Asymptotic Counting in Conformal Dynamical Systems

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Abstract

In this monograph we consider the general setting of conformal graph directed Markov systems modeled by countable state symbolic subshifts of finite type. We deal with two classes of such systems: attracting and parabolic. The latter being treated by means of the former.

We prove fairly complete asymptotic counting results for multipliers and diameters associated with preimages or periodic orbits ordered by a natural geometric weighting. We also prove the corresponding Central Limit Theorems describing the further features of the distribution of their weights.

These results have direct applications to a wide variety of examples, including the case of Apollonian Circle Packings, Apollonian Triangle, expanding and parabolic rational functions, Farey maps, continued fractions, Mannenville-Pomeau maps, Schottky groups, Fuchsian groups, and many more. This gives a unified approach which both recovers known results and proves new results.

Our new approach is founded on spectral properties of complexified Ruelle–Perron–Frobenius operators and Tauberian theorems as used in classical problems of prime number theory.

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CHAPTER 1

Introduction

1.1. Short General Introduction

We begin with a simple problem formulated for general iterated function systems acting on a compact metric space X. Let

$$\big(\varphi_e : X \to X\big)_{e\in E},$$

be a countable, either finite or infinite, family of $C^{1+\alpha}$ contracting maps on a metric space. We can associate to a point $\xi \in X$ the images

$$\varphi_\omega(\xi) := \varphi_{\omega_1} \circ \cdots \circ \varphi_{\omega_n}(\xi)$$

where $\omega = (\omega_1, \cdots, \omega_n) \in E^n$, $n \geq 1$, and then we associate two natural weights

$$\lambda_\xi(\omega) := -\log|(\varphi_\omega)'(\xi)|$$

and

$$\Delta_\xi(\omega) := -\log \operatorname{diam}(\varphi_\omega(X)).$$

Since there is no natural way to order and count these images in terms of their combinatorial weight (i.e., the length n of $\omega = (\omega_1, \cdots, \omega_n)$), we use instead the two weights introduced above: namely, $\lambda_\xi(\omega)$ and $\Delta_\xi(\omega)$.

Under mild natural hypotheses, we show that there exist two constants $C_1, C_2 > 0$ (and we provide explicit dynamical expressions for them) and $\delta \in (0, +\infty)$ such that

$$\lim_{T\to+\infty} \frac{\#\{\omega : \lambda_\xi(\omega) \leq T\}}{e^{\delta T}} = C_1$$

and

$$\lim_{T\to+\infty} \frac{\#\{\omega : \Delta_\xi(\omega) \leq T\}}{e^{\delta T}} = C_2.$$

These are perhaps the highlights of our results which are simplest to present; but we actually prove more. For example, we also provide the corresponding asymptotic results when, in addition, one requires that the points $\varphi_\omega(\xi)$ are to fall into a prescribed ball B in X. We also count the corresponding multipliers if the points $\varphi_\omega(\xi)$ are replaced by periodic points of the system, i.e. by unique fixed points x_ω of the maps φ_ω, which exists because all the maps φ_ω are (with our current hypotheses) contractions of the space X into itself. To this end, we can denote

$$\lambda_p(\omega) = -\log|(\varphi_\omega)'(x_\omega)|$$

and then there exists $C_3 > 0$ such that

$$\lim_{T\to+\infty} \frac{\#\{\omega : \lambda_p(\omega) \leq T\}}{e^{\delta T}} = C_3.$$

A fuller description of our results is provided below in further subsections of this introduction and in complete detail in appropriate technical sections of the manuscript.

There are natural and instructive parallels of our work and the classical approach to the prime number theorem, as well as with known results on the Patterson-Sullivan orbit counting technology and the asymptotics of Apollonian circles. There are also natural counting problems in both expanding and parabolic rational functions, complex continued fractions, Farey maps, Manneville-Pomeau maps, Schottky groups, Fuchsian groups, including Hecke groups, and more examples. We apply our general results to all of them, thus giving a unified approach which yields both new results and a new approach to established results.

All of these are based on our current results for conformal graph directed Markov systems over a countable alphabet. These, i. e. such directed Markov systems, form the core of the manuscript, and are objects of ultimate results of Part 1 and Part 2. Their more detailed informal description is presented below in Section 1.2, entitled Asymptotic Counting Results; Section 1.3 is devoted to the, above mentioned, classes of examples.

Our counting results (on the symbolic level) are close in spirit to those of Steve Lalley from [**37**]. These would directly apply to our counting on the symbolic level if the graph directed Markov systems we considered had finite alphabets. However, we need to deal with those systems with a countable alphabet and we obtain our counting results via the study of spectral properties of complexified Ruelle–Perron–Frobenius operators, as used by William Parry and the first–named author, rather than the renewal theory approach of Lalley. It is worth mentioning that our results on the symbolic level could have been formulated and proved with no real additional difficulties in terms of ergodic sums of summable Hölder continuous potentials rather than merely the functions $\lambda_\xi(\omega)$ from the next subsection.

We would also like to add that our work was partly inspired by counting results of Kontorovich and Oh for Apollonian packings from [**36**] (see also [**56**]–[**58**]), which in our monograph are recovered and ultimately follow from our more general results for conformal graph directed Markov systems. Nevertheless, the approach and the level of generality of our approach is entirely different than that of Kontorovich and Oh. We have recently received an interesting preprint [**31**] of Byron Heersink where he studies the counting problems for the Farey map, Gauss map, and closed geodesics on the modular surface. We would also like to note that a part of the classical work of the first named author and William Parry (including [**68**], [**69**], [**62**], [**61**]), the method of the complex Perron–Frobenius operator to approach various counting problems in geometry and dynamics, has been used by several authors including [**50**], [**52**], [**72**], [**3**].

We now discuss our results below in more detail.

1.2. Asymptotic Counting Results

In Sections 2.2, and 3.1, we will recall from [**47**] the respective concepts of attracting and parabolic countable alphabet conformal graph directed Markov systems. This symbolic viewpoint is a convenient framework for keeping track of the quantities we want to count. We begin by recalling enough notation to allow us to formulate versions of our main results, beginning with the family of contractions we will study, referring the read to the appropriate later sections for more details.

In contrast to the simple family of contractions described in Subsection 1.1, we will need to consider a more general "Markovian structure" for our family of contractions, so as to accommodate the examples we wish to apply them to (see Subsection 1.3). A *directed multigraph* consists of a finite set V of vertices, a countable (either finite or infinite) set E of directed edges, two functions

$$i, t : E \longrightarrow V,$$

and an *incidence matrix* $A : E \times E \to \{0, 1\}$ for (V, E, i, t) such that

$$A_{ab} = 1 \quad \text{implies} \quad t(b) = i(a).$$

Now suppose that in addition, we have a collection of nonempty compact metric spaces $\{X_v\}_{v\in V}$ and a number $\kappa \in (0, 1)$, such that for every $e \in E$, we have a one-to-one contraction $\varphi_e : X_{t(e)} \to X_{i(e)}$ with Lipschitz constant (bounded above by) κ. Then the collection

$$\mathcal{S} = \{\varphi_e : X_{t(e)} \to X_{i(e)}\}_{e\in E}$$

is called an *attracting graph directed Markov system* (or GDMS). The GDMS is called an *attracting iterated function system* (or IFS) if the set V of vertices is a singleton and all the entries of the incidence matrix A are 1s. We will explain these definitions in greater detail in Section 2.2.

We denote by $E_A^\infty \subseteq E^{\mathbb{N}}$ the subshift of finite type associated to the alphabet E and the matrix A, and we denote by E_A^* the collection of finite words admissible by the matrix A. We say that the incidence matrix A is *finitely irreducible* if there exists a finite set $\Omega \subset E_A^*$ such that for all $a, b \in E$ there exists a word $\omega \in \Omega$ such that the concatenation $a\omega b$ is in E_A^*. We then also call the system $\mathcal{S}$ irreducible. We extend the functions $i, t : E \to V$ in a natural way to E_A^* as follows:

$$t(\omega) := t\big(\omega_{|\omega|}\big) \quad \text{and} \quad i(\omega) := i(\omega_1).$$

For every word $\omega \in E_A^*$, say $\omega \in E_A^n$, $n \geq 0$, let us denote

$$\varphi_\omega := \varphi_{\omega_1} \circ \cdots \circ \varphi_{\omega_n} : X_{t(\omega)} \to X_{i(\omega)}.$$

This symbolic setting is particularly useful for our analysis (in particular, the introduction of a transfer operator).

Now, we define the natural coding map

$$\pi_{\mathcal{S}} = \pi : E_A^\infty \longrightarrow X := \coprod_{v \in V} X_v,$$

by

$$\{\pi_{\mathcal{S}}(\omega)\} := \bigcap_{n \in \mathbb{N}} \varphi_{\omega|_n}\left(X_{t(\omega_n)}\right)$$

where $\omega \in E_A^\infty$ and $\coprod_{v \in V} X_v$ is the disjoint union of the compact topological spaces X_v, $v \in V$. The set

$$J = J_{\mathcal{S}} = \pi_{\mathcal{S}}(E_A^\infty)$$

is called the *limit set* of the GDMS $\mathcal{S}$. We will describe these objects in greater detail in Section 2.2.

To be able to study geometrical features of $\mathcal{S}$ we need to impose some additional hypotheses. We call a GDMS $\mathcal{S}$ *conformal* if for some $d \in \mathbb{N}$, the following conditions are satisfied.

