1$, the lower bound for Hausdorff dimension follows from Theorem 1.1.

1.5. **Starkov's conjecture.** In [59, p.213], Starkov asked whether there exists a singular vector (i.e. $m \times 1$ singular matrix) which is not very well approximable. Here, we recall that a matrix A is called *very well approximable* if for some $\omega > \frac{n}{m}$, there exist infinitely many pairs $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n$ such that

$$\|A\mathbf{q} + \mathbf{p}\| \leq \|\mathbf{q}\|^{-\omega}, \tag{1.3}$$

or equivalently in terms of the Dani correspondence principle, a matrix A is very well approximable if $\limsup_{t\to\infty} -\frac{1}{t}\log\lambda_1(g_t u_A \mathbb{Z}^d) > 0$. This question was answered affirmatively by Cheung [16, Theorem 1.4] in the case $m = 2$. In fact, Cheung showed that if ψ is any function such that $q^{1/2}\psi(q) \to 0$ as $q \to \infty$, then there exists a 2×1 singular vector which is not ψ-approximable. Here, a matrix A is called *ψ-approximable* if there exist infinitely many pairs $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n$ such that $\mathbf{q} \neq \mathbf{0}$ and

$$\|A\mathbf{q} + \mathbf{p}\| \leq \psi(\|\mathbf{q}\|).$$

The following theorem improves on Cheung's result both by generalizing it to the case of arbitrary m, n (i.e. to the matrix approximation framework), and also by computing the dimension of the set of matrices with the given property:

Theorem 1.12. *If ψ is any function such that $q^{n/m}\psi(q) \to 0$ as $q \to \infty$, then the set of $m \times n$ singular matrices that are not ψ-approximable has Hausdorff dimension $\delta_{m,n}$. Equivalently, if ϕ is any function such that $\phi(t) \to \infty$ as $t \to \infty$, then the set of $m \times n$ singular matrices A such that $-\log\lambda_1(g_t u_A \mathbb{Z}^d) \leq \phi(t)$ for all t sufficiently large has Hausdorff dimension $\delta_{m,n}$. The same is true for the packing dimension.*

Note that this theorem is optimal in the sense that if $\psi(q) \geq cq^{-n/m}$ for some constant c, then it is easy to check that every singular $m \times n$ matrix is ψ-approximable.

1.6. **Schmidt's conjecture.** In [56, p.273], Schmidt conjectured that for all $2 \leq k \leq m$, there exists an $m \times 1$ matrix A such that

$$\lambda_{k-1}(g_t u_A \mathbb{Z}^d) \to 0 \text{ and } \lambda_{k+1}(g_t u_A \mathbb{Z}^d) \to \infty \text{ as } t \to \infty. \tag{1.4}$$

This conjecture was proven by Moshchevitin [48], who constructed an $m \times 1$ matrix A satisfying (1.4) and not contained in any rational hyperplane[4] (see also [36, 52]). To extend this discussion to the matrix framework, we make the following definition.

Definition 1.13. An $m \times n$ matrix A is *k-singular* for $2 \leq k \leq m+n-1$ if

$$\lambda_{k-1}(g_t u_A \mathbb{Z}^d) \to 0 \text{ and } \lambda_{k+1}(g_t u_A \mathbb{Z}^d) \to \infty \text{ as } t \to \infty. \tag{1.5}$$

(Note that any matrix satisfying (1.5) is singular by Theorem 1.2.)

We improve Moshchevitin's result by computing a lower bound on the Hausdorff dimension of the set of matrices witnessing Schmidt's conjecture in the matrix framework:

Theorem 1.14. *For all $(m,n) \neq (1,1)$ and for all $2 \leq k \leq m+n-1$, the Hausdorff dimension of the set of matrices A that satisfy* (1.5) *is at least*

$$\max(f_{m,n}(k), f_{m,n}(k-1))$$

where

$$f_{m,n}(k) \stackrel{\text{def}}{=} mn - \frac{k(m+n-k)mn}{(m+n)^2} - \left\{\frac{km}{m+n}\right\}\left\{\frac{kn}{m+n}\right\}. \tag{1.6}$$

Here $\{x\}$ denotes the fractional part of a real number x. The same formula is valid for the set of matrices A that satisfy (1.5) *and are not trivially singular.*

Remark. The function $f_{m,n}$ satisfies $f_{m,n}(m+n-k) = f_{m,n}(k)$ and $f_{m,n}(1) = f_{m,n}(m+n-1) = \delta_{m,n}$. Moreover, for all $1 \leq k \leq m+n-1$ we have $f_{m,n}(k) \leq \delta_{m,n}$. It follows that when $k=2$ or $m+n-1$, the Hausdorff and packing dimensions of the set of matrices A that satisfy (1.5) are both equal to $\delta_{m,n}$.

Remark. When $m=1$ or $n=1$, the fractional parts appearing in (1.6) can be computed explicitly, leading to the formula

$$f_{m,n}(k) = mn - \frac{k(m+n-k)}{m+n}.$$

However, this formula is not valid when $m,n \geq 2$.

We conjecture that the lower bound in Theorem 1.14 is optimal for both the Hausdorff and packing dimensions (see Conjecture 3.1 below).

[4]As observed by Moshchevitin [48, Corollary 2], proving Schmidt's conjecture by constructing an $m \times 1$ matrix A satisfying (1.4) which is contained in a rational hyperplane is actually trivial: let $A = (\mathbf{x}, \mathbf{0})$ where $\mathbf{x} \in \mathbb{R}^{k-1}$ or $\mathbf{x} \in \mathbb{R}^{k-2}$ is a badly approximable vector. We assume that if Schmidt had noticed this example, he would have included in his conjecture the requirement that A should not be contained in a rational hyperplane.

2. The variational principle

2.1. **Successive minima functions and templates.** All the theorems in the previous section (with the exception of Theorems 1.2 and 1.3) are consequences of a single *variational principle* in the parametric geometry of numbers. This variational principle is a quantitative analogue of theorems due to Schmidt and Summerer [57, §2] and Roy [49, Theorem 1.3]. However, we will state their results in language somewhat different from the language used in their papers, due to the fact that the fundamental object we consider is the one-parameter family of unimodular lattices $(g_t u_A \mathbb{Z}^d)_{t\geq 0}$ used by the Dani correspondence principle, rather than a one-parameter family of (non-unimodular) convex bodies as is done in [57, 49]. We leave it to the reader to verify that the theorems we attribute below to [57] and [49] are indeed faithful translations of their results to our setting.

The fundamental question of our version of the parametric geometry of numbers will be as follows: given a matrix A, what does the function $\mathbf{h} = \mathbf{h}_A = (h_1, \ldots, h_d) : [0, \infty) \to \mathbb{R}^d$ defined by the formula

$$h_i(t) \stackrel{\text{def}}{=} \log \lambda_i(g_t u_A \mathbb{Z}^d) \tag{2.1}$$

look like? The function $\mathbf{h}_A$ will be called the *successive minima function* of the matrix A. The Dani correspondence principle shows that many interesting Diophantine questions about the matrix A are equivalent to questions about its successive minima function. Thus the dictionary in §1.1 may be translated as follows.

Diophantine properties of A	**Asymptotic properties of** $h_{A,1}$
A is *badly approximable*	$\limsup_{t\to\infty} -h_{A,1}(t) < \infty$
A is *singular*	$\liminf_{t\to\infty} -h_{A,1}(t) = \infty$
A is *very well approximable*	$\limsup_{t\to\infty} \dfrac{-h_{A,1}(t)}{t} > 0$

The main restriction on the successive minima function comes from an application of Minkowski's second theorem on successive minima (see Theorem 7.1 below) to certain subgroups of the lattice $g_t u_A \mathbb{Z}^d$. Specifically, fix $j = 1, \ldots, d-1$ and let I be an interval such that $h_j(t) < h_{j+1}(t)$ for all $t \in I$. For each $t \in I$, let[5]

$$V_{j,t} \stackrel{\text{def}}{=} \langle \mathbf{r} \in \mathbb{Z}^d : \|g_t u_A \mathbf{r}\| \leq \lambda_j(g_t u_A \mathbb{Z}^d) \rangle \subseteq \mathbb{R}^d.$$

Then the map $t \mapsto V_{j,t}$ is continuous, and therefore constant, on I. By Minkowski's second theorem (Theorem 7.1), we have

$$\sum_{i \leq j} h_i(t) \asymp_+ F_{j,I}(t) \stackrel{\text{def}}{=} \log \|g_t u_A(V_{j,t} \cap \mathbb{Z}^d)\|,$$

where $\|\Gamma\|$ denotes the covolume of a discrete group $\Gamma \subseteq \mathbb{R}^d$ (relative to its linear span $\mathbb{R}\Gamma$). Now an argument based on the exterior product formula for covolume and the definition of g_t (see Lemma 8.8) shows that $F_{j,I} \asymp_+ G_{j,I}$ for some convex, piecewise linear function $G_{j,I}$ whose slopes are in the set

$$Z(j) \stackrel{\text{def}}{=} \left\{ \tfrac{L_+}{m} - \tfrac{L_-}{n} : L_\pm \in [0, d_\pm]_\mathbb{Z},\ \ L_+ + L_- = j \right\}, \tag{2.2}$$

where for convenience we write $d_+ = m$, $d_- = n$, and $[a,b]_\mathbb{Z} = [a,b] \cap \mathbb{Z}$. This suggests that $\mathbf{h}$ can be approximated by a piecewise linear function $\mathbf{f}$ such that whenever $f_j < f_{j+1}$ on an interval I, the function $F_j := \sum_{i \leq j} f_i$ is convex and piecewise linear on I with slopes in $Z(j)$. Moreover, it is obvious that $h_1 \leq \cdots \leq h_d$, and the formula for g_t implies that for all i, we have $-\frac{1}{n} \leq h_i' \leq \frac{1}{m}$ wherever h_i is differentiable. We therefore make the following definition:

Definition 2.1. An $m \times n$ *template* is a piecewise linear[6] map $\mathbf{f} : [0, \infty) \to \mathbb{R}^d$ with the following properties:

(I) $f_1 \leq \cdots \leq f_d$.
(II) $-\frac{1}{n} \leq f_i' \leq \frac{1}{m}$ for all i.
(III) For all $j = 1, \ldots, d$ and for every interval I such that $f_j < f_{j+1}$ on I, the function $F_j \stackrel{\text{def}}{=} \sum_{i \leq j} f_i$ is convex and piecewise linear on I with slopes in $Z(j)$. Here we use the convention that $f_0 = -\infty$ and $f_{d+1} = +\infty$. We will call the assertion that F_j is convex the *convexity condition*, and the assertion that its slopes are in $Z(j)$ the *quantized slope condition*.

When $m = 1$, templates are a slight generalization of reparameterized versions of the *rigid systems* of [49]. We denote the space of $m \times n$ templates by $\mathcal{T}_{m,n}$.

A template $\mathbf{f}$ will be called *balanced* if $F_d = f_1 + \ldots + f_d = 0$. Note that every template is equal to a constant plus a balanced template, since by condition (III), F_d is piecewise

[5]Here, $V_{j,t}$ is the smallest subspace containing $\{\mathbf{r} \in \mathbb{Z}^d : \|g_t u_A \mathbf{r}\| \leq \lambda_j(g_t u_A \mathbb{Z}^d)\}$. See Convention 4.

[6]In this paper, *piecewise linear* functions are assumed to be continuous.

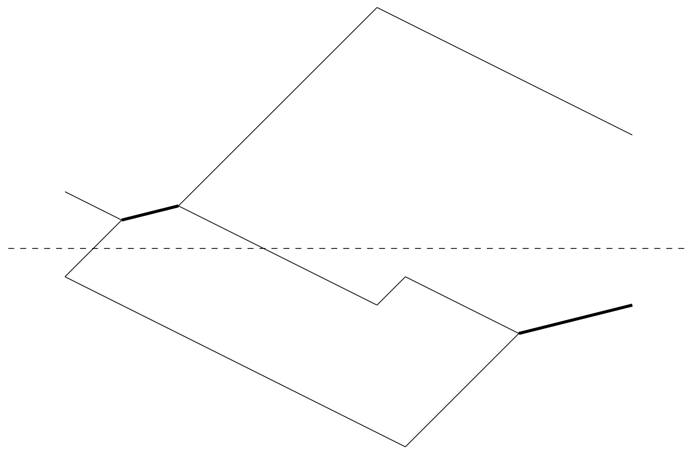

FIGURE 2. The joint graph of a 1×2 partial template $\mathbf{f} = (f_1, f_2, f_3)$, where the *joint graph* of a template is the union of the graphs of its component functions.

linear with slopes in $Z(d) = \{0\}$, and thus constant. So for most purposes the distinction between balanced and unbalanced templates is irrelevant, but in some places it will make a difference. A *partial template* is a piecewise linear map $\mathbf{f}$ satisfying (I)-(III) whose domain is a closed, possibly infinite, subinterval of $[0, \infty)$. An example of a (partial) template is shown in Figure 2.

The fundamental relation between templates and successive minima functions is given as follows:

Theorem 2.2.

(i) *For every $m \times n$ matrix A, there exists an $m \times n$ template $\mathbf{f}$ such that $\mathbf{h}_A \asymp_+ \mathbf{f}$.*
(ii) *For every $m \times n$ template $\mathbf{f}$, there exists an $m \times n$ matrix A such that $\mathbf{h}_A \asymp_+ \mathbf{f}$.*

In the case $m = 1$, Theorem 2.2 follows from [49, Theorem 1.3] (cf. [50, Corollary 4.7] for part (ii)).

Theorem 2.2(ii) asserts that for every template $\mathbf{f}$, the set

$$\mathcal{M}(\mathbf{f}) \stackrel{\text{def}}{=} \{A : \mathbf{h}_A \asymp_+ \mathbf{f}\}$$

is nonempty. It is natural to ask how big this set is in terms of Hausdorff and packing dimension. Moreover, given a collection of templates $\mathcal{F}$, we can ask the same question about the set

$$\mathcal{M}(\mathcal{F}) \stackrel{\text{def}}{=} \bigcup_{\mathbf{f} \in \mathcal{F}} \mathcal{M}(\mathbf{f}).$$

It turns out to be easier to answer the second question than the first, assuming that the collection of templates $\mathcal{F}$ is closed under finite perturbations. Here, $\mathcal{F}$ is said to be *closed under finite perturbations* if whenever $\mathbf{g} \asymp_+ \mathbf{f} \in \mathcal{F}$, we have $\mathbf{g} \in \mathcal{F}$.

Theorem 2.3 (Variational principle, version 1)**.** *Let $\mathcal{F}$ be a collection of templates closed under finite perturbations. Then*

$$\dim_H(\mathcal{M}(\mathcal{F})) = \sup_{\mathbf{f}\in\mathcal{F}} \underline{\delta}(\mathbf{f}), \qquad \dim_P(\mathcal{M}(\mathcal{F})) = \sup_{\mathbf{f}\in\mathcal{F}} \overline{\delta}(\mathbf{f}), \tag{2.3}$$

where the functions $\underline{\delta}, \overline{\delta} : \mathcal{T}_{m,n} \to [0, mn]$ are as in Definition 2.5 below.

Corollary 2.4. *With $\mathcal{F}$ as above, we have*

$$\dim_H(\mathcal{M}(\mathcal{F})) = \sup_{\mathbf{f}\in\mathcal{F}} \dim_H(\mathcal{M}(\mathbf{f})), \qquad \dim_P(\mathcal{M}(\mathcal{F})) = \sup_{\mathbf{f}\in\mathcal{F}} \dim_P(\mathcal{M}(\mathbf{f})). \tag{2.4}$$

However, note that Theorem 2.3 does not imply that $\dim_H(\mathcal{M}(\mathbf{f})) = \underline{\delta}(\mathbf{f})$ for an individual template $\mathbf{f}$, since the family $\{\mathbf{f}\}$ is not closed under finite perturbations. And indeed, since the function $\underline{\delta}$ is sensitive to finite perturbations, the formula $\dim_H(\mathcal{M}(\mathbf{f})) = \underline{\delta}(\mathbf{f})$ cannot hold for all $\mathbf{f} \in \mathcal{T}_{m,n}$.

Definition 2.5. We define the lower and upper average contraction rate of a template $\mathbf{f}$ as follows. Let I be an open interval on which $\mathbf{f}$ is linear. For each $q = 1, \ldots, d$ such that $f_q < f_{q+1}$ on I, let $L_\pm = L_\pm(\mathbf{f}, I, q) \in [0, d_\pm]_\mathbb{Z}$ be chosen to satisfy $L_+ + L_- = q$ and

$$F_q' = \sum_{i=1}^q f_i' = \frac{L_+}{m} - \frac{L_-}{n} \text{ on } I, \tag{2.5}$$

as guaranteed by (III) of Definition 2.1. An *interval of equality* for $\mathbf{f}$ on I is an interval $(p, q]_\mathbb{Z}$, where $0 \leq p < q \leq d$ satisfy

$$f_p < f_{p+1} = \cdots = f_q < f_{q+1} \text{ on } I. \tag{2.6}$$

As before, we use the convention that $f_0 = -\infty$ and $f_{d+1} = +\infty$. Note that the collection of intervals of equality forms a partition of $[1, d]_\mathbb{Z}$. If $(p, q]_\mathbb{Z}$ is an interval of equality for $\mathbf{f}$ on I, then we let $M_\pm(p, q) = M_\pm(\mathbf{f}, I, p, q)$, where

$$M_\pm(\mathbf{f}, I, p, q) = L_\pm(\mathbf{f}, I, q) - L_\pm(\mathbf{f}, I, p), \tag{2.7}$$

and we let

$$S_+ = S_+(\mathbf{f}, I) = \bigcup_{(p,q]_\mathbb{Z}} \big(p, p + M_+(p,q)\big]_\mathbb{Z} \tag{2.8}$$

$$S_- = S_-(\mathbf{f}, I) = \bigcup_{(p,q]_\mathbb{Z}} \big(p + M_+(p,q), q\big]_\mathbb{Z} \tag{2.9}$$

where the unions are taken over all intervals of equality for $\mathbf{f}$ on I. Note that S_+ and S_- are disjoint and satisfy $S_+ \cup S_- = [1,d]_\mathbb{Z}$, and that $\#(S_+) = m$ and $\#(S_-) = n$. Next, let

$$\delta(\mathbf{f}, I) = \#\{(i_+, i_-) \in S_+ \times S_- : i_+ < i_-\} \in [0, mn]_\mathbb{Z}, \tag{2.10}$$

and note that

$$mn - \delta(\mathbf{f}, I) = \#\{(i_+, i_-) \in S_+ \times S_- : i_+ > i_-\}. \tag{2.11}$$

The *lower and upper average contraction rates* of $\mathbf{f}$ are the numbers

$$\underline{\delta}(\mathbf{f}) \stackrel{\text{def}}{=} \liminf_{T\to\infty} \Delta(\mathbf{f}, T), \qquad \overline{\delta}(\mathbf{f}) \stackrel{\text{def}}{=} \limsup_{T\to\infty} \Delta(\mathbf{f}, T), \tag{2.12}$$

where

$$\Delta(\mathbf{f}, T) \stackrel{\text{def}}{=} \frac{1}{T}\int_0^T \delta(\mathbf{f}, t)\, \mathrm{d}t.$$

Here we abuse notation by writing $\delta(\mathbf{f}, t) = \delta(\mathbf{f}, I)$ for all $t \in I$. We will also have occasion later to use the notations

$$\Delta(\mathbf{f}, [T_1, T_2]) = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \delta(\mathbf{f}, t)\, \mathrm{d}t$$

and

$$\delta(T_+, T_-) = \#\{(i_+, i_-) \in T_+ \times T_- : i_+ < i_-\} \in [0, mn]_\mathbb{Z}. \tag{2.13}$$

Note that according to (2.13), $\delta(\mathbf{f}, I) = \delta(S_+, S_-)$.

Definition 2.5 can be understood intuitively in terms of a simple version of one-dimensional physics with sticky collisions and conservation of momentum; cf. Figure 3. Suppose that we observe particles $P_1, \ldots, P_d$ travelling along trajectories $f_1, \ldots, f_d$ during a time interval I along which $\mathbf{f}$ is linear, and we want to infer the velocities of these particles before they collided, based on the following background information: before the collision m of the particles were travelling upwards at a speed of $\frac{1}{m}$, and n of the particles were travelling downwards at a speed of $\frac{1}{n}$. When particles collide (that is, when the velocities of the particles of lower index are more upwards than the velocities of the particles of higher index at the same location), they join forces to move as a unit, and

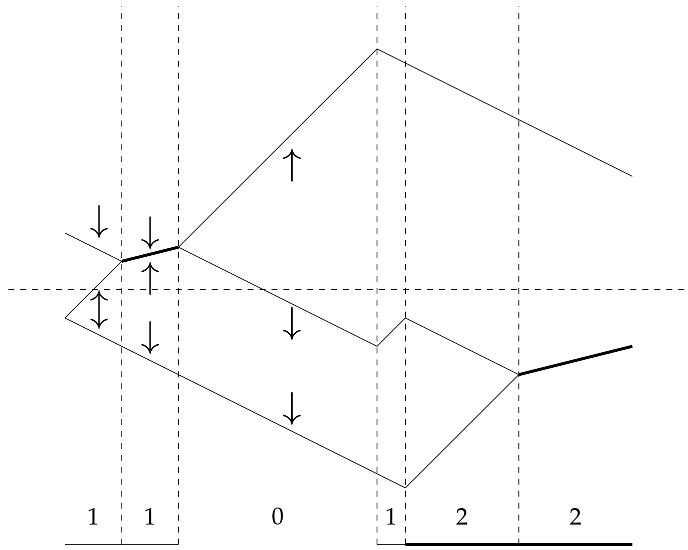

FIGURE 3. The joint graph in Figure 2, with an illustration of the sets $S_\pm(\mathbf{f}, I)$ and the contraction rates $\delta(\mathbf{f}, I)$ for each interval of linearity I. The "one-dimensional physics" interpretation of templates can be seen in this picture as follows: first one particle is going up while two are going down; then the top two collide into each other and their new veolocity is determined by conservation of momentum; then they split apart again. Given this interpretation of the motion occurring in I as being the result of "collisions" between m particles going up and n particles going down, $\delta(\mathbf{f}, I)$ counts the number of particle pairs that are "moving towards" each other (including particles "colliding" with each other).

their new velocity is determined by conservation of momentum. However, we can still think of the group as being composed of a certain number of "upwards" particles and a certain number of "downwards" particles.

The equations (2.8) and (2.9) can be understood as suggesting a particular solution to this problem of inference: assume that within each group, all of the upwards-travelling particles started out below all of the downwards-travelling particles. This is not the only possible solution but it is the nicest one for certain purposes. Specifically, we can imagine a force of "gravity" attempting to bring all of the particles together, which acts between any two particles by imposing a fixed energy cost if the two particles are travelling away from each other.[7] The total energy cost is then the codimension $mn - \delta(\mathbf{f}, I)$ defined by (2.11). The equations (2.8) and (2.9) can then be thought of as giving the solution that minimizes this cost.

[7]This is of course unlike real gravity, which imposes an energy cost that varies with respect to distance.

The idea of codimension as an energy cost is also useful for computing the suprema (2.3) in certain circumstances, since it suggests principles like the conservation of energy. However, one needs to be careful since the stickiness of collisions means that some naive formulations of conservation of energy are violated.

In most cases of interest, the collection $\mathcal{F}$ in Theorem 2.3 is defined by some Diophantine condition. In this case, generally rather than $\mathcal{M}(\mathcal{F})$ the set we are really interested in is the set of all matrices whose corresponding successive minima functions satisfy the same Diophantine condition. Although these two sets are *a priori* different, Theorem 2.2(i) implies that they are the same and thus Theorem 2.3 is equivalent modulo Theorem 2.2(i) to the following:

Theorem 2.6 (Variational principle, version 2)**.** *Let $\mathcal{S}$ be a collection of functions from $[0,\infty)$ to $\mathbb{R}^d$ which is closed under finite perturbations, and let*

$$\mathcal{M}(\mathcal{S}) = \{A : \mathbf{h}_A \in \mathcal{S}\}. \tag{2.14}$$

Then

$$\dim_H(\mathcal{M}(\mathcal{S})) = \sup_{\mathbf{f}\in\mathcal{S}\cap\mathcal{T}_{m,n}} \underline{\delta}(\mathbf{f}), \qquad \dim_P(\mathcal{M}(\mathcal{S})) = \sup_{\mathbf{f}\in\mathcal{S}\cap\mathcal{T}_{m,n}} \overline{\delta}(\mathbf{f}). \tag{2.15}$$

In fact, Theorem 2.6 will be the version of the variational principle that we prove.

Proof of equivalence. Theorem 2.6 implies Theorem 2.3 since we can take $\mathcal{S} = \{\mathbf{g} : \mathbf{g} \asymp_+ \mathbf{f} \in \mathcal{F}\}$. Conversely, Theorem 2.3 implies Theorem 2.6 modulo Theorem 2.2(i) since we can take $\mathcal{F} = \mathcal{S} \cap \mathcal{T}_{m,n}$. □

Theorem 2.6 can be thought of as a quantitative strengthening of Theorem 2.2, as shown by the following equivalent formulation:

Theorem 2.7 (Variational principle, version 3)**.**

(i) *Let S be a set of $m \times n$ matrices of Hausdorff (resp. packing) dimension $> \delta$. Then there exist a matrix $A \in S$ and a template $\mathbf{f} \asymp_+ \mathbf{h}_A$ whose lower (resp. upper) average contraction rate is $> \delta$.*

(ii) *Let $\mathbf{f}$ be a template whose lower (resp. upper) average contraction rate is $> \delta$. Then there exists a set S of $m \times n$ matrices of Hausdorff (resp. packing) dimension $> \delta$, such that $\mathbf{h}_A \asymp_+ \mathbf{f}$ for all $A \in S$.*

Proof of equivalence. Part (i) is equivalent to the $\leq$ direction of (2.15), and part (ii) to the $\geq$ direction. For the first equivalence, for the forwards direction take $S = \{A : \mathbf{h}_A \in \mathcal{S}\}$, and for the backwards direction take $\mathcal{S} = \{\mathbf{g} : \mathbf{g} \asymp_+ \mathbf{h}_A,\ A \in S\}$. For the second equivalence, for the backwards direction take $S = \mathcal{M}(\mathbf{f})$ and $\mathcal{S} = \{\mathbf{g} : \mathbf{g} \asymp_+ \mathbf{f}\}$. $\square$

It is worth stating the special case of Theorem 2.6 that occurs when the collection $\mathcal{S}$ is defined by the Diophantine conditions defining $\mathrm{Sing}_{m,n}(\omega)$ and $\mathrm{Sing}^*_{m,n}(\omega)$ for some $\omega \geq \frac{n}{m}$. Thus, we define the *uniform dynamical exponent* of a map $\mathbf{f} : [0,\infty) \to \mathbb{R}^d$ to be the number

$$\widehat{\tau}(\mathbf{f}) \stackrel{\text{def}}{=} \liminf_{t\to\infty} \frac{-1}{t} f_1(t).$$

Similarly, $\mathbf{f}$ is said to be *trivially singular* if $f_{j+1}(t) - f_j(t) \to \infty$ as $t \to \infty$ for some $j = 1,\ldots,d-1$. Letting $\mathcal{S} = \{\mathbf{f} : \widehat{\tau}(\mathbf{f}) = \tau\}$ or $\mathcal{S} = \{\mathbf{f} : \widehat{\tau}(\mathbf{f}) = \tau,\ \mathbf{f} \text{ not trivially singular}\}$ in Theorem 2.6 yields the following result:

Theorem 2.8 (Special case of variational principle). *For all $\omega \geq \frac{n}{m}$, we have*

$$\begin{aligned}
\dim_H(\mathrm{Sing}_{m,n}(\omega)) &= \sup\{\underline{\delta}(\mathbf{f}) : \mathbf{f} \in \mathcal{T}_{m,n},\ \ \widehat{\tau}(\mathbf{f}) = \tau\}\\
\dim_P(\mathrm{Sing}_{m,n}(\omega)) &= \sup\{\overline{\delta}(\mathbf{f}) : \mathbf{f} \in \mathcal{T}_{m,n},\ \ \widehat{\tau}(\mathbf{f}) = \tau\}\\
\dim_H(\mathrm{Sing}^*_{m,n}(\omega)) &= \sup\{\underline{\delta}(\mathbf{f}) : \mathbf{f} \in \mathcal{T}_{m,n},\ \ \widehat{\tau}(\mathbf{f}) = \tau,\ \mathbf{f} \text{ not trivially singular}\}\\
\dim_P(\mathrm{Sing}^*_{m,n}(\omega)) &= \sup\{\overline{\delta}(\mathbf{f}) : \mathbf{f} \in \mathcal{T}_{m,n},\ \ \widehat{\tau}(\mathbf{f}) = \tau,\ \mathbf{f} \text{ not trivially singular}\}
\end{aligned}$$

where τ is as in (1.1)*.*

Theorem 2.6 can also be used to compute the dimensions of the set

$$\widetilde{\mathrm{Sing}}^*_{m,n}(\omega) \stackrel{\text{def}}{=} \{A : \widehat{\omega}(A) \geq \omega,\ A \text{ not trivially singular}\} = \bigcup_{\omega' \geq \omega} \mathrm{Sing}^*_{m,n}(\omega').$$

Theorem 2.9 (Special case of variational principle). *For all $\omega \geq \frac{n}{m}$, we have*

$$\begin{aligned}
\dim_H(\widetilde{\mathrm{Sing}}^*_{m,n}(\omega)) &= \sup_{\omega' \geq \omega} \dim_H(\mathrm{Sing}^*_{m,n}(\omega'))\\
\dim_P(\widetilde{\mathrm{Sing}}^*_{m,n}(\omega)) &= \sup_{\omega' \geq \omega} \dim_P(\mathrm{Sing}^*_{m,n}(\omega')).
\end{aligned}$$

(Theorem 2.9 is also true with the stars removed, but in that case it is not as interesting because $\dim_H(\mathrm{Sing}_{m,n}(\infty))$ is “too large”, whereas $\dim_H(\mathrm{Sing}^*_{m,n}(\infty))$ is the “correct” size according to §1.2.1.)

It is natural to expect that the map $\omega \mapsto \dim_H(\mathrm{Sing}^*_{m,n}(\omega))$ is monotonically decreasing, in which case Theorem 2.9 would imply that

$$\dim_H(\widetilde{\mathrm{Sing}}^*_{m,n}(\omega)) = \dim_H(\mathrm{Sing}^*_{m,n}(\omega)).$$

2.2. **New proofs of old results.** In addition to our new results, our techniques now provide a uniform framework to prove classical results in metric Diophantine approximation. The following result was proven in the one-dimensional setting by Jarník (1928) and in the matrix setting by Schmidt (1969).

Theorem 2.10 (Jarník–Schmidt, [32, 54])**.** *The Hausdorff dimension of the set of badly approximable matrices is mn.*

Recall that for each $\omega > \frac{n}{m}$, we say that a matrix A is ω-approximable if

$$\limsup_{|\mathbf{q}|\to\infty} \sup_{\mathbf{p}\in\mathbb{Z}^m} \frac{-\log\|A\mathbf{q}-\mathbf{p}\|}{\log\|\mathbf{q}\|} \geq \omega.$$

It follows from the Dani correspondence principle that A is ω-approximable if and only if

$$\limsup_{t\to\infty} \frac{-h_{A,1}(t)}{t} \geq \tau$$

where τ is as in (1.1).

The following theorem was proven in the one-dimensional case independently by Jarník (1929) and Besicovitch (1934), and in the matrix case by Bovey and Dodson (1986).

Theorem 2.11 (Jarník–Besicovitch–Bovey–Dodson, [33, 6, 8])**.** *The Hausdorff dimension of the set of ω-approximable matrices is $mn(1-\tau)$. In particular, the Hausdorff dimension of the very well approximable matrices is mn.*

We provide proofs of these theorems in Sections 28 and 29 respectively.

3. Directions to further research

We conclude our introduction with a small sample of open problems sampled from a range of potential research directions, which we hope will illustrate the wide scope available for future exploration.

3.1. **Quantitative Schmidt's conjecture.** We conjecture that the inequality in Theorem 1.14 is actually an equality:

Conjecture 3.1. *For $2 \leq k \leq m+n-1$, the Hausdorff and packing dimensions of the set of k-singular $m\times n$ matrices (see Definition 1.13) are both equal to*

$$\max(f_{m,n}(k), f_{m,n}(k-1)), \text{ where}$$

$$f_{m,n}(k) \stackrel{\text{def}}{=} mn - \frac{kmn}{m+n}\left(1-\frac{k}{m+n}\right) - \left\{\frac{km}{m+n}\right\}\left\{\frac{kn}{m+n}\right\}.$$

Here, $\{x\}$ denotes the fractional part of a real number x.

Remark 3.2. When $k = 2$ or $m + n - 1$, the Hausdorff and packing dimensions of the set of k-singular matrices are both equal to $\delta_{m,n}$.

3.2. Regularity of dimension functionals.

Problem 3.3. Determine when/whether the functions

$$\omega \mapsto \dim_H(\mathrm{Sing}_{m,n}(\omega)), \qquad \omega \mapsto \dim_P(\mathrm{Sing}_{m,n}(\omega))$$

are decreasing and continuous.

Although it is natural to suspect that these functions are in fact decreasing and continuous, Theorem 1.9 seems to suggest otherwise: it suggests that the function $\tau \mapsto \dim_P(\mathrm{Sing}_{m,1}(\tau))$ may have a discontinuity at $\tau = 1/m^2$ for all $m \geq 3$. Indeed, the proof of Theorem 1.9 gives us no reason to suspect that the inequality is strict in Theorem 1.8 for τ slightly greater than $1/m^2$. If in fact equality holds for such τ, then there *is* a discontinuity! If this were the case, it would show that the conjecture we made in the announcement of this paper [20, Conjecture 2.10] was too optimistic.

3.3. Intersecting standard and uniform exponent level sets.

Let $\omega(A)$ and $\widehat{\omega}(A)$ denote the standard and uniform exponents of irrationality of a matrix A, respectively:

$$\widehat{\omega}(A) \stackrel{\text{def}}{=} \liminf_{Q\to\infty} \sup_{0<\|\mathbf{q}\|\leq Q} \sup_{\mathbf{p}} \frac{-\log\|A\mathbf{q}+\mathbf{p}\|}{\log Q}$$

$$\omega(A) \stackrel{\text{def}}{=} \limsup_{Q\to\infty} \sup_{0<\|\mathbf{q}\|\leq Q} \sup_{\mathbf{p}} \frac{-\log\|A\mathbf{q}+\mathbf{p}\|}{\log Q}$$

The Hausdorff dimensions of the levelsets of ω are well-known, and we have provided many results on the Hausdorff dimensions of the levelsets of $\widehat{\omega}$. However, it is natural to ask about the dimension of the intersection of two such sets:

Question 3.4. What is the behavior of the function

$$(\omega, \widehat{\omega}) \mapsto \dim_H(\{A : \omega(A) = \omega,\ \widehat{\omega}(A) = \widehat{\omega}\})?$$

3.4. Metric theory for ε-Dirichlet improvable matrices.

Given $0 < \varepsilon < 1$, an $m \times n$ matrix A is called *ε-Dirichlet improvable* (see [21]) if for all sufficiently large Q, there exists $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n}$ such that

$$\|A\mathbf{q} - \mathbf{p}\| \leq \varepsilon Q^{-n/m} \quad \text{and} \quad 0 < \|\mathbf{q}\| < Q.$$

An $m \times n$ matrix A is *Dirichlet improvable* if it is ε-Dirichlet improvable for some $0 < \varepsilon < 1$. Singular matrices are ε-Dirichlet improvable for *all* $0 < \varepsilon < 1$.

Question 3.5. How do the Hausdorff and packing dimensions of the set of ε-Dirichlet improvable $m \times n$ matrices vary as functions of ε?

3.5. **Weighted singular matrices and general diagonal flows.** In the parametric geometry of numbers and the Dani correspondence principle we are generally concerned with the (g_t) flow as defined in §1.1. What happens if the (g_t) flow is replaced by some other diagonal flow (h_t), for example

$$h_t = \mathrm{diag}(e^{a_1 t}, \ldots, e^{a_m t}, e^{-b_1 t}, \ldots, e^{-b_n t}) \in \mathrm{SL}_{m+n}(\mathbb{R})$$

where $a_1, \ldots, a_m, b_1, \ldots, b_n$ are positive real numbers? For example, is it possible to compute the Hausdorff and packing dimensions of the set of $m \times n$ matrices A such that the trajectory $(h_t u_A \mathbb{Z}^{m+n})_{t \geq 0}$ is divergent as a function of $a_1, \ldots, a_m, b_1, \ldots, b_n$? When $m = 2$ and $n = 1$, this question in case of the Hausdorff dimension has been addressed by Liao, Shi, Solan, and Tamam [43].

3.6. **Inhomogeneous Diophantine approximation.** Our results fall within the domain of homogeneous Diophantine approximation. It would be of interest to investigate analogues of our results in the frameworks of inhomogeneous approximation, see [12, 13, 42]. In this setting, given an $m \times n$ matrix A and $\mathbf{x} \in \mathbb{R}^m$, the pair $(A, \mathbf{x})$ is called *singular* if for all $\varepsilon > 0$, there exists Q_ε such that for all $Q \geq Q_\varepsilon$, there exist integer vectors $\mathbf{p} \in \mathbb{Z}^m$ and $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q} + \mathbf{p} + \mathbf{x}\| \leq \varepsilon Q^{-n/m} \quad \text{and} \quad 0 < \|\mathbf{q}\| \leq Q.$$

It is also natural to study the inhomogeneous approximation frameworks where we fix one coordinate of the pair $(A, \mathbf{x})$ and let the other vary. Extending our variational principle (Theorem 2.6) and its corollaries to such inhomogeneous frameworks would be a natural next step.

