RANDOM NON-HYPERBOLIC EXPONENTIAL MAPS

MARIUSZ URBAŃSKI AND ANNA ZDUNIK

Abstract. We consider random iteration of exponential entire functions, i.e. of the form $\mathbb{C} \ni z \mapsto f_\lambda(z) := \lambda e^z \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$. Assuming that λ is in a bounded closed interval $[A, B] \subset \mathbb{R}$ with $A > 1/e$, we deal with random iteration of the maps f_λ governed by an invertible measurable map $\theta : \Omega \to \Omega$ preserving a probability ergodic measure m on Ω, where Ω is a measurable space. The link from Ω to exponential maps is then given by an arbitrary measurable function $\eta : \Omega \longmapsto [A, B]$. We in fact work on the cylinder space $Q := \mathbb{C}/\sim$, where $\sim$ is the natural equivalence relation: $z \sim w$ if and only if $w - z$ is an integral multiple of $2\pi i$. We prove that then for every $t > 1$ there exists a unique random conformal measure $\nu^{(t)}$ for the random conformal dynamical system on Q. We further prove that this measure is supported on the, appropriately defined, radial Julia set. Next, we show that there exists a unique random probability invariant measure $\mu^{(t)}$ absolutely continuous with respect to $\nu^{(t)}$. In fact $\mu^{(t)}$ is equivalent with $\nu^{(t)}$. Then we turn to geometry. We define an expected topological pressure $\mathcal{E}\mathrm{P}(t) \in \mathbb{R}$ and show that its only zero h coincides with the Hausdorff dimension of m–almost every fiber radial Julia set $J_r(\omega) \subset Q$, $\omega \in \Omega$. We show that $h \in (1, 2)$ and that the omega–limit set of Lebesgue almost every point in Q is contained in the real line $\mathbb{R}$. Finally, we entirely transfer our results to the original random dynamical system on $\mathbb{C}$. As our preliminary result, we show that all fiber Julia sets coincide with the entire complex plane $\mathbb{C}$.

Contents

M. Urbański was supported in part by the NSF Grant DMS 1361677. A. Zdunik was supported in part by the Grant NCN grant 2014/13/B/ST1/04551.
We wish to thank the anonymous referee whose remarks, comments, and suggestions allowed us to improve the final exposition of the paper.

1. Introduction

The study of the dynamics of entire transcendental functions of the complex plane has begun with the foundational research of Pierre Fatou ([14]) in the third decade of 20th century. For some decades since then I. N. Baker ([4], [2] and [3] for example), was the sole champion of the research in this field. The breakthrough has come in 1981 when Michal Misiurewicz ([29]) proved that the Julia set of the exponential function $\mathbb{C} \ni z \mapsto e^z \in \mathbb{C}$ is the whole complex plane $\mathbb{C}$. This positively affirmed Fatou's Conjecture from [14] and opened up the gates for new extensive research. Indeed, the early papers such as [31], [20], [28], [12], [13] have appeared. These concerned topological and measurable (Lebesgue) aspects of the dynamics of entire functions. However, already McMullen's paper [28] also touched on Hausdorff dimension, providing deep and unexpected results. The theme of Hausdorff dimension for entire functions was taken up in a series of papers by G. Stallard (see for ex. [34]–[38] and in [39] and [40]). These two latter papers concerned hyperbolic exponential functions, i.e. those of the form

$$\mathbb{C} \ni z \longmapsto f_\lambda(z) := \lambda e^z \in \mathbb{C},$$

where λ is such that the map f_λ is hyperbolic, i.e. it has an attracting periodic orbit. Although it did not concern entire functions but meromorophic ones (tangent family in fact), we would like to mention here the seminal paper of K. Barański ([5]) where for the first time the thermodynamic formalism was applied to study transcendental functions. The papers [39] and [40] also used the ideas of thermodynamic formalism and, particularly, of conformal measures. This is in these papers where the concept of a radial (called also conical) Julia set, denoted by $J_r(f)$, occurred. This is the set of points z in the Julia set $J(f)$ for which infinitely many holomorphic pullbacks from $f^n(z)$ to z are defined on balls centered at points $f^n(z)$ and having radii larger than zero independently of n. For hyperbolic functions f_λ this is just the set of points that do not escape to infinity under the action of the map f_λ. What we have discovered in [39] and [40] is that $\text{HD}(J_r(f_\lambda)) < 2$ for hyperbolic exponential functions f_λ defined above. This is in stark contrast with McMullen's results from [28] asserting that $\text{HD}(J_r(f_\lambda)) = 2$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Note that the set $J_r(f_\lambda)$ is dynamically significant as for example, because of Poincaré's Recurrence Theorem, every finite Borel f_λ–invariant measure on $\mathbb{C}$ is supported on this set. In addition we proved in [39] and [40] that its Hausdorff dimension $\text{HD}(J_r(f_\lambda))$ is equal to the unique zero of the pressure function $t \mapsto \text{P}(t)$ defined absolutely independently of $J_r(f_\lambda)$.

The study of geometric (Hausdorff dimension) and ergodic (invariant measures absolutely continuous with respect to conformal ones) properties of transcendental entire functions by means of thermodynamic formalism followed. It is impossible to cite here all of them, we just mention only [19], [23], [24], [26], [7] and [8]. Related papers include [33], [9], [6] and many more.

We would like to pay particular attention to the paper [41], where a fairly full account of ergodic theory and conformal measures was provided for a large class of non-hyperbolic exponential functions f_λ, namely those for which the number 0 escapes to infinity "really fast"; it includes all maps for which λ is real and larger than $1/e$. Our current work stems from this one and provides a systematic account of ergodic theory and conformal measures for randomly iterated functions f_λ, where $\lambda > 1/e$. The theory of random dynamical systems is a large fast developing subfield of dynamical systems with a specific variety of methods, tools, and goals. We just mention the classical works of Yuri Kifer, [16], [17] and of Ludwig Arnold ([1]), see also [18]. Our present work in this respect stems from [22], [27], and [25].

Our first result, whose proof occupies Section 3, is that if a sequence $(a_n)_{n=1}^\infty$ of real numbers in $[A, B]$ is taken, where $A > 1/e$, then the Julia set of the compositions

$$f_{a_n} \circ f_{a_{n-1}} \circ \ldots \circ f_{a_2} \circ f_{a_1} : \mathbb{C} \to \mathbb{C}$$

is equal to the entire complex plane $\mathbb{C}$. This is a far going generalization of, already mentioned above, Misiurewicz's result from [29]. In addition, our proof is a simplification of Misiurewicz's one, even in the autonomous (just one map) case. The above compositions are said to constitute a non–autonomous dynamical system as the "rule of evolution" depends on time.

Our main focus in this paper are random dynamical systems. These are objects lying somewhat in between autonomous and non–autonomous systems, sharing many dynamical and geometrical features with both of them. As in [1], [11], [22], [26], and [27] the randomness for us is modeled by a measure preserving invertible dynamical system $\theta : \Omega \to \Omega$, where $(\Omega, \mathcal{F}, m)$ is a complete probability measurable space, and θ is a measurable invertible map, with θ^{-1} measurable, preserving the measure m. Fix some real constants $B > A > 1/e$ and let

$$\eta : \Omega \longmapsto [A, B]$$

be measurable function. Furthermore, to each $\omega \in \Omega$ there is associated the exponential map $f_\omega := f_{\eta(\omega)} : \mathbb{C} \to \mathbb{C}$; precisely

$$f_\omega(z) := \eta(\omega)e^z.$$

Consequently, for every $z \in \mathbb{C}$, the map

$$\Omega \ni \omega \longmapsto f_{\eta(\omega)}(z) \in \mathbb{C}$$

is measurable. In order to avoid ambiguity and confusion about what is f_ω and what is $f_{\eta(\omega)}$, we assume without loss of generality that Ω is disjoint from $[A, B]$. If however Ω happened to intersect $[A, B]$, we could always replace it for example by $\Omega \times \{0\}$ to achieve the required disjointness. Another option: to introduce different letters/symbols for f_ω and $f_{\eta(\omega)}$ would be just too cumbersome and would make the whole exposition less readable.

We consider the dynamics of random iterates of exponentials:

$$f_\omega^n := f_{\theta^{n-1}\omega} \circ \cdots \circ f_{\theta\omega} \circ f_\omega : \mathbb{C} \longrightarrow \mathbb{C}.$$

The sextuple

$$f := \big(\Omega, \mathcal{F}, m;\ \theta : \Omega \to \Omega;\ \eta : \Omega \to [A, B]; f_\eta : \mathbb{C} \to \mathbb{C}\big)$$

and induced by it random dynamics

$$\left(f_\omega^n : \mathbb{C} \longrightarrow \mathbb{C}\right)_{n=0}^{\infty}, \quad \omega \in \Omega,$$

will be referred to in the sequel as *random exponential dynamical system*. Obviously, this generates also a global dynamics (skew product) $f : \Omega \times \mathbb{C} \to \Omega \times \mathbb{C}$ defined as $f(\omega, x) = (\theta(\omega), f_\omega(x))$.

We define the equivalence relation $\sim$ on the complex plane $\mathbb{C}$ by saying that $Z \sim W$ if there exists $k \in \mathbb{Z}$ such that

$$Z - W = 2\pi i k.$$

We denote the quotient space $\mathbb{C}/\sim$ by Q. So, Q is conformally an infinite cylinder. We denote by π the natural projection $\pi : \mathbb{C} \to Q$, i.e.,

$$\pi(Z) = [Z]$$

is the equivalence class of z with respect to relation $\sim$. Since both maps $f_\eta : \mathbb{C} \to \mathbb{C}$ and $\pi \circ f_\eta : \mathbb{C} \to Q$, $\eta \in \mathbb{C}^*$, are constant on equivalence classes, they canonically induce conformal maps $f_\eta : Q \to \mathbb{C}$ and

$$F_\eta : Q \to Q.$$

So, F_η can be represented as

$$F_\eta = \pi \circ f_\eta \circ \pi^{-1}.$$

In Sections 4–11 we will be exclusively interested in the sextuple

$$F := \big(\Omega, \mathcal{F}, m;\ \theta : \Omega \to \Omega;\ \eta : \Omega \to [A, B]; F_\eta : Q \to Q\big),$$

induced by it random dynamics

$$\left(F_\omega^n := F_{\theta^{n-1}\omega} \circ \cdots \circ F_{\theta\omega} \circ F_\omega : Q \longrightarrow Q\right)_{n=0}^{\infty}, \quad \omega \in \Omega$$

and the global dynamics (skew product) $F : \Omega \times Q \to \Omega \times Q$ defined as $F(\omega, x) = (\theta(\omega), F_\omega(x))$. All our technical work in Sections 4–11 will concern the sextuple F acting on the cylinder Q. The main results for this sextuple will be obtained in Sections 8–11. In Sections 12 and 13 we will fully transfer them for the case of sextuple

$$f := \big(\Omega, \mathcal{F}, m;\ \theta : \Omega \to \Omega;\ \eta : \Omega \to [A, B]; f_\eta : \mathbb{C} \to \mathbb{C}\big)$$

and induced by it random dynamics.

We now describe our results for the sextuple F. Let $X = \Omega \times Q$ and let $\pi_1 : X \to \Omega$ be the projection onto the first coordinate, i.e., $\pi_1(\omega, z) = \omega$.

Following [11] we consider random measures (with respect to the measure m). Let $\mathcal{M}_m \subset \mathcal{M}(X)$ be the set of all non-negative probability measures on X that project onto m under the map $\pi_1 : X \to \Omega$, i.e.

$$\mathcal{M}_m = \big\{\mu \in \mathcal{M}(X) : \mu \circ \pi_1^{-1} = m\big\}.$$

The members of $\mathcal{M}_m$ are called random measures with respect to m. Their disintegration measures μ_ω, $\omega \in \Omega$, with respect to the partition of X into sets $\{\omega\} \times \mathbb{C}$, are called fiber-wise random measures, and frequently, abusing slightly terminology, these are (also) called just random measures. We are interested in conformal random measures, their existence, uniqueness, and geometrical and dynamical properties. Such measures are characterized by the property that

$$\nu_{\theta\omega}(F_\omega(A)) = \lambda_{t,\omega} \int_A \left|(F_\omega)'\right|^t d\nu_\omega$$

for m–a.e. $\omega \in \Omega$ and for every Borel set $A \subset Q$ such that $F_\omega|_A$ is 1–to–1,where $\lambda_t : \Omega \to (0, +\infty)$ is some measurable function. Our first main result is about the existence of conformal random measures. Indeed, we proved the following.

Theorem 1 (Existence of conformal measures). *For every $t > 1$ there exists $\nu^{(t)}$, a random t–conformal measure, for the map $F : Q \to Q$.*

The proof of this theorem is much more involved than its deterministic counterpart of [41]; the whole Sections 5–8 are entirely devoted to this task. There are many reasons for that. One of them, notorious for random dynamics, is the difficulty to control upper and lower bounds of the measurable function λ_t. In the deterministic case there is just one number $e^{\mathrm{P}(t)}$. Here, we have an apriori uncontrolled function λ_t. We overcome this difficulty by starting of with good class of random measures: the sets $\mathcal{P}$ and $\hat{\mathcal{P}}$ of Sections 5–8. We also must carefully control the trajectories of 0, the singularity of f_η^{-1} for every $\eta \in [A, B]$ and points approaching these trajectories. This is the more difficult in the random case that we now have the trajectory of 0 for every $\omega \in \Omega$. There are more subtle and involved issues.

We then turned our attention to the problem of F-invariant random measures absolutely continuous with respect to the random conformal measure of Theorem 1. This was done in Section 9. Its full outcome is contained in the following.

Theorem 2. *For every $t > 1$ there exists a unique Borel probability F–invariant random measure $\mu^{(t)}$ absolutely continuous with respect to $\nu^{(t)}$, the random t–conformal measure of Theorem 1. In fact, $\mu^{(t)}$ is equivalent with $\nu^{(t)}$ and ergodic.*

Note that in terms of fiberwise invariant measures, F–invariance of the measure $\mu^{(t)}$ means that

$$\mu_\omega^{(t)} \circ F_\omega^{-1} = \mu_{\theta\omega}^{(t)}$$

for m–a.e. $\omega \in \Omega$.
Note that we do not claim that the measure $\mu^{(t)}$ is absolutely continuous with respect to any random t–conformal measure for the map F. We claim this only for the measure $\nu^{(t)}$ resulting from the proof of Theorem 1, i.e. Theorem 38. The proof of Theorem 2 is done "globally" and requires very subtle estimates of fiberwise random conformal measures of various balls and inverse images of measurable sets under all iterates.

Turning to geometry, we have defined random counterpart of radial (conical) Julia sets $J_r(\omega)$ and global radial Julia set $J_r(F)$. The precise definition is provided in Section 9 as Definition 55.

This definition of radial sets differs a little bit from the standard one. What we mean is that, when applied to deterministic systems, it produces the sets J_r that are different than, though contained in, those introduced in [39], comp. ex. [40], [41], [23], [32] and [24]. Therein one merely required that the sets $N_\omega(z, N)$ are infinite. For "truly random" systems we do however need such a more involved definition, the one which naturally matches with the random structure.

With HD denoting Hausdorff dimension, we proved in Section 10 the following theorem about the geometric structure of the random radial Julia sets $J_r(\omega)$.

Theorem 3. *For $t > 1$ put*

$$\mathcal{E}\mathrm{P}(t) := \int_\Omega \log \lambda_{t,\omega} dm(\omega).$$

Then

(1) $\mathcal{E}\mathrm{P}(t) < +\infty$ *for all* $t > 1$,

(2) *The function* $(1, +\infty) \ni t \mapsto \mathcal{E}\mathrm{P}(t)$ *is strictly decreasing, convex, and thus continuous,*

(3) $\lim_{t\to 1} \mathcal{E}\mathrm{P}(t) = +\infty$ *and* $\mathcal{E}\mathrm{P}(2)) \leq 0$.

(4) *(Bowen's formula) Let $h > 1$ be the unique value $t > 1$ for which $\mathcal{E}\mathrm{P}(t) = 0$. Then*

$$\mathrm{HD}(J_r(\omega)) = h$$

for m–a.e.$\omega \in \Omega$.

A remarkable fact of this theorem is that the Hausdorff dimension of random radial Julia sets $J_r(\omega)$, $\omega \in \Omega$, is expressed in terms (zero of the expected pressure $\mathcal{E}\mathrm{P}(t)$) that have nothing to do with these sets. Another remarkable observation about these sets, is their dynamical significance, which follows from the fact, which we proved, that

$$\mu(J_r(F)) = 1$$

for every F–invariant random measure on X.

As a matter of fact, we proved even more about geometry of random radial Julia sets $J_r(\omega)$ than Theorem 3. Namely:

Theorem 4. *The Hausdorff dimension $h = \mathrm{HD}(J_r(\omega))$ of the random radial Julia set $J_r(\omega)$, is constant for m–a.e. $\omega \in \Omega$ and satisfies $1 < h < 2$. In particular, the 2–dimensional Lebesgue measure of m–a.e. $\omega \in \Omega$ set $J_r(\omega)$ is equal to zero.*

As its, almost immediate, corollary, we obtain the following result about trajectories of (Lebesgue) typical points.

Theorem 5 (Trajectory of a (Lebesgue) typical point I)**.** *For m–almost every $\omega \in \Omega$ there exists a subset $Q_\omega \subset Q$ with full Lebesgue measure such that for all $z \in Q_\omega$ the following holds.*

$$\begin{aligned} &\forall \delta > 0 \ \exists n_z(\delta) \in \mathbb{N} \ \forall n \geq n_z(\delta) \ \exists k = k_n(z) \geq 0 \\ &\qquad |F^n_\omega(z) - F^k_{\theta^{n-k}\omega}(0)| < \delta \quad or \quad |F^n_\omega(z)| \geq 1/\delta. \end{aligned} \tag{1.1}$$

In addition, $\limsup_{n\to\infty} k_n(z) = +\infty$.

As an immediate consequence of this theorem we get the following.

Corollary 6 (Trajectory of a (Lebesgue) typical point II)**.** *For m–almost every $\omega \in \Omega$ there exists a subset $Q_\omega \subset Q$ with full Lebesgue measure such that for all $z \in Q_\omega$, the set of accumulation points of the sequence*

$$\left(F^n_\omega(z)\right)_{n=0}^\infty$$

is contained in $[0, +\infty] \cup \{-\infty\}$ *and contains* $+\infty$.

These last two properties are truly astonishing and were first time observed for the exponential map $\mathbb{C} \ni z \mapsto e^z \in \mathbb{C}$ in [31] and [20] and then extended to many other exponential functions in [41]. Our approach to establish these two properties is different than those of [31] and [20] and relies on investigation of h–dimensional packing measure Q.

As it is explained in detail in Sections 12 and Section 13, dealing with the sextuple

$$F := \big(\Omega, \mathcal{F}, m;\ \theta : \Omega \to \Omega;\ \eta : \Omega \to [A, B]; F_\eta : Q \to Q\big)$$

and induced by it random dynamics $\big(F_\omega^n : Q \longrightarrow Q\big)_{n=0}^{\infty}$ is entirely equivalent to dealing with the sextuple

$$f := \big(\Omega, \mathcal{F}, m;\ \theta : \Omega \to \Omega;\ \eta : \Omega \to [A, B]; f_\eta : \mathbb{C} \to \mathbb{C}\big)$$

and induced by it random dynamics $\big(f_\omega^n : \mathbb{C} \longrightarrow \mathbb{C}\big)_{n=0}^{\infty}$, if the derivatives of the maps f_ω^n are calculated with respect to the conformal Riemannian metric

$$|dz|/|z|.$$

This metric pops up naturally in Section 12 and coincides with the metric dealt with in [23] and [24]. In Sections 12 and 13 we fully transfer all the main results proven for the sextuple F to the case of the sextuple f.

2. Preliminaries

2.1. **The Quotient Cylinder and the Quotient Maps.** We define the equivalence relation $\sim$ on the complex plane $\mathbb{C}$ by saying that $Z \sim W$ if there exists $k \in \mathbb{Z}$ such that

$$Z - W = 2\pi i k.$$

We denote the quotient space $\mathbb{C}/\sim$ by Q. So, Q is conformally an infinite cylinder. We denote by π the natural projection $\pi : \mathbb{C} \to Q$, i.e., $\pi(Z) = [Z]$ is the equivalence class of Z with respect to relation $\sim$. Since both maps $f_\eta : \mathbb{C} \to \mathbb{C}$ and $\pi \circ f_\eta : \mathbb{C} \to Q$ are constant on equivalence classes, they canonically induce conformal maps $f_\eta : Q \to \mathbb{C}$ and $F_\eta : Q \to Q$. So, F_η can be represented as

$$F_\eta = \pi \circ f_\eta \circ \pi^{-1},$$

precisely meaning that for every point in Q, its image under $\pi \circ f_\eta \circ \pi^{-1}$ is a singleton and the above equality holds. Although, formally, Q is the set of equivalence classes $[z]$, we shall often use the notation $z \in Q$, whenever this does not lead to a confusion.

We will also use occasionally the natural identification

$$Q \sim \{Z \in \mathbb{C} : 0 \le \mathrm{Im} Z < 2\pi\},$$

when this does not lead to a confusion. For $z \in Q$ we denote

$$|z| := \inf\{|Z| : Z \in \pi^{-1}(z)\}.$$

Similarly, for $z \in Q$ we denote by $\mathrm{Re} z$ the common value $\mathrm{Re} Z$ for $Z \in \pi^{-1}(z)$.

We denote by Y_M the set

$$Y_M := \{z \in Q : |\mathrm{Re}(z)| > M\}.$$

This set splits naturally as $Y_M^+ \cup Y_M^-$,where

$$Y_M^+ := \{z \in Q : \mathrm{Re}(z) > M\} \quad \text{and} \quad Y_M^- := \{z \in Q : \mathrm{Re}(z) < -M\}.$$

We also denote:

$$Q_M := \{z \in Q : |\mathrm{Re} z| \le M\}.$$

For positive variables A, B, depending on a collection of parameters, we write $A \preceq B$ if there exists a constant C independent of the parameters such that

$$A \le C \cdot B.$$

Similarly, we write $A \succeq B$ if $B \preceq A$. We write

$$A \asymp B \quad \text{if and only if} \quad A \preceq B \ \text{ and } \ A \succeq B.$$

2.2. **Koebe's Distortion Theorems.** For every $\xi \in \mathbb{C}$ and every $r > 0$ let

$$B(\xi, r) := \{z \in \mathbb{C} : |z - \xi| < r\}$$

be the open disk (ball) centered at the point ξ with radius r. We abbreviate

$$\overline{B}(\xi, r) := \overline{B(\xi, r)}.$$

We record the following classical Koebe's distortion theorems; for proofs see e.g., [15].

Theorem (Koebe's Distortion Theorem)**.** *Let $\xi \in \mathbb{C}$ and let $r > 0$. Let $g : \overline{B}(\xi, r) \to \mathbb{C}$ be a univalent holomorphic map. Then for every $t \in [0, 1)$ and every $z \in \overline{B}(\xi, tr)$ we have*

$$\frac{1-t}{(1+t)^3} \le \frac{|g'(z)|}{|g'(\xi)|} \le \frac{1+t}{(1-t)^3},$$

$$\frac{tr}{(1+t)^2}|g'(\xi)| \le |g(z) - g(\xi)| \le \frac{tr}{(1-t)^2}|g'(\xi)|.$$

and

Theorem (Koebe's 1/4 Theorem)**.** *Let $\xi \in \mathbb{C}$ and let $r > 0$. If $g : \mathbb{D}(\xi, r) \to \mathbb{C}$ is a univalent holomorphic map, then*

$$g(\mathbb{D}(z_0, r)) \supset \mathbb{D}\left(g(\xi), \frac{1}{4}|g'(\xi)| \cdot r\right).$$

We shall often refer to these results as to *standard distortion estimates*. From now on throughout the paper, for every $t \in [0, 1)$ we set

$$K_t := \max\left\{\frac{1+t}{(1-t)^3}, \frac{(1+t)^3}{1-t}\right\} \ge 1$$

and

$$K := K_{1/2}.$$

We will often make use of Bloch's theorem (for a proof see e.g., [15]), which does not require the map to be univalent:

Theorem (Bloch's Theorem)**.** *Let $f\mathbb{D} : \mathbb{C}$ be a holomorphic map defined on the unit disc $\mathbb{D}$. If $|f'(0)| = 1$, then there is a region (open connected set) $U \subset \mathbb{D}$ which is mapped by f univalently onto a disc of radius $r \ge 1/72$.*

3. Julia Sets of Non–Autonomous Iterations of exponential Maps

As in the introduction, for $\eta \neq 0$ we denote by $f_\eta : \mathbb{C} \to \mathbb{C}$ the entire map defined by

$$f_\eta(z) = \eta e^z.$$

Fix two real numbers $A \le B$ with $A > 1/e$. Put

$$\mathbf{A} := [A, B]^{\mathbb{N}}.$$

For every infinite sequence of numbers in $[A, B]$, i.e., every element $\mathbf{a} = \{a_1, a_2, \ldots\}$ of the infinite product $[A, B]^{\mathbb{N}}$, define the non–autonomous dynamical system by the following formula:

$$f_{\mathbf{a}}^n := f_{a_n} \circ f_{a_{n-1}} \circ \cdots \circ f_{a_2} \circ f_{a_1} : \mathbb{C} \longrightarrow \mathbb{C}.$$

For every $\mathbf{a} \in \mathbf{A}$ the respective Fatou and Julia sets $F_{\mathbf{a}}$ and $J_{\mathbf{a}}$ are then defined analogously as in the deterministic case:

$$F_{\mathbf{a}} := \Big\{ z \in \mathbb{C} : f_{\mathbf{a}}^n|_U \text{ is normal for some neighborhood } U \text{ of } z \Big\}$$

and

$$J_{\mathbf{a}} := \mathbb{C} \setminus F_{\mathbf{a}}.$$

Denote by $\sigma : \mathbf{A} \longrightarrow \mathbf{A}$ the left shift, i.e., the map

$$\sigma(a_1, a_2, a_3 \ldots) = (a_2, a_3, a_4 \ldots).$$

Note that both these sets $F_{\mathbf{a}}$ and $J_{\mathbf{a}}$ are invariant by the dynamics. More precisely:

$$f_{\mathbf{a}}^1(J_{\mathbf{a}}) = f_{a_1}(J_{\mathbf{a}}) \subset J_{\sigma(\mathbf{a})} \quad \text{and} \quad f_{\mathbf{a}}^1(F_{\mathbf{a}}) = f_{a_1}(F_{\mathbf{a}}) \subset F_{\sigma(\mathbf{a})}.$$

Our next theorem extends to the non–autonomous case the celebrated result of Michał Misiurewicz (see [29]) which was conjectured by Pierre Fatou already in 1926 (see [14]). The proof we provide is simple and it constitutes a substantial simplification also for deterministic maps.

Theorem 7. *For every* $\mathbf{a} \in \mathbf{A}$*, we have that*

$$J_{\mathbf{a}} = \mathbb{C}.$$

The proof of Theorem 7 will consist of several lemmas.

Lemma 8. *For every* $\mathbf{a} \in \mathbf{A}$*,*

$$J_{\mathbf{a}} \supset \mathbb{R}.$$

Proof. First, observe that if $x \in \mathbb{R}$, then

$$\lim_{n\to\infty} f_{\mathbf{a}}^n(x) \to +\infty.$$

Now, if $w \in \mathbb{R} \setminus J_{\mathbf{a}}$, then there exists a neighborhood $V \subset \mathbb{C}$ of the point w in $\mathbb{C}$ such that the family $\left(f_{\mathbf{a}}^n|_V\right)_{n=0}^{\infty}$ is normal. So, since also $f_{\mathbf{a}}^n|_{\mathbb{R}\cap V} \to \infty$, as $n \to \infty$, we conclude that $f_{\mathbf{a}}^n$ converges to infinity uniformly on compact subsets of V as $n \to \infty$. Remember that for this specific family f_η we have $f_\eta = f_\eta'$. So, if $\overline{B}(w, r) \subset V$, then

$$\big|(f_{\mathbf{a}}^n)'\big|\big|_{B(w,r)} \to \infty$$

uniformly as $n \to \infty$. Thus, by virtue of Bloch's Theorem, the image $f_{\mathbf{a}}^n(B(w,r))$ contains a ball of radius 2π for all $n \geq 0$ sufficiently large. This implies that there exists a sequence of points $z_n \in B(w,r)$, $n \geq 0$ large enough, such that

$$\lim_{n\to\infty} \left|\mathrm{Re}(f_{\mathbf{a}}^n(z_n))\right| = +\infty,$$

and

$$\mathrm{Im} f_{\mathbf{a}}^n(z_n)) \in \pi + 2\pi\mathbb{Z}.$$

Then $f_{\mathbf{a}}^{n+1}(z_n) \in (-\infty, 0)$ and, consequently, $|f_{\mathbf{a}}^{n+2}(z_n)| < B$, where, we recall, B is the number fixed, along with A, at the beginning of this section. Thus, $f_{\mathbf{a}}^n|_{B(w,r)}$ does not tend to infinity as $n \to \infty$. This contradiction finishes the proof. □

As an immediate consequence of this lemma we get the following.

Corollary 9. *If $V \subset \mathbb{C}$ is an open set and $V \cap J_{\mathbf{a}} = \emptyset$, then $V \cap \mathbb{R} = \emptyset$. Furthermore,*

(1)
$$\mathbb{R} \cap \bigcup_{n=0}^{\infty} f_{\mathbf{a}}^n(V) = \emptyset,$$
and, more generally,

(2)
$$\left(\bigcup_{k\in\mathbb{Z}} \mathbb{R} + k\pi i\right) \cap \bigcup_{n=0}^{\infty} f_{\mathbf{a}}^n(V) = \emptyset.$$

The next lemma and its proof come as minor modifications from [29].

Lemma 10. *For every $z \in \mathbb{C}$ and every integer $n \geq 1$,*
$$|(f_{\mathbf{a}}^n)'(z)| \geq |\mathrm{Im} f_{\mathbf{a}}^n(z)|.$$

Proof. $f_\eta(z) = \eta e^z = \eta e^x \cos y + i\eta e^x \sin y$. Since $|\sin y| \leq |y|$, we thus have that $|\mathrm{Im} f_\eta(z)| \leq \eta e^x |y| = |f_\eta(z)||\mathrm{Im}(z)|$. So,

$$\frac{|\mathrm{Im} f_\eta(z)|}{|\mathrm{Im}(z)|} \leq |f_\eta(z)|. \tag{3.1}$$

Therefore,

$$\begin{aligned}
|\mathrm{Im} f_{\mathbf{a}}^n(z)| &= \frac{|\mathrm{Im} f_{\mathbf{a}}^n(z)|}{|\mathrm{Im} f_{\mathbf{a}}^{n-1}(z)|} \cdot \frac{|\mathrm{Im} f_{\mathbf{a}}^{n-1}(z)|}{|\mathrm{Im} f_{\mathbf{a}}^{n-2}(z)|} \cdot \ldots \cdot \frac{|\mathrm{Im} f_{\mathbf{a}}^2(z)|}{|\mathrm{Im} f_{\mathbf{a}}(z)|} \cdot |\mathrm{Im} f_{\mathbf{a}}(z)| \\
&\leq |f_{\mathbf{a}}^n(z)| \cdot |f_{\mathbf{a}}^{n-1}(z)| \cdot \ldots \cdot |f_{\mathbf{a}}^2(z)| \cdot |\mathrm{Im} f_{\mathbf{a}}(z)| \\
&\leq |f_{\mathbf{a}}^n(z)| \cdot |f_{\mathbf{a}}^{n-1}(z)| \cdot \ldots \cdot |f_{\mathbf{a}}^2(z)| \cdot |f_{\mathbf{a}}(z)| \\
&= |(f_{\mathbf{a}}^n)'(z)|.
\end{aligned}$$

□

Remark 1. The above computation, although very simple, reflects the following phenomenon: Denoting by $\mathbb{H}^+$ and $\mathbb{H}^-$, respectively, the upper and lower half-plane, we see that the branches f_η^{-1} of the inverse map are well-defined in $\mathbb{H}^+$ and $\mathbb{H}^-$, and each of them map $\mathbb{H}^\pm$ into $\mathbb{H}^+$ or $\mathbb{H}^-$. Since the hyperbolic metric in $\mathbb{H}^\pm$ is given by $\frac{|dz|}{|\mathrm{Im}(z)|}$, the inequality (3.1) just expresses the fact that f_η^{-1} are contractions in the hyperbolic metric.