(a) For every vertex $v \in V$, X_v is a compact connected subset of $\mathbb{R}^d$, and $X_v = \overline{\mathrm{Int}(X_v)}$.
(b) (Open Set Condition) For all $a, b \in E$ such that $a \neq b$,
$$\varphi_a(\mathrm{Int}(X_{t(a)})) \cap \varphi_b(\mathrm{Int}(X_{t(b)})) = \emptyset.$$
(c) (Conformality) There exists a family of open connected sets $W_v \subset X_v$, $v \in V$, such that for every $e \in E$, the map φ_e extends to a C^1 conformal diffeomorphism from $W_{t(e)}$ into $W_{i(e)}$ with Lipschitz constant $\leq \kappa$.
(d) (Bounded Distortion Property (BDP)) There are two constants $L \geq 1$ and $\alpha > 0$ such that for every $e \in E$ and every pair of points $x, y \in X_{t(e)}$,
$$\left| \frac{|\varphi_e'(y)|}{|\varphi_e'(x)|} - 1 \right| \leq L\|y - x\|^\alpha,$$
where $|\varphi_\omega'(x)|$ denotes the scaling factor of the derivative $\varphi_\omega'(x) : \mathbb{R}^d \to \mathbb{R}^d$ which is a similarity map.

From now on through this introduction and, actually, through the entire manuscript we assume that the system $\mathcal{S}$ is finitely irreducible, i.e. that the incidence matrix A is finitely irreducible. For our counting results we need one natural hypothesis more. We call the system $\mathcal{S}$ *strongly regular* if there exists $s \in [0, +\infty)$ such that

$$0 < \mathrm{P}(s) < +\infty$$

where for $s \geq 0$, we let

$$\mathrm{P}(s) := \lim_{n \to +\infty} \frac{1}{n} \log \left(\sum_{|\omega| = n} \|\varphi_\omega'\|_\infty^s \right),$$

and $\|\varphi'\|_\infty$ denotes the supremum norm of the derivative of a conformal map φ over its domain. For example, every non trivial finite GDMS is strongly regular. In particular, every finite IFS with the alphabet E having at least two elements is strongly regular.

Finally, we want to introduce a standard form of non-degeneracy condition on the derivatives. First,

$$E_p^* := \{\omega \in E_A^* : A_{\omega_{|\omega|}\omega_1} = 1\}.$$

Further, for all $t, a \in \mathbb{R}$ we denote by $G_a(t)$ multiplicative subgroup of positive reals $(0, +\infty)$ that is generated by the set

$$\left\{e^{-a|\omega|}|\varphi_\omega'(x_\omega)|^t : \omega \in E_p^*\right\} \subseteq (0, +\infty)$$

where x_ω is the only fixed point for $\varphi_\omega : X_{i(\omega_1)} \to X_{i(\omega_1)}$. Let $\mathcal{S} = \{\varphi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS then call a parameter $t \in \mathbb{R}$ *strongly $\mathcal{S}$–generic* if there exists **no** $a \in \mathbb{R}$ such that $G_a(t)$ is generated by $e^{2\pi k}$ for some $k \in \mathbb{N}_0$. We call the system $\mathcal{S}$ *D–generic* if each parameter $t \in \mathbb{R} \setminus \{0\}$ is $\mathcal{S}$–generic.

In order to formulate an equidistribution result we need to introduce an appropriate reference measure. There is (see [**47**], comp. [**42**]) a natural ambient Borel probability measure $m_{\delta_\mathcal{S}}$ on the shift space E_A^∞ occasionally called the *the symbolic conformal measure*, and which satisfies the following Gibbs property: For every $\omega \in E_A^*$, we have that

$$C_{\delta_\mathcal{S}}^{-1}\|\varphi_\omega'\|_\infty^{\delta_\mathcal{S}} \le m_{\delta_\mathcal{S}}([\omega]) \le C_{\delta_\mathcal{S}}\|\varphi_\omega'\|_\infty^{\delta_\mathcal{S}} \tag{1.1}$$

where $\delta_\mathcal{S}$ is the Hausdorff Dimension of the limit set $J_\mathcal{S}$, $C_{\delta_\mathcal{S}} \in (0, +\infty)$ is a constant independent of ω and we denote

$$[\omega] := \left\{\tau \in E_A^\infty : \tau|_{|\omega|} = \omega\right\}$$

is the cylinder generated by the word ω. In here $|\omega|$ is the length of the finite word ω and $\tau|_n$ is the word formed by the first n terms of τ.

There is also (see again[**47**], comp. [**42**]) $\mu_{\delta_\mathcal{S}}$, a unique Borel probability shift invariant measure on E_A^∞ absolutely continuous with respect to $m_{\delta_\mathcal{S}}$. In fact $\mu_{\delta_\mathcal{S}}$ and $m_{\delta_\mathcal{S}}$ are equivalent and the corresponding Radon–Nikodym derivatives are bounded.

$\widetilde{m}_{\delta_\mathcal{S}} := m_{\delta_\mathcal{S}} \circ \pi_\mathcal{S}^{-1}$, the image of the measure $m_{\delta_\mathcal{S}}$ under the projection $\pi_\mathcal{S}$, is then supported on $J_\mathcal{S}$ and is called the ($\delta_\mathcal{S}$–) *conformal measure* on $J_\mathcal{S}$. It is characterized (see [**47**], comp. [**42**]) by the following two properties. Firstly,

$$\widetilde{m}_{\delta_\mathcal{S}}(\varphi_\omega(F)) = \int_F |\varphi_\omega'|^{\delta_\mathcal{S}}\, d\widetilde{m}_{\delta_\mathcal{S}}$$

for every $\omega \in E_A$ and every Borel set $F \subseteq X_{t(\omega)}$, and secondly,

$$\widetilde{m}_{\delta_\mathcal{S}}\big(\varphi_\alpha(X_{t(\alpha)}) \cap \varphi_\beta(X_{t(\beta)})\big) = 0$$

whenever α, β are incomparable elements of E_A^*. We also denote

$$\widetilde{\mu}_{\delta_\mathcal{S}} := \widetilde{\mu}_{\delta_\mathcal{S}} \circ \pi_\mathcal{S}^{-1},$$

the image of the invariant measure $\mu_{\delta_\mathcal{S}}$ under the projection $\pi_\mathcal{S}$. We will return to these definitions again in Section 2.2 and Section 3.1.

An equally important role for us is played by parabolic conformal GDMSs. These are somewhat the same as finite alphabet attracting systems with one exemption that some moduli of derivatives at some fixed points can be equal to 1. This apparent small change in definition yields however quite transparently visible differences in dynamical and geometric properties. This can be readily seen from our exposition in Section 3.1, particularly in what concerns invariant measures. Furthermore, some counting results for parabolic systems are strikingly different than those for attracting ones as the content of Theorem 1.2.2 readily shows.

We are now in a position to formulate our first counting and equidistribution results. Let $\pi_{\mathcal{S}}(\rho) \in J \subset X$ be a reference point coded by an infinite sequence $\rho \in E_A^\infty$. Fix any non-empty Borel set $B \subset X$. Then for all $T > 0$ we define:

$$N_\rho(B,T) := \#\left\{\omega \in E_\rho^* : \varphi_\omega(\pi_{\mathcal{S}}(\rho)) \in B \text{ and } \lambda_{\pi_{\mathcal{S}}(\rho)}(\omega) \le T\right\}$$
$$\text{and}$$
$$N_p(B,T) := \#\left\{\omega \in E_p^* : x_\omega \in B \text{ and } \lambda_p(\omega) \le T\right\},$$

where

$$E_\rho^* := \{\omega \in E_A^* : \omega\rho \in E_A^\infty\},$$

and, we recall,

$$E_p^* = \{\omega \in E_A^* : A_{\omega_{|\omega|}\omega_1} = 1\},$$

are finite words of symbols, i.e. we count the number of words $\omega \in E_i^*$ for which the weight $\lambda_i(\omega)$ does not exceed T and, additionally, the image $\varphi_\omega(\pi_{\mathcal{S}}(\rho))$ is in B if $i = \rho$, or the fixed point x_ω of φ_ω is in B if $i = p$.

The following result is based on Theorem 2.4.9 for attracting conformal GDMSs and Theorem 3.3.2 for parabolic systems.

Theorem 1.2.1 (Asymptotic Equidistribution Formula for Multipliers). *Suppose that $\mathcal{S}$ is either a strongly regular finitely irreducible D-generic attracting conformal GDMS or an irreducible parabolic conformal GDMS. Let $\delta = \delta_{\mathcal{S}} = \mathrm{HD}(J_{\mathcal{S}})$ be the Hausdorff dimension of the associated limit set $J_{\mathcal{S}}$.*

Fix $\rho \in E_A^\infty$. Let $B \subset X$ be a Borel set such that $m_{\delta_{\mathcal{S}}}(\partial B) = 0$, then

$$\lim_{T\to+\infty} \frac{N_\rho(B,T)}{e^{\delta T}} = \frac{\psi_\delta(\rho)}{\delta \chi_{\mu_\delta}} \widetilde{m}_\delta(B)$$

and

$$\lim_{T\to+\infty} \frac{N_p(B,T)}{e^{\delta T}} = \frac{1}{\delta \chi_{\mu_\delta}} \widetilde{\mu}_\delta(B),$$

where $\psi_\delta = d\mu_\delta/dm_d$ and χ_{μ_δ} is the Lyapunov exponent of the measure μ_δ.

This result, and essentially all counting results which follow, can be rephrased in terms of weak–star convergence of appropriately defined and normalized counting measures.

We will formulate more counting results in the present subsection and in the next one we will discuss representative examples of conformal dynamical systems where the appropriate counting results will be obtain by associating to them either attracting or parabolic GDMSs and applying the above theorem.