3.7. **Parametric geometry of numbers in arbitrary characteristic.** It would be of interest to develop the technology introduced in this work to study questions of Diophantine approximation in the function field setting, see Roy and Waldschmidt [51].

Part 2. Dimension games

4. Preliminaries on measures and dimensions

We first recall the basics of Hausdorff and packing measures and dimensions, [7, 27]. Hausdorff measure and dimension were introduced in 1918 by Hausdorff [30], while packing measure and dimension were introduced by Tricot in 1982 [62]. Sullivan independently re-invented packing measures and dimensions when studying the limit sets of geometrically finite Kleinian groups in 1984 [60].

The *s-dimensional Hausdorff measure* of a set $A \subseteq \mathbb{R}^d$ is

$$\mathcal{H}^s(A) \stackrel{\text{def}}{=} \lim_{\varepsilon \searrow 0} \inf \left\{ \sum_{i=1}^\infty (\mathrm{diam}(U_i))^s : \begin{array}{l} (U_i)_1^\infty \text{ is a countable cover of } A \\ \text{with } \mathrm{diam}(U_i) \leq \varepsilon \ \ \forall i \end{array} \right\}.$$

Dual to the Hausdorff measure, which is defined via economical coverings by small balls, it is natural to define a measure in terms of dense packings by small disjoint balls. This leads to the notion of the *s-dimensional packing measure* of a set $A \subseteq \mathbb{R}^d$, which is defined by the formulas

$$\widetilde{\mathcal{P}}^s(A) \stackrel{\text{def}}{=} \lim_{\varepsilon \searrow 0} \sup \left\{ \sum_{j=1}^\infty (\mathrm{diam}(B_j))^s : \begin{array}{l} (B_j)_1^\infty \text{ is a countable disjoint collection of balls} \\ \text{with centers in } A \text{ and with } \mathrm{diam}(B_j) \leq \varepsilon \ \ \forall j \end{array} \right\}$$

$$\mathcal{P}^s(A) \stackrel{\text{def}}{=} \inf \left\{ \sum_{i=1}^\infty \widetilde{\mathcal{P}}^s(A_i) : A \subseteq \bigcup_{i=1}^\infty A_i \right\}.$$

Given the measures defined above, we define the *Hausdorff dimension* and *packing dimension* of a set $A \subseteq \mathbb{R}^d$ as follows:

$$\dim_H(A) \stackrel{\text{def}}{=} \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}$$

$$\dim_P(A) \stackrel{\text{def}}{=} \inf\{s : \mathcal{P}^s(A) = 0\} = \sup\{s : \mathcal{P}^s(A) = \infty\}.$$

We recall two basic facts (see [27, § 3.2 and § 3.5]) about these dimensions. First, they are both *monotonic*, i.e. if $E \subseteq F \subseteq \mathbb{R}^d$, then $\dim_H(E) \leq \dim_H(F)$ and $\dim_P(E) \leq \dim_P(F)$. Second, the packing dimension is bounded below by the Hausdorff dimension, i.e. for $F \subseteq \mathbb{R}^d$, we have $\dim_H(F) \leq \dim_P(F)$.

In the sequel we will also apply the following corollary to the Rogers–Taylor–Tricot density theorem [58, Theorem 2.1], a method of computing the Hausdorff and packing dimensions of a set in terms of local geometric-measure-theoretic information, which is stated below for the reader's convenience. For each point $\mathbf{x} \in \mathbb{R}^d$ define the lower and upper pointwise dimensions of a measure μ at $\mathbf{x}$ by

$$\underline{\dim}_{\mathbf{x}}(\mu) \stackrel{\text{def}}{=} \liminf_{\rho\to 0} \frac{\log \mu(B(\mathbf{x},\rho))}{\log \rho} \quad \text{and} \quad \overline{\dim}_{\mathbf{x}}(\mu) \stackrel{\text{def}}{=} \limsup_{\rho\to 0} \frac{\log \mu(B(\mathbf{x},\rho))}{\log \rho}.$$

Theorem 4.1 (Rogers–Taylor–Tricot [26, Proposition 2.3])**.** *Fix $d \in \mathbb{N}$ and let μ be a locally finite Borel measure on $\mathbb{R}^d$. Then for every Borel set $A \subseteq \mathbb{R}^d$,*

- *If $\underline{\dim}_{\mathbf{x}}(\mu) \geq s$ for all $x \in A$ and $\mu(A) > 0$, then $\dim_H(A) \geq s$.*
- *If $\underline{\dim}_{\mathbf{x}}(\mu) \leq s$ for all $x \in A$, then $\dim_H(A) \leq s$.*
- *If $\overline{\dim}_{\mathbf{x}}(\mu) \geq s$ for all $x \in A$ and $\mu(A) > 0$, then $\dim_P(A) \geq s$.*
- *If $\overline{\dim}_{\mathbf{x}}(\mu) \leq s$ for all $x \in A$, then $\dim_P(A) \leq s$.*

The results for packing dimension follow from work of Taylor and Tricot in 1985 [61, Theorem 2.1], while a short proof of the results for Hausdorff dimension can be found in [7, Theorem 4.3.3]. Meanwhile, [45, §8] contains a generalization of both the Hausdorff and packing results to the setting of arbitrary metric spaces.

5. A characterization of Hausdorff and packing dimensions using games

Schmidt's game is a two-player topological game introduced in a seminal paper of Wolfgang M. Schmidt in 1966 [53] as a technique to analyze Diophantine sets that are exceptional with respect to both measure and category. Schmidt's paper led to a plethora of applications at the interface of dynamical systems, Diophantine approximation and fractal geometry, which often involve various modifications of his eponymous game. For a small sample of such research, see [19, 46, 41, 9, 15, 1, 4, 28, 2].

The proof of our variational principle is based on a new variant of Schmidt's game which is in principle capable of computing the Hausdorff and packing dimensions of any set. In Schmidt's original game, players take turns choosing a descending sequence of balls and compete to determine whether or not the intersection point of these balls is in a certain target set. The key feature of our new variant is that instead of requiring the rate at which the players' moves contribute information to the game to be constant, the new variant allows the rate of information transfer to be variable, with the first player, Alice, getting to choose the rate of information transfer. However, Alice is penalized if she exerts too much control over the game over long periods of time without giving her opponent Bob a chance to exert control over the game.

Definition 5.1. Given $0 < \beta < 1$, Alice and Bob play the *δ-dimensional Hausdorff (resp. packing) β-game* as follows:

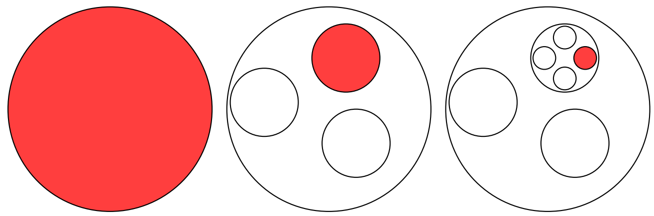

FIGURE 4. Three consecutive rounds of the Hausdorff/packing game. On each round Alice presents Bob with a set of balls to choose between (represented by the set of centers of those balls), and Bob chooses one of the balls, which are colored/shaded above.

- The turn order is alternating, with Alice playing first. Thus, Bob's kth turn occurs after Alice's kth turn and before Alice's $(k+1)$st turn.
- Alice begins by choosing a starting radius $\rho_0 > 0$.
- On the kth turn, Alice chooses a nonempty $3\rho_k$-separated set $A_k \subseteq \mathbb{R}^d$, and Bob responds by choosing a ball $B_k = B(\mathbf{x}_k, \rho_k)$, where $\mathbf{x}_k \in A_k$ and $\rho_k = \beta^k \rho_0$. (We can think of Alice's choice A_k as representing the collection of balls $\{B(\mathbf{x}, \rho_k) : \mathbf{x} \in A_k\}$ from which Bob chooses his ball.)
- On the first (0th) turn, Alice's choice A_0 can be any finite set, but on subsequent turns she must choose it to satisfy

$$A_{k+1} \subseteq B(\mathbf{x}_k, (1-\beta)\rho_k). \tag{5.1}$$

Note that this condition guarantees (see Figure 4) that

$$B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$$

After infinitely many turns have passed, the point

$$\mathbf{x}_\infty = \lim_{k\to\infty} \mathbf{x}_k \in \bigcap_{k=0}^{\infty} B_k \tag{5.2}$$

is computed (note that the right-hand side is always a singleton). It is called the *outcome* of the game. Also, we let $\mathcal{A} = (A_k)_{k\in\mathbb{N}}$, and we compute the numbers

$$\underline{\delta}(\mathcal{A}) \stackrel{\text{def}}{=} \liminf_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k} \frac{\log \#(A_i)}{-\log(\beta)} \tag{5.3}$$

and

$$\overline{\delta}(\mathcal{A}) \stackrel{\text{def}}{=} \limsup_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k} \frac{\log \#(A_i)}{-\log(\beta)}, \tag{5.4}$$

which represent Alice's *score* in the Hausdorff and packing games, respectively. Alice's goal will be to ensure that the outcome is in a certain set S, called the *target set*, and simultaneously to guarantee that her score is at least δ. To be precise, a set $S \subseteq \mathbb{R}^d$ is said to be *δ-dimensionally Hausdorff (resp. packing) β-winning* if Alice has a strategy to simultaneously ensure that the outcome $\mathbf{x}_\infty$ is in S, and that her score in the Hausdorff (resp. packing) game is at least δ. The set S is said to be *δ-dimensionally Hausdorff (resp. packing) winning* if it is δ-dimensionally Hausdorff (resp. packing) β-winning for all sufficiently small $\beta > 0$. (Equivalently, we could say that Alice's score is automatically set equal to zero whenever $\mathbf{x}_\infty \notin S$, in which case we would say that S is δ-dimensionally Hausdorff β-winning if Alice has a strategy to ensure that her score is at least δ.)

The following result is one of the key ingredients in the proof of the variational principle:

Theorem 5.2. *The Hausdorff (resp. packing) dimension of a Borel set $S \subseteq \mathbb{R}^d$ is the supremum of δ such that S is δ-dimensionally Hausdorff (resp. packing) winning.*

Remark 5.3. The theorem remains true (with the same proof) if $\mathbb{R}^d$ is replaced by any doubling metric space.[8]

Proof. We prove the theorem for the case of Hausdorff dimension; the argument in the case of packing dimension is nearly identical.

We begin by proving the lower bound. Suppose that S is δ-dimensionally Hausdorff winning, and we must show that $\dim_H(S) \geq \delta$. Fix $\beta > 0$ such that S is δ-dimensionally Hausdorff β-winning, and consider a strategy for Alice to win the δ-dimensional Hausdorff β-game with target set S. Now for each $k \geq 0$, let E_k denote the union of all sets A_k that Alice might choose according to her strategy in response to some possible sequence of moves that Bob could play, and let $\rho_k = \beta^k \rho_0$. Then the set

$$C \stackrel{\text{def}}{=} \bigcap_{k=0}^{\infty} \bigcup_{\mathbf{x}_k \in E_k} B(\mathbf{x}_k, \rho_k)$$

is the set of all possible outcomes of the game when Alice plays her winning strategy. It is a closed and totally disconnected set, contained entirely in S.

To bound the Hausdorff dimension of C, we introduce a probability measure on C by considering the scenario where Alice plays according to her winning strategy and Bob

[8] A metric space is *doubling* if there exists constants C, r_0 such that every ball of radius $0 < r \leq r_0$ can be covered by at most C balls of radius $r/2$.

plays randomly: on the kth turn, Bob chooses the point $\mathbf{x}_k \in A_k$ uniformly at random, independently of all previous choices. This yields a random game whose outcome is distributed according to some probability measure μ on C. Now fix $\mathbf{x} \in C$, and for each $k \geq 0$ let $\mathbf{x}_k \in E_k$ be chosen so that $\mathbf{x} \in B(\mathbf{x}_k, \rho_k)$. By induction and since A_k is $3\rho_k$-separated, for all k, if Bob plays in a way such that the final outcome is in $B(\mathbf{x}, \rho_k)$, then on the kth turn he must choose the ball $B(\mathbf{x}_k, \rho_k)$. It follows that

$$B(\mathbf{x}, \rho_k) \cap C \subseteq B(\mathbf{x}_k, \rho_k)$$

and thus

$$\mu(B(\mathbf{x}, \rho_k)) \leq \mu(B(\mathbf{x}_k, \rho_k)) = \left(\prod_{i=0}^{k} \#(A_i)\right)^{-1}.$$

So the lower pointwise dimension of μ at $\mathbf{x}$ is

$$\begin{aligned}
\underline{\dim}_{\mathbf{x}}(\mu) &\stackrel{\text{def}}{=} \liminf_{\rho \to 0} \frac{\log \mu(B(\mathbf{x}, \rho))}{\log \rho} \\
&= \liminf_{k \to \infty} \frac{\log \mu(B(\mathbf{x}, \rho_k))}{\log \rho_k} && (\text{since } \rho_k = \beta^k \rho_0) \\
&\geq \liminf_{k \to \infty} \frac{\log \mu(B(\mathbf{x}_k, \rho_k))}{\log \rho_k} \\
&= \liminf_{k \to \infty} \frac{-\sum_{i=0}^{k} \log \#(A_i)}{k \log \beta + \log \rho_0} \\
&= \underline{\delta}(\mathcal{A}) \geq \delta
\end{aligned}$$

since Alice is using a winning strategy. Since $\mathbf{x} \in C$ was arbitrary and $\mu(C) = 1$, applying the Rogers–Taylor–Tricot Theorem 4.1 proves the lower bound $\dim_H(S) \geq \delta$.

To prove the upper bound, suppose that S is not δ-dimensionally Hausdorff winning, and we will show that $\dim_H(S) \leq \delta$. Fix $0 < \beta \leq 1/2$ small enough so that S is not δ-dimensionally Hausdorff β-winning. Then Alice does not have a winning strategy for the δ-dimensional Hausdorff β-game with target set S. Since by assumption S is Borel, by the Borel determinacy theorem [44] we know that Bob must have a winning strategy for this game, which we now fix.

Fix a radius $\rho_0 > 0$, and for each $k \in \mathbb{N}$

- let E_k be a maximal $\frac{1}{2}\beta^k \rho_0$-separated subset of $\mathbb{R}^d$, and
- let $E_k^{(1)}, \ldots, E_k^{(p)}$ be disjoint $3\beta^k \rho_0$-separated subsets of E_k such that $E_k = \bigcup_{i=1}^{p} E_k^{(i)}$.

Since $\mathbb{R}^d$ is a doubling metric space (see Footnote 8), it is possible to choose p to be independent of k and β. We define a family of strategies for Alice as follows. Consider the kth turn for some $k \in \mathbb{N}$, and if $k \geq 1$ then let $B_{k-1} = B(\mathbf{x}_{k-1}, \rho_{k-1})$ be the move that Bob just played. Let

$$\widetilde{B}_{k-1} = \begin{cases} B(\mathbf{x}_{k-1}, (1-\beta)\rho_{k-1}) & k \geq 1 \\ B(\mathbf{0}, \kappa + \rho_0) & k = 0 \end{cases},$$

where $\kappa > 0$ is a large constant. Next let

$$X_k^{(i)} = E_k^{(i)} \cap \widetilde{B}_{k-1}, \quad N_k^{(i)} = \#(X_k^{(i)}), \quad A_k^{(i, N_k^{(i)})} = X_k^{(i)}.$$

From then on, we define the moves $A_k^{(i,j)}$ and $B_k^{(i,j)}$ by backwards recursion as follows:

- if $A_k^{(i,j)}$ is defined for some $j \geq 1$, then
$$B_k^{(i,j)} = B(\mathbf{x}_k^{(i,j)}, \rho_k)$$
is Bob's response if Alice plays $A_k^{(i,j)}$.
- if $A_k^{(i,j)}$ and $B_k^{(i,j)} = B(\mathbf{x}_k^{(i,j)}, \rho_k)$ are both defined for some $j \geq 1$, then
$$A_k^{(i,j-1)} \stackrel{\text{def}}{=} A_k^{(i,j)} \setminus \{\mathbf{x}_k^{(i,j)}\}.$$

Note that $\#(A_k^{(i,j)}) = j$ for all $j = 0, \ldots, N_k^{(i)}$.

Now consider the scenario where Bob plays according to his winning strategy and Alice plays randomly: on the kth turn, Alice chooses a move $A_k^{(i_k,j_k)}$ where the integers i_k and j_k are chosen randomly with respect to a probability distribution satisfying

$$\mathscr{P}(i_k = i,\ j_k = j) \geq c j^{-(1+\varepsilon)}, \tag{5.5}$$

where $\varepsilon > 0$ is fixed and $c > 0$ is a constant depending on ε and p. This yields a random sequence of plays whose outcome is distributed according to some probability measure μ on $\mathbb{R}^d$.

Now fix $\mathbf{x} \in S \cap B(\mathbf{0}, \kappa)$. There is some sequence $(\mathbf{x}_k)_{k \geq 0}$ such that for each $k \in \mathbb{N}$, we have $\mathbf{x}_k \in E_k$ and $\mathbf{x}_{k+1} \in B(\mathbf{x}_k, (1-\beta)\rho_k)$. Also, we have $\mathbf{x}_0 \in B(\mathbf{0}, \kappa + \rho_0)$. It follows that Alice can guarantee that the outcome is equal to $\mathbf{x}$ by playing the move $A_k^{(I_k,J_k)}$ on the kth turn for some sequences of integers $(I_k)_{k \in \mathbb{N}}$, $(J_k)_{k \in \mathbb{N}}$. Since Bob's strategy is winning and $\mathbf{x} \in S$, it follows Alice's score is less than δ, i.e.

$$\underline{\delta}(\mathcal{A}) < \delta.$$

Let G_1 denote the sequence of plays described above, and let G_2 be a sequence of plays where on the kth turn, Alice chooses a set $A_k^{(i_k,j_k)}$, and Bob responds according to his winning strategy, such that $i_k = I_k$ and $j_k = J_k$ for all $k \in \{0, \ldots, \ell\}$. Then the ℓth ball of

G_2 is equal to the ℓth ball of G_1, and thus the outcome of G_2 is within $2\rho_\ell$ of the outcome of G_1, i.e. $\mathbf{x}$. Thus if we think of G_2 as being chosen randomly, then

$$\begin{aligned}\mu\big(B(\mathbf{x},2\rho_\ell)\big) &\geq \mathscr{P}\big(i_k = I_k,\ j_k = J_k\ \ \forall k \leq \ell\big)\\ &\geq \prod_{k=0}^{\ell} c J_k^{-(1+\varepsilon)}\\ &= c^\ell \exp\left(-(1+\varepsilon)\sum_{k=0}^{\ell}\log\#(A_k)\right)\end{aligned}$$

and so

$$\begin{aligned}\underline{\dim}_{\mathbf{x}}(\mu) &= \liminf_{\ell\to\infty}\frac{\log\mu(B(\mathbf{x},2\rho_\ell))}{\log(2\rho_\ell)}\\ &\leq \liminf_{\ell\to\infty}\frac{\ell\log(c) + (1+\varepsilon)\sum_{k=0}^{\ell} - \log\#(A_k)}{\ell\log\beta + \log(2\rho_0)}\\ &= \frac{\log(c)}{\log(\beta)} + (1+\varepsilon)\underline{\delta}(\mathcal{A})\\ &< \frac{\log(c)}{\log(\beta)} + (1+\varepsilon)\delta.\end{aligned}$$

Since $\mathbf{x} \in S$ was arbitrary, applying the Rogers–Taylor Theorem 4.1 again yields

$$\dim_H(S) \leq \frac{\log(c)}{\log(\beta)} + (1+\varepsilon)\delta.$$

Letting $\beta, \varepsilon \to 0$ completes the proof. □

6. Playing games with Diophantine targets

In practice, when we play the Hausdorff or packing game with a target set defined in terms of the parametric geometry of numbers, it is helpful to use a different formalism to encode Alice and Bob's moves. First of all, note that for each k, the ball $B_k = B(\mathbf{x}_k, \rho_k)$ is homeomorphic to the unit ball $B(\mathbf{0}, 1)$ via the similarity transformation

$$T_k(\mathbf{z}) = \mathbf{x}_k + \rho_k\mathbf{z}.$$

By replacing A_{k+1} and $\mathbf{x}_{k+1}$ by their preimages under T_k, we can see that we can make the following changes to the rules of the δ-dimensional Hausdorff (resp. packing) β-game without affecting the existence of winning strategies for either player:

- For $k \geq 1$, instead of requiring that Alice's chose A_k is $3\rho_k$-separated, we require that it is 3β-separated.

- Instead of (5.1), Alice must choose A_{k+1} to satisfy

$$A_{k+1} \subseteq B(\mathbf{0}, 1-\beta). \tag{6.1}$$

- The outcome of the game, instead of being computed by (5.2), is computed by the formula

$$\mathbf{x}_\infty \stackrel{\text{def}}{=} \mathbf{x}_0 + \sum_{k=1}^{\infty} \beta^k \rho_0 \mathbf{x}_k. \tag{6.2}$$

We will call the version of the Hausdorff (resp. packing) game resulting from these rule changes the *modified Hausdorff (resp. packing) game*. It will be the version we use in the proof of Theorem 2.6 (Variational principle, version 2).

Now let us assume that the target set is of the form $S = \mathcal{M}(\mathcal{S})$ for some collection $\mathcal{S}$ of functions from $[0,\infty)$ to $\mathbb{R}$ closed under finite perturbations (i.e. if whenever $\mathbf{f} \in \mathcal{F}$ and $\mathbf{g} \asymp_+ \mathbf{f}$, we have $\mathbf{g} \in \mathcal{F}$). In this case, we can track the "progression" of the game by associating a unimodular lattice to each turn of the game. Specifically, for each $k \geq 0$ let

$$\Lambda_{k+1} \stackrel{\text{def}}{=} g_{-\alpha\log(\beta^{k+1}\rho_0)} u_{Y_k} \mathbb{Z}^{m+n}, \text{ where } \alpha \stackrel{\text{def}}{=} \frac{mn}{m+n} \text{ and } Y_k \stackrel{\text{def}}{=} X_0 + \sum_{i=1}^{k} \beta^i \rho_0 X_i. \tag{6.3}$$

Here, we use uppercase letters (X, Y) instead of bold letters $(\mathbf{x}, \mathbf{y})$ because we are working with matrices rather than with vectors. Then for $k \geq 1$, Λ_k and Λ_{k+1} are related by the formula

$$\Lambda_{k+1} = g_{-\alpha\log(\beta)} u_{X_k} \Lambda_k.$$

Notation 6.1. To simplify the notation in the previous displayed equation, we let

$$\gamma = -\alpha\log(\beta) > 0, \qquad g = g_\gamma,$$

so that

$$\Lambda_{k+1} = g u_{X_k} \Lambda_k.$$

Intuitively, this means that Λ_k is well-defined at the start of turn k, and that Alice and Bob's choices on turn k can be thought of as a process of choosing Λ_{k+1} indirectly by choosing X_k.

The significance of the sequence of lattices $(\Lambda_k)_1^\infty$ is given by the following lemma:

Lemma 6.2. *Let $\mathbf{f} : [0,\infty) \to \mathbb{R}^d$ be the function defined on $\gamma\mathbb{Z}$ by the formula*

$$\mathbf{f}(k\gamma) = \mathbf{h}(\Lambda_k)$$

and extended to $[0,\infty)$ *via linear interpolation. Then* $\mathbf{f} \asymp_+ \mathbf{h}_{X_\infty}$. *In particular,* $X_\infty \in \mathcal{M}(\mathcal{S})$ *if and only if* $\mathbf{f} \in \mathcal{S}$.

Proof. Fix k, and write

$$Z_k = \sum_{i=0}^{\infty} \beta^i X_{k+i}.$$

Then

$$u_{Z_k}\Lambda_k = g_{-\alpha\log(\rho_0)+k\gamma} u_{X_\infty}\mathbb{Z}^{m+n}.$$

Since $Z_k \in B(\mathbf{0},1)$, this implies that

$$\mathbf{f}(k\gamma) = \mathbf{h}(\Lambda_k) \asymp_+ \mathbf{h}(u_{Z_k}\Lambda_k) = \mathbf{h}(g_{-\alpha\log(\rho_0)+k\gamma} u_{X_\infty}\mathbb{Z}^{m+n}) = \mathbf{h}_{X_\infty}(-\alpha\log(\rho_0)+k\gamma)$$

and thus $\mathbf{f} \asymp_+ \mathbf{h}_{X_\infty}$. Since $\mathcal{S}$ is closed under finite perturbations, it follows that $\mathbf{f}$ is in $\mathcal{S}$ if and only if $\mathbf{h}_{X_\infty}$ is, i.e. if and only if $X_\infty \in \mathcal{M}(\mathcal{S})$. □

Part 3. **Proof of the variational principle**

7. Preliminaries

This section collects various notation and lemmata employed in our proof of the variational principle, viz. Theorem 2.6. Though some of these results may be considered elementary by experts familiar with the geometry of numbers, we include such for the benefits of self-containment. Thus, for instance, we begin by recalling Minkowski's second theorem on successive minima for the reader's convenience.

Theorem 7.1 (Minkowski, [14, Theorem V in §VIII.4.3])**.** *Let* Λ *be a lattice in a vector space* $V \subseteq \mathbb{R}^d$. *Then*

$$\prod_{j=1}^{\dim(V)} \lambda_j(\Lambda) \asymp \|\Lambda\|,$$

where $\|\Lambda\|$ *denotes the covolume of* Λ, *and* $\lambda_j(\Lambda)$ *is the jth successive minimum of* Λ.

Definition 7.2. Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice. A subspace $V \subseteq \mathbb{R}^d$ is called Λ-*rational* if $V \cap \Lambda$ is a lattice in V. Denote the set of all q-dimensional Λ-rational subspaces of $\mathbb{R}^d$ by $\mathcal{V}_q(\Lambda)$.

Notation 7.3. If V is a Λ-rational subspace, we denote the covolume of $V \cap \Lambda$ in V by $\|V\|$. Although this notation is misleading since $\|V\|$ depends on Λ and not just on V, in practice this should not be a problem as it should generally be clear what Λ is.

Notation 7.4. We denote the subspace of $\mathbb{R}^d$ contracted by the (g_t) flow (defined in § 1.1) by $\mathcal{L}$, i.e.

$$\mathcal{L} \stackrel{\text{def}}{=} \{\mathbf{0}\} \times \mathbb{R}^n.$$

The conical ε-neighborhood of a subspace $V \subseteq \mathbb{R}^d$ will be denoted

$$\mathcal{C}(V, \varepsilon) = \{\mathbf{r} : d(\mathbf{r}, V) \leq \varepsilon \|\mathbf{r}\|\}.$$

Given $0 < \beta < 1$ in the definition of the δ-dimensional Hausdorff (resp. packing) β-game (see Definition 5.1), and following Notation 6.1 and (6.3) from Section 6, we write

$$\gamma = -\frac{mn}{m+n}\log(\beta), \qquad\qquad g = g_\gamma.$$

Following Definition 2.1, we write

$$F_q = \sum_{i=1}^{q} f_i.$$

The following lemmas will be used in the proof of Theorem 2.6.

Lemma 7.5. *Let $\Lambda \leq \mathbb{R}^d$ be a lattice. Then there exists a basis $\{\mathbf{r}_1, \ldots, \mathbf{r}_d\}$ of Λ such that if $V_q = \sum_{i=1}^{q} \mathbb{R}\mathbf{r}_i$, then*

$$\log \|V_q\| \asymp_+ \sum_{i=1}^{q} \log \lambda_i(\Lambda).$$

Moreover,

$$\log \|\mathbf{r}_i\| \asymp_+ \log \lambda_i(\Lambda) \text{ for all } i. \tag{7.1}$$

Proof. Let $\{\mathbf{r}_1, \ldots, \mathbf{r}_d\}$ be a Minkowski reduced basis of Λ (see [31, Proposition 5.3]). Now let h be the change of basis matrix changing $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ into $\{\mathbf{r}_1, \ldots, \mathbf{r}_d\}$ and write $h = kan$ where $k \in \mathrm{SO}(d)$, a is a diagonal matrix, and n is an upper triangular matrix. Since $\{\mathbf{r}_1, \ldots, \mathbf{r}_d\}$ is a Minkowski reduced basis, n is bounded and thus $g = k(ana^{-1})$ is also bounded. Note that $\mathbf{r}_i = ga\mathbf{e}_i$ for all i. Then for all i,

$$\|\mathbf{r}_i\| \asymp a_i = \lambda_i(a\mathbb{Z}^d) \asymp \lambda_i(ga\mathbb{Z}^d).$$

On the other hand, for each $q = 1, \ldots, d$, we have

$$\log \|V_q\| \asymp_+ \log \|aE_q\| = \log(a_1 \ldots a_q) = \sum_{i=1}^{q} \log(a_i) \asymp_+ \sum_{i=1}^{q} \log \lambda_i(\Lambda),$$

where $E_q = \sum_{i=1}^{q} \mathbb{R}\mathbf{e}_i$. $\square$

Lemma 7.6. *Let $\Lambda \leq \mathbb{R}^d$ be a lattice, and let V_q be as in Lemma 7.5. Then*

$$\log \|V_q'\| \gtrsim_+ \sum_{i=1}^{q-1} \log \lambda_i(\Lambda) + \log \lambda_{q+1}(\Lambda) \text{ for all } V_q' \in \mathcal{V}_q(\Lambda) \setminus \{V_q\}. \tag{7.2}$$

Proof. Fix $V_q' \in \mathcal{V}_q(\Lambda) \setminus \{V_q\}$. By Minkowski's second theorem (Theorem 7.1), we have

$$\log \|V_q'\| \asymp_+ \sum_{i=1}^q \lambda_i(\Lambda \cap V_q'). \tag{7.3}$$

For all $i = 1, \ldots, q-1$, we have

$$\lambda_i(\Lambda \cap V_q') \geq \lambda_i(\Lambda). \tag{7.4}$$

For the $i = q$ term, we use a different argument to get a better bound. Let E (resp. E') be a spanning set for $\Lambda \cap V_q$ (resp. $\Lambda \cap V_q'$). Then $E \cup E'$ is a spanning set for $\Lambda \cap (V_q + V_q')$. Since $\dim(V_q + V_q') \geq q+1$, it follows that

$$\max_{\mathbf{r} \in E \cup E'} \|\mathbf{r}\| \geq \lambda_{q+1}(\Lambda).$$

Taking the infimum over all E, E' gives

$$\max\left(\lambda_q(\Lambda \cap V_q), \lambda_q(\Lambda \cap V_q')\right) \geq \lambda_{q+1}(\Lambda).$$

On the other hand, it follows from (7.1) that $\lambda_q(\Lambda \cap V_q) \asymp_+ \lambda_q(\Lambda) \leq \lambda_q(\Lambda \cap V_q')$. Thus,

$$\lambda_q(\Lambda \cap V_q') \gtrsim_+ \lambda_{q+1}(\Lambda).$$

Combining with (7.3) and (7.4) yields (7.2). □

Lemma 7.7. *Recall that $d_+ = m$ and $d_- = n$. Fix $L_\pm \in [0, d_\pm] \cap \mathbb{Z}$ and let $q \stackrel{\text{def}}{=} L_+ + L_-$. Let Λ be a lattice and let V be a q-dimensional Λ-rational subspace such that*

$$L_- \geq \sup_{\|Y\| \leq \beta} \dim(u_Y V \cap \mathcal{L}).$$

Then for all $t \geq 0$,

$$\log \|g_t V\| - \log \|V\| \gtrsim_{+,\beta} \left(\frac{L_+}{m} - \frac{L_-}{n}\right) t.$$

The reverse inequality holds if $\dim(V \cap \mathcal{L}) = L_-$.

Proof. Define a linearly independent sequence $(\mathbf{r}_i)_1^k$ in V recursively as follows: if $\mathbf{r}_i = (\mathbf{p}_i, \mathbf{q}_i) \in V$ has been defined for $i = 1, \ldots, j$, then let $\mathbf{r}_{j+1} = (\mathbf{p}_{j+1}, \mathbf{q}_{j+1}) \in W_j \stackrel{\text{def}}{=} V \cap \bigcap_1^j (\mathbf{p}_i, \mathbf{0})^\perp$ be chosen so that $\|\mathbf{p}_{j+1}\| \geq \beta \|\mathbf{q}_{j+1}\|$. Continue until it is not possible to continue further; then for all $\mathbf{r} = (\mathbf{p}, \mathbf{q}) \in W_k$, we have $\|\mathbf{p}\| \leq \beta \|\mathbf{q}\|$. It follows that there exists $\|Y\| \leq \beta$ such that $u_Y W_k \subseteq \mathcal{L}$, which implies that

$$q - k = \dim(W_k) = \dim(u_Y W_k) \leq \dim(u_Y V \cap \mathcal{L}) \leq L_-.$$

Rearranging gives $k \geq L_+$. Finally, let $(\mathbf{r}_i)_{k+1}^q$ be an arbitrary basis of W_k. Then there exists a constant $\alpha > 0$ such that

$$\|V\| = \alpha \|\mathbf{r}_1 \wedge \cdots \wedge \mathbf{r}_q\|$$

and

$$\|g_t V\| = \alpha \|g_t \mathbf{r}_1 \wedge \cdots \wedge g_t \mathbf{r}_q\|.$$

In particular

$$\|V\| \leq \alpha \|\mathbf{r}_1\| \cdots \|\mathbf{r}_k\| \cdots \|\mathbf{r}_{k+1} \wedge \cdots \mathbf{r}_q\| \lesssim_\beta \alpha \|\mathbf{p}_1\| \cdots \|\mathbf{p}_k\| \cdots \|\mathbf{r}_{k+1} \wedge \cdots \mathbf{r}_q\|$$

while

$$\|g_t V\| = \alpha \|e^{t/m}[(\mathbf{p}_1, \mathbf{0}) + o(1)] \wedge \cdots \wedge e^{t/m}[(\mathbf{p}_k, \mathbf{0}) + o(1)] \wedge g_t \mathbf{r}_{k+1} \wedge \cdots \wedge g_t \mathbf{r}_q\|.$$

Since $(\mathbf{p}_i)_1^k$ are orthogonal to each other and also to $\mathbf{r}_{k+1}, \ldots, \mathbf{r}_q$, it follows that

$$\begin{aligned} \|g_t V\| &\gtrsim \alpha e^{kt/m} \|\mathbf{p}_1\| \cdots \|\mathbf{p}_k\| \cdot \|g_t \mathbf{r}_{k+1} \wedge \cdots \wedge g_t \mathbf{r}_q\| \\ &\gtrsim \alpha e^{kt/m} \|\mathbf{p}_1\| \cdots \|\mathbf{p}_k\| \cdot e^{-(q-k)t/n} \|\mathbf{r}_{k+1} \wedge \cdots \wedge \mathbf{r}_q\| \\ &\gtrsim_\beta e^{kt/m-(q-k)t/n} \|V\|. \end{aligned}$$

Since $k \geq L_+$, this completes the proof. ☐

We finish with an elementary observation about the slopes of line segments appearing in templates (see Definition 2.1) that is used in proving Lemma 8.14, which in turn is needed in the proof of the lower bound of the variational principle.

Observation 7.8. If $\mathbf{f}$ is a template then for all $t \geq 0$ we have

$$f_j'(t) - f_i'(t) \in \tfrac{1}{q}\mathbb{Z} \text{ for some } q \leq mnd^2.$$

Proof. For all i, t we have

$$f_i'(t) = \frac{p}{mnq}$$

for some $p \in \mathbb{Z}$ and $q = 1, \ldots, d$. So we have

$$f_j'(t) - f_i'(t) = \frac{p_2}{mnq_2} - \frac{p_1}{mnq_1} = \frac{p}{mnq_1q_2}$$

and we have $mnq_1q_2 \leq mnd^2$. ☐

8. Proof of Theorem 2.6, lower bound

Let $\mathbf{f}$ be a template. We must show that

$$\dim_H(\mathcal{M}(\mathbf{f})) \geq \underline{\delta}(\mathbf{f}), \qquad \dim_P(\mathcal{M}(\mathbf{f})) \geq \overline{\delta}(\mathbf{f}). \tag{8.1}$$

To this end, we will play the modified Hausdorff and packing games with target set $S = \mathcal{M}(\mathbf{f})$. It turns out that the same strategy will work for Alice in both games.

The proof can be divided into four basic stages:

1. *Reduction*: We can without loss of generality assume that the template $\mathbf{f}$ appearing in the statement of the theorem is in a special form which is convenient to the later argument.
2. *Mini-strategy*: For any template $\mathbf{g}$ (not necessarily the same as the $\mathbf{f}$ appearing in the theorem), Alice can guarantee that if A is the outcome of the game, then the successive minima function $\mathbf{h}_A$ remains close to $\mathbf{g}$ for a certain interval of time before diverging from it. This interval can be an interval of linearity of $\mathbf{g}$, or the union of any fixed number of intervals of linearity. However, the upper bound on $|\mathbf{h}_A - \mathbf{g}|$ rapidly grows as the allowed number of intervals of linearity increases.
3. *Error correction*: If the value of the successive minima function $\mathbf{h}_A$ at a certain time t is slightly off from the value of $\mathbf{f}$ at t, then we can perturb $\mathbf{f}$ into a partial template $\mathbf{g}$ such that $\mathbf{g}(t) = \mathbf{h}_A(t)$. Alice can then follow the perturbed template $\mathbf{g}$ rather than the original template $\mathbf{f}$.
4. *Uniform error bounds*: The error correction techniques from stage (3) are sufficient to guarantee that the final successive minima function $\mathbf{h}_A$ remains a bounded distance from the desired template $\mathbf{f}$, and that the inequalities $\underline{\delta}(\mathcal{A}) \geq \underline{\delta}(\mathbf{f}) - \varepsilon$ and $\overline{\delta}(\mathcal{A}) \geq \overline{\delta}(\mathbf{f}) - \varepsilon$ are satisfied.