Lemma 11. *If $V \subset \mathbb{C}$ is an open connected set and $V \subset \overline{V} \subset \mathbb{C} \setminus J_{\mathbf{a}}$, then there exists an integer $N \geq 0$ such that for all $n \geq N$,*

$$f_{\mathbf{a}}^n(V) \subset S := \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \pi\}.$$

Proof. By Corollary 9, for every $n \in \mathbb{N}$, either the set $f^n(V)$ is contained in S, or it is disjoint from S. If $f_{\mathbf{a}}^n(V) \cap S = \emptyset$ for infinitely many integers $n \geq 1$ then, using Lemma 10 and the Chain Rule, we conclude that

$$\limsup_{n\to\infty} |(f_{\mathbf{a}}^n)'|_{|V} = +\infty.$$

This (using e.g. Bloch's Theorem) implies that for infinitely many integers $n \geq 1$ the set $f_{\mathbf{a}}^n(V)$ contains a ball of radius 2π. So, for all such n, $f_{\mathbf{a}}^n(V) \cap \left(\bigcup_{k\in\mathbb{Z}} \mathbb{R} + k\pi i\right) \neq \emptyset$. This however contradicts Corollary 9, and we are done. □

Write S as $S = S^+ \cup S^- \cup \mathbb{R}$, where

$$S^+ := \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \pi\} \text{ and } S^- := \{z \in \mathbb{C} : -\pi < \mathrm{Im}(z) < 0\}.$$

For $\mathbf{a} \in \mathbf{A}$ denote by $g_{\mathbf{a}}$ the holomorphic branch of $f_{\mathbf{a}}^{-1}$ defined on S^+ and mapping S^+ into S^+. More generally, for every $\eta \in [A, B]$, the map g_η denotes the holomorphic branch of f_η^{-1} mapping S^+ into S^+. Denote by ρ the hyperbolic metric in S^+.

Lemma 12. *For every $\eta \in [A, B]$ and for all $z, w \in S^+$, we have that*

$$\rho(g_\eta(z), g_\eta(w)) \leq \rho(z, w). \tag{3.2}$$

Also, for every compact subset $K \subset S^+$ there exists $\kappa \in (0, 1)$ such that for every $\eta \in [A, +\infty)$ and for all $z, w \in K$, we have that

$$\rho(g_\eta(z), g_\eta(w)) \leq \kappa\rho(z, w). \tag{3.3}$$

Proof. The formula (3.2) is a straightforward consequence of Schwarz Lemma. Since the map $g_\eta : S^+ \to S^+$ is not bi–holomorphic, it also follows from Schwarz Lemma that

$$\rho(g_\eta(z), g_\eta(w)) < \rho(z, w) \tag{3.4}$$

whenever $z, w \in S^+$ and $z \neq w$, and in addition,

$$\limsup_{\substack{z,w\to\xi \\ z\neq w}} \frac{\rho(g_\eta(z), g_\eta(w)}{\rho(z, w)} < 1 \tag{3.5}$$

for every $\xi \in S^+$. In order to prove (3.3), fix $\eta_2 > \eta_1 \geq A$. Since $g_{\eta_2}(z) = g_{\eta_1}(z) - \log \frac{\eta_2}{\eta_1}$ and $g_{\eta_2}(w) = g_{\eta_1}(w) - \log \frac{\eta_2}{\eta_1}$, and since the metric ρ is invariant under the horizontal translation, we have

$$\rho(g_{\eta_2}(z), g_{\eta_2}(w)) = \rho(g_{\eta_1}(z), g_{\eta_1}(w)).$$

So, it is enough to check the estimate (3.3) for f_A. But this follows immediately from (3.4), (3.5), and compactness of the set K. □

Lemma 13 below will complete the proof of Theorem 7.

Lemma 13. *The interior of the set*

$$\Lambda := \bigcap_{n=0}^{\infty} f_{\mathbf{a}}^{-n}(S)$$

is empty.

Proof. Since

$$f_{\mathbf{a}}(S^+) = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}, \;\; f_{\mathbf{a}}(S^-) = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\},$$

and

$$f_{\mathbf{a}}(\mathbb{R}) = (0, +\infty),$$

it follows that

$$\bigcap_{n=0}^{\infty} f_{\mathbf{a}}^{-n}(S) = \bigcap_{n=0}^{\infty} f_{\mathbf{a}}^{-n}(S^+) \cup \bigcap_{n=0}^{\infty} f_{\mathbf{a}}^{-n}(S^-) \cup \mathbb{R}.$$

We shall prove that the set $\bigcap_{n=0}^{\infty} f_{\mathbf{a}}^{-n}(S^+)$ has empty interior; the set $\bigcap_{n=0}^{\infty} f_{\mathbf{a}}^{-n}(S^-)$ can be dealt with in the same way.

So, suppose that there exists $V \subset \mathbb{C}$, a nonempty, open, connected, bounded with

$$V \subset \overline{V} \subset \bigcap_{n=0}^{\infty} f_{\mathbf{a}}^{-n}(S^+).$$

Then, obviously, the family $\left(f_{\mathbf{a}}^n|_V\right)_{n=0}^{\infty}$ is normal. Now, fix a non-empty open connected set W (e.g.: a disk) contained, together with its closure, in V. Put

$$\delta := \mathrm{dist}(W, \partial V) > 0.$$

Let $N \geq 1$ be so large integer that

$$\left(\frac{\pi}{2}\right)^N \cdot \frac{\delta}{72} > 2\pi.$$

Now, seeking a contradiction, assume that there exists $\xi \in W$ such that for at least N integers $n_1, \dots, n_N \geq 0$ we have that

$$f_\omega^n(\xi) \in \{z \in \mathbb{C} : \mathrm{Im} z > \pi/2\}.$$

Then $|(f_\omega^{n_N})'(\xi)| > (\pi/2)^N$, and again Bloch's Theorem implies that $f_\omega^n(W)$ contains some ball of radius 2π. Since $f^{n_N}(W)$ does not intersect the Julia set $J_{\theta^n \mathbf{a}}$, this is a contradiction, as $J_{\theta^n \mathbf{a}} \supset \mathbb{R} + 2\pi i \mathbb{Z}$.

We therefore conclude that the trajectory $f_{\mathbf{a}}^n(z)$ of every point $z \in W$ visits the domain $\{z \in \mathbb{C} : \mathrm{Im} z > \pi/2\}$ at most N times. For every integer $k \geq 0$ let

$$W_k := W \cap \bigcap_{n=k}^{\infty} f_{\mathbf{a}}^{-n}\big(\{z \in \mathbb{C} : \mathrm{Im} z \leq \pi/2\}\big).$$

Each set W_k, $k \geq 0$, is closed (with respect to the relative topology) in W and, as we have just proved, that

$$\bigcup_{k=0}^{\infty} W_k = W.$$

Since W is an open subset of $\mathbb{C}$, it is completely metrizable, and therefore the Baire Category Theorem holds for it. Thus, there exits $q_1 \geq 0$ such that

$$W^* := \text{Int}_{\mathbb{C}}(W_k) \neq \emptyset.$$

This means that for all integers $n \geq q_1$, we have that

$$f_{\mathbf{a}}^n(W^*) \subset \{z \in \mathbb{C} : \text{Im} z \leq \pi/2\}.$$

Since all sets $f_{\mathbf{a}}^n(W^*)$, $n \geq 0$, are open in $\mathbb{C}$ and contained in S^+, we thus conclude that

$$f_{\mathbf{a}}^n(W^*) \subset \{z \in \mathbb{C} : 0 < \text{Im} z < \pi/2\} \tag{3.6}$$

for all integers $n \geq q_1$. Consequently,

$$f_{\mathbf{a}}^n(W^*) \subset \{z \in \mathbb{C} : \text{Re} z > 0\}$$

for all $n \geq q_1 + 1$. Moreover, observe that there exists a constant $M > 0$ such that, if $\text{Re} z \geq M$, $\text{Im} z \in (0, \pi/2)$, and $f_\eta(z) \in S$, $\eta \in [A, B]$, then

$$\text{Re} f_\eta(z) > \text{Re} z + 1.$$

This, in turn, implies that if $f_{\mathbf{a}}^j(W^*) \cap \{z \in \mathbb{C} : \text{Re} z \geq M\} \neq \emptyset$ for some integer $j \geq q_1 + 1$, then the sequence $\left(f_\omega^n|_{W^*}\right)_{n=q_1+1}^\infty$ converges uniformly to ∞. But since then the sequence $\left((f^n)'|_{W^*}\right)_{n=q_1+1}^\infty$ also converges uniformly to ∞, this possibility is again excluded by the conjunction of Bloch's Theorem and (3.6).

So, we have concluded that

$$f_\omega^n(W^*) \subset \{z \in \mathbb{C} : 0 < \text{Re} z < M \text{ and } 0 < \text{Im} z < \pi/2\}$$

for all integers $n \geq q_1$. Since the family $\left(f_\omega^n|_{W^*}\right)_{n=q_1+1}^\infty$ is normal, its every subsequence contains a subsequence converging uniformly in W^* to some limit holomorphic function. Since all the maps $f_\omega^n|_{W^*}$, $n \geq q_1 + 1$, expand the hyperbolic metric ρ, there are no constant limits of subsequences $\left(f_{\mathbf{a}}^{n_k}|_{W^*}\right)_{k=1}^\infty$ with values in S^+.

So, let g be a non-constant limit of some subsequence $\left(f_{\mathbf{a}}^{n_k}|_{W^*}\right)_{k=1}^\infty$ converging uniformly. Shrinking W^* if necessary, one can assume that $g(W^*)$ is contained in some compact subset $K \subset S^+$. Putting

$$\tilde{K} := \{z \in S^+ : \rho(z, K) \leq 1\},$$

we see that there is $q_2 \geq q_1 + 1$ such that for every $k \geq q_2$

$$f_\omega^{n_k}(W^*) \subset \tilde{K}.$$

Note that $\tilde{K}$ has finite hyperbolic diameter, in fact is compact, and put $D := \text{diam}_\rho(\tilde{K}) < \infty$. Record that for all $k > q_2$, we have that

$$f_{\mathbf{a}}^{n_k} = f_{\theta^{n_{k-1}}\mathbf{a}}^{n_k - n_{k-1}} \circ \cdots \circ f_{\theta^{n_{q_2}}\mathbf{a}}^{n_{q_2+1} - n_{q_2}} \circ f_{\mathbf{a}}^{n_{q_2}}.$$

Let $z, w \in W^*$ with $z \neq w$. Then, using (3.3) and (3.2), we see that $\rho(z, w) \leq \kappa^{k-q_2} D$ for every $k \geq q_2$, which is a contradiction.

So, the sequence $\left(f_{\mathbf{a}}^n|_{W^*}\right)_{n=0}^\infty$ has no subsequence with a non–constant limit.

Since all limit functions of subsequences of the sequence $\left(f_{\mathbf{a}}^n|_{W^*}\right)_{n=0}^\infty$ with values in S^+ have been also already excluded, we arrive at the following conclusion:

For every $\theta > 0$ there exists an integer $n_\theta \geq 0$ such that
$$f_{\mathbf{a}}^n(W^*) \subset \{z \in \mathbb{C} : 0 < \mathrm{Im} z < \theta\} \cap \{z \in \mathbb{C} : 0 < \mathrm{Re} z < M\}.$$
for every $n \geq n_\theta$.

In order to complete the proof, we now shall show that the above is impossible. This can be deduced immediately from the following lemma. Its proof is an easy calculation and will be omitted.

Lemma 14. *Let $\delta > 0$ be so small that $(1-\delta) > \frac{1}{Ae}$. Then for every $\eta \geq A$ and for every $z \in \mathbb{C}$ with $\cos \mathrm{Im} z > 1-\delta$, we have that*
$$\mathrm{Re} f_\eta(z) > \mathrm{Re}(z) + Ae(1-\delta).$$
In particular, the map f_η moves the region $\{z \in S^+ : \cos \mathrm{Im} z > 1-\delta)\}$ by ε to the right.

□

4. Random Exponential Dynamics and Random Measures in Q

As in [1], [11], [22], [26], and [27] the randomness is modeled by a measure preserving invertible dynamical system
$$\theta : \Omega \to \Omega,$$
where $(\Omega, \mathcal{F}, m)$ is a complete probability measurable space, and θ is a measurable invertible map, with θ^{-1} measurable, preserving the measure m. As in the previous section, fix some real constants A, B with $A > 1/e$. Let
$$\eta : \Omega \longmapsto [A, B]$$
be a measurable function. Furthermore, to each point $\omega \in \Omega$ associate the exponential map
$$f_\omega := f_{\eta(\omega)} : \mathbb{C} \longrightarrow \mathbb{C}$$
given by the formula
$$f_\omega(z) = \eta(\omega) e^z.$$
Consequently, for every $z \in \mathbb{C}$, the map
$$\Omega \ni \omega \longmapsto f_{\eta(\omega)}(z) \in \mathbb{C}$$
is measurable.

We consider the dynamics of random iterates of exponential functions:
$$f_\omega^n := f_{\theta^{n-1}\omega} \circ \cdots \circ f_{\theta\omega} \circ f_\omega : \mathbb{C} \longrightarrow \mathbb{C}.$$
The quintuple
$$\big(\Omega, \mathcal{F}, m;\, \theta : \Omega \to \Omega;\, \eta : \Omega \to [A, B]\big)$$
and induced by it random dynamics
$$\big(f_\omega^n : \mathbb{C} \to \mathbb{C}\big)_{n=0}^\infty, \quad \omega \in \Omega,$$
will be referred to in the sequel as *random exponential dynamical system*. As we have explained it in the introduction, we will in fact do all of our investigations for the maps projected to the cylinder Q. More precisely, for every $\omega \in \Omega$, we consider the map
$$F_\omega = \pi \circ f_\omega \circ \pi^{-1},$$

and the corresponding random dynamical system

$$F_\omega^n := F_{\theta^{n-1}\omega} \circ \cdots \circ F_{\theta\omega} \circ F_\omega : Q \longrightarrow Q.$$

As it was indicated in the introduction, and explained in detail in Section 12, and in Section 13, this is entirely equivalent to dealing with the random dynamical system (f_ω^n) with derivatives calculated with respect to the conformal Riemannian metric

$$\frac{|dz|}{|z|}.$$

Recall from [11] that a function $g : \Omega \times Q \to \mathbb{C}$, $g(\omega, z) = g_\omega(z)$, is called a random continuous function if, for every $\omega \in \Omega$ the function

$$Q \ni z \longmapsto g_\omega(z) \in \mathbb{C}$$

is continuous and bounded, and, in addition, for every $z \in Q$ the function

$$\Omega \ni \omega \longmapsto g(\omega, z) \in \mathbb{C}$$

is measurable. It then follows (see, e.g., Lemma 1.1 in[11]) that every random continuous function is measurable with respect to the product σ–algebra $\mathcal{F} \otimes \mathcal{B}$, where $\mathcal{B}$ is the Borel σ–algebra in Q. Moreover, the map

$$\Omega \ni \omega \longmapsto \|g_\omega\|_\infty \in \mathbb{R}$$

is measurable and, $m-$ integrable. The vector space of all real–valued random continuous functions is denoted by $C_b(\Omega \times Q)$. Equipped with the norm

$$\|g\| := \int_\Omega \|g_\omega\|_\infty dm(\omega)$$

it becomes a Banach space.

The simplest example of such a random map is obtained just by putting $\Omega := \Pi_{-\infty}^{\infty}[A, B]$, equipped with some (completed) product measure, and putting, for $\omega = (\ldots \eta_{-1}, \eta_0, \eta_1 \ldots)$ $\eta(\omega) := \eta_0$.

Put

$$X := \Omega \times Q.$$

Denote by $\mathcal{M}(X)$ the space of all those signed measures ν defined on the σ-algebra $\mathcal{F} \otimes \mathcal{B}$ for which

$$\|\nu\|_\infty := \text{esssup}\{|\nu_\omega| : \omega \in \Omega\} < +\infty,$$

where ν_ω, $\omega \in \Omega$, is the corresponding disintegration of ν and, for each $\omega \in \Omega$ the number $|\nu_\omega|$ is the total variation norm of ν_ω.

These measures, i.e. the members of $\mathcal{M}(X)$, can be canonically identified with linear continuous functionals on the Banach space $C_b(\Omega \times Q)$.

Let $\pi_1 : X \to \Omega$ be the projection onto the first coordinate, i.e.,

$$\pi_1(\omega, z) = \omega.$$

Let $\mathcal{M}_m \subset \mathcal{M}(X)$ be the set of all non-negative probability measures on X that project onto m under the map $\pi_1 : X \to \Omega$, i.e.

$$\mathcal{M}_m = \{\mu \in \mathcal{M}(X) : \mu \circ \pi_1^{-1} = m\}.$$

A map $\mu : \Omega \times \mathcal{B} \to [0,1]$, $(\omega, B) \longmapsto \mu_\omega(B)$, is called a random probability measure on Q if

- For every set $B \in \mathcal{B}$ the function $\Omega \ni \omega \longmapsto \mu_\omega(B) \in [0,1]$ is measurable,
- For m-almost every $\omega \in \Omega$ the map $\mathcal{B} \ni B \mapsto \mu_\omega(B) \in [0,1]$ is a Borel probability measure.

A random measure μ will be frequently denoted as $\{\mu_\omega\}_{\omega\in\Omega}$ or $\{\mu_\omega : \omega \in \Omega\}$.

The set $\mathcal{M}_m(X)$ can be canonically identified with the collection of all random probability measures on Q as follows.

Proposition 15 (see Propositions 3.3 and 3.6 in[11])**.** *With the above notation, for every measure $\mu \in \mathcal{M}_m(X)$ there exists a unique random measure $\{\mu_\omega\}_{\omega\in\Omega}$ on Q such that*

$$\int_{\Omega\times Q} h(\omega, z)\, d\mu(\omega, z) = \int_\Omega \left(\int_Q h(\omega, z)\, d\mu_\omega(z) \right) dm(\omega)$$

for every bounded measurable function $h : \Omega \times Q \to \mathbb{R}$.

Conversely, if $\{\mu_\omega\}_{\omega\in\Omega}$ is a random measure on Q, then for every bounded measurable function $h : \Omega \times Q \to \mathbb{R}$ the function $\Omega \ni \omega \longmapsto \int_Q h(\omega, z) d\mu_\omega(z)$ is measurable, and the assignment

$$\mathcal{F} \otimes \mathcal{B} \ni A \longmapsto \int_\Omega \int_Q \mathbb{1}_A(\omega, z) d\mu_\omega(z) dm(\omega),$$

defines a probability measure $\mu \in \mathcal{M}_m(Q)$.

Both sets $\mathcal{M}(X)$ are $\mathcal{M}_m$ are equipped in [11] with a topology called therein as a narrow topology. This topology is on $\mathcal{M}(X)$ generated by the following local bases of neighborhoods of elements $\nu \in \mathcal{M}(X)$:

$$U_{g_1,\dots g_k;\delta}(\nu) := \left\{ \mu \in \mathcal{M}(X) : \left| \int g_j d\mu - \int g_j d\nu \right| < \delta \right\},$$

where $g_1, \dots g_k$ is an arbitrary collection of random continuous functions and δ is some positive number. The space $\mathcal{M}_m$ is then endowed with the subspace topology of the narrow topology on $\mathcal{M}(X)$. This topology is in general non–metrizable neither on $\mathcal{M}(X)$ nor on $\mathcal{M}_m$.

A subset $\mathcal{R} \subset \mathcal{M}_m$ is said to be tight if for every $\varepsilon > 0$ there exists $M > 0$ such that for every $\nu \in \mathcal{R}$ we have that

$$\nu(\Omega \times Q_M) \geq 1 - \varepsilon.$$

We recall Theorem 4.4 in [11]:

Theorem 16 (Crauel's Prokhorov Compactness Theorem)**.** *A set $\mathcal{R} \subset \mathcal{M}_m$ is tight if an only if it is relatively compact with respect to the narrow topology. In this case, $\mathcal{R}$ is also relatively sequentially compact.*

5. Random Conformal Measures for Random Exponential Functions; a Preparatory Step

In this section, after short preparation, we define random t–conformal measures, and our main goal in it is to prove their existence for every $t > 1$. In order to do this we introduce a subspace of random measures for our random dynamics of exponentials. After defining a properly chosen convex and compact subset $\mathcal{P} \subset \mathcal{M}_m$, with respect to the narrow topology, we will check that this set is invariant under an appropriate continuous map. The existence of a random conformal measure will be then deduced from the Schauder–Tichonov Fixed Point Theorem.

In this section, and in the next Sections 6, 7, 8, 9, we work with an arbitrary, but fixed, $t > 1$. So, the space $\mathcal{P}$ and the estimates depend on t.

Definition 17. We define a family of operators $\mathcal{L}_{t,\omega}$, $t > 1$, $\omega \in \Omega$, by

$$\mathcal{L}_{t,\omega}(g)(z) := \sum_{w \in F_\omega^{-1}(z)} g(w) \cdot |F_\omega'(w)|^{-t} \in \mathbb{R},$$

where $g : Q \to \mathbb{R}$ ranges over bounded continuous functions. Note that the series above converges indeed since $t > 1$; this is not difficult to check and can be done in exactly the same way as in [41], in fact it can be seen immediately from the formula (5.2) below.

Furthermore, we define the global transfer operator $\mathcal{L}_t$ on the space $C_b(\Omega \times Q)$ as follows: for $(\omega, z) \in X = \Omega \times Q$ and a random continuous function g, we put

$$(\mathcal{L}_t)_\omega g(z)) := \mathcal{L}_{t,\theta^{-1}\omega}(g_{\theta^{-1}\omega})(z).$$

Note that $\mathcal{L}_t$ does not act on the space $C_b(\Omega \times Q)$, i.e. its image is not contained in $C_b(\Omega \times Q)$. The point is that for each $\omega \in \Omega$ the function $\mathcal{L}_{t,\omega}(\mathbb{1})$ is unbounded. However, we shall check that for each random continuous function $g : X \to \mathbb{R}$ and suitably chosen family of random measures ν, the integral

$$\int \mathcal{L}_{t,\omega}(g_\omega) d\nu_{\theta\omega}$$

is well defined. This will follow from integrability of the functions

$$Q \ni z \longmapsto \mathcal{L}_{t,\omega}(\mathbb{1})(z) \in \mathbb{R},$$

$\omega \in \Omega$, with respect to the measures $\nu_{\theta\omega}$. Verifying this will allow us to define formally the measures $\mathcal{L}_{t,\omega}^*\nu_{\theta\omega}$, $\omega \in \Omega$, as

$$\mathcal{L}_{t,\omega}^*\nu_{\theta\omega}(g) := \int \mathcal{L}_{t,\omega} g_\omega d\nu_{\theta\omega}.$$

The random measure $\left(\nu_\omega\right)_{\omega\in\Omega}$ is then said to be t–conformal if

$$\mathcal{L}_{t,\omega}^*(\nu_{\theta\omega}) = \lambda_{t,\omega}\nu_\omega$$

for m–a.e. $\omega \in \Omega$, where $\lambda_t : \Omega \to (0, +\infty)$ is some measurable function. A straightforward calculation shows that t–conformality is also characterized by the property that

$$\nu_{\theta\omega}(F_\omega(A)) = \lambda_{t,\omega} \int_A \left|\left(F_\omega\right)'\right|^t d\nu_\omega$$

for m–a.e. $\omega \in \Omega$ and for every Borel set $A \subset Q$ such that $F_\omega|_A$ is 1–to–1, where $\lambda_t : \Omega \to (0, +\infty)$ is some measurable function.

Our task now, in the upcoming sections, is to prove the existence of random t–conformal measures for every $t > 1$. Let $\mathcal{P} \subset \mathcal{M}_m$. We want to define a map $\Phi : \mathcal{P} \to \mathcal{M}_m$ by the following formula/requirement:

$$(\Phi(\nu))_\omega := \frac{\mathcal{L}^*_{t,\omega}(\nu_{\theta\omega})}{\mathcal{L}^*_{t,\omega}(\nu_{\theta\omega})(1\!\!1)}, \tag{5.1}$$

i.e., the measure $\Phi(\nu)$ is the only measure in $\mathcal{M}_m$, with disintegration $\Phi(\nu)_\omega$ given by (5.1). We look for a sufficient condition under which the map Φ is well defined on $\mathcal{P}$. We first prove a technical lemma and then provide such sufficient condition in Proposition 19 following it.

Lemma 18. *Fix $\varepsilon > 0$ arbitrary. Let $\mathcal{C}_\varepsilon \subset C_b(\Omega \times Q)$ be the set of all random continuous functions defined on $\Omega \times Q$ that vanish in*

$$\Omega \times \{z \in Q : \mathrm{Re}(z) < \log \varepsilon\}.$$

Then

$$\mathcal{L}_t g \in C_b(\Omega \times Q)$$

for each $g \in \mathcal{C}_\varepsilon$.

Proof. In order to prove that $\mathcal{L}_t g \in C_b(\Omega \times Q)$, we need to check

- measurability of the function $\Omega \ni \omega \longmapsto \mathcal{L}_{t,\omega}(g_\omega)(z)$, with fixed $z \in Q$,
- continuity of the function $Q \ni z \longmapsto \mathcal{L}_{t,\omega}(g_\omega)(z)$ with fixed $\omega \in \Omega$, and finally,
- the bound

$$\int_\Omega \|\mathcal{L}_{t,\omega}(g_\omega)\|_\infty dm(\omega) < \infty.$$

Recall the definition:

$$\mathcal{L}_{t,\omega}(g_\omega)(z) = \sum_{w \in F_\omega^{-1}(z)} g_\omega(w) \cdot |F'_\omega(w)|^{-t}.$$

The preimages $w \in F_\omega^{-1}(z)$ can be easily calculated, using the equation $\eta(\omega)\exp(w_k) = z + 2k\pi i$, so,

$$w_k = w_k(\omega) = \mathrm{Log}\left(\frac{z + 2k\pi i}{\eta(\omega)}\right)$$

where we denoted by $\mathrm{Log}(Z)$ the only $W \in Q$ such that $\exp(W) = Z$.

With z fixed, the measurability with respect to ω is now easily seen from the above explicit formula. Note also, that we can write even more explicitly:

$$\mathcal{L}_{t,\omega}(g_\omega)(z) = \sum_{w_k(\omega)} g_\omega(w_k) \cdot \left|\frac{\eta(\omega)}{z + 2k\pi i}\right|^t. \tag{5.2}$$

Since $t > 1$, the above series of continuous functions converges uniformly in a neighborhood of any point $z \in Q$, $z \neq 0$, thus defining a continuous function. It remains to prove continuity at 0. But, since we assumed that $g \in \mathcal{C}_\varepsilon$, it follows that in a sufficiently small neighborhood of $z = 0$ the summand corresponding to the integer $k = 0$ vanishes, and the sum in (5.2) is taken only over all $k \neq 0$; then the previous argument, i.e. the one for points $z \neq 0$ applies.

Finally, the formula (5.2) also shows that in some neighborhood U_ε of 0 we have the following bound:

$$|\mathcal{L}_{t,\omega}(g_\omega)(z)| \leq \sum_{k\in\mathbb{Z},k\neq 0} \left|\frac{\eta(\omega)}{z+2k\pi i}\right|^t \cdot ||g_\omega||_\infty$$

while, outside U_ε,

$$|\mathcal{L}_{t,\omega}(g_\omega)(z)| \leq \sum_{k\in\mathbb{Z}} \left|\frac{\eta(\omega)}{z+2k\pi i}\right|^t \cdot ||g_\omega||_\infty$$

Thus, there exists a constant D_ε such that

$$\|\mathcal{L}_{t,\omega}(g_\omega)\|_\infty \leq D_\varepsilon \cdot ||g_\omega||_\infty.$$

We conclude that $\mathcal{L}_t g = (\mathcal{L}_{t,\omega}(g_\omega))_{\omega\in\Omega}$ is a random continuous function. □

Proposition 19. *Let $\mathcal{P} \subset \mathcal{M}_m$. Assume that there exist $\rho > 0$ and a monotone increasing continuous function $\varphi : (0, \rho) \to [0, +\infty)$ such that $\lim_{\varepsilon\to 0^+} \varphi(\varepsilon) = 0$ and for each $\nu \in \mathcal{P}$, every $\omega \in \Omega$ and $\varepsilon \in (0, \rho)$ we have that*

$$\int_{B(0,\varepsilon)} \mathcal{L}_{t,\omega}(\mathbb{1})(z)\, d\nu_{\theta\omega}(z) \leq \varphi(\varepsilon). \tag{5.3}$$

Assume also that there are constants $P \geq p > 0$ such that

$$p \leq \int \mathcal{L}_{t,\omega}(\mathbb{1})(z)\, d\nu_{\theta\omega}(z) \leq P \tag{5.4}$$

for all $\nu \in \mathcal{P}$ and each $\omega \in \Omega$.

Then, the map $\Omega \ni \omega \longmapsto \mathcal{L}^*_{t,\omega}\nu_{\theta\omega}$, given by the formula

$$\mathcal{L}^*_{t,\omega}\nu_{\theta\omega}(g) := \nu_{\theta\omega}\big(\mathcal{L}_{t,\omega}g\big), \tag{5.5}$$

where $g \in C_b(Q)$, is well defined, making $\mathcal{L}^*_{t,\omega}\nu_{\theta\omega}$ a finite Borel measure on Q. Moreover, the global measure

$$\mathcal{L}^*_t\nu := \big(\mathcal{L}^*_{t,\omega}\nu_{\theta\omega}\big)_{\omega\in\Omega} \tag{5.6}$$

is well defined and it belongs to $\mathcal{M}(X)$.

Furthermore, the map Φ, given by (5.1), is well defined, meaning that

- *The collection $\{\Phi(\nu)_\omega\}_{\omega\in\Omega}$ forms a random measure on Q; equivalently:*
- *$\Phi(\mathcal{P}) \subset \mathcal{M}_m$;*
- *Furthermore, the map $\Phi : \mathcal{P} \longrightarrow \mathcal{M}_m$ is continuous with respect to the narrow topology on $\mathcal{M}_m$.*

Proof. The first part of the proposition, i.e. the one pertaining to the formula (5.5), is immediate from (5.4). Passing to the part pertaining to (5.6), let $\nu \in \mathcal{P}$. First, we need to check that for every random continuous function $g : \Omega \times Q \to \mathbb{R}$, the function

$$\Omega \ni \omega \longmapsto \int_Q g_\omega d\big(\mathcal{L}^*_{t,\omega}\nu_{\theta\omega}\big) \in \mathbb{R}$$

is measurable. Equivalently, we need to check the measurability of the function:

$$\Omega \ni \omega \longmapsto \int \mathcal{L}_{t,\omega}(g_\omega)\, d\nu_{\theta\omega}. \tag{5.7}$$

Since ν is a random measure, for every random continuous function

$$h(\omega, z) = h_\omega(z),$$

the function $\Omega \ni \omega \longmapsto \int h_\omega d\nu_{\theta\omega} \in \mathbb{R}$ is measurable. However, the function $\Omega \ni \omega \longmapsto \mathcal{L}_{t,\omega}(g_\omega)$, particularly the function $\Omega \ni \omega \longmapsto \mathcal{L}_{t,\theta\omega}(1\!\!1)$, is not a random continuous function. This is so because the function $\Omega \ni \omega \longmapsto \mathcal{L}_{t,\omega}(g_\omega)$ is unbounded unless $g_\omega(z) \to 0$ as $\mathrm{Re} z \to -\infty$.

In order to overcome this difficulty, we invoke Lemma 18. Indeed, it follows from this lemma that for every $g \in \mathcal{C}_\varepsilon$ the function

$$\Omega \ni \omega \longmapsto \int \mathcal{L}_{t,\omega}(g_\omega)\, d\nu_{\theta\omega}$$

is measurable and finite. Since the constant function $1\!\!1$ is a pointwise limit of a monotone (increasing) sequence of functions in $\mathcal{C}_\varepsilon$ (with ε converging to 0), and since the integrals $\int \mathcal{L}_{t,\omega}(1\!\!1) d\nu_{\theta\omega}$ are uniformly bounded with respect to $\nu \in \mathcal{P}$, we conclude that the function

$$\Omega \ni \omega \longmapsto \int \mathcal{L}_{t,\omega}(1\!\!1) d\nu_{\theta\omega}$$

is measurable and bounded, as a pointwise limit of an increasing sequence of measurable and uniformly bounded functions. In fact the monotonicity property (increasing sequence) and boundedness were inessential in this argument, and the same reasoning shows that the function

$$\Omega \ni \omega \longmapsto \int \mathcal{L}_{t,\omega}(g_\omega) d\nu_{\theta\omega}$$

is measurable for every $\nu \in \mathcal{P}$ and any random continuous function $g : X \to \mathbb{R}$. Measurability of the function defined by (5.7) is thus proved, and the part of (5.6) is established.