Our proof of Theorem 1.2.1 for attracting systems is based on following five steps:

(1) Describing the spectrum of an associated complexified Ruelle-Perron-Frobenius (RPF) operator; done at the symbolic level, culminating in the results of Section 2.3,
(2) Using this information on the RPF operator in order to find meromorphic extensions of associated complex η functions, i.e., Poincaré functions (or series), see Section 2.5,
(3) Using the information on the domain of the Poincaré series to deduce the asymptotic formulae (Theorem 2.4.8) for $\lambda_\omega(\xi)$ on the mixture of the symbolic level (the words $\omega\rho$ are required to belong to a symbolic cylinder $[\tau]$ rather than $\varphi_\omega(\pi_{\mathcal{S}}(\rho))$ or x_ω to belong to B) and GDMS level, by classical methods from prime number theory based on Tauberian theorems.
(4) Having (3) derive the asymptotic formulae for $-\log|\varphi_\omega'(x_\omega)|$; i.e. for periodic points x_ω of φ_ω by means of sufficiently fine approximations.
(5) Deducing the asymptotic formulae for the Borel sets $B \subseteq X$ (Theorem 2.4.9) from those at the symbolic level (Theorem 2.4.8).

We can leverage our results for attracting systems to prove the corresponding results for the more delicate case of parabolic systems. This is done by associating with a parabolic system (by a form of inducing) a countable alphabet attracting GDMSs and expressing the corresponding Poincaré series for parabolic systems as infinite sums of the Poincaré series for those associated attracting systems. The rewards for this extra work is that our results then apply to a wide class of interesting examples (see next subsection).

It is interesting to note that whereas the D-generic hypothesis of Theorem 1.2.1 needed for attracting systems is very mild, in the case of parabolic systems, or more precisely the attracting systems naturally associated to them, they are automatically D–generic (see Theorem 3.1.7), so no genericity hypothesis is needed for them at all.

We would like to stress again that parabolic systems are of equal importance to the attracting systems. Indeed, many of the applications, such as to Farey maps or Apollonian packings for example, are based on parabolic GDMSs. The parabolic systems frequently generate more complex and intriguing counting phenomena, particularly in regard to counting diameters, which we will now address.

We now describe the corresponding results for asymptotic counting of diameters. These are more geometrical and more complex than those for multipliers, and counting multipliers is intrinsically more of a "dynamical process".

We bring up the appropriate counting definitions related to diameters of sets. We fix $\rho \in E_A^\infty$, put $\xi = \pi_{\mathcal{S}}(\rho)$ and fix a set $Y \subseteq X_{i(\rho)}$. We denote

$$\Delta(\omega) = \Delta_Y(\omega) := -\log \operatorname{diam}(\varphi_\omega(Y)), \quad \omega \in E_\rho^*,$$

with the natural convention that for $\omega = \varepsilon$, being the empty (neutral) word:

$$\Delta_Y(\varepsilon) = -\log \operatorname{diam}(Y),$$

and further, for any $T > 0$,

$$\mathcal{D}_Y^\rho(B,T) := \{\omega \in E_\rho^* : \Delta_Y(\omega) \le T \text{ and } \varphi_\omega(\xi) \in B\},$$
$$D_Y^\rho(B,T) := \#\mathcal{D}_Y^\rho(B,T).$$

Also

$$\mathcal{E}_Y^\rho(B,T) := \{\omega \in E_\rho^* : \Delta_Y(\omega) \le T \ \text{ and } \ \varphi_\omega(Y) \cap B \ne \emptyset\}$$

and

$$E_Y^\rho(B,T) := \#\mathcal{E}_Y^\rho(B,T).$$

We refer the reader to the appropriate sections for further relevant definitions and concepts, and to the next subsection for, already mentioned, examples of conformal dynamical systems. However, for the present, we note that Ω denotes the set of all parabolic elements of E, that for every $e \in E$,

$$\Omega_e := \{a \in \Omega : A_{ae} = 1\}$$

and that

$$\Omega_\infty = \Omega_\infty(\mathcal{S}) := \left\{a \in \Omega : \frac{2p_a}{p_a+1} \ge \delta_{\mathcal{S}}\right\}.$$

The following theorem comprises Theorem 2.7.1, Theorem 2.7.4, Remark 2.7.5, Theorem 3.4.1, Theorem 3.4.2, and Remark 3.4.3.

THEOREM 1.2.2 (Asymptotic Equidistribution Formula for Diameters). *Suppose that $\mathcal{S}$ is either a strongly regular finitely irreducible D-generic attracting conformal GDMS or an irreducible parabolic conformal GDMS.*

Denote by δ the Hausdorff dimension of the limit set $J_{\mathcal{S}}$. Fix $\rho \in E_A^\infty$ and then a set $Y \subseteq X_{i(\rho)}$ having at least two elements. If $B \subset X$ is a Borel set such that $\widetilde{m}_\delta(\partial B) = 0$ then,

$$\lim_{T\to+\infty} \frac{D_Y^\rho(B,T)}{e^{\delta T}} = C_{\rho_1}(Y)\widetilde{m}_\delta(B)$$

and

$$\lim_{T\to+\infty}\frac{E_Y^{\rho}(B,T)}{e^{\delta T}}=C_{\rho_1}(Y)\widetilde{m}_\delta(B),$$

where $C_{\rho_1}(Y)\in(0,+\infty]$ is a constant depending only on the system $\mathcal{S}$, the letter ρ_1 and the set Y.

In addition $C_{\rho_1}(Y)$ is finite if and only if either

(1) $\overline{Y}\cap\Omega_\infty=\emptyset$ *or*
(2) $\delta>\max\left\{p(a):a\in\Omega_{\rho_1}\ \text{ and }\ x_a\in\overline{Y}\right\}$.

In particular $C_{\rho_1}(Y)$ is finite if the system $\mathcal{S}$ is attracting.

The proofs of the results in Theorem 1.2.2 for diameters are based on those for multipliers. The subtlety in the attracting case is that the basic bounded distortion property alone does not suffice to pass from the case of multipliers to the case of diameters; one needs additional approximating steps. For parabolic systems, even the basic bounded distortion property is weaker and more involved and a careful analysis of parabolic behavior is needed.

It is worth emphasizing once again the importance of parabolic systems for many applications and classes of examples, including that of Apollonian packings. This is even more transparent in the case of diameters than multipliers, since the diameters often appear more frequently in the geometric setting.

1.3. Examples

Now we would like to describe some classes of conformal dynamical systems to which we can apply Theorem 1.2.1 and Theorem 1.2.2. Often applying these results requires some non-trivial preparation.

Our first class of examples is formed by conformal expanding repellers, see Definition 5.1.1. The appropriate consequences of Theorem 1.2.1 and Theorem 1.2.2 are stated as Theorem 5.1.8. The primary examples of non-linear conformal expanding repellers are formed by expanding rational functions of the Riemann sphere $\widehat{\mathbb{C}}$. The consequences of Theorem 1.2.1 and Theorem 1.2.2 in this context, are given by Theorem 5.1.22.

Perhaps the the most obvious example related to attracting GDMSs are the Gauss map

$$G(x)=\frac{1}{x}-\left[\frac{1}{x}\right],$$

and the corresponding Gauss IFS $\mathcal{G}$ consisting of the maps

$$[0,1]\ni x\longmapsto g_n(x):=\frac{1}{x+n},\quad n\in\mathbb{N}.$$

Theorem 5.1.15 summarizes the consequences of Theorem 1.2.1 and Theorem 1.2.2 stated for the Gauss map G itself.

Now let describe some well known parabolic GDMSs to which our results apply. We start with 1-dimensional systems. Our primary classes of such systems, defined and analyzed in Subsection 5.2, are illustrated by following.

a) Manneville–Pomeau maps $f_\alpha:[0,1]\to[0,1]$, where $\alpha>0$ is a fixed number, defined by

$$f_\alpha(x)=x+x^{1+\alpha}\ (\text{mod } 1),$$

and the Farey map $f:[0,1]\to[0,1]$ defined by

$$f(x)=\begin{cases}\frac{x}{1-x} & \text{if } 0\le x\le\frac{1}{2}\\ \frac{1-x}{x} & \text{if } \frac{1-x}{x}\le x\le 1.\end{cases}$$

The appropriate asymptotic counting results, stemming from Theorem 1.2.1 and Theorem 1.2.2, are provided by Theorem 5.2.1 and Theorem 5.2.2.

b) A large class of conformal parabolic systems is provided by parabolic rational functions of the Riemann sphere $\widehat{\mathbb{C}}$. These are those rational functions (see Subsection 5.2.2) that have no critical points in the Julia sets but do have rationally indifferent periodic points. The appropriate asymptotic counting results, consequences of Theorem 1.2.1 and Theorem 1.2.2, are stated as Corollary 5.2.10. Probably the best known example of a parabolic rational function is the polynomial

$$\widehat{\mathbb{C}} \ni z \longmapsto f_{1/4}(z) := z^2 + \frac{1}{4} \in \widehat{\mathbb{C}}.$$

It has only one parabolic point, namely $z = 1/2$. In fact this is a fixed point of $f_{1/4}$ and $f'_{1/4}(1/2) = 1$. Another explicit class of such functions is given by the maps of the form

$$\widehat{\mathbb{C}} \ni z \longmapsto 2 + 1/z + t$$

where $t \in \mathbb{R}$.

c) A separate large class of examples is provided by some classes of Kleinian groups, namely by finitely generated classical Schottky groups and essentially all finitely generated Fuchsian groups.

Convex co-compact (no tangencies) Schottky groups are described and analyzed in detail in Section 6.1 while general Schottky groups (allowing tangencies) are dealt with in Subsection 6.2. The appropriate asymptotic counting results, stemming from Theorem 1.2.1 and Theorem 1.2.2, are provided by Theorem 6.1.10 and Theorem 6.2.3.

As a particularly famous example, the counting problem of circles in a full Apollonian packing reduces to an appropriate counting problem for a finitely generated classical Schottky group with tangencies. The full presentation of asymptotic counting in this context, stemming from Theorem 1.2.1 and Theorem 1.2.2, is given by Corollary 6.2.9. We present below a more restricted form (see Theorem 6.2.13) involving only the counting of diameters; it recovers results from [**36**] (see also [**56**]–[**58**]), obtained by entirely different methods.