Stage 2 is in some sense the most important one because it makes the connection between the parametric geometry of numbers and the theory of templates. In the other stages, for the most part we do not deal with parametric geometry of numbers directly.

8.1. Reduction.

There are two key features we would like to assume of our template $\mathbf{f}$: its corner points[9] should be appropriately spaced, and each corner point should have only one "purpose".

Definition 8.1. Given $\eta > 0$, a template $\mathbf{f}$ is *η-integral* if

(I) its corner points are multiples of η, and
(II) for all $t \in \eta\mathbb{N}$ we have $f_i(t) \in \frac{\eta}{mnd!}\mathbb{Z}$.

[9] I.e. points where the derivative of $\mathbf{f}$ is undefined.

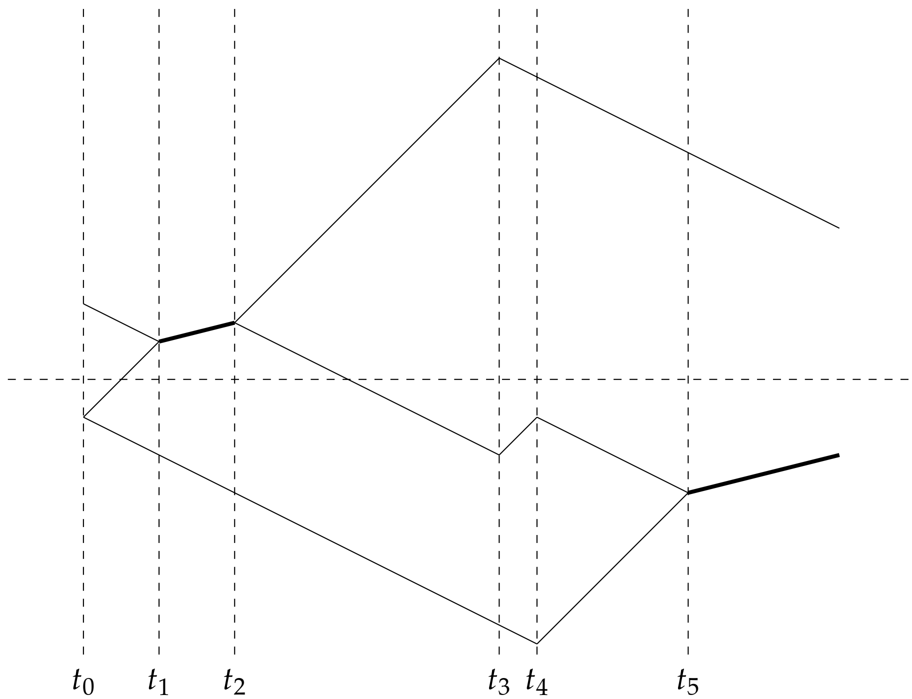

FIGURE 5. In this figure of a portion of an arbitrary 1×2 template, the corner points t_0 and t_2 are splits, t_1 and t_5 are merges, and t_3 and t_4 are transfers.

Definition 8.2 (Cf. Figure 5)**.** Let $\mathbf{f}$ be a template, let $t > 0$ be a corner point of $\mathbf{f}$, and let I_-, I_+ be the two intervals of linearity for $\mathbf{f}$ such that $I_- = (t_-, t)$ and $I_+ = (t, t_+)$ for some $t_- < t < t_+$.

- We call t a *split* (resp. *merge*) if there exists $q = 1, \ldots, d-1$ such that $f_q(t) = f_{q+1}(t)$, but $f_q < f_{q+1}$ on I_+ (resp. on I_-).
- We call t a *transfer* if there exists $q = 1, \ldots, d-1$ such that $f_q(t) < f_{q+1}(t)$ and $L_+(\mathbf{f}, I_+, q) > L_+(\mathbf{f}, I_-, q)$ (equiv. $F_q'(I_+) < F_q'(I_-)$).

Finally, we call the template $\mathbf{f}$ *simple* if the sets of splits, merges, and transfers are pairwise disjoint.

Remark 8.3. In any template, every corner point is either a split, a merge, or a transfer.

We now show that we can assume without loss of generality that the template $\mathbf{f}$ appearing in the statement of Theorem 2.6 is both simple and integral.

Lemma 8.4. *For every $\eta > 0$ and for every template $\mathbf{f}$, there exists a simple η-integral template $\mathbf{g}$ which approximates $\mathbf{f}$ to within an additive constant, i.e. satisfies $\mathbf{g} \asymp_+ \mathbf{f}$. The implied constant* *depends on η but not on $\mathbf{f}$. Moreover, $\mathbf{g}$ can be chosen so that for all q, t, t' such that $g_q(t) < g_{q+1}(t)$ and $|t' - t| \leq \eta$, we have $f_{q+1}(t') - f_q(t') \geq \eta$ and $G_q'(t) \geq F_q'(t')$. Consequently,*

$$\underline{\delta}(\mathbf{g}) \geq \underline{\delta}(\mathbf{f}), \qquad \overline{\delta}(\mathbf{g}) \geq \overline{\delta}(\mathbf{f}). \tag{8.2}$$

Proof. Since a similar argument will be needed for the proof of the upper bound of Theorem 2.6 (specifically, showing that a successive minima function can always be approximated by a template (cf. Lemma 8.8 below), we prove this lemma in slightly greater generality than may appear to be necessary. Fix $\eta > 0$, and let $\mathbf{f} : [0, \infty) \to \mathbb{R}^d$ be a map (not necessarily a template) satisfying the following conditions:

(I) $f_1 \leq \cdots \leq f_d$.
(II) For all $t_1 < t_2$ and $i = 1, \ldots, d$ we have
$$-\frac{1}{n} \leq \frac{f_i(t_2) - f_i(t_1)}{t_2 - t_1} \leq \frac{1}{m}.$$
(III) For all $q = 1, \ldots, d$ and for every interval I such that
$$f_{q+1} > f_q \text{ on } I, \tag{8.3}$$
there exists a convex, piecewise linear function $F_{q,I} : I \to \mathbb{R}$ with slopes in $Z(q)$ (cf. (2.2)) which satisfies
$$F_q \asymp_+ F_{q,I} \text{ on } I \tag{8.4}$$
and if $\mathbf{f}$ is a template, then
$$F_{q,I}' \geq F_q' \text{ on } I. \tag{8.5}$$

Note that any template satisfies these conditions (and in fact, one can take $F_{q,I} = F_q$ in (III)).

Fix a large constant $\delta_* \in \eta\mathbb{N}$, and let $\delta_{**} = mnd^{4d}\delta_*$. Fix $q = 1, \ldots, d$, and let $\mathcal{I}_q$ be the collection of all intervals satisfying (8.3) whose endpoints are in $\delta_{**}\mathbb{N} \cup \{\infty\}$, and which are maximal with respect to these two properties. For each $I \in \mathcal{I}_q$, let $F_{q,I} : I \to \mathbb{R}$ be a convex, piecewise linear function as in (III). By first moving the corner points of $F_{q,I}$ to the left and then increasing $F_{q,I}$ by an additive constant, we may without loss of generality suppose that the following hold:

(IV) the corner points of $F_{q,I}$ are all integer multiples of δ_{**}; and
(V) the values of $F_{q,I}$ at integer multiples of δ_{**} are all in the set $(d^{4(d-q)} + d^{4d}\mathbb{Z})\delta_*$. (The displacement term $d^{4(d-q)}$ will help us guarantee that the resulting template $\mathbf{g}$ is simple.)

Here, we have used the fact that $Z(q) \subseteq \frac{1}{mn}\mathbb{Z}$ for all q, to ensure that the conditions are not inconsistent. Moving the corner points to the left rather than to the right guarantees that (8.5) is still satisfied. We can also assume that $F_{d,I_*} \equiv 0$, where $I_* = [0,\infty)$ is the unique element of $\mathcal{I}_d$.

Claim 8.5. *In addition to the properties listed above, we can without loss of generality assume the following: There exists a constant $C_1 > 0$ such that for all $q_1 < q_2$, $I_1 \in \mathcal{I}_{q_1}$, and $I_2 \in \mathcal{I}_{q_2}$, if $|I_1|, |I_2| > C_1$, then*

$$-\frac{1}{n} \leq \frac{F'_{q_2,I_2} - F'_{q_1,I_1}}{q_2 - q_1} \leq \frac{1}{m} \text{ on } I_1 \cap I_2. \tag{8.6}$$

Proof. Fix a constant $C_2 > 0$ to be determined, and let $C_1 = 2d^2C_2$. Let $\mathcal{J}_q = \{I \in \mathcal{I}_q : |I| > C_1\}$ and $S = \bigcup_q \bigcup_{I\in\mathcal{J}_q} S_{q,I}$, where $S_{q,I}$ is the set of corner points and end points of $F_{q,I}$. Let $\sigma : S \to S$ be defined as follows: $\sigma(t)$ is the smallest element of S that can be reached from t by jumps of size $\leq C_2$. For each $q = 1,\ldots,d$ and $I \in \mathcal{J}_q$, let $\widetilde{F}_{q,I}$ be a piecewise linear function such that if $(a,b) \subseteq I$ is a maximal interval of linearity for $F_{q,I}$ of slope z, then $(\sigma(a),\sigma(b))$ is a maximal interval of linearity for $\widetilde{F}_{q,I}$ of slope z.

We claim that $|\sigma(a) - a| \leq C_1$ for all $a \in S$. Indeed, otherwise there exist points $t_0 < \ldots < t_{2d^2}$ in S such that $t_{i+1} - t_i \leq C_2$ for all i. By the pigeonhole principle there exists $q = 1,\ldots,d$ such that $\#(\bigcup_{I\in\mathcal{J}_q} S_{q,I} \cap [t_0, t_{2d^2}]) \geq 2d+1$. Since $\#(S_{q,I}) \leq \#(Z(q)) + 1 = \min(q, d-q) + 1 \leq d$ for all $I \in \mathcal{I}_q$, applying the pigeonhole principle again shows that there exist at least 3 intervals $I \in \mathcal{J}_q$ such that $S_{q,I} \cap [t_0, t_{2d^2}] \neq \varnothing$. But then the middle interval is a subset of $[t_0, t_{2d^2}]$, which contradicts $I \in \mathcal{J}_q$.

It follows that $\widetilde{F}_{q,I} \asymp_+ F_{q,I}$ for all q, I. Thus, (8.4) holds with F replaced by $\widetilde{F}$. Moreover, since $\sigma(S) \subseteq S \subseteq \delta_{**}\mathbb{Z}$, condition (IV) holds for $\widetilde{F}$. Also, by translating each $\widetilde{F}_{q,I}$ by a constant if necessary, we can without loss of generality assume that condition (V) holds for $\widetilde{F}$.

Finally, we need to show that (8.6) holds for $\widetilde{F}$. Indeed, let $(\sigma(a),\sigma(b)) \subseteq I_1 \cap I_2$ be a maximal interval of linearity for $\widetilde{F}_{q,I_2} - \widetilde{F}_{q,I_1}$. Since $\sigma(a) < \sigma(b)$, we have $b - a \geq C_2$. Let $k = q_2 - q_1$. Then by condition (II) we have

$$-\frac{k}{n} \leq \sum_{i=q_1+1}^{q_2} \frac{f_i(b) - f_i(a)}{b-a} \leq \frac{k}{m}$$

and thus if z is the constant value of $F'_{q_2,I_2} - F'_{q_1,I_1}$ on (a,b), then

$$-\frac{k}{n} - \frac{4C_3}{C_2} \leq z \leq \frac{k}{m} + \frac{4C_3}{C_2}$$

where C_3 is the implied constant of (8.4). On the other hand, we have $z \in \frac{1}{mn}\mathbb{Z}$ by condition (III). So if we choose $C_2 > 4mnC_3$, then we get $-k/n \leq z \leq k/m$, completing the proof. $\triangleleft$

Next, let $F_{q,*} : [0,\infty) \to \mathbb{R} \cup \{*\}$ be defined by the formula

$$F_{q,*}(t) = \begin{cases} F_{q,I}(t) & \text{if } t \in I \text{ for some } I \in \mathcal{J}_q \\ * & \text{otherwise,} \end{cases}$$

and let $G_{q,*}(t) = F_{q,*}(t) + q(d-q)C_4$, where $C_4 \in d^{4d}\delta_*\mathbb{N}$ is large to be determined. Let $G_{0,*}(t) = 0 \in (d^{4(d-0)} + d^{4d}\mathbb{Z})\delta_*$ for all t, and note that $G_{d,*}(t) = \delta_*$ for all t by our condition on F_{d,I_*}.

At this point, the intuitive idea is to try to define the template $\mathbf{g}$ by solving the equations

$$G_{q,*}(t) = \begin{cases} \sum_{i=1}^q g_i(t) & \text{if } g_q(t) < g_{q+1}(t) \\ * & \text{if } g_q(t) = g_{q+1}(t). \end{cases} \tag{8.7}$$

However, the formula (8.7) is not necessarily solvable with respect to $\mathbf{g}$, due to the fact that the natural candidate for a solution does not necessarily satisfy $g_1(t) \leq \cdots \leq g_d(t)$. To address this issue, we introduce the concept of the convex hull function of a set:

Definition 8.6. The *convex hull function* of a set $\Gamma \subseteq \mathbb{R}^2$ is the largest convex function $h : I \to \mathbb{R}$ such that $h(x) \leq y$ for all $(x,y) \in \Gamma$, where I is the smallest interval containing the projection of Γ onto the first coordinate.

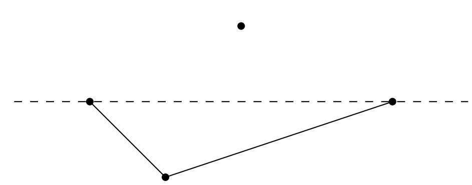

FIGURE 6. The convex hull function h of the set $\{(0,0),(1,-1),(2,1),(4,0)\}$. Since $1 > h(2) = -2/3$, the convex hull function does not change when the point $(2,1)$ is removed.

We can now define $\mathbf{g}$ via the formula

$$g_q(t) = h_t(q) - h_t(q-1),$$

where $h_t : [0,d] \to \mathbb{R}$ is the convex hull function of the set

$$\Gamma(t) = \{(q, G_{q,*}(t)) : q = 0,\ldots,d,\ G_{q,*}(t) \neq *\}.$$

To complete the proof, we must show

- that $\mathbf{g} = (g_1, \ldots, g_d)$ is a simple η-integral template,
- that $\mathbf{g} \asymp_+ \mathbf{f}$,
- that if $\mathbf{f}$ is a template, then for all q, t, t' such that $g_q(t) < g_{q+1}(t)$ and $|t' - t| \leq \eta$, we have $f_{q+1}(t') - f_q(t') \geq \eta$ and $G_q'(t) \geq F_q'(t')$, and consequently (8.2) holds.

Claim 8.7. *For all $q = 1, \ldots, d-1$ and $t \geq 0$ such that $G_{q,*}(t) = *$, we have $f_q(t) \asymp_+ f_{q+1}(t)$.*

Proof. Indeed, let $\ell = \lceil C_1/\eta \rceil + 1 \geq 2$, and write $t \in I = (k\eta, (k+\ell)\eta)$ for some $k \in \mathbb{N}$. If (8.3) holds, then there exists $J \in \mathcal{I}_q$ such that $I \subseteq J$. Since $|J| \geq |I| = \ell\eta > C_1$, we have $J \in \mathcal{J}_q$, which contradicts $G_{q,*}(t) = *$. Thus (8.3) does not hold, and so there exists $t' \in I$ such that $f_q(t') = f_{q+1}(t')$. By condition (II), this implies $f_q(t) \asymp_+ f_{q+1}(t)$. $\triangleleft$

We next show that $\mathbf{g}$ is continuous. From this it is easy to see that it is piecewise linear, the first step to proving that it is a template. Fix $t > 0$, and write $h(t^\pm) = \lim_{s \to t^\pm} h(s)$. Let

$$\Gamma(t^\pm) = \lim_{s \to t^\pm} \Gamma(s) = \{(q, G_{q,*}(t^\pm)) : q = 0, \ldots, d,\ G_{q,*}(t^\pm) \neq *\}.$$

We need to show that $\Gamma(t^-)$ and $\Gamma(t^+)$ have the same convex hull function. For this purpose, it suffices to show that any point in one of these sets but not the other is not an element of the graph of the corresponding convex hull function (which implies that the convex hull function does not change when the point is removed).

Indeed, fix $q = 1, \ldots, d-1$ and suppose that $G_{q,*}(t^+) \neq *$ but $G_{q,*}(t^-) = *$. Let $0 \leq p < q$ and $d \geq r > q$ be maximal and minimal, respectively, such that $G_{p,*}(t^+), G_{r,*}(t^+) \neq *$. Then by Claim 8.7 we have

$$f_{p+1}(t) \asymp_+ \ldots \asymp_+ f_q(t) \asymp_+ f_{q+1}(t) \asymp_+ \ldots \asymp_+ f_r(t)$$

and thus by (8.4),

$$\frac{F_{q,*}(t^+) - F_{p,*}(t^+)}{q-p} \asymp_+ f_q(t) \asymp_+ f_{q+1}(t) \asymp_+ \frac{F_{r,*}(t^+) - F_{q,*}(t^+)}{r-q}.$$

It follows that

$$\begin{aligned}\frac{G_{r,*}(t^+) - G_{q,*}(t^+)}{r-q} - \frac{G_{q,*}(t^+) - G_{p,*}(t^+)}{q-p} &\asymp_+ \left[\frac{r(d-r) - q(d-q)}{r-q} - \frac{q(d-q) - p(d-p)}{q-p}\right] C_4 \\ &= -(r-p)C_4 \leq -2C_4.\end{aligned}$$

So if C_4 is sufficiently large, then

$$\frac{G_{r,*}(t^+) - G_{q,*}(t^+)}{r-q} < \frac{G_{q,*}(t^+) - G_{p,*}(t^+)}{q-p}$$

i.e. the slope of the line from $(q, G_{q,*}(t^+))$ to $(r, G_{r,*}(t^+))$ is less than the slope of the line from $(p, G_{p,*}(t^+))$ to $(q, G_{q,*}(t^+))$. It follows that $(q, G_{q,*}(t^+))$ lies above the graph of the convex hull function of $\Gamma(t^+)$. Since q was arbitrary, this shows that $\Gamma(t^-)$ and $\Gamma(t^+)$ have the same convex hull function. Thus $\mathbf{g}(t^-) = \mathbf{g}(t^+)$, and $\mathbf{g}$ is continuous at t.

We next demonstrate that $\mathbf{g}$ satisfies conditions (I)-(III) of Definition 2.1. (I) follows from the fact that convex hull functions are convex, while (II) follows from Claim 8.5. To demonstrate (III), fix $q = 1, \ldots, d$ and let I be an interval of linearity for $\mathbf{g}$ such that $g_q < g_{q+1}$ on I. Fix $t \in I$. Since $h_t(q) - h_t(q-1) < h_t(q+1) - h_t(q)$, the point $(q, h_t(q))$ is an extreme point of the convex hull of $\Gamma(t)$ and thus $(q, h_t(q)) \in \Gamma(t)$, i.e. $G_{q,*}(t) = h_t(q)$. It follows that

$$\sum_{i=1}^{q} g_i = G_{q,*} \text{ on } I. \tag{8.8}$$

Since $G_{q,*} \upharpoonright I$ is convex and piecewise linear with slopes in $Z(q)$, it follows that the same is true for $\sum_1^q g_i \upharpoonright I$. Thus, $\mathbf{g}$ is a template.

To show that $\mathbf{g}$ is simple and η-integral, we first observe that by construction, all transfers occur at integer multiples of δ_{**}. Let t be a split or a merge with corresponding index q. Then $(q, G_{q,*}(s))$ is an extreme point of the convex hull of $\Gamma(s)$ when s approaches t from one side, but not from the other side. So there exist $0 \leq p < q < r \leq d$ such that the point $(q, G_{q,*}(t))$ lies on the line segment connecting $(p, G_{p,*}(t))$ and $(r, G_{r,*}(t))$. Thus, we have $\Phi(t) = 0$ where

$$\Phi(s) = (r-q)G_{p,*}(s) + (q-p)G_{r,*}(s) - (r-p)G_{q,*}(s).$$

Write $t = t' + t''$ where t' is a multiple of δ_{**} and $0 \leq t'' < \delta_{**}$. Then by assumption $G_j(t') \in (d^{4(d-j)} + d^{4d}\mathbb{Z})\delta_*$ for all j. Thus $\frac{1}{\delta_*}\Phi(t') \in \mathbb{Z}$, and furthermore

$$\begin{aligned} \tfrac{1}{\delta_*}\Phi(t') &\equiv (r-q)d^{4(d-p)} + (q-p)d^{4(d-r)} - (r-p)d^{4(d-q)} \\ &\equiv (q-p)d^{4(d-r)} \\ &\not\equiv 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{modulo } d^{4(d-q)}). \end{aligned}$$

In particular $\Phi(t') \neq 0 = \Phi(t)$, so $t'' > 0$ and thus t is not a transfer. Thus, the set of splits and the set of merges are both disjoint from the set of transfers.

Since $G_{p,*}, G_{q,*}, G_{r,*}$ are linear on $[t', t' + \delta_{**}]$, so is Φ. Let z denote the constant value of Φ' on $[t', t' + \delta_{**}]$, and note that

$$\begin{aligned} 0 \neq z &= \left(\tfrac{1}{m} + \tfrac{1}{n}\right) \Big[(r-q)L_+(p) + (q-p)L_+(r) - (r-p)L_+(q)\Big] \\ &\in \left(\tfrac{1}{m} + \tfrac{1}{n}\right) \{-(r-p)q, \ldots, (r-q)q + (q-p)r\} \subseteq \left(\tfrac{1}{m} + \tfrac{1}{n}\right) \{-d^2, \ldots, d^2\}. \end{aligned}$$

Thus

$$t'' = -\frac{\Phi(t')}{z} \in \frac{\delta_* \mathbb{Z}}{\left(\frac{1}{m} + \frac{1}{n}\right)(d^2)!} \subseteq \frac{\delta_* \mathbb{Z}}{d \cdot (d^2)!},$$

so by letting $\delta_* = d \cdot (d^2)! \eta$ we can guarantee that $t'' \in \mathbb{Z}\eta$. Since transfers also occur at integer multiples of η, this implies that condition (I) of Definition 8.1 is satisfied. To check condition (II), note that we have $G_{q,*}(t) \in \delta_* \mathbb{N}$ whenever $t \in \delta_{**}\mathbb{N}$, and thus since $G_{q,*}$ has slopes in $Z(q) \subseteq \frac{1}{mn}\mathbb{Z}$, we have $G_{q,*}(t) \in \frac{\eta}{mn}\mathbb{N}$ whenever $t \in \eta\mathbb{N}$. Now for each $q = 1, \ldots, d$ and $t \in \eta\mathbb{N}$, there exist $p < q \leq r$ such that

$$g_q(t) = \frac{G_{r,*}(t) - G_{p,*}(t)}{r - p} \in \frac{\eta}{mnd!}\mathbb{N}.$$

Thus $\mathbf{g}$ is η-integral.

Next, since $t'' > 0$, it follows that Φ is linear in a neighborhood of t, and thus there exist points near t for which Φ is strictly negative. At these points, we have $g_q = g_{q+1}$. It follows that t is not both a split and a merge with respect to the same index q.

By contradiction, suppose that t is both a split and a merge, with corresponding indices $q_1 \neq q_2$. We can apply the above argument twice: for each $i = 1, 2$ we get indices $0 \leq p_i < q_i < r_i \leq d$, a function Φ_i, and a slope z_i. We have

$$-\frac{\Phi_1(t')}{z_1} = t'' = -\frac{\Phi_2(t')}{z_2}$$

and thus

$$\frac{\Phi_1(t')}{\delta_*} \cdot \frac{z_2}{\frac{1}{m} + \frac{1}{n}} = \frac{\Phi_2(t')}{\delta_*} \cdot \frac{z_1}{\frac{1}{m} + \frac{1}{n}}.$$

So there exist $a_1, a_2 \in \{-d^2, \ldots, d^2\} \setminus \{0\}$ such that

$$\begin{aligned} &a_1[(r_1 - q_1)d^{4(d-p_1)} + (q_1 - p_1)d^{4(d-r_1)} - (r_1 - p_1)d^{4(d-q_1)}] \\ \equiv\; &a_2[(r_2 - q_2)d^{4(d-p_2)} + (q_2 - p_2)d^{4(d-r_2)} - (r_2 - p_2)d^{4(d-q_2)}] \qquad (\text{modulo } d^{4d}). \end{aligned}$$

Comparing the base d^4 expansions of both sides shows that $(p_1, q_1, r_1) = (p_2, q_2, r_2)$, contradicting that $q_1 \neq q_2$. Thus, the set of splits and the set of merges are disjoint.

We next show that $\mathbf{g} \asymp_+ \mathbf{f}$. Indeed, fix $t \geq 0$. Let h_1, h_2, and h_3 be the convex hull functions of $\Gamma(t)$, $\{(q, F_q(t)) : G_{q,*}(t) \neq *\}$, and $\{(q, F_q(t)) : q = 0, \ldots, d\}$, respectively. Since $G_{q,*}(t) \asymp_+ F_q(t)$ for all q such that $G_{q,*}(t) \neq *$, we have $h_1 \asymp_+ h_2$, and by Claim 8.7, we have $h_2 \asymp_+ h_3$. Moreover, since $f_1(t) \leq \cdots \leq f_d(t)$, the map $q \mapsto F_q(t)$ is convex and thus $h_3(q) = F_q(t)$. But then $g_q(t) = h_1(q) - h_1(q-1) \asymp_+ h_3(q) - h_3(q-1) = f_q(t)$ for all q, i.e. $\mathbf{g}(t) \asymp_+ \mathbf{f}(t)$.

Next, suppose that $\mathbf{f}$ is a template, and fix q, t, t' such that $g_q(t) < g_{q+1}(t)$ and $|t' - t| \leq \eta$. We will show that $f_{q+1}(t') - f_q(t') \geq \eta$ and $G_q'(t) \geq F_q'(t')$. Indeed, by (8.8) we have

$$G_q(t) = G_{q,*}(t) = F_{q,*}(t) + q(d-q)C_4 = F_q(t) + q(d-q)C_4$$

and on the other hand $G_{q\pm 1}(t) \leq G_{q\pm 1,*}(t) = F_{q\pm 1}(t) + (q \pm 1)(d - q \mp 1)C_4$. Consequently,

$$\begin{aligned} f_{q+1}(t) - f_q(t) &= F_{q+1}(t) + F_{q-1}(t) - 2F_q(t) \\ &\geq G_{q+1}(t) + G_{q-1}(t) - 2G_q(t) + 2C_4 \geq 2C_4. \end{aligned}$$

It follows that $f_{q+1}(t') - f_q(t') \geq 2C_4 - \eta$. Choosing $C_4 \geq \eta$, we get $f_{q+1}(t') - f_q(t') \geq \eta$. On the other hand, since $F_{q,*}' = G_{q,*}'$ near t, by (8.5) we have $G_i'(t) \geq F_q'(t)$.

Finally, to demonstrate (8.2), let I be an interval on which both $\mathbf{f}$ and $\mathbf{g}$ are linear. For all q such that $g_q < g_{q+1}$ on I, the previous argument gives $G_i' \geq F_q'$ on I, and thus $L_+(\mathbf{g}, I, q) \geq L_+(\mathbf{f}, I, q)$ (the right-hand side being well-defined since $f_q < f_{q+1}$ on I). It follows that

$$\#\big(S_+(\mathbf{g}, I) \cap (0, q]_{\mathbb{Z}}\big) \geq \#\big(S_+(\mathbf{f}, I) \cap (0, q]_{\mathbb{Z}}\big) \tag{8.9}$$

for all q such that $g_q < q_{q+1}$ on I (cf. Definition 2.5). Combining with (2.8) shows that (8.9) holds for all $q = 1, \ldots, d$, and thus since

$$\delta(\mathbf{f}, I) = \sum_{q=1}^{d-1} \#\big(S_+(\mathbf{f}, I) \cap (0, q]_{\mathbb{Z}}\big) - \binom{m}{2},$$

we have $\delta(\mathbf{g}, I) \geq \delta(\mathbf{f}, I)$. Since I was arbitrary, we get (8.2). □

Lemma 8.8. *If Λ is a unimodular lattice in $\mathbb{R}^d$, then the successive minima function $\mathbf{h} = (h_1, \ldots, h_d)$, where*

$$h_i(t) = \log \lambda_i(g_t \Lambda),$$

satisfies conditions (I)-(III) *appearing in the proof of Lemma 8.4, meaning that it can be approximated by a template.*

Proof. Condition (I) is immediate from the definition, while condition (II) follows from some simple calculations which we leave to the reader. To demonstrate property (III), fix $j = 1, \ldots, d-1$ and an interval $[T_1, T_2]$ such that $f_{j+1}(t) > f_j(t)$ for all $t \in [T_1, T_2]$. For each $t \in [T_1, T_2]$ let[10]

$$V_j(t) = \langle \mathbf{r} \in \Lambda : \|g_t \mathbf{r}\| \le \lambda_j \rangle = \langle \mathbf{r} \in \Lambda : \|g_t \mathbf{r}\| < \lambda_{j+1} \rangle.$$

The assumption on $[T_1, T_2]$ guarantees that the map $t \mapsto V_j(t)$ is continuous on this interval, and since this map takes only rational values, it is therefore constant. So $V_j(t)$ is independent of t. By Minkowski's second theorem (Theorem 7.1), for all $t \in [T_1, T_2]$ we have

$$\prod_{i=1}^{j} \lambda_i(g_t \Lambda) \asymp \mathrm{Covol}(g_t V_j(t)) = \mathrm{Covol}(g_t V_j).$$

To continue further, we use the exterior product formula for covolume:

$$\mathrm{Covol}(g_t V_j) = \|g_t v_1 \wedge \cdots \wedge g_t v_j\|$$

where $v_1, \ldots, v_j$ is a basis of $V_j \cap \Lambda$. The expression on the right-hand side is a member of the space $\bigwedge^j \mathbb{R}^d$, which has a basis of the form $\{e_S : S \subseteq \{1 \ldots, d\}, \#(S) = j\}$. Thus,

$$\mathrm{Covol}(g_t V_j) \asymp \max_{\#(S)=j} \langle g_t v_1 \wedge \cdots \wedge g_t v_j, e_S \rangle = \max_{\#(S)=j} \mathrm{Covol}(\pi_S g_t V_j),$$

where π_S denotes the coordinate projection from $\mathbb{R}^d$ to $\mathbb{R}^S$. The logarithm of the right-hand side is the maximum of linear maps whose slopes are in the set $Z(j)$. Thus, the function

$$g_j(t) \stackrel{\text{def}}{=} \max_{\#(S)=j} \log \mathrm{Covol}(\pi_S g_t V_j)$$

satisfies the appropriate conditions. □

8.2. **Mini-strategy.** Suppose that Alice and Bob have played the first k turns of the modified Hausdorff game, and that Alice wants to play so as to guarantee that the successive minima function of the outcome will be close to a given template $\mathbf{g}$ for some short period of time starting at $k\gamma$. Whether or not she can do this depends both on the template $\mathbf{g}$ and on the lattice Λ_k given by (6.3). Intuitively, we expect that she can do it if $\mathbf{h}(\Lambda_k)$ is close to $\mathbf{g}(k\gamma)$, and Λ_k is "positioned in a way so as to allow Alice to continue this correspondence for larger values of k". If the lattice Λ_k is positioned appropriately, we will call it a *C-match* for $\mathbf{g}$ at time $k\gamma$. We give the formal definition as follow:

[10]Here, $V_j(t)$ is the smallest subspace containing $\{\mathbf{r} \in \Lambda : \|g_t \mathbf{r}\| \le \lambda_j\}$. See Convention 4.

Definition 8.9. Fix $k_\delta \in \mathbb{N}$, and let $\delta = k_\delta \gamma$. Let $\mathbf{g}$ be a δ-integral partial template, and fix $0 < C \leq C_\delta \stackrel{\text{def}}{=} \frac{\delta}{4mnd!}$. A lattice $\Lambda \subseteq \mathbb{R}^d$ is a *C-match* for $\mathbf{g}$ at time $t \in \delta\mathbb{N}$ if

(I) We have

$$\|\mathbf{h}(\Lambda) - \mathbf{g}(t)\| < C. \tag{8.10}$$

(II) For all $q = 0, \ldots, d$ such that $g_q(t) < g_{q+1}(t)$, if $V_q = V_q(\Lambda)$ is the linear span of $\{\mathbf{r} \in \Lambda : \|\mathbf{r}\| \leq \lambda_q(\Lambda)\}$, then

$$\dim(V_q \cap \mathcal{L}) \geq L_-(\mathbf{g}, I, q), \tag{8.11}$$

where I is an interval of linearity for $\mathbf{g}$ whose left endpoint is t.

We now show that if Λ_{k_1} is a C_1-match for $\mathbf{g}$ at time $t_1 = k_1\gamma$, then it is possible for Alice to follow $\mathbf{g}$ for any fixed number of intervals of linearity:

Lemma 8.10. *Fix $k_1, k_2 \in \mathbb{N}$ and let $t_i = k_i\gamma$. Let $\mathbf{g} : (t_1, \infty] \to \mathbb{R}^d$ be a partial template, and let N be the number of maximal intervals of linearity of the function $\mathbf{g} \upharpoonright (t_1, t_2)$. Suppose that on the k_1th turn of the dynamical game, Λ_{k_1} is a C_1-match for $\mathbf{g}$ at time t_1. Then Alice has a strategy for turns $k_1, \ldots, k_2 - 1$ of the dynamical game guaranteeing the following:*

(i) *For all k,*

$$\mathbf{h}(\Lambda_k) \asymp_{+,C_1,N} \mathbf{g}(k\gamma). \tag{8.12}$$

(ii) *If k_δ is sufficiently large (depending on C_1 and N), then the final lattice Λ_{k_2} is a C_2-match for $\mathbf{g}$ at time t_2, where C_2 is a constant depending only on C_1 and N.*

(iii) *We have*

$$\Delta(\mathcal{A}, [k_1, k_2]) = \delta(\mathbf{g}, [t_1, t_2]) + O\left(\frac{1}{\gamma} + \frac{1}{k_\delta}\right),$$

where the implied constant may depend on C_1 and N. Here we use the notation

$$\Delta(\mathcal{A}, [k_1, k_2]) \stackrel{\text{def}}{=} \frac{1}{k_2 - k_1} \sum_{k=k_1}^{k_2} \frac{\log \#(A_k)}{-\log(\beta)}.$$

Proof. By induction, it suffices to prove the lemma in the case where $N = 1$, i.e. where $\mathbf{g}$ is linear on $I = (t_1, t_2)$.

Let

$$Q' = \{q : g_q(t_1) < g_{q+1}(t_1)\}, \qquad Q = \{q : g_q < g_{q+1} \text{ on } I\},$$

and for each $q \in Q'$, let V_q be as in Definition 8.9. Note that $Q' \subseteq Q$. Moreover, for all $q \in Q'$, since $\mathbf{g}$ is δ-integral, we have $g_{q+1}(t_1) - g_q(t_1) \geq 2C_\delta \geq 2C$ and thus by (8.10) we have $h_q(\Lambda_{k_1}) < h_{q+1}(\Lambda_{k_1})$, so $\dim(V_q) = q$ for all $q \in Q'$. Let $\Lambda = \Lambda_{k_1}$.