Then, the assignment

$$C_b(\Omega \times Q) \ni g \longmapsto \left(\frac{\int \mathcal{L}_{t,\omega}(g_\omega) d\nu_{\theta\omega}}{\int \mathcal{L}_{t,\omega}(1\!\!1) d\nu_{\theta\omega}} \right)_{\omega \in \Omega}$$

defines a linear function from the Banach space $C_b(\Omega \times Q)$ into $\mathbb{R}$, and moreover, by virtue of (5.4), this function is continuous.

We thus conclude that $\Phi(\nu) \in \mathcal{M}(X)$, and, since for each $\omega \in \Omega$, $(\Phi(\nu))_\omega$, is a probability measure, we get that

$$\Phi(\nu) \in \mathcal{M}_m,$$

i.e., $\Phi(\nu)$ is a random probability measure.

In order to prove continuity of the map Φ, it is enough to show that the map

$$\mathcal{P} \ni \nu \longmapsto \mathcal{L}_t^* \nu \in \mathcal{M}(X)$$

is continuous with respect to the narrow topology.

So, let $V \subset \mathcal{M}(X)$ be an open set, and assume that

$$\tilde{\nu} := \mathcal{L}_t^* \nu \in V$$

where ν is some measure in $\mathcal{P}$. We need to show that $\mathcal{L}_t^{*-1}(V)$ contains some neighborhood of ν in the narrow topology on $\mathcal{M}(X)$.

We can assume without loss of generality that V is taken from the the local base of neighborhoods of $\tilde{\nu}$, i.e.

$$V = \{\tilde{\mu} \in \mathcal{M} : |\tilde{\mu}(g_i) - \tilde{\nu}(g_i)| < \delta,\, i = 1, \dots k\}$$

with some integer $k \geq 1$, some d > 0, and g_i, $i = 1, 2, , \dots, k$ some functions from the space $C_b(\Omega \times Q)$. Now, we can further assume with no loss of generality that $k = 1$, so that

$$V = \{\tilde{\mu} \in \mathcal{M} : |\tilde{\mu}(g) - \tilde{\nu}(g)| < \delta\}.$$

where g is some function in $C_b(\Omega \times Q)$. Thus,

$$\mathcal{L}_t^{*-1}(V) = \big\{\mu \in \mathcal{P} : |\mu(\mathcal{L}_t g) - \nu(\mathcal{L}_t g)| < \delta\big\}.$$

By the assumptions of our proposition, imposed on φ, there exists $\varepsilon > 0$ so small that

$$\varphi(\varepsilon)\|g\| < \delta/3. \tag{5.8}$$

Next, let $h_\varepsilon : Q \to [0, 1]$ be a continuous function such that

- $h_\varepsilon(z) = 1$ whenever $z \in Q$ and $|B \exp(z)| < \varepsilon/2$, and
- $h_\varepsilon(z) = 0$ whenever $z \in Q$ and $|B \exp(z)| > \varepsilon$.

Then, for every $\omega \in \Omega$, the function $\mathcal{L}_{t,\omega}(h_\varepsilon)$ is non-zero only in the ball $B(0, \varepsilon)$. Define an auxiliary function

$$g_\varepsilon := (1 - h_\varepsilon)g.$$

Then $g_\varepsilon \in \mathcal{C}_{\varepsilon/2B}$. Put

$$U := \big\{\mu \in \mathcal{P} : |\mu(\mathcal{L}_t g_\varepsilon) - \nu(\mathcal{L}_t g_\varepsilon)| < \delta/3\big\}.$$

Then, by Lemma 18, U is an open neighborhood of ν in $\mathcal{P}$. If $\mu \in U$, then

$$|\mathcal{L}_t^*\mu(g) - \mathcal{L}_t^*\nu(g)| \leq |\mathcal{L}_t^*\mu(g) - \mathcal{L}_t^*\mu(g_\varepsilon)| + |\mathcal{L}_t^*\mu(g_\varepsilon) - \mathcal{L}_t^*\nu(g_\varepsilon)| + |\mathcal{L}_t^*\nu(g_\varepsilon) - \mathcal{L}_t^*\nu(g)|.$$

The second summand is just equal to $|\mu(\mathcal{L}_t g_\varepsilon) - \nu(\mathcal{L}_t g_\varepsilon)|$, so it can be estimated above by $\delta/3$. The third summand is equal to

$$\int_\Omega \int_Q \mathcal{L}_{t,\omega}(h_\varepsilon \cdot g)\, d\nu_\omega dm(\omega),$$

so its absolute value can be estimated above by

$$\int_\Omega \|g_\omega\|_\infty \int_Q \mathcal{L}_{t,\omega}(h_\varepsilon)\, d\nu_\omega dm(\omega) \leq \int_\Omega \|g_\omega\|_\infty \int_{B(0,\varepsilon)} \mathcal{L}_{t,\omega}(1\!\!1)\, d\nu_\omega dm(\omega) \leq \varphi(\varepsilon)\|g\| < \delta/3,$$

where the second inequality sign "$\leq$" comes from (5.3) and the third (last) one is due to (5.8). Since $\mu \in \mathcal{P}$, exactly the same estimate applies to the first summand. Summing up, we conclude that $U \subset \mathcal{L}_t^{*-1}(V)$. Since U is open in $\mathcal{P}$, the proof continuity of $\mathcal{L}_t^*$, and simultaneously of the whole Proposition 19, is complete. □

Our goal is to apply the general scheme described above, to a properly chosen set $\mathcal{P}$.

First, we fix a number

$$r_0 \in \left(0, \frac{1}{2K}\right). \tag{5.9}$$

Next, we formulate the following straightforward estimate. Its proof is omitted.

Lemma 20. *There exist constants $D > d > 0$ (depending on $t > 1$) such that, for every $\omega \in \Omega$ and every $z \in Q$,*

$$\frac{d}{|z|^{t-1}} \le \mathcal{L}_{t,\omega}(1\!\!1)(z) \le \frac{D}{|z|^{t-1}}.$$

Put $c := d/2$, where d comes from Lemma 20. For $M_0 > 0$ define the constants:

$$C(M_0) := \frac{M_0^{t-1}}{c} \tag{5.10}$$

and

$$c(M_0) := 2DC(M_0) \tag{5.11}$$

where D comes from Lemma 20.

Definition 21 (Definition of the space $\mathcal{P}$)**.** Fix some $t > 1$. Suppose that $\mathcal{P} \subset \mathcal{M}_m$ is such a set for which there exists $M_0 > 0$, with $c(M_0) > 0$, and $C(M_0) > 0$ defined as in (5.10), (5.11), such that the the following are satisfied:

$$\nu_\omega(Q_{M_0}) \ge 1/2 \quad \text{for all} \quad \omega \in \Omega, \tag{5.12}$$

$$\nu_\omega(Y_M^+) \le c(M_0) e^{\frac{M}{2}(1-t)} \quad \text{for all } \omega \in \Omega \text{ and all } M > 0, \tag{5.13}$$

and, for every integer $n \ge 0$ the following *Condition* W_n holds:

Condition W_n**.** *For every $\omega \in \Omega$ and every $j \in \mathbb{N} \cup \{0\}$ the following bounds hold:*

$$\nu_{\theta^j(\omega)}(F^{-n}_{\theta^j\omega,*}(B(F^{n+j}_\omega(0), r_0)) \le a_{j,n}(\omega) \cdot b_{n+j}(\omega), \tag{5.14}$$

where

$$a_{j,n}(\omega) := (K \cdot C(M_0))^n |F^{j+1}_\omega(0)|^{-t} \cdot |F^{j+2}_\omega(0)|^{-t} \cdot \dots \cdot |F^{j+n}_\omega(0)|^{-t}, \tag{5.15}$$

$$a_{j,0}(\omega) := 1,$$

and

$$b_k(\omega) := \left(\frac{Kr_0}{2\pi}\right) c(M_0) \cdot C(M_0) \cdot |F^{k+1}_\omega(0)|^{1-t} \cdot e^{-\frac{(t-1)}{4}|F^{k+1}_\omega(0)|}, \tag{5.16}$$

where $F^{-n}_{\theta^j\omega,*}$ is the holomorphic branch of $F^{-n}_{\theta^j\omega}$, defined in $B(F^{n+j}_\omega(0), r_0)$ and mapping $F^{n+j}_\omega(0)$ back to $F^j_\omega(0)$.

6. The Map Φ is Well Defined on $\mathcal{P}$

Our goal in this section is to show that if the constant $M_0 > 0$, together with $c(M_0) > 0$, and $C(M_0) > 0$ as in (5.11), (5.10), is properly selected, then there exist numbers $\rho > 0$, $P > 0$, and $p > 0$ and a function $\varphi(\varepsilon)$ such that for any set $\mathcal{P} \subset \mathcal{M}_m$ fulfilling the requirements of Definition 21, the hypothesis of Proposition 19 are satisfied. In particular, the map Φ is well defined on $\mathcal{P}$. In the next section, we will show that

$$\Phi(\mathcal{P}) \subset \mathcal{P}. \tag{6.1}$$

So, our strategy is to fix a non–empty set $\mathcal{P} \subset \mathcal{M}_m$ fulfilling the requirements of Definition 21 with some, undetermined yet, constant $M_0 > 0$, and to work out such sufficient conditions for this constant that the hypothesis of Proposition 19 will be satisfied, and later, in the next section, to show that formula (6.1) holds.

Now, given $\omega \in \Omega$, we define a sequence of radii $\big(r_n(\omega)\big)_{n=1}^{\infty}$, converging to 0 as $n \to \infty$. Put

$$r_n = r_n(\omega) := \frac{1}{4} r_0 \big(|F_\omega(0)| \dots |F_\omega^n(0)|\big)^{-1}.$$

Then, by Koebe's $\frac{1}{4}$–Theorem,

$$\tilde{B}_{n,\omega} := F_\omega^{-n}\big(B(F_\omega^n(0), r_0)\big) \supset B\left(0, \frac{1}{4} r_0 \big(|F_\omega(0)| \dots |F_\omega^n(0)|\big)^{-1}\right) = B(0, r_n).$$

Lemma 22. *Put $s = 3t+7$. Then there exist a constant $C \in (0, +\infty)$, independent of M_0, such that if ν is any random measure in $\mathcal{P}$, then for every radius $r \in (0, r_0/4)$ we have that*

$$\nu_\omega(B(0,r)) \le C \cdot (M_0^{t-1})^{n_\omega(r)+2} r^s, \tag{6.2}$$

where $n_\omega(r)$ is the unique integer $n \ge 0$ for which $r_{n+1}(\omega) \le r < r_n(\omega)$.

Proof. Denote $n_\omega(r)$ by n. Using condition W_n we get

$$\nu_\omega\big(F_{\omega,*}^{-n}\big(B(F_\omega^n(0), r_0)\big)\big) \le a_{0,n}(\omega) b_n(\omega). \tag{6.3}$$

Using this condition again one can easily deduce that

$$\nu_\omega(B(0,r)) \le \nu_\omega(B(0,r_n)) \le a_{0,n}(\omega) b_n(\omega) \le \mathrm{Const}(M_0^{t-1})^{n+2} \exp\left(\frac{F_\omega^{n+1}(0)}{8}(1-t)\right),$$

where the constants are independent of ω, n and M_0. Now, there exists an integer $N \ge 1$ such that for all $n \ge N$, all $\omega \in \Omega$, and all $r_{n+1}(\omega) \le r < r_n(\omega)$,

$$\exp\left(\frac{F_\omega^{n+1}(0)}{8}(1-t)\right) \le \left(\frac{1}{4} r_0 \big(|F_\omega(0)| \dots |F_\omega^n(0)| \cdot |F_\omega^{n+1}(0)\big)\right)^{-s} = r_{n+1}^s \le r^s.$$

If $n < N$, we still have (6.2), by increasing the constant C if needed. The proof is complete. □

Lemma 23. *We have that*

$$\lim_{r \to 0} \frac{n_\omega(r)}{\ln \ln \frac{1}{r}} = 0$$

uniformly with respect to $\omega \in \Omega$.

Proof. As in the proof of the previous lemma put $n := n_\omega(r)$. Then we have

$$\frac{1}{r} > \frac{1}{r_n} = \frac{4}{r_0} \cdot |F_\omega(0)| \cdot \dots \cdot |F_\omega^n(0)|.$$

So, using $F_\omega^k(0) = \eta(\theta^{k-1}\omega) \exp\big(F_\omega^{k-1}(0)\big) > \frac{1}{e} \exp\big(F_\omega^{k-1}(0)\big)$, true for every $k \geq 1$, we get that

$$\begin{aligned}\ln \frac{1}{r} &> \ln 4 - \ln r_0 + \ln |F_\omega(0)| + \dots + \ln |F_\omega^{n-1}(0)| \\ &> \ln 4 - \ln r_0 + |F_\omega(0)| + \dots |F_\omega^{n-1}(0)| - n \\ &> |F_\omega^{n-1}(0)|\end{aligned}$$

for all n large enough, and so, also for all n large enough: $\ln \ln \frac{1}{r} > \ln F_\omega^{n-1}(0) \geq n^2$, and the lemma follows. □

Lemma 24. *There exist $u \geq 2t + 7$ and $\rho \in (0, r_0/4)$ (ρ depends on M_0) such that, for every measure $\nu \in \mathcal{P}$, we have that*

$$\nu_\omega(B(0, r)) \leq r^u$$

for all $r < \rho$ and m–a.e. $\omega \in \Omega$.

Proof. The estimate (6.2) says that $\nu_\omega(B(0, r)) \leq C \cdot (M_0^{t-1})^{n_\omega(r)+2} r^s$ where $s = 3t + 7 > 3t + 6 > 2t + 7$. Thus, invoking Lemma 23 and the definition of $r_n(\omega)$, we see that the required estimate follows, with $u := 2t + 7$. □

For every $\varepsilon \in (0, r_0/4)$ let $k(\varepsilon) \geq 0$ be the least non–negative integer k such that

$$A \exp\big(-M_0(k + 1)\big) < \varepsilon.$$

Then, define the function $\tilde{\varphi}(\varepsilon)$

$$\tilde{\varphi}(\varepsilon) := DB^{t+8} \sum_{k=k(\varepsilon)}^{\infty} \exp\big(-M_0(t + 8)k\big), \tag{6.4}$$

Conforming to our general strategy, thus aiming to apply Proposition 19, we shall prove the following.

Lemma 25. *If $\nu \in \mathcal{P}$, then for m–a.e. $\omega \in \Omega$ and every $\varepsilon \in (0, \rho)$, we have that*

$$\int_{B(0,\varepsilon)} \mathcal{L}_{t,\omega}(\mathbb{1})(z)\, d\nu_{\theta\omega}(z) \leq \tilde{\varphi}(\varepsilon). \tag{6.5}$$

Proof. For every $\omega \in \Omega$, let

$$B(\omega) =: \big\{z \in Q : |z| < \eta(\omega)e^{-M_0}\big\}.$$

Partition the ball $B(\omega)$ into annuli

$$P_k(\omega) := \big\{z \in Q : \eta(\omega)e^{-(k+1)M_0} < |z| \leq \eta(\omega)e^{-kM_0}\big\}.$$

We define $k_\omega(\varepsilon) \geq 0$ to be the only non–negative integer such that $\varepsilon \in P_{k_\omega(\varepsilon)}(\omega)$. Of course $k(\varepsilon) \leq k_\omega(\varepsilon)$. Therefore, using also Lemma 20 and Lemma 24, we can estimate as follows.

$$\begin{aligned}
\int_{B(0,\varepsilon)} \mathcal{L}_{t,\omega}(1\!\!1)(z)d\nu_{\theta\omega}(z) &\leq \sum_{k=k_\omega(\varepsilon)}^{\infty} \int_{P_k(\omega)} \mathcal{L}_{t,\omega}(1\!\!1)(z)d\nu_{\theta\omega}(z) \leq D \sum_{k=k_\omega(\varepsilon)}^{\infty} \int_{P_k(\omega)} |z|^{1-t} d\nu_{\theta\omega}(z) \\
&\leq DB^{1-t} \sum_{k=k_\omega(\varepsilon)}^{\infty} \exp\big(M_0(t-1)(k+1)\big)\nu_{\theta\omega}\big(P_k(\omega)\big) \\
&\leq DB^{1-t} \sum_{k=k_\omega(\varepsilon)}^{\infty} \exp\big(M_0(t-1)(k+1)\big)B^{2t+7}\exp\big(-M_0(2t+7)k\big) \\
&= DB^{t+8} \sum_{k=k_\omega(\varepsilon)}^{\infty} \exp\big(-M_0(t+8)\big) \\
&\leq DB^{t+8} \sum_{k=k(\varepsilon)}^{\infty} \exp\big(-M_0(t+8)\big) \\
&= \tilde{\varphi}(\varepsilon).
\end{aligned}$$

□

Since the function $(0,\rho) \ni \varepsilon \longmapsto \tilde{\varphi}(\varepsilon) \in (0,+\infty)$, is monotone increasing and $\lim_{\varepsilon\to 0^+} \tilde{\varphi}(\varepsilon) = 0$, there exists a monotone increasing continuous function $(0,\rho) \ni \varepsilon \longmapsto \varphi(\varepsilon) \in (0,+\infty)$ such that

$$\tilde{\varphi}(\varepsilon) \leq \varphi(\varepsilon)$$

for all $\varepsilon \in (0,\rho)$ and

$$\lim_{\varepsilon\to 0^+} \varphi(\varepsilon) = 0.$$

Therefore, as an immediate consequence of Lemma 25, we get the following.

Lemma 26. *If $\nu \in \mathcal{P}$, then for m–a.e. $\omega \in \Omega$ and every $\varepsilon \in (0,\rho)$, we have that*

$$\int_{B(0,\varepsilon)} \mathcal{L}_{t,\omega}(1\!\!1)(z)d\nu_{\theta\omega}(z) \leq \varphi(\varepsilon). \tag{6.6}$$

In this Lemma, both ρ, and the function $\varphi(\varepsilon)$ depend on the choice of M_0. As an immediate consequence of this lemma and Lemma 20, we get the following.

Lemma 27. *If $\nu \in \mathcal{P}$, then there exists $P \in (0,+\infty)$ such that for m–a.e. $\omega \in \Omega$, we have that*

$$\nu_{\theta\omega}\big(\mathcal{L}_{t,\omega}1\!\!1\big) = \int_Q \mathcal{L}_{t,\omega}(1\!\!1)(z)d\nu_{\theta\omega}(z) \leq P. \tag{6.7}$$

and, again, the constant P depends on the choice of M_0.

Using Lemma 20 again, we will obtain a common, i.e. good for all $\omega \in \Omega$, lower bound on $\nu_{\theta\omega}\big(\mathcal{L}_{t,\omega}1\!\!1\big)$.

Lemma 28. *For every measure $\nu \in \mathcal{P}$ the following holds:*

$$\nu_{\theta\omega}\big(\mathcal{L}_{t,\omega}1\!\!1\big) = \int_Q \mathcal{L}_{t,\omega}(1\!\!1)(z)d\nu_{\theta\omega}(z) \geq \frac{c}{M_0^{t-1}} = \frac{1}{C(M_0)}, \tag{6.8}$$

where, we recall, $C(M_0) > 0$ is of the form (5.10)*, thus, satisfying in particular,* (5.12)*.*

Proof. Using Lemma 20, we obtain

$$\nu_{\theta\omega}\big(\mathcal{L}_{t,\omega}1\!\!1\big) \geq \nu_{\theta\omega}(Q_{M_0}) \cdot \inf_{z \in Q_{M_0}} \mathcal{L}_{t,\omega}(1\!\!1) \geq d\nu_{\theta\omega}(Q_{M_0}) \cdot \inf_{z \in Q_{M_0}} \frac{1}{|z|^{t-1}} \geq \frac{c}{M_0^{t-1}}.$$

□

7. Invariance of the Space $\mathcal{P}$ Under the Map Φ: $\Phi(\mathcal{P}) \subset \mathcal{P}$

Having Lemmas 25, 27, and 28 proved, we can apply Proposition 19 and take all its fruits. In particular, the measures $\mathcal{L}_t^*\nu$ and $\Phi(\nu)$ are well defined for all measures $\nu \in \mathcal{P}$.

Lemma 29. *If $\nu \in \mathcal{P}$ (thus, in particular, ν satisfies* (5.12)*) then the measure $\Phi(\nu)$ satisfies the estimate* (5.13)*, with the constant*

$$(DM_0^{t-1}/c + C) = (D \cdot C(M_0) + C),$$

where $C > 0$ is some absolute constant, depending on $t > 1$ but independent of M_0. Therefore, if M_0 is sufficiently large, then the condition (5.13) *is satisfied.*

Proof. We have

$$\begin{aligned}
\mathcal{L}_{t,\omega}^*\nu_{\theta\omega}(Y_M^+) &= \int \mathcal{L}_{t,\omega}(1\!\!1_{Y_M^+})(z)d\nu_{\theta\omega}(z) = \\
&= \int_{|\mathrm{Re} z| < e^{M/2}} \mathcal{L}_{t,\omega}(1\!\!1_{Y_M^+})(z)d\nu_{\theta\omega}(z) + \int_{|\mathrm{Re} z| \geq e^{M/2}} \mathcal{L}_{t,\omega}(1\!\!1_{Y_M^+})(z)d\nu_{\theta\omega}(z) \\
&\leq \int_{|\mathrm{Re} z| < e^{M/2}} \mathcal{L}_{t,\omega}(1\!\!1_{Y_M^+})(z)d\nu_{\theta\omega}(z) + \nu_{\theta\omega}(\{z : |\mathrm{Re} z| \geq e^{M/2}\}) \cdot \sup_{|\mathrm{Re} z| \geq e^{M/2}} \mathcal{L}_{t,\omega}(1\!\!1)(z) \\
&= \Sigma_1 + \Sigma_2.
\end{aligned}$$

It follows from Lemma 20 that for z with $|\mathrm{Re} z| \geq e^{M/2}$, we have that $\mathcal{L}_{t,\omega}1\!\!1(z) \leq De^{\frac{M}{2}(1-t)}$ and, consequently

$$\Sigma_2 \leq De^{\frac{M}{2}(1-t)}. \tag{7.1}$$

Now, we estimate Σ_1. For $z = [a + bi]$ with $|\mathrm{Re} z| < e^{M/2}$, Lemma 20 yields:

$$\mathcal{L}_{t,\omega}(1\!\!1)(z) \geq de^{\frac{M}{2}(1-t)}.$$

Also:

$$\mathcal{L}_{t,\omega}(1\!\!1_{Y_M^+})(z) = \sum_k \frac{1}{|a + bi + 2k\pi i|^t} \leq Ce^{M(1-t)}, \tag{7.2}$$

with another positive constant C, where the sum is taken over all such integers k for which $\log|a+bi+2k\pi i| - \log\eta(\omega) > M$. Therefore, we can write, for $|z| < e^{M/2}$, the following estimate for the summand I, with possibly modified constant C:

$$\frac{\mathcal{L}_{t,\omega}(\mathbb{1}_{Y_M^+})(z)}{\mathcal{L}_{t,\omega}(\mathbb{1})(z)} \le Ce^{\frac{M}{2}(1-t)} \tag{7.3}$$

or, equivalently,

$$\mathcal{L}_{t,\omega}(\mathbb{1}_{Y_M^+})(z) \le Ce^{\frac{M}{2}(1-t)}\mathcal{L}_{t,\omega}(\mathbb{1})(z). \tag{7.4}$$

We are close to the end of the proof of (5.13). Using the lower estimate of Proposition 28, together with estimates (7.1) and (7.4), we can write

$$\begin{aligned}\mathcal{L}^*_{t,\omega}(\nu_{\theta\omega})(Y_M^+) &\le \Sigma_2 + \Sigma_1 \le De^{\frac{M}{2}(1-t)} + Ce^{\frac{M}{2}(1-t)}\int_Q \mathcal{L}_{t,\omega}(\mathbb{1})(z)d\nu_{\theta\omega} \\ &= De^{\frac{M}{2}(1-t)} + Ce^{\frac{M}{2}(1-t)}\mathcal{L}^*_{t,\omega}\nu_{\theta\omega}(\mathbb{1}).\end{aligned}$$

Now, using the definition of the map Φ, we obtain

$$\Phi(\nu)_\omega(Y_M^+) = \frac{\mathcal{L}^*_{t,\omega}(\nu_{\theta\omega})(Y_M^+)}{\mathcal{L}^*_{t,\omega}\nu_{\theta\omega}(\mathbb{1})} \le \frac{M_0^{t-1}}{c}De^{\frac{M}{2}(1-t)} + Ce^{\frac{M}{2}(1-t)} = (DC(M_0)+C)e^{\frac{M}{2}(1-t)},$$

Since D and C are absolute constants, and $C(M_0)\to\infty$ as $M_0\to\infty$, it is clear that for all M_0 sufficiently large $DC(M_0)+C < 2DC(M_0) = c(M_0)$. The proof is complete. □

At this stage of the paper we have all the constants of Definition 21 except M_0. Now, we will determine its value.

First of all, we require M_0 to be large enough as to satisfy Lemma 29. Next, let us note the following direct consequence of Lemma 29.

Corollary 30. *If $M_0 > 0$ is large enough, then for every random measure $\nu \in \mathcal{P}$ (thus, in particular, satisfying condition* (5.12)*), the measures $\Phi(\nu)_\omega$, $\omega\in\Omega$, satisfy the following:*

$$\Phi(\nu)_\omega(Y_{M_0}^+) < 1/4.$$

From now on, we also assume that $M_0 > 0$ is large enough to satisfy Corollary 30.

Proposition 31. *If $\nu\in\mathcal{P}$, then, for every $j \ge 0$, we have that*

$$(\Phi(\nu))_{\theta^j\omega}(B(F_\omega^j(0), r_0)) \le b_j(\omega),$$

where

$$b_j(\omega) := \frac{Kr_0}{2\pi}c(M_0)C(M_0)\cdot|F_\omega^{j+1}(0)|^{1-t}\cdot e^{\frac{(1-t)}{4}|F_\omega^{j+1}(0)|}.$$

In other words, the measure $\Phi(\nu)$ satisfies condition W_0.

Proof. Since

$$F_{\theta^j\omega}\big(B(F_\omega^j(0), r_0)\big) \subset B\big(F_\omega^{j+1}(0), Kr_0|F_\omega^{j+1}(0)|\big).$$

and since

$$Kr_0|F_\omega^{j+1}(0)| \le 1/2|F_\omega^{j+1}(0)|,$$

we conclude that

$$F_{\theta^j\omega}(B(F_\omega^j(0), r_0)) \subset Y_M^+$$

with $M = \frac{1}{2}|F_\omega^{j+1}(0)| = \frac{1}{2}\mathrm{Re}F_\omega^{j+1}(0)$.

Now, using Lemma 29, i.e. using formula (5.13) that it yields, with $M = \frac{1}{2}|F_\omega^{j+1}(0)|$, and the fact that the image $F_{\theta^j\omega}(B(F_\omega^j(0), r_0))$ covers Y_M^+ at most $\frac{r_0 F_\omega^{j+1}(0)\cdot K}{2\pi}$ times, i.e. every point in $F_{\theta^j\omega}(B(F_\omega^j(0), r_0))$ has at most $\frac{r_0 F_\omega^{j+1}(0)\cdot K}{2\pi}$ preimages in $B(F_\omega^j(0), r_0))$, we can estimate the measure $\mathcal{L}^*_{t,\theta^j\omega}\nu(B(F_\omega^j(0), r_0))$ as follows:

$$\begin{aligned}\mathcal{L}^*_{t,\theta^j\omega}\nu(B(F_\omega^j(0), r_0) &\le c(M_0)\exp\left(\frac{1}{4}F_\omega^{j+1}(0)(1-t)\right)\cdot|F_\omega^{j+1}(0)|^{-t}\cdot\frac{|F_\omega^{j+1}(0)|Kr_0}{2\pi}\\ &= \left(\frac{Kr_0}{2\pi}\right)c(M_0)\cdot|F_\omega^{j+1}(0)|^{-t}\cdot\exp\left(\frac{1}{4}(1-t)F_\omega^{j+1}(0)\right)\cdot|F_\omega^{j+1}(0)|\end{aligned}$$

and, using in addition the lower bound provided in Proposition 28,

$$(\Phi(\nu))_{\theta^j\omega}(B(F_\omega^j(0), r_0) \le \left(\frac{Kr_0}{2\pi}\right)c(M_0)\cdot C(M_0)\cdot\exp\left(\frac{1-t}{4}F_\omega^{j+1}(0)\right)\cdot|F_\omega^{j+1}(0)|^{1-t}.$$

□

Proposition 32. *If ν is a random measure in $\mathcal{P}$, then the measure $\Phi(\nu)$ satisfies all the conditions W_n, $n \ge 0$.*

Proof. It was proved in Proposition 31 that then $\Phi(\nu)$ satisfies the condition W_0. So, below, we prove that all the conditions W_n, $n \ge 1$, are satisfied. We estimate as follows:

$$\begin{aligned}&\mathcal{L}^*_{t,\theta^j\omega}\nu_{\theta^{j+1}\omega}\big(F^{-n}_{\theta^j\omega,*}(B(F_\omega^{n+j}(0), r_0))\big) =\\ &= \nu_{\theta^{j+1}\omega}\left(\mathcal{L}_{t,\theta^j\omega}1\!\!1_{F^{-n}_{\theta^j\omega,*}\left(B(F_\omega^{n+j}(0,r_0))\right)}\right) = \int_{F^{-(n-1)}_{\theta^{j+1}\omega,*}\left(B(F_\omega^{n+j}(0),r_0)\right)}\big|(F^{-1}_{\theta^j\omega,*})'(y)\big|^t d\nu_{\theta^{j+1}\omega}(y)\\ &= \int_{F^{-(n-j)}_{\theta^{j+1}\omega,*}\left(B(F_\omega^{n+j}(0),r_0)\right)}|y|^{-t}d\nu_{\theta^{j+1}\omega}(y)\\ &\le K|F_\omega^{j+1}(0)|^{-t}\nu_{\theta^{j+1}\omega}\big(F^{-(n-1)}_{\theta^{j+1}\omega}(B(F_\omega^{n+j}(0), r_0))\big).\end{aligned}$$

Thus, using the fact that $\mathcal{L}^*_{t,\omega}(\nu_{\theta\omega})(1\!\!1) \ge 1/C(M_0)$, known from Lemma 28, together with the estimate W_n applied to the measure ν, we get that

$$\begin{aligned}\Phi(\nu)_{\theta^j\omega}(F^{-n}_{\theta^j\omega,*}\big(B(F_\omega^{n+j}(0), r_0)\big) &\le KC(M_0)|F_\omega^{j+1}(0)|^{-t}\nu_{\theta^{j+1}\omega}\big(F^{-(n-1)}_{\theta^{j+1}\omega,*}(B(F_\omega^{n+j}(0), r_0))\big)\\ &\le KC(M_0)|F_\omega^{j+1}(0)|^{-t}a_{j+1,n-1}(\omega)b_{j+1+n-1}(\omega)\\ &= K\cdot C(M_0)|F_\omega^{j+1}(0)|^{-t}a_{j+1,n-1}(\omega)b_{j+n}(\omega)\\ &= \kappa\cdot|F_\omega^{j+1}(0)|^{-t}\cdot a_{j+1,n-1}(\omega)b_{j+n}(\omega)\\ &= a_{j,n}(\omega)b_{j+n}(\omega).\end{aligned}$$

Thus, the measure $\Phi(\nu)$ satisfies all conditions W_n, $n \ge 1$, and the proof is complete. □

Before stating the next proposition, let us recall that the definition of the space $\mathcal{P}$ depends on the constant M_0, which we assumed to be large enough to for the hypotheses of Corollary 30 to be satisfied. Proposition 33 below will impose one more condition on M_0.

Proposition 33. *If $M_0 > 0$ is large enough then for every $\nu \in \mathcal{P}$ and m–a.e. $\omega \in \Omega$, we have that*

$$\Phi(\nu)_\omega(Y_{M_0}^-) \le (C/c)M_0^{2t-1}(\eta(\omega))^{2t+7}e^{M_0 t}\sum_{k=1}^{\infty}\exp(-(2t+7)kM_0)(M_0^{t-1})^{\log k+\log M_0+2} < 1/4,$$

where the constant C comes from Lemma 22.

Proof. Given $\omega \in \Omega$, we have

$$F_\omega\big(Y_{M_0}^-\big) = B(\omega) = \big\{z \in Q : |z| < \eta(\omega)e^{-M_0}\big\}.$$

Note also that for every point $z \in B(\omega)$ the set

$$F_\omega^{-1}(z) \cap Y_{M_0}^-$$

is a singleton. Denote it by w and note that $F'_\omega(w) = z$.