THEOREM 1.3.1. *Let C_1, C_2, C_3 be three mutually tangent circles in the Euclidean plane having mutually disjoint interiors. Let C_4 be the circle tangent to all the circles C_1, C_2, C_3 and having all of them in its interior; we then refer to the configuration (C_1, C_2, C_3, C_4) as bounded. Let $\mathcal{A}$ be the corresponding circle packing.*

Let $\delta = 1.30561\ldots$ be the Hausdorff dimension of the residual set of $\mathcal{A}$ and let m_δ be the Patterson-Sullivan measure of the corresponding parabolic Schottky group Γ.

If $N_{\mathcal{A}}(T)$ denotes the number of circles in $\mathcal{A}$ of diameter at least $1/T$ then the limit

$$\lim_{T\to+\infty} \frac{N_{\mathcal{A}}(T)}{e^{\delta T}}$$

exists, is positive, and finite. Moreover, there exists a constant $C \in (0, +\infty)$ such that if $N_{\mathcal{A}}(T; B)$ denotes the number of circles in $\mathcal{A}$ of diameter at least $1/T$ and lying in B, then

$$\lim_{T\to+\infty} \frac{N_{\mathcal{A}}(T; B)}{e^{\delta T}} = Cm_\delta(B)$$

for every open ball $B \subset \mathbb{C}$.

Closely related to $\mathcal{A}$ is the curvilinear triangle $\mathcal{T}$ (or Apollonian triangle) formed by the three edges joining the three tangency points of C_1, C_2, C_3 and lying on these circles. The collection

$$\mathcal{G} := \{C \in \mathcal{A} : C \subset \mathcal{T}\}$$

is called the Apollonian gasket generated by the circles C_1, C_2, C_3. As a consequence of Theorem 1.3.1, taking $B = \mathcal{T}$, we get the following (see Corollary 6.2.14); it overlaps with results from [**36**] (see also [**56**]–[**58**]), obtained with entirely different methods.

COROLLARY 1.3.2. *Let C_1, C_2, C_3 be three mutually tangent circles in the Euclidean plane having mutually disjoint interiors. Let C_4 be the circle tangent to all the circles C_1, C_2, C_3 and having all of them in its interior; we then refer to the configuration C_1, C_2, C_3, C_4 as bounded. Let $\mathcal{A}$ be the corresponding circle packing.*

If $\mathcal{T}$ is the curvilinear triangle formed by C_1, C_2 and C_3, then the limit

$$\lim_{T\to+\infty} \frac{N_{\mathcal{A}}(T;\mathcal{T})}{e^{\delta T}}$$

exists, is positive, and finite and counts the elements of $\mathcal{G}$. Moreover, there exists a constant $C \in (0,+\infty)$, in fact the one of Theorem 6.2.13, such that

$$\lim_{T\to+\infty} \frac{N_{\mathcal{A}}(T;B)}{e^{\delta T}} = Cm_\delta(B)$$

for every Borel set $B \subset \mathcal{T}$ with $m_\delta(\partial B) = 0$.

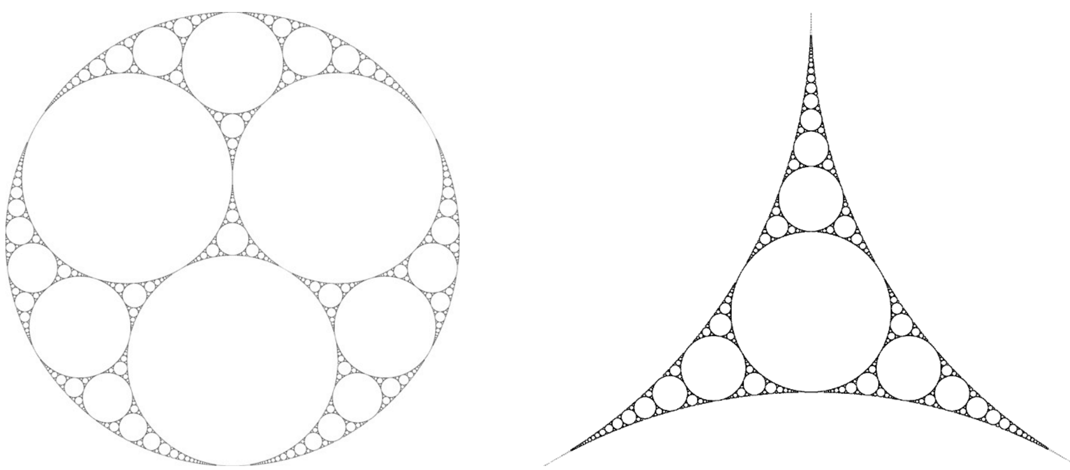

FIGURE 1. (i) The Standard Apollonian Packing; (ii) The Apollonian Gasket

In fact we can provide a more direct proof of Corollary 1.3.2, by appealing directly to the theory of parabolic conformal IFSs and avoiding the intermediate step of parabolic Schottky groups. Indeed, it follows immediately from Theorem 3.4.6.

In the context of limit sets, such as circle packings, there is scope for finding error terms in the above asymptotic formulae, see ex. [**39**] and [**60**]. It could be also done using the techniques worked out in our present manuscript. However, in the general setting of conformal graph directed Markov systems quite delicate technical hypotheses might well be required.

1.4. Statistical Results

A second aim of this monograph is to consider the statistical properties of the distribution of the different weights $\lambda_\rho(\omega)$ and $\mathrm{diam}(\varphi_\omega(X))$ corresponding to words ω with the same length n. This is a very specific mathematical problem, but is set against the backdrop of a vast literature dealing with different statistical properties of dynamical systems.

The classical Central Limit Theorem for Gibbs measures and uniformly hyperbolic dynamical systems (originally due to Sinai, Ratner, etc.) were inspired by the classical theorems for independent identically distributed random variables. In particular, in this context there are two particularly fruitful approaches: Firstly, the spectral approach based on perturbation theory for the maximal eigenvalue; and, secondly, the martingale method of Gordin [**22**]. An excellent account of Central Limit Theorems in this setting appears in [**14**]. Stronger results based on invariance principles were pioneered by Denker-Philipp [**15**].

In the broader setting of non-uniformly hyperbolic systems and natural invariant probability measures there have been a number of important contributions by different authors, including Young [**94**], [**95**], Sarig [**80**] Liverani-Saussol-Vaienti [**40**] and Gouëzel [**25**]. In the case of transformations with only a sigma finite natural invariant measures there are results on stable limit laws, see [**97**] and the references therein.

Since our aim is to develop Central Limit Theorems to deal specifically with the distribution of diameters of sets, not only typical points in a measurable sense (Theorem 1.4.2) and also in terms of counting averages (Theorem 1.4.4), we cannot apply the results above directly, but they provide a key blueprint for us to follow.

There are many other statistical properties that might be considered (e.g., Berry-Essen estimates, Shrinking targets, Large Deviations, Local Limit Theorems, Extremal theory, Multifractal analysis, etc.) but these are beyond the scope of this monograph.

In the context of attracting and parabolic GDMSs we have the following Central Limit Theorem, see Part 4. We refer the reader to the appropriate section for a detailed definitions of the hypothesis.

THEOREM 1.4.1. *If $\mathcal{S}$ is either a strongly regular finitely irreducible D–generic conformal GDMS or a finite alphabet irreducible parabolic GDMS with $\delta_{\mathcal{S}} > \frac{2p_{\mathcal{S}}}{p_{\mathcal{S}}+1}$* [1]*, then there exists $\sigma^2 > 0$ such that if $G \subset \mathbb{R}$ is a Lebesgue measurable set with $\mathrm{Leb}(\partial G) = 0$, then*

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{\omega \in E_A^\infty : \frac{-\log\left|\varphi'_{\omega|_n}(\pi_{\mathcal{S}}(\sigma^n(\omega)))\right| - \chi_{\mu_\delta} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{\omega \in E_A^\infty : \alpha \le \frac{-\log\left|\varphi'_{\omega|_n}(\pi_{\mathcal{S}}(\sigma^n(\omega)))\right| - \chi_{\mu_\delta} n}{\sqrt{n}} \le \beta\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

The following result is an alternative Central Limit Theorem considering instead the logarithms of the diameters of the images of reference sets.

THEOREM 1.4.2. *Suppose that there $\mathcal{S}$ is either a strongly regular finitely irreducible D–generic conformal GDMS or a finite alphabet irreducible parabolic GDMS with $\delta_{\mathcal{S}} > \frac{2p_{\mathcal{S}}}{p_{\mathcal{S}}+1}$. Let $\sigma^2 := \mathrm{P}''(0)(\neq 0)$. For every $v \in V$ let $Y_v \subset X_v$ be a set with at least two points. If $G \subset \mathbb{R}$ is a Lebesgue measurable set with $\mathrm{Leb}(\partial G) = 0$, then*

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{\omega \in E_A^\infty : \frac{-\log \operatorname{diam}(\varphi_{\omega|_n}(Y_{t(\omega_n)})) - \chi_{\mu_\delta} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{\omega \in E_A^\infty : \alpha \le \frac{-\log \operatorname{diam}(\varphi_{\omega|_n}(Y_{t(\omega_n)})) - \chi_{\mu_\delta} n}{\sqrt{n}} \le \beta\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

There are more theorems in this vein proven in Part 4, for example the Law of Iterated Logarithm. In order to formulate other statistical results of a slightly different flavor, we define the following measures

$$\mu_n(H) := \frac{\sum_{\omega\in H} e^{-\delta\lambda_\rho(\omega)}}{\sum_{\omega\in E_\rho^n} e^{-\delta\lambda_\rho(\omega)}}$$

for integers $n \ge 1$ and $H \subset E_\rho^n$. We also consider the function $\Delta_n : E_A^n \to \mathbb{R}$ given by

$$\Delta_n(\omega) = \frac{\lambda_\rho(\omega) - \chi_\delta n}{\sqrt{n}}.$$

[1]this hypothesis means that the corresponding invariant measure μ_δ is finite, thus a probability after normalization

THEOREM 1.4.3. *If $\mathcal{S}$ is either a finitely irreducible strongly regular conformal GDMS or a finite alphabet irreducible parabolic GDMS with $\delta_{\mathcal{S}} > \frac{2p_{\mathcal{S}}}{p_{\mathcal{S}}+1}$, then for every $\rho \in E_A^\infty$ we have that*

$$\lim_{n\to+\infty} \int_{E_\rho^n} \frac{\lambda_\rho}{n} d\mu_n = \chi_{\mu_\delta} = -\int_{E_\rho^\infty} \log\left|\varphi'_{\omega_1}\left(\pi_{\mathcal{S}}(\sigma(\omega))\right)\right| d\mu_\delta(\omega).$$

The following theorem describes precisely the magnitude of deviations in this convergence, and is another form of Central Limit Theorem.