Claim 8.11. *If β is sufficiently small, then there exists a family of Λ-rational subspaces $(V_q)_{q\in Q}$ extending $(V_q)_{q\in Q'}$ with the following properties:*

(i) $\dim(V_q) = q$ *for all* $q \in Q$.
(ii) $V_p \subseteq V_q$ *for all* $p, q \in Q$ *such that* $p < q$.
(iii) $\log \|V_q\| \asymp_+ \sum_1^q g_i(t_1)$ *for all* $q \in Q$*, where the implied constant may depend on* C_1.
(iv) *There exists* $X \in B_{\mathcal{M}}(\mathbf{0}, 1-\beta)$ *such that for all* $q \in Q$*,*

$$\dim(u_X V_q \cap \mathcal{L}) = L_-(q) \stackrel{\text{def}}{=} L_-(\mathbf{g}, I, q).$$

and

$$\dim(u_{X+Y} V_q \cap \mathcal{L}) \leq L_-(q) \text{ for all } \|Y\| \leq 2\beta^{1/2}.$$

Proof. Fix $\varepsilon > 0$ small, and let $S_\pm = S_\pm(\mathbf{g}, I)$. We will define the family $(V_q)_{q\in Q}$ and a sequence of lattice vectors $(\mathbf{r}_i)_{i\in S_-}$ by simultaneous recursion: Fix $j \in S_-$ and suppose that $\mathbf{r}_i$ has been defined for all $i \in S_-(j) := \{i \in S_- : i < j\}$. Let $q \in Q$ and $r \in Q'$ be maximal and minimal, respectively, such that $q < j \leq r$. If V_q has not been defined yet, then let $V_q \subseteq V_r$ be a Λ-rational subspace of dimension q such that

$$V_p \subseteq V_q \quad \forall p < q, \qquad\qquad \mathbf{r}_i \in V_q \quad \forall S_- \ni i \leq q, \tag{8.13}$$

chosen so as to minimize $\|V_q\|$ subject to these restrictions. Then

$$\dim(V_r \cap \mathcal{L}) \underset{(8.11)}{\geq} L_-(r) = \#(S_-(r+1)) > \#(S_-(j)), \quad \dim(V_r) > \dim\left(V_q + \textstyle\sum_{i\in S_-(j)} \mathbb{R}\mathbf{r}_i\right)$$

and thus it is possible to choose $\mathbf{r}_j \in \Lambda \cap V_r$ such that[11]

$$\measuredangle(\mathbf{r}_j, \mathcal{L}) \leq \varepsilon, \qquad \measuredangle(\mathbf{r}_i, \mathbf{r}_j) \geq \pi/2 - \varepsilon \quad \forall i \in S_-(j), \tag{8.14}$$

$$\measuredangle\left(\mathbf{r}_j, V_q + \textstyle\sum_{i\in S_-(j)} \mathbb{R}\mathbf{r}_i\right) \geq \varepsilon^2, \qquad \log\|\mathbf{r}_j\| \lesssim_+ g_j(t_1). \tag{8.15}$$

To demonstrate (iii), first we observe that it holds for $q \in Q'$ by Minkowski's second theorem (Theorem 7.1). By induction, suppose that (iii) holds for all $p < q$, where $q \in Q \setminus Q'$, and let $p \in Q$ and $r \in Q'$ be maximal and minimal, respectively, such that $p < q \leq r$. Then by (8.15), we have

$$\log\left\|V_p + \sum_{i\in S_-(p,q)} \mathbb{R}\mathbf{r}_i\right\| \leq \log\|V_p\| + \sum_{i\in S_-(p,q)} \log\|\mathbf{r}_i\| \lesssim_+ \sum_{i\leq p} g_i(t_1) + \sum_{i\in S_-(p,q)} g_i(t_1),$$

[11] In the equations below $\measuredangle$ denotes the angle between two vectors, or between a vector and a vector subspace.

where $S_-(p,q) = \{i \in S_- : p < i \leq q\}$. Thus by (8.10), it is possible to choose a Λ-rational subspace $V_q \subseteq V_r$ satisfying (8.13) such that $\log\|V_q\| \lesssim_+ \sum_{i\leq q} g_i(t_1)$. The reverse inequality follows directly from Minkowski's second theorem (Theorem 7.1).

To demonstrate (iv), let $\mathcal{L}' = \sum_{j\in S_-} \mathbb{R}\mathbf{r}_j$. Then (8.14) implies that $d_{\mathcal{G}}(\mathcal{L},\mathcal{L}') = O(\varepsilon)$, where $d_{\mathcal{G}}$ denotes distance in the Grassmannian variety. It follows that if ε is sufficiently small, then there exists $X \in B_{\mathcal{M}}(\mathbf{0}, 1-\beta)$ such that $u_X\mathcal{L} = \mathcal{L}'$. Then for all $q \in Q$, by (8.13) we have

$$\dim(u_{-X}V_q \cap \mathcal{L}) = \dim(V_q \cap \mathcal{L}') \geq \#\{i \in S_- : i \leq q\} = L_-(q).$$

Conversely, fix $\|Y\| \leq 2\beta^{1/2}$ and consider

$$\dim(u_{-X+Y}V_q \cap \mathcal{L}) = \dim(u_Y V_q \cap \mathcal{L}') \leq L_-(q) + \dim\left(u_Y V_q \cap \sum_{S_-\ni i>q} \mathbb{R}\mathbf{r}_i\right).$$

So by contradiction, suppose that there exists

$$\mathbf{0} \neq \mathbf{r} \in u_Y V_q \cap \sum_{S_-\ni i>q} \mathbb{R}\mathbf{r}_i.$$

Write $\mathbf{r} = \sum_{S_-\ni i>q} c_i\mathbf{r}_i$ for some constants $c_i \in \mathbb{R}$. Let $S_- \ni j > q$ be chosen so as to maximize $\delta^{-j}|c_j|\cdot\|\mathbf{r}_j\|$, where $\delta > 0$ is small. Then

$$\mathbf{r}_j = \frac{1}{c_j}\left(\mathbf{r} - \sum_{i\neq j} c_i\mathbf{r}_i\right)$$

and thus

$$\begin{aligned}
\frac{1}{\|\mathbf{r}_j\|} d\left(\mathbf{r}_j, V_q + \textstyle\sum_{i\in S_-(j)} \mathbb{R}\mathbf{r}_i\right) &\leq \frac{1}{|c_j|\cdot\|\mathbf{r}_j\|}\left[\|\mathbf{r} - u_{-Y}\mathbf{r}\| + \sum_{i>j}|c_i|\cdot\|\mathbf{r}_i\|\right] \\
&\lesssim \frac{\|Y\|\cdot\|\mathbf{r}\|}{|c_j|\cdot\|\mathbf{r}_j\|} + \sum_{i>j}\delta^{i-j} \lesssim 2\beta^{1/2}\max_i \frac{|c_i|\cdot\|\mathbf{r}_i\|}{|c_j|\cdot\|\mathbf{r}_j\|} + \delta \leq \delta^{1-d}\beta^{1/2} + \delta.
\end{aligned}$$

Letting $\delta = \beta^{1/(2d)} = \varepsilon^3$ gives

$$\measuredangle\left(\mathbf{r}_j, V_q + \textstyle\sum_{i\in S_-(j)} \mathbb{R}\mathbf{r}_i\right) \lesssim \varepsilon^3,$$

which contradicts the first half of (8.15) if ε (or equivalently β) is sufficiently small. $\triangleleft$

Now fix $k = k_1, \ldots, k_2 - 1$, and suppose that the game has progressed to turn k, so that the matrices $X_{k_1}, \ldots, X_{k-1} \in B_{\mathcal{M}}(\mathbf{0}, 1-\beta)$ have all been defined. For each $q \in Q$ let

$$V_q^{(k)} = (gu_{X_{k-1}})\cdots(gu_{X_{k_1}})V_q,$$

where $g = g_\gamma$, and note that $V_q^{(k)}$ is a Λ_k-rational subspace of $\mathbb{R}^d$.

Now let $(p,q]_\mathbb{Z}$ be an interval of equality for $\mathbf{g}$ on I, and consider the lattice $\Gamma_k = \Lambda_k \cap V_q^{(k)}/V_p^{(k)}$. Let $(\mathbf{r}_i^{(k)})_1^{q-p}$ be a basis for Γ_k such that

$$\|\mathbf{r}_i^{(k)}\| \asymp \lambda_i(\Gamma_k) \text{ for all } i.$$

For each $j = 1, \ldots, q-p$ let

$$V_{p+j}^{(k)} = V_p^{(k)} + \sum_{i=1}^{j} \mathbb{R}\mathbf{r}_i^{(k)}.$$

Next, a matrix X will be called *good* on turn k if for all $j = 1, \ldots, d$ we have

$$\dim\left(u_X V_j^{(k)} \cap \mathcal{L}\right) = L_-(j) \stackrel{\text{def}}{=} \#(S_- \cap [1,j])$$

and

$$\dim\left(u_{X+Y} V_j^{(k)} \cap \mathcal{L}\right) \leq L_-(j) \text{ for all } \|Y\| \leq 2\beta^{1/2}.$$

Alice's strategy on turn k can now be given as follows: Let A_k be a maximal 3β-separated subset of the set of matrices in $B_\mathcal{M}(\mathbf{0}, 1-\beta)$ that are good on turn k.

Now consider a possible sequence of responses from Bob, i.e. a sequence $(X_k)_{k=k_1}^{k_2-1}$ such that for each k, we have $X_k \in A_k$. Let

$$X = \sum_{k=k_1}^{k_2-1} \beta^{k-k_1} X_k \in B(\mathbf{0}, 1),$$

so that for all $q \in Q$ and $k = k_1, \ldots, k_2$, we have

$$V_q^{(k)} = u_Y g^{k-k_1} u_X V_q$$

for some $Y = Y_k \in B_\mathcal{M}(\mathbf{0}, 1)$. Now fix $q \in Q$, and let $L_\pm = L_\pm(\mathbf{g}, I, q)$. Fix $k = k_1 + 1, \ldots, k_2$. Since X_{k-1} is good, we have

$$\dim(u_X V_q \cap \mathcal{L}) = \dim(V_q^{(k)} \cap \mathcal{L}) = L_-$$

and since X_{k_1+1} is good, we have

$$\dim\left(u_{X+Y} V_q \cap \mathcal{L}\right) \leq L_- \text{ for all } \|Y\| \leq \beta^2.$$

By Lemma 7.7, these two formulas imply that

$$\begin{aligned}\log\|V_q^{(k)}\| - \log\|V_q\| &\asymp_+ \log\|g^{k-k_1}u_X V_q\| - \log\|u_X V_q\|\\ &\asymp_+ \left(\frac{L_+}{m} - \frac{L_-}{n}\right)(k-k_1)\gamma\\ &= \sum_{i=1}^q g_i(k\gamma) - \sum_{i=1}^q g_i(t_1). && \text{(by (2.5))}\end{aligned}$$

Combining with condition (iii) of Claim 8.11 shows that

$$\log\|V_q^{(k)}\| \asymp_+ \sum_{i=1}^q g_i(k\gamma). \tag{8.16}$$

Now let $(p,q]_\mathbb{Z}$ be an interval of equality for $\mathbf{g}$ on I, and let $\Gamma_k = \Lambda_k \cap V_q^{(k)}/V_p^{(k)}$ as above.

Claim 8.12. *We have*

$$\log\lambda_j(\Gamma_k) \asymp_+ g_{p+j}(k\gamma)$$

for all $j = 1,\ldots,q-p$ and $k = k_1,\ldots,k_2$.

Proof. Write

$$\eta_j(k) \stackrel{\text{def}}{=} \log\lambda_j(\Gamma_k) - g_{p+j}(k\gamma).$$

By (8.16) and Minkowski's second theorem (Theorem 7.1), we have

$$\sum_{i=1}^{q-p} \eta_i(k) \asymp_+ \log\|\Gamma_k\| - \sum_{i=p+1}^q g_i(k\gamma) \asymp_+ 0. \tag{8.17}$$

First suppose that $M_- = 0$, where $M_\pm = M(\mathbf{g}, I, p, q)$. Then for all j,k we have

$$g_{p+j}(k\gamma) - g_{p+j}(k_1\gamma) = \frac{(k-k_1)\gamma}{m} \geq \log\lambda_j(\Gamma_k) - \log\lambda_j(\Gamma_{k_1}),$$

and (8.17) implies that approximate equality holds. Similar logic works if $M_+ = 0$.

So suppose that $M_+, M_- > 0$. Let K be a large constant. We will prove by induction that

$$-\frac{K}{M_+} \leq \eta_j(k) \leq \frac{K}{M_-} \tag{8.18}$$

for all $j = 1,\ldots,q-p$ and $k = k_1,\ldots,k_2$. Indeed, suppose that (8.18) holds for k, and we will prove that it holds for $k' = k + \ell_0$, where ℓ_0 is a large integer. By (8.17), we have

$$j\eta_j(k) \geq \sum_{i=1}^j \eta_i(k) \asymp_+ -\sum_{i=j+1}^{q-p} \eta_i(k) \geq -\frac{(q-p-j)K}{M_-}.$$

Letting $j = M_+ + 1$ shows that

$$\eta_{M_+ +1}(k) \gtrsim_+ -\frac{M_- - 1}{M_-}\frac{K}{M_+ + 1} = -\frac{K}{M_+} + \alpha K,$$

where $\alpha > 0$ is a positive constant.

Note that since $g_{p+j}(t) = g_q(t)$ for all $t \in I$ and $j = 1, \ldots, q-p$, we have $\eta_1 \leq \cdots \leq \eta_{q-p}$, so if (8.18) fails for $k' = k + \ell_0$, then either $\eta_1(k') < -K/M_+$ or $\eta_{q-p}(k') > K/M_-$. By contradiction suppose that $\eta_1(k') < -K/M_+$ (the other case is similar). Then $\eta_1(k) \asymp_{+,\ell_0} -K/M_+$ and thus

$$\eta_{M_+ +1}(k) - \eta_1(k) \gtrsim_{+,\ell_0} \alpha K.$$

If K is sufficiently large in comparison to ℓ_0, then it follows that there exists $j = 1, \ldots, M_+$ such that

$$\eta_{j+1}(k) - \eta_j(k) \geq \frac{\alpha K}{M_+ + 1}. \tag{8.19}$$

It follows from (8.19) that

$$V_j^{(\ell)} = b_{k,\ell} V_j^{(k)} \text{ for all } \ell = k, \ldots, k'$$

where $b_{k,\ell} = (g u_{X_{\ell-1}}) \cdots (g u_{X_k})$. Thus since $X_k, \ldots, X_{\ell-1}$ are good, Lemma 7.7 shows that

$$\log \|V_{p+j}^{(k')}\| - \log \|V_{p+j}^{(k)}\| \asymp_+ (k'-k)\gamma\left(\frac{L_+(p+j)}{m} - \frac{L_-(p+j)}{n}\right).$$

Subtracting (8.16) (with $q = p$) and using the asymptotic

$$\log \|V_{p+j}^{(\ell)}\| - \log \|V_p^{(\ell)}\| \asymp_+ \sum_{i=1}^{j} \log \lambda_i(\Gamma_\ell)$$

and the relations

$$L_+(p+j) = L_+(p) + j, \qquad\qquad L_-(p+j) = L_-(p)$$

(valid since $j \leq M_+$) show that

$$\sum_{i=1}^{j} \log \lambda_i(\Gamma_{k'}) - \sum_{i=1}^{j} \log \lambda_i(\Gamma_k) \asymp_+ (k'-k)\gamma\frac{j}{m}.$$

On the other hand, since $\log \|b_{k,k'}\| \lesssim_+ (k'-k)\gamma/m$, we have

$$\log \lambda_i(\Gamma_{k'}) - \log \lambda_i(\Gamma_k) \lesssim_+ (k'-k)\gamma\frac{1}{m}$$

and thus

$$\log\lambda_i(\Gamma_{k'}) - \log\lambda_i(\Gamma_k) \asymp_+ (k'-k)\gamma\frac{1}{m} \text{ for all } i = 1,\ldots,j.$$

In particular

$$\eta_1(k') - \eta_1(k) \asymp_+ (k'-k)\gamma\left[\frac{1}{m} - \frac{1}{M_+ + M_-}\left(\frac{M_+}{m} - \frac{M_-}{n}\right)\right].$$

The right-hand side is strictly positive, so if ℓ_0 is sufficiently large, then the left-hand side is also positive. But this contradicts our assumption that $\eta_1(k') < -K/M_+ \leq \eta_1(k)$, thus demonstrating (8.18). ☐

Now applying the inequality

$$\lambda_i(\Lambda_k \cap V_q^{(k)}/V_p^{(k)}) \lesssim \lambda_{p+i}(\Lambda_k) \lesssim \max_{\substack{(p',q']\\ q'\leq q}} \lambda_{q'-p'}(\Lambda_k \cap V_{q'}^{(k)}/V_{p'}^{(k)}),$$

where the maximum is taken over all intervals of equality $(p',q']$ for $\mathbf{g}$ that satisfy $q' \leq q$, completes the proof of (8.12), i.e. condition (i) of Lemma 8.10. We proceed to prove conditions (ii) and (iii).

By (8.12), (8.10) holds with $\Lambda = \Lambda_{k_2}$, $t = t_2$, and $C = C_2$, where C_2 is the implied constant of (8.12). Now suppose that δ is large enough so that $C_2 < C_\delta$, and fix q such that $g_q(t_2) < g_{q+1}(t_2)$. Since $\mathbf{g}$ is δ-integral, we have

$$g_{q+1}(t_2) - g_q(t_2) \geq 2C_\delta \tag{8.20}$$

and thus by (8.12), we have $h_q(\Lambda_{k_2}) < h_{q+1}(\Lambda_{k_2})$.

Claim 8.13. $V_q^{(k_2)} = V_q(\Lambda_{k_2})$.

Proof. Suppose not. Then

$$\log\|V_q^{(k_2)}\| \gtrsim_+ \sum_{i=1}^{q-1}\lambda_i(\Lambda_{k_2}) + \lambda_{q+1}(\Lambda_{k_2}) \asymp_+ \sum_{i=1}^{q-1} g_i(t_2) + g_{q+1}(t_2)$$

and combining with (8.16) gives

$$g_q(t_2) \asymp_+ g_{q+1}(t_2).$$

By (8.20), this is a contradiction if γ_* is sufficiently large in comparison to the implied constant. ☐

Thus

$$\dim(V_q(\Lambda_{k_2}) \cap \mathcal{L}) = \dim(V_q^{(k_2)} \cap \mathcal{L}) \geq L_-(\mathbf{g}, I, q) \geq L_-(\mathbf{g}, I_+, q),$$

where I_+ is the interval of linearity for $\mathbf{g}$ whose left endpoint is t_2. Note that the last inequality is due to the assumption of convexity in (III) of Definition 2.1. It follows that condition (II) of Definition 8.9 holds with $\Lambda = \Lambda_{k_2}$ and $t = t_2$, which completes the proof of (ii).

To demonstrate (iii), it suffices to show that

(a) $\#(A_{k_1}) \geq 1$, and
(b) $\#(A_k) \gtrsim \beta^{-\delta}$ for all $k > k_1$, where $\delta = \delta(\mathbf{g}, I)$.

Note that (a) is true by part (iv) of Claim 8.11. To demonstrate (b), fix $k > k_1$, and observe that since X_{k-1} is good on turn $k-1$, for all $q \in Q$ we have

$$\dim(V_q^{(k)} \cap \mathcal{L}) = L_-(q) \tag{8.21}$$

and

$$\dim(u_Y V_q^{(k)} \cap \mathcal{L}) \leq L_-(q) \quad \forall Y \in B_{\mathcal{M}}(\mathbf{0}, \beta^{-1/2}).$$

We now construct a basis of $\mathbb{R}^d$ as follows. Let $(p,q]_{\mathbb{Z}}$ be an interval of equality for $\mathbf{g}$ on I, and let $M_\pm = M_\pm(p,q)$. Let $(\mathbf{r}_i)_{p+1}^{p+M_+}$ be an orthonormal basis of

$$V_q^{(k)} \cap (V_p^{(k)})^\perp \cap (V_q^{(k)} \cap \mathcal{L})^\perp$$

and let $(\mathbf{r}_i)_{p+M_++1}^{q}$ be an orthonormal basis of

$$V_q^{(k)} \cap \mathcal{L} \cap (V_p^{(k)} \cap \mathcal{L})^\perp.$$

Such bases exist because (8.21) allows us to compute the dimensions of these spaces. Now $(\mathbf{r}_i)_1^d$ is an almost orthonormal basis of $\mathbb{R}^d$ (meaning that $\mathbf{r}_i \cdot \mathbf{r}_j = \delta_{ij} + o(1)$ as $\beta \to 0$ for all i,j), and $(\mathbf{r}_i)_{i \in S_-}$ is an orthonormal basis of $\mathcal{L}$.

Let $\mathcal{Z}$ be the space of all $d \times d$ matrices X such that for all i,j such that $X_{i,j} \neq \delta_{i,j}$, we have $i > j$, $i \in S_-$, and $j \in S_+$. Evidently, $\dim(\mathcal{Z}) = \delta(\mathbf{g}, I)$. Now let $\mathbf{R}$ be the matrix whose column vectors are $\mathbf{r}_1, \ldots, \mathbf{r}_d$. Then for all $X \in \mathcal{Z}$, the matrix $\mathbf{R} \cdot X \cdot \mathbf{R}^{-1}$ preserves the subspaces $(V_q^{(k)})_{q \in Q}$. Now define a map $\Phi : \mathcal{Z} \to \mathcal{M}$ as follows: for each $X \in \mathcal{Z}$, $X = \Phi(X)$ is the unique matrix such that

$$u_X \mathcal{L} = \mathbf{R} \cdot X \cdot \mathbf{R}^{-1} \mathcal{L}.$$

It is easy to check that in a neighborhood of the origin, Φ is a bi-Lipschitz embedding with bi-Lipschitz constant depending only on $\mathbf{R}$ and $\mathbf{R}^{-1}$. But since the basis $(\mathbf{r}_i)_1^d$ is almost orthonormal, there is a uniform bound this constant.

Thus, let C be the bi-Lipschitz constant of Φ. Let A_k' be a maximal $3C\beta$-separated subset of $B_{\mathcal{Z}}(\mathbf{0}, 1/C)$. Then $A_k = \Phi(A_k')$ is a 3β-separated subset of $B(0,1)$ consisting entirely of matrices good on turn k. It follows that

$$\#(A_k) = \#(A'_k) \asymp \beta^{-\delta}. \qquad \square$$

8.3. **Error correction.** Fix $\eta > 0$, let $\mathbf{f}$ be a simple η-integral template, fix $t_0 \in \eta\mathbb{N}$, and let $\mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{R}^d$ be a vector such that $b_i \leq b_{i+1}$ for all q such that $f_i(t_0) = f_{i+1}(t_0)$. Such a vector will be called a *perturbation vector*. For convenience, for each $k \in \mathbb{N}$ let $t_k = t_0 + k\eta$. We define the function $\mathbf{a} : \mathbb{N} \cup \{-1\} \to \mathbb{R}^d$ recursively as follows:

- $\mathbf{a}(-1) = \mathbf{b}$.
- Fix $k \geq 0$ such that $\mathbf{a}(k-1)$ has been defined, and let $I_k = (t_k, t_{k+1})$. If $(p, q]_\mathbb{Z}$ is an interval of equality for $\mathbf{f}$ on I_k (cf. Definition 2.5), then for all $i = p+1, \ldots, q$, we let

$$a_i(k) = \begin{cases} a_i(k-1) & \text{if } f'_{p+1} = \ldots = f'_q \in \{\frac{1}{m}, -\frac{1}{n}\} \text{ on } (t_0, t_{k+1}) \\ \dfrac{1}{q-p} \displaystyle\sum_{j=p+1}^{q} a_j(k-1) & \text{otherwise.} \end{cases} \tag{8.22}$$

The idea is that we will construct a new template by displacing $\mathbf{f}$ by $\mathbf{a}(k)$ on each interval I_k, and then changing the resulting function into a template by modifying it slightly to deal with the issues that arise near multiples of η. The motivation for the equation (8.22) will become apparent when we analyze when it is possible to perform such a modification. Note that by induction, for all k we have

$$\|\mathbf{a}(k)\| \leq \|\mathbf{b}\| \tag{8.23}$$

and

$$a_i(k) \leq a_{i+1}(k) \text{ whenever } f_i = f_{i+1} \text{ on } I_k. \tag{8.24}$$

Lemma 8.14. *Let the notation be as above. If $\|\mathbf{b}\| < C_\eta = \dfrac{\eta}{2mnd!}$, then there exists a partial template $\mathbf{g} : [t_0, \infty) \to \mathbb{R}^d$ such that*

$$\mathbf{g}(t_0) = \mathbf{f}(t_0) + \mathbf{b} \tag{8.25}$$

and such that for all k, we have

$$\mathbf{g} = \mathbf{f} + \mathbf{a}(k) \text{ on } \widetilde{I}_k := \big(t_k + s, t_{k+1} - s\big), \tag{8.26}$$

where $\mathbf{a}$ is as above, and

$$s = 2mnd^2\|\mathbf{b}\|.$$

Moreover, we have

$$\delta(\mathbf{g}, t) = \delta(\mathbf{f}, t) \text{ for } t \in \widetilde{I}_k. \tag{8.27}$$

The partial template $\mathbf{g}$ constructed in the proof below will be called the $\mathbf{b}$-*perturbation* of $\mathbf{f}$ at t_0.

Proof. We will first show that for all $k \geq 0$, if $\mathbf{g}$ is any function satisfying (8.26), then $\mathbf{g} \upharpoonright \widetilde{I}_k$ is a partial template. Indeed, since $\mathbf{g}$ is linear on $\widetilde{I}_k$, it suffices to check conditions (I) and (II) of Definition 2.1, along with the following weakening of condition (III):

(III′) For all $j = 1, \ldots, d-1$ such that $g_j < g_{j+1}$ on $\widetilde{I}_k$, we have $G_j'(\widetilde{I}_k) \in Z(j)$.

Condition (II) is obvious, so we check (I) and (III′).

Proof of (I). Fix $i = 1, \ldots, d-1$, and we will show that $g_i \leq g_{i+1}$ on $\widetilde{I}_k$. There are three cases:

- If $f_i = f_{i+1}$ on I_k, then by (8.24) we have $a_i(k) \leq a_{i+1}(k)$ and thus $g_i \leq g_{i+1}$ on $\widetilde{I}_k$.
- If $f_i(t_k) = f_{i+1}(t_k)$ but $f_i < f_{i+1}$ on I_k, then we have $f_i'(I_k) < f_{i+1}'(I_k)$, and thus

$$\begin{aligned} f_{i+1} - f_i &> (f_{i+1}'(I_k) - f_i'(I_k))s && \text{(on } \widetilde{I}_k\text{)} \\ &\geq \tfrac{1}{mnd^2} s && \text{(by Observation 7.8)} \\ &= 2\|\mathbf{b}\| \\ &\geq |a_{i+1}(k) - a_i(k)|, && \text{(by (8.23))} \end{aligned}$$

  so $g_i < g_{i+1}$ on $\widetilde{I}_k$. Similar logic applies if $f_i(t_{k+1}) = f_{i+1}(t_{k+1})$ but $f_i < f_{i+1}$ on I_k.
- If $f_i(t_k) < f_{i+1}(t_k)$ and $f_i(t_{k+1}) < f_{i+1}(t_{k+1})$, then since $\mathbf{f}$ is η-integral we have

$$\begin{aligned} f_{i+1} - f_i &\geq \min\big(f_{i+1}(t_k) - f_i(t_k), f_{i+1}(t_{k+1}) - f_i(t_{k+1})\big) && \text{(on } I_k\text{)} \\ &\geq \tfrac{\eta}{mnd!} && \text{(by Definition 8.1)} \\ &= 2C_\eta \\ &> 2\|\mathbf{b}\| && \text{(by hypothesis)} \\ &\geq |a_{i+1}(k) - a_i(k)| && \text{(by (8.23))} \end{aligned}$$

  and thus $g_i < g_{i+1}$ on $\widetilde{I}_k$. ◁

Proof of (III′). Fix $j = 1, \ldots, d-1$ such that $g_j < g_{j+1}$ on $\widetilde{I}_k$. There are two cases:

- If $f_j < f_{j+1}$ on I_k, then $G_j'(\widetilde{I}_k) = F_j'(I_k) \in Z(j)$.
- If $f_j = f_{j+1}$ on I_k, then $a_j(k) < a_{j+1}(k)$. Moreover, j and $j+1$ are in the same interval of equality $(p, q]_{\mathbb{Z}} \ni j, j+1$ for $\mathbf{f}$ on I_k. By (8.22) we have $f_{p+1}'(I_k) = \ldots = f_q'(I_k) \in \{\frac{1}{m}, -\frac{1}{n}\}$. Without loss of generality suppose that $f_{p+1}'(I_k) = \ldots =$

$f_q'(I_k) = \frac{1}{m}$. Then we have

$$G_j'(\widetilde{I}_k) = F_j'(I_k) = \frac{L_+(\mathbf{f}, I_k, p) + (j-p)}{m} - \frac{L_-(\mathbf{f}, I_k, p)}{n} \in Z(j).$$

(The intuition behind this calculation is that $\frac{1}{m}$ and $-\frac{1}{n}$ are "free slopes" that can be used by an individual f_j without the need for averaging; cf. the model of "particle physics" described in the paragraph below Definition 2.5.) $\triangleleft$

Next, we demonstrate (8.27). Let $(p,q]_{\mathbb{Z}}$ be an interval of equality for $\mathbf{f}$ on I_k. By the proof of (I) above, we have $g_p < g_{p+1}$ and $g_q < g_{q+1}$ on $\widetilde{I}_k$. Let

$$M_\pm = M_\pm(\mathbf{f}, I_k, p, q) = L_\pm(\mathbf{g}, \widetilde{I}_k, q) - L_\pm(\mathbf{g}, \widetilde{I}_k, p).$$

If $M_+ > 0$ and $M_- > 0$, then $a_{p+1}(k) = \ldots = a_q(k)$ and thus $(p,q]_{\mathbb{Z}}$ is an interval of equality for $\mathbf{g}$ on $\widetilde{I}_k$, which implies that $S_+(\mathbf{f}, I_k) \cap (p,q]_{\mathbb{Z}} = S_+(\mathbf{g}, \widetilde{I}_k) \cap (p,q]_{\mathbb{Z}}$. On the other hand, if $M_+ = 0$, then $S_+(\mathbf{f}, I_k) \cap (p,q]_{\mathbb{Z}} = \varnothing = S_+(\mathbf{g}, \widetilde{I}_k) \cap (p,q]_{\mathbb{Z}}$, and if $M_- = 0$, then $S_+(\mathbf{f}, I_k) \cap (p,q]_{\mathbb{Z}} = (p,q]_{\mathbb{Z}} = S_+(\mathbf{g}, \widetilde{I}_k) \cap (p,q]_{\mathbb{Z}}$. Since $(p,q]_{\mathbb{Z}}$ was arbitrary we have $S_+(\mathbf{f}, I_k) = S_+(\mathbf{g}, \widetilde{I}_k)$ and thus $\delta(\mathbf{f}, I_k) = \delta(\mathbf{g}, \widetilde{I}_k)$.

Finally, we describe how to define $\mathbf{g}$ on an interval of the form

$$J_k = \begin{cases} [t_k - s, t_k + s] & \text{if } k > 0 \\ [t_0, t_0 + s] & \text{if } k = 0 \end{cases}$$

We now consider two cases:

Case 1. If $\mathbf{a}(k-1) = \mathbf{a}(k)$, then we can continue to use the formula $\mathbf{g} = \mathbf{f} + \mathbf{a}(k)$ on J_k. Minor modifications to the previous argument show that $\mathbf{g} \upharpoonright \widetilde{I}_{k-1} \cup J_k \cup \widetilde{I}_k$ is a partial template (where we use the convention that $\widetilde{I}_{-1} = \varnothing$).

Case 2. Suppose that $\mathbf{a}(k-1) \neq \mathbf{a}(k)$. By (8.22), this means that t_k is either a merge, a transfer, or t_0. We restrict our attention to the case where t_k is a merge; the other cases are similar. Define $\mathbf{g}$ on J_k as follows: Let $(p,q]_{\mathbb{Z}}$ be an interval of equality for $\mathbf{f}$ on I_k which is not an interval of equality for $\mathbf{f}$ on I_{k-1}, and let $M_\pm = M_\pm(\mathbf{f}, I_k, p, q)$, so that $M_+ + M_- = q - p$. Note that $M_+, M_- > 0$, as otherwise we would have $f_{p+1}' = f_q' \in \{\frac{1}{m}, -\frac{1}{n}\}$ on I_{k-1}, and thus $(p,q]_{\mathbb{Z}}$ would be an interval of equality for $\mathbf{f}$ on I_{k-1}. We define the functions $g_{p+1}, \ldots, g_q$ on J_k by imposing the following conditions:

- We have
$$\mathbf{g}(\min(J_k)) = \mathbf{f}(\min(J_k)) + \mathbf{a}(k-1).$$

- We have

$$\sum_{i=p+1}^{q} g_i' = \sum_{i=p+1}^{q} f_i' = \frac{M_+}{m} - \frac{M_-}{n} \text{ on } J_k \tag{8.28}$$

(the second equality holds because t_k cannot be a transfer).
- For all $p < i \leq p + M_+$ and $t \in J_k$, we have $g_i'(t) = \frac{1}{m}$ unless $g_i(t) = g_{p+M_++1}(t)$, in which case $g_i'(t) = z(t)$.
- For all $p + M_+ < i \leq q$ and $t \in J_k$, we have $g_i'(t) = -\frac{1}{n}$ unless $g_i(t) = g_{p+M_+}(t)$, in which case $g_i'(t) = z(t)$.

The number $z(t)$ appearing in the last two conditions can be computed by plugging the values of g_i' appearing in those conditions into (8.28) and then solving for $z(t)$. In all of the above formulas, derivatives should be assumed to be taken from the right.

It is easy to check that these conditions uniquely determine the functions $g_{p+1}, \ldots, g_q$ on the interval J_k. To ensure that this does not lead to an inconsistency with (8.26), we need to check that

$$\mathbf{g}(\max(J_k)) = \mathbf{f}(\max(J_k)) + \mathbf{a}(k). \tag{8.29}$$

Since t_k is not a split, by (8.22) the map $i \mapsto f_i(\max(J_k))a_i(k)$ is constant on $(p,q]_{\mathbb{Z}}$, and thus by (8.28), we just need to show that the map $i \mapsto g_i(\max(J_k))$ is also constant on $(p,q]_{\mathbb{Z}}$. Suppose not. Then either there exists $p < i \leq p + M_+$ such that $g_i(t) < g_{p+M_++1}(t)$ for all $t \in J_k$, or there exists $p + M_+ < i \leq q$ such that $g_i(t) > g_{p+M_+}(t)$ for all $t \in J_k$. Without loss of generality suppose the first case holds. Then $g_{p+1}(t) < g_{p+M_++1}(t)$ for all $t \in J_k$, and thus $g_{p+1}'(t) = \frac{1}{m}$ for all $t \in J_k$. Now let

$$F(t) = \sum_{i=p+1}^{q} \left[f_i(t) - f_{p+1}(t)\right],$$

$$G(t) = \sum_{i=p+1}^{q} \left[g_i(t) - g_{p+1}(t)\right].$$

Then $F(t), G(t) \geq 0$, $F(t_k) = 0$, and

$$F'(t) \geq G'(t) = -Z \stackrel{\text{def}}{=} \left[\frac{M_+}{m} - \frac{M_-}{n}\right] - \frac{q-p}{m} = -M_- \left[\frac{1}{m} + \frac{1}{n}\right].$$

It follows that

$$\begin{aligned}0 &\le G(\max(J_k)) = G(\min(J_k)) - Z|J_k| \\ &\le F(\min(J_k)) + 2d\|\mathbf{b}\| - Z|J_k| \\ &\le F(t_k) + Z\big[k > 0\big]s + 2d\|\mathbf{b}\| - Z|J_k| = 2d\|\mathbf{b}\| - Zs < 0,\end{aligned}$$

where the last inequality follows from the definition of s and the inequality $M_- \ge 1$. This is a contradiction. Thus (8.29) holds, and so $\mathbf{g}$ is continuous in a neighborhood of $\max(J_k)$. We leave the verification of the other conditions of Definition 2.1 as an exercise to the reader. □

Now we combine the concept of perturbation vectors with the concept of C-matches introduced in §8.2. The following lemma shows that by perturbing a template, it is possible to improve the constant C appearing in Definition 8.9:

Lemma 8.15. *Let Λ be a C_η-match for an η-integral template $\mathbf{f}$ at $t_0 \in \eta\mathbb{N}$, and let $\mathbf{g}$ be the $\mathbf{b}$-perturbation of $\mathbf{f}$ at t_0, where $\mathbf{b} \in (d^3)!\delta\mathbb{Z}^d$ is a perturbation vector such that*

$$\big\|\mathbf{h}(\Lambda) - [\mathbf{f}(t_0) + \mathbf{b}]\big\| < C_1 \tag{8.30}$$

for some constant $C_1 \le C_\eta$. Suppose that t_0 is not a split with respect to $\mathbf{f}$. Then Λ is a C_1-match for $\mathbf{g}$ at t_0.

Proof. Since $\mathbf{f}$ is η-integral and $\mathbf{b} \in (d^3)!\delta\mathbb{Z}^d$, analyzing the proof of Lemma 8.14 shows that $\mathbf{g}$ is δ-integral.

Since $\mathbf{g}(t_0) = \mathbf{f}(t_0) + \mathbf{b}$, (8.30) implies that condition (I) of Definition 8.9 holds with $C = C_1$. Now fix $j = 1, \ldots, d-1$ and let V_j be the linear span of $\{\mathbf{r} \in \Lambda : \|\mathbf{r}\| \le \lambda_j(\Lambda)\}$. Let I_+ be an interval of linearity for both $\mathbf{f}$ and $\mathbf{g}$ whose left endpoint is t_0, and let $(p,q]_\mathbb{Z} \ni j$ be an interval of equality for $\mathbf{f}$ on I_+. Then $f_q(t) < f_{q+1}(t)$, so since $\mathbf{f}$ is η-integral, by (8.30) we have $h_q(\Lambda) < h_{q+1}(\Lambda)$. It follows that

$$\dim(V_q) - \dim(V_j) \le q - j$$

and thus

$$\begin{aligned}\dim(V_j \cap \mathcal{L}) &\ge \max\big(\dim(V_p \cap \mathcal{L}), \dim(V_q \cap \mathcal{L}) - (q-j)\big) \\ &\ge \max\big(L_-(\mathbf{f}, I_+, p), L_-(\mathbf{f}, I_+, q) - (q-j)\big) = L_-(\mathbf{f}, I_+, j) = L_-(\mathbf{g}, I_+, j),\end{aligned}$$

where the second-to-last equality follows from the assumption that t is not a split for $\mathbf{f}$. □

8.4. **Uniform error bounds.** We are now ready to complete the proof of (8.1). First, by Lemma 8.4 we can without loss of generality assume that $\mathbf{f}$ is simple and that its corner points are all multiples of 2η, where $\eta = k_\eta\delta$, $k_\eta \in \mathbb{N}$ is large to be determined, and δ is as above. After translating by η, we can assume that the corner points are at odd multiples of η instead of even multiples. We can now define Alice's strategy as follows: Fix $\ell \in \mathbb{N}$ and let $k_\ell = 2\ell k_\eta$, and suppose that the game has progressed to turn k_ℓ. This means that the lattice $\Lambda^{(\ell)} := \Lambda_{k_\ell}$ has already been defined.