Utilizing the annuli $P_k(\omega)$, introduced in the proof of Lemma 25, and using Lemma 23, we may assume $M_0 > 0$ to be so large that, if $z \in P_k(\omega)$, then $n_{\theta\omega}(|z|) < \log k + \log M_0 - 2$. So, applying (6.2) and Lemma 22, we can thus estimate as follows:

$$\begin{aligned}
\mathcal{L}^*\nu_\omega(Y_{M_0}^-) = \nu_{\theta\omega}\big(\mathcal{L}_\omega(\mathbb{1}_{Y_{M_0}^-})\big) &= \sum_{k=1}^{\infty}\int_{P_k(\omega)}\frac{1}{|z|^t}d\nu_{\theta\omega} \le \sum_{k=1}^{\infty}\sup_{z\in P_k(\omega)}\frac{1}{|z|^t}\cdot\nu_{\theta\omega}(P_k(\omega)) \\
&\le C(\eta(\omega))^{s-t}e^{M_0 t}\sum_{k=1}^{\infty}\exp\big(-k(s-t)M_0\big)\big(M_0^{t-1}\big)^{(n_{\theta\omega}(\eta(\omega))+2)} \\
&\le C(\eta(\omega))^{s-t}e^{M_0 t}\sum_{k=1}^{\infty}\exp\big(-k(s-t)M_0\big)\big(M_0^{t-1}\big)^{\log k+\log M_0} \\
&= C(\eta(\omega))^{2t+7}e^{M_0 t}\sum_{k=1}^{\infty}\exp\big(-(2t+7)kM_0\big)\big(M_0^{t-1}\big)^{\log k+\log M_0}.
\end{aligned}$$

Therefore, invoking now Lemma 28, we get

$$\Phi(\nu)_\omega(Y_{M_0}^-) \le (C/c)B^{2t+7}M_0^{t-1}e^{M_0 t}\sum_{k=1}^{\infty}\exp\big(-(2t+7)kM_0\big)\big(M_0^{t-1}\big)^{\log k+\log M_0} < 1/4,$$

the last inequality holding provided that $M_0 > 0$ is large enough. □

Now, fix $M_0 > 0$ so large as to satisfy all the above estimates. Let us summarize the above sequence of propositions:

- Let $\nu \in \mathcal{P}$.
- Then Lemma 29 shows that $\Phi(\nu)$ satisfies the estimate (5.13).
- Next, Corollary 30 and Proposition 33 guarantee that $\Phi(\nu)$ satisfies condition (5.12).
- Finally, Propositions 31 and 32 guarantee that the conditions $W_0, W_1, \dots, W_n, \dots$ hold for $\Phi(\nu)$.

The final conclusion of this section is thus the following.

Proposition 34. *If $\mathcal{P}$ is the set of all measures in $\mathcal{M}_m$ satisfying the conditions of Definition 21, with the appropriate constants M_0, $c(M_0)$, and $C(M_0)$, determined in the last two sections, then*

$$\Phi(\mathcal{P}) \subset \mathcal{P}.$$

8. Random Conformal Measures for Random Exponential Functions; the Final Step

Since for every integer $l \geq 1$, we have $lM_0 \geq M_0$, Proposition 33 entails the following.

Proposition 35. *If $\nu \in \mathcal{P}$, where $\mathcal{P}$ comes from Proposition 34, then for every $l \in \mathbb{N}$, we have that*

$$(\Phi(\nu)_\omega)(Y^-_{lM_0}) \leq S(l) \tag{8.1}$$

where

$$S(l) := (C/c)B^{2t+7}(M_0 l)^{t-1}e^{M_0 tl}\sum_{k=1}^{\infty}\exp\big(-(2t+7)kM_0 l\big)(M_0^{t-1})^{\log k+\log M_0+l}$$

and

$$\lim_{l\to\infty} S(l) = 0. \tag{8.2}$$

If $\mathcal{P}$ is the set produced in Proposition 34, then we denote by $\hat{\mathcal{P}}$ its subset consisting of all those measures ν for which

$$\nu_\omega(Y^-_{lM_0}) \leq S(l) \tag{8.3}$$

for m–a.e. $\omega \in \Omega$ and all integers $l \geq 1$. Because of Proposition 34 and Proposition 35, we have the following.

Proposition 36. *If $\mathcal{P}$ is the set produced in Proposition 34, then*

$$\Phi(\mathcal{P}) \subset \hat{\mathcal{P}}$$

and

$$\Phi(\hat{\mathcal{P}}) \subset \hat{\mathcal{P}}.$$

Proposition 37. *If $\mathcal{P}$ is the set produced in Proposition 34, then the set $\hat{\mathcal{P}}$ is nonempty, convex and compact with respect to the narrow topology on $\mathcal{M}_m$.*

Proof. First, we shall prove that the set $\mathcal{P}$ produced in Proposition 34 is non-empty. Indeed, define ν in the following way: for every $\omega \in \Omega$ consider the set

$$Z_\omega := Q_{M_0} \setminus \bigcup_{j=0}^{\infty} B\big(F^j_{\theta^{-j}\omega}(0), r_0\big).$$

Let ν_ω be just the normalized Lebesgue measure on Z_ω. Since $\text{supp}(\nu_\omega) \subset Q_{M_0}$, the conditions (5.12) and (5.13) are trivially satisfied. Since, for every $j \in \mathbb{Z}$ and every $n \geq 0$,

$$F^{-n}_{\theta^j\omega,*}\big(B(F^{n+j}_\omega(0), r_0\big) \subset B(F^j_\omega(0), r_0),$$

all the conditions W_n, $n \geq 0$, are also trivially satisfied. So $\nu \in \mathcal{P}$, hence $\mathcal{P} \neq \emptyset$. Then $\hat{\mathcal{P}} \neq \emptyset$ because of the first part of Proposition 36.

Convexity of $\hat{\mathcal{P}}$ follows immediately from its definition. The uniform estimates provided by formula (5.13) and (8.3) along with (8.2) show that the family $\hat{\mathcal{P}}$ is tight, thus relatively compact according to Theorem 16.

Finally, the set $\hat{\mathcal{P}}$ is closed with respect to the narrow topology on $\mathcal{M}_m$ because for every measurable set $A \subset \Omega \times Q$ and all measurable functions $g : \Omega \to [0, +\infty)$, both the sets

$$\big\{\nu \in \mathcal{M}_m : \nu_\omega(A_\omega) \leq g(\omega) \ \text{ for all } \omega \in \Omega\big\}$$

and

$$\big\{\nu \in \mathcal{M}_m : \nu_\omega(A_\omega) \geq g(\omega) \ \text{ for all } \omega \in \Omega\big\}$$

are closed in $\mathcal{M}_m$ with respect to the narrow topology. □

Now, we can prove the main theorem of this section.

Theorem 38 (Existence of (ω, t) conformal measures ν_ω)**.** *For every $t > 1$ there exists a random t–conformal measure $\nu^{(t)} \in \hat{\mathcal{P}}$. Recall that t–conformality means that*

$$\mathcal{L}^*_{t,\omega}(\nu^{(t)}_{\theta\omega}) = \lambda_{t,\omega}\nu^{(t)}_\omega \tag{8.4}$$

for every $\omega \in \Omega$, where $\lambda_{t,\omega} := \mathcal{L}^*_{t,\omega}\nu^{(t)}_{\theta\omega}(\mathbb{1})$.

Proof. Because of the second part of Proposition 36 and because of Proposition 37, the Schauder–Tichonov Fixed Point Theorem applies to the continuous map $\Phi : \hat{\mathcal{P}} \to \hat{\mathcal{P}}$, thus yielding a fixed point of Φ in $\hat{\mathcal{P}}$. This just means that formula (8.4) holds. □

Also recall that a, very useful in calculations, property equivalent to (8.4), which will be frequently used in the sequel, is that

$$\nu^{(t)}_{\theta\omega}(F_\omega(A)) = \lambda_{t,\omega} \int_A \big|\big(F_\omega\big)'\big|^t \, d\nu^{(t)}_\omega \tag{8.5}$$

for every $\omega \in \Omega$ and for every Borel set $A \subset Q$ such that $F_\omega|_A$ is 1–to–1. By an immediate induction, we then get for every integer $n \geq 0$ the following.

$$\nu^{(t)}_{\theta\omega}(F^n_\omega(A)) = \lambda^n_{t,\omega} \int_A \big|\big(F^n_\omega\big)'\big|^t \, d\nu^{(t)}_\omega \tag{8.6}$$

for every $\omega \in \Omega$ and for every Borel set $A \subset Q$ such that $F_\omega|_A$ is 1–to–1. Lemmas 27, and 28 can be now reformulate as follows. There are two constants $0 < p, P < +\infty$ such that

$$1/p \leq \lambda_{t,\omega} \leq P \tag{8.7}$$

for m–a.e. $\omega \in \Omega$. Let us record the following property of the measure $\nu^{(t)}$.

Proposition 39. *For m–a.e $\omega \in \Omega$ we have that*

$$\operatorname{supp}(\nu^{(t)}_\omega) = Q.$$

Moreover, for all numbers $x > 0$, $R > 0$, and $\varepsilon \in (0, 1)$ there exists a constant $\xi = \xi(x, R, \varepsilon) > 0$ and a measurable set $\Omega(x, R, \varepsilon)$ such that

$$m\big(\Omega(x, R, \varepsilon)\big) > 1 - \varepsilon$$

and for every $\omega \in \Omega(x, R, \varepsilon)$ and every $z \in Q_x$, we have that

$$\nu_\omega^{(t)}(B(z, R)) \geq \xi.$$

Proof. Let $z \in Q$, $r > 0$. We need to check that

$$\nu_\omega^{(t)}(B(z, r)) > 0.$$

Since $J(f_\omega) = \mathbb{C}$, there exists an integer $n = n(\omega, z, r) \geq 0$ such that $f_\omega^n(B(z, r)) \cap \mathbb{R} \neq \emptyset$. So, there exists $z' \in B(z, r)$ such that $f_\omega^n(z') \in \mathbb{R}$. Since for every $\omega \in \Omega$ and every $w \in \mathbb{R}$,

$$\lim_{k\to\infty} \big(f_\omega^k\big)'(w) = \lim_{k\to\infty} f_\omega^k(w) = +\infty,$$

and since each map f_η is 1–to–1 on each open ball with radius π, we first conclude that for all integers $k \geq 0$ large enough

$$f_\omega^k(B(z, r)) \supset B\big(f_\omega^k(z'), \pi\big).$$

Having this, using the above, we then immediately conclude that for given $S > 0$, we have that

$$f_\omega^k(B(z, r)) \supset B\big(f_\omega^k(z'), S\big)$$

for all integers $k \geq 0$ large enough. Then the sets $f_\omega^{k+1}(B(z, r))$ contain annuli centered at the origin with the ratio of the outer and inner radii as large as one wishes. These annuli in turn will contain some set of the form

$$Q_{M_0} + 2l\pi i,$$

where $l \in \mathbb{Z}$. This yields

$$F_\omega^{k+1}(B(z, r)) \supset Q_{M_0} \tag{8.8}$$

for all integers $k \geq 0$ large enough. On the other hand, if $\nu_\omega^{(t)}(B(z, r)) = 0$ then, using conformality of the measures ν_γ, $\gamma \in \Omega$, i.e. using (8.6), we conclude that

$$\nu_{\theta^{k+1}\omega}^{(t)}(F_\omega^{k+1}(B(z, r))) = 0.$$

This contradicts (5.12) and (8.8), finishing the proof of the first part of Proposition 39.

In order to prove the second statement first note that in view of its first part, we have that for every radius $r > 0$ and m–a.e. $\omega \in \Omega$,

$$\xi_r(\omega) := \inf\big\{\nu_\omega(B(z, r)) : z \in Q_x\big\} > 0. \tag{8.9}$$

Now, fix a countable dense subset Γ of Q_x. Then the function

$$\Omega \ni \omega \longmapsto \xi_R^*(\omega) := \inf\big\{\nu_\omega(B(z, R/2)) : z \in \Gamma\big\} \in [0, 1]$$

is measurable and

$$\xi_{R/2}(\omega) \leq \xi_R^*(\omega) \leq \xi_R(\omega). \tag{8.10}$$

In particular $\xi_R^*(\omega) > 0$ for m–a.e. $\omega \in \Omega$. Therefore, there exists $\xi > 0$ so small that

$$m\big((\xi_R^*)^{-1}((\xi, +\infty)\big) > 1 - \varepsilon.$$

Hence, taking

$$\Omega(x, R, \varepsilon) := m\big((\xi_R^*)^{-1}((\xi, +\infty)\big)$$

and taking into account the right–hand part of completes the proof. ☐

Now we shall prove a lemma which is of more restricted scope than Proposition 39 but which gives estimates uniform with respect to all $\omega \in \Omega$. We will then derive some of its consequences and will use them later in the paper.

Lemma 40. *For every radius $r > 0$ there exists $\Delta(r) \in (0, +\infty)$ such that*

$$\nu_\omega^{(t)}(B(0,r)) \geq \Delta(r)$$

for m–a.e. $\omega \in \Omega$.

Proof. Proceeding in the same way as at the beginning of the proof of Proposition 39, we conclude that there exists an integer $k = k(r) \geq 0$ such that

$$F_\omega^k(B(0,r)) \supset Q_{M_0}$$

for m–a.e. $\omega \in \Omega$. Because of the right hand side of (8.7) and because of (5.12), we get that

$$\frac{1}{2} \leq \nu_{\theta^k\omega}^{(t)}\big(F_\omega^k(B(0,r))\big) \leq \lambda_{t,\omega}^k \int_{B(0,r)} \left|\big(F_\omega^k\big)'\right|^t d\nu_\omega^{(t)} \leq P^k\big(f_B^k(0)\big)^t \nu_\omega^{(t)}(B(0,r)).$$

Hence,

$$\nu_\omega^{(t)}(B(0,r)) \geq \frac{1}{2} P^{-k}\big(f_B^k(0)\big)^{-t} > 0,$$

and the proof is complete. □

Corollary 41. *For every $r > 0$ there exist $r_* > 0$ and $\Delta^*(r) > 0$ such that*

$$\nu_\omega(B(0,r) \setminus B(0,r_*)) \geq \Delta^*(r)$$

for m–a.e. $\omega \in \Omega$.

Proof. Fix $u > 0$ produced in Lemma 24. Take then $r_* \in (0,r)$ so small that $r_*^u < \frac{1}{2}\Delta(r)$. It then follows from Lemma 40 and Lemma 24 that

$$\nu_\omega(B(0,r) \setminus B(0,r_*)) = \nu_\omega(B(0,r)) - \nu_\omega(B(0,r_*)) \geq \Delta(r) - r_*^u \geq \Delta(r) - \frac{1}{2}\Delta(r) = \frac{1}{2}\Delta(r) > 0$$

So, taking $\Delta^*(r) := \frac{1}{2}\Delta(r)$ completes the proof. □

Corollary 42. *For every $M > 0$ there exist $M_+ \in (M, +\infty)$ and $\Delta_-(M) > 0$ such that*

$$\nu_\omega\big(Y_M^- \setminus Y_{M_+}^-\big) \geq \Delta_-(M)$$

for m–a.e. $\omega \in \Omega$.

Proof. Take $M_+ \in (M, +\infty)$ so large that

$$Be^{-M_+} < (Ae^{-M})_*,$$

where $(Ae^{-M})_*$ comes from Corollary 41. Using this corollary and the right-hand side of (8.7) again, we obtain

$$F_\omega\big(Y_M^- \setminus Y_{M_+}^-\big) \supset B\big(0, Ae^{-M}\big) \setminus B\big(0, Be^{-M_+}\big) \supset B\big(0, Ae^{-M}\big) \setminus B\big(0, (Ae^{-M})_*\big)$$

and

$$\Delta^*(Ae^{-M}) \leq \nu_{\theta\omega}\big(F_\omega\big(Y_M^- \setminus Y_{M_+}^-\big)\big) \leq \lambda_{t,\omega} \int_{Y_M^- \setminus Y_{M_+}^-} \left|\big(F_\omega\big)'\right|^t d\nu_\omega \leq PB^t \nu_\omega\big(Y_M^- \setminus Y_{M_+}^-\big).$$

Hence,

$$\nu_\omega\big(Y_M^- \setminus Y_{M_+}^-\big) \ge P^{-1}B^{-t}\Delta^*(Ae^{-M}),$$

and the proof is complete. □

9. Random Invariant Measures Equivalent to Random Conformal Measures

From now on until explicitly stated otherwise, we fix $t > 1$ and the random t–conformal measure $\nu := \nu^{(t)}$, with disintegrations $\big(\nu_\omega\big)_{\omega\in\Omega}$ constructed in the previous section. Recall that we denote

$$\lambda_{t,\omega} = \mathcal{L}_{t,\omega}^*\nu_{\theta\omega}(1\!\!1) \tag{9.1}$$

for all $\omega \in \Omega$. We will also use the notation

$$\lambda_{t,\omega}^n := \prod_{j=0}^{n-1}\lambda_{t,\theta^j\omega}.$$

We introduce normalized operators

$$\hat{\mathcal{L}}_{t,\omega} := \lambda_{t,\omega}^{-1}\mathcal{L}_{t,\omega} \ \text{ and } \ \hat{\mathcal{L}}_{t,\omega}^n := (\lambda_{t,\omega}^n)^{-1}\mathcal{L}_{t,\omega}^n,$$

so that

$$\hat{\mathcal{L}}_{t,\omega}(\nu_{\theta\omega}) = \nu_\omega.$$

Our purpose in this section is to prove the following.

Theorem 43. *There exists a random measure μ, i.e. one belonging to $\mathcal{M}_m$, such that for all $\omega \in \Omega$ the fiber measures μ_ω and ν_ω are equivalent, and the random measure μ is F–invariant. The latter meaning that*

$$\mu \circ F^{-1} = \mu,$$

or equivalently:

$$\mu_\omega \circ F_\omega^{-1} = \mu_{\theta\omega}$$

for m–a.e. $\omega \in \Omega$.

The proof of Theorem 43 will follow from Proposition 52. We start with some estimates. Fix some numbers $u > 2t + 7$ and $\rho > 0$ satisfying Lemma 24. Also, because of Lemma 20 there exists $M_1 \ge M_0$ large enough so that

$$\frac{1}{p}\sup\big\{\mathcal{L}_{t,\omega}1\!\!1(z) : z \in Y_{M_1}\big\} < \frac{1}{2}. \tag{9.2}$$

The need for such choice of M_1 will become clear in the course of the proof of Proposition 51. Note that there exists an integer $N \ge 1$ large enough that for all ω

$$Q_{M_1} \cap \bigcup_{j=N+1}^{\infty} B\big(F_{\theta^{-j}\omega}^j(0), r_0\big) = \emptyset.$$

Since also $\nu_\omega(Q_{M_1}) > 1/2$ for m–a.e. $\omega \in \Omega$, decreasing $r_0 > 0$ if necessary, we can assume without loss of generality that $0 < r_0 < \rho$ and

$$\nu_\omega\left(Q_{M_1} \setminus \bigcup_{j=0}^{\infty} B\big(F^j_{\theta^{-j}\omega}(0), r_0\big)\right) > 1/2$$

for m–a.e. $\omega \in \Omega$.

Lemma 44. *If $n \geq 0$ is an integer and*

$$A \subset Q_{M_1} \setminus \bigcup_{j=0}^{N} B\big(F^j_{\theta^{n-j}\omega}(0), r_0\big) \tag{9.3}$$

is a Borel set, then

$$\nu_\omega(F_\omega^{-n}(A)) \leq c(M_1, r_0)\nu_{\theta^n\omega}(A), \tag{9.4}$$

where $c(M_1, r_0) > 0$ is some constant depending on M_1 and r_0, but independent of ω.

Proof. Notice that by partitioning the set

$$Q_{M_1} \setminus \bigcup_{j=0}^{N} B\big(F^j_{\theta^{n-j}\omega}(0), r_0\big)$$

into a finite disjoint union of Borel sets with diameters smaller than $r_0/4$, we may assume without loss of generality that $\mathrm{diam}(A) < r_0/4$. Then we further notice that holomorphic branches of F_ω^{-n}, labeled as $F_{\omega,*}^{-n}$ are well–defined on A, in fact on a ball with radius $r_0/2$ centered at a point of A, with distortion bounded by K, meaning that

$$\frac{|(F_{\omega,*}^{-n})'(x)|}{|(F_{\omega,*}^{-n})'(y)|} \leq K$$

for all $x, y \in A$. We have

$$\nu_\omega(F_\omega^{-n}(A)) = \int_A \hat{\mathcal{L}}^n_{t,\omega}(\mathbb{1})(z)d\nu_{\theta^n\omega}(z) \leq \sup_A \big(\hat{\mathcal{L}}^n_{t,\omega}(\mathbb{1})\big)\nu_{\theta^n\omega}(A). \tag{9.5}$$

In order to establish the upper bound for $\sup_A \big(\hat{\mathcal{L}}^n_{t,\omega}(\mathbb{1})\big)$ notice that

$$\begin{aligned}\nu_\omega\Big(F_\omega^{-n}\Big(Q_{M_1}\setminus\bigcup_{j=0}^{N} B\big(F^j_{\theta^{n-j}\omega}(0), r_0\big)\Big)\Big) &= \\ &= \int_{Q_{M_1}\setminus\bigcup_{j=0}^{N} B\left(F^j_{\theta^{n-j}\omega}(0), r_0\right)} \hat{\mathcal{L}}^n_{t,\omega}(\mathbb{1})(z)d\nu_{\theta^n\omega}(z) \\ &\geq \inf_{Q_{M_1}} \big(\hat{\mathcal{L}}^n_{t,\omega}(\mathbb{1})\big)\nu_{\theta^n\omega}\Big(Q_{M_1} \setminus \bigcup_{j=0}^{N} B\big(F^j_{\theta^{n-j}\omega}(0), r_0\big)\Big).\end{aligned}$$

Now, again by distortion estimates, there exists a constant $c(M_1, r_0) > 0$ such that

$$\begin{aligned}(9.6)\quad &\inf\Big(\hat{\mathcal{L}}_{t,\omega}^n(\mathbb{1})(z) : z \in Q_{M_1} \setminus \bigcup_{j=0}^N B\big(F^j_{\theta^{n-j}\omega}(0), r_0\big)\Big) \geq \\ &\geq \frac{2}{c(M_1, r_0)} \sup\Big(\hat{\mathcal{L}}_{t,\omega}^n(\mathbb{1})(z) : z \in Q_{M_1} \setminus \bigcup_{j=0}^N B\big(F^j_{\theta^{n-j}\omega}(0), r_0\big)\Big).\end{aligned}$$

Thus,

$$\begin{aligned}1 &\geq \nu_\omega\Big(F_\omega^{-n}\Big(Q_{M_1} \setminus \bigcup_{j=0}^N B\big(F^j_{\theta^{n-j}\omega}(0), r_0\big)\Big)\Big) \\ &\geq \frac{1}{2}\frac{2}{c(M_1, r_0)} \sup\Big(\hat{\mathcal{L}}_{t,\omega}^n(\mathbb{1})(z) : z \in Q_{M_1} \setminus \bigcup_{j=0}^N B\big(F^j_{\theta^{n-j}\omega}(0), r_0\big)\Big) \\ &= \frac{1}{c(M_1, r_0)} \sup\Big(\hat{\mathcal{L}}_{t,\omega}^n(\mathbb{1})(z) : z \in Q_{M_1} \setminus \bigcup_{j=0}^N B\big(F^j_{\theta^{n-j}\omega}(0), r_0\big)\Big),\end{aligned}$$

i.e.

$$(9.7)\quad \sup\Big(\hat{\mathcal{L}}_{t,\omega}^n(\mathbb{1})(z) : z \in Q_{M_1} \setminus \bigcup_{j=0}^N B\big(F^j_{\theta^{n-j}\omega}(0), r_0\big)\Big) \leq c(M_1, r_0).$$

So, inserting this estimate to (9.5), we obtain $\nu_\omega(F_\omega^{-n}(A)) \leq c(M_1, r_0)\nu_{\theta^n\omega}(A)$, as required. □

Given $\omega \in \Omega$, $n \in \mathbb{N}$, and $0 \leq j \leq N$, set

$$\beta_{n,j}(\omega) := F^j_{\theta^{n-j}\omega}(0).$$

Now let

$$A \subset B\big(\beta_{n,j}(\omega), r_0\big)$$

be an arbitrary Borel set. Consider all connected components C of $F_\omega^{-n}\big(B\big(\beta_{n,j}(\omega), r_0\big)\big)$. We say that such a C is good if there exists a holomorphic branch of F_ω^{-n} defined on $B\big(\beta_{n,j}(\omega), r_0\big))$ and mapping $B\big(\beta_{n,j}(\omega), r_0\big))$ onto C. Otherwise, we say that C is bad. Note that C is bad if and only if $0 \in f_{\theta^{k+1}\omega}(F^k_\omega(C))$ for some $0 \leq k \leq n-1$. Equivalently, C is bad if and only if C is unbounded. Now, the set $F_\omega^{-n}(A)$ splits into the disjoint union

$$F_\omega^{-n}(A) = F^{-n}_{\omega,B}(A) \cup F^{-n}_{\omega,G}(A),$$

where $F^{-n}_{\omega,B}(A)$ is the intersection of $F_\omega^{-n}(A)$ with the union of all bad components of $F_\omega^{-n}\big(B\big(\beta_{n,j}(\omega), r_0\big)\big)$ and $F^{-n}_{\omega,G}(A)$ is the intersection of $F_\omega^{-n}(A)$ with the union of all good components of $F_\omega^{-n}\big(B\big(\beta_{n,j}(\omega), r_0\big)\big)$

The next lemma is proved in an analogous way as Lemma 44, with possibly modified constant $c(M_1, r_0)$, still independent of $\omega \in \Omega$.

Lemma 45. *If $\omega \in \Omega$, $n \in \mathbb{N}$, $0 \leq j \leq N$, and $A \subset B\big(\beta_{n,j}(\omega), r_0\big)$ is an arbitrary Borel set, then*

$$\nu_\omega(F^{-n}_{\omega,G}(A)) \leq c(M_1, r_0) \cdot \nu_{\theta^n\omega}(A).$$

In Lemma 49, we will provide estimates for bad components of $F_\omega^{-n}(A)$. In order to do this, we start with the following.

Lemma 46. *There exits a constant $\gamma > 0$ such that for all radii $0 < r \le r_0$, all integers $n \ge 0$, and all m–a.e. $\omega \in \Omega$, we have that*

$$\nu_\omega(F_{\omega,B}^{-n}(B(0,r)) \preceq r^\gamma.$$

Proof. First note that the only bad component of $F_{\theta^{n-1}\omega,B}^{-1}(B(0,r))$ is of the form $\pi \circ f_{\theta^{n-1}\omega}^{-1}(B(0,r))$ where the latter $B(0,r)$ is considered as a subset of $\mathbb{C}$, and $\pi : \mathbb{C} \to Q$ is the canonical projection. Thus,

$$F_{\theta^{n-1}\omega,B}^{-1}(B(0,r)) = Y_M^- \tag{9.8}$$

for some $M \in [\ln(1/r) + \ln A, \ln(1/r) + \ln B]$. Next, using the estimate from Lemma 24 and (8.7), we easily conclude that for $0 < r \le r_0$, we have that

$$\nu_{\theta^{n-1}\omega}\big(F_{\theta^{n-1}\omega,B}^{-1}(B(0,r))\big) \le Cr^{-t}r^u = Cr^{u-t}. \tag{9.9}$$

with some constant $C \in (0, +\infty)$. Now, every component of $F_{\omega,B}^{-n}(B(0,r))$ is of the form $F_*^{-(n-1)}(Y_M^-)$ where $F_*^{-(n-1)}(Y_M^-)$ is some connected component of $F^{-(n-1)}(Y_M^-)$. Let us note that the set $f_{\theta^{(n-2)}\omega}^{-1}(\{Z \in \mathbb{C} : \mathrm{Re}Z < -M\})$ is a union of (repeated periodically, with period $2\pi i$) unbounded components, each being bounded by some curve of the form

$$f_{\theta^{(n-2)}\omega}^{-1}(\{Z \in \mathbb{C} : \mathrm{Re}Z = -M\}).$$

Since the projection onto Q identifies these components, the set

$$\mathcal{C}_M := F_{\theta^{(n-2)}\omega}^{-1}(Y_M^-) \subset Q$$

is connected, and the map F restricted to $\mathcal{C}_M$ is infinite–to–one. Similarly, the set

$$F_{\theta^{(n-2)}\omega}^{-1}(Y_1^-) \supset \mathcal{C}_M$$

is connected, and the map $F_{\theta^{(n-2)}\omega}$ restricted to $\mathcal{C}_1$ is infinite–to–one.

Now, the holomorphic branches of $F_\omega^{-(n-2)}$ are all well defined on $\mathcal{C}_1$ and the restriction of these branches to $\mathcal{C}_M$ produces all bad connected components of $F_\omega^{-n}(B(0,r))$, i.e., the set $F_{\omega,B}^{-n}(B(0,r))$. Denote

$$Y(*) := Y_1^- \setminus Y_{1_+}^- = \{z \in Q : \mathrm{Re}z \in [-1_+, -1]\},$$

and partition the set $\mathcal{C}_1$ into subsets $\mathcal{C}_1^k$ by defining

$$\mathcal{C}_1^k := \big\{z \in \mathcal{C}_1 : \mathrm{Im}f_{\theta^{(n-2)}\omega}(z) \in [2k\pi, 2(k+1)\pi)\big\}.$$

Similarly, let

$$\mathcal{C}_M^k := C_M \cap \mathcal{C}_1^k = \big\{z \in \mathcal{C}_M : \mathrm{Im}f_{\theta^{(n-2)}\omega}(z) \in [2k\pi, 2(k+1)\pi)\big\}.$$

Then for each $k \in \mathbb{Z}$ the function $f_{\theta^{n-2}\omega}$ maps $\mathcal{C}_1^k$ bijectively onto the region

$$\big\{Z \in \mathbb{C} : \mathrm{Re}Z < -1 \text{ and } \mathrm{Im}Z \in [2k\pi, 2(k+1)\pi)\big\},$$

which we identify with Y_1^-. Denote by G_k^* the corresponding inverse map. Then the holomorphic map

$$Z \longmapsto G_k(z) := G_k^*(Z + 2k\pi i)$$

is in fact defined and univalent on $\{Z\in\mathbb{C}:\mathrm{Re}(Z)<-1\}$, and maps the region

$$\big\{Z\in\mathbb{C}:\mathrm{Re}(Z)<-1 \text{ and } \mathrm{Im}Z\in[0,2\pi)\big\},$$

which we identify with Y_1^-, onto $\mathcal{C}_1^k$, while it maps the region

$$\big\{Z\in\mathbb{C}:\mathrm{Re}(Z)<-M \text{ and } \mathrm{Im}Z\in[0,2\pi)\big\},$$

which we identify with Y_M^-, onto $\mathcal{C}_M^k$.

Still keeping the identification $Q\simeq\{Z\in\mathbb{C}:0\le \mathrm{Im}Z<2\pi\}$, we thus see that the inverse–image

$$F_{\omega,B}^{-n}(B(0,r))=F_\omega^{-(n-1)}(Y_M^-)$$

can be expressed as

$$\bigcup_{k\in\mathbb{Z}}\bigcup_g g\circ G_k(Y_M^-),$$

where, the second union is taken over all holomorphic branches g of $F_\omega^{-(n-2)}$ defined on $\mathcal{C}_1$.

Since, as we see, each such branch $g\circ G_k$ has a univalent holomorphic extension to the whole left half-plane $\{Z\in\mathbb{C}:\mathrm{Re}(Z)<-1\}$, we can use Koebe's Distortion Theorem to compare the measure $\nu_\omega\big(g\circ G_k(Y_1^-\setminus Y_{1_+}^-)\big)$ and $\nu_\omega\big(g\circ G_k(Y_M^-)\big)$. Applying this theorem separately for each composition $g\circ G_k$ and then summing up, with using also (8.7), we obtain that

$$\frac{\nu_\omega(F_\omega^{-(n-1)}(Y_M^-))}{\nu_\omega\big(F_\omega^{-(n-1)}\big(Y_1^-\setminus Y_{1_+}^-\big)\big)}\preceq |M|^3\frac{\nu_{\theta^{n-1}\omega}(Y_M^-)}{\nu_{\theta^{n-1}\omega}\big(Y_1^-\setminus Y_{1_+}^-\big)}.$$

By virtue of (9.8) and (9.9), this gives

$$\begin{aligned}\nu_\omega((F_{\omega,B}^{-n}(B(0,r)))=\nu_\omega(F_\omega^{-(n-1)}(Y_M^-))&\preceq \nu_\omega\big(F_\omega^{-(n-1)}\big(Y_1^-\setminus Y_{1_+}^-\big)\big)\frac{|M|^3r^{u-t}}{\nu_{\theta^{n-1}\omega}\big(Y_1^-\setminus Y_{1_+}^-\big)}\\ &\le \frac{|M|^3r^{u-t}}{\nu_{\theta^{n-1}\omega}\big(Y_1^-\setminus Y_{1_+}^-\big)}.\end{aligned}$$

The proof is now completed by invoking the bounds $\ln(1/r)+\ln A\le M\le \ln(1/r)+\ln B$ along with Corollary 42 which gives

$$\nu_\omega\big(Y_1^-\setminus Y_{1_+}^-\big)\ge \Delta_-(1)>0.$$

□

As a consequence of Lemma 24, Lemma 45, and Lemma 46, we get the following.