THEOREM 1.4.4. *If $\mathcal{S}$ is either a strongly regular finitely irreducible D–generic attracting conformal graph directed Markov system or a finite alphabet irreducible parabolic GDMS with $\delta_{\mathcal{S}} > \frac{2p_{\mathcal{S}}}{p_{\mathcal{S}}+1}$, then the sequence of random variables $(\Delta_n)_{n=1}^\infty$ converges in distribution to the normal (Gaussian) distribution $\mathcal{N}_0(\sigma)$ with mean value zero and the variance $\sigma^2 = \mathrm{P}''(\delta)$. Equivalently, the sequence $(\mu_n \circ \Delta_n^{-1})_{n=1}^\infty$ converges weakly to the normal distribution $\mathcal{N}_0(\sigma^2)$. This means that for every Borel set $F \subset \mathbb{R}$ with* $\mathrm{Leb}(\partial F) = 0$, *we have*

$$\lim_{n\to+\infty} \frac{\sum_{\omega\in E_\rho^n} \left|\varphi'_\omega(\pi_{\mathcal{S}}(\rho)\right|^\delta \mathbb{1}_F\left(\frac{\lambda_\rho(\omega|_n)-\chi_\delta n}{\sqrt{n}}\right)}{\sum_{\omega\in E_\rho^n}\left|\varphi'_\omega(\pi_{\mathcal{S}}(\rho|)\right|^\delta} = \lim_{n\to+\infty} \mu_n(\Delta_n^{-1}(F)) = \frac{1}{\sqrt{2\pi}\sigma}\int_F e^{-t^2/2\sigma^2} dt.$$

In particular all these theorems hold for all classes of examples described in subsection 1.3, in the case of parabolic systems under the additional hypothesis that $\delta > \frac{2p}{p+1}$, which ensures that the corresponding invariant measure μ_δ is finite, thus probabilistic after normalization. In the case of continued fractions these take on exactly the same form, in the case of Kleinian groups, including Apollonian circle packings, as for associated GDMSs.

However, in giving statements of the Central Limit Theorems for examples, we have chosen rational functions to best illustrate them. The first result is a Central Limit Theorem for the distribution of the derivatives of an expanding rational function along orbits.

THEOREM 1.4.5. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be either an expanding rational function of the Riemann sphere $\widehat{\mathbb{C}}$ or a parabolic rational function of $\widehat{\mathbb{C}}$ with $\delta_{\mathcal{S}} > \frac{2p_{\mathcal{S}}}{p_{\mathcal{S}}+1}$. Then there exists $\sigma^2 > 0$ such that if $G \subset \mathbb{R}$ is a Lebesgue measurable set with* $\mathrm{Leb}(\partial G) = 0$, *then*

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in J(f) : \frac{\log\left|(f^n)'(z)\right| - \chi_{\mu_\delta} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}} dt.$$

In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in J(f) : \alpha \le \frac{\log\left|(f^n)'(z)\right| - \chi_{\mu_\delta} n}{\sqrt{n}} \le \beta\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}} dt.$$

The second result is a Central limit Theorem describing the diameter of the preimages of reference sets.

THEOREM 1.4.6. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be either an expanding rational function of the Riemann sphere $\widehat{\mathbb{C}}$ or a parabolic rational function of $\widehat{\mathbb{C}}$ with $\delta_{\mathcal{S}} > \frac{2p_{\mathcal{S}}}{p_{\mathcal{S}}+1}$. Then for every $e \in F$ let $Y_e \subset R_e$ be a set with at least two points. If $G \subset \mathbb{R}$ is a Lebesgue measurable set with* $\mathrm{Leb}(\partial G) = 0$, *then*

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in J(f) : \frac{-\log \operatorname{diam}\left(f_x^{-n}(Y_{e(z,n)})\right) - \chi_{\mu_\delta} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}} dt$$

where f_x^{-n} is a local inverse for f^n in a neighborhood of $x = f^n(z)$. In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in J(f) : \alpha \le \frac{-\log \operatorname{diam}\left(f_x^{-n}(Y_{e(z,n)})\right) - \chi_{\mu_\delta} n}{\sqrt{n}} \le \beta\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}} dt.$$

THEOREM 1.4.7. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is either an expanding rational function of the Riemann sphere $\widehat{\mathbb{C}}$ or a parabolic rational function of $\widehat{\mathbb{C}}$ with $\delta_{\mathcal{S}} > \frac{2p_{\mathcal{S}}}{p_{\mathcal{S}}+1}$, then for every $\xi \in J(f)$, we have that*

$$\lim_{n\to+\infty} \int_{f^{-n}(\xi)} \frac{\log|(f^n)'|}{n} d\mu_n = \chi_\delta.$$

The final result concerning central limit theorems is a Central Limit Theorem which describes the distribution of preimages of a reference point.

THEOREM 1.4.8. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is either an expanding rational function of the Riemann sphere $\widehat{\mathbb{C}}$ or a parabolic rational function of $\widehat{\mathbb{C}}$ with $\delta_{\mathcal{S}} > \frac{2p_{\mathcal{S}}}{p_{\mathcal{S}}+1}$, then the sequence of random variables $(\Delta_n)_{n=1}^\infty$ converges in distribution to the normal (Gaussian) distribution $\mathcal{N}_0(\sigma)$ with mean value zero and the variance $\sigma^2 > 0$. Equivalently, the sequence $(\mu_n \circ \Delta_n^{-1})_{n=1}^\infty$ converges weakly to the normal distribution $\mathcal{N}_0(\sigma^2)$. This means that for every Borel set $F \subset \mathbb{R}$ with* $\mathrm{Leb}(\partial F) = 0$, *we have*

$$\lim_{n\to+\infty} \frac{\sum_{z\in f^{-n}(\xi)} |(f^n)'(z)|^{-\delta} 1\!\!1_F\left(\frac{\log|(f^n)'(z)|-\chi_\delta n}{\sqrt{n}}\right)}{\sum_{z\in f^{-n}(\xi)} |(f^n)'(z)|^{-\delta}} = \lim_{n\to+\infty} \mu_n(\Delta_n^{-1}(F)) = \frac{1}{\sqrt{2\pi}\sigma} \int_F e^{-t^2/2\sigma^2} dt.$$

We complete this section by putting our results in Parts 4 and 5 into context.

1.5. Historical Overview of Applications and Examples

At the heart of this monograph is a new general method, based on the concept of conformal graph directed Mrkov systems of [**47**], which serves to provide a unified approach to both counting problems (Theorem 1.2.1 and Theorem 1.2.2) and statistical results (Theorem 1.4.1 and Theorem 1.4.2), which can then be applied to many different examples. Although many of our applications are new, it is only to be expected that some of these touch upon the work of others, particularly for some of the better known examples. For the benefit of the reader, and to place our results into context, in this subsection we briefly describe how our results relate to the existing literature.

In subsection 1.3 (and later in Example 5.1.14) we began with the historically important examples of the uniformly expanding Gauss map and non-uniformly expanding Manneville-Pomeau map, and our asymptotic counting results for these appear as Theorems 5.2.1 and 5.2.2. Indirectly, one could relate the counting results for periodic orbits for these maps to those for closed geodesics on the Modular surface, by the use of appropriate sections to the flow [**48**]. Then the corresponding asymptotic counting results for closed geodesics are well known by use of the Selberg trace formula (see [**30**]). In fact, the results for this special example are even stronger in that they also have error terms for the counting function, something we have not considered. There is an alternative dynamical approach for counting closed geodesics in [**52**], [**53**]. A version of the metric central limit theorem (Theorem 5.2.4) and Law of the Iterated Logarithm Theorem (Theorem 5.2.4) for the Manneville-Pomeau map can be found in the classical works of Philipp [**64**]and Doeblin [**19**]. We are not aware of earlier work on the statistical results for closed orbits and preimages of the Manneville-Pomeau map in Theorem 5.2.5.

In the same subsection (subsection 1.3, and later in subsection 5.2.2) we consider the example of the parabolic rational functions. In this case a metric Central Limit Theorem (related to Theorem 1.4.5) appears, for example, in the paper [**18**] for Gibbs measures. An earlier version for hyperbolic rational function follows from the work in Bowen's book [**4**] with the aid of Markov partitions. There are various results on the equidistribution of preimages, starting with Lyubich's result [**41**]. However, we do not know of any previous results related to Theorem 1.4.8, Corollary 5.2.10 or the subsequent results.