- If $\Lambda^{(\ell)}$ is not a C_η-match for $\mathbf{f}$ at $t_\ell := k_\ell\delta = 2\ell\delta$, then Alice resigns on turn k_ℓ.
- Suppose that $\Lambda^{(\ell)}$ is a C_η-match for $\mathbf{f}$ at t_ℓ. Let $\mathbf{b} = \mathbf{b}^{(\ell)}$ be the element of $(d^3)!\delta\mathbb{Z}^d$ closest to $\mathbf{h}(\Lambda^{(\ell)}) - \mathbf{f}(t_\ell)$ (using any tiebreaking mechanism). Then $\mathbf{b}$ is a perturbation vector satisfying (8.30) with $\Lambda = \Lambda^{(\ell)}$, $t_0 = t_\ell$, and $C_1 = (1/2)(d^3)!\delta$. Let $\mathbf{g} = \mathbf{g}^{(\ell)}$ be the $\mathbf{b}$-perturbation of $\mathbf{f}$ at t_ℓ. Then by Lemma 8.15, $\mathbf{g}$ is a C_1-match for $\Lambda^{(\ell)}$. This allows us to apply Lemma 8.10, and on turns $k_\ell, \ldots, k_{\ell+1} - 1$ Alice plays the strategy given by this lemma.

Let $t'_\ell = (2\ell+1)\eta$. Since $\mathbf{f}$ is linear on $I_0^{(\ell)} = [t_\ell, t'_\ell]$ and $I_1^{(\ell)} = [t'_\ell, t_{\ell+1}]$, analyzing the proof of Lemma 8.14 shows that the perturbation $\mathbf{g} \upharpoonright I_\ell$ has at most $2d$ intervals of linearity. In particular we have $N \leq 2d$ in Lemma 8.10.

To compute the relation between $\mathbf{b}^{(\ell)}$ and $\mathbf{b}^{(\ell+1)}$, we let $\mathbf{a}^{(\ell)} : \mathbb{N} \cup \{-1\} \to \mathbb{R}^d$ be the function defined in §8.3, so that $\mathbf{a}^{(\ell)}(-1) = \mathbf{b}^{(\ell)}$. Then we have

$$\mathbf{g}^{(\ell)} = \mathbf{f} + \mathbf{a}^{(\ell)}(0) \text{ on } \widetilde{I}_0^{(\ell)} = [t_\ell + s, t'_\ell - s],$$
$$\mathbf{g}^{(\ell)} = \mathbf{f} + \mathbf{a}^{(\ell)}(1) \text{ on } \widetilde{I}_1^{(\ell)} \cup J_2^{(\ell)} \cup \widetilde{I}_2^{(\ell)} = [t'_\ell + s, t'_{\ell+1} - s].$$

The second equality follows from the fact that $t_2^{(\ell)} = t_{\ell+1}$ is not a corner point, so $\mathbf{a}^{(\ell)}(1) = \mathbf{a}^{(\ell)}(2)$ and thus Case 1 of the proof of Lemma 8.14 applies. In particular, we have

$$\mathbf{g}^{(\ell)}(t_{\ell+1}) = \mathbf{f}(t_{\ell+1}) + \mathbf{a}^{(\ell)}(1).$$

On the other hand, according to part (i) of Lemma 8.10, $\Lambda^{(\ell+1)} = \Lambda_{k_{\ell+1}}$ is a C_2-match for $\mathbf{g}^{(\ell)}$ at $t_{\ell+1}$, where C_2 is a constant depending only on C_1. Thus,

$$\left\|\mathbf{h}(\Lambda^{(\ell+1)}) - [\mathbf{f}(t_{\ell+1}) + \mathbf{a}^{(\ell)}(1)]\right\| \leq C_2$$

and so by the definition of $\mathbf{b}^{(\ell+1)}$, we have

$$\|\mathbf{b}^{(\ell+1)} - \mathbf{a}^{(\ell)}(1)\| \leq C_2 + C_1, \tag{8.31}$$

assuming that Alice does not resign on turn $k_{\ell+1}$.

Assume now that there exists a constant $B > 0$ such that

$$\|\mathbf{b}^{(\ell)}\| \leq B \text{ for all } \ell \text{ such that Alice does not resign on or before turn } k_\ell. \tag{8.32}$$

Fix ℓ such that Alice does not resign on or before turn k_ℓ. Then $\Lambda^{(\ell+1)}$ is a C_2-match for $\mathbf{g}^{(\ell)}$ at $t_{\ell+1}$, and is therefore a $(C_2 + B)$-match for $\mathbf{f}$ at $t_{\ell+1}$, since $\|\mathbf{a}^{(\ell)}(1)\| \leq \|\mathbf{b}^{(\ell)}\| \leq B$. Letting $\delta \geq 4mnd!(C_2 + B)$, we see that $\Lambda^{(\ell+1)}$ is a C_η-match for $\mathbf{f}$ at $t_{\ell+1}$, and thus Alice does not resign on turn $k_{\ell+1}$. So by induction Alice never resigns.

So for all ℓ, $\Lambda^{(\ell)}$ is a C-match for $\mathbf{f}$ at t_ℓ, where $C = C_2 + B$. It follows that the final outcome A_∞ is in the target set $\mathcal{M}(\mathbf{f})$. To compute Alice's score, we use the second part of Lemma 8.10 to get that

$$\begin{aligned}\Delta(\mathcal{A}, [0, k_\ell]) = \frac{1}{\ell}\sum_{j=0}^{\ell-1} \Delta(\mathcal{A}, [k_j, k_{j+1}]) &= \frac{1}{\ell}\sum_{j=0}^{\ell-1} \Delta(\mathbf{f}, [t_j, t_{j+1}]) + O\left(\tfrac{1}{\delta} + \tfrac{1}{k_\eta}\right)\\ &= \Delta(\mathbf{f}, [0, t_\ell]) + O\left(\tfrac{1}{\delta} + \tfrac{1}{k_\eta}\right)\end{aligned}$$

and thus

$$\underline{\delta}(\mathcal{A}) = \underline{\delta}(\mathbf{f}) + O\left(\tfrac{1}{\delta} + \tfrac{1}{k_\eta}\right).$$

Given $\varepsilon > 0$, we can choose δ small enough and k_η large enough so that the last term is less than ε, which shows that $\underline{\delta}(\mathcal{A}) \geq \underline{\delta}(\mathbf{f}) - \varepsilon$, and thus $\mathcal{M}(\mathbf{f})$ is $(\underline{\delta}(\mathbf{f}) - \varepsilon)$-dimensionally Hausdorff β-winning. Applying Theorem 5.2 shows that $\dim_H(\mathcal{M}(\mathbf{f})) \geq \underline{\delta}(\mathbf{f}) - \varepsilon$, and a similar argument shows that $\dim_P(\mathcal{M}(\mathbf{f})) \geq \overline{\delta}(\mathbf{f}) - \varepsilon$. This completes the proof assuming (8.32). In what follows we will prove (8.32).

Given $q = 0, \ldots, d$, an interval $[\ell_1, \ell_2]$ will be called a *q-interval* if either

$$f_q < f_{q+1} \text{ on } (t'_{\ell_1 - 1}, t'_{\ell_2})$$

or

$$f'_q = f'_{q+1} \in \{\tfrac{1}{m}, -\tfrac{1}{n}\} \text{ on } (t'_{\ell_1-1}, t'_{\ell_2}).$$

Note that every interval is both a 0-interval and a d-interval (according to our convention that $f_0 = -\infty$ and $f_{d+1} = +\infty$).

Claim 8.16. *Fix $q = 1, \ldots, d-1$ and let $[\ell_1, \ell_2]$ be a q-interval. Then there exists a constant $\alpha = \alpha(q, \ell_1, \ell_2)$ such that for all $\ell = \ell_1, \ldots, \ell_2$, we have*

$$\sum_{i=1}^{q} b_i^{(\ell)} \asymp_+ \alpha(q, \ell_1, \ell_2). \tag{8.33}$$

Proof. First suppose that $f_q < f_{q+1}$ on $(t'_{\ell_1-1}, t'_{\ell_2})$. Since $\Lambda^{(\ell)}$ is a C_2-match for $\mathbf{g}^{(\ell)}$ at t_ℓ, we have

$$\sum_{i=1}^q b_i^{(\ell)} \asymp_+ \sum_{i=1}^q h_i(\Lambda^{(\ell)}) - F_q(t_\ell).$$

Let $V^{(\ell)}$ denote the $\Lambda^{(\ell)}$-rational subspace minimizing $\|V^{(\ell)}\|$, and note that

$$\sum_{i=1}^q h_i(\Lambda^{(\ell)}) \asymp_+ \log\|V^{(\ell)}\|.$$

Since $f_q < f_{q+1}$ on $(t'_{\ell_1-1}, t'_{\ell_2})$, we have $gu_{X_\ell}V^{(\ell)} = V^{(\ell+1)}$ for all ℓ. By modifying the proof of Lemma 8.10, we can see that

$$\log\|V^{(\ell')}\| - \log\|V^{(\ell)}\| \asymp_+ (t_{\ell'} - t_\ell)z$$

for any $\ell < \ell'$ such that $F'_q = z$ on $(t_\ell, t_{\ell'})$. Since F_q is piecewise linear on $(t'_{\ell_1-1}, t'_{\ell_2})$ with a bounded number of intervals of linearity, it follows that

$$\log\|V^{(\ell)}\| - \log\|V^{(\ell_1}\| \asymp_+ \int_{t_{\ell_1}}^{t_\ell} F'_q = F_q(t_\ell) - F_q(t_{\ell_1}).$$

Thus, letting

$$\alpha(q, \ell_1, \ell_2) = \log\|V^{(\ell_1)}\| - F_q(t_{\ell_1})$$

completes the proof.

Now suppose that $f'_q = f'_{q+1} \in \{\frac{1}{m}, -\frac{1}{n}\}$ on $(t'_{\ell_1-1}, t'_{\ell_2})$. For each interval of linearity I and for each $t \in I$, let $(p(t), r(t)]$ be the interval of equality for I that contains q. If $f'_q = f'_{q+1} = \frac{1}{m}$, then p, r are increasing functions, while if $f'_q = f'_{q+1} = -\frac{1}{n}$, then they are decreasing functions. Since p, r are integer-valued, it follows that $(t'_{\ell_1-1}, t'_{\ell_2})$ can be decomposed into a bounded number of intervals on which p, r are constant. So we can without loss of generality suppose that p and r are constant. But then by the same logic as before we have

$$\log\|V_p^{(\ell')}\| - \log\|V_p^{(\ell)}\| \asymp_+ (t_{\ell'} - t_\ell)z_p, \log\|V_r^{(\ell')}\| - \log\|V_r^{(\ell)}\| \quad \asymp_+ (t_{\ell'} - t_\ell)z_r,$$

while on the other hand $z_r - z_p \in (r-p)\{\frac{1}{m}, -\frac{1}{n}\}$. It then follows from geometric considerations that

$$\log\|V_q^{(\ell')}\| - \log\|V_q^{(\ell)}\| \asymp_+ (t_{\ell'} - t_\ell)z_q,$$

and the proof can be continued in the same way as above. □

Now fix $\ell \in \mathbb{N}$ and $q = 1, \ldots, d-1$. If ℓ is contained in a q-interval then we let

$$\alpha(q, \ell) = \alpha(q, \ell_1, \ell_2),$$

where $[\ell_1, \ell_2]$ is the longest q-interval containing ℓ. Otherwise, we let $\alpha(q,\ell) = *$. Next, we let $\mathbf{c}^{(\ell)} \in \mathbb{R}^d$ be the unique vector such that

$$\sum_{i=1}^{q} c_i^{(\ell)} = \alpha(q,\ell) \qquad \text{when } \alpha(q,\ell) \in \mathbb{R}, \tag{8.34}$$

$$c_q^{(\ell)} = c_{q+1}^{(\ell)} \qquad \text{when } \alpha(q,\ell) = *. \tag{8.35}$$

Then by (8.22) and (8.33), we have $\mathbf{c}^{(\ell)} \asymp_+ \mathbf{b}^{(\ell)}$.

For convenience, we introduce a slightly modified version of intervals of equality. We call an interval $(p,q]_\mathbb{Z}$ an *interval of mixing* for $\mathbf{f}$ on I if either

- $(p,q]_\mathbb{Z}$ is an interval of equality for $\mathbf{f}$ on I, and $f_q' \notin \{\frac{1}{m}, -\frac{1}{n}\}$ on I, or
- $q = p+1$ and $f_q' \in \{\frac{1}{m}, -\frac{1}{n}\}$ on I.

Note that if $(p,q]_\mathbb{Z}$ is an interval of mixing for $\mathbf{f}$ on $I_\ell := (t'_{\ell-1}, t'_\ell)$, then $[\ell,\ell]$ is both a p-interval and a q-interval.

Let $(p,q]_\mathbb{Z}$ be an interval of mixing for $\mathbf{f}$ on I_ℓ. Then by (8.22), we have $a_i^{(\ell-1)}(1) = a$ for all $i \in (p,q]_\mathbb{Z}$, where a is a constant. By (8.31), we have $|b_i^{(\ell)} - a| \leq C_2 + C_1$ for all $i \in (p,q]_\mathbb{Z}$, and thus by (8.22), we have $|a_i^{(\ell)}(0) - a| \leq C_2 + C_1$ for all $i \in (p,q]_\mathbb{Z}$. On the other hand, for $i = 1,\ldots,d$ such that $f_i' \in \{\frac{1}{m}, -\frac{1}{n}\}$ on I_ℓ, (8.22) implies that $a_i^{(\ell)}(0) = b_i$. Thus

$$\|\mathbf{a}^{(\ell)}(0) - \mathbf{b}^{(\ell)}\| \leq 2(C_2 + C_1)$$

and consequently $\mathbf{a}^{(\ell)}(0) \asymp_+ \mathbf{c}^{(\ell)}$. On the other hand, by (8.31) we have $\mathbf{a}^{(\ell)}(1) \asymp_+ \mathbf{c}^{(\ell+1)}$, so by (8.22), for every interval of mixing $(p,q]_\mathbb{Z}$ for $\mathbf{f}$ on $I_{\ell+1}$, we have

$$c_i^{(\ell+1)} \asymp_+ \frac{1}{q-p} \sum_{j=p+1}^{q} c_j^{(\ell)} \tag{8.36}$$

Let B be a large number, fix $\varepsilon \in \{\pm 1\}$, and write $\overline{c}_i^{(\ell)} = B + \varepsilon c_i^{(\ell)}$. We claim that there exist constants $C(1),\ldots,C(d) \geq 0$, independent of B, such that if $B \geq \max_i C(i)/i$, then for all $j < k$ and $\ell \in \mathbb{N}$ we have

$$\sum_{i=j+1}^{k} \overline{c}_i^{(\ell)} \geq C(k-j). \tag{8.37}$$

Indeed, when $\ell = 0$, we have $\mathbf{c}^{(0)} = \mathbf{0}$ and thus $\sum_{i=j+1}^k \overline{c}_i^{(\ell)} = (k-j)B \geq C(k-j)$. For the inductive step, fix $\ell \in \mathbb{N}$ and suppose that (8.37) holds for all $j < k$. Fix $j < k$, and we will show that

$$\sum_{i=j+1}^{k} \overline{c}_i^{(\ell+1)} \geq C(k-j). \tag{8.38}$$

Case 1. Suppose that $[\ell, \ell+1]$ is both a j-interval and a k-interval. Then $\alpha(j,\ell) = \alpha(j,\ell+1) \in \mathbb{R}$ and similarly for k. So

$$\sum_{i=j+1}^{k} c_i^{(\ell)} = \alpha(k,\ell) - \alpha(j,\ell) = \alpha(k,\ell+1) - \alpha(j,\ell+1) = \sum_{i=j+1}^{k} c_i^{(\ell+1)}$$

and thus

$$\sum_{i=j+1}^{k} \overline{c}_i^{(\ell+1)} = \sum_{i=j+1}^{k} \overline{c}_i^{(\ell)} \geq C(k-j).$$

Case 2. Suppose that $[\ell, \ell+1]$ is a j-interval but not a k-interval. Let $(p,q]_{\mathbb{Z}} \ni k$ be an interval of mixing for $\mathbf{f}$ on either I_ℓ or $I_{\ell+1}$. Then by (8.36), we have

$$c_i^{(\ell+1)} \asymp_+ \frac{1}{q-p} \sum_{i=p+1}^{q} c_i^{(\ell)} \text{ for all } i \in (p,q]_{\mathbb{Z}}. \tag{8.39}$$

In the latter case this follows directly from (8.36), while if $(p,q]_{\mathbb{Z}}$ is an interval of mixing for $\mathbf{f}$ on I_ℓ, then by (8.36) we have $c_i^{(\ell)} \asymp_+ c$ for all $i \in (p,q]_{\mathbb{Z}}$ for some constant c, and applying (8.36) again gives (8.39). On the other hand, since $[\ell, \ell+1]$ is a j-interval we have $j \leq p$, and thus the previous case gives

$$\sum_{i=j+1}^{p} \overline{c}_i^{(\ell)} = \sum_{i=j+1}^{p} \overline{c}_i^{(\ell+1)}.$$

So

$$\sum_{i=j+1}^{k} \overline{c}_i^{(\ell+1)} \asymp_+ \sum_{i=j+1}^{p} \overline{c}_i^{(\ell)} + \frac{k-p}{q-p} \sum_{i=p+1}^{q} \overline{c}_i^{(\ell)} \geq \frac{1}{d} \sum_{i=j+1}^{k+1} \overline{c}_i^{(\ell)} \geq \frac{1}{d} C(k+1-j).$$

Let C denote the implied constant of the asymptotic, and let $C(1), \ldots, C(d)$ be defined by the recursive formula

$$C(1) = 0, \qquad\qquad C(k+1) = d(C(k) + C).$$

Then we have demonstrated (8.38), completing the inductive step.

Case 3. If $[\ell, \ell+1]$ is a k-interval but not a j-interval, or is neither a j-interval nor a k-interval, then the proof is similar to Case 2. We leave the details to the reader.

This completes the proof of (8.37), which in turn implies (8.32), thereby completing the proof of (8.1). Thus, we have completed proving the lower bounds in Theorem 2.6.

9. Proof of Theorem 2.6, upper bound

Let $\mathcal{S}$ be a class of functions from $[0,\infty)$ to $\mathbb{R}^d$ closed under finite perturbations. We claim that

$$\dim_H(S) \leq \sup_{\mathbf{f}\in\mathcal{S}\cap\mathcal{T}_{m,n}} \underline{\delta}(\mathbf{f}), \qquad \dim_P(S) \leq \sup_{\mathbf{f}\in\mathcal{S}\cap\mathcal{T}_{m,n}} \overline{\delta}(\mathbf{f}),$$

where $S = \mathcal{M}(\mathcal{S})$ is as in (2.14). As in the proof of the lower bounds, we will play the modified Hausdorff and packing games with target set S. This time, we will define a strategy for Bob.

Definition of the strategy. Suppose that the game has progressed to turn k, with corresponding lattice Λ_k as in §6. Let $\{\mathbf{r}_1,\ldots,\mathbf{r}_d\}$ be a Minkowski basis of Λ_k (cf. Lemma 7.5), and for each $q = 0,\ldots,d$ let $V_q = \sum_{i=1}^q \mathbb{R}\mathbf{r}_i$. Essentially, Bob's strategy will be to "push the subspaces V_q away from $\mathcal{L}$ as much as possible given Alice's move". To make this precise, fix $X \in B_{\mathcal{M}}(\mathbf{0}, 1-\beta)$, and for each $q = 0,\ldots,d$ let

$$L_q^- = L_q^-(k,X) \stackrel{\text{def}}{=} \sup_{\|Y\|\leq 2\beta} \dim(u_{X+Y}V_q \cap \mathcal{L}), \qquad L_q^+ \stackrel{\text{def}}{=} q - L_q^-. \tag{9.1}$$

Let

$$S_+ = S_+(k,X) \stackrel{\text{def}}{=} \{q = 1,\ldots,d : L_q^+ = L_{q-1}^+ + 1 \text{ and } L_q^- = L_{q-1}^-\},$$
$$S_- = S_-(k,X) \stackrel{\text{def}}{=} \{q = 1,\ldots,d : L_q^- = L_{q-1}^- + 1 \text{ and } L_q^+ = L_{q-1}^+\},$$

and note that $S_+ \cup S_- = (0,d]_{\mathbb{Z}}$, $\#(S_\pm) = d_\pm$, and $d_+ = m$, $d_- = n$. Also note that $L_q^\pm = \#(S_\pm \cap (0,q]_{\mathbb{Z}})$ for all $q = 1,\ldots,d$. Finally, let $\delta(k,X) = \delta(S_+,S_-)$, where as in (2.13),

$$\delta(T_+,T_-) \stackrel{\text{def}}{=} \#\{(i_+,i_-) \in T_+ \times T_- : i_+ > i_-\}.$$

Bob's strategy on turn k can now be given as follows: If Alice makes the move $A_k \subseteq B_{\mathcal{M}}(\mathbf{0},1-\beta)$, then Bob responds by choosing $X_k \in A_k$ so as to maximize $\delta(k,X_k)$. Note that larger values of $\delta(k,X_k)$ correspond to larger values of L_q^+ and correspondingly smaller values of L_q^-, which in turn correspond to the intuitive idea of "pushing V_q away from $\mathcal{L}$ (by a distance of at least 2β)".

The following claim will be used to relate scores in the Hausdorff and packing games with the dimensions of templates.

Claim 9.1. *For all k we have*

$$\#(A_k) \lesssim \beta^{-\delta(k,X_k)}.$$

Proof. Let $\delta = \delta(k, X_k)$. Clearly,

$$A_k \subseteq \{X : \delta(k,X) \leq \delta\} \subseteq \bigcup_{T_+,T_-} \{X : L_q^-(X) \geq \#(T_- \cap (0,q]_\mathbb{Z}) \text{ for all } q = 1,\ldots,d\},$$

where the union is taken over all sets $T_+, T_- \subseteq (0,d]_\mathbb{Z}$ such that $T_+ \cap T_- = \varnothing$, $T_+ \cup T_- = (0,d]_\mathbb{Z}$, $\#(T_\pm) = d_\pm$, and

$$\delta(T_+, T_-) \leq \delta.$$

Fix T_+, T_- as above, and for each $q = 1,\ldots,d$ let $\widehat{L}_q^- = \#(T_- \cap (0,q]_\mathbb{Z})$. We need to estimate the size of the set

$$A_k(T_+,T_-) \stackrel{\text{def}}{=} \{X \in A_k : L_q^-(X) \geq \widehat{L}_q^- \text{ for all } q\}.$$

Since $A_k = \bigcup_{T_+,T_-} A_k(T_+,T_-)$, to complete the proof it suffices to show that

$$\#(A_k(T_+,T_-)) \lesssim \beta^{-\delta}.$$

Note that for each $X \in B_\mathcal{M}(\mathbf{0}, 1-\beta)$ and $q = 1,\ldots,d$, we have $L_q^-(X) \geq \widehat{L}_q^-$ if and only if X is in the 2β-neighborhood of the algebraic set

$$\mathcal{Z}_q = \{X : \dim(u_X V_q \cap \mathcal{L}) \geq \widehat{L}_q^-\} \subseteq \mathcal{M}.$$

Thus,

$$A_k(T_+,T_-) \subseteq \bigcap_{q=1}^d \mathcal{N}(\mathcal{Z}_q, 2\beta).$$

Let $\mathcal{Z} = \bigcap_{q=1}^d \mathcal{Z}_q$. We claim that

$$\bigcap_{q=1}^d \mathcal{N}(\mathcal{Z}_q, 2\beta) \subseteq \mathcal{N}(\mathcal{Z}, C\beta) \tag{9.2}$$

for some uniform constant C. Indeed, fix $X \in \bigcap_q \mathcal{N}(\mathcal{Z}_q, 2\beta)$. For each $q = 1,\ldots,d$, choose $X_q \in \mathcal{Z}_q \cap B(X, 2\beta)$, and let

$$\widehat{V}_q = u_{X_q} V_q \cap \mathcal{L}.$$

Next, for $q = 1,\ldots,d$ we recursively define

$$W_q = \widehat{V}_q \cap \widehat{W}_{q-1}^\perp, \quad \widehat{W}_q = W_1 + \ldots + W_q,$$

with the understanding that $\widehat{W}_0 = \{\mathbf{0}\}$. Note that since $X_q \in \mathcal{Z}_q$,

$$\dim(\widehat{W}_q) = \dim\left(\widehat{W}_{q-1} + (\widehat{V}_q \cap \widehat{W}_{q-1}^\perp)\right) \geq \dim(\widehat{V}_q) \geq \widehat{L}_q^-.$$

Let Z be the unique matrix such that $u_{-Z}\mathbf{v} = u_{-X_q}\mathbf{v}$ for all $q = 1,\ldots,d$ and $\mathbf{v} \in W_q$. Since $\mathcal{L} = \widehat{W}_d = W_1 + \ldots + W_d$ is an orthogonal decomposition, such a Z exists, and we have $\|Z - X\| \lesssim \beta$. Now fix $q = 1,\ldots,d$. For all $p = 1,\ldots,q$ and $\mathbf{v} \in W_p$, we have $u_{-Z}\mathbf{v} = u_{-X_p}\mathbf{v} \in V_p \subseteq V_q$. This implies that $u_{-Z}\widehat{W}_q \subseteq V_q$ and thus

$$\dim(u_Z V_q \cap \mathcal{L}) \geq \dim(\widehat{W}_q) \geq \widehat{L}_q^-,$$

so $Z \in \mathcal{Z}_q$. Since q was arbitrary, we have $Z \in \mathcal{Z}$, and thus $X \in \mathcal{N}(\mathcal{Z}, C\beta)$. This completes the proof of (9.2).

So $A_k(T_+, T_-) \subseteq \mathcal{N}(\mathcal{Z}, C\beta)$, where $A_k(T_+, T_-)$ is a 3β-separated set and $\mathcal{Z}$ is an algebraic set whose diagram in the sense of [63, Definition 4.2] is constant (i.e. independent of k and β). By [63, Corollary 5.7], it follows that

$$\#(A_k(T_+, T_-)) \lesssim \beta^{-\dim(\mathcal{Z})},$$

whereas we wish to show that $\#(A_k(T_+, T_-)) \lesssim \beta^{-\delta}$. So to complete the proof we must show that $\dim(\mathcal{Z}) \leq \delta$.

Consider first the case where the subspaces V_q ($q = 1,\ldots,d$) are all coordinate subspaces, i.e. $V_q = \sum_{i \in I_q} \mathbb{R}\mathbf{e}_i$ for some $I_q \subseteq (0,d]_\mathbb{Z}$, and where $\dim(V_q \cap \mathcal{L}) = \widehat{L}_q^-$ for all q. In this case, we write $I^- = \{m+1,\ldots,d\}$, so that $\mathcal{L} = \sum_{i \in I^-} \mathbb{R}\mathbf{e}_i$. Let σ be the unique permutation of $(0,d]_\mathbb{Z}$ such that for each $q = 1,\ldots,d$, we have $I_q = \{\sigma(1),\ldots,\sigma(q)\}$. Then since

$$\#(I_q \cap I_-) = \widehat{L}_q^- = \#(T_- \cap (0,q]_\mathbb{Z}) \text{ for all } q,$$

we have $I_- = \sigma(T_-)$.

It is readily verified that $X \in \mathcal{Z}$ if and only if $X_{i,j} = 0$ for all $i = 1,\ldots,m$ and $j = 1,\ldots,n$ such that

$$\sigma^{-1}(i) > \sigma^{-1}(m+j).$$

Thus, $\dim(\mathcal{Z})$ is equal to the number of pairs $(i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n\}$ such that $\sigma^{-1}(i) < \sigma^{-1}(m+j)$, or equivalently the number of pairs $(i_+, i_-) \in T_+ \times T_-$ such that $i_+ < i_-$. In other words, $\dim(\mathcal{Z}) = \delta(T_+, T_-) \leq \delta$.

For the general case, note that the map $X \mapsto u_{-X}\mathcal{L}$ is a coordinate chart for the Grassmannian variety $\mathcal{G} = \mathcal{G}(d,n)$ of n-dimensional subspaces of $\mathbb{R}^d$. So it suffices to show that $\dim(\mathcal{Z}') \leq \delta$, where

$$\mathcal{Z}' = \{W \in \mathcal{G} : \dim(V_q \cap W) \geq \widehat{L}_q^- \text{ for all } q\}.$$

Let W be a smooth point of $\mathcal{Z}'$. Then $\dim(V_q \cap W) = \widehat{L}_q^-$ for all q. Moreover, there is a basis of $\mathbb{R}^d$ such that the subspaces V_q ($q = 1, \ldots, d$) and W are all coordinate subspaces with respect to this basis. So from the previous argument, it follows that $\dim(\mathcal{Z}' \cap U) = \delta$, where U is a neighborhood of W (depending on the basis). Since W was an arbitrary smooth point, this shows that $\dim(\mathcal{Z}') = \delta$. □

Now suppose that the game is played according to Bob's strategy, let A denote the outcome, and suppose that the corresponding successive minima function $\mathbf{h}_A$ is in $\mathcal{S}$. By Lemma 8.4, there exists a template $\mathbf{g}$ such that $\mathbf{g} \asymp_+ \mathbf{h}_A$. Fix a large constant $C_1 \geq \gamma$. Applying Lemma 8.4 again, there exists a template $\mathbf{f}$ such that $\mathbf{f} \asymp_{+,C_1} \mathbf{g}$ and such that for all q, t, t' such that $f_q(t) < f_{q+1}(t)$ and $|t' - t| \leq \eta$, we have $g_{q+1}(t') - g_q(t') \geq \eta$ and $F_q'(t) \geq G_q'(t')$. Since $\mathbf{h}_A \in \mathcal{S}$ and $\mathcal{S}$ is closed under finite perturbations, we have $\mathbf{f} \in \mathcal{S}$.

Claim 9.2. *We have*

$$\underline{\delta}(\mathbf{f}) \geq \underline{\delta} - O(1/\log\beta), \qquad \overline{\delta}(\mathbf{f}) \geq \overline{\delta} - O(1/\log\beta),$$

where $\underline{\delta}$ and $\overline{\delta}$ denote Alice's scores in the Hausdorff and packing games, respectively.

Proof. It suffices to show that for all $k \in \mathbb{N}$ and $t' \in [k\gamma, (k+1)\gamma]$,

$$\delta(\mathbf{f}, t') \geq \frac{\log \#(A_k) - O(1)}{-\log(\beta)}.$$

Indeed, fix such k, t', and let $t = k\gamma$. By Claim 9.1, we have

$$\delta(k, X_k) \geq \frac{\log \#(A_k) - O(1)}{-\log(\beta)},$$

so to complete the proof it suffices to show that

$$\delta(\mathbf{f}, t') \geq \delta(k, X_k).$$

Indeed, fix $q = 1, \ldots, d-1$ such that $f_q(t') < f_{q+1}(t')$, and we will show that

$$L_+(\mathbf{f}, t', q) \geq L_q^+. \tag{9.3}$$

Indeed, first note that by assumption, and since $C_1 \geq \gamma$, the inequality $f_q(t') < f_{q+1}(t')$ implies that $g_{q+1}(t) - g_q(t) \geq C_1$. Now by the definition of $\mathbf{g}$ and Lemma 6.2, we have

$$\mathbf{g}(t) \asymp_+ \mathbf{h}_A(t) \asymp_+ \mathbf{h}(\Lambda_k)$$

and thus we in fact get $\log\lambda_{q+1}(\Lambda_k) - \log\lambda_q(\Lambda_k) \gtrsim_+ C_1$.

Now let

$$Z_k = \sum_{\ell=k}^{\infty} \beta^{\ell-k} X_\ell \in B_{\mathcal{M}}(X_k, \beta) \subseteq B_{\mathcal{M}}(\mathbf{0}, 1).$$

By (9.1), we have

$$\sup_{\|Y\|\leq\beta} \dim(u_{Z_k+Y}V_q \cap \mathcal{L}) \leq L_q^-.$$

Thus by Lemma 7.7, for all $s \geq 0$, we have

$$\log \|g_s u_{Z_k} V_q\| \gtrsim_{+,\beta} \log \|V_q\| + \left(\frac{L_q^+}{m} - \frac{L_q^-}{n}\right) s. \tag{9.4}$$

On the other hand, since $\log \lambda_{q+1}(\Lambda_k) - \log \lambda_q(\Lambda_k) \gtrsim_+ C_1$, for all $V_q' \in \mathcal{V}_q(\Lambda_k) \setminus \{V_q\}$, by Lemma 7.6 we have

$$\log \|V_q'\| - \log \|V_q\| \gtrsim_+ C_1$$

and thus for all $0 \leq s \leq \frac{mn}{qd} C_1$, since $\log \|g_s^{-1}\| \leq s/n$, we have

$$\begin{aligned}\log \|g_s u_{Z_k} V_q'\| &\gtrsim_+ \log \|V_q'\| - \frac{q}{n}s \gtrsim_+ \log \|V_q\| + C_1 - \frac{q}{n}s \\ &\geq \log \|V_q\| + \frac{q}{m}s \geq \log \|V_q\| + \left(\frac{L_q^+}{m} - \frac{L_q^-}{n}\right) s.\end{aligned}$$

Combining with (9.4) gives

$$\inf_{V_q' \in \mathcal{V}_q(\Lambda_k)} \log \|g_s u_{Z_k} V_q'\| \gtrsim_{+,\beta} \log \|V_q\| + \left(\frac{L_q^+}{m} - \frac{L_q^-}{n}\right) s.$$

On the other hand, since $\mathbf{g} \asymp_+ \mathbf{h}_A$, by Lemmas 7.5 and 6.2, we have

$$\log \|V_q\| \asymp_+ \sum_{i=1}^q \log \lambda_i(\Lambda_k) \asymp_+ G_q(t),$$

$$\inf_{V_q' \in \mathcal{V}_q(\Lambda_k)} \log \|g_s u_{Z_k} V_q'\| \asymp_+ \sum_{i=1}^q \log \lambda_i(g_s u_{Z_k} \Lambda_k) \asymp_+ G_q(t+s),$$

so

$$G_q(t+s) \gtrsim_{+,\beta} G_q(t) + \left(\frac{L_q^+}{m} - \frac{L_q^-}{n}\right) s.$$

Rearranging gives

$$\int_t^{t+s} G_q' \gtrsim_{+,\beta} \left(\frac{L_q^+}{m} - \frac{L_q^-}{n}\right) s.$$

Suppose that $G_q' < \frac{L_q^+}{m} - \frac{L_q^-}{n}$ on $[t, t+s]$. Then since $\mathbf{g}$ is a template,

$$G_q' \leq \frac{L_q^+ - 1}{m} - \frac{L_q^- + 1}{n} \text{ on } [t, t+s]$$

and thus
$$\left(\frac{L_q^+ - 1}{m} - \frac{L_q^- + 1}{n}\right) s \gtrsim_{+,\beta} \left(\frac{L_q^+}{m} - \frac{L_q^-}{n}\right) s$$
which implies $s \asymp_{+,\beta} 0$, i.e. $|s| \leq C_2$ for some constant C_2. Let C_1, s be chosen so that $C_2 < s \leq \min(C_1, \frac{mn}{qd} C_1)$ and $\gamma \leq C_1$. Then the inequality $|s| \leq C_2$ contradicts the definition of s, so the hypothesis that $G_q' < \frac{L_q^+}{m} - \frac{L_q^-}{n}$ on $[t, t+s]$ must be incorrect, i.e. we must have $G_q'(t'') \geq \frac{L_q^+}{m} - \frac{L_q^-}{n}$ for some $t'' \in [t, t+s]$. Now since $t', t'' \in [t, t+C_1]$, we have $|t'' - t'| \leq C_1$, and thus by our assumptions on $\mathbf{f}$ we have
$$\frac{L_+(\mathbf{f}, t', q)}{m} - \frac{L_-(\mathbf{f}, t', q)}{n} = F_q'(t') \geq G_q'(t'') \geq \frac{L_q^+}{m} - \frac{L_q^-}{n}$$
demonstrating (9.3).

To summarize, we have
$$\#\big(S_+(\mathbf{f}, t') \cap (0, q]_{\mathbb{Z}}\big) \geq \#\big(S_+(k, X_k) \cap (0, q]_{\mathbb{Z}}\big)$$
for all q such that $f_q(t') < f_{q+1}(t')$. It follows from (2.8) that the same inequality holds for all $q = 1, \ldots, d-1$. Since
$$\delta(T_+, T_-) = \sum_{q=1}^{d-1} \#\big(T_+ \cap (0, q]\big) - \binom{m}{2},$$
(where δ is as in (2.13)), we have
$$\delta(\mathbf{f}, t') = \delta(S_\pm(\mathbf{f}, t')) \geq \delta(S_\pm(k, X_k)) = \delta(k, X_k). \qquad \square$$

Fix $\delta > \sup_{\mathbf{f} \in \mathcal{S} \cap \mathcal{T}_{m,n}} \underline{\delta}(\mathbf{f})$. Then by the previous Claim 9.2, we have $\delta > \underline{\delta}$ as long as β is sufficiently small. So by Theorem 5.2, we have $\delta \geq \dim_H(S)$. Since δ was arbitrary, we have
$$\dim_H(S) \leq \sup_{\mathbf{f} \in \mathcal{S} \cap \mathcal{T}_{m,n}} \underline{\delta}(\mathbf{f}).$$
A similar argument gives the bound for the packing dimension, thereby completing the proof of the upper bounds in Theorem 2.6.