Lemma 47. *We have that*

$$\nu_\omega(F_\omega^{-n}\big(B(0,r))\big)\preceq r^\gamma$$

for every integer $n\ge 0$, all $\omega\in\Omega$ and every $r\in(0,r_0]$.

We shall prove the following.

Lemma 48. *There exists $\beta>0$ such that for every Borel set $A\subset B(0,r_0)$ and all integers $n\ge 0$ we have that*

$$\nu_\omega(F_\omega^{-n}(A))\preceq \nu_{\theta^n\omega}^\beta(A).$$

Proof. By Lemma 24 and Lemma 47 there exist constants $C \in (0,+\infty)$ and $D \in (0,+\infty)$ such that

$$\nu_\omega(B(0,r)) \le Cr^u$$

and

$$\nu_\omega\big(F_\omega^{-n}\big(B(0,r)\big)\big) \le Dr^\gamma.$$

for all $r \in (0,r_0]$, almost all $\omega \in \Omega$ and all integers $n \ge 0$. So, Since $u > 6$, given such r_0, ω, and n, there exists $r \in (0,r_0]$ such that

$$\nu_{\theta^n\omega}(A) = Cr^6.$$

Then,

$$\nu_\omega(F^{-n}(A \cap B(0,r))) \le \nu_\omega\big(F_\omega^{-n}(B(0,r))\big) \le Dr^\gamma = D\left(\frac{\nu_{\theta^n\omega}(A)}{C}\right)^{\gamma/6} = DC^{-\gamma/6}\nu_{\theta^n\omega}^{\gamma/6}(A),$$

while using (9.7), we get

$$\begin{aligned}\nu_\omega\big(F_\omega^{-n}(A \setminus B(0,r))\big) &\le \sup\big\{\hat{\mathcal{L}}_{t,\omega}^n(\mathbb{1})(z) : z \in A \setminus B(0,r)\big\}\,\nu_{\theta^n\omega}(A \setminus B(0,r)) \\ &\preceq r^{-3}\sup\Big\{\hat{\mathcal{L}}_{t,\omega}^n(\mathbb{1})(z) : z \in Q_{M_1} \setminus \bigcup_{j=0}^{N} B(\beta_{n,j}(\omega),r_0)\Big\}\,\nu_{\theta^n\omega}(A) \\ &\preceq c(M_1,r_0)\nu_{\theta^n\omega}^{-1/2}(A)\nu_{\theta^n\omega}(A) \\ &= c(M_1,r_0)\nu_{\theta^n\omega}^{1/2}(A).\end{aligned}$$

Thus, the statement holds with $\beta := \min(\gamma/6, 1/2)$. □

Lemma 49. *There exists $\beta > 0$ such that, if $\omega \in \Omega$, $j \le N$, $n \in \mathbb{N}$, and $A \subset B(\beta_{n,j}(\omega),r_0)$ is an arbitrary Borel set, then*

$$\nu_\omega(F_{\omega,B}^{-n}(A)) \preceq \nu_{\theta^n\omega}^\beta(A).$$

Proof. Recall that the bad components of $F_\omega^{-n}(B(\beta_{n,j}(\omega),r_0))$ are all the connected components of the set

$$F_\omega^{-(n-j)}(F_{\theta^{n-j}\omega,*}^{-j}(B(\beta_{n,j}(\omega),r_0))),$$

where $F_{\theta^{n-j}\omega,*}^{-j}$ is the branch of $F_{\theta^{n-j}\omega}^{-j}$ mapping $B(\beta_{n,j}(\omega),r_0)$ into $B(0,r_0)$, and $F_{\omega,B}^{-n}(A))$ is the union of all these components intersected with $F_\omega^{-n}(A))$. Since, using (8.7), we obtain

$$\nu_{\theta^{n-j}\omega}(F_{\theta^{n-j}\omega,*}^{-j}(A)) \le \max_{0\le k\le N}\big\{K^t|(F_{\theta^{n-j}\omega}^k)'(0)|^{-t}p^k\big\}\nu_{\theta^n\omega}(A),$$

we thus conclude the proof by applying Lemma 48. □

We summarize the above Lemmas 44, 45, 48, 49 in the following.

Lemma 50. *There exists $\beta > 0$ such that for every Borel set $A \subset Q_{M_1}$ and for every $n \ge 0$*

$$\nu_\omega(F_\omega^{-n}(A)) \preceq \nu_{\theta^n\omega}^\beta(A).$$

The next proposition deals with sets contained in the complement of Q_{M_1}.

Lemma 51. *There exists $\beta > 0$ such that for every Borel set $A \subset Y_{M_1}$ and for every $n \ge 0$*

$$\nu_\omega(F_\omega^{-n}(A)) \preceq \nu_{\theta^n\omega}^\beta(A).$$

Proof. First, let us notice that using (9.2) and the bounds on $\lambda_{t,\omega}$, see (8.7), we can estimate as follows.

$$\begin{aligned}\nu_{\theta^{n-1}\omega}(F^{-1}_{\theta^{n-1}\omega}(A)) &= \int_A \hat{\mathcal{L}}_{t,\theta^{n-1}\omega}(\mathbb{1})(z)d\nu_{\theta^n\omega}(z)\\ &\le \frac{1}{p}\sup\left\{\mathcal{L}_{t,\theta^{n-1}\omega}(\mathbb{1})(z) z\in Y_{M_1}\right\}\nu_{\theta^n\omega}(A) < \frac{1}{2}\nu_{\theta^n\omega}(A).\end{aligned} \tag{9.10}$$

Write

$$F^{-1}_{\theta^{n-1}\omega}(A) = A_1\cup A_2 = (F^{-1}_{\theta^{n-1}\omega}(A)\cap Q_{M_1})\cup(F^{-1}_{\theta^{n-1}\omega}(A)\setminus Q_{M_1})$$

and

$$F_\omega^{-n}(A) = F_\omega^{-(n-1)}(A_1)\cup F_\omega^{-(n-1)}(A_2).$$

Using (9.10) and Lemma 50, we have, with some positive constant C_1 guaranteed by Lemma 50:

$$\nu_\omega(F_\omega^{-(n-1)}(A_1)) \le C_1\nu^\beta_{\theta^{n-1}\omega}(A_1) \le C_1\nu^\beta_{\theta^{n-1}\omega}(F^{-1}_{\theta^{n-1}\omega}(A)) \le 2^{-\beta}C_1\nu^\beta_{\theta^n\omega}(A),$$

while, again,

$$F^{-1}_{\theta^{n-2}\omega}(A_2) = A_{21}\cup A_{22} = (F^{-1}_{\theta^{n-2}\omega}(A_2)\cap Q_{M_1})\cup(F^{-1}_{\theta^{n-2}\omega}(A_2)\setminus Q_{M_1}),$$

and

$$F_\omega^{-(n-1)}(A_2) = F_\omega^{-(n-2)}(A_{21})\cup F_\omega^{-(n-2)}(A_{22}),$$

where

$$\nu_{\theta^{n-2}\omega}(A_{21}) \le \frac{1}{2}\nu_{\theta^{n-1}\omega}(A_2) \le \left(\frac{1}{2}\right)^2\nu_{\theta^n\omega}(A)$$

and, similarly,

$$\nu_{\theta^{n-2}\omega}(A_{22}) \le \left(\frac{1}{2}\right)^2\nu_{\theta^n\omega}(A).$$

Proceeding inductively we thus obtain the following splitting.

$$F_\omega^{-n}(A) = F_\omega^{-(n-1)}(A_1)\cup F_\omega^{-(n-2)}(A_{21})\cup F_\omega^{-(n-3)}(A_{221})\cdots\cup F_\omega^{-1}(A_{22\ldots1})\cup A_{22\ldots2},$$

where

$$\nu_{\theta^{n-k}\omega}(A_{22\ldots1}) \le \left(\frac{1}{2}\right)^k\nu_{\theta^n\omega}(A).$$

Since for all sets $A_{22\ldots1}$ Lemma 50 applies, we conclude that

$$\nu_\omega(F_\omega^{-n}(A)) \le C_1\nu^\beta_{\theta^n\omega}(A)(1+2^{-\beta}+2^{-2\beta}+\cdots+2^{-n\beta}) + \left(\frac{1}{2}\right)^n\nu_{\theta^n\omega}(A).$$

This ends the proof, with possibly modified constant c_1, and the same β as in Lemma 50. □

We summarize the above lemmas as follows.

Proposition 52. *There exist constants $\beta>0$ and $C>0$ such that if $A\subset Q$ is a Borel set then for every $n\in\mathbb{N}$ and for m–a.e. $\omega\in\Omega$,*

$$\nu_\omega(F_\omega^{-n}(A)) \le C\nu^\beta_{\theta^n\omega}(A).$$

Now we are position to prove the following.

Theorem 53. *For every $t > 1$ there exists a random Borel probability F–invariant measure $\mu = \mu^{(t)}$ absolutely continuous with respect to $\nu^{(t)}$, the t-conformal random measure for $F : \Omega \times \mathbb{C} \to \Omega \times \mathbb{C}$, produced in Theorem 38. Furthermore,*

$$\mu^{(t)}(A) = \ell_B\big((\nu^{(t)} \circ F^{-n}(A))_{n=0}^{\infty}\big),$$

where $\ell_B : \ell_\infty \to \mathbb{R}$ is a (fixed) Banach limit on ℓ_∞.

Proof. It is well–known in abstract ergodic theory that all assertions of Theorem 53, perhaps except that $\mu \in \mathcal{M}_m$, would follow from uniform absolute continuity of measures $\big(\nu \circ F^{-n}\big)_{n=0}^{\infty}$ with respect to measure ν. In order to prove this uniform continuity, fix $\varepsilon > 0$ and suppose that $A \subset \Omega \times Q$ is a measurable set such that $\nu(A) < \varepsilon^2$. We then get for every integer $n \geq 0$ that

$$\begin{aligned}\nu(F^{-n}(A)) &= \int_\Omega \nu_\omega(F_\omega^{-n}(A_{\theta^n(\omega)}))\, dm(\omega)\\ &= \int_{\Omega_0} \nu_\omega(F_\omega^{-n}(A_{\theta^n(\omega)}))\, dm(\omega) + \int_{\Omega_0^c} \nu_\omega(F_\omega^{-n}(A_{\theta^n(\omega)}))\, dm(\omega),\end{aligned}$$

where

$$\Omega_0 := \big\{\omega \in \Omega : \nu_{\theta^n(\omega)}(A_{\theta^n(\omega)}) \geq \varepsilon\big\}.$$

But then

$$m(\Omega_0) = m\big(\{\omega \in \Omega : \nu_\omega(A_\omega) \geq \varepsilon\}\big) \leq \nu(A)/\varepsilon.$$

So, applying Proposition 52, we get that

$$\nu(F^{-n}(A)) \leq \frac{\nu(A)}{\varepsilon} + C\varepsilon^\beta \leq \varepsilon + \varepsilon^\beta,$$

and the required uniform absolute continuity has been proved. In order to see that $\mu \in \mathcal{M}_m$, let Γ be an arbitrary Borel subset of Ω. Then

$$\begin{aligned}\mu(\Gamma \times \mathbb{C}) &= \ell_B\big((\nu \circ F^{-n}(\Gamma \times \mathbb{C}))_{n=0}^{\infty}\big) = \ell_B\big((\nu(\theta^{-n}(\Gamma) \times \mathbb{C}))_{n=0}^{\infty}\big)\\ &= \ell_B\big((m(\theta^{-n}(\Gamma)))_{n=0}^{\infty}\big) = \ell_B\big((m(\Gamma))_{n=0}^{\infty}\big) = m(\Gamma).\end{aligned}$$

This means that $\mu \in \mathcal{M}_m$, and the proof is complete. □

We can prove more about the invariant measure $\mu^{(t)}$. Namely:

Theorem 54. *Let $t > 1$. If $\nu^{(t)}$ is the t-conformal random measure for $F : \Omega \times \mathbb{C} \to \Omega \times \mathbb{C}$, produced in Theorem 38, then the Borel probability F–invariant measure $\mu = \mu^{(t)} \in \mathcal{M}_m$ absolutely continuous with respect to $\nu^{(t)}$, produced in Theorem 53, is in fact equivalent with $\nu^{(t)}$.*

Proof. To simplify notation, we write again $\nu := \nu^{(t)}$. Since $\lim_{n\to\infty} F_\omega^n(0) = +\infty$ uniformly with respect to $\omega \in \Omega$ and since each measure μ_ω is a probability one satisfying, by virtue of F-invariance, $\mu_{\theta(\omega)}(F_\omega(A)) \geq \mu_\omega(A)$ for every $\omega \in \Omega$ and every Borel set $A \subset \mathbb{C}$, we have that

$$\mu_\omega\big(\big\{F^n_{\theta^{-n}(\omega)}(0) : n \geq 0\big\}\big) = 0$$

for m–almost all $\omega \in \Omega$. Therefore,

$$\mu\left(\bigcup_{\omega\in\Omega} \{\omega\} \times \big\{F^n_{\theta^{-n}(\omega)}(0) : n \geq 0\big\}\right) = 0$$

Hence, there exists $R \in (0, r_0/2)$ so small that

$$\mu\left(\bigcup_{\omega\in\Omega}\{\omega\}\times\bigcup_{n=0}^\infty B\big(F^n_{\theta^{-n}(\omega)}(0),2R\big)\right) < 1/8.$$

Hence, there exists a measurable set $\Omega_0 \subset \Omega$ such that

$$m(\Omega_0) \geq 1/2 \tag{9.11}$$

and

$$\mu_\omega\left(\bigcup_{n=0}^\infty B\big(F^n_{\theta^{-n}(\omega)}(0),2R\big)\right) < 1/4 \quad \text{for all} \quad \omega \in \Omega_0. \tag{9.12}$$

Now, there exists a constant $M > 0$ so large that $\mu(\Omega \times Y_M) < 1/8$, and therefore there exists a measurable set $\Omega_1 \subset \Omega_0$ such that

$$m(\Omega_1) \geq 1/4$$

and

$$\mu_\omega(Q_M) \geq 1/2 \quad \text{for all} \quad \omega \in \Omega_1.$$

Combining this along with (9.12), we conclude that there exists $\alpha > 0$ and for every $\omega \in \Omega_1$ there exists

$$\xi_\omega \in Q_M \cap \left(\mathbb{C}\setminus\bigcup_{n=0}^\infty B\big(F^n_{\theta^{-n}(\omega)}(0),2R\big)\right)$$

such that

$$\mu_\omega(B(\xi_\omega, R)) \geq \alpha \tag{9.13}$$

and the choice $\Omega_1 \ni \omega \mapsto \xi_\omega$ is measurable. Let

$$\Gamma := \bigcup_{\omega\in\Omega_1}\{\omega\}\times B(\xi_\omega, R).$$

Of course, $\mu(\Gamma) \geq \alpha/8 > 0$. We shall prove the following

Claim 1^0: If $A \subset \Gamma$ is a measurable set and $\nu(A) > 0$, then $\mu(A) > 0$.

Proof. Because of our definition of the set Γ, for every $\omega \in \Omega_1$, every integer $n \geq 0$, and every $\xi \in F^{-n}_{\theta^{-n}(\omega)}(\xi_\omega)$, we have that

$$\begin{aligned}\nu_{\theta^{-n}(\omega)}\big(F^{-n}_{\theta^{-n}(\omega),\xi}(A_\omega)\big) &= \lambda^{-n}_{\theta^{-n}(\omega)}\int_{A_\omega}\big|\big(F^{-n}_{\theta^{-n}(\omega),\xi}\big)'\big|^t\,d\nu_\omega \\ &\geq K^{-t}\lambda^{-n}_{\theta^{-n}(\omega)}\big|\big(F^n_{\theta^{-n}(\omega)}\big)'(\xi)\big|^{-t}\nu_\omega(A_\omega),\end{aligned}$$

while

$$\nu_{\theta^{-n}(\omega)}\big(F^{-n}_{\theta^{-n}(\omega),\xi}(B(\xi_\omega,R))\big) \leq K^t\lambda^{-n}_{\theta^{-n}(\omega)}\big|\big(F^n_{\theta^{-n}(\omega)}\big)'(\xi)\big|^{-t}\nu_\omega(B(\xi_\omega,R)).$$

Therefore,

$$\begin{aligned}\nu_{\theta^{-n}(\omega)}\big(F^{-n}_{\theta^{-n}(\omega),\xi}(A_\omega)\big) &\geq K^{-2t}\nu_{\theta^{-n}(\omega)}\big(F^{-n}_{\theta^{-n}(\omega),\xi}(B(\xi_\omega,R))\big)\frac{\nu_\omega(A_\omega)}{\nu_\omega(B(\xi_\omega,R))} \\ &\geq K^{-2t}\nu_{\theta^{-n}(\omega)}\big(F^{-n}_{\theta^{-n}(\omega),\xi}(B(\xi_\omega,R))\big)\nu_\omega(A_\omega)\end{aligned}$$

Now notice that if

$$\Omega_* := \left\{\omega \in \Omega_1 : \nu_\omega(A_\omega) \geq \frac{1}{2}\nu(A)\right\},$$

then $m(\Omega_*) > 0$.

Therefore, for every $\omega \in \Omega_*$, we get

$$\begin{aligned}
\nu_{\theta^{-n}(\omega)}\big(F^{-n}_{\theta^{-n}(\omega)}(A_\omega)\big) &= \sum_{\xi \in F^{-n}_{\theta^{-n}(\omega)}(\xi_\omega)} \nu_{\theta^{-n}(\omega)}\big(F^{-n}_{\theta^{-n}(\omega),\xi}(A_\omega)\big) \\
&\geq \frac{1}{2}K^{-2t}\nu(A) \sum_{\xi \in F^{-n}_{\theta^{-n}(\omega)}(\xi_\omega)} \nu_{\theta^{-n}(\omega)}\big(F^{-n}_{\theta^{-n}(\omega),\xi}(B(\xi_\omega, R))\big) \\
&= \frac{1}{2}K^{-2t}\nu(A)\nu_{\theta^{-n}(\omega)}\big(F^{-n}_{\theta^{-n}(\omega)}(B(\xi_\omega, R))\big)
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\nu(F^{-n}(A)) &= \int_\Omega \nu_{\theta^{-n}(\omega)}(F^{-n}_{\theta^{-n}(\omega)}(A_\omega))\, dm(\omega) \\
&\geq \int_{\Omega_*} \nu_{\theta^{-n}(\omega)}(F^{-n}_{\theta^{-n}(\omega)}(A_\omega))\, dm(\omega) \\
&\geq \frac{1}{2}K^{-2t}\nu(A) \int_{\Omega_*} \nu(A)\nu_{\theta^{-n}(\omega)}\big(F^{-n}_{\theta^{-n}(\omega)}(B(\xi_\omega, R)\big) \\
&= \frac{1}{2}K^{-2t}\nu(A)\left(F^{-n}\left(\bigcup_{\omega\in\Omega_*}\{\omega\} \times B(\xi_\omega, R)\right)\right).
\end{aligned}$$

Finally, using (9.13), we get

$$\begin{aligned}
\mu(A) &= \ell_B\big((\nu(F^{-n}(A)))_{n=0}^\infty\big) \\
&\geq \frac{1}{2}K^{-2t}\nu(A)\ell_B\left(\left(\nu\left(F^{-n}\left(\bigcup_{\omega\in\Omega_*}\{\omega\} \times B(\xi_\omega, R)\right)\right)\right)_{n=0}^\infty\right) \\
&= \frac{1}{2}K^{-2t}\mu\left(\bigcup_{\omega\in\Omega_*}\{\omega\} \times B(\xi_\omega, R)\right)\nu(A) \geq \frac{1}{2}K^{-2t}\alpha\nu(A) > 0,
\end{aligned}$$

and the Claim is proved. □

Now we conclude the proof of Theorem 54. So, let $D \subset \Omega \times \mathbb{C}$ be an arbitrary Borel set with $\nu(D) > 0$. Then there exist a measurable set $\Omega_2 \subset \Omega$ and $\eta \in (0, 1/2)$ such that $m(\Omega_2) > 0$ and for every $\omega \in \Omega_2$ there exists $x_\omega \in A(0; 2\eta, 1/\eta)$, depending measurably on ω, such that

$$\nu_\omega(D_\omega \cap B(x_\omega, \eta)) > 0. \tag{9.14}$$

Denote the ball $B(x_\omega, \eta)$ just by B_ω. From our hypotheses on the functions f_ω, $\omega \in \Omega$, there exists an integer $N \geq 0$ such that

$$F^n_\omega(B(z, R)) \supset \bigcup_{x\in A(0;2\eta,1/\eta)} B(x, \eta)$$

for all $\omega \in \Omega$, all $n \geq N$, and all $z \in Q_M$. Since, $m(\Omega_1), m(\Omega_2) > 0$ and since the map $\theta : \Omega \to \Omega$ is ergodic, there exists $n \geq N$ such that $m(\Omega_1 \cap \theta^{-n}(\Omega_2)) > 0$. Then

$$m(\theta^n(\Omega_1) \cap \Omega_2) > 0, \tag{9.15}$$

and

$$F^n(\Gamma) \supset F^n\left(\bigcup_{\omega \in \Omega_1 \cap \theta^{-n}(\Omega_2)} \{\omega\} \times B(\xi_\omega, R)\right)$$
$$\supset \bigcup_{\omega \in \theta^n(\Omega_1) \cap \Omega_2} \{\omega\} \times B_\omega \supset \bigcup_{\omega \in \theta^n(\Omega_1) \cap \Omega_2} \{\omega\} \times (D_\omega \cap B_\omega).$$

Therefore there exists a measurable set $H \subset \Gamma$ such that

$$F^n(H) = \bigcup_{\omega \in \theta^n(\Omega_1) \cap \Omega_2} \{\omega\} \times (D_\omega \cap B_\omega) \subset D. \tag{9.16}$$

But then, because of (9.14) and (9.15), we have that $\nu(F^n(H)) > 0$. This in turn, by conformality of ν, yields $\nu(H) > 0$. Since $H \subset \Gamma$, it then follows from Claim 1^0 that $\mu(H) > 0$. Hence, by virtue of (9.16), we get that $\mu(D) \geq F^n(H) \geq \mu(H) > 0$. The proof of Theorem 54 is thus complete. □

We shall prove more about measures $\mu^{(t)}$: their ergodicity and uniqueness. This requires some preparation.

As promised in the Introduction, we now give the definition of random radial Julia sets.

Fix $(\omega, z) \in \Omega \times Q$. Let $N \in \mathbb{N}$. Define $N_\omega(z, N)$ to be the set of all integers $n \geq 0$ such that there exists a (unique) holomorphic inverse branch

$$F_{\omega,z}^{-n} : B(F_\omega^n(z), 2/N) \to Q$$

of $F_\omega^n : Q \to Q$ sending $F_\omega^n(z)$ to z and such that $|F_\omega^n(z)| \leq N$. Following a number theory tradition, given a set $A \subset \mathbb{N}$, we denote by $\underline{\rho}(A)$ and $\overline{\rho}(A)$ respective lower and upper densities of the set A. Precisely,

$$\underline{\rho}(A) := \varliminf_{n\to\infty} \frac{1}{N} \#(A \cap \{1, 2, \ldots, N\})$$

and

$$\overline{\rho}(A) := \varlimsup_{n\to\infty} \frac{1}{N} \#(A \cap \{1, 2, \ldots, N\})$$

Having the above concepts introduced, we can now define random radial Julia sets, as follows:

Definition 55 (Random radial Julia sets $J_r(\omega)$)**.** We define

$$J_r(\omega) := \left\{z \in Q : \lim_{N\to\infty} \underline{\rho}(N_\omega(z, N)) = 1\right\}. \tag{9.17}$$

$J_r(\omega)$ is said to be the set of radial (or conical) points of F at ω. We further denote:

$$J_r(F) := \bigcup_{\omega \in \Omega} \{\omega\} \times J_r(\omega),$$

and call it the set of all radial points of F.

This definition of radial sets differs a little bit from the standard one. What we mean is that, when applied to deterministic systems, it produces the sets J_r that are different than, though contained in, those introduced in [39], comp. ex. [40], [41], [23], [32] and [24]. Therein one merely required that the sets $N_\omega(z,N)$ are infinite.

We will need some sufficient conditions for a point (ω, z) to be radial. In order to formulate it, we need an auxiliary subset $\tilde{N}_\omega(z,N)$ of $N_\omega(z,N)$. It consists of all integers $n \ge 0$ such that for every integer $0 \le k \le n$,

$$F^k_{\theta^{n-k}(\omega)}(0) \notin B\big(F^n_\omega(z), 2/N\big) \quad \text{and} \quad |F^n_\omega(z)| \le N.$$

Of course

$$\tilde{N}_\omega(z,N) \subset N_\omega(z,N). \tag{9.18}$$

Also, $n \in \tilde{N}_\omega(z,N)$ if and only if

$$F^n_\omega(z) \in Q_N$$

and

$$F^n_\omega(z) \notin \bigcup_{k=0}^n B\big(F^k_{\theta^{n-k}(\omega)}(0), 2/N\big) = \bigcup_{k=0}^n B\big(F^k_{\theta^{-k}(\theta^n(\omega))}(0), 2/N\big).$$

Therefore, if we denote

$$J^*_N(F) := \bigcup_{\omega\in\Omega} \{\omega\} \times \left(Q_N \setminus \bigcup_{k=0}^\infty B\left(F^k_{\theta^{-k}(\omega)}(0), 2/N\right)\right)$$

and

$$\tilde{N}^*_\omega(z,N) := \big\{n \ge 0 : F^n_\omega(z) = F^n(\omega, z) \in J^*_N(F)\big\},$$

then

$$\tilde{N}^*_\omega(z,N) \subset \tilde{N}_\omega(z,N). \tag{9.19}$$

The first significance of the set of radial points comes from the following.

Proposition 56. *If $\mu \in \mathcal{M}_m$ is F–invariant, then $\mu(J_r(F)) = 1$.*

Proof. By considering ergodic decomposition, we may assume without loss of generality that measure μ is ergodic. By virtue of (9.18) and (9.19) it suffices to show that

$$\lim_{N\to\infty} \mu\big(J^*_N(F)\big) = 1.$$

And indeed, let

$$J^*_N(F)^c := (\Omega \times Q) \setminus J^*_N(F)$$

be the complement of $J^*_N(F)$ in $\Omega \times Q$. Then

$$J^*_N(F)^c = \bigcup_{\omega\in\Omega} \{\omega\} \times \left(Y_N \cup \bigcup_{k=0}^\infty B\left(F^k_{\theta^{-k}(\omega)}(0), 2/N\right)\right)$$

and $\big(J^*_N(F)^c\big)_{N=1}^\infty$ is a descending sequence of measurable sets with

$$\bigcap_{N=1}^\infty J^*_N(F)^c = \bigcup_{\omega\in\Omega} \{\omega\} \times \big\{F^k_{\theta^{-k}(\omega)}(0) : k \ge 0\big\}. \tag{9.20}$$

But

$$F\left(\bigcap_{N=1}^{\infty} J_N^*(F)^c\right) = \bigcup_{\omega\in\Omega}\{\theta(\omega)\}\times\left\{F_{\theta^{-k}(\omega)}^{k+1}(0) : k\ge 0\right\}$$
$$= \bigcup_{\omega\in\Omega}\{\theta(\omega)\}\times\left\{F_{\theta^{-(k+1)}(\theta(\omega))}^{k+1}(0) : k\ge 0\right\} \subset \bigcap_{N=1}^{\infty} J_N^*(F)^c,$$

hence by ergodicity of μ,

$$\mu\left(\bigcap_{N=1}^{\infty} J_N^*(F)^c\right)\in\{0,1\}.$$

If the above measure is equal to zero, we are done. So, suppose that

$$\mu\left(\bigcap_{N=1}^{\infty} J_N^*(F)^c\right) = 1. \tag{9.21}$$

Then for m–a.e. $\omega\in\Omega$, say $\omega\in\Omega^*$, with Ω^* being θ-invariant,

$$\mu_\omega\left(\left\{F_{\theta^{-k}(\omega)}^{k}(0) : k\ge 0\right\}\right) = 1.$$

But as $\mu_\omega\circ F_\omega^{-1} = \mu_{\theta(\omega)}$, we then get that

$$\mu_{\theta(\omega)}\left(\left\{F_{\theta^{-(k+1)}(\theta(\omega))}^{k+1}(0) : k\ge 0\right\}\right) = \mu_{\theta(\omega)}\left(\left\{F_\omega\left(F_{\theta^{-k}(\omega)}^{k}(0) : k\ge 0\right\}\right)\right)$$
$$\ge \mu_\omega\left(\left\{F_{\theta^{-k}(\omega)}^{k}(0) : k\ge 0\right\}\right) = 1.$$

Hence, $\mu_\omega(F_{\theta^{-1}(\omega)}(0)) = 0$ for all $\omega\in\Omega^*$. Proceeding in the same way by induction, we deduce that

$$\mu_\omega\left(F_{\theta^{-k}(\omega)}^{k}(0)\right) = 0$$

for every integer $k\ge 0$ and all $\omega\in\Omega^*$. Thus

$$\mu_\omega\left(\left\{F_{\theta^{-k}(\omega)}^{k}(0) : k\ge 0\right\}\right) = 0$$

for all $\omega\in\Omega^*$. By (9.20) this entails

$$\mu\left(\bigcap_{N=1}^{\infty} J_N^*(F)^c\right) = 0,$$

contrary to (9.21). The proof of Proposition 56 is complete. □

We now pass to consider random conformal measures and we do this with their relations to the set of radial points. Let $t>1$ and suppose we are given two t-conformal measures $\nu^{(1)}$ and $\nu^{(2)}$. Denote by $\lambda_\omega^{(1)}$ and $\lambda_\omega^{(2)}$ the corresponding normalizing factors coming from the definition of a conformal measure. For every $l>0$ and $\omega\in\Omega$ let

$$L_\omega(l) := \left\{n\ge 1 : \frac{\lambda_\omega^{(1)n}}{\lambda_\omega^{(2)n}}\le l\right\}\subset\mathbb{N}. \tag{9.22}$$

Let $\hat{\Omega}_l$ be the set of all points $\omega \in \Omega$ such that the set $L_\omega(l) \subset \mathbb{N}$ has positive upper density. Finally let

$$\hat{\Omega} := \bigcup_{l=1}^{\infty} \hat{\Omega}_l.$$

We shall prove the following.