Finally, we considered the case of Kleinian Schottky groups Γ. In this context, in much the same was as in the case of the Gauss map, some of the counting results can be reformulated in terms of closed geodesics, this time on the manifold $\mathbb{H}^{d+1}/\Gamma$ with all sectional curvatures equal to -1. Unfortunately, most of the known counting results where Γ is a lattice (where $\mathbb{H}^{d+1}/\Gamma$ has finite volume) due to Huber, Selberg and others (see [**30**]) do not apply. In the case of a classical hyperbolic Schottky group some of the easier counting and distribution results from Theorem 6.1.10 for fixed points could probably be deduced from counting closed orbits for Axiom A flows (see [**61**]), and the simpler results for displacements might be derived from work in [**37**] or [**71**]. In the case of convex cocompact groups there is also a metric Central Limit Theorem, which essentially comes from the work of Ratner [**76**]. (Ratner's statement is for Anosov flows, but since the proof uses symbolic dynamics the same approach works for hyperbolic flows and thus applies here). For the case of lattices the metric Central Limit Theorem was established in [**38**]. However, the Central Limit theorem in Theorem 6.1.12 appears new.

The model example of the Apollonian Circle packing introduced in subsection 1.3, and described in subsection 6.2.2, has received considerable attention in recent years. Kontorovich and Oh [**36**] proved the original asymptotic counting result for circles (Theorem 1.3.1) and our contribution is an alternative approach. There are generalizations and refinements due to Oh and Shah [**56**], [**57**] and others, including error terms by Lee and Oh for the counting fuctions, which again we have not considered [**39**]. An alternative approach to the equidistribution results appears in the [**63**] which, in common with[**36**], works with the dynamics in $\mathbb{H}^{d+1}$, in contrast to our approach which works on the boundary. We are not aware of any previous Central Limit Theorems or other related statistical properties in this context.

We would like to say that apart from many other results of our monograph, several of which mentioned above, all the results of Parts 1–4, concerning asymptotic distribution and statistics of diameters, are up to our best knowledge, new. Also, all the counting results, for multipliers and diameters alike, of Parts 1 and 2 (conformal graph directed Markov systems), seem to us to be formulated and proved here for the first time.

Now, we present our systematic exposition of the above mentioned (an more) results along with their proofs. We start with thermodynamic formalism for countable alphabet subshifts of finite type.

CHAPTER 2

Attracting Conformal Graph Directed Markov Systems

2.1. Thermodynamic Formalism of Subshifts of Finite Type with Countable Alphabet; Preliminaries

In this section we introduce more completely than in the introduction the symbolic setting in which we will be working. Furthermore, we will describe the fundamental thermodynamic concepts, ideas and results, particularly those related to the associated Ruelle-Perron-Frobenius operators, which will play a crucial role throughout the monograph.

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of all positive integers and let E be a countable set, either finite or infinite, called in the sequel an *alphabet*. Let

$$\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$$

be the shift map, i.e. cutting off the first coordinate and shifting one place to the left. It is given by the formula

$$\sigma\big((\omega_n)_{n=1}^{\infty}\big) = \big((\omega_{n+1})_{n=1}^{\infty}\big).$$

We also set

$$E^* = \bigcup_{n=0}^{\infty} E^n.$$

to be the set of finite strings. For every $\omega \in E^*$, we denote by $|\omega|$ the unique integer $n \geq 0$ such that $\omega \in E^n$. We call $|\omega|$ the length of ω. We make the convention that $E^0 = \{\emptyset\}$. If $\omega \in E^{\mathbb{N}}$ and $n \geq 1$, we put

$$\omega|_n = \omega_1 \ldots \omega_n \in E^n.$$

If $\tau \in E^*$ and $\omega \in E^* \cup E^{\mathbb{N}}$, we define the concatenation of τ and ω by:

$$\tau\omega := \begin{cases} \tau_1 \ldots \tau_{|\tau|}\omega_1\omega_2 \ldots \omega_{|\omega|} & \text{if } \omega \in E^*, \\ \tau, \ldots \tau_{|\tau|}\omega_1\omega_2 \ldots & \text{if } \omega \in E^{\mathbb{N}}, . \end{cases}$$

Given $\omega, \tau \in E^{\mathbb{N}}$, we define $\omega \wedge \tau \in E^{\mathbb{N}} \cup E^*$ to be the longest initial block common to both ω and τ. For each $\alpha > 0$, we define a metric d_α on $E^{\mathbb{N}}$ by setting

$$d_\alpha(\omega, \tau) = \mathrm{e}^{-\alpha|\omega\wedge\tau|}. \tag{2.1}$$

All these metrics induce the same topology, known to be the product (Tichonov) topology. A real or complex valued function defined on a subset of $E^{\mathbb{N}}$ is uniformly continuous with respect to one of these metrics if and only if it is uniformly continuous with respect to all of them. Also, this function is Hölder with respect to one of these metrics if and only if it is Hölder with respect to all of them although, of course, the Hölder exponent depends on the metric. If no metric is specifically mentioned, we take it to be d_1.

Now consider an arbitrary matrix $A : E \times E \to \{0, 1\}$. Such a matrix will be called the incidence matrix in the sequel. Set

$$E_A^{\infty} := \{\omega \in E^{\mathbb{N}} : \ A_{\omega_i\omega_{i+1}} = 1 \text{ for all } i \in \mathbb{N}\}.$$

Elements of E_A^∞ are called *A-admissible*. We also set

$$E_A^n := \{\omega \in E^{\mathbb{N}} :\ A_{\omega_i\omega_{i+1}} = 1 \text{ for all } 1 \le i \le n-1\},\ n \in \mathbb{N},$$

and

$$E_A^* := \bigcup_{n=0}^\infty E_A^n.$$

The elements of these sets are also called A-admissible. For every $\omega \in E_A^*$, we put

$$[\omega] := \{\tau \in E_A^\infty :\ \tau_{|_{|\omega|}} = \omega\}.$$

The set $[\omega]$ is called the cylinder generated by the word ω. The collection of all such cylinders forms a base for the product topology relative to E_A^∞. The following fact is obvious.

PROPOSITION 2.1.1. *The set E_A^∞ is a closed subset of $E^{\mathbb{N}}$, invariant under the shift map $\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$, the latter meaning that*

$$\sigma(E_A^\infty) \subseteq E_A^\infty.$$

We recall that the matrix A is said to be *finitely irreducible* if there exists a finite set $\Lambda \subseteq E_A^*$ such that for all $i, j \in E$ there exists $\omega \in \Lambda$ for which $i\omega j \in E_A^*$. If all elements of some Λ are of the same length, then A is called *finitely primitive* (or aperiodic).

The topological pressure of a continuous function $f : E_A^\infty \to \mathbb{R}$ with respect to the shift map $\sigma : E_A^\infty \to E_A^\infty$ is defined to be

$$\mathrm{P}(f) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in E_A^n} \exp\left(\sup_{\tau\in[\omega]} \sum_{j=0}^{n-1} f(\sigma^j(\tau))\right). \tag{2.2}$$

The existence of this limit, following from the observation that the "log" above forms a subadditive sequence, was established in [**46**], comp. [**47**]. Following the common usage we abbreviate

$$S_n f := \sum_{j=0}^{n-1} f \circ \sigma^j$$

and call $S_n f(\tau)$ the nth Birkhoff's sum of f evaluated at a word $\tau \in E_A^\infty$.

Observe that a function $f : E_A^\infty \to \mathbb{R}$ is (locally) *Hölder continuous with an exponent* $\alpha > 0$ if and only if

$$V_\alpha(f) := \sup_{n\ge1}\{V_{\alpha,n}(f)\} < +\infty,$$

where

$$V_{\alpha,n}(f) = \sup\{|f(\omega) - f(\tau)|e^{\alpha(n-1)} : \omega, \tau \in E_A^\infty \text{ and } |\omega \wedge \tau| \ge n\}.$$

Observe further that $\mathrm{H}_\alpha(A)$, the vector space of all bounded Hölder continuous functions $f : E_A^\infty \to \mathbb{R}$ (or $\mathbb{C}$) with an exponent $\alpha > 0$ becomes a Banach space with the norm $||\cdot||_\alpha$ defined as follows:

$$||f||_\alpha := ||f||_\infty + V_\alpha(f).$$

The following theorem has been proved in [**46**], comp. [**47**], for the class of acceptable functions defined there. Since Hölder continuous ones are among them, we have the following.

THEOREM 2.1.2 (Variational Principle). *If the incidence matrix $A : E\times E \to \{0, 1\}$ is finitely irreducible and if $f : E_A^\infty \to \mathbb{R}$ is Hölder continuous, then*

$$\mathrm{P}(f) = \sup\Big\{\mathrm{h}_\mu(\sigma) + \int f\, d\mu\Big\},$$

where the supremum is taken over all σ-invariant (ergodic) Borel probability measures μ such that $\int f\, d\mu > -\infty$.

We would like also to mention that this theorem was proved in [**78**] for Hölder continuous functions f though with a different definition of topological pressure.