Part 4. Proof of main theorems using the variational principle

10. Leitfaden to Part 4

In this part we prove Theorem 2.2 from Section 2 and all the theorems of Section 1, with the exception of Theorems 1.2 and 1.3, making full use of the variational principle that has been proven in Part 3.

For reference, the following theorems are proven in the following subsections:

- Theorem 2.2 is proven in §27.
- Theorems 1.1 and 1.4 are proved in §11 and §12.

  To prove Theorems 1.1 and 1.4 it suffices[12] to show that

$$\dim_P(\mathrm{Sing}(m,n)) \leq \delta_{m,n}, \tag{10.1}$$

$$\dim_H(\mathrm{VSing}(m,n)) \geq \delta_{m,n}. \tag{10.2}$$

  We prove these inequalities first (in §11 and §12 respectively), since their proofs provide the best basic illustration of our techniques.
- Theorem 1.8 is proven in §13 and §16.

  The packing dimension upper bound (valid for $n \geq 2$) is proven in §13. The packing dimension lower bound is proven in §16.
- Theorem 1.9 is proven in §17.
- Theorem 1.5 is proven in §11, §12, §14, §15, and §16.

  In §11, after proving (10.1), we obtain the upper bound for packing dimension in Theorem 1.5. The packing dimension lower bound in Theorem 1.8 (proven in §16) implies the packing dimension lower bound in Theorem 1.5. This completes the proof for the packing dimension asymptotic formula. Regarding the Hausdorff dimension, there are two asymptotic formulas that have to be proved. For the first case of Theorem 1.5: the lower bound for Hausdorff dimension is obtained in §12, after proving (10.2); and the upper bound for Hausdorff dimension is proven in §14. For the second case of Theorem 1.5: the lower bound for Hausdorff dimension is proven in §15; and the upper bound for Hausdorff dimension follows from that for packing dimension (proven in §11).
- Theorem 1.6 is proven in §13 §16, §18, and §20.

  Theorem 1.8 (proven in §13 and §16) implies the packing dimension formula in Theorem 1.6. The upper and lower bounds for the Hausdorff dimension formula in Theorem 1.6 are proven in §18 and §20, respectively.
- Theorem 1.7 is proven in §17,§19, §21, and §22.

  The packing dimension upper bound in Theorem 1.7 is proven in §22. The packing dimension lower bound is implied by Theorem 1.9 (proven in §17). The lower and upper bounds for the Hausdorff dimension formula in Theorem 1.7 are proven in §19 and §21, respectively.

[12]This follows from the monotonicity of the Hausdorff and packing dimensions, and the fact that the latter is bounded below by the former (see Section 4).

- Theorem 1.10 is proven in §23.
- Theorem 1.11 is proven in §24.
- Theorem 1.12 is proven in §25.
- Theorem 1.14 is proven in §26.

11. Proof of (10.1) + Theorem 1.5, upper bound for packing dimension

In some sense, the variational principle means that it is harder to prove upper bounds on dimension than lower bounds: for a lower bound one only needs to exhibit a template or sequence of templates with the appropriate dimension properties, while for an upper bound one needs to prove something about all possible templates. This is in contrast to the usual situation in which it is easier to prove upper bounds. Our technique for proving upper bounds is based on continuing the analogy with physics by defining a function that measures the "potential energy" of any configuration of particles: the potential energy is larger the farther apart the particles are. We then prove an inequality relating the change in potential energy and the contraction rate. Integrating this inequality gives a relation between the potential energy at a given point in time, which is always positive, and the average contraction rate up to that time. This then yields a bound on the average contraction rate up to any point in time.

Let $\mathbf{f} : [0, \infty) \to \mathbb{R}^d$ be a balanced template. We define the "potential energy of $\mathbf{f}$ at time t" to be the number

$$\phi(t) = \phi_{\mathbf{f}}(t) = \max\left(\frac{m^2 n}{m+n}|f_1(t)|, \frac{mn^2}{m+n}|f_d(t)|\right). \tag{11.1}$$

Note that $\phi(t) \geq 0$ for all $t \geq 0$. The motivation for the definition of ϕ is the following lemma:

Lemma 11.1. *Let I be an interval of linearity*[13] *for $\mathbf{f}$ such that $\phi'(t)$ is well-defined for all $t \in I$, and such that $\mathbf{f}(t) \neq \mathbf{0}$ for all $t \in I$. Then*

$$\phi'(t) \leq \delta_{m,n} - \delta(\mathbf{f}, I) \tag{11.2}$$

for $t \in I$. Equality holds in precisely the following cases:

1. $S_+(\mathbf{f}, I) = \{1, \ldots, m\}$;
2. $S_+(\mathbf{f}, I) = \{1, \ldots, m-1, m+1\}$, *and* $f_1 = \ldots = f_m$ *and* $f_{m+1} = \ldots = f_{m+n}$ *on* I *(and in particular* $m|f_1| = n|f_d|$ *on* I*)*;

3a. $S_+(\mathbf{f}, I) = \{2, \ldots, m+1\}$, *and* $m|f_1| \geq n|f_d|$ *on* I;

[13] I.e. an interval on which $\mathbf{f}$ is linear. If I is an interval of linearity for $\mathbf{f}$, we will denote the constant value of $\mathbf{f}'$ on I by $\mathbf{f}'(I)$.

3b. *$S_+(\mathbf{f}, I) = \{1, \ldots, m-1, m+n\}$, and $n|f_d| \geq m|f_1|$ on I.*

If equality does not hold, then the difference between the two sides of (11.2) *is at least* $1/\max(m,n)$.

Note that when $m = 1$, cases 2 and 3a are equivalent, and when $n = 1$, cases 2 and 3b are equivalent.

Proof. Note that the cases 3a and 3b are symmetric with respect to the operation of replacing the $m \times n$ template $\mathbf{f}$ by the $n \times m$ template $-\mathbf{f}$, while the other two cases are individually symmetric with respect to this operation. Thus, we may without loss of generality suppose that

$$\phi = \frac{m^2 n}{m+n}|f_1| \quad \text{i.e.} \quad m|f_1| \geq n|f_d| \tag{11.3}$$

on I. Let $j \geq 1$ be the largest number such that

$$f_j = f_1 \text{ on } I.$$

Note that since $\mathbf{f}$ is balanced, (11.3) implies that $j \leq m$. Since I is an interval of linearity for $\mathbf{f}$, it follows that $f_j < f_{j+1}$ on I. Accordingly, let $L_\pm = L_\pm(\mathbf{f}, I, j)$ and $S_\pm = S_\pm(\mathbf{f}, I)$. Then by (11.3) and (2.5) we have

$$\phi'(t) = \frac{m^2 n}{m+n}\frac{-F_j'(t)}{j} = \frac{m^2 n}{(m+n)j}\left[\frac{L_-}{n} - \frac{L_+}{m}\right]$$

and on the other hand, by (2.11) we have

$$mn - \delta(\mathbf{f}, I) \geq \#\big(S_- \cap (0,j]\big) \cdot \#\big(S_+ \cap (j,d]\big) = L_-(m - L_+) \tag{11.4}$$

and thus

$$\delta_{m,n} - \delta(\mathbf{f}, I) \geq L_-(m - L_+) - \frac{mn}{m+n}.$$

So to demonstrate (11.2) it suffices to show that

$$\frac{m^2 n}{(m+n)j}\left[\frac{L_-}{n} - \frac{L_+}{m}\right] \leq L_-(m - L_+) - \frac{mn}{m+n}.$$

Indeed, since $L_+ + L_- = j$, we have

$$\frac{m^2 n}{(m+n)j}\left[\frac{L_-}{n} - \frac{L_+}{m}\right] = \frac{m^2 n}{(m+n)j}\left[L_-\left(\frac{1}{m} + \frac{1}{n}\right) - \frac{j}{m}\right] = \frac{L_- m}{j} - \frac{mn}{m+n}$$

so we need to show that

$$\frac{L_- m}{j} \leq L_-(m - L_+). \tag{11.5}$$

If $L_- = 0$, then this inequality is trivial (and equality holds). So suppose that $L_- > 0$. Since $j = L_- + L_+ \leq m$, we have $L_+ < j \leq m$ and thus

$$m \leq j + (m - L_+) - 1 \leq j(m - L_+), \tag{11.6}$$

and rearranging yields (11.5). This completes the proof of (11.2).

Now suppose that equality holds in (11.2). The equality in (11.4) implies that

$$S_+ = \{1, \ldots, L_+\} \cup \{j+1, \ldots, j+m-L_+\}.$$

On the other hand, the equality in (11.5) implies that either $L_- = 0$, or equality holds in (11.6). In the latter case we have $L_- = j - L_+ = 1$, and either $j = 1$ or $m - L_+ = 1$, from the left and right hand sides of (11.6), respectively. So there are three cases:

1. If $L_- = 0$, then $S_+ = \{1, \ldots, m\}$.
2. If $L_- = 1$ and $m - L_+ = 1$, then $S_+ = \{1, \ldots, m-1, m+1\}$. In this case $j = m$, i.e. $f_1 = \ldots = f_m$ on I. Combining with (11.3) and using the fact that $\mathbf{f}$ is balanced shows that $f_{m+1} = \ldots = f_{m+n}$ on I.

3a. If $L_- = 1$ and $j = 1$, then $S_+ = \{2, \ldots, m+1\}$.

Note that the case 3b does not appear in this list due to the fact that we made the assumption (11.3) without loss of generality, using the fact that 3a and 3b are symmetric. The converse direction can be proved similarly.

Finally, suppose that equality does not hold in (11.2). Note that the difference between the two sides of (11.2) is the sum of the difference between the two sides of (11.4) and those of (11.5), i.e. $mn - \delta(\mathbf{f}, I) - L_- m/j$. Since this is a positive rational number with denominator j, it must be $\geq 1/j \geq 1/\max(m,n)$. □

Now suppose that the template $\mathbf{f}$ is singular, i.e. satisfies $f_1(t) \to -\infty$ as $t \to \infty$. Then $\mathbf{f}(t) \neq \mathbf{0}$ for all sufficiently large t. So by Lemma 11.1, (11.2) holds for almost all sufficiently large t, and thus

$$0 \lesssim_+ \phi(T) - \phi(0) \lesssim_+ \int_0^T [\delta_{m,n} - \delta(\mathbf{f}, t)]\, \mathrm{d}t = T[\delta_{m,n} - \Delta(\mathbf{f}, T)].$$

It follows that

$$\overline{\delta}(\mathbf{f}) = \limsup_{T\to\infty} \Delta(\mathbf{f}, T) \leq \limsup_{T\to\infty} \left[\delta_{m,n} + O(1/T) - \frac{\phi(T)}{T}\right] = \delta_{m,n} - \liminf_{T\to\infty} \frac{\phi(T)}{T} \leq \delta_{m,n},$$

and applying Theorem 2.7 to the set

$$\mathcal{S} = \{\mathbf{f} : f_1(t) \to -\infty \text{ as } t \to \infty\}$$

yields (10.1). Note that if $\mathbf{f}$ is τ-singular, then $\phi(T) \geq \frac{m^2 n}{m+n}\tau T$ for all T, and thus

$$\overline{\delta}(\mathbf{f}) \leq \delta_{m,n} - \frac{m^2 n}{m+n}\tau.$$

Applying Theorem 2.8 yields the upper bound of the packing dimension assertion of Theorem 1.5.

12. Proof of (10.2) + Theorem 1.5, first formula, lower bound for Hausdorff dimension

Lemma 11.1 provides motivation for how to construct a template yielding the lower bound (10.2). Namely, the template $\mathbf{f}$ should be constructed in a way such that most of the time, one of the four cases for the possible value of $S_+(\mathbf{f}, I)$ listed in Lemma 11.1 holds. For example, there may be two consecutive intervals of linearity I_1 and I_2 such that $S_+(\mathbf{f}, I_1) = \{2, \ldots, m+1\}$ and $S_+(\mathbf{f}, I_2) = \{1, \ldots, m\}$; cf. Figure 7.

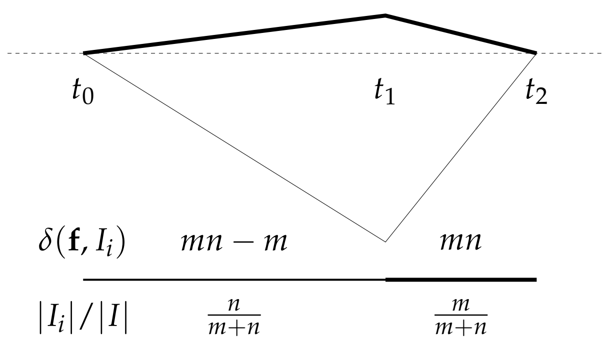

Figure 7. The joint graph of a partial template $\mathbf{f}$ such that $S_+(\mathbf{f}, I_1) = \{2, \ldots, m+1\}$ and $S_+(\mathbf{f}, I_2) = \{1, \ldots, m\}$, where $I_1 = (t_0, t_1)$ and $I_2 = (t_1, t_2)$. In this picture we have $\mathbf{f}(t_0) = \mathbf{f}(t_2) = \mathbf{0}$, and thus $|I_1| = \frac{n}{m+n}|I|$ and $|I_2| = \frac{m}{m+n}|I|$, where $I = (t_0, t_2)$. Consequently,

$$\frac{1}{|I|}\int_I \delta(\mathbf{f}, t)\, \mathrm{d}t = \frac{n}{m+n}(mn - m) + \frac{m}{m+n}(mn) = \delta_{m,n}$$

i.e. the average contraction rate of $\mathbf{f}$ over I is $\delta_{m,n}$. Note that this partial template is exactly the standard template defined by the points $(t_0, 0)$ and $(t_2, 0)$ (cf. Definition 12.1).

In contrast to the picture in Figure 7, if we want the template $\mathbf{f}$ to be singular then we need $\mathbf{f}(t) \neq \mathbf{0}$ for all t, so we will need to "cut off" a small part of the picture. By "gluing" infinitely many of these pictures together we will get a singular template of large Hausdorff dimension; cf. Figure 8.

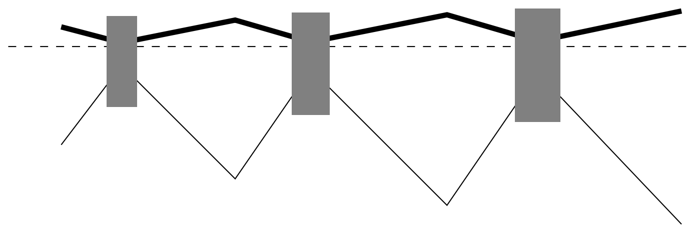

FIGURE 8. The joint graph of a template $\mathbf{f}$ designed to be a singular template of large Hausdorff dimension. The gray regions represent intervals where the precise value of the template is irrelevant; what matters is that the template stays away from $\mathbf{0}$ on these regions.

To make the idea conveyed in Figure 8 rigorous, we introduce the notion of the *standard template* defined by two points $(t_k, -\varepsilon_k)$ and $(t_{k+1}, -\varepsilon_{k+1})$. The idea is that $\mathbf{f} : [t_k, t_{k+1}] \to \mathbb{R}^d$ should satisfy $f_k(t_i) = f_{k+1}(t_i) = -\varepsilon_i$ for $i = k, k+1$, and f_k should be as small as possible given this restriction. Finally, the template should be chosen so that f_d is as small as possible, given the previous restrictions. Formally we make the following definition:

Definition 12.1. Fix $0 \leq t_k < t_{k+1}$ and $\varepsilon_k, \varepsilon_{k+1} \geq 0$ and let $\Delta t = \Delta t_k = t_{k+1} - t_k$ and $\Delta\varepsilon = \Delta\varepsilon_k = \varepsilon_{k+1} - \varepsilon_k$. Assume that the following formulas hold:

$$-\tfrac{1}{m}\Delta t \leq \Delta\varepsilon \leq \tfrac{1}{n}\Delta t, \tag{12.1}$$

$$\Delta\varepsilon \geq -\tfrac{n-1}{2n}\Delta t \text{ if } m = 1 \text{ and } \Delta\varepsilon \leq \tfrac{m-1}{2m}\Delta t \text{ if } n = 1, \tag{12.2}$$

$$\text{either } (n-1)\left(\tfrac{1}{n}\Delta t - \Delta\varepsilon\right) \geq d\varepsilon_k \text{ or } (m-1)\left(\tfrac{1}{m}\Delta t + \Delta\varepsilon\right) \geq d\varepsilon_{k+1}. \tag{12.3}$$

Then the *standard template* defined by the two points $(t_k, -\varepsilon_k)$ and $(t_{k+1}, -\varepsilon_{k+1})$ is the partial template $\mathbf{f} : [t_k, t_{k+1}] \to \mathbb{R}^d$ defined as follows:

- Let $g_1, g_2 : [t_k, t_{k+1}] \to \mathbb{R}$ be piecewise linear functions such that $g_i(t_j) = -\varepsilon_j$, and g_i has two intervals of linearity: one on which $g_i' = \frac{1}{m}$ and another on which $g_i' = -\frac{1}{n}$. For $i = 1$ the latter interval comes first while for $i = 2$ the former interval comes first; cf. Figure 9. The existence of such functions g_1 and g_2 is guaranteed by (12.1). Finally, let $g_3 = \ldots = g_d$ be chosen so that $g_1 + \ldots + g_d = 0$.
- For each $t \in [t_k, t_{k+1}]$ let $\mathbf{f}(t) = \mathbf{g}(t)$ if $g_2(t) \leq g_3(t)$; otherwise let $f_1(t) = g_1(t)$ and let $f_2(t) = \ldots = f_d(t)$ be chosen so that $f_1 + \ldots + f_d = 0$.

We will sometimes denote the standard template defined by $(t_k, -\varepsilon_k)$ and $(t_{k+1}, -\varepsilon_{k+1})$ by $\mathbf{s}[(t_k, -\varepsilon_k), (t_{k+1}, -\varepsilon_{k+1})]$.

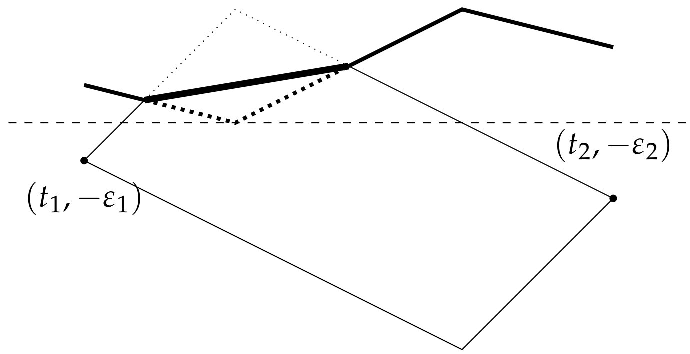

FIGURE 9. The joint graphs of $\mathbf{f}$ and $\mathbf{g}$ on the interval $[t_1, t_2]$. $\mathbf{g}$ is shown dotted while $\mathbf{f}$ is shown solid.

Lemma 12.2. *A standard template is indeed a balanced partial template.*

Proof. We show where the formulas (12.2) and (12.3) are needed, leaving the rest of the proof as an exercise to the reader. The condition (12.3) is equivalent to the assertion that $g_2(t) \geq g_3(t)$ where t is the location of the maximum of g_2. This implies that $f_2(t) = f_3(t)$, guaranteeing that the convexity condition is satisfied at t. The condition (12.2) is equivalent to the assertion that there is no interval on which $f_1' = f_2' = \pm 1$. If such an interval exists, then $\mathbf{f}$ cannot be a template because if it were, we would have $\{1, 2\} \subseteq S_\pm$ but $\#(S_\pm) = d_\pm = 1$, a contradiction. Conversely, if there is no such interval then the sets $S_\pm$ can be computed in a consistent way on any interval of linearity for $\mathbf{f}$. $\square$

Example 12.3. The inequalities (12.1)-(12.3) always hold when $\varepsilon_k = \varepsilon_{k+1} = 0$. In this case, the standard template $\mathbf{f}$ defined by the points $(t_k, 0)$ and $(t_{k+1}, 0)$ has only two intervals of linearity, and the average value of $\delta(\mathbf{f}, \cdot)$ on $[t_k, t_{k+1}]$ is equal to $\delta_{m,n}$; see Figure 7.

Definition 12.4. Let $(t_k)_0^\infty$ be an increasing sequence of nonnegative real numbers, and let $\varepsilon_k \geq 0$ for each k. The *standard template* defined by the sequence of points $(t_k, -\varepsilon_k)$ is the partial template produced by gluing together the standard templates defined by the pairs of points $(t_k, -\varepsilon_k)$ and $(t_{k+1}, -\varepsilon_{k+1})$ for each k. The *standard template* defined by two parameters $\tau \geq 0$ and $\lambda > 1$, denoted $\mathbf{f}[\tau, \lambda]$, is the one defined by the sequence of points $(t_k, \varepsilon_k)_{k \in \mathbb{Z}}$, where $t_k = \lambda^k$ and $\varepsilon_k = \tau t_k$ for all k. Note that in this case, (12.1)-(12.3) become

$$\tau \leq \tfrac{1}{n}, \tag{12.4}$$

$$\tau \leq \tfrac{m-1}{2m} \text{ if } n = 1, \tag{12.5}$$

$$\text{either } (n-1)\left(\tfrac{1}{n} + \tau\right) \geq \tfrac{1}{\lambda-1} d\tau \text{ or } (m-1)\left(\tfrac{1}{m} + \tau\right) \geq \tfrac{\lambda}{\lambda-1} d\tau. \tag{12.6}$$

Now fix $\tau > 0$ small and let $\lambda = 1 + \sqrt{\tau}$ (or more generally $\lambda = 1 + \Theta(\sqrt{\tau})$), and note that (12.4)-(12.6) hold. Let $\mathbf{f} = \mathbf{f}[\tau, \lambda]$ and t_k, ε_k be as above. Now since the map $(\varepsilon_1, \varepsilon_2) \mapsto \Delta(\mathbf{s}[(0, -\varepsilon_1), (1, -\varepsilon_2)], 1)$ is Lipschitz continuous, it follows that

$$\begin{aligned}
\Delta(\mathbf{f}, [t_k, t_{k+1}]) &= \Delta(\mathbf{s}[(t_k, -\varepsilon_k), (t_{k+1}, -\varepsilon_{k+1})], [t_k, t_{k+1}]) \\
&= \Delta\left(\mathbf{s}\left[\left(0, -\tfrac{\varepsilon_k}{\Delta t_k}\right), \left(1, -\tfrac{\varepsilon_{k+1}}{\Delta t_k}\right)\right], 1\right) \\
&= \Delta(\mathbf{s}[(0,0), (1,0)], 1) - O\left(\tfrac{\max(\varepsilon_k, \varepsilon_{k+1})}{\Delta t_k}\right) \\
&= \delta_{m,n} - O\left(\tfrac{\tau}{\lambda-1}\right) = \delta_{m,n} - O(\sqrt{\tau})
\end{aligned}$$

and thus for sufficiently large k

$$\Delta(\mathbf{f}, t_k) = \delta_{m,n} - O(\sqrt{\tau}).$$

Given T large, let k be chosen so that $t_k \leq T < t_{k+1}$. Then

$$\Delta(\mathbf{f}, T) - \Delta(\mathbf{f}, t_k) = O\left(\tfrac{T - t_k}{t_k}\right) = O(\lambda - 1) = O(\sqrt{\tau})$$

and thus

$$\Delta(\mathbf{f}, T) = \delta_{m,n} - O(\sqrt{\tau}).$$

Taking the limit as $T \to \infty$ shows that

$$\dim_H(\mathrm{Sing}_{m,n}(\tau)) \geq \underline{\delta}(\mathbf{f}) = \delta_{m,n} - O(\sqrt{\tau}),$$

and taking the limit as $\tau \to 0$ completes the proof of (10.2), as well as of the lower bound for Hausdorff dimension in the first case of Theorem 1.5.

Remark 12.5. The $O(\sqrt{\tau})$ term in the above proof comes from combining two sources of error: one of size $O(\lambda - 1)$ and another of size $O(\frac{\tau}{\lambda-1})$. We chose $\lambda = 1 + \Theta(\sqrt{\tau})$ so as to minimize the sum of these two error terms.

Remark 12.6. Via a more careful argument one could exactly compute $\underline{\delta}(\mathbf{f}[\tau, \lambda])$ in terms of τ and λ for the template $\mathbf{f}$ described above. Using calculus one could then optimize over the variable λ to get a lower bound which is the best possible using this technique.

13. Proof of Theorem 1.8, upper bound

To prove the upper bound in Theorem 1.8, we need a different definition of "potential energy". Let $\mathbf{f} : [0, \infty) \to \mathbb{R}^d$ be a balanced $m \times n$ template. For each $t \geq 0$ let

$$\psi(t) = \psi_{\mathbf{f}}(t) = \max\left(\frac{mn}{m+n}\big|(m+1)f_1(t) + (d-1)f_2(t)\big|, \frac{mn^2}{m+n}|f_d(t)|\right).$$

Note that since $\mathbf{f}$ is balanced,

$$(m+1)f_1(t) + (d-1)f_2(t) \leq (m+1)f_1(t) + f_2(t) + \ldots + f_d(t) = mf_1(t) \leq 0$$

and thus $\psi(t) \geq \phi(t) \geq 0$ for all $t \geq 0$. The analogous result to Lemma 11.1 is stated as follows:

Lemma 13.1. *Suppose that $n \geq 2$. Let I be an interval of linearity for $\mathbf{f}$ such that $\psi'(t)$ is well-defined for all $t \in I$, and such that $\mathbf{f}(t) \neq \mathbf{0}$ for all $t \in I$. Then*

$$\psi'(t) \leq \delta_{m,n} - \delta(\mathbf{f}, t) \tag{13.1}$$

for $t \in I$. Equality holds in the following (non-exhaustive) cases:

1. *when $f_1 < f_2 = f_d$ on I,*
2. *when $f_1 < f_2 < f_3 = f_d$, and $f_2' = -1/n$ on I.*

Note that there is no symmetry here, unlike in the proof of Lemma 11.1, since ψ is not symmetric with respect to $\mathbf{f} \mapsto -\mathbf{f}$.

Proof. The proof is similar to that of Lemma 11.1. We can suppose that

$$\big|(m+1)f_1(t) + (d-1)f_2(t)\big| \geq n|f_d(t)| \tag{13.2}$$

for $t \in I$, since otherwise $\psi = \phi$ on I and Lemma 11.1 implies the conclusion. Let $j \geq 2$ be the largest number such that

$$f_2 = f_j \text{ on } I.$$

Since I is an interval of linearity for $\mathbf{f}$, we have $f_j < f_{j+1}$ on I. Let $L_\pm = L_\pm(\mathbf{f}, I, j)$ and $S_\pm = S_\pm(\mathbf{f}, I)$.

Suppose first that $f_1 < f_2$ on I. Let $A_\pm = L_\pm(\mathbf{f}, I, 1)$ and $B_\pm = L_\pm - A_\pm$. By (2.5), on I we have

$$\begin{aligned}
F_j' &= \frac{j}{m} - \frac{m+n}{mn} L_- \\
f_1' &= \frac{1}{m} - \frac{m+n}{mn} A_- \\
\psi' &= -\frac{mn}{m+n} \left((m+1) f_1' + \frac{d-1}{j-1} (F_j' - f_1') \right) \\
&= -\frac{(m+d)n}{d} + (m+1) A_- + \frac{d-1}{j-1} B_-
\end{aligned}$$

and on the other hand, by (2.11) we have

$$\begin{aligned}
mn - \delta(\mathbf{f}, t) &\geq \#\big(S_- \cap \{1\}\big) \cdot \#\big(S_+ \cap (1, d]\big) + \#\big(S_- \cap (1, j]\big) \cdot \#\big(S_+ \cap (j, d]\big) \\
&= m A_- + B_- (m - L_+)
\end{aligned} \tag{13.3}$$

and thus

$$\delta_{m,n} - \delta(\mathbf{f}, t) \geq m A_- + B_-(m - L_+) - \frac{mn}{d}.$$

So to demonstrate (13.1), it suffices to show that

$$-n + (m+1) A_- + \frac{d-1}{j-1} B_- \leq m A_- + B_-(m - L_+).$$

Rearranging gives the equivalent formulation

$$\frac{d-1}{j-1} B_- \leq (n - A_-) + B_-(m - L_+).$$

If $B_- = 0$ this is obvious (and since $n \geq 2$ by assumption, the inequality is strict in this case), so assume that $B_- > 0$. Then we can rearrange again to get

$$\frac{d-1}{B_+ + B_-} \leq \frac{n - A_-}{B_-} + m - L_+,$$

and subtracting 1 from both sides gives

$$\frac{(n - L_-) + (m - L_+)}{B_+ + B_-} \leq \frac{n - L_-}{B_-} + m - L_+. \tag{13.4}$$

This formula is obviously true, so backtracking shows that (13.1) is true as well. If $f_2 = f_d$ on I, then $j = d$ and thus $L_+ = m$, $L_- = n$ and so equality holds. Similarly, if $f_1 < f_2 < f_3 = f_d$ and $f_2' = -1/n$ on I, then $j = 2$ and $B_+ = 0$, so $B_- = B_+ + B_- = j - 1 = 1$ and thus equality holds.

Next suppose that $f_1 = f_2$ on I. Then on I we have

$$\psi' = -\frac{mn}{m+n}\frac{m+d}{j}F_j' = -\frac{(m+d)n}{d} + \frac{m+d}{j}L_-$$

and on the other hand, as in (11.4) we have

$$\delta_{m,n} - \delta(\mathbf{f}, I) \geq L_-(m - L_+) - \frac{mn}{d} \tag{13.5}$$

so to demonstrate (13.1), it suffices to show that

$$-n + \frac{m+d}{j}L_- \leq L_-(m - L_+).$$

If $L_- = 0$ this is obvious (and the inequality is strict), so assume that $L_- > 0$. Then rearranging gives the equivalent formulation

$$\frac{2m+n}{L_+ + L_-} \leq \frac{n}{L_-} + m - L_+.$$

Write $M_+ = m - L_+$ and $M_- = n - L_-$. Then subtracting 1 from both sides gives

$$\frac{L_+ + 2M_+ + M_-}{L_+ + L_-} \leq \frac{M_-}{L_-} + M_+$$

and multiplying by $L_+ + L_-$ and rearranging gives

$$L_+ \leq \frac{M_- L_+}{L_-} + M_+(L_+ + L_- - 2). \tag{13.6}$$

We now demonstrate (13.6). First, note that since $L_+ + L_- = j \geq 2$, both terms on the right-hand side are nonnegative. So if either term is individually at least L_+, then (13.6) holds. In particular, if $L_- \leq M_-$, then the first term is $\geq L_+$, and if $L_- \geq 2$ and $M_+ \geq 1$, then the second term is $\geq L_+$. Also, if $L_+ = 0$ then (13.6) obviously holds. So assume that $L_+ > 0$, that $L_- > M_-$, and that either $L_- \leq 1$ or $M_+ = 0$.

If $L_- \leq 1$, then since $L_- > M_-$, we have $M_- = 0$. But since $n = L_- + M_-$, this contradicts our assumption that $n \geq 2$.

If $M_+ = 0$, then

$$j = L_+ + L_- > L_+ + \frac{L_- + M_-}{2} = \frac{2L_+ + 2M_+ + L_- + M_-}{2} = \frac{2m+n}{2}$$

and thus $\frac{n}{d-j} > \frac{m+d}{j}$. Since $\mathbf{f}$ is balanced, this implies

$$\begin{aligned} nf_d + (m+1)f_1 + (d-1)f_2 &= nf_d + (m+d)f_j \\ &\geq \frac{n}{d-j}(f_{j+1} + \ldots + f_d) + \frac{m+d}{j}(f_1 + \ldots + f_j) > 0, \end{aligned}$$

contradicting (13.2). Thus neither $L_- \leq 1$ nor $M_+ = 0$ can hold, and so (13.6) holds, and backtracking yields (13.1). $\square$

We are now ready to prove the upper bound in Theorem 1.8. Let $\mathbf{f} \in \mathrm{Sing}_{m,n}(\omega)$ be a balanced template, and let T be a time such that $\delta(T) > mn - m$. Note that this implies that $1 \in S_+(\mathbf{f}, T)$. Let T' be the largest time such that $f_1' = 1/m$ on (T, T'). If $T' > T$, then the convexity condition implies that $f_1(T') = f_2(T')$. On the other hand, if $T = T'$, then $f_1'(T) < 1/m$, and since $1 \in S_+(\mathbf{f}, T)$, this implies that $f_1(T) = f_2(T)$. So either way $f_1(T') = f_2(T')$.

Let $\mathbf{g} : [0, T'] \to \mathbb{R}^d$ be the standard template defined by the points $(0,0)$ and $(T', f_1(T'))$. Then $f_1(T) = g_1(T)$ while $f_2(T) \leq g_2(T)$. Since $\mathbf{f}$ is balanced, using the definition of $\mathbf{g}$ this implies that $f_d(T) \geq g_d(T)$. Consequently $\psi_{\mathbf{f}}(T) \geq \psi_{\mathbf{g}}(T)$ and hence

$$\Delta(\mathbf{f}, T) \leq \delta_{m,n} - \psi_{\mathbf{f}}(T) \leq \delta_{m,n} - \psi_{\mathbf{g}}(T) = \Delta(\mathbf{g}, T) = \delta\left(\frac{-f_1(T')}{T'}\right).$$

The first equality holds because for $\mathbf{g}$ defined as above, on each interval of linearity one of the conditions 1,2 is satisfied (cf. Table 1), and the second equality is a restatement of (16.1).

Thus for all T such that $\delta(T) > mn - m$, we have

$$\Delta(\mathbf{f}, T) \leq \max\left(mn - m,\ \max_{T' \geq T} \delta\left(\frac{-f_1(T')}{T'}\right)\right),$$

and it follows that the same is true for all T. Taking the limsup gives

$$\overline{\delta}(\mathbf{f}) \leq \max\big(mn - m, \delta(\tau)\big),$$

where τ is as in (1.1). Taking the supremum over all $\mathbf{f}$ and applying Theorem 2.8 completes the proof.

14. Proof of Theorem 1.5, upper bound for Hausdorff dimension

Let $\mathbf{f}$ be a τ-singular template such that $\underline{\delta}(\mathbf{f}) > \delta_{m,n} - z$, where $z > 0$ is small. We aim to show that $\tau = O(z^2)$ if $(m,n) \neq (2,2)$. Indeed, let ϕ be as in (11.1), and let

$$E = \{t \geq 0 : \phi'(t) < \delta_{m,n} - \delta(\mathbf{f}, t)\}.$$

By Lemma 11.1, we have

$$\phi'(t) \leq \delta_{m,n} - \delta(\mathbf{f}, t) - \max(m,n)^{-1}[t \in E]$$

for all t sufficiently large. Integrating over $[0, T]$ gives

$$\phi(T) - \phi(0) \leq T\big(\delta_{m,n} - \Delta(\mathbf{f}, T)\big) - \max(m,n)^{-1}\lambda\big(E \cap [0,T]\big),$$

where λ denotes Lebesgue measure. On the other hand, since $\underline{\delta}(\mathbf{f}) > \delta_{m,n} - z$, we have $\Delta(\mathbf{f}, T) \geq \delta_{m,n} - z$ for all sufficiently large T, and thus rearranging the previous equation and using the fact that $\phi(T) \geq 0$ gives

$$\lambda\big(E \cap [0,T]\big) = O(zT) \tag{14.1}$$

and

$$\phi(T) = O(zT) \tag{14.2}$$

assuming T is sufficiently large. The trick now is that we also know $\phi(T) = \Omega(\tau T)$ since $\mathbf{f}$ is τ-singular. So the question is what kind of templates satisfy both an upper bound and a lower bound for ϕ, but do not have many exceptional points. The answer is given by the following lemma, in which the problem has been rescaled so that the upper bound for ϕ is just 1:

Lemma 14.1. *Suppose that $(m,n) \neq (2,2)$, and fix $x > 0$. Let $\mathbf{f} : I = [t_-, t_+] \to \mathbb{R}^d$ be a partial template such that $x \leq \phi(t) \leq 1$ for all $t \in I$. Then if $|I|$ is sufficiently large depending on m, n, then*

$$\lambda(E) \gtrsim x.$$

Proof. Let $y = \lambda(E)$; we need to show that either $\phi(t) \lesssim y$ for some $t \in I$, or else $|I| = O(1)$.

Throughout this proof, we will call an interval J a Type 1 interval if case 1 of Lemma 11.1 holds along it; we define Type 2/3a/3b intervals similarly. The basic idea is to reduce to the case of a Type 2 interval to the left of a Type 3 interval to the left of a Type 1 interval, modulo a small perturbation. Since $\mathbf{f}$ cannot be static on any interval of fixed Type, the bound on ϕ implies a bound on the length of each interval and thus on the length of the whole interval I.

Suppose first that there is some Type 1 interval which is to the left of a Type 2/3a/3b interval. Without loss of generality, we may assume that there are no Type (1/2/3a/3b) intervals between them. It follows that if the two intervals are $I_1 = (t_1, t_2)$ and $I_2 = (t_3, t_4)$, respectively, then we have $0 \leq t_3 - t_2 \leq y$.

If I_2 is Type 2, then $f_1(t_3) = \ldots = f_m(t_3)$ and $f_{m+1}(t_3) = \ldots = f_{m+n}(t_3)$. On the other hand, by the convexity condition we have $f_m(s) = f_{m+1}(s)$ for some $s \in [t_2, t_3]$. It follows that $|f_{m+1}(t_3) - f_m(t_3)| \lesssim y$ and thus $\phi(t_3) \asymp |\mathbf{f}(t_3)| \lesssim y$.