Lemma 57. *If $t > 1$ and two t-conformal measures $\nu^{(1)}$ and $\nu^{(2)}$ are given, then for every m-a.e. $\omega \in \hat{\Omega}$, the fiber measure $\nu_\omega^{(2)}|_{J_r(\omega)}$ is absolutely continuous with respect to the fiber measure $\nu_\omega^{(1)}|_{J_r(\omega)}$.*

Proof. Fix an integer $l \geq 1$ and then an integer $q \geq 1$. By t–conformality and quasi topological exactness of the map $F : \Omega \times Q \to \Omega \times Q$, each measure $\nu_\omega^{(i)}$, $i = 1, 2$, $\omega \in \Omega$, has full topological support, i.e. it is positive on all non-empty open subsets of Q. Therefore, for every $N \in \mathbb{N}$, we have that

$$M_N^{(i)}(\omega) := \inf \left\{ \nu_\omega^{(i)}\big(B(z, (4KN)^{-1})\big) : z \in Q_N \right\} > 0,$$

and the function

$$\Omega \ni \omega \longmapsto M_N^{(i)}(\omega) \in (0, +\infty)$$

is measurable. Hence, for every integer $k \geq 1$ there exists $\varepsilon_{N,k}^{(i)} > 0$ so small that

$$m\left(M_N^{(i)-1}\big((\varepsilon_{N,k}^{(i)}, +\infty)\big) \right) > 1 - \frac{1}{2k}.$$

By Birkhoff's Ergodic Theorem, for m–a.e. $\omega \in \Omega$, say ω in some θ–invariant set $\Omega_{N,k}^{(i)}$ with measure m equal to 1, we have that

$$\rho\left(\Lambda_{N,k}^{(i)}(\omega) \right) = m\left(M_N^{(i)-1}\big((\varepsilon_{N,k}^{(i)}, +\infty)\big) \right) > 1 - \frac{1}{2k}, \tag{9.23}$$

where

$$\Lambda_{N,k}^{(i)}(\omega) := \left\{ n \geq 0 : \theta^n(\omega) \in M_N^{(i)-1}\big((\varepsilon_{N,k}^{(i)}, +\infty)\big) \right\} \subset \mathbb{N}.$$

Let $\hat{\Omega}_{l,q}$ be the set of all points $\omega \in \Omega$ such that the set $\overline{\rho}(L_\omega(l)) \geq 1/q$. Of course

$$\hat{\Omega}_l = \bigcup_{q=1}^{\infty} \hat{\Omega}_{l,q}.$$

It therefore suffices to prove our lemma with the set $\hat{\Omega}$ replaced by $\hat{\Omega}_{l,q}$. In order to do this we shall estimate from above the limit

$$\varliminf_{r \to 0} \frac{\nu_\omega^{(2)}(B(z, r))}{\nu_\omega^{(1)}(B(z, r))}$$

for all $\omega \in \hat{\Omega}_{l,q}$ and all $z \in J_r(\omega)$. So, fix $N_q \geq 1$ so large that

$$\underline{\rho}\big(N_\omega(z, N_q)\big) > 1 - \frac{1}{2q}. \tag{9.24}$$

It then follows from $\frac{1}{4}$–Koebe's Distortion Theorem, Koebe's Distortion Theorem, and t-conformality of measure $\nu^{(1)}$ that for every $n \in N_\omega(z, N_q) \cap L_\omega(l)$, we have that

$$
\begin{aligned}
&\nu_\omega^{(2)}\left(B\left(z, \frac{1}{4}\frac{1}{N_q}\left|\left(F_\omega^n\right)'(z)\right|^{-1}\right)\right) \le \nu_\omega^{(2)}\left(F_{\omega,z}^{-n}\left(B\left(F_\omega^n(z), 1/N_q\right)\right)\right) \\
&\le K^t \lambda_\omega^{(2)-n}\left|\left(F_\omega^n\right)'(z)\right|^{-t}\nu_{\theta^n(\omega)}^{(2)}\left(B\left(F_\omega^n(z), 1/N_q\right)\right) \le K^t \lambda_\omega^{(2)-n}\left|\left(F_\omega^n\right)'(z)\right|^{-t}
\end{aligned}
\tag{9.25}
$$

By the same token,

$$
\begin{aligned}
\nu_\omega^{(1)}\left(B\left(z, \frac{1}{4}\frac{1}{N_q}\left|\left(F_\omega^n\right)'(z)\right|^{-1}\right)\right) &\ge \nu_\omega^{(1)}\left(F_{\omega,z}^{-n}\left(B\left(F_\omega^n(z), (4KN_q)^{-1}\right)\right)\right) \\
&\ge K^{-t} \lambda_\omega^{(1)-n}\left|\left(F_\omega^n\right)'(z)\right|^{-t}\nu_{\theta^n(\omega)}^{(1)}\left(B\left(F_\omega^n(z), (4KN_q)^{-1}\right)\right).
\end{aligned}
\tag{9.26}
$$

Now assume in addition that

$$\omega \in \Omega_{N_q,q}^{(1)}.$$

Then, we deduce from (9.24) and (9.23) that

$$\overline{\rho}\left(N_\omega(z, N_q) \cap L_\omega(l) \cap \Lambda_{N_q,q}^{(1)}\right) > 0.$$

Therefore, for ever $n \in N_\omega(z, N_q) \cap L_\omega(l) \cap \Lambda_{N_q,q}^{(1)}$, we get that

$$
\begin{aligned}
\frac{\nu_\omega^{(2)}\left(B\left(z, \frac{1}{4}\frac{1}{N_q}\left|\left(F_\omega^n\right)'(z)\right|^{-1}\right)\right)}{\nu_\omega^{(1)}\left(B\left(z, \frac{1}{4}\frac{1}{N_q}\left|\left(F_\omega^n\right)'(z)\right|^{-1}\right)\right)} &\le K^t\left(\nu_{\theta^n(\omega)}^{(1)}\left(B\left(F_\omega^n(z), (4KN_q)^{-1}\right)\right)\right)^{-1}\frac{\lambda_\omega^{(1)n}}{\lambda_\omega^{(2)n}} \\
&\le K^t\left(\varepsilon_{N_q,q}^{(1)}\right)^{-1} l.
\end{aligned}
\tag{9.27}
$$

Consequently,

$$\varlimsup_{r \to 0}\frac{\nu_\omega^{(2)}(B(z,r))}{\nu_\omega^{(1)}(B(z,r))} \le \varlimsup_{n \to \infty}\frac{\nu_\omega^{(2)}\left(B\left(z, \frac{1}{4}\frac{1}{N_q}\left|\left(F_\omega^n\right)'(z)\right|^{-1}\right)\right)}{\nu_\omega^{(1)}\left(B\left(z, \frac{1}{4}\frac{1}{N_q}\left|\left(F_\omega^n\right)'(z)\right|^{-1}\right)\right)} \le K^t\left(\varepsilon_{N_q,q}^{(1)}\right)^{-1} l.$$

This implies that for each $\omega \in \hat{\Omega}_{l,q} \cap \Omega_{N_q,q}^{(1)}$, the measure $\nu_\omega^{(2)}|_{J_r(\omega)}$ is absolutely continuous with respect to $\nu_\omega^{(1)}|_{J_r(\omega)}$, and the proof of Lemma 57 is complete. □

Our ultimate theorem about conformal and invariant measures is this.

Theorem 58. *Let $t > 1$. If $\nu^{(t)}$ is the t-conformal random measure for $F : \Omega \times \mathbb{C} \to \Omega \times \mathbb{C}$, produced in Theorem 38, then the Borel probability F–invariant measure $\mu = \mu^{(t)} \in \mathcal{M}_m$ absolutely continuous with respect to $\nu^{(t)}$, produced in Theorem 53, is in fact*

(a) *Equivalent with $\nu^{(t)}$,*

(b) *Ergodic,*

(c) *It is the only Borel probability F–invariant measure in $\mathcal{M}_m$ absolutely continuous with respect to $\nu^{(t)}$.*

Proof. Item (a) is just Theorem 54. In order to prove ergodicity of μ, i.e. item (b) of Theorem 58, assume for a contradiction that there are two disjoint totally F-invariant measurable sets $A, B \subset \Omega \times \mathbb{C}$ such that

$$0 < \mu(A),\, \mu(B) < 1.$$

Since $\theta : \Omega \to \Omega$ is ergodic with respect to measure m, we have that $0 < \mu_\omega(A_\omega), \mu_\omega(B_\omega) < 1$ for m–ae. $\omega \in \Omega$. Therefore, also

$$0 < \nu_\omega(A_\omega),\, \nu_\omega(B_\omega) < 1$$

for m–ae. $\omega \in \Omega$. Define two random measures measures $\hat{\nu}_A$ and $\hat{\nu}_B$ by demanding that their fiber measures $\hat{\nu}_{A,\omega}$ and $\hat{\nu}_{B,\omega}$ are respective conditional measures of the measure ν_ω on the sets A_ω and B_ω. By this very definition both $\hat{\nu}_A$ and $\hat{\nu}_B$ belong to $\mathcal{M}_m$. It is easy to verify that these two measures are also t–conformal with respective generalized eigenvalues equal to

$$\lambda_{A,\omega} = \lambda_\omega \frac{\nu_\omega(A_\omega)}{\nu_\omega(A_{\theta(\omega)})} \quad \text{and} \quad \lambda_{B,\omega} = \lambda_\omega \frac{\nu_\omega(B_\omega)}{\nu_\omega(B_{\theta(\omega)})}.$$

But then

$$\lambda^n_{A,\omega} = \lambda^n_\omega \frac{\nu_\omega(A_\omega)}{\nu_\omega\big(A_{\theta^n(\omega)}\big)} \quad \text{and} \quad \lambda^n_{B,\omega} = \lambda^n_\omega \frac{\nu_\omega(B_\omega)}{\nu_\omega\big(B_{\theta^n(\omega)}\big)}$$

for every integer $n \geq 0$. Therefore

$$\frac{\lambda^n_{A,\omega}}{\lambda^n_{B,\omega}} = \frac{\nu_\omega(A_\omega)}{\nu_\omega(B_\omega)} \cdot \frac{\nu_\omega\big(B_{\theta^n(\omega)}\big)}{\nu_\omega\big(A_{\theta^n(\omega)}\big)} \leq \frac{1}{\nu_\omega(B_\omega)} \cdot \frac{1}{\nu_\omega\big(A_{\theta^n(\omega)}\big)}.$$

Now, since $\nu(A) > 0$, there exists $\varepsilon > 0$ such that

$$m\big(\{\omega \in \Omega : \nu_\omega(A_\omega) \geq \varepsilon\}\big) > 0.$$

Denote this, just defined, subset of Ω by Ω^*. By Birkhoff's Ergodic Theorem and ergodicity of the measure m with respect to the map $\theta : \Omega \to \Omega$, we have for m–a.e. $\omega \in \Omega$, say $\omega \in \Omega^+$, that

$$\rho\big(\{n \geq 0 : \theta^n(\omega) \in \Omega^*\}\big) = m(\Omega^*) > 0.$$

For every $k \geq 1$ let

$$\Omega_k := \{\omega \in \Omega : \nu_\omega(B_\omega) \geq 1/k\}.$$

Then $\Omega_k \cap \Omega^+ \subset \hat{\Omega}_{k/\varepsilon} \subset \hat{\Omega}$. Hence

$$\bigcup_{k=1}^\infty \Omega_k \cap \Omega^+ \subset \hat{\Omega}.$$

Since also $m\big(\bigcup_{k=1}^\infty \Omega_k \cap \Omega^+\big) = 1$, it thus follows from Lemma 57 that the fiber measure $\hat{\nu}_{B,\omega}|_{J_r(\omega)}$ is absolutely continuous with respect to the fiber measure $\hat{\nu}_{A,\omega}|_{J_r(\omega)}$ for m–a.e. $\omega \in \Omega$. But because of Proposition 56 and Theorem 54, $\nu_\omega(J_r(\omega)) = 1$ for m–a.e. $\omega \in \Omega$; consequently $\hat{\nu}_{B,\omega}(J_r(\omega)) = \hat{\nu}_{A,\omega}(J_r(\omega))) = 1$ for m–a.e. $\omega \in \Omega$. We thus obtained that the fiber measure $\hat{\nu}_{B,\omega}$ is absolutely continuous with respect to the fiber measure $\hat{\nu}_{A,\omega}$ for m–a.e. $\omega \in \Omega$. This contradicts the fact that $A_\omega \cap B_\omega = \emptyset$ for m–a.e. $\omega \in \Omega$, and finishes the proof of item (b), i.e. ergodicity of the measure μ.

The proof of item (c) is now straightforward. Assume for a contradiction that there exists an F-invariant Borel probability measure on $\Omega \times Q$ absolutely continuous with respect to ν

and different from μ. Then there also exists an ergodic measure η with all such properties. But then by (a), η is absolutely continuous with respect to μ. As both measures η and μ are ergodic, we thus conclude that $\mu = \eta$. This contradiction finishes the proof of item (c) and simultaneously the whole proof of Theorem 58. $\square$

As an immediate consequence of Proposition 56 and Theorem 58, we get the following.

Corollary 59. *For every $t > 1$ we have that $\nu^{(t)}(J_r(F)) = 1$.*

As the last important fact in this section, we shall prove the following.

Proposition 60. *For every $t > 1$ the global Lyapunov exponent*

$$\chi_{\mu^{(t)}} := \int_{\Omega\times Q} \log|F'_\omega(z)|\, d\mu^{(t)}(\omega, z) = \int_{\Omega\times Q} \log|f'_\omega(z)|\, d\mu^{(t)}(\omega, z)$$

is finite and positive.

Proof. We first note that

$$\chi_{\mu^{(t)}} = \int_{\Omega\times Q} \log|f'_\omega(z)|\, d\mu^{(t)}(\omega, z) = \int_{\Omega\times Q} \big(\log \eta(\omega) + \mathrm{Re}(z)\big)\, d\mu^{(t)}(\omega, z).$$

Since $\log A \le \log \eta(\omega) \le \log B$ for all $\omega \in \Omega$ and since $\mu^{(t)}$ is a probability measure, we are thus to show that

$$\int_{\Omega\times Q} |\mathrm{Re}(z)|\, d\mu^{(t)}(\omega, z) < +\infty.$$

In order to do this, we will provide sufficiently good upper estimates for $\mu_\omega^{(t)}(Y_M^\pm)$ for all $M \ge 0$ and all $\omega \in \Omega$. First, using (5.13) and Proposition 52, we have

$$\nu_\omega^{(t)}\big(F_\omega^{-n}(Y_M^+)\big) \le (c(M_0))^\beta e^{\frac{\beta M}{2}(1-t)}$$

for every integer $n \ge 0$ and every real number $M > 0$. Second, by Proposition 52 again and by Proposition 35 there are two constants $D > 0$ and $\gamma > 0$ such that

$$\nu_\omega^{(t)}\big(F_\omega^{-n}(Y_M^-)\big) \le De^{-\gamma M}$$

for every integer $n \ge 0$ and every real number $M > 0$. Therefore,

$$\nu^{(t)}\big(F^{-n}(\Omega \times Y_M^+)\big) = \int_\Omega \nu_\omega^{(t)}\big(F_\omega^{-n}(Y_M^+)\big)\, dm(\omega) \le c^\beta(M_0)e^{\frac{\beta M}{2}(1-t)}$$

and likewise,

$$\nu^{(t)}\big(F^{-n}(\Omega \times Y_M^-)\big) \le De^{-\gamma M}.$$

It therefore follows from Theorem 53 and basic properties of Banach limits that

$$\mu^{(t)}(\Omega \times Y_M^+) \le c^\beta(M_0)e^{\frac{\beta M}{2}(1-t)} \quad \text{and} \quad \mu^{(t)}(\Omega \times Y_M^-) \le De^{-\gamma M}. \tag{9.28}$$

Hence, by straightforward calculation:

$$\int_{\Omega\times Y_1^+} |\mathrm{Re}(z)|\, d\mu^{(t)}(\omega, z) < +\infty.$$

In the same way, based on the right–hand side of (9.28), we get

$$\int_{\Omega\times Y_1^-} |\mathrm{Re}(z)|\, d\mu^{(t)}(\omega, z) < +\infty.$$

Since obviously,

$$\int_{\Omega\times Q_1} |\mathrm{Re}(z)|\, d\mu^{(t)}(\omega,z) \le 1,$$

we thus conclude that

$$\int_{\Omega\times Q} |\mathrm{Re}(z)|\, d\mu^{(t)}(\omega,z) < +\infty,$$

and the proof of finiteness of the global Lyapunov exponent $\chi_{\mu^{(t)}}$ is complete.

So, we now pass to the proof that $\chi_{\mu^{(t)}} > 0$. The first observation is that for each $\omega\in\Omega$ the set

$$D_\omega := \{z\in Q : |\mathrm{Im}(f_\omega(z))| > 2\}$$

is non-empty and open. Therefore $\mu^{(t)}(D) > 0$, where

$$D := \bigcup_{\omega\in\Omega} \{\omega\}\times D_\omega.$$

It thus follows from ergodicity of the global map $F : \Omega\times Q \to \Omega\times Q$ with respect to $\mu^{(t)}$ (Theorem 58) and from Birkhoff's Ergodic Theorem that there exists a measurable set $\Gamma\subset\Omega\times Q$ such that $\mu^{(t)}(\Gamma) = 1$ and

$$\lim_{n\to\infty} \frac{1}{n}\#\big\{0\le j\le n-1 : F_\omega^j(z)\in D\big\} = \mu^{(t)}(D) > 0 \tag{9.29}$$

for every $(\omega,z)\in\Gamma$. Since $|(F_\omega^k)'(z)| = |f_\omega^k(z)| \ge |\mathrm{Im}(f_\omega^k(z))|$ for each $k\ge 1$, it follows from Lemma 10, formula (9.29), the Chain Rule, and the definition of the set D, that

$$\liminf_{n\to\infty} \frac{1}{n}\log|(F_\omega^n)'(z)| \ge \mu^{(t)}(D)\log 2.$$

Since, by Birkhoff's Ergodic Theorem again,

$$\chi_{\mu^{(t)}} = \lim_{n\to\infty}\frac{1}{n}\log|(F_\omega^n)'(z)|$$

(in particular the limit exists) for $\mu^{(t)}$–a.e. $(\omega,z)\in\Omega\times Q$, we thus obtain that

$$\chi_{\mu^{(t)}} \ge \mu^{(t)}(D)\log 2 > 0,$$

and the proof of Proposition 60 is complete. □

10. Bowen's formula

In this section we prove a formula holds that determines the value of the Hausdorff dimension of radial Julia sets. We refer to it as Bowen's formula. Precisely, we prove the following.

Theorem 61. *For $t > 1$ put*

$$\mathcal{E}\mathrm{P}(t) := \int_\Omega \log\lambda_{t,\omega}\, dm(\omega).$$

Then

(1) $\mathcal{E}\mathrm{P}(t) < +\infty$ *for all* $t > 1$,

(2) *The function* $(1,+\infty)\ni t\mapsto \mathcal{E}\mathrm{P}(t)$ *is strictly decreasing, convex, and thus continuous,*

(3) $\lim_{t\to 1}\mathcal{E}\mathrm{P}(t)=+\infty$ *and* $\mathcal{E}\mathrm{P}(2))\le 0$.

(4) *Let* $h>1$ *be the unique value* $t>1$ *for which* $\mathcal{E}\mathrm{P}(t)=0$. *Then*

$$\mathrm{HD}(J_r(\omega))=h$$

for m*–a.e.*$\omega\in\Omega$.

The proof of this theorem will be deduced from a series of lemmas.

Lemma 62. $\mathcal{E}\mathrm{P}(2)\le 0$.

Proof. Assume for a contrary that $\mathcal{E}\mathrm{P}(2)>0$. It then follows from Birkhoff's Ergodic Theorem that

$$\lim_{n\to\infty}\lambda_{2,\omega}^n=+\infty$$

for m–a.e. $\omega\in\Omega$, and in fact, the rate of divergence is exponential. Then using Definition 9.17 (of the set $J_r(\omega)$), conformality of the measure $\nu^{(2)}$ produced in Theorem 53, and Koebe's Distortion Theorem, we can write for m–almost every $\omega\in\Omega$ and for $\nu_\omega^{(2)}$– almost every $z\in J_r(\omega)$, every integer $N\ge 1$ and all $n\in N_\omega(z,N)$, that

$$\nu_\omega^{(2)}(F_{\omega,z}^{-n}(B(F_\omega^n(z),1/N)))\le C(N)\frac{1}{\lambda_{\omega,2}^n}\mathrm{diam}^2(F_{\omega,z}^{-n}(B(F_\omega^n(z),1/N))),$$

with some constant $C(N)\in(0,+\infty)$ depending only on N. Using Koebe's Distortion Theorem again, we thus conclude that

$$\liminf_{r\to 0}\frac{\nu_\omega^{(2)}(B(z,r))}{r^2}=0. \tag{10.1}$$

But since $\mathrm{Leb}(B(z,r))=\pi r^2$ for all $r>0$ small enough independently of z, where Leb denotes the 2–dimensional Lebesgue measure on Q, formula (10.1) implies (standard in geometric measure theory, see e.g., Lemma 2.13 in [21] or [30]) that $\nu_\omega^{(2)}(J_r(\omega))=0$. This contradicts Corollary 59 and finishes the proof. □

Lemma 63. *For every* $t>1$ *the expected pressure* $\mathcal{E}\mathrm{P}(t)$ *is finite and the function*

$$(1,+\infty)\ni t\mapsto \mathcal{E}\mathrm{P}(t)\in\mathbb{R}$$

is convex, thus continuous.

Proof. First note that finiteness of the expected pressure follows immediately from the bounds on $\lambda_{t,\omega}$ provided in (8.7). The constants p,P in this estimate depend on t but they are independent of ω.

Let us fix some $t_1,t_2>1$ and $\alpha\in[0,1]$. Put $t_3=\alpha t_1+(1-\alpha)t_2$. We are to show that

$$\mathcal{E}\mathrm{P}(t_3)\le\alpha\mathcal{E}\mathrm{P}(t_1)+(1-\alpha)\mathcal{E}\mathrm{P}(t_2).$$

For every $\omega\in\Omega$ denote:

$$E_\omega:=Q_{M_1}\setminus\bigcup_{j\in\mathbb{N}}B\big(F_{\theta^{n-j}\omega}^j(0),r_0\big),$$

with $M_1>0$ and $r_0>0$ produced in the course of the proof of Theorem 43). Increasing M_1 and decreasing r_0 if necessary, we can assume that

$$\nu_{\theta^n\omega}^{(t)}(E_\omega)>1/2 \tag{10.2}$$

for m–a.e. $\omega \in \Omega$ and for all $t \in \{t_1, t_2, t_3\}$. Since

$$\lim_{n\to\infty} F_\omega^n(z) = +\infty$$

uniformly with respect to $\omega \in \Omega$ on $\mathbb{R} \cap \Omega_{M_1}$, and since each measure μ_ω is a probability one satisfying, by virtue of its fiberwise F–invariance, $\mu_{\theta(\omega)}(F_\omega(A)) \ge \mu_\omega(A)$ for every $\omega \in \Omega$ and every Borel set $A \subset \mathbb{C}$, we conclude that

$$\mu_\omega^{(t)}(\mathbb{R} \cap Q_{M_1}) = 0 \tag{10.3}$$

for m–almost all $\omega \in \Omega$. This is a stronger statement than the one obtained at the very beginning of the proof of Theorem 54. Denote

$$E := \big\{z \in Q_{M_1} : |\mathrm{Im}(z)| > r_0\big\}.$$

Obviously,

$$E \subset E_\omega \tag{10.4}$$

for each $\omega \in \Omega$. We conclude from (10.3) that if $r_0 > 0$ is small enough, then there exists a measurable set $\Omega' \subset \Omega$, with $m(\Omega') > 0.999$ and such that

$$\mu_\omega^{(t)}(E) > 0.999$$

for all $t \in \{t_1, t_2, t_3\}$ and all $\omega \in \Omega'$. It follows from (9.6) and (9.7) that

$$\frac{1}{\lambda_{t,\omega}^n}\mathcal{L}_{t,\omega}^n(z) \asymp \frac{\nu_\omega^{(t)}(F_\omega^{-n}(E_\omega)}{\nu_{\theta^n\omega}^{(t)}(E_\omega)} \tag{10.5}$$

for all $t \in \{t_1, t_2, t_3\}$ with comparability constants witnessing to the comparability sign "$\asymp$" above being independent of $\omega \in \Omega$ and $z \in E_\omega$. Because of (10.2), we have that

$$\nu_\omega^{(t)}(F_\omega^{-n}(E)) \le \nu_\omega^{(t)}(F_\omega^{-n}(E_\omega)) \le \frac{\nu_\omega^{(t)}(F_\omega^{-n}(E_\omega))}{\nu_{\theta^n\omega}^{(t)}(E_\omega)} \le 2\nu_\omega^{(t)}(F_\omega^{-n}(E_\omega)) \le 2 \tag{10.6}$$

Our goal now is to find a lower bound for $\nu_\omega^{(t)}(F_\omega^{-n}(E))$, for some measurable set, with positive measure m, of ω's and for a sequence of infinitely many n's that may depend on ω. Put

$$A := \Omega' \times E.$$

Then

$$\mu^{(t)}(A) > 0.999 \cdot 0.999 > 0.99$$

for all $t \in \{t_1, t_2, t_3\}$. Recall that

$$\mu^{(t)}(A) = \ell_B\big(\nu^{(t)}(F^{-n}(A))_{n=0}^\infty\big) \tag{10.7}$$

Since

$$\mu^{(t_1)}(A) + \mu^{(t_2)}(A) + \mu^{(t_3)}(A) > 2.97,$$

it follows from (10.7) that

$$\limsup_{n\to\infty} \big(\nu^{(t_1)}(F^{-n}(A)) + \nu^{(t_2)}(F^{-n}(A)) + \nu^{(t_3)}(F^{-n}(A)\big) > 2.97$$

So, we conclude that there exists an infinite set $\big(n_k\big)_{k=1}^\infty$ such that

$$\nu^{(t)}(F^{-n_k}(A)) > 0.97$$

for all $t \in \{t_1, t_2, t_3\}$. Using the fact that $\nu_\omega^{(t)}(F_\omega^{-n_k}(E)) \le 1$ for all $\omega \in \Omega$, it is straightforward to conclude that for each such n_k and all $t \in \{t_1, t_2, t_3\}$,

$$m\big(\{\omega \in \Omega : \nu_\omega^{(t)}(F_\omega^{-n_k}(E)) < 0.1\}\big) < \frac{1}{30}.$$

So, for each such n_k there exists a measurable subset $\Omega_k \subset \Omega$ with

$$m(\Omega_k) \ge 0.9$$

and such that

$$\nu_\omega^{(t)}(F_\omega^{-n_k}(E)) \ge 0.1 \tag{10.8}$$

for each $\omega \in \Omega_k$ and $t \in \{t_1, t_2, t_3\}$. Next, put

$$\Omega_* := \bigcap_{m \in \mathbb{N}} \bigcup_{k \ge m} \Omega_k$$

i.e., Ω_* is the set of all elements $\omega \in \Omega$ that belong to Ω_k for infinitely many integers k. Arrange the set of these integers k into an increasing sequence $\big(k_j(\omega)\big)_{j=1}^\infty$. We immediately see that

$$m(\Omega_*) \ge 0.9.$$

By Birkhoff's Ergodic Theorem a measurable set $\Omega_{**} \subset \Omega_*$ such that

$$m(\Omega_{**}) = m(\Omega_*) \ge 0.9$$

and the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \lambda_{t,\omega}^n$$

exists and is equal to $\mathcal{E}\mathrm{P}(t)$ for all $t \in \{t_1, t_2, t_3\}$ and all $\omega \in \Omega_{**}$. For all $\omega \in \Omega_{**}$, denoting $n_{k_j(\omega)}$ by $s_j(\omega)$ for all integers $j \ge 1$, and using (10.5), (10.6), end (10.8), we see that

$$0.1 \preceq \frac{1}{\lambda_{t,\omega}^{s_j(\omega)}} \mathcal{L}_{t,\omega}^{s_j(\omega)}(z) \preceq 2$$

for all $t \in \{t_1, t_2, t_3\}$ and all $\omega \in \Omega_{**}$. So, for all $t \in \{t_1, t_2, t_3\}$, all $\omega \in \Omega_{**}$, and all $z \in E_\omega$, we have that

$$\mathcal{E}\mathrm{P}(t) = \lim_{k\to\infty} \frac{1}{s_j(\omega)} \log \lambda_{t,\omega}^{s_j(\omega)} = \lim_{k\to\infty} \frac{1}{s_j(\omega)} \log \mathcal{L}_{t,\omega}^{s_j(\omega)}(z). \tag{10.9}$$

A direct application of Hölder inequality gives now the following:

$$\begin{aligned}
\mathcal{E}\mathrm{P}(t_3) &= \lim_{j\to\infty} \frac{1}{s_j(\omega)} \log \mathcal{L}_{\alpha t_1 + (1-\alpha)t_2,\omega}^{s_j(\omega)}(z) \\
&\le \lim_{j\to\infty} \frac{1}{n_k} \Big(\alpha \log \mathcal{L}_{t_1,\omega}^{s_j(\omega)}(z) + (1-\alpha) \log \mathcal{L}_{t_2,\omega}^{s_j(\omega)}(z)\Big) \\
&= \alpha \lim_{j\to\infty} \frac{1}{s_j(\omega)} \log \mathcal{L}_{t_2,\omega}^{s_j(\omega)}(z) + (1-\alpha) \lim_{k\to\infty} \frac{1}{s_j(\omega)} \log \mathcal{L}_{t,\omega}^{s_j(\omega)}(z) \\
&= \alpha \mathcal{E}\mathrm{P}(t_2) + (1-\alpha)\mathcal{E}\mathrm{P}(t_2).
\end{aligned}$$

The proof is complete. □

Lemma 64. *The function $(1, \infty) \ni t \longmapsto \mathcal{E}\mathrm{P}(t) \in \mathbb{R}$ is strictly decreasing.*

Proof. Seeking contradiction suppose that

$$\mathcal{E}\mathrm{P}(t) \geq \mathcal{E}\mathrm{P}(s) \tag{10.10}$$

for some $1 < s < t$. It follows from Corollary 59, Theorem 58, Proposition 60, and Birkhoff's Ergodic Theorem, that there exist $\chi > 0$, an integer $q_0 \geq 1$, a measurable set $\Omega_0 \subset \Omega$ with $m(\Omega_0) > 1/2$, and for each $\omega \in \Omega_0$, a measurable set $J_r^0(\omega) \subset J_r(\omega)$ such that

$$\nu_\omega^{(t)}(J_r^0(\omega)) \geq 1/2,$$

and

$$|(F_\omega^n)'(z)| \geq e^{\chi n}$$

for every $\omega \in \Omega_0$, every $z \in J_r^0(\omega)$, and every integer $n \geq q_0$. It furthermore follows from Birkhoff's Ergodic Theorem and (10.10) that there are an integer $q_1 \geq q_0$, a measurable set $\Omega_1 \subset \Omega_0$ with $m(\Omega_1) > 1/4$, and

$$\frac{\lambda_{t,\omega}^{-n}}{\lambda_{s,\omega}^{-n}} \leq e^{\frac{1}{2}\chi(t-s)n}.$$

for every $\omega \in \Omega_1$ and every integer $n \geq q_1$. Fix such $\omega \in \Omega_1$ and $z \in J_r^0(\omega)$. By the definition of $J_r(\omega)$ there exists an integer $N \geq 1$ such that $\underline{\rho}(N_\omega(z,N)) > 3/4$. So, there exist an integer $N \geq 1$ depending on ω and z and an unbounded increasing sequence $(n_k)_{k=1}^\infty$ of integers $\geq q_1$ with lower density $\geq 3/4$ such that for every $k \geq 1$ there exists a holomorphic branch $F_{\omega,z}^{-n_k} : B(F_\omega^{n_k}(z), 2/N) \to Q$ of $F_{\omega,z}^{-n_k}$ that maps $F_\omega^{n_k}(z)$ back to z and

$$|F_\omega^{n_k}(z)| \leq N.$$

By Birkhoff's Ergodic Theorem and Proposition 39 there exist a measurable set $\Omega_2 \subset \Omega_1$ such that $m(\Omega_2) > 1/8$ and

$$\theta^n\omega \in \Omega\big(N, (4KN)^{-1}, 1/4\big)$$

for all $\omega \in \Omega_2$ and a set of integers $n \geq 0$ of lower density $\geq 3/4$, where the set $\Omega\big(N, (4KN)^{-1}, 1/4\big)$ comes from Proposition 39. Passing to a subsequence we may therefore assume that

$$\theta^{n_k}\omega \in \Omega\big(N, (4KN)^{-1}, 1/4\big)$$

for all $\omega \in \Omega_2$ and every integer $k \geq 1$.