We call a σ-invariant probability measure μ on E_A^∞ an *equilibrium state of a Hölder continuous function* $f : E_A^\infty \to \mathbb{R}$ if $\int -f\, d\mu < +\infty$ and

$$\mathrm{h}_\mu(\sigma) + \int f\, d\mu = \mathrm{P}(f). \tag{2.3}$$

If $f : E_A^\infty \to \mathbb{R}$ is a Hölder continuous function, then following [**46**], and [**47**] a Borel probability measure μ on E_A^∞ is called a *Gibbs state* for f (comp. also [**4**], [**29**], [**74**], [**77**], [**81**], [**92**] and [**91**]) if there exist constants $Q \geq 1$ and $\mathrm{P}_\mu \in \mathbb{R}$ such that for every $\omega \in E_A^*$ and every $\tau \in [\omega]$

$$Q^{-1} \leq \frac{\mu([\omega])}{\exp\big(S_{|\omega|}f(\tau) - \mathrm{P}_\mu|\omega|\big)} \leq Q. \tag{2.4}$$

If additionally μ is shift-invariant, it is then called an *invariant Gibbs state*. It is readily seen from this definition that if a Hölder continuous function $f : E_A^\infty \to \mathbb{R}$ admits a Gibbs state μ, then

$$\mathrm{P}_\mu = \mathrm{P}(f).$$

From now on throughout this section $f : E_A^\infty \to \mathbb{R}$ is assumed to be a Hölder continuous function with an exponent $\alpha > 0$, and it is also assumed to satisfy the following requirement

$$\sum_{e\in E} \exp(\sup(f|_{[e]})) < +\infty. \tag{2.5}$$

Functions f satisfying this condition are called (see [**46**], and [**47**]) in the sequel *summable*. We note that if f has a Gibbs state, then f is summable. This requirement of summability, allows us to define the *Ruelle–Perron–Frobenius operator*

$$\mathcal{L}_f : C_b(E_A^\infty) \to C_b(E_A^\infty),$$

acting on the space of bounded continuous functions $C_b(E_A^\infty)$ endowed with $\|\cdot\|_\infty$, the supremum norm, as follows:

$$\mathcal{L}_f(g)(\omega) := \sum_{e\in E: A_{e\omega_1}=1} \exp(f(e\omega)g(e\omega).$$

Then $\|\mathcal{L}_f\|_\infty \leq \sum_{e\in E} \exp(\sup(f|_{[e]})) < +\infty$ and for every $n \geq 1$

$$\mathcal{L}_f^n(g)(\omega) = \sum_{\tau\in E_A^n: A_{\tau_n\omega_1}=1} \exp\big(S_n f(\tau\omega)\big)g(\tau\omega).$$

The conjugate operator $\mathcal{L}_f^*$ acting on the space $C_b^*(E_A^\infty)$ has the following form:

$$\mathcal{L}_f^*(\mu)(g) := \mu(\mathcal{L}_f(g)) = \int \mathcal{L}_f(g)\, d\mu.$$

Observe that the operator $\mathcal{L}_f$ preserves the space $\mathrm{H}_\alpha(A)$, of all Hölder continuous functions with an exponent $\alpha > 0$. More precisely

$$\mathcal{L}_f(\mathrm{H}_\alpha(A)) \subseteq \mathrm{H}_\alpha(A).$$

We now provide a brief account of those elements of the spectral theory that we will need and use in the sequel. Let B be a Banach space and let $L : B \to B$ be a bounded linear operator. A point $\lambda \in \mathbb{C}$ is said to belong to the spectral set (spectrum) of the operator L if the operator $\lambda I_B - L : B \to B$ is not invertible, where $I_B : B \to B$ is the identity operator on B. The spectral radius $r(L)$ of L is defined to be the supremum of moduli of all elements in the spectral set of L. It is known that $r(L)$ is finite and

$$r(L) = \lim_{n\to\infty} \|L^n\|^{1/n}.$$

A point λ of the spectrum of L is said to belong to the essential spectral set (essential spectrum) of the operator L if λ is not an isolated eigenvalue of L of finite multiplicity. The essential spectral radius $r_{\mathrm{ess}}(L)$

of L is defined to be the supremum of moduli of all elements in the essential spectral set of L. It is known (see [**54**]) that

$$r_{\mathrm{ess}}(L) = \varlimsup_{n\to\infty} \inf \{ \|L^n - K\|^{1/n} \},$$

where for every $n \geq 1$ the infimum is taken over all compact operators $K : B \to B$. The operator $L : B \to B$ is called quasi-compact if either $r(L) = 0$ or

$$r_{\mathrm{ess}}(L) < r(L).$$

The proof of the following theorem can be found in [**46**] and [**47**]. For the items (a)–(f) see also Corollary 4.3.8 in [**8**].

THEOREM 2.1.3. *Suppose that $f : E_A^\infty \to \mathbb{R}$ is a Hölder continuous summable function and the incidence matrix A is finitely irreducible. Then*

(a) *There exists a unique Borel probability eigenmeasure m_f of the conjugate Perron-Frobenius operator $\mathcal{L}_f^*$ and the corresponding eigenvalue is equal to $e^{\mathrm{P}(f)}$.*

(b) *The eigenmeasure m_f is a Gibbs state for f.*

(c) *The function $f : E_A^\infty \to \mathbb{R}$ has a unique σ-invariant Gibbs state μ_f.*

(d) *The measure μ_f is ergodic, equivalent to m_f and if $\psi_f = d\mu_f/dm_f$ is the Radon–Nikodym derivative of μ_f with respect to m_f, then $\log \psi_f$ is uniformly bounded.*

(e) *If $\int -f\, d\mu_f < +\infty$, then the σ-invariant Gibbs state μ_f is the unique equilibrium state for the potential f.*

(f) *In case the incidence matrix A is finitely primitive, the Gibbs state μ_f is completely ergodic.*

(g) *The spectral radius of the operator $\mathcal{L}_f$ considered as acting either on $C_b(E_A^\infty)$ or $\mathrm{H}_\alpha(A)$ is in both cases equal to $e^{\mathrm{P}(f)}$.*

(h) *In either case of* (g) *the number $e^{\mathrm{P}(f)}$ is a simple (isolated in the case of $\mathrm{H}_\alpha(A)$) eigenvalue of $\mathcal{L}_f$ and the Radon–Nikodym derivative $\psi_f \in \mathrm{H}_\alpha(A)$ generates its eigenspace.*

(i) *The remainder of the spectrum of the operator $\mathcal{L}_f : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ is contained in a union of finitely many eigenvalues of finite multiplicity (different from $e^{\mathrm{P}(f)}$) of modulus $e^{\mathrm{P}(f)}$ and a closed disk centered at* 0 *with radius strictly smaller than $e^{\mathrm{P}(f)}$.*
In particular, the operator $\mathcal{L}_f : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ is quasi-compact.

In the case where the incidence matrix A is finitely primitive a stronger statement holds: namely, apart from $e^{\mathrm{P}(f)}$, the spectrum of $\mathcal{L}_f : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ is contained in a closed disk centered at 0 *with radius strictly smaller than $e^{\mathrm{P}(f)}$.*
In particular, the operator $\mathcal{L}_f : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ is quasi-compact.

We are indeed concerned with Gibbs states and these suffice for us in this monograph. Theorem 2.1.3 gives us a full power of thermodynamic formalism resulting from a spectral gap. For this we do assume finite irreducibility. Indeed, we would like to add that Omri Sarig proved in [**81**] that finite irreducibility is also necessary for the existence of Gibbs states. Other papers of Sarig on countable shifts include [**78**], [**79**], [**82**]. The reader may also consult [**11**] and [**10**]. We are far from claiming that the above list of the works on the subject of countable shift is complete.

2.2. Attracting Conformal Countable Alphabet Graph Directed Markov Systems (GDMSs) and Countable Alphabet Attracting Iterated Function Systems (IFSs); Preliminaries

In this monograph we consider conformal countable alphabet graph directed Markov system (abbr. GDMS) as defined and extensively studied in [**47**]. These are quite far going generalizations of conformal countable alphabet iterated function systems (abbr. IFS) of [**42**], which in turn generalize the finite alphabet ones. All of them contain appropriate similarity systems and each step of the above generalizations gives rise to new dynamical and geometric phenomena.

The highest level of flexibility, the countable alphabet GDMSs, are interested on their own, of course in this manuscript with respect to the counting phenomena, and are well suited to modeling the dynamical examples in which we are interested. In later sections we will prove the results in this context and explain how they can be used to derive different geometric and dynamical results, such as those already mentioned in the introduction.

Let us define a graph directed Markov system (abbr. GDMS) relative to a directed multigraph (V, E, i, t) and an incidence matrix $A : E \times E \to \{0, 1\}$. As said, such systems have been defined and first studied at length in [**42**] and [**47**]. We recall that *directed multigraph* consists of a finite set V of vertices, a countable (either finite or infinite) set E of directed edges, two functions

$$i, t : E \to V,$$

and an *incidence matrix* $A : E \times E \to \{0, 1\}$ on (V, E, i, t) such that

$$A_{ab} = 1 \quad \text{implies} \quad t(a) = i(b).$$

Now suppose that in addition, we have a collection of nonempty compact metric spaces $\{X_v\}_{v\in V}$ and a number $\kappa \in (0, 1)$, such that for every $e \in E$, we have a one-to-one contraction $\varphi_e : X_{t(e)} \to X_{i(e)}$ with Lipschitz constant (bounded above by) κ. We recall that the collection

$$\mathcal{S} = \{\varphi_e : X_{t(e)} \to X_{i(e)}\}_{e\in E}$$

is called an *attracting graph directed Markov system* (or GDMS). We will frequently refer to it just as a graph directed Markov system or GDMS. We will however always keep the adjective "parabolic" when, in later sections, we will also speak about parabolic graph directed Markov systems. We extend the functions $i, t : E \to V$ in a natural way to E_A^* as follows:

$$t(\omega) := t\big(\omega_{|\omega|}\big) \quad \text{and} \quad i(\omega) := i(\omega_1).$$

For every word $\omega \in E_A^*$, say $\omega \in E_A^n$, $n \geq 0$, let us denote

$$\varphi_\omega := \varphi_{\omega_1} \circ \cdots \circ \varphi_{\omega_n} : X_{t(\omega)} \to X_{i(\omega)}.$$

We now describe the limit set, also frequently called the attractor, of the system $\mathcal{S}$. For any $\omega \in E_A^\infty$, the sets $\{\varphi_{\omega|_n}\left(X_{t(\omega_n)}\right)\}_{n\geq 1}$ form a descending sequence of nonempty compact sets and therefore $\bigcap_{n\geq 1} \varphi_{\omega|_n}\left(X_{t(\omega_n)}\right) \neq \emptyset$. Since for every $n \geq 1$,

$$\operatorname{diam}\left(\varphi_{\omega|_n}\left(X_{t(\omega_n)}\right)\right) \leq \kappa^n \operatorname{diam}\left(X_{t(\omega_n)}\right) \leq \kappa^n \max\{\operatorname{diam}(X_v) : v \in V\},$$

we conclude that the intersection

$$\bigcap_{n\in\mathbb{N}} \varphi_{\omega|_n}\left(X_{t(\omega_n)}\right)$$

is a singleton and we denote its only element by $\pi_{\mathcal{S}}(\omega)$ or simpler, by $\pi(\omega)$. In this way we have defined a map

$$\pi_{\mathcal{S}} := \pi : E_A^\infty \longrightarrow X := \coprod_{v\in V} X_v,$$

where $\coprod_{v\in V} X_v$ is the disjoint union of the compact topological spaces X_v, $v \in V$. The map π is called the *coding map*, and the set

$$J = J_{\mathcal{S}} := \pi(E_A^\infty)$$

is called the *limit set* of the GDMS $\mathcal{S}$. The sets

$$J_v = \pi(\{\omega \in E_A^\infty : i(\omega_1) = v\}), \;\; v \in V,$$

are called the *local limit sets* of $\mathcal{S}$.