If I_2 is Type 3a, then $m|f_1(t_3)| \geq n|f_d(t_3)|$. On the other hand, by the convexity condition, for each $j = 1, \ldots, m$ there exists $s_j \in [t_2, t_3]$ such that $f_j(s_j) = f_{j+1}(s_j)$. It follows that $|f_{j+1}(t_3) - f_j(t_3)| \lesssim y$, so

$$(m+1)|f_1(t_3)| = -\sum_{i=1}^{m+1} f_i(t_3) + O(y) = \sum_{i=m+2}^{m+n} f_i(t_3) + O(y)$$
$$\leq n|f_d(t_3)| + O(y) \leq m|f_1(t_3)| + O(y)$$

and thus $\phi(t_3) \asymp |\mathbf{f}(t_3)| \lesssim y$. A similar argument applies if I_2 is Type 3b.

Thus, we may assume that no Type 1 interval is to the left of any Type 2/3a/3b interval. Now let

$$\psi(t) = \big| m|f_1(t)| - n|f_d(t)| \big|$$

and let J be a Type 2/3a/3b interval. If J is Type 2, then $\psi = 0$ on J and thus $\psi' = 0$. Suppose that J is Type 3a. Then on J we have $m|f_1| \geq n|f_d|$, $f_1' = -\frac{1}{n}$, and $f_d' \leq \frac{1}{n(m+n-1)}$, from which it follows that

$$\psi' \geq c_{m,n} \stackrel{\text{def}}{=} \frac{m}{n} - \frac{1}{m+n-1}.$$

Note that $c_{m,n} > 0$ unless $m = 1$, in which case $c_{m,n} = 0$. Similar logic shows that if J is Type 3b, then $\psi' \geq c_{n,m}$ on J.

Now let A_i denote the union of the Type i intervals. Note that $A_1 \cup A_2 \cup A_3 \cup E = I$ except for finitely many points. We can assume that $t_0 \stackrel{\text{def}}{=} \sup(A_2) \leq t_1 \stackrel{\text{def}}{=} \inf(A_1)$ and $\sup(A_3) \leq t_1$, as otherwise we are done by the preceding argument. Since t_0 is the endpoint of a Type 2 interval, we have $\psi(t_0) = 0$. On the other hand, we have

$$\psi(t_0) \geq \int_{t_-}^{t_0} \psi'(t)\,\mathrm{d}t \geq c_{m,n}\lambda\big([t_-, t_0] \cap A_{3\mathrm{a}}\big) + c_{n,m}\lambda\big([t_-, t_0] \cap A_{3\mathrm{b}}\big) - O(y)$$

so we have $\lambda\big([t_-, t_0] \cap A_{3\mathrm{a}}\big) = O(y)$ if $m \geq 2$ and $\lambda\big([t_-, t_0] \cap A_{3\mathrm{b}}\big) = O(y)$ if $n \geq 2$, respectively. On the other hand, if $m = 1$ then $A_{3\mathrm{a}} = A_2$ and if $n = 1$ then $A_{3\mathrm{b}} = A_2$. Consequently

$$\lambda\big([t_-, t_0] \cap A_3 \setminus A_2\big) = O(y)$$

and thus since $\phi' = \delta_{m,n} - (mn-1)$ on A_2, we have

$$\begin{aligned}
0 &\asymp_+ \phi(t_0) - \phi(t_-) = \int_{t_-}^{t_0} \phi'(t)\,\mathrm{d}t \\
&\geq [\delta_{m,n} - (mn-1)]\lambda\big([t_-, t_0] \cap A_2\big) - O\left(\lambda\big([t_-, t_0] \cap E \cup A_3 \setminus A_2\big)\right) \\
&\geq \left(1 - \frac{mn}{m+n}\right)(t_0 - t_-) - O(y)
\end{aligned}$$

Since $(m,n) \neq (2,2)$ by assumption, we have $(m-1)(n-1) \neq 1$ and thus

$$1 - \frac{mn}{m+n} \neq 0$$

and thus $t_0 - t_- = O(1)$. Similarly, since $\phi' = \delta_{m,n} - (mn - m) = \frac{m^2}{m+n}$ on $A_{3\mathrm{a}}$ and $\phi' = \delta_{m,n} - (mn - n) = \frac{n^2}{m+n}$ on $A_{3\mathrm{b}}$, and since $(t_0, t_1) \subseteq A_3 \cup E$, we have

$$0 \gtrsim_+ \frac{\min(m,n)^2}{m+n}(t_1 - t_0) - O(y).$$

Since $\phi' = \delta_{m,n} - mn = -\frac{mn}{m+n}$ on A_1, and since $(t_1, t_+) \subseteq A_1 \cup E$, we have

$$0 \asymp_+ -\frac{mn}{m+n}(t_+ - t_1) + O(y).$$

Thus $t_1 - t_0 = O(1)$ and $t_+ - t_1 = O(1)$, so combining gives $|I| = t_+ - t_- = O(1)$. $\square$

Let $C > 0$ be the constant such that Lemma 14.1 is true whenever $|I| \geq C$. Notice that any partial template whose domain has length $\geq C$ can be split up into partial templates whose domains have length $= C$ which cover the majority of the original domain. It follows that in the context of Lemma 14.1, in general we have

$$\lambda(E) \gtrsim x|I| \text{ as long as } |I| \geq C,$$

where I is the domain of a partial template $\mathbf{f}$ satisfying $x \leq \phi \leq 1$. Applying a scaling argument yields:

Lemma 14.2. *Suppose that $(m,n) \neq (2,2)$, and fix $0 < x_0 \leq x_1$ and $I \subseteq \mathbb{R}$ such that $|I| \geq Cx_1$. Let $\mathbf{f} : I \to \mathbb{R}^d$ be a partial template such that $x_0 \leq \phi(t) \leq x_1$ for all $t \in I$. Then*

$$\lambda(E) \gtrsim \frac{x_0|I|}{x_1}.$$

Now fix T large, let $I = [T/2, T]$, and let $x_0 = \inf_I \phi$, $x_1 = \sup_I \phi$. Since $\mathbf{f}$ is τ-singular we have $x_0 \gtrsim \tau T$, while by (14.2) we have $x_1 = O(zT)$. In particular, if z is sufficiently small then $T \geq Cx_1$. Consequently, by Lemma 14.2 and (14.1),

$$\tau T^2 = O(x_0 T) = O\big(x_1 \lambda(E \cap I)\big) = O(zT)^2,$$

which implies $\tau = O(z^2)$.

It follows that if $\mathbf{f}$ is a τ-singular template, then $\underline{\delta}(\mathbf{f}) \leq \delta_{m,n} - \Theta(\sqrt{\tau})$, since otherwise we can take $z = 2(\delta_{m,n} - \underline{\delta}(\mathbf{f}))$ and apply the above argument. Taking the supremum over $\mathbf{f}$ and applying Theorem 2.8 shows that

$$\dim_H\Big(\mathrm{Sing}_{m,n}(\tau)\Big) \leq \delta_{m,n} - \Theta(\sqrt{\tau}) \text{ if } (m,n) \neq (2,2).$$

When $(m,n) = (2,2)$, the upper bound for Hausdorff dimension follows from the upper bound for packing dimension which we proved in §11.

15. Proof of Theorem 1.5, second formula, lower bound for Hausdorff dimension

Let $m = n = 2$, and fix $0 < \tau < \frac{1}{n} = \frac{1}{2}$. Fix $\lambda > 1$ and let $t_k = \lambda^k$ and $\varepsilon_k = \tau t_k$. However, rather than letting $\mathbf{f} = \mathbf{f}[\tau,\lambda]$, we will introduce a new parameter $\gamma \in [1 + 6\tau + 2\lambda\tau, \lambda]$. We define $\mathbf{f}$ as follows:

- On $[1,\gamma]$, we have $\mathbf{f} = \mathbf{s}[(1,-\tau),(\gamma,-\lambda\tau)]$. Note that (12.3) is satisfied due to the lower bound on γ.
- Extend $\mathbf{f}$ to $[\gamma,\lambda]$ via the requirement that $\mathbf{f}$ is constant on $[\gamma,\lambda]$: $f_1 = f_2 = -\lambda\tau$ and $f_3 = f_4 = \lambda\tau$ on $[\gamma,\lambda]$.
- Extend $\mathbf{f}$ to $[0,\infty)$ via exponential equivariance, i.e. so that $\mathbf{f}(\tau t) = \tau\mathbf{f}(t)$ for some $\tau > 1$ (in this case $\tau = \gamma^{-1}\lambda$).

For simplicity of calculation, we set $\gamma = 1 - 2\tau + 10\lambda\tau$ (this is possible as long as $1 - 2\tau + 10\lambda\tau \leq \lambda$), since this means that $\mathbf{f}$ has only three intervals of linearity on $[1,\gamma]$ (otherwise $\mathbf{f}$ has four intervals of linearity on $[1,\gamma]$):

$$\mathbf{f}'(t) = \begin{cases} (-\frac{1}{2},\frac{1}{2},0,0) & 1 < t < 1+4\tau \\ (-\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}) & 1+4\tau < t < 1-2\tau+6\lambda\tau \\ (\frac{1}{2},-\frac{1}{2},0,0) & 1-2\tau+6\lambda\tau < t < 1-2\tau+10\lambda\tau \end{cases}$$

(cf. Figure 10). It follows that

$$\delta(\mathbf{f},t) = \begin{cases} 2 & 1 < t < 1-2\tau+6\lambda\tau \\ 3 & 1-2\tau+6\lambda\tau < t < \lambda \end{cases}$$

and thus if we let $r = 1 - 2\tau + 6\lambda\tau$, then

$$\begin{aligned} \underline{\delta}(\mathbf{f}) &= \Delta(\mathbf{f},r) = \Delta(\mathbf{f},[\lambda^{-1}r,r]) \\ &= \frac{3(1-\lambda^{-1}r) + 2(r-1)}{r-\lambda^{-1}r} = 3 - \frac{r-1}{r-\lambda^{-1}r} \\ &= 3 - \frac{6\lambda\tau - 2\tau}{(1-\lambda^{-1})(1-2\tau+6\lambda\tau)} \\ &= 3 - \Theta(\tau), \end{aligned}$$

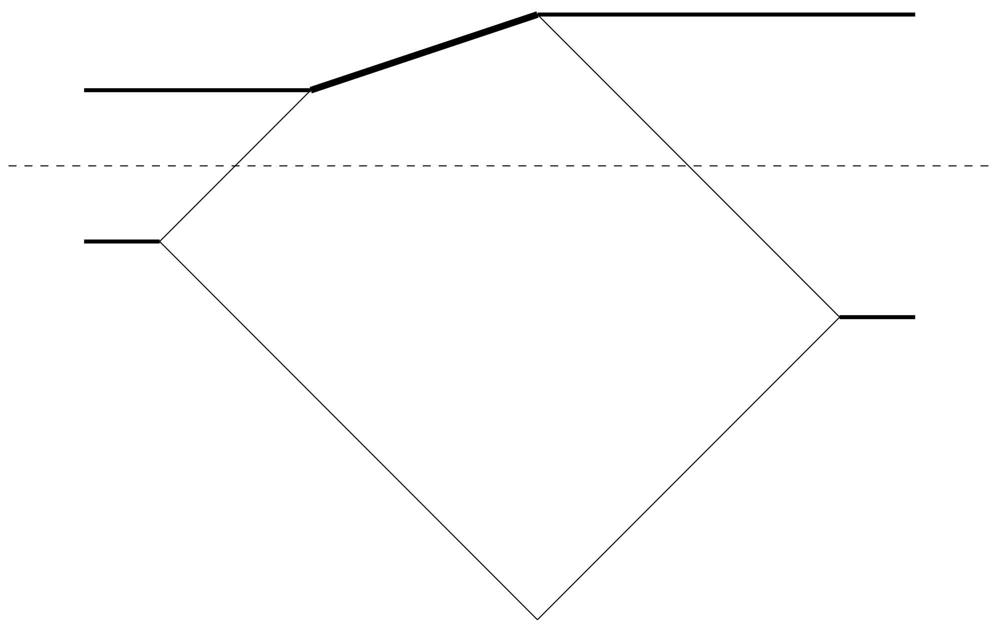

FIGURE 10. A period of an exponentially periodic 2×2 template, with $\gamma = 1 - 2\tau + 10\lambda\tau$.

where the implied constant of Θ can depend on λ. This completes the proof. Note that as in §12, one can optimize over the parameter λ to get the best possible bound using templates of this form, but we omit the required calculations.

16. PROOF OF THEOREM 1.8, LOWER BOUND

Fix $0 < \tau < 1/n$, such that $\tau < \frac{m-1}{2m}$ if $n = 1$. Now if λ is sufficiently large, then (12.4)-(12.6) are satisfied (the left half of (12.6) if $n \geq 2$, and the right half if $n = 1$), and thus there is a standard template $\mathbf{f} = \mathbf{f}_\lambda = \mathbf{f}[\tau, \lambda]$ defined by the sequence of points $(t_k, -\varepsilon_k)_0^\infty$, where $t_k = \lambda^k$ and $\varepsilon_k = \tau t_k$. On the other hand, let $\mathbf{g} = \mathbf{s}[(0,0),(1,-\tau)]$. We claim that as $\lambda \to \infty$, the upper average contraction rate of $\mathbf{f}_\lambda$ tends to

(16.1)
$$\sup_{0<T\leq 1} \Delta(\mathbf{g}, T) = \delta(\tau) \stackrel{\text{def}}{=} \max\left(mn - m,\ \delta_{m,n} - \frac{mn}{m+n}(d+m)\tau,\ mn - \frac{mn}{m+n}\frac{1+m\tau}{1 - \frac{mn}{m-1}\tau}\right).$$

Indeed, first let $\gamma > 0$ be small enough so that $\mathbf{g}' = (-\frac{1}{n}, \frac{1}{n(d-1)}, \ldots, \frac{1}{n(d-1)})$ on $(0, 2\gamma)$; the definition of $\mathbf{g}$ guarantees that such γ exists. Now $\overline{\delta}(\mathbf{f}_\lambda) = \sup_{T\in[\gamma,\lambda\gamma]} \Delta(\mathbf{f}_\lambda, T)$, and $\Delta(\mathbf{f}_\lambda, \cdot) \to \Delta(\mathbf{g}, \cdot)$ uniformly on $[\gamma, \lambda\gamma]$, where we extend $\mathbf{g}$ to $[0,\infty)$ by stipulating that

$g_1' = -\frac{1}{n}$ on $[1,\infty)$ and then defining $g_2,\ldots,g_d$ on $[1,\infty)$ in the same way as for standard templates. Thus

$$\dim_P(\mathrm{Sing}_{m,n}(\tau)) \geq \lim_{\lambda\to\infty} \overline{\delta}(\mathbf{f}_\lambda) = \sup_{T\in[\gamma,\infty)} \Delta(\mathbf{g},T).$$

But since $\delta(\mathbf{g},t) = mn - m$ for all $t \in [0,\gamma]\cup[1,\infty)$, it follows that $\Delta(\mathbf{g},T) \leq \max(mn - m, \Delta(\mathbf{g},1)) = \max(\Delta(\mathbf{g},\gamma),\Delta(\mathbf{g},1))$ for all $T \in [0,\gamma]\cup[1,\infty)$, and thus

$$\sup_{T\in[\gamma,\infty)} \Delta(\mathbf{g},T) = \sup_{0<T\leq 1} \Delta(\mathbf{g},T).$$

To complete the proof, we need to show that (16.1) holds. Indeed, from the definition of $\mathbf{g}$, it follows that there exist intervals $I_i = (t_i,t_{i+1})$, $i = 0,1,2$, with $t_0 = 0$, $t_3 = 1$, as follows:

	(g_1',g_2')	$S_+(\mathbf{g},\cdot)$	$mn-\delta(\mathbf{g},\cdot)$
I_0	$(-\frac{1}{n},\frac{1}{n(d-1)})$	$\{2,\ldots,m+1\}$	m
I_1 (case 1)	$(-\frac{1}{n},-\frac{1}{n})$	$\{3,\ldots,m+2\}$	$2m$
I_1 (case 2)	$(\frac{1}{m},-\frac{1}{m(d-1)})$	$\{1,\ldots,m\}$	0
I_2	$(\frac{1}{m},-\frac{1}{n})$	$\{1,3,\ldots,m+1\}$	$m-1$

TABLE 1. Two cases for the intervals of linearity of $\mathbf{g}$. See Figure 11.

Here case 1 holds when $\tau \geq \frac{m-1}{n(d+m-1)}$, while case 2 holds when $\tau \leq \frac{m-1}{n(d+m-1)}$. (When equality holds, I_1 is empty and so the cases are compatible.) Now let $0 < T \leq 1$ be maximal such that $\Delta(\mathbf{g},\cdot)$ attains its maximum at T. Then $\delta(\mathbf{g},t) \geq \Delta(\mathbf{g},T)$ for t slightly less than T, while $\delta(\mathbf{g},t) < \Delta(\mathbf{g},T)$ for t slightly greater than T. Thus $T = t_i$ for some $i = 1,2,3$. But if case 1 holds, then $\Delta(\mathbf{g},t_2) < mn - m = \Delta(\mathbf{g},t_1)$, so if $T = t_2$ then case 2 holds. Now it can be checked by direct calculation[14] that in this case,

$$\begin{aligned}
\Delta(\mathbf{g},t_1) &= mn - m,\\
\Delta(\mathbf{g},t_2) &= mn - \frac{mn}{m+n}\frac{1+m\tau}{1-\frac{mn}{m-1}\tau} \text{ if case 2 holds,}\\
\Delta(\mathbf{g},t_3) &= \delta_{m,n} - \frac{mn}{m+n}(d+m)\tau,
\end{aligned}$$

[14]The calculation of $\Delta(\mathbf{g},t_3)$ is somewhat tedious and it is easier to use the equality case of Lemma 13.1 below instead of performing a direct computation, since $\psi_{\mathbf{g}}(1) = \frac{mn}{m+n}(d+m)\tau$. Some other formulas useful for the calculations: when case 2 of Table 1 holds we have $t_1 = \frac{n}{m+n}(1+m\tau)$ and $t_2 = 1 - \frac{mn}{m-1}\tau$.

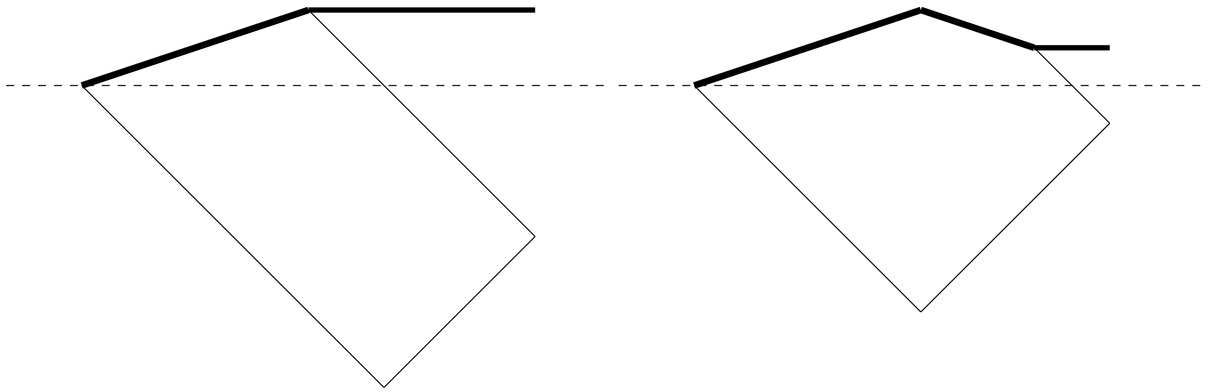

FIGURE 11. The joint graph of $\mathbf{g}$ in Case 1 and Case 2, respectively. Note that the slope of the last top segment may be either negative or positive according to whether $m < n$ or $m > n$, respectively (in the picture we have $m = n$ which corresponds to a horizontal slope).

which implies (16.1), since if $\tau \geq \frac{m-1}{n(d+m-1)}$ then $mn - \frac{mn}{m+n}\frac{1+m\tau}{1-\frac{mn}{m-1}\tau} \leq mn - m$, and thus when case 1 holds, the last term on the right-hand side of (16.1) does not contribute to the maximum. When $m = 1$, case 1 holds for all $\tau \geq 0$ and thus again the last term can be ignored.

17. PROOF OF THEOREM 1.9

Assume $n = 1$. First fix $0 < \tau < \frac{m-1}{2m}$. Fix $\lambda > 1$ and let $t_k = \lambda^k$ and $\varepsilon_k = \tau t_k$. However, rather than letting $\mathbf{f} = \mathbf{f}[\tau, \lambda]$, we will introduce a new parameter $\gamma > 0$ (which we think of as being independent of λ), small enough so that $\mathbf{s}[(\gamma, -\varepsilon), (1, -\tau)]$ is well-defined for all $0 \leq \varepsilon \leq \frac{m-1}{2m}\gamma$ (it suffices to take $\gamma \leq \frac{4m}{m^2-1}(\frac{m-1}{2m} - \tau)$). Let

$$\varepsilon = \left(\frac{\tau + (\lambda - 1)\frac{m-1}{2m}}{\lambda}\right)\gamma.$$

We define $\mathbf{f}$ as follows:

- On $[\gamma, 1]$, we have $\mathbf{f} = \mathbf{s}[(\gamma, -\varepsilon), (1, -\tau)]$.
- Extend $\mathbf{f}$ to $[1, \gamma\lambda]$ via the requirements that $f_1' = f_2' = -\frac{m-1}{2m}$ and $f_3 = \ldots = f_d$ on $[1, \gamma\lambda]$.
- Extend $\mathbf{f}$ to $[0, \infty)$ via exponential equivariance. This is possible by the definition of ε.

Now since $\delta(\mathbf{f},\cdot) = 1$ on $[1,\gamma\lambda]$, we have

$$\Delta(\mathbf{f},\gamma\lambda) \geq \tfrac{\gamma\lambda-1}{\gamma\lambda}.$$

Taking the supremum over $\mathbf{f}$ and applying Theorem 2.8 yields

$$\dim_P(\mathrm{Sing}_{m,n}(\tau)) \geq \tfrac{\gamma\lambda-1}{\gamma\lambda}.$$

Taking $\lambda \to \infty$ completes the proof.

Now suppose that $\tau < \frac{1}{m^2}$, and let $\tau' = \frac{(m-1)\tau}{1-m\tau} < \frac{1}{m}$. For each $\lambda > 1$ let $\mathbf{f}_\lambda = \mathbf{f}[\tau',\lambda]$ be the standard $1 \times m$ template defined by τ' and λ. The proof of Theorem 1.8 shows that

$$\lim_{\lambda\to\infty} \overline{\delta}(\mathbf{f}_\lambda) = \delta_{1,m}(\tau') \geq mn - n = m - 1.$$

Now the $m \times 1$ template $-\mathbf{f}_\lambda$ has the same upper average contractivity as $\mathbf{f}_\lambda$. Thus to complete the proof, it suffices to show that

$$\tau(-\mathbf{f}_\lambda) = \tau$$

for all sufficiently large λ. Indeed,

$$\tau(-\mathbf{f}_\lambda) = \liminf_{t\to\infty} \tfrac{1}{t} f_d(t) = \tfrac{1}{t_0} f_d(t_0),$$

where $t_0 > 1$ is the smallest time such that $f_2(t_0) = f_3(t_0)$. Since $\mathbf{f}(1) = (-\tau, -\tau, \frac{2}{m-1}\tau, \ldots, \frac{2}{m-1}\tau)$ and $\mathbf{f}' = (-\frac{1}{m}, 1, -\frac{1}{m}, \ldots, -\frac{1}{m})$ on $(1, t_0)$ (cf. Figure 9), we have

$$\begin{aligned}
f_d(t_0) &= -\tau' + (t_0 - 1) = \tfrac{2}{m-1}\tau' - \tfrac{1}{m}(t_0 - 1)\\
t_0 &= 1 + \tfrac{m}{m-1}\tau'\\
f_d(t_0) &= \tfrac{1}{m-1}\tau'\\
\tau(-\mathbf{f}(\lambda)) &= \frac{\frac{1}{m-1}\tau'}{1 + \frac{m}{m-1}\tau'} = \tau.
\end{aligned}$$

This completes the proof in the case $\tau < \frac{1}{m^2}$. Finally, the equality cases $\tau = \frac{m-1}{2m}$ and $\tau = \frac{1}{m^2}$ require some trivial limiting arguments which we omit.

18. Proof of Theorem 1.6, lower bound for Hausdorff dimension

Assume $n \geq 2$, and fix $0 < \tau < \frac{1}{n}$. Modifying the argument of §16 shows that

$$\dim_H(\mathrm{Sing}_{m,n}(\tau)) \geq \inf_{0<T\leq 1} \Delta(\mathbf{g}, T),$$

where $\mathbf{g} = \mathbf{s}[(0,0),(1,-\tau)]$. Now $\delta(\mathbf{g},t) \geq mn - 2m$ for all t, and $\delta(\mathbf{g},t) \geq mn - m$ for all $t \leq \frac{n(d-1)}{d}[\frac{1}{n} - \tau]$. It follows that

$$\Delta(\mathbf{g},T) \geq mn - 2m + \frac{mn(d-1)}{d}\left[\frac{1}{n} - \tau\right]$$

for all $0 < T \leq 1$.

19. Proof of Theorem 1.7, lower bound for Hausdorff dimension

Assume $n = 1$, and fix $0 < \tau < \frac{m-1}{2m}$, and let λ be minimal such that (12.6) holds, i.e.

$$\lambda = 1 + \tfrac{d\tau}{2}\left(\tfrac{m-1}{2m} - \tau\right)^{-1},$$

As usual we let $t_k = \lambda^k$, $\varepsilon_k = \tau t_k$, and $\mathbf{f} = \mathbf{f}[\tau,\lambda]$. Now since

$$f_1'(t) \leq -1 + \tfrac{m+1}{m}\delta(\mathbf{f},t) \text{ for all } t,$$

we have

$$-\tfrac{m-1}{2m} < -\tau = f_1(1) \leq -1 + \tfrac{m+1}{m}\Delta(\mathbf{f},1),$$

and rearranging gives

$$\Delta(\mathbf{f},1) > \tfrac{1}{2}.$$

It follows that $\Delta(\mathbf{f},T) \geq \frac{1}{2T} \geq \frac{1}{2\lambda}$ for all $T \in [1,\lambda]$. The exponential equivariance of $\mathbf{f}$ then implies that $\Delta(\mathbf{f},T) \geq \frac{1}{2\lambda}$ for all $T > 0$. So

$$\underline{\delta}(\mathbf{f}) \geq \tfrac{1}{2\lambda} = \Theta\left(\tfrac{m-1}{2m} - \tau\right)$$

and applying Theorem 2.8 completes the proof.

20. Proof of Theorem 1.6, upper bound for Hausdorff dimension

Let $\mathbf{f}$ be a τ-singular template which is not trivially singular. Then we have $f_1 = f_2$ infinitely often. Fix T such that $f_1(T) = f_2(T)$. Since $\mathbf{f}$ is τ-singular, we have $f_2(T) = f_1(T) \leq -\tau T$.

For all t such that $f_1'(t) = f_2'(t) = -\frac{1}{n}$, we have

$$mn - \delta(\mathbf{f},t) \geq 2m$$

and for all t such that $f_i'(t) > -\frac{1}{n}$ for some $i = 1,2$, we have

$$f_i'(t) \geq \frac{1}{n+1}\left[\frac{1}{m} - \frac{n}{n}\right] = -\frac{1}{n} + \frac{m+n}{mn(n+1)}$$

and thus

$$f_1'(t) + f_2'(t) \geq -\frac{2}{n} + \frac{m+n}{mn(n+1)}$$

and at the same time $mn - \delta(\mathbf{f}, t) \geq 0$. Combining these two cases we have

$$mn - \delta(\mathbf{f}, t) \geq 2m - \frac{2m^2 n(n+1)}{m+n} \left[f_1'(t) + f_2'(t) + \frac{2}{n} \right]$$

and averaging over the interval $[0, T]$ gives

$$\begin{aligned} mn - \Delta(\mathbf{f}, T) &\geq 2m - \frac{2m^2 n(n+1)}{m+n} \left[\frac{f_1(T)}{T} + \frac{f_2(T)}{T} + \frac{2}{n} \right] \\ &\geq 2m - \frac{4m^2 n(n+1)}{m+n} \left[\frac{1}{n} - \tau \right]. \end{aligned}$$

Taking the liminf as $T \to \infty$ completes the proof.

21. Proof of Theorem 1.7, upper bound for Hausdorff dimension

Let $\mathbf{f}$ be a τ-singular template. First suppose that both $f_1 = f_2$ and $f_2 = f_3$ infinitely often.

Fix $T_1 > 0$ such that $f_2(T_1) = f_3(T_1)$, and let $T \geq T_1$ be minimal such that $f_1(T) = f_2(T)$. Let $x = \frac{m-1}{2m} - \tau > 0$. For each t, let $j(t)$ denote the unique element of $S_-(\mathbf{f}, t)$. Then

$$f_1'(t) + f_2'(t) \begin{cases} = \frac{1}{m} - 1 & j(t) = 1, 2 \\ \geq \frac{1}{m} - 1 + \alpha & j(t) > 2 \end{cases}$$

where $\alpha > 0$ is a constant. On the other hand,

$$\tfrac{1}{T}\big(f_1(T) + f_2(T)\big) \leq -2\tau = \frac{1}{m} - 1 + 2x.$$

It follows that

$$\lambda(\{t \leq T : j(t) > 2\}) = O(xT)$$

where λ is Lebesgue measure. Consequently $f_i(t) = \frac{t}{m} + O(xT)$ for all $i > 2$ and $t \in [0, T]$. On the other hand, since $f_2' \geq -1$ we have

$$f_2(t) \leq f_2(T) + T - t \leq -\left(\frac{m-1}{2m} - x \right) T + T - t = \frac{m+1}{2m} T - t + O(xT)$$

for all $t \in [0, T]$, and thus we have $f_2 < f_3$ for all $t \in I := (T/2 + cxT, T)$, where $c > 0$ is a constant. In particular we have $T_1 \leq T/2 + cxT$. By the minimality of T, it follows that $f_1 < f_2$ on I. Using the convexity condition it is possible to prove that $j(t) = 2$ for all $t \in I$. Thus $f_1' = \frac{1}{m}$ on I and thus

$$f_1(T/2) = f_1(T) - \tfrac{1}{m}(T/2) + O(xT) \leq -\tau T - \tfrac{1}{m}(T/2) + O(xT) = -(T/2) + O(xT).$$

Consequently,

$$\lambda(\{t \leq T/2 : j(t) > 1\}) = O(xT) \tag{21.1}$$

and thus $\Delta(\mathbf{f}, T/2) = O(xT)$.

Now if $f_1 < f_2$ for all time, then $j(t) = 1$ for all time, and thus $\tau = 1$ and $\underline{\delta}(\mathbf{f}) = 0$.

If $f_2 < f_3$ for all time, then $j(t) \leq 2$ for all time, and thus

$$2f_1(t) \leq f_1(t) + f_2(t) = -\frac{m-1}{m}t$$

demonstrating that $\tau \geq \frac{m-1}{2m}$. Since equality holds infinitely often, we have $\tau = \frac{m-1}{2m}$. Thus for $\tau < \frac{m-1}{2m}$, we have $f_2 = f_3$ infinitely often.

22. Proof of Theorem 1.7, upper bound for packing dimension

Let $T_1 > 0$ be a local maximum of $\Delta(\mathbf{f}, \cdot)$, and by contradiction suppose that $\Delta(\mathbf{f}, T_1) > 1$. Then $\delta(\mathbf{f}, I) > 1$, where I is the interval of linearity for $\mathbf{f}$ whose right endpoint is T_1. Equivalently, $j > 2$ on I, where j is as in §21. Let T be as in §21. Since $f_1 < f_2$ on (T_1, T), by the convexity condition we have $j > 1$ on (T_1, T) and thus by (21.1) we have $T_1 = T/2 + O(xT)$. But then by the argument of §21, we have

$$\Delta(\mathbf{f}, T_1) = \Delta(\mathbf{f}, T/2) + O(x) = O(x)$$

and thus if x is sufficiently small, then $\Delta(\mathbf{f}, T_1) < 1$, a contradiction.

23. Proof of Theorem 1.10

Note that the packing dimension formula in Theorem 1.10 follows immediately from Theorem 1.8. Thus, we prove only the Hausdorff dimension formula. However, note that the first part of the proof could apply to the computation of packing dimension as well.

Fix $\tau > 0$, and let $\mathbf{f}$ be a 1×2 template which satisfies $\tau(\mathbf{f}) = \tau$ but is not trivially singular. We claim that

$$\underline{\delta}(\mathbf{f}) \leq \underline{\delta}(\tau), \tag{23.1}$$

where $\underline{\delta}(\tau)$ is the right-hand side of the first formula of Theorem 1.10. This will prove the upper bound of that formula. Indeed, since $\mathbf{f}$ is not trivially singular, the sets $F_- = \{t \geq 0 : f_1(t) = f_2(t)\}$ and $F_+ = \{t \geq 0 : f_2(t) = f_3(t)\}$ are both unbounded. Since $\mathbf{f}$ is piecewise linear, we can write $F_- \cup F_+$ as the union of a sequence of intervals $[s_1, t_1] < [s_2, t_2] < \ldots$

Claim 23.1. *We can assume without loss of generality that*

$$F_+ = [s_1, t_1] \cup [s_3, t_3] \cup \ldots \text{ and } F_- = [s_2, t_2] \cup [s_4, t_4] \cup \ldots$$

Proof. First, since F_- and F_+ are disjoint, for each k we have either $[s_k, t_k] \subseteq F_-$ or $[s_k, t_k] \subseteq F_+$. Now let $\mathbf{g} : [0, \infty) \to \mathbb{R}^3$ be defined by the formulas

$$\mathbf{g}(t) = \begin{cases} \left(-\frac{1}{2}f_3(t), -\frac{1}{2}f_3(t), f_3(t), \ldots, f_3(t)\right) & \text{if } t \in (t_k, s_{k+1}),\ [s_k, t_k], [s_{k+1}, t_{k+1}] \subseteq F_- \\ \left(f_1(t), -\frac{1}{2}f_1(t), \ldots, -\frac{1}{2}f_1(t)\right) & \text{if } t \in (t_k, s_{k+1}),\ [s_k, t_k], [s_{k+1}, t_{k+1}] \subseteq F_+ \\ \mathbf{f}(t) & \text{otherwise.} \end{cases}$$

Then $\delta(\mathbf{g}, t) \geq \delta(\mathbf{f}, t)$ for all $t \geq 0$, so $\underline{\delta}(\mathbf{g}) \geq \underline{\delta}(\mathbf{f})$ and $\overline{\delta}(\mathbf{g}) \geq \overline{\delta}(\mathbf{f})$. Moreover, since the minima of the functions

$$t \mapsto \frac{-f_1(t)}{t} \text{ and } t \mapsto \frac{-g_1(t)}{t}$$

on an interval of the form $[t_k, s_{k+1}]$ are always attained at one of the endpoints of the interval, we have $\tau(\mathbf{g}) = \tau(\mathbf{f})$. So it suffices to prove (23.1) with $\mathbf{f}$ replaced by $\mathbf{g}$. Now the corresponding sets F_- and F_+ defined in terms of $\mathbf{g}$ are clearly of the desired form, with the exception that the roles of F_- and F_+ may be switched; this exception can be dealt with by truncating the template from the left so as to cut out the interval $[s_1, t_1]$. $\square$

We observe that f_1 and f_2 "split" at times t_{2k} and "merge" at times s_{2k}, while f_2 and f_3 "split" at times t_{2k+1} and "merge" at times s_{2k+1}. Consequently

$$\begin{aligned} f_1'(t_{2k}^+) &< f_2'(t_{2k}^+), & f_2'(t_{2k+1}^+) &< f_3'(t_{2k+1}^+), \\ f_1'(s_{2k}^-) &> f_2'(s_{2k}^-), & f_2'(s_{2k+1}^-) &> f_3'(s_{2k+1}^-). \end{aligned}$$

It follows that if $j(t)$ denotes the unique element of $S_+(\mathbf{f}, t)$, then

$$\begin{aligned} j(s_{2k}^+) = j(t_{2k}^-) &= 1, & j(t_{2k}^+) = j(s_{2k+1}^-) &= 2, \\ j(s_{2k+1}^+) = j(t_{2k+1}^-) &= 2, & j(t_{2k+1}^+) = 3 > j(s_{2k+2}^-) &= 1. \end{aligned}$$

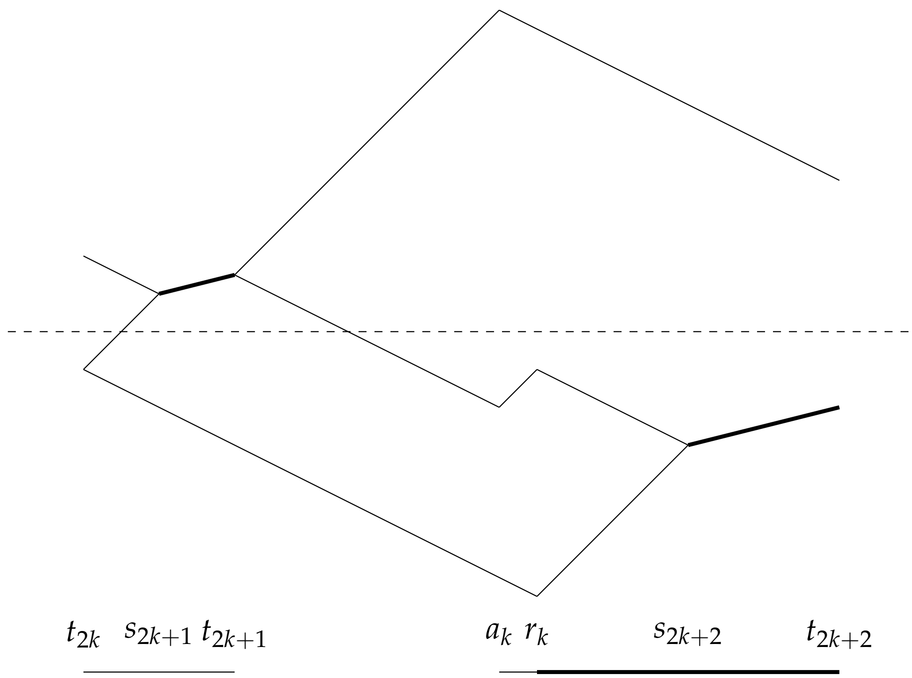

FIGURE 12. A piece of an arbitrary 1×2 template.