Using all the above, Koebe's Distortion Theorems, conformality of the measures $\nu_\omega^{(t)}$ and $\nu_\omega^{(s)}$, and at the end Proposition 39 (the constant $\xi = \xi\big(N, (4KN)^{-1}, 1/4\big) > 0$ below comes from it), we obtain

$$\begin{aligned}\frac{\nu_\omega^{(t)}\big(B(z,(4N)^{-1}|(F_\omega^{n_k}(z)'|^{-1})\big)}{\nu_\omega^{(s)}\big(B(z,(4N)^{-1}|(F_\omega^{n_k}(z)'|^{-1})\big)} &\le \frac{\nu_\omega^{(t)}\big(F_{\omega,z}^{-n_k}\big(B(F_\omega^{n_k}(z),N^{-1})\big)\big)}{\nu_\omega^{(s)}\big(F_{\omega,z}^{-n_k}\big(B(F_\omega^{n_k}(z),(4KN)^{-1})\big)\big)} \le \\ &\le K^{t-s}|(F_\omega^{n_k})'(z)|^{s-t}\frac{\nu_{\theta^{n_k}\omega}^{(t)}\big(B(F_\omega^{n}(z),N^{-1})\big)}{\nu_{\theta^{n_k}\omega}^{(s)}\big(B(F_\omega^{n_k}(z),(4KN)^{-1})\big)}\frac{\lambda_{t,\omega}^{-n_k}}{\lambda_{s,\omega}^{-n_k}} \\ &\le K^{t-s}\exp(\chi(s-t)n_k)\big(\nu_{\theta^{n_k}\omega}^{(s)}\big(B(F_\omega^{n_k}(z),(4KN)^{-1})\big)\big)^{-1}\exp\left(\frac{1}{2}\chi(t-s)n_k\right) \\ &= K^{t-s}\exp\left(\frac{1}{2}\chi(s-t)n_k\right)\big(\nu_{\theta^{n_k}\omega}^{(s)}\big(B(F_\omega^{n_k}(z),(4KN)^{-1})\big)\big)^{-1} \\ &\le \xi^{-1}K^{t-s}\exp\left(\frac{1}{2}\chi(s-t)n_k\right).\end{aligned}$$

Therefore

$$\begin{aligned}\varliminf_{r\to 0}\frac{\nu_\omega^{(t)}(B(z,r))}{\nu_\omega^{(s)}(B(z,r))} &\le \varliminf_{k\to\infty}\frac{\nu_\omega^{(t)}\big(B\big(z,(4N)^{-1}|(F_\omega^{n_k}(z)'|^{-1}\big)\big)}{\nu_\omega^{(s)}\big(B\big(z,(4N)^{-1}|(F_\omega^{n_k}(z)'|^{-1}\big)\big)} \\ &\le \xi^{-1}K^{t-s}\varliminf_{k\to\infty}\exp\left(\frac{1}{2}\chi(s-t)n_k\right)=0.\end{aligned}$$

This, in a standard way, implies that

$$\nu_\omega^{(t)}\left(\bigcup_{\omega\in\Omega_1}\{\omega\}\times J_r^0(\omega)\right)=0.$$

But, on the other hand, from the very definition of the sets Ω_2 and $J_r^0(\omega)$, we have that

$$\nu_\omega^{(t)}\left(\bigcup_{\omega\in\Omega_1}\{\omega\}\times J_r^0(\omega)\right)\ge 1/8>0.$$

This contradiction ends the proof of Lemma 64. □

Lemma 65. *There exist constants $C>0$ and $t_0>1$ such that for every $t\in(1,t_0)$ there exists $M=M_t>0$ such that for every $z\in Q_M$ and every $\omega\in\Omega$, we have that*

$$\mathcal{L}_{t,\omega}(\mathbb{1}_{Q_M})(z)\ge\frac{C}{t-1}.$$

Proof. Take an arbitrary point $z\in Q_M$, $z\ne 0$, and its representative in $x+iy\in\mathbb{C}$ with $y\in(-\pi,\pi]$. We assume that $y\ge 0$, the other case being treated identically. Then

$$\mathcal{L}_{t,\omega}(\mathbb{1}_{Q_M})(z)=\sum_{k\in K_z}\frac{1}{(x^2+(y+2k\pi)^2)^{t/2}}\ge\sum_{k\in K_z}\frac{1}{(|x|+y+2k\pi)^t}.$$

where K_z is the set of integers k for which $\big|\log|x+iy+2k\pi i|-\log\eta(\omega)\big|\le M$, i.e.,

$$\log|x+iy+2k\pi i|\le M+\log\eta(\omega) \tag{10.11}$$

and

$$\log|x+iy+2k\pi i|\ge -M+\log\eta(\omega). \tag{10.12}$$

Let

$$K := \frac{Ae^M - M}{2\pi} - \frac{1}{2}. \tag{10.13}$$

A straightforward calculation shows that if $0 \le k \le K$, then k satisfies inequality (10.11) for every $z \in Q_M$. On the other hand, if $-M + \log \eta(\omega) \le 0$, then inequality (10.11) holds for every integer $k \ge 1$. So, assuming that $M > \log B$, which we do from now on, we will have that (10.12) holds for every integer $k \ge 1$ and every point $z \in Q_M$. Hence, for every $z \in Q_M$,

$$\{k \in \mathbb{Z} : 1 \le k \le K\} \subset K_z.$$

Therefore,

$$\mathcal{L}_{t,\omega}(\mathbb{1}_{Q_M})(z) \ge \sum_{1 \le k \le K} \frac{1}{(|x| + y + 2k\pi)^t} = \frac{1}{(2\pi)^t} \sum_{0 \le k \le K-1} \frac{1}{\left(\frac{|x|+y+2\pi}{2\pi} + k\right)^t} \tag{10.14}$$

We are thus to estimate from below the sum $\sum_{k=0}^{K-1} \frac{1}{(a+k)^t}$ with $a = \frac{|x|+y+2\pi}{2\pi}$. The bound is given by the integral

$$\int_0^K \frac{1}{(a+s)^t} ds = \frac{1}{t-1}\left(\frac{1}{a^{t-1}} - \frac{1}{(a+K)^{t-1}}\right) \tag{10.15}$$

We now want to find $M > \log B$ so large that the estimate

$$\frac{1}{(a+K)^{t-1}} < \frac{1}{2}\frac{1}{a^{t-1}} \quad \text{or, equivalently,} \quad a + K > 2^{\frac{1}{t-1}} a \tag{10.16}$$

holds. So, for (10.16) to be satisfied it is enough to have $K/a > 2^{\frac{1}{t-1}}$. Assuming that $M > 3\pi$, invoking the formula (10.13) which defines K, and using the inequality $2\pi a \le M + 1 + 2\pi$, we see that for (10.16) to hold it is enough to have tho following inequality:

$$\frac{Ae^M - 2M}{2M} > 2^{\frac{1}{t-1}}. \tag{10.17}$$

A straightforward estimate shows that, if

$$M \ge M_t := \frac{2\log 2}{t-1}$$

and t is sufficiently close to 1, then (10.17), and, consequently, also (10.16) holds. We can this estimate (10.14)

$$\begin{aligned}\int_0^K \frac{1}{(a+s)^t} &\ge \frac{1}{2(t-1)}\frac{1}{a^{t-1}} \ge \frac{1}{2(t-1)}\left(\frac{2\pi}{M_t+3\pi}\right)^{t-1} \\ &= \frac{1}{t-1}\frac{(2\pi)^{t-1}}{2}\left(\frac{M_t}{M_t+3\pi}\right)^{t-1}\frac{1}{M_t^{t-1}}.\end{aligned}$$

Now, since $M_t \ge 3\pi$, we have that

$$\left(\frac{M_t}{M_t+3\pi}\right)^{t-1} \ge \frac{1}{2}$$

for all $t \in (1,2]$. Invoking the formula defining M_t and using, in the last step, the inequality $x^x \geq \exp(-1/e)$ for all $x \in (0,1]$), we thus have for $z \in Q_{M_t}$ that

$$\mathcal{L}_{t,\omega}(1\!\!1_{Q_{M_t}})(z) \geq \frac{1}{t-1} \cdot \frac{(2\pi)^{t-1}}{4} \cdot \frac{1}{(2\pi)^t} \cdot \frac{(t-1)^{t-1}}{(2\log 2)^{t-1}} \geq C\frac{1}{t-1}$$

with

$$C = \frac{1}{8\pi(\log 2)^2}\exp(-1/e).$$

The proof is complete. □

Corollary 66.

$$\lim_{t\to 1^+} \mathcal{E}\mathrm{P}(t) = +\infty.$$

Proof. Indeed, it follows from Lemma 63 and (10.9) that for each $t > 1$ and every M_1 sufficiently large, there exists a sequence n_k and a positive measure set Ω_{**} such that for $\omega \in \Omega_{**}, z \in E_\omega \subset Q_{M_1}$

$$\mathcal{E}\mathrm{P}(t) = \lim_{k\to\infty} \frac{1}{n_k} \log \mathcal{L}_{t,\omega}^{n_k}(1\!\!1)(z).$$

Obviously, one can take $M_1 \geq M_t$, where M_t comes from Lemma 65. Then, by applying this lemma we obtain the required estimate. □

As a consequence of the above lemmas and the last corollary, we get the following.

Corollary 67. *There exists a unique value $h \in (1,2]$ such that $\mathcal{E}\mathrm{P}(h) = 0$.*

Now, in order to conclude the proof of Theorem 61, we are only left to establish its item (4). Towards this end, we shall prove the following auxiliary result.

Lemma 68. *For every $\omega \in \Omega$ and every integer $N \geq 2$ there exists $q(\omega, N)$ such that if $z \in Q_N$ and if $q(\omega,N) \leq n \in N_\omega(z,N)$, then*

$$|(F_\omega^n)'(z)| \geq 2.$$

Proof. Fix an integer $N \geq 2$. Notice that then there exists an integer $q_N \geq 0$ such that

$$f_\omega^n(\mathbb{R}) \subset [4N, +\infty) \tag{10.18}$$

for all $\omega \in \Omega$ and all $n \geq q_N$. Now fix also $\omega \in \Omega$. Assume for a contrary that there exist a strictly increasing sequence $(n_l)_{l=1}^\infty$ of integers, all greater than or equal to q_N, and a sequence $(z_l)_{l=1}^\infty$ of points in Q_N such that

$$n_l \in N_\omega(z_l, N) \ \text{ and } \ |(F_\omega^{n_l})'(z_l)| \leq 2$$

for every $l \geq 1$. Using compactness of Q_N we can replace the sequence $(n_l)_{l=1}^\infty$ by its increasing subsequence for which there exist a point $\xi \in Q_N$ such that

$$z_l \in B(\xi, 1/(16Nl)).$$

It then follows from Koebe's $\frac{1}{4}$–Distortion Theorem that

$$B(\xi, 1/(16N)) \subset F_{\omega,z_l}^{-n_l}\big(B(F_\omega^{n_l}(z_l), 1/N)\big) \tag{10.19}$$

now seeking contradiction assume that

$$f_\omega^k\big(B(\xi,2^{-4}N^{-1})\big)\cap\left(\bigcup_{j\in\mathbb{Z}}\mathbb{R}+j\pi i\right)\neq\emptyset$$

for some integer $k\geq 0$. Then for for every integer $l\geq 1$,

$$f_\omega^k\big(F_{\omega,z_l}^{-n_l}\big(B(F_\omega^{n_l}(z_l),1/N)\big)\big)\cap\left(\bigcup_{j\in\mathbb{Z}}\mathbb{R}+j\pi i\right)\neq\emptyset.$$

Fix $l\geq 1$ so large that $n_l-k\geq q_N+1$. Then invoking (10.18), we conclude that

$$B(F_\omega^{n_l}(z_l),1/N)\cap[4N,+\infty)\neq\emptyset.$$

Hence, $F_\omega^{n_l}(z_l)\notin Q_N$ contrary to the fact that $n_l\in N_\omega(z_l,N)$. So,

$$f_\omega^k\big(F_{\omega,z_l}^{-n_l}\big(B(F_\omega^{n_l}(z_l),1/N)\big)\big)\cap\left(\bigcup_{j\in\mathbb{Z}}\mathbb{R}+j\pi i\right)=\emptyset \tag{10.20}$$

for every integer $k\geq 0$. Now, as at the beginning of the paper, keep S to denote the set

$$\{z\in\mathbb{C}:|\mathrm{Im}(z)|<\pi\}.$$

If the set

$$A_\omega(N):=\big\{k\geq 0: f_\omega^k\big(B(\xi,2^{-4}N^{-1})\big)\cap S=\emptyset\big\}$$

is infinite, then $\overline{\lim}_{j\to\infty}|(f_\omega^j)'(\xi)|=+\infty$ by lemma 10 and the Chain Rule. This and (10.20) would however contradict Bloch's Theorem, proving that the set $A_\omega(N)$ is finite. But if $f_\omega^k\big(B(\xi,2^{-4}N^{-1})\big)\cap S\neq\emptyset$, then

$$f_\omega^k\big(B(\xi,2^{-4}N^{-1})\big)\subset S \tag{10.21}$$

by (10.20) again. Therefore, (10.34) holds for all but finitely many k's. This however contradicts Lemma 13, finishing the proof of Lemma 68. ☐

Now, within the framework of Lemma 68, let $q_0(\omega,N)$ denote the least number $q(\omega,N)$ produced by this lemma. We immediately observe the following.

Observation 69. *For every integer $N\geq 2$ the function*

$$\Omega\ni\omega\longmapsto q_0(\omega,N)\in\mathbb{N}$$

is measurable.

Lemma 70.

$$\mathrm{HD}(J_r(\omega))=h$$

for m–a.e. $\omega\in\Omega$.

Proof. The beginning of this proof is similar to the proof of Lemma 64. It follows from Corollary 59, Theorem 58, Proposition 60, and Birkhoff's Ergodic Theorem, that there exist $\chi>0$, a measurable set $\Omega_0\subset\Omega$ with $m(\Omega_0)=1$, and for each $\omega\in\Omega_0$, a measurable set $J_r^0(\omega)\subset J_r(\omega)$ such that

$$\nu_\omega^{(h)}(J_r^0(\omega))=1$$

and

$$\lim_{n\to\infty}\frac{1}{n}\log|(F_\omega^n)'(z)|=\chi \tag{10.22}$$

for every $\omega\in\Omega_0$ and every $z\in J_r^0(\omega)$. It furthermore follows from Birkhoff's Ergodic Theorem that there exists a measurable set $\Omega_1\subset\Omega_0$ with $m(\Omega_1)=1$, and such that

$$\lim_{n\to\infty}\frac{1}{n}\log\lambda_{h,\omega}^n=0 \tag{10.23}$$

for every $\omega\in\Omega_1$. Fix such $\omega\in\Omega_1$ and $z\in J_r^0(\omega)$. Fix $\eta\in(0,1/2)$ arbitrary. By the definition of $J_r(\omega)$ there exists an integer $N_\eta\ge 1$ such that

$$\underline{\rho}(N_\omega(z,N_\eta))>1-\eta. \tag{10.24}$$

For every $r\in(0,1/N_\eta)$ let $k:=k(z,r)$ be the largest integer $n\in N_\omega(z,N_\eta)$ such that

$$F_{\omega,z}^{-n}\big(B(F_\omega^n(z),1/N_\eta)\big)\supset B(z,r). \tag{10.25}$$

Let $s=s_k$ be the largest integer $\ge k+1$ belonging to $N_\omega(z,N_\eta)$. It follows from (10.24) that

$$\lim_{r\to 0}\frac{k(z,r)}{s_{k(z,r)}}\ge 1-\eta. \tag{10.26}$$

Applying conformality of the measure $\nu^{(h)}$ and Koebe's Distortion Theorem, we now conclude from (10.25) and the definition of k that

$$\begin{aligned}\nu_\omega^{(h)}\big(B(z,r)\big)&\le\nu_\omega^{(h)}\big(F_{\omega,z}^{-k}\big(B(F_\omega^k(z),1/N_\eta)\big)\big)\\&\le K^h\lambda_{\omega,h}^{-k}\big|\big(F_\omega^k\big)'(z)\big|^{-h}\nu_{\theta^k\omega}^{(h)}\big(B(F_\omega^k(z),1/N_\eta)\big)\\&\le K^h\lambda_{\omega,h}^{-k}\big|\big(F_\omega^k\big)'(z)\big|^{-h}.\end{aligned} \tag{10.27}$$

On the other hand $B(z,r)\not\subset F_{\omega,z}^{-s}\big(B(F_\omega^s(z),1/N_\eta)\big)$. But since, by $\frac14$-Koebe's Distortion Theorem,

$$F_{\omega,z}^{-s}\big(B(F_\omega^s(z),1/N_\eta)\big)\supset B\left(z,\frac14\big|\big(F_\omega^s\big)'(z)\big|^{-1}N_\eta^{-1}\right),$$

we thus get that $r\ge\frac14\big|\big(F_\omega^s\big)'(z)\big|^{-1}N_\eta^{-1}$. Equivalently,

$$\big|\big(F_\omega^s\big)'(z)\big|^{-1}\le 4N_\eta r.$$

By inserting this into (10.27) and using also the Chain Rule, we obtain

$$\nu_\omega^{(h)}\big(B(z,r)\big)\le(4KN_\eta)^hr^h\lambda_{\omega,h}^{-k}\big|\big(F_{\theta^k\omega}^{s-k}\big)'(F_\omega^k(z))\big|^h.$$

Equivalently,

$$\begin{aligned}\frac{\log\nu_\omega^{(h)}(B(z,r))}{\log r}&\ge h+\frac{\log(4KN_\eta)}{\log r}-\frac{\log\lambda_{\omega,h}^k}{\log r}+h\frac{\log\big|\big(f_{\theta^k\omega}^{s-k}\big)'(F_\omega^k(z))\big|}{\log r}\\&=h-\frac{k}{\log(1/r)}\frac{\log(4KN_\eta)}{k}+\frac{k}{\log(1/r)}\frac1k\log\lambda_{\omega,h}^k-\\&\qquad-h\frac{k}{\log(1/r)}\frac1k\log\big|\big(f_{\theta^k\omega}^{s-k}\big)'(F_\omega^k(z))\big|.\end{aligned} \tag{10.28}$$

Now, Koebe's Distortion Theorem yields

$$F_{\omega,z}^{-k}\big(B(F_\omega^k(z),1/N_\eta)\big)\subset B\big(z,K\big|\big(F_\omega^k\big)'(z)\big|^{-1}N_\eta^{-1}\big).$$

Along with (10.25) and the definition of k this gives $r\le K\big|\big(F_\omega^k\big)'(z)\big|^{-1}N_\eta^{-1}$. Equivalently:

$$-\log r\ge \log(N_\eta/K)+\log\big|\big(F_\omega^k\big)'(z)\big|. \tag{10.29}$$

Therefore, invoking (10.22), we get that

$$\limsup_{r\to 0}\frac{k(z,r)}{\log(1/r)}\le 1/\chi. \tag{10.30}$$

Also, formula (10.22) along with (10.26) gives

$$\lim_{r\to 0}\frac{1}{k}\log\big|\big(F_{\theta^k\omega}^{s-k}\big)'(F_\omega^k(z))\big|\le\frac{\chi}{1-\eta}-\chi=\frac{\eta}{1-\eta}\chi. \tag{10.31}$$

Inserting now (10.30), (10.23), and (10.31) to (10.28), we obtain

$$\liminf_{r\to 0}\frac{\log\nu_\omega^{(h)}(B(z,r))}{\log r}\ge h\left(1-\frac{\eta}{1-\eta}\right)=\frac{1-2\eta}{1-\eta}h.$$

Since $\eta\in(0,1/2)$ was arbitrary, this yields

$$\liminf_{r\to 0}\frac{\log\nu_\omega^{(h)}(B(z,r))}{\log r}\ge h, \tag{10.32}$$

Therefore

$$\mathrm{HD}(J_r(\omega))\ge \mathrm{HD}(J_r^0(\omega))\ge h, \tag{10.33}$$

and one side of the equation from Lemma 70 is thus established.

For the opposite inequality set $\eta:=1/4$ and

$$N:=N_{1/4}.$$

By Lemma 68, Observation 69, and Proposition 39, there exists an integer $q\ge 1$ such that

$$m\big(\big\{\omega\in\Omega: q_0(\omega,N)\le q\big\}\cap\Omega(N,1/N,1/8)\big)>5/8. \tag{10.34}$$

It therefore follows from Birkhoff's Ergodic Theorem that there exists a measurable set $\hat\Omega\subset\Omega$ such that $m(\hat\Omega)=1$ and

$$\rho\big(\big\{n\ge 0: q_0(\theta^n\omega,N)\le q\ \text{ and }\ \theta^n\omega\in\Omega(N,1/N,1/8)\big\}\big)>5/8$$

for all $\omega\in\hat\Omega$. Then

$$\rho\big(N_\omega(z,N)\cap\big\{n\ge 0: q_0(\theta^n\omega,N)\le q\ \text{ and }\ \theta^n\omega\in\Omega(N,1/N,1/8)\big\}\big)>3/8. \tag{10.35}$$

Fix now an arbitrary element $\omega\in\hat\Omega$ and $z\in J_r(\omega)$. S There thus exists an integer $l_0\ge 1$ so large that if l is an integer $\ge l_0$ and if u_l is the lth element of the set

$$N_\omega(z,N)\cap\big\{n\ge 0: q_0(\theta^n\omega,N)\le q\ \text{ and }\ \theta^n\omega\in\Omega(N,1/N,1/8\big\},$$

then

$$\frac{l}{u_l}\ge 3/8.$$

So, applying Lemma 68, we thus get that

$$|(F_\omega^{u_l})'(z)| \ge 2^{l/q} \ge 2^{3u_l/8q} \tag{10.36}$$

Let $r_l > 0$ be the least radius such that

$$F_\omega^{-u_l}\big(B(F_\omega^{u_l}(z), 1/N)\big) \subset B(z, r_l). \tag{10.37}$$

But, by Koebe's Distortion Theorem, $F_\omega^{-u_l}\big(B(F_\omega^{u_l}(z),1/N)\big) \subset B\big(z, KN^{-1}|(F_\omega^{u_l})'(z)|^{-1}\big)$; hence

$$r_l \le KN^{-1}|(F_\omega^{u_l})'(z)|^{-1}. \tag{10.38}$$

Formula (10.37) along with Koebe's Distortion Theorem and (10.38), and Proposition 39 (the constant $\xi = \xi(N, 1/N, 1/8) > 0$ below comes from it), yield

$$\begin{aligned} \nu_\omega^{(h)}\big(B(z,r_l)\big) &\ge \nu_\omega^{(h)}\big(F_\omega^{-u_l}\big(B(F_\omega^{u_l}(z),1/N)\big)\big) \\ &\ge K^{-h}\lambda_{\omega,h}^{-u_l}\big|\big(F_\omega^{u_l}\big)'(z)\big|^{-h}\nu_{\theta^{u_l}\omega}\big(B(F_\omega^{u_l}(z),1/N)\big) \\ &\ge K^{-h}\xi\lambda_{\omega,h}^{-u_l}\big|\big(F_\omega^{u_l}\big)'(z)\big|^{-h} \\ &\ge (K^{-2}N)^h\xi\lambda_{\omega,h}^{-u_l}r_l^h. \end{aligned} \tag{10.39}$$

Therefore,

$$\frac{\log \nu_\omega^{(h)}(B(z,r_l))}{\log r_l} \le h + \frac{h\log(N/K^2)}{\log r_l} - \frac{\log \lambda_{\omega,h}^{u_l}}{\log r_l} + \frac{\xi}{\log r_l}. \tag{10.40}$$

Formula (10.38) equivalently means that

$$-\log r_l \ge \log|(F_\omega^{u_l})'(z)| + \log(N/K). \tag{10.41}$$

Hence, invoking (10.36), we get that

$$-\log r_l \ge \frac{3\log 2}{8q}u_l + \log(N/K). \tag{10.42}$$

Inserting this to (10.40) and using (10.23), we get

$$\liminf_{r\to 0}\frac{\log \nu_\omega^{(h)}(B(z,r))}{\log r} \le \liminf_{l\to\infty}\frac{\log \nu_\omega^{(h)}(B(z,r_l))}{\log r_l} \le h.$$

Therefore,

$$\mathrm{HD}(J_r(\omega)) \le h,$$

and along with (10.33) this finishes the proof of Lemma 70. $\square$

11. Hausdorff Dimension of the radial Julia set is smaller than 2

Lemma 71. *Let $(Y, \mathfrak{F}, \mu)$ be a probability space and let $T : Y \to Y$ be a measure preserving ergodic transformation. Assume that $\varphi : Y \to \mathbb{R}$ is an integrable function with $\int \varphi d\mu = 0$. Assume further that there exist a set $A \in \mathfrak{F}$ with $\mu(A) > 0$ and a constant $C > 0$ such that for all $y \in A$*

$$\sup_{k\geq 1}\{S_k\varphi(y)\} < C.$$

Then for μ-a.e. $y \in A$ the following implication holds:

$$T^n(y) \in A \implies S_n\varphi(y) > -2C. \tag{11.1}$$

Proof. Assume that (11.1) does not hold. Then there exists a measurable subset $B \subset A$ with $\mu(B) > 0$ and such that for every $y \in B$ there exists an integer $n \geq 1$ for which

$$T^n(y) \in A \quad \text{and} \quad S_n\varphi(y) \leq -2C. \tag{11.2}$$

Replacing B by its subset, still of positive measure, we can assume that there exists an integer $k \geq 1$ such that (11.2) holds for integers n being the kth returns of y to A. Now, let us consider the map $\hat{T}_B^{(k)} : B \to B$ being the kth return from B to B. For μ–almost every $x \in B$ denote by $n_B(x)$ the first return time of x to B and by $n_B^{(k)}(x)$ the kth return time of x to B.

Kac's lemma applied for the kth return map $\hat{T}^{(k)}$ and for the function φ thus gives

$$\int_B S_{n_B^{(k)}(x)}\varphi(x)d\mu(x) = k\int_B S_{n_B(x)}\varphi(x)d\mu(x) = k\int_X \varphi(x)d\mu(x) = 0. \tag{11.3}$$

Still for μ–almost every $x \in B$ denote by $n_A^{(k)}(x)$ the k-th entrance time of x to A and notice an obvious inequality $n_B^{(k)}(x) \geq n_A^{(k)}(x)$. Writing

$$S_{n_B^{(k)}(x)}\varphi(x) = S_{n_A^{(k)}(x)}\varphi(x) + S_{n_B^{(k)}(x)-n_A^{(k)}(x)}\varphi\big(T^{n_A^{(k)}(x)}(x)\big),$$

we see that

$$S_{n_B^{(k)}(x)}\varphi(x) < -2C + C = -C$$

for μ–almost all $x \in B$. But this contradicts (11.3) and finishes the proof of our lemma. □

In Section 10 we proved that the dimension of the radial random Julia set $J_r(\omega)$ is almost surely equal to the only value h such that the expected pressure at h, i.e.

$$\mathcal{E}\mathrm{P}(h) = \int \log \lambda_{h,\omega} dm(\omega) = 0.$$

As in Section 9, we denote

$$\lambda_{h,\omega}^n := \lambda_{h,\omega} \cdot \lambda_{h,\theta\omega} \cdot \lambda_{h,\theta^{n-1}\omega}.$$

Our goal now is to prove that $h < 2$. The crucial technical ingredient is the following.

Proposition 72. *For m–a.e. $\omega \in \Omega$ and for $\nu_\omega^{(h)}$–almost every point $z \in Q$ we have that*

$$\liminf_{r\to 0} \frac{\nu_\omega^{(h)}(B(z,r))}{r^h} = 0. \tag{11.4}$$

Proof. We consider two separate cases.

Case 1^0**.** Partial sums

$$\log \lambda_{h,\omega}^n = \sum_{j=0}^{n-1} \log \lambda_{h,\theta^j\omega}$$

are bounded above for a measurable set of points $\omega \in \Omega$ with positive measure m. This means that there exist a measurable set $A \subset \Omega$ with $m(A) > 0$ and a constant $C < +\infty$ such that

$$\log \lambda_{h,\omega}^n < C \tag{11.5}$$

for all $\omega \in A$. By ergodicity of the map $F : \Omega \times Q \to \Omega \times Q$ with respect to the measure $\mu^{(h)}$ (see Theorem 58) and by Birkhoff's Ergodic Theorem there exists a measurable set $\Gamma_1 \subset \Omega \times Q$ with $\mu^{(h)}(\Gamma_1) = \mu^{(h)}(\Gamma_1) = 1$ and such that for every point $(\omega, z) \in \Gamma_1$ there exists an integer $k_1(\omega, z) \geq 0$ such that

$$F^{k_1(\omega,z)}(\omega, z) \in A \times Q. \tag{11.6}$$

Fix an integer $N \geq 1$ and consider the set

$$A \times \big(Y_{N+2}^+ \setminus (\mathbb{R} \times (-2, 2))\big).$$

Since $\mu^{(h)}\big(A \times \big(Y_{N+2}^+ \setminus (\mathbb{R} \times (-2, 2))\big)\big) > 0$, again by ergodicity of the map $F : \Omega \times Q \to \Omega \times Q$ with respect to the measure $\mu^{(h)}$ (see Theorem 58) and by Birkhoff's Ergodic Theorem there exists a measurable set $\Gamma_2 \subset A \times Q$ with $\mu^{(h)}(\Gamma_2) = \mu^{(h)}(A \times Q)$ and such that for every point $(\tau, \xi) \in \Gamma_2$ there exists an integer $k_2(\tau, \xi; N) \geq 0$ such that

$$F^{k_2(\tau,\xi;N)}(\tau, \xi) \in A \times \big(Y_{N+2}^+ \setminus (\mathbb{R} \times (-2, 2))\big). \tag{11.7}$$

In conclusion, there exists a measurable set $\Gamma_3(N) \subset \Gamma_1$ such that $\mu^{(h)}(\Gamma_3(N)) = 1$ and

$$F^{k_1(\omega,z)}(\omega, z) \in \Gamma_2 \tag{11.8}$$

for all points $(\omega, z) \in \Gamma_3(N)$. In particular $k_2\big(F^{k_1(\omega,z)}(\omega, z); N\big)$ is well defined and finite. For every point $(\omega, z) \in \Gamma_3(N)$ set

$$\ell_N(\omega, z) := k_1(\omega, z) + k_2\big(F^{k_1(\omega,z)}(\omega, z); N\big). \tag{11.9}$$

Denote

$$\Gamma_3(\infty) := \bigcap_{N=1}^{\infty} \Gamma_3(N).$$

Then

$$\mu^{(h)}(\Gamma_3(\infty)) = 1$$

and the number $\ell_N(\omega, z)$ is well defined for all points $(\omega, z) \in \Gamma_3(\infty)$ and all integers $N \geq 1$. Fix such (ω, z) and N. Then

$$\log \lambda_{h,\omega}^{\ell_N(\omega,z)} = \log \lambda_{h,\omega}^{k_1(\omega,z)} + \log \lambda_{h,\theta^{k_1(\omega,z)}\omega}^{k_2\left(F^{k_1(\omega,z)}(\omega,z);N\right)} > \log \lambda_{h,\omega}^{k_1(\omega,z)} - 2C \tag{11.10}$$

by (11.6)–(11.9) and Lemma 71.