We call $\mathcal{S}$ *maximal* if for all $a, b \in E$, we have $A_{ab} = 1$ if and only if $t(b) = i(a)$. In [**47**], a maximal GDMS was called a *graph directed system* (abbr. GDS). Finally, we call a maximal GDMS $\mathcal{S}$ an *iterated function system* (or IFS) if V, the set of vertices of $\mathcal{S}$, is a singleton. Equivalently, a GDMS is an IFS if and only if the set of vertices of $\mathcal{S}$ is a singleton and all entries of the incidence matrix A are equal to 1.

Definition 2.2.1. We call the GDMS $\mathcal{S}$ and its incidence matrix A *finitely irreducible* if there exists a finite set $\Omega \subset E_A^*$ such that for all $a, b \in E$ there exists a word $\omega \in \Omega$ such that the concatenation $a\omega b$ is in E_A^*. $\mathcal{S}$ and A are called *finitely primitive* if the set Ω may be chosen to consist of words all having the same length. If such a set Ω exists but is not necessarily finite, then $\mathcal{S}$ and A are called irreducible and primitive, respectively. Note that all IFSs are symbolically irreducible.

Remark 2.2.2. For every integer $q \geq 1$ define $\mathcal{S}^q$, the qth iterate of the system $\mathcal{S}$, to be

$$\{\varphi_\omega : X_{t(\omega)} \to X_{i(\omega)} : \omega \in E_A^q\}$$

and its alphabet is E_A^q. All the theorems proved in this monograph hold under the formally weaker hypothesis that all the elements of some iterate $\mathcal{S}^q$, $q \geq 1$, of the system $\mathcal{S}$, are uniform contractions. This in particular pertains to the Gauss system of Example 5.1.14 for which $q = 2$ works.

With the aim of moving on to geometric applications, and following [**47**], we recall that we called a GDMS *conformal* if for some $d \in \mathbb{N}$, the following conditions were satisfied.

(a) For every vertex $v \in V$, X_v is a compact connected subset of $\mathbb{R}^d$, and $X_v = \overline{\mathrm{Int}(X_v)}$.
(b) (Open Set Condition) For all $a, b \in E$ such that $a \neq b$,

$$\varphi_a(\mathrm{Int}(X_{t(a)})) \cap \varphi_b(\mathrm{Int}(X_{t(b)})) = \emptyset.$$

(c) (Conformality) There exists a family of open connected sets $W_v \subset X_v$, $v \in V$, such that for every $e \in E$, the map φ_e extends to a C^1 conformal diffeomorphism from $W_{t(e)}$ into $W_{i(e)}$ with Lipschitz constant $\leq \kappa$.
(d) (Bounded Distortion Property (BDP)) There are two constants $L \geq 1$ and $\alpha > 0$ such that for every $e \in E$ and every pair of points $x, y \in X_{t(e)}$,

$$\left| \frac{|\varphi_e'(y)|}{|\varphi_e'(x)|} - 1 \right| \leq L\|y - x\|^\alpha,$$

where $|\varphi_\omega'(x)|$ denotes the scaling factor of the derivative $\varphi_\omega'(x) : \mathbb{R}^d \to \mathbb{R}^d$ which is a similarity map.

Remark 2.2.3. When $d = 1$ the conformality is automatic. If $d \geq 2$ and a family $\mathcal{S} = \{\varphi_e\}_{e \in E}$ satisfies the conditions (a) and (c), then it also satisfies condition (d) with $\alpha = 1$. When $d = 2$ this is due to the well-known Koebe's Distortion Theorem (see for example, [**9**, Theorem 7.16], [**9**, Theorem 7.9], or [**32**, Theorem 7.4.6]). When $d \geq 3$ it is due to [**47**] depending heavily on Liouville's representation theorem for conformal mappings; see [**34**] for a detailed development of this theorem leading up to the strongest current version, and also including exhaustive references to the historical background.

For every real number $s \geq 0$, let (see [**42**] and [**47**])

$$\mathrm{P}(s) := \lim_{n \to +\infty} \frac{1}{n} \log \left(\sum_{|\omega| = n} \|\varphi_\omega'\|_\infty^s \right),$$

where $\|\varphi'\|_\infty$ denotes the supremum norm of the derivative of a conformal map φ over its domain; in our context these domains will be always the sets X_v, $v \in V$. The above limit always exists because the corresponding sequence is clearly subadditive. The number $\mathrm{P}(s)$ is called the topological pressure of the parameter s. Because of the Bounded Distortion Property (i.e., Property (d)), we have also the following characterization of topological pressure:

$$\mathrm{P}(s) := \lim_{n\to+\infty} \frac{1}{n} \log \left(\sum_{|\omega|=n} |\varphi'_\omega(z_\omega)|^s \right),$$

where $\{z_\omega : \omega \in E_A^*\}$ is an entirely arbitrary set of points such that $z_\omega \in X_{t(\omega)}$ for every $\omega \in E_A^*$. Let $\zeta : E_A^\infty \to \mathbb{R}$ be defined by the formula

$$\zeta(\omega) := \log |\varphi'_{\omega_1}(\pi(\sigma(\omega))|. \tag{2.1}$$

The following proposition is easy to prove; see [**47**, Proposition 3.1.4] for complete details.

Proposition 2.2.4. *For every real $s \geq 0$ the function $s\zeta : E_A^\infty \to \mathbb{R}$ is Hölder continuous and*

$$\mathrm{P}(s\zeta) = \mathrm{P}(s).$$

Definition 2.2.5. We say that a nonnegative real number s belongs to $\Gamma_\mathcal{S}$ if

$$\sum_{e\in E} \|\varphi'_e\|_\infty^s < +\infty. \tag{2.2}$$

Let us record the following immediate observation.

Observation 2.2.6. A nonnegative real number s belongs to $\Gamma_\mathcal{S}$ if and only if the Hölder continuous potential $s\zeta : E_A^\infty \to \mathbb{R}$ is summable.

We recall from [**42**] and [**47**] the following definitions:

$$\gamma_\mathcal{S} := \inf \Gamma_\mathcal{S} = \inf \left\{ s \geq 0 : \sum_{e\in E} \|\varphi'_e\|_\infty^s < +\infty \right\}.$$

The proofs of the following two statements can be found in [**47**].

Proposition 2.2.7. *If $\mathcal{S}$ is an irreducible conformal GDMS, then for every $s \geq 0$ we have that*

$$\Gamma_\mathcal{S} = \{s \geq 0 : \mathrm{P}(s) < +\infty\}$$

In particular,

$$\gamma_\mathcal{S} := \inf \{s \geq 0 : \mathrm{P}(s) < +\infty\}.$$

Theorem 2.2.8. *If $\mathcal{S}$ is a finitely irreducible conformal GDMS, then the function $\Gamma_\mathcal{S} \ni s \mapsto \mathrm{P}(s) \in \mathbb{R}$ is*

(1) *strictly decreasing,*
(2) *real-analytic,*
(3) *convex, and*
(4) $\lim_{s\to+\infty} \mathrm{P}(s) = -\infty$.

We denote

$$\mathcal{L}_s := \mathcal{L}_{s\zeta}$$

acting either on $C_b(E_A^\infty)$ or on $\mathrm{H}_a(A)$. Because of Proposition 2.2.4 and Observation 2.2.6, our Theorem 2.1.3 applies to all functions $s\zeta : E_A^\infty \to \mathbb{R}$ giving the following.

Theorem 2.2.9. *Suppose that the system $\mathcal{S}$ is finitely irreducible and $s \in \Gamma_\mathcal{S}$. Then*

(a) *There exists a unique Borel probability eigenmeasure* m_s *of the conjugate Perron-Frobenius operator* $\mathcal{L}_s^*$ *and the corresponding eigenvalue is equal to* $e^{\mathrm{P}(s)}$.

(b) *The eigenmeasure* m_s *is a Gibbs state for* $s\zeta$.

(c) *The function* $s\zeta : E_A^\infty \to \mathbb{R}$ *has a unique* σ*-invariant Gibbs state* μ_s.

(d) *The measure* μ_s *is ergodic, equivalent to* m_s *and if* $\psi_s = d\mu_s/dm_s$ *is the Radon–Nikodym derivative of* μ_s *with respect to* m_s*, then* $\log \psi_s$ *is uniformly bounded.*

(e) *If* $\chi_{\mu_s} := -\int \zeta \, d\mu_s < +\infty$*, then the* σ*-invariant Gibbs state* μ_s *is the unique equilibrium state for the potential* $s\zeta$.

(f) *In case the the system* $\mathcal{S}$ *is finitely primitive, the Gibbs state* μ_s *is completely ergodic.*

(g) *The spectral radius of the operator* $\mathcal{L}_s$ *considered as acting either on* $C_b(E_A^\infty)$ *or* $\mathrm{H}_\alpha(A)$ *is in both cases equal to* $e^{\mathrm{P}(s)}$.

(h) *In either case of* (g