Thus by the convexity condition, there exists sequences of numbers $t_{2k+1} < a_k \leq r_k < s_{2k+2}$ such that

$$\mathbf{f}'(t) = \begin{cases} \left(-\frac{1}{2}, 1, -\frac{1}{2}\right) & t_{2k} < t < s_{2k+1} \\ \left(-\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) & s_{2k+1} < t < t_{2k+1} \\ \left(-\frac{1}{2}, -\frac{1}{2}, 1\right) & t_{2k+1} < t < a_k \\ \left(-\frac{1}{2}, 1, -\frac{1}{2}\right) & a_k < t < r_k \\ \left(1, -\frac{1}{2}, -\frac{1}{2}\right) & r_k < t < s_{2k+2} \\ \left(\frac{1}{4}, \frac{1}{4}, -\frac{1}{2}\right) & s_{2k+2} < t < t_{2k+2} \end{cases}$$

(cf. Figure 12). Evidently, we have

$$\delta(\mathbf{f}, t) = 3 - j(\mathbf{f}, t) = \begin{cases} 1 & t_{2k} < t < t_{2k+1} \\ 0 & t_{2k+1} < t < a_k \\ 1 & a_k < t < r_k \\ 2 & r_k < t < t_{2k+2}. \end{cases}$$

Now let $A_k, B_k, C_k, D_k \in \mathbb{R}$ be chosen so that

$$f_1(t) = A_k - \tfrac{1}{2}t \text{ for all } t \in [t_{2k}, r_k],$$
$$f_1(t) = B_k + t \text{ for all } t \in [r_k, s_{2k+2}],$$
$$f_3(t) = C_k + t \text{ for all } t \in [t_{2k+1}, a_k],$$
$$f_3(t) = D_k - \tfrac{1}{2}t \text{ for all } t \in [a_k, s_{2k+3}].$$

Then the set of parameters

$$\Big(A_k, B_k, C_k, D_k\Big)_{k\in\mathbb{N}}$$

is a necessary and sufficient set of parameters for $\mathbf{f}$ in the following sense: the map sending $\mathbf{f}$ to this set of parameters is injective, and its image is the set of all sequences of parameters that satisfy the following inequalities:

$$s_k \leq t_k < s_{k+1}, \tag{23.2}$$

$$t_{2k+1} < a_k \leq r_k < s_{2k+2} \tag{23.3}$$

where s_k, t_k, a_k, r_k are defined by the equations

$$0 = (A_k - \tfrac{1}{2}r_k) - (B_k + r_k) \tag{23.4}$$

$$0 = (C_k + a_k) - (D_k - \tfrac{1}{2}a_k) \tag{23.5}$$

$$0 = 2\big(A_k - \tfrac{1}{2}t_{2k}\big) + \big(D_{k-1} - \tfrac{1}{2}t_{2k}\big) \tag{23.6}$$

$$0 = 2\big(B_k + s_{2k+2}\big) + \big(D_k - \tfrac{1}{2}s_{2k+2}\big) \tag{23.7}$$

$$0 = \big(A_k - \tfrac{1}{2}t_{2k+1}\big) + 2\big(C_k + t_{2k+1}\big) \tag{23.8}$$

$$0 = \big(A_k - \tfrac{1}{2}s_{2k+1}\big) + 2\big(D_{k-1} - \tfrac{1}{2}s_{2k+1}\big) \tag{23.9}$$

The idea now is to take a function $\mathbf{f}$ defined by a sequence of parameters satisfying (23.2)-(23.3), and to replace it by a function $\widetilde{\mathbf{f}}$ defined by a sequence of parameters

$$\Big(\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k\Big)_{k\in\mathbb{N}}.$$

If we can show that $\Delta(\widetilde{\mathbf{f}}, T) \geq \Delta(\mathbf{f}, T)$ for all T, while $\widehat{\tau}(\widetilde{\mathbf{f}}) = \widehat{\tau}(\mathbf{f})$, then it suffices to prove (23.1) for $\widetilde{\mathbf{f}}$. A change that satisfies this inequality will be called an *allowable change*. Note that if a change only affects the value of $\delta(\mathbf{f}, \cdot)$ on two intervals I_1, I_2 such that $\max(I_1) < \min(I_2)$, increasing it on I_1 and decreasing it on I_2, with greater total area for the effect on I_1, then the change is allowable. We now show that we can make some allowable changes to simplify the structure of the template $\mathbf{f}$.

Claim 23.2. *We can without loss of generality assume that $a_k = r_k$ for all k.*

Proof. We claim that decreasing C_k by ε while leaving all other parameters fixed is an allowable change. Indeed, this change will have the effect of increasing t_{2k+1} by $\frac{4}{3}\varepsilon$ while increasing a_k by $\frac{2}{3}\varepsilon$. This means that $\delta(\mathbf{f},\cdot)$ is increased by 1 on an interval of length $\frac{4}{3}\varepsilon$ around t_{2k+1}, but decreased by 1 on an interval of length $\frac{2}{3}\varepsilon$ around a_k. Thus, the change is allowable, and applying the maximum value of $\varepsilon = \frac{3}{2}(r_k - a_k)$ completes the proof. □

From now on we will not treat C_k as an independent parameter, but rather assume that it is given by (23.5) together with the formula $a_k = r_k$. Note that in this case, (23.4), (23.5), and (23.8) combine to form the equation

$$0 = \left(A_k - \tfrac{1}{2}t_{2k+1}\right) + 2\left(D_k - A_k + B_k + t_{2k+1}\right). \tag{23.10}$$

Claim 23.3. *The following set of parameter changes is allowable:*

$$\begin{aligned}\widetilde{A}_k &= A_k + \varepsilon\\ \widetilde{B}_{k-1} &= B_{k-1} + \varepsilon\\ \widetilde{D}_{k-1} &= D_{k-1} - \varepsilon\end{aligned}$$

Proof. These changes lead to the following changes to t_k, r_k:

- no change to t_{2k-1}
- decrease r_{k-1} by $\frac{2}{3}\varepsilon$ (thus increasing $\delta(\mathbf{f},\cdot)$ by 2 on an interval of this length)
- increase t_{2k} by $\frac{2}{3}\varepsilon$ (thus increasing $\delta(\mathbf{f},\cdot)$ by 1 on an interval of this length)
- increase t_{2k+1} by $\frac{2}{3}\varepsilon$ (thus increasing $\delta(\mathbf{f},\cdot)$ by 1 on an interval of this length)
- increase r_k by $\frac{2}{3}\varepsilon$ (thus decreasing $\delta(\mathbf{f},\cdot)$ by 2 on an interval of this length)

The changes to s_k can be ignored as they do not affect $\delta(\mathbf{f},\cdot)$, except to note that $\Delta s_k = \widetilde{s}_k - s_k$ is always negative and so $\widetilde{t}_k - \widetilde{s}_k \geq t_k - s_k \geq 0$. The only decreasing effect, due to the change on r_k, is dominated by the increasing effect due to the change on r_{k-1}. Thus the changes are allowable. □

Now for each k, choose the maximum value of ε such that the changes lead to parameters satisfying (23.2)-(23.3) as well as the inequality

$$f_1(t) \leq -\tau t \text{ for all } t,$$

where $\tau < \widehat{\tau}(\mathbf{f})$ is arbitrary. Note that by piecewise linearity, this inequality is equivalent to saying that for all k we have

$$f_1(t_{2k}) \leq -\tau t_{2k}. \tag{23.11}$$

Then after the changes, (23.11) will be satisfied with equality for every k. Equivalently,

$$A_k - \tfrac{1}{2}t_{2k} = -\tau t_{2k}. \tag{23.12}$$

Let $u_k = t_{2k}$, and note that

$$\mathbf{f}(u_k) = (-\tau u_k, -\tau u_k, 2\tau u_k)\,.$$

This equality implies that for each k, we can define a template $\mathbf{g}^{(k)}$ by letting $\mathbf{g}^{(k)} = \mathbf{f}$ on $[u_k, u_{k+1}]$ and then extending by exponential equivariance:

$$\mathbf{g}^{(k)}(\lambda t) = \lambda \mathbf{g}^{(k)}(t) \text{ where } \lambda = u_{k+1}/u_k.$$

Note that clearly, $\tau(\mathbf{g}^{(k)}) = \tau$ for all k. From now on we will specialize to the Hausdorff dimension case of Theorem 1.10.

Claim 23.4. *We have*

$$\underline{\delta}(\mathbf{f}) \leq \sup_k \underline{\delta}(\mathbf{g}^{(k)}). \tag{23.13}$$

Proof. Fix $\varepsilon > 0$. Then there exist infinitely many k such that $\Delta(\mathbf{f}, u_{k+1}) \geq \Delta(\mathbf{f}, u_k) - \varepsilon$. For such a k, we have

$$\Delta(\mathbf{g}^{(k)}, u_k) = \Delta(\mathbf{f}, [u_k, u_{k+1}]) \geq \Delta(\mathbf{f}, u_k) - O(\varepsilon)$$

since u_{k+1}/u_k is bounded away from 1. Thus

$$\inf_{T\in[u_k,u_{k+1}]} \Delta(\mathbf{f}, T) \leq \inf_{T\in[u_k,u_{k+1}]} \Delta(\mathbf{g}^{(k)}, T) + O(\varepsilon) = \underline{\delta}(\mathbf{g}^{(k)}) + O(\varepsilon).$$

Taking the liminf over k and then letting $\varepsilon \to 0$ gives (23.13). ☐

Thus, we can without loss of generality assume that $\mathbf{f}$ is exponentially equivariant, i.e. that

$$A_k = \lambda^k A, \qquad B_k = \lambda^k B, \qquad D_k = \lambda^k D \tag{23.14}$$

for some $A, B, D > 0$ and $\lambda > 1$. Now by rescaling, we can without loss of generality assume that $u_0 = 1$. Plugging $k = 0$ into the formulas (23.4)-(23.9), (23.10), and (23.12), and solving for the appropriate variables yields

$$\begin{aligned}
A &= \tfrac{1}{2} - \tau \\
D &= \lambda(\tfrac{3}{2} - 2A) = \lambda(\tfrac{1}{2} + 2\tau) \\
t_0 &= 1 \\
s_1 &= \tfrac{2}{3}(A + 2\lambda^{-1}D) = 2 - 2A = 1 + 2\tau \\
t_1 &= \tfrac{2}{3}(A - 2D - 2B) \\
r_0 &= \tfrac{2}{3}(A - B) \\
s_2 &= -\tfrac{2}{3}(2B + D) \\
t_2 &= \lambda.
\end{aligned}$$

On the interval $[u_0, u_1] = [1, \lambda]$, the behavior of $\delta(\mathbf{f}, \cdot)$ is as follows:

$$\delta(\mathbf{f}, t) = \begin{cases} 1 & 1 < t < t_1 \\ 0 & t_1 < t < r_0 \\ 2 & r_0 < t < \lambda. \end{cases} \tag{23.15}$$

Now consider the change $\widetilde{B} = B - \varepsilon$. This change increases t_1 by $\frac{4}{3}\varepsilon$ and increases r_0 by $\frac{2}{3}\varepsilon$, this increasing $\delta(\mathbf{f}, \cdot)$ by 1 on an interval of length $\frac{4}{3}\varepsilon$ around t_1 and decreasing $\delta(\mathbf{f}, \cdot)$ by 2 on an interval of length $\frac{2}{3}\varepsilon$ around r_0. Thus the change is allowable, and by taking the maximum possible value of $\varepsilon = \frac{3}{4}(t_2 - s_2)$, we can without loss of generality assume that $s_2 = t_2$, or equivalently that

$$B = -\tfrac{3}{4}\lambda - \tfrac{1}{2}D = \lambda(A - \tfrac{3}{2}) = -\lambda(1 + \tau)$$

(cf. Figure 13). Note that this implies

$$t_1 = \tfrac{2}{3}A + \tfrac{4}{3}\lambda A.$$

Now it is a problem of one-variable calculus: λ is the only free parameter, and we must optimize $\underline{\delta}(\mathbf{f})$. Note that λ is subject to the restriction

$$\lambda \geq \frac{3/2 - 2A}{A} = \frac{1/2 + 2\tau}{1/2 - \tau}$$

which comes from the inequality $s_1 \leq t_1$. Now from (23.15), we have

$$\underline{\delta}(\mathbf{f}) = \Delta(\mathbf{f}, r_0) = \Delta(\mathbf{f}, [\lambda^{-1}r_0, r_0]) = \frac{1(t_1 - 1) + 2(1 - \lambda^{-1}r_0)}{r_0 - \lambda^{-1}r_0}.$$

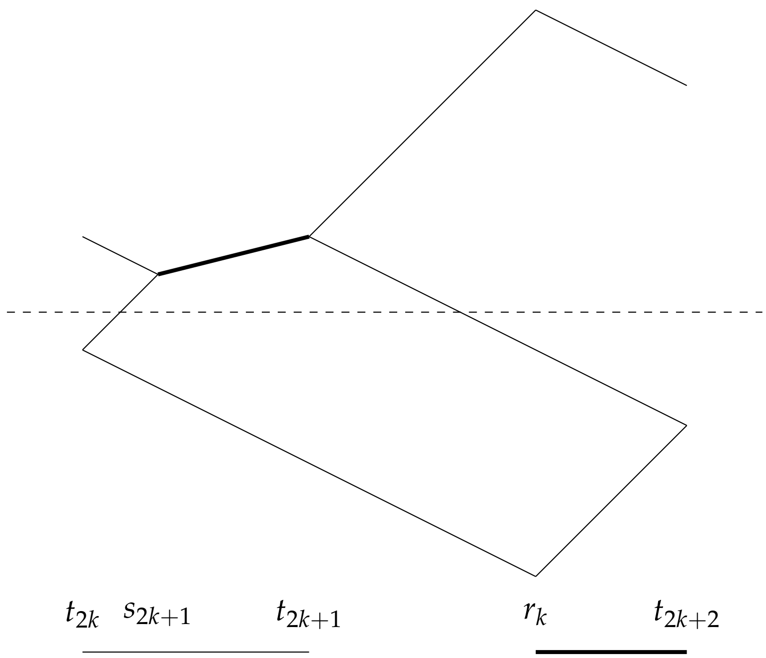

FIGURE 13. A period of an exponentially periodic 1×2 template, simplified using the arguments of this section.

On the other hand,

$$t_1 = \tfrac{2}{3}A + \tfrac{4}{3}\lambda A, \qquad r_0 = \tfrac{2}{3}A - \tfrac{2}{3}\lambda A + \lambda.$$

Let $x = \tau$ and $y = \frac{1}{3}(\lambda - 1)$. Then

$$\begin{aligned} t_1 &= (\tfrac{1}{3} - \tfrac{2}{3}x)(3 + 6y) = (1 - 2x)(1 + 2y), \\ r_0 &= 1 + 3y - (\tfrac{1}{3} - \tfrac{2}{3}x)(3y) = 1 + (2 + 2x)y, \\ \underline{\delta}(\mathbf{f}) &= \frac{\lambda(t_1 - 1) + 2(\lambda - r_0)}{r_0(\lambda - 1)} \\ &= \frac{(1 + 3y)(-2x + (2 - 4x)y) + (2 - 4x)y}{(1 + (2 + 2x)y)(3y)} \\ &= f_x(y) \stackrel{\text{def}}{=} \frac{2}{3} \cdot \frac{-x + (2 - 7x)y + (3 - 6x)y^2}{y + (2 + 2x)y^2}. \end{aligned}$$

We now need to find the maximum of the function f_x on the interval $[\frac{x}{1/2-x}, \infty)$, assuming that $0 < x < 1/2$. The function f_x has two critical points, given by the formulas[15]

$$\begin{aligned} 0 &= x + (4x+4x^2)y + (-1+4x+14x^2)y^2 \\ y &= \frac{\varepsilon\sqrt{x-6x^3+4x^4}+2x+2x^2}{1-4x-14x^2} = \frac{x}{\varepsilon\sqrt{x-6x^3+4x^4}-2x-2x^2} \\ f_x(y) &= \frac{4}{3} - \frac{4}{3}\varepsilon\sqrt{x-6x^3+4x^4} - 2x + \frac{8}{3}x^2 \end{aligned}$$

where $\varepsilon = \pm 1$. Note that since the critical point corresponding to $\varepsilon = -1$ is negative, it is not in the domain and so can be ignored. The critical point corresponding to $\varepsilon = +1$ is positive if and only if $1 - 4x - 14x^2 > 0$, which in turn is true if and only if $x < \frac{3\sqrt{2}-2}{14}$. In this case, it is easy to check that this critical point is in the domain of f_x, and that the critical point is a maximum. Thus in this case

$$\sup_y f_x(y) = f_x(y_{\mathrm{crit}}) = \frac{4}{3} - \frac{4}{3}\sqrt{x-6x^3+4x^4} - 2x + \frac{8}{3}x^2.$$

On the other hand, if $x \geq \frac{3\sqrt{2}-2}{14}$, then this critical point is negative or undefined, and thus f_x has no critical points on its domain. It can be verified that f_x is increasing in this case, so its supremum is equal to its limiting value:

$$\sup_y f_x(y) = \lim_{y\to\infty} f_x(y) = \frac{2}{3}\cdot\frac{3-6x}{2+2x} = \frac{1-2x}{1+x}.$$

Since $\underline{\delta}(\mathbf{f}) \leq \sup_y f_x(y)$, this completes the proof of the upper bound. To prove the lower bound, note that if $y \in [\frac{x}{1/2-x}, \infty)$, then there is a unique exponentially periodic template $\mathbf{f}$ satisfying the formulas appearing in the above proof, and this template satisfies $\underline{\delta}(\mathbf{f}) = f_x(y)$. Thus $\dim_H(\mathrm{Sing}_{1,2}(\omega)) \geq f_x(y)$, and taking the supremum over y proves the lower bound. Note that the exponentially periodic template $\mathbf{f}$ is the same as the standard

[15]Note that we found it easier to do these calculations first for the general case

$$f(y) = \frac{-A+By+Cy^2}{Dy+Ey^2},$$

then plug in the values $A = x$, $B = 2-7x$, $C = 3-6x$, $D = 1$, and $E = 2+2x$, and finally multiply by $\frac{2}{3}$. In the general case the formulas are

$$\begin{aligned} 0 &= AD + 2AEy - (BE-CD)y^2 \\ y &= \frac{\varepsilon\sqrt{Q}+AE}{BE-CD} = \frac{AD}{\varepsilon\sqrt{Q}-AE} \qquad \text{where } Q = (AE)^2 + (AD)(BE-CD) \\ f(y) &= \frac{1}{D^2}\big(2AE + BD - 2\varepsilon\sqrt{A^2E^2+ABDE-ACD^2}\big). \end{aligned}$$

template defined by the sequence of points $(t_k, -\varepsilon_k) = (\lambda^k, -\tau\lambda^k)$, where $\tau = x$ and $\lambda = 1 + 3y$.

24. Proof of Theorem 1.11

Let $\mathbf{f}$ be a template, and let ϕ be as in §11. We claim that

$$\phi'(t) \leq \delta_{m,n} - \delta(\mathbf{f}, t) + \frac{mn}{m+n} g(t),$$

where $g(t) = 1$ if $\mathbf{f}(t) = \mathbf{0}$ and $g(t) = 0$ otherwise. Indeed, when $\mathbf{f}(t) \neq \mathbf{0}$, this follows from Lemma 11.1, and when $\mathbf{f}(t) = \mathbf{0}$ it follows from direct calculation using the fact that $\phi'(t) = 0$ and $\delta(\mathbf{f}, t) = mn$. Now fix $T > 0$. Integrating over $[0, T]$ gives

$$0 \lesssim_+ \phi(T) - \phi(0) \leq \int_0^T \left[\delta_{m,n} - \delta(\mathbf{f}, t) + \frac{mn}{m+n} g(t)\right] \,\mathrm{d}t = T\left[\delta_{m,n} - \Delta(\mathbf{f}, T) + \frac{mn}{m+n} G(T)\right],$$

where $G(T)$ is the average of g on $[0, T]$. It follows that

$$\overline{\delta}(\mathbf{f}) \leq \limsup_{T\to\infty} \left[\delta_{m,n} + \frac{mn}{m+n} G(T)\right] = \delta_{m,n} + \frac{mn}{m+n} \limsup_{T\to\infty} G(T) = \mathcal{P}(\mathbf{f})\delta_{m,n} + (1 - \mathcal{P}(\mathbf{f}))mn,$$

where

$$\mathcal{P}(\mathbf{f}) = \liminf_{T\to\infty} (1 - G(T))$$

is the proportion of time spent near infinity. Applying Theorem 2.6 gives

$$\dim_H(\{A : \mathcal{P}(A) = p\}) \leq \dim_P(\{A : \mathcal{P}(A) = p\}) \leq p\delta_{m,n} + (1 - p)mn.$$

For the reverse direction, fix p and $\varepsilon > 0$ small. Define $\mathbf{f}$ on $[1, 1 + \varepsilon]$ as follows:

- Let $\mathbf{f} = \mathbf{g}[(0, 0), (1 + p\varepsilon, 0)]$ on $[1, 1 + p\varepsilon]$
- Let $\mathbf{f}(t) \equiv \mathbf{0}$ on $[1 + p\varepsilon, 1 + \varepsilon]$

and extend by exponential equivariance. It is easy to see that $\mathcal{P}(\mathbf{f}) = p$ and

$$\dim_P(\mathcal{M}(\mathbf{f})) \geq \dim_H(\mathcal{M}(\mathbf{f})) \geq p\delta_{m,n} + (1 - p)mn - O(\varepsilon).$$

This completes the proof.

25. Proof of Theorem 1.12

Let ϕ be a function such that $\phi(t) \to \infty$ as $t \to \infty$, and without loss of generality suppose that ϕ is increasing. Let $(t_k, -\varepsilon_k)$ be a sequence of points such that:

(i) $\Delta t_k \leq \frac{1}{2}\phi(t_k)$ for all k;
(ii) $\varepsilon_k \leq \frac{1}{2}\phi(t_k)$ for all k;
(iii) $\varepsilon_k \to \infty$ as $k \to \infty$;

(iv) $\varepsilon_k/\Delta t_k \to 0$ and $\varepsilon_{k+1}/\Delta t_k \to 0$ as $k \to \infty$.

Then let $\mathbf{f}$ be the standard template defined by the sequence of points $(t_k, -\varepsilon_k)$. Conditions (i) and (ii) imply that $f_1(t) \geq -\phi(t_k) \geq -\phi(t)$ for all $k \in \mathbb{N}$ and $t \in [t_k, t_{k+1}]$. Condition (iii) implies that $\mathbf{f}$ is singular. Finally, condition (iv) implies that $\underline{\delta}(\mathbf{f}) = \delta_{m,n}$, since

$$\begin{aligned}\Delta(\mathbf{f}, [t_k, t_{k+1}]) &= \Delta\left(\mathbf{s}\left[\left(0, -\frac{\varepsilon_k}{\Delta t_k}\right), \left(0, -\frac{\varepsilon_{k+1}}{\Delta t_k}\right)\right], 1\right) \\ &\to \Delta(\mathbf{s}[(0,0),(1,0)],1) = \delta_{m,n} \text{ as } k \to \infty.\end{aligned}$$

26. Proof of Theorem 1.14

Fix $2 \leq k \leq d-1$ and $j \in \{k-1, k\}$, and let $\mathbf{f}$ be a template with the following properties:

$$f_{k-1}(t) \to -\infty \text{ as } t \to \infty, \tag{26.1}$$

$$f_{k+1}(t) \to +\infty \text{ as } t \to \infty, \tag{26.2}$$

$$\frac{1}{t}\mathbf{f}(t) \to 0 \text{ as } t \to \infty, \tag{26.3}$$

$$\frac{1}{T}\lambda\big([0,T] \cap (S_j^+ \cup S_j^-)\big) \to 1 \text{ as } T \to \infty, \tag{26.4}$$

where S_j^+ (resp. S_j^-) is the set of all times $t \geq 0$ such that the following hold:

- $f_1(t) = \ldots = f_j(t) < f_{j+1}(t) = \ldots = f_d(t)$,
- $(L_+, L_-) = (\lceil \frac{jm}{d} \rceil, \lfloor \frac{jn}{d} \rfloor)$ (resp. $(L_+, L_-) = (\lfloor \frac{jm}{d} \rfloor, \lceil \frac{jn}{d} \rceil)$), where $L_\pm = L_\pm(\mathbf{f}, t, j)$.

Such a template can be constructed by alternating long $S_j^\pm$ intervals with short intervals along which f_k crosses 0 and returns, in a manner consistent with the rule on changes of slopes (cf. Figure 14). The key point is that if $t \in S_j^+$ then $f_1'(t) \geq 0$, but if $t \in S_j^-$ then $f_1'(t) \leq 0$ (with equality if and only if $\frac{jm}{d}$ is an integer). Note that the template $\mathbf{f}$ is not trivially singular.

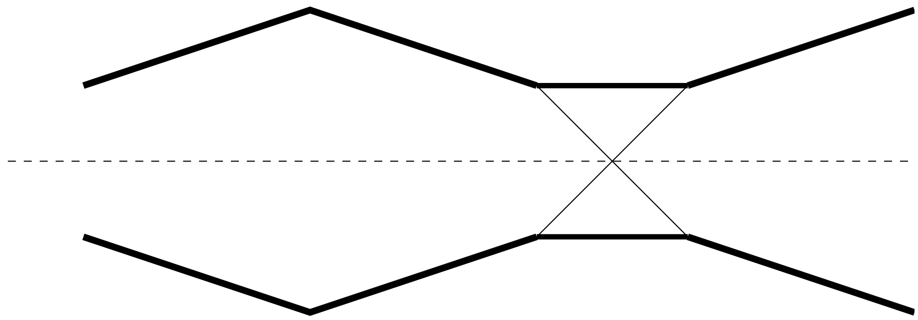

FIGURE 14. A piece of a template $\mathbf{f}$ with the desired properties, as described in §26 (Proof of Theorem 1.14). The hourglass portion of the figure can be made arbitrarily small in proportion to the rest.

To compute the lower contractivity of $\mathbf{f}$, we observe that for $t \in S_j^+$, we have

$$\begin{aligned}
f_1'(S_j^+) \stackrel{\text{def}}{=} f_1'(t) &= \frac{1}{j}\left[\frac{\lceil \frac{jm}{d} \rceil}{m} - \frac{\lfloor \frac{jn}{d} \rfloor}{n}\right] \\
&= \frac{1}{j}\left[\frac{\frac{jm}{d} + \{-\frac{jm}{d}\}}{m} - \frac{\frac{jn}{d} - \{\frac{jn}{d}\}}{n}\right] \\
&= \frac{1}{j}\frac{m+n}{mn}\left\{\frac{jn}{d}\right\} \\
mn - \delta(\mathbf{f}, S_j^+) \stackrel{\text{def}}{=} mn - \delta(\mathbf{f}, t) &= L_-(m - L_+) = \left\lfloor \frac{jn}{d} \right\rfloor \left(m - \left\lceil \frac{jm}{d} \right\rceil\right) \\
&= \left(\frac{jn}{d} - \left\{\frac{jn}{d}\right\}\right)\left(m - \frac{jm}{d} - \left\{-\frac{jm}{d}\right\}\right) \\
&= \frac{j(d-j)mn}{d^2} - \frac{(d-j)m + jn}{d}\left\{\frac{jn}{d}\right\} + \left\{\frac{jn}{d}\right\}^2.
\end{aligned}$$

Similarly, for $t \in S_j^-$ we have

$$\begin{aligned}
f_1'(S_j^-) \stackrel{\text{def}}{=} f_1'(t) &= -\frac{1}{j}\frac{m+n}{mn}\left\{\frac{jm}{d}\right\} \\
mn - \delta(\mathbf{f}, S_j^-) \stackrel{\text{def}}{=} mn - \delta(\mathbf{f}, t) &= \frac{j(d-j)mn}{d^2} + \frac{(d-j)m + jn}{d}\left\{\frac{jm}{d}\right\} + \left\{\frac{jm}{d}\right\}^2.
\end{aligned}$$

On the other hand, for $t \notin S_j^+ \cup S_j^-$ we have $-\frac{1}{n} \leq f_1'(t) \leq \frac{1}{m}$ and $0 \leq \delta(\mathbf{f}, t) \leq mn$. If $\frac{jm}{d}$ is an integer, then by (26.4) we have

$$\delta(\mathbf{f}, S_j^+) = \delta(\mathbf{f}, S_j^-) = f_{m,n}(j)$$

and we are done. Otherwise, by (26.3) and (26.4) we have

$$\frac{1}{T}\lambda\big([0,T] \cap S_j^{\pm}\big) \to \alpha^{\pm} \text{ as } T \to \infty,$$

where $\alpha^+ + \alpha^- = 1$ and

$$\alpha^+ f_1'(S_j^+) + \alpha^- f_1'(S_j^-) = 0.$$

It follows that

$$\alpha^+ = \left\{\frac{jm}{d}\right\}, \qquad \alpha^- = \left\{\frac{jn}{d}\right\},$$

and thus

$$\underline{\delta}(\mathbf{f}) = \alpha^+ \delta(\mathbf{f}, S_j^+) + \alpha^- \delta(\mathbf{f}, S_j^-) = f_{m,n}(j).$$

This completes the proof.

27. Proof of Theorem 2.2

Part (i) follows directly from Lemma 8.8, since we can take $\Lambda = u_A \mathbb{Z}^d$ where A is the matrix in question. To prove part (ii), consider the template $\mathbf{f}$ that we need to approxiomate by a successive minima function $\mathbf{h}_A$. If $\overline{\delta}(\mathbf{f}) > 0$, then by Theorem 2.6, the packing dimension of $\mathcal{M}(\mathbf{f})$ is positive and thus $\mathcal{M}(\mathbf{f})$ is nonempty. If we take $A \in \mathcal{M}(\mathbf{f})$, then $\mathbf{h}_A \asymp_+ \mathbf{f}$. On the other hand, suppose that $\overline{\delta}(\mathbf{f}) = 0$, and consider the set

$$Z = \{t \geq 0 : \Delta(\mathbf{f}, t) > 0\}$$

Then the density of Z is zero, i.e. $\lim_{T\to\infty} \frac{1}{T}|Z \cap [0,T]| = 0$, where $|\cdot|$ denotes 1-dimensional Lebesgue measure. On the other hand, for all $t \notin Z$ we must have $\mathbf{f}'(t) = (-\frac{1}{n}, \ldots, -\frac{1}{n}, \frac{1}{m}, \ldots, \frac{1}{m})$. It follows that $f_n(t) < f_{n+1}(t)$ for all sufficiently large t. Then the convexity and quantized slope conditions (see Definition 2.1) imply that F_n must be piece-wise linear with only finitely many intervals of linearity. Now it follows, using the fact that Z has zero density, that $F_n'(t) = -1$ for all sufficiently large t, which in turn implies that $\mathbf{f}(t) \asymp_+ (-\frac{1}{n}, \ldots, -\frac{1}{n}, \frac{1}{m}, \ldots, \frac{1}{m})t$. Now there exist matrices A such that $\mathbf{h}_A(t) \asymp_+ (-\frac{1}{n}, \ldots, -\frac{1}{n}, \frac{1}{m}, \ldots, \frac{1}{m})t$ (for example, matrices with rational entries) and so this completes the proof.

28. Proof of Theorem 2.10

A matrix A is badly approximable if and ony if its successive minima function $\mathbf{h}_A$ is bounded. Thus, by Theorem 2.6, the Hausdorff dimension of the set of badly approximable matrices is equal to the supremum of $\underline{\delta}$ over bounded templates. Since $\underline{\delta}(\mathbf{0}) = mn$ and $\underline{\delta}(\mathbf{f}) \leq mn$ for all templates $\mathbf{f}$, this supremum is equal to mn.

29. Proof of Theorem 2.11

Analogously to the uniform dynamical exponent, we define the regular (non-uniform) dynamical exponent of a map $\mathbf{f} : [0, \infty) \to \mathbb{R}^d$ to be the number

$$\tau(\mathbf{f}) \stackrel{\text{def}}{=} \limsup_{t \to \infty} \frac{-1}{t} f_1(t).$$

Now let $\mathbf{f}$ be a template with $\tau(\mathbf{f}) = \tau$ and consider the potential function

$$\phi(t) = \phi_{\mathbf{f}}(t) = mn|f_1(t)|.$$

Lemma 29.1. *Let I be an interval of linearity for $\mathbf{f}$. Then*

$$\phi'(I) \leq mn - \delta(I),$$

with equality in the following cases:

- $\mathbf{f} = \mathbf{0}$ *on* I
- $f_1' = -\frac{1}{n}$ *and* $f_2 = f_d$ *on* I.

Proof. Let j be the largest value such that $f_1 = f_j$ on I, and let $L_\pm = L_\pm(\mathbf{f}, I, j)$. Then

$$\phi'(I) = mn\frac{1}{j}\left[\frac{L_-}{n} - \frac{L_+}{m}\right]$$

while

$$mn - \delta(I) \geq L_-(m - L_+).$$

So we need to show that

$$\frac{1}{j}[mL_- - nL_+] \leq L_-(m - L_+).$$

Indeed, since $L_- \leq n$ and $j \geq 1$, we have

$$\frac{1}{j}[mL_- - nL_+] \leq \frac{1}{j}[mL_- - L_-L_+] = \frac{1}{j}L_-(m - L_+) \leq L_-(m - L_+).$$

Equality holds when $L_+ = m$ and $L_- = n$, and when $L_+ = 0$ and $L_- = 1$. $\triangleleft$

Integrating gives

$$\phi(T) = mn|f_1(T)| \leq T(mn - \Delta(T)).$$

Dividing by T and then taking the limsup gives

$$mn\tau \leq mn - \underline{\delta}(\mathbf{f}).$$

Rearranging gives

$$\underline{\delta}(\mathbf{f}) \leq mn(1-\tau).$$

It is not hard to see that equality holds for some values of $\mathbf{f}$. Thus, by Theorem 2.6, we have

$$\dim_H(\{\omega\text{-approximable matrices}\}) = \sup_{\mathbf{f}:\tau(\mathbf{f})=\tau} \underline{\delta}(\mathbf{f}) = mn(1-\tau).$$

Part 5. **Appendix and references**

Appendix A. Translating between Schmidt–Summerer's notation and ours

This appendix explains the relations between certain concepts and notation in our paper and in Schmidt–Summerer's [57] to provide a guide for readers of both.

Schmidt–Summerer are working in the framework of simultaneous approximation, so $n = 1$ for them, and further: their n is our $d = m+1$, their y is our r, their ξ is our A. In particular, note that they have $r = (q,p)$ instead of $r = (p,q)$. Their $\Lambda(\xi)$ would translate to $u_A\mathbb{Z}^d$ in our paper, and what they call $\mathcal{K}(Q)$ is what we would call $g_{-t}B$, where $Q = e^t$ and $B = [-1,1]^d$. Finally, their T is our g_{-1}.

Schmidt–Summerer's set-up encodes the same geometric information as ours since

$$\lambda_i(g_t u_A \mathbb{Z}^d, B) = \lambda_i(u_A\mathbb{Z}^d, g_{-t}B).$$

Therefore, in their notation the right-hand side is $\lambda_i(\Lambda(\xi), \mathcal{K}(Q))$. Similarly, $L_i(q)$ in their notation is the same as $h_i(t)$ in our notation, where $q = t/(n-1)$. The connection between our notion of a *template* (see Definition 2.1) and Schmidt–Summerer's (n,γ)-*systems* (see [57, §2]) is as follows: if P is an $(n,0)$-system then

$$\mathbf{h}(t) = \frac{n}{n-1}P(t) - \frac{t}{n-1}$$

is an $(n-1)\times 1$ template.

We further remark that after Schmidt–Summerer consider the limiting case of an $(n,0)$-system in [57, §3], they go on, in [57, §4, pg. 62], to conjecture that the study of these systems should suffice to determine the spectra of the family of exponents of approximation that they are interested in. The rest of their paper develops a theory of covers of an (n,γ)-system, which is then applied to prove relations between several exponents of approximation.

Interested readers are also referred to Roy's paper [49] for translating between Schmidt–Summerer's notation and his. In contrast to Schmidt–Summerer who work in the simultaneous approximation framework, Roy works in the dual framework of approximation by linear forms. Roy defines the notion of a *rigid system* (a special case of $(n, 0)$-systems) in the introduction of [49] and goes on to prove that every (n, γ)-system can be approximated by a rigid system up to bounded additive difference (see [49, Theorem 1.3]). Roy's rigid systems translate to our *η-integral templates* (see Definition 8.1).

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