Now, since $F_\omega^{\ell_N(\omega,z)}(z)\big) \in Y_{N+2}^+ \setminus (\mathbb{R} \times (-2, 2))$, the holomorphic inverse branch $F_\omega^{-\ell_N(\omega,z)} : B\big(F_\omega^{\ell_N(\omega,z)}(z), 2\big) \to Q$, sending $F_\omega^{\ell_N(\omega,z)}(z)$ back to z, is well defined,

$$B\big(F_\omega^{\ell_N(\omega,z)}(z), 1\big) \subset Y_N^- \tag{11.11}$$

and, with a use of Koebe's Distortion Theorem,

$$B\left(z, \frac{1}{4}\big|\big(F_\omega^{\ell_N(\omega,z)}\big)'(z)\big|^{-1}\right) \subset F_\omega^{-\ell_N(\omega,z)}\big(B\big(F_\omega^{\ell_N(\omega,z)}(z), 1\big)\big).$$

Set

$$r_N(\omega, z) := \frac{1}{4}\big|\big(F_\omega^{\ell_N(\omega,z)}\big)'(z)\big|^{-1}.$$

Then, using, (11.11), (11.10), and (5.13), we get

$$\begin{aligned}
\frac{\nu_\omega^{(h)}\big(B(z, r_N(\omega, z))\big)}{r_N^h(\omega, z)} &\le \frac{\nu_\omega^{(h)}\big(F_\omega^{-\ell_N(\omega,z)}\big(B\big(F_\omega^{\ell_N(\omega,z)}(z), 1\big)\big)\big)}{r_N^h(\omega, z)} \\
&\le \frac{\lambda_{h,\omega}^{-\ell_N(\omega,z)} K^h \big|\big(F_\omega^{\ell_N(\omega,z)}\big)'(z)\big|^{-h} \nu_{\theta^{\ell_N(\omega,z)}\omega}^{(h)}\big(B\big(F_\omega^{\ell_N(\omega,z)}(z), 1\big)\big)}{r_N^h(\omega, z)} \\
&= \frac{(4K)^h \lambda_{h,\omega}^{-\ell_N(\omega,z)} r_N^h(\omega, z) \nu_{\theta^{\ell_N(\omega,z)}\omega}^{(h)}\big(B\big(F_\omega^{\ell_N(\omega,z)}(z), 1\big)\big)}{r_N^h(\omega, z)} \\
&= (4K)^h \lambda_{h,\omega}^{-\ell_N(\omega,z)} \nu_{\theta^{\ell_N(\omega,z)}\omega}^{(h)}\big(B\big(F_\omega^{\ell_N(\omega,z)}(z), 1\big)\big) \\
&\le (4K)^h \lambda_{h,\omega}^{-\ell_N(\omega,z)} \nu_{\theta^{\ell_N(\omega,z)}\omega}^{(h)}(Y_N^+) \\
&\le (4K)^h e^{2C} \lambda_{h,\omega}^{-k_1(\omega,z)} C(M_0) e^{(1-t))\frac{N}{2}}.
\end{aligned}$$

Since $1 - t < 0$, this yields

$$\varliminf_{r\to 0} \frac{\nu_\omega^{(h)}(B(z, r)}{r^h} \le \varliminf_{N\to\infty} \frac{\nu_\omega^{(h)}\big(B(z, r_N(\omega, z))\big)}{r_N^h(\omega, z)} \le (4K)^h e^{2C} \lambda_{h,\omega}^{-k_1(\omega,z)} C(M_0) \lim_{N\to\infty} e^{(1-t))\frac{N}{2}} = 0.$$

Case 2. For m–a.e $(\omega, z) \in \Omega \times Q$

$$\limsup_{n\to\infty} \lambda_\omega^n = +\infty.$$

Let (ω, z) be a point for which the above upper limit is equal to $+\infty$. There then exists a strictly increasing sequence $\big(n_j(\omega, z)\big)_{j=1}^\infty$ of positive integers such that

$$\lim_{n\to\infty} \lambda_\omega^{n_j(\omega,z)} = +\infty. \tag{11.12}$$

Fix a radius $0 < s < \min\{1, \rho\}/4$, where ρ comes from Lemma 24. Fix an integer $j \ge 1$. If

$$B(F_\omega^{n_j(\omega,z)}(z), 2s) \cap \big\{F_{\theta^k\omega}^{n_j(\omega,z)-k}(0) : k = 1, \dots n_j - 1\big\} = \emptyset,$$

then there exists a holomorphic branch $F_{\omega,z}^{-n_j(\omega,z)}$ defined on $B\big(F^{n_j(\omega,z)}(z), 2s\big)$ and sending $F_\omega^{n_j}(z)$ back to z. Analogously as in the previous case, put

$$r_j(\omega, z) := \frac{1}{4}\big|\big(F_\omega^{n_j(\omega,z)}\big)'(z)\big|^{-1} s.$$

Then by the same token as in the previous case, we get

$$\frac{\nu_\omega^{(h)}\big(B(z, r_j(\omega, z))\big)}{r_j^h(\omega, z)} \le (4K)^h \lambda_{h,\omega}^{-n_j(\omega,z)} \nu_{\theta^{n_j(\omega,z)}\omega}^{(h)}\big(B\big(F_\omega^{n_j(\omega,z)}(z), r\big)\big) \le (4K)^h \lambda_{h,\omega}^{-n_j(\omega,z)}. \tag{11.13}$$

Finally, consider the case when the ball $B\big(F_\omega^{n_j(\omega,z)}(z),2s\big)$ contains some point from the set $\big\{F_{\theta^k\omega}^{n_j-k}(0) : k=1,\dots n_j(\omega,z)-1\big\}$. Fix such $k\in\{0,\dots n_j-1\}$ with the smallest distance between $F_\omega^{n_j(\omega,z)}(z)$ and $F_{\theta^k\omega}^{n_j(\omega,z)-k}(0)$ in addition. Denote this distance by $2\hat s$. Then $2\hat s<2s<\rho/2$ and

$$B\big(F_\omega^{n_j(\omega,z)(\omega,z)}(z),\hat s\big)\subset B\big(F_{\theta^k\omega}^{n_j(\omega,z)-k}(0),3\hat s\big).$$

It thus follows from Lemma 24 that

$$\nu^{(h)}_{\theta^{n_j(\omega,z)}\omega}\Big(B\big(F_\omega^{n_j(\omega,z)(\omega,z)}(z),\hat s\big)\Big)\le \nu^{(h)}_{\theta^{n_j(\omega,z)}\omega}\Big(B\big(F_{\theta^k\omega}^{n_j(\omega,z)-k}(0),3\hat s\big)\Big)\preceq \hat r^u\le r^h.$$

It also follows from the definition of $\hat r$ that there exists a unique holomorphic branch $F_{\omega,z}^{-n_j(\omega,z)}$ defined on $B\big(F^{n_j(\omega,z)}(z),2\hat r\big)$ and sending $F_\omega^{n_j}(z)$ back to z. Analogously as in the previous case, put

$$\hat r_j(\omega,z):=\frac14\big|\big(F_\omega^{n_j(\omega,z)}\big)'(z)\big|^{-1}\hat r.$$

Then, in the same way as (11.13), we get

$$\frac{\nu_\omega^{(h)}\big(B(z,\hat r_j(\omega,z))\big)}{\hat r_j^h(\omega,z)}\le (4K)^h\lambda_{h,\omega}^{-n_j(\omega,z)}\hat r^{-h}\nu^{(h)}_{\theta^{n_j(\omega,z)}\omega}\Big(B\big(F_\omega^{n_j(\omega,z)}(z),\hat r\big)\Big)\preceq \lambda_{h,\omega}^{-n_j(\omega,z)}. \tag{11.14}$$

Along with formula (11.12), formulas (11.13) and (11.14) respectively imply that $\lim_{j\to\infty}r_j(\omega,z)=\lim_{j\to\infty}\hat r_j(\omega,z)=0$ and

$$\liminf_{r\to0}\frac{\nu_\omega^{(h)}(B(z,r))}{r^h}=0.$$

The proof of Proposition 72 is complete. ☐

Theorem 73. *The Hausdorff dimension $h=\mathrm{HD}(J_r(\omega))$ of the random radial Julia set $J_r(\omega)$, is constant for m–a.e. $\omega\in\Omega$ and satisfies $1<h<2$. In particular, the 2–dimensional Lebesgue measure of m–a.e. $\omega\in\Omega$ set $J_r(\omega)$ is equal to zero.*

Proof. The fact that the function $\Omega\ni\omega\mapsto \mathrm{HD}(J_r(\omega))$ is constant for m–a.e. $\omega\in\Omega$, and the inequality $h>1$ is just item (4) of Theorem 61.

Because of Proposition 72, h–dimensional packing measure of Q is locally infinite for m–a.e. $\omega\in\Omega$. Since 2–dimensional packing measure is just the (properly rescaled) 2–dimensional Lebesgue measure, it is locally finite. Thus $h<2$.

☐

As a corollary, we obtain the following result about trajectories of (Lebesgue) typical points.

Theorem 74 (Trajectory of a (Lebesgue) typical point I)**.** *For m–almost every $\omega\in\Omega$ there exists a subset $Q_\omega\subset Q$ with full Lebesgue measure such that for all $z\in Q_\omega$ the following holds.*

$$\begin{aligned}&\forall\delta>0\ \exists n_z(\delta)\in\mathbb N\ \forall n\ge n_z(\delta)\ \exists k=k_n(z)\ge0\\ &\qquad |F_\omega^n(z)-F^k_{\theta^{n-k}\omega}(0)|<\delta\quad\text{or}\quad |F_\omega^n(z)|\ge 1/\delta.\end{aligned}\tag{11.15}$$

In addition, $\limsup_{n\to\infty}k_n(z)=+\infty$.

Proof. For every $\omega \in \Omega$, the set of points with trajectories described (11.15) contains the complement of the radial set Julia set $J_r(\omega)$. So, now the first assertion follows immediately from the last assertion of Theorem 73. The second assertion is obvious. □

As an immediate consequence of this theorem we get the following.

Corollary 75 (Trajectory of a (Lebesgue) typical point II)**.** *For m–almost every $\omega \in \Omega$ there exists a subset $Q_\omega \subset Q$ with full Lebesgue measure such that for all $z \in Q_\omega$, the set of accumulation points of the sequence*

$$\left(F_\omega^n(z)\right)_{n=0}^{\infty}$$

is contained in $[0, +\infty] \cup \{-\infty\}$ and contains $+\infty$.

12. Random Dynamics on the Complex Plane: the Original Random Dynamical system f_ω^n

In this section we will show that both random dynamical systems

$$f_\omega^n := f_{\theta^{n-1}\omega} \circ \cdots \circ f_{\theta\omega} \circ f_\omega : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$

and

$$F_\omega^n := F_{\theta^{n-1}\omega} \circ \cdots \circ F_{\theta\omega} \circ F_\omega : Q \longrightarrow Q,$$

$\omega \in \Omega$ are conjugate via conformal (bi-holomorphic) homeomorphisms. Start with a single exponential map $f_\eta : \mathbb{C} \longrightarrow \mathbb{C}$ given by the formula

$$f_\eta(z) = \eta e^z.$$

Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Since $\exp : \mathbb{C} \longrightarrow \mathbb{C}^*$ is a quotient map and f_η is constant on each set $\exp^{-1}(z)$, $z \in \mathbb{C}^*$, the map f_η induces a unique continuous map

$$\tilde{F}_\eta : \mathbb{C}^* \to \mathbb{C}^*$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f_\eta} & \mathbb{C} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ \mathbb{C}^* & \xrightarrow{\tilde{F}_\eta} & \mathbb{C}^* \end{array}$$

i.e.

$$\tilde{F}_\eta(\exp(z)) = \exp(f_\eta(z)). \tag{12.1}$$

The map $\tilde{F}_\eta$ can be easily calculated:

$$\tilde{F}_\eta(z) = \exp(f_\eta(\exp^{-1}(z))) = \exp(\eta z).$$

Let $H_\eta : \mathbb{C} \to \mathbb{C}$ be the similarity map given by the formula $H_\eta(z) = z/\eta$. Then

$$\tilde{F}_\eta \circ H_\eta(z) = \exp(\eta(z/\eta)) = \exp(z) = \frac{1}{\eta} f_\eta(z) = H_\eta \circ f_\eta(z). \tag{12.2}$$

This means that the maps $\tilde{F}_\eta$ and f_η are conjugate via H_η. Consequently,

Proposition 76. *For every integer $n \geq 1$,*

$$\tilde{F}_\eta^n \circ \exp = \exp \circ f_\eta^n, \tag{12.3}$$

i.e. the following diagram commutes

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f_\eta^n} & \mathbb{C} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ \mathbb{C}^* & \xrightarrow{\tilde{F}_\eta^n} & \mathbb{C}^* \end{array}$$

and

$$\tilde{F}_\eta^n \circ H_\eta = H_\eta \circ f_\eta^n, \tag{12.4}$$

i.e. the following diagram commutes

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f_\eta^n} & \mathbb{C} \\ {\scriptstyle H_\eta}\downarrow & & \downarrow{\scriptstyle H_\eta} \\ \mathbb{C} & \xrightarrow{\tilde{F}_\eta^n} & \mathbb{C} \end{array}$$

We now pass to the non–autonomous case. This means that we fix an element $\mathbf{a} \in [A, B]^{\mathbb{N}}$ and we consider the non–autonomous compositions

$$f_{\mathbf{a}}^n := f_{a_{n-1}} \circ f_{a_{n-2}} \circ \cdots \circ f_{a_1} \circ f_{a_0} : \mathbb{C} \longrightarrow \mathbb{C}.$$

and likewise with $\tilde{F}_{\mathbf{a}}^n$. Iterating (non-autonomously) (12.1) and doing straightforward calculations based on (12.2), we get the following.

Proposition 77. *For every integer $n \geq 1$,*

$$\tilde{F}_{\mathbf{a}}^n \circ \exp = \exp \circ f_{\mathbf{a}}^n. \tag{12.5}$$

and

$$\tilde{F}_{\mathbf{a}}^n \circ H_{a_1} = H_{a_n} \circ f_{\sigma(\mathbf{a})}^n \tag{12.6}$$

We need two more "little" results. First recall that the map $\exp : Q \to \mathbb{C}^*$ naturally defined from the cylinder Q to $\mathbb{C}^*$ is indeed well defined and is holomorphic:

Proposition 78. *The map* $\exp : Q \longrightarrow \mathbb{C}^*$

(1) is a conformal/holomorphic homeomorphism;
(2) transfers the Euclidean metric on Q to the conformal metric on $\mathbb{C}^*$: $|d\rho| := \frac{|dz|}{|z|}$
(3) conjugates $\tilde{F}_\eta$ and F_η, i.e.

$$\tilde{F}_\eta \circ \exp = \exp \circ F_\eta \quad \textit{and}$$

(4)

$$\tilde{F}_\eta^n \circ \exp = \exp \circ F_\eta^n$$

for every integer $n \geq 0$. In other words, the following diagram commutes

$$\begin{array}{ccc} Q & \xrightarrow{F_\eta^n} & Q \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ \mathbb{C}^* & \xrightarrow{\tilde{F}_\eta^n} & \mathbb{C}^* \end{array}$$

(5)

$$\tilde{F}_{\mathbf{a}}^n \circ \exp = \exp \circ F_{\mathbf{a}}^n$$

for every $\mathbf{a} \in [A,B]^{\mathbb{N}}$*. In other words, the following diagram commutes*

$$\begin{array}{ccc} Q & \xrightarrow{F_{\mathbf{a}}^n} & Q \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ \mathbb{C}^* & \xrightarrow{\tilde{F}_{\mathbf{a}}^n} & \mathbb{C}^* \end{array}$$

Proof. Item (1) is obvious. In order to prove item (3), we calculate

$$\exp(F_\eta([w]) = \exp(\pi \circ f_\eta \circ \pi^{-1}([w]) = \exp(\pi \circ f_\eta(w)) = \exp(f_\eta(w) = \tilde{F}\eta(z) = \tilde{F}_\eta(\exp([w]).$$

So,

$$\exp \circ F_\eta = \tilde{F}_\eta \circ \exp .$$

Item (4) is a standard consequence of item (3). Likewise, item (5), follows by a straightforward inductive argument based on (3).

Now, we shall prove item (2). If $[w] \in Q$ and v is a tangent vector at $[w]$ with Euclidean length 1 then it is mapped by exp to a tangent vector at the point $z = \exp([w])$, whose Euclidean length is equal to $|\exp'([w]) = |z|$. So, the conformal metric on $\mathbb{C}^*$ which makes the bijection $\exp : Q \to \mathbb{C}^*$ an isometry is exactly the one

$$d\rho := \frac{|dz|}{|z|}.$$

□

13. Random Ergodic Theory and Geometry on the Complex Plane: the Original Random Dynamical system f_ω^n

We shall now transfer all our results concerning random conformal measures, random invariant measures, Hausdorff dimension of fiber radial Julia sets, and asymptotic behavior of Lebesgue typical points to the case of original random system

$$\big(\Omega, \mathcal{F}, m; \, \theta : \Omega \to \Omega; \, \eta : \Omega \to [A,B]\big)$$

and induced by it random dynamics

$$\big(f_\omega^n : \mathbb{C} \to \mathbb{C}\big)_{n=0}^{\infty}, \quad \omega \in \Omega,$$

given by the formula:

$$f_\omega^n := f_{\theta^{n-1}\omega} \circ \cdots \circ f_{\theta\omega} \circ f_\omega : \mathbb{C} \longrightarrow \mathbb{C}.$$

Lemma 79. *Fix $t > 1$. If $\nu = \big(\nu_\omega\big)_{\omega\in\Omega}$ is a random conformal measure for the random conformal system*

$$F_\omega^n : Q \longrightarrow Q,\ \omega\in\Omega,\ n\ge 0,$$

with a measurable function $\lambda:\Omega\longrightarrow(0,+\infty)$ and the standard Euclidean metric, then the random measure

$$\tilde\nu := \big(\nu_\omega\circ\exp^{-1}\big)_{\omega\in\Omega}$$

is a random conformal measure for the random conformal system

$$\tilde F_\omega^n : \mathbb{C}^* \longrightarrow \mathbb{C}^*,\ \omega\in\Omega,\ n\ge 0,$$

with the same measurable function $\lambda:\Omega\longrightarrow(0,+\infty)$ and the Riemannian metric ρ given by formula (2) of Proposition 78. The converse is also true.

Proof. Using formula (5) of Proposition 78 and, of course, the definition of the random measure $\tilde\nu$, we get for every $\omega\in\Omega$, every integer $n\ge 1$ and every Borel set $A\subset\mathbb{C}^*$ such that the restricted map $F_\omega^n|_A$ is 1–to–1, that

$$\begin{aligned}
\tilde\nu\big(\tilde F_\omega^n(A)\big) &= \nu\big(\exp^{-1}(\tilde F_\omega^n(A))\big) = \nu\big(F_\omega^n(\exp^{-1}(A)))\big) \\
&= \int_{\exp^{-1}(A)} \lambda_\omega^n\big|\big(F_\omega^n\big)'\big|^t\,d\nu_\omega = \int_A \lambda_\omega^n\big|\big(F_\omega^n\big)'\circ\exp^{-1}\big|^t\,d\tilde\nu_\omega \\
&= \lambda_\omega^n\int_A \big|\tilde F_\omega^n(z)\big|^{-t}\big|\big(\tilde F_\omega^n\big)'(z)\big|^t|z|^t\,d\tilde\nu_\omega(z) \\
&= \lambda_\omega^n\int_A \big|\big(\tilde F_\omega^n\big)'\big|_\rho^t\,d\tilde\nu_\omega.
\end{aligned} \tag{13.1}$$

An analogous calculation gives the converse. ☐

Lemma 80. *Fix $t > 1$. If $\tilde\nu = \big(\nu_\omega\big)_{\omega\in\Omega}$ is a random conformal measure for the random conformal system*

$$\tilde F_\omega^n : \mathbb{C} \longrightarrow \mathbb{C},\ \omega\in\Omega,\ n\ge 0,$$

with a measurable function $\lambda:\Omega\longrightarrow(0,+\infty)$ and the Riemannian metric ρ given by formula (2) of Proposition 78, then the random measure

$$\hat\nu := \big(\tilde\nu_\omega\circ H_{\theta^{-1}\omega}\big)_{\omega\in\Omega}$$

is a random conformal measure for the random conformal system

$$f_\omega^n : \mathbb{C} \longrightarrow \mathbb{C},\ \omega\in\Omega,\ n\ge 0,$$

with the same measurable function $\lambda:\Omega\longrightarrow(0,+\infty)$ and the same Riemannian metric ρ. The converse is also true.

Proof. First note that if $s\in\mathbb{C}^*$ and $H_s:\mathbb{C}\longrightarrow\mathbb{C}$ is the map given by the formula

$$H_s(z) = s^{-1}z,$$

then

$$|(H_s)'(z)|_\rho = |H_s(z)|^{-1}|\cdot|(H_s)'(z)|\cdot|z| = \frac{|s|}{|z|}\cdot\frac{1}{|s|}\cdot|z|. = 1.$$

Using this formula, the definition of the random measure $\hat{\nu}$, and formula (12.6) of Proposition 77, we get for every $\omega \in \Omega$, every integer $n \geq 1$, and every Borel set $A \subset \mathbb{C}$ such that the restricted map $f^n_{\theta\omega}|_A$ is 1–to–1, that

$$\begin{aligned}\hat{\nu}_{\theta^{n+1}\omega}\big(f^n_{\theta^n\omega}(A)\big) &= \tilde{\nu}_{\theta^n\omega}\big(H_{\theta^n\omega}\big(f^n_{\theta^n\omega}(A)\big)\big) = \tilde{\nu}_{\theta^n\omega}\big(\tilde{F}^n_\omega(H_\omega(A)\big) \\ &= \lambda^n_\omega \int_{H_\omega(A)} \big|\big(\tilde{F}^n_\omega\big)'\big|^t_\rho\big|\, d\tilde{\nu}_\omega = \lambda^n_\omega \int_A \big|\big(\tilde{F}^n_\omega\big)'\big|^t_\rho \circ H_\omega \, d\hat{\nu}_{\theta\omega} \\ &= \lambda^n_\omega \int_A \big|\big(f^n_{\theta\omega}\big)'\big|^t_\rho \, d\hat{\nu}_{\theta\omega}.\end{aligned} \tag{13.2}$$

□

Now we pass to transferring of invariant random measures. This is even easier. We shall prove the following two lemmas.

Lemma 81. *If $\mu = \big(\mu_\omega\big)_{\omega\in\Omega}$ is an invariant random measure for the random conformal system*

$$F^n_\omega : Q \longrightarrow Q,\ \omega \in \Omega,\ n \geq 0,$$

then the random measure

$$\tilde{\mu} := \big(\mu_\omega \circ \exp^{-1}\big)_{\omega\in\Omega}$$

is an invariant random measure for the random conformal system

$$\tilde{F}^n_\omega : \mathbb{C}^* \longrightarrow \mathbb{C}^*,\ \omega \in \Omega,\ n \geq 0,$$

The converse is also true.

Proof. The proof is an immediate consequence of Proposition 78 (5). □

Lemma 82. *If $\tilde{\mu} = \big(\tilde{\mu}_\omega\big)_{\omega\in\Omega}$ is an invariant random measure for the random conformal system*

$$\tilde{F}^n_\omega : \mathbb{C} \longrightarrow \mathbb{C},\ \omega \in \Omega,\ n \geq 0,$$

then the random measure

$$\hat{\mu} := \big(\tilde{\mu}_\omega \circ H_{\theta^{-1}\omega}\big)_{\omega\in\Omega}$$

is an invariant random measure for the random conformal system

$$f^n_\omega : \mathbb{C} \longrightarrow \mathbb{C},\ \omega \in \Omega,\ n \geq 0,$$

The converse is also true.

Proof. The proof is carried through by an explicate direct calculation based on formula (12.6) of Proposition 77.

$$\hat{\mu}_{\theta\omega} \circ f^{-n}_{\theta\omega} = \tilde{\mu}_\omega \circ H_\omega \circ f^{-n}_{\theta\omega} = \tilde{\mu}_{\theta^n\omega} \circ H_{\theta^n\omega} = \hat{\mu}_{\theta^{n+1}\omega}.$$

□

As a consequence of the lemmas and Theorem 38 along with Theorem 58, we get the following.

Theorem 83. *For every $t > 1$ there exists a random t–conformal measure $\hat{\nu}^{(t)}$, the one resulting from Theorem 8.4, Lemma 79 and Lemma 80, for the random conformal system*

$$f_\omega^n : \mathbb{C} \longrightarrow \mathbb{C},\ \omega \in \Omega,\ n \geq 0,$$

with with respect to the Riemannian metric ρ defined in item (2) of Proposition 78. This means that formula (13.2) *holds.*

Furthermore, there exists a Borel probability f–invariant measure $\hat{\mu} = \hat{\mu}^{(t)}$ absolutely continuous with respect to $\hat{\nu}^{(t)}$. It has the following further properties.

(a) *$\hat{\mu}^{(t)}$ is equivalent to $\hat{\nu}^{(t)}$,*
(b) *$\hat{\mu}^{(t)}$ is ergodic.*
(c) *$\hat{\mu}^{(t)}$ is is the only Borel probability f–invariant measure absolutely continuous with respect to $\hat{\nu}^{(t)}$.*

Turning to geometry, we now define random radial (conical) Julia sets on the complex plane $\mathbb{C}$ for the random conformal system

$$f_\omega^n : \mathbb{C} \longrightarrow \mathbb{C},\ \omega \in \Omega,\ n \geq 0.$$

These sets are defined analogously as the radial random sets for the random conformal system F_ω^n:

$$J_r(f)(\omega) := \big\{z \in \mathbb{C} : \underline{\lim}_{N\to\infty} \rho(N_\omega(z, N)) = 1\big\}, \tag{13.3}$$

where $N_\omega(z, N)$ is the set of all integers $n \geq 0$ such that there exists a (unique) holomorphic inverse branch

$$f_{\omega,z}^{-n} : B(f_\omega^n(z), 2/N) \longrightarrow \mathbb{C}$$

of $f_\omega^n : \mathbb{C} \to \mathbb{C}$ sending $f_\omega^n(z)$ to z and such that $|F_\omega^n(z)| \leq N$. The set $J_r(f)(\omega)$ is said to be the set of radial (or conical) points of f at ω. Based on the propositions proved in this section, it is easy to prove that for every $\omega \in \Omega$,

$$J_r(f)(\omega) = H_{\theta^{-1}\omega}^{-1} \circ \exp\big(J_r(\theta^{-1}\omega)\big). \tag{13.4}$$

Having this and all the propositions proved in this section, as an immediate consequence of Theorem 61, Theorem 73, Theorem 74, and Corollary 75, we get the following.

Theorem 84. *For the random conformal system*

$$f_\omega^n : \mathbb{C} \longrightarrow \mathbb{C},\ \omega \in \Omega,\ n \geq 0.$$

we have that

(1) $\mathrm{HD}\big(J_r(f)(\omega)\big) = h$ *for m–a.e.$\omega \in \Omega$, where $h \in (1, 2)$ is the number coming from item (4) of Theorem 61. In particular:*
(2) The 2–dimensional Lebesgue measure of m–a.e. $\omega \in \Omega$ set $J_r(\omega)$ is equal to zero.
(3) For m–almost every $\omega \in \Omega$ there exists a subset $\mathbb{C}_\omega \subset \mathbb{C}$ with full Lebesgue measure such that for all $z \in \mathbb{C}_\omega$ the following holds.

$$\forall \delta > 0\ \exists n_z(\delta) \in \mathbb{N}\ \forall n \geq n_z(\delta)\ \exists k = k_n(z) \geq 0$$
$$|f_\omega^n(z) - f_{\theta^{n-k}\omega}^k(0)| < \delta \quad or \quad |f_\omega^n(z)| \geq 1/\delta.$$

In addition, $\limsup_{n\to\infty} k_n(z) = +\infty$. *In consequence,* *(4) The set of accumulation points of the sequence*

$$\left(f_\omega^n(z)\right)_{n=0}^{\infty}$$

is contained in $[0, +\infty] \cup \{-\infty\}$ *and contains* $+\infty$.

References

[1] Ludwig Arnold. *Random dynamical systems.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. 1, 4

[2] I. N. Baker. Limit functions and sets of non-normality in iteration theory. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 467:11, 1970. 1

[3] I. N. Baker. The domains of normality of an entire function. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 1(2):277–283, 1975. 1

[4] I. N. Baker. Wandering domains in the iteration of entire functions. *Proc. London Math. Soc. (3)*, 49(3):563–576, 1984. 1

[5] Krzysztof Barański. Hausdorff dimension and measures on Julia sets of some meromorphic maps. *Fund. Math.*, 147(3):239–260, 1995. 1

[6] Krzysztof Barański, Núria Fagella, Xavier Jarque, and Bogusł awa Karpińska. Absorbing sets and Baker domains for holomorphic maps. *J. Lond. Math. Soc. (2)*, 92(1):144–162, 2015. 1

[7] Krzysztof Barański, Bogusława Karpińska, and Anna Zdunik. Hyperbolic dimension of Julia sets of meromorphic maps with logarithmic tracts. *Int. Math. Res. Not. IMRN*, (4):615–624, 2009. 1

[8] Krzysztof Barański, Bogusława Karpińska, and Anna Zdunik. Bowen's formula for meromorphic functions. *Ergodic Theory Dynam. Systems*, 32(4):1165–1189, 2012. 1

[9] Walter Bergweiler. Iteration of meromorphic functions. *Bull. Amer. Math. Soc. (N.S.)*, 29(2):151–188, 1993. 1

[10] John Conway. Functions of One Complex Variable. *Graduate Texts in Mathematics, Vol 11, v. 1, 2nd Edition*, Springer 1978.

[11] Hans Crauel. *Random probability measures on Polish spaces*, volume 11 of *Stochastics Monographs*. Taylor & Francis, London, 2002. 1, 4, 15, 4

[12] Robert L. Devaney and MichałKrych. Dynamics of exp(z). *Ergodic Theory Dynam. Systems*, 4(1):35–52, 1984. 1

[13] A. È. Erëmenko and M. Yu. Lyubich. Dynamical properties of some classes of entire functions. *Ann. Inst. Fourier (Grenoble)*, 42(4):989–1020, 1992. 1

[14] Pierre Fatou. Sur l'iteration des fonctions transcendentales entières. *Acta Math.*, 47:337–370, 1926. 1, 3

[15] Einar Hille. *Analytic function theory. Vol. 1.* Introduction to Higher Mathematics. Ginn and Company, Boston, 1959. 2.2

[16] Yuri Kifer. *Ergodic theory of random transformations*, volume 10 of *Progress in Probability and Statistics*. Birkhäuser Boston, Inc., Boston, MA, 1986. 1

[17] Yuri Kifer. *Random perturbations of dynamical systems*, volume 16 of *Progress in Probability and Statistics*. Birkhäuser Boston, Inc., Boston, MA, 1988. 1

[18] Yuri Kifer and Pei-Dong Liu. Random dynamics. In *Handbook of dynamical systems. Vol. 1B*, pages 379–499. Elsevier B. V., Amsterdam, 2006. 1

[19] Janina Kotus and Mariusz Urbański. Geometry and ergodic theory of non-recurrent elliptic functions. *J. Anal. Math.*, 93:35–102, 2004. 1

[20] M. Yu. Lyubich. The measurable dynamics of the exponential. *Sibirsk. Mat. Zh.*, 28(5):111–127, 1987. 1, 1

[21] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, volume 44 of *Cambridge Studies in Advanced Mathematics*. 10

[22] Volker Mayer, Bartł omiej Skorulski, and Mariusz Urbański. Regularity and irregularity of fiber dimensions of non-autonomous dynamical systems. *Ann. Acad. Sci. Fenn. Math.*, 38(2):489–514, 2013. 1, 4

[23] Volker Mayer and Mariusz Urbański. Geometric thermodynamic formalism and real analyticity for meromorphic functions of finite order. *Ergodic Theory Dynam. Systems*, 28(3):915–946, 2008. 1, 1, 1, 9

[24] Volker Mayer and Mariusz Urbański. Thermodynamical formalism and multifractal analysis for meromorphic functions of finite order. *Mem. Amer. Math. Soc.*, 203(954):vi+107, 2010. 1, 1, 1, 9

[25] Volker Mayer and Mariusz Urbański. Countable alphabet random subhifts of finite type with weakly positive transfer operator. *J. Stat. Phys.*, 160(5):1405–1431, 2015. 1

[26] V. Mayer and M. Urbański. Thermodynamic formalism and integral means spectrum of asymptotic tracts for transcendental entire functions. *ArXiv e-prints*, September 2017. 1, 4

[27] Volker Mayer and Mariusz Urbański. Random dynamics of transcendental functions. *J. Anal. Math.*, 134(1):201–235, 2018. 1, 4

[28] Curt McMullen. Area and Hausdorff dimension of Julia sets of entire functions. *Trans. Amer. Math. Soc.*, 300(1):329–342, 1987. 1

[29] Michał Misiurewicz. On iterates of e^z. *Ergodic Theory Dynamical Systems*, 1(1):103–106, 1981. 1, 3, 3

[30] Feliks Przytycki and Mariusz Urbański. *Conformal fractals: ergodic theory methods*, volume 371 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2010. 10

[31] Mary Rees. The exponential map is not recurrent. *Math. Z.*, 191(4):593–598, 1986. 1, 1

[32] Lasse Rempe. Hyperbolic dimension and radial julia sets of transcendental functions. *Proceedings of the American Mathematical Society*, 137(4):14111420, 2009. 1, 9

[33] Lasse Rempe-Gillen. Hyperbolic entire functions with full hyperbolic dimension and approximation by Eremenko-Lyubich functions. *Proc. Lond. Math. Soc. (3)*, 108(5):1193–1225, 2014. 1

[34] Gwyneth M. Stallard. The Hausdorff dimension of Julia sets of entire functions. *Ergodic Theory Dynam. Systems*, 11(4):769–777, 1991. 1

[35] Gwyneth M. Stallard. The Hausdorff dimension of Julia sets of meromorphic functions. *J. London Math. Soc. (2)*, 49(2):281–295, 1994.

[36] Gwyneth M. Stallard. The Hausdorff dimension of Julia sets of entire functions. II. *Math. Proc. Cambridge Philos. Soc.*, 119(3):513–536, 1996.

[37] Gwyneth M. Stallard. The Hausdorff dimension of Julia sets of entire functions. III. *Math. Proc. Cambridge Philos. Soc.*, 122(2):223–244, 1997.

[38] Gwyneth M. Stallard. The Hausdorff dimension of Julia sets of hyperbolic meromorphic functions. *Math. Proc. Cambridge Philos. Soc.*, 127(2):271–288, 1999. 1

[39] Mariusz Urbański and Anna Zdunik. The finer geometry and dynamics of the hyperbolic exponential family. *Michigan Math. J.*, 51(2):227–250, 2003. 1, 1, 9

[40] Mariusz Urbański and Anna Zdunik. Real analyticity of Hausdorff dimension of finer Julia sets of exponential family. *Ergodic Theory Dynam. Systems*, 24(1):279–315, 2004. 1, 1, 9

[41] Mariusz Urbański and Anna Zdunik. Geometry and ergodic theory of non-hyperbolic exponential maps. *Trans. Amer. Math. Soc.*, 359(8):3973–3997, 2007. 1, 1, 1, 1, 17, 9

Mariusz Urbański, University of North Texas, Department of Mathematics, 1155 Union Circle #311430, Denton, TX 76203-5017, USA

E-mail address: urbanski@unt.edu

Anna Zdunik, Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warszawa, Poland

E-mail address: A.Zdunik@mimuw.edu.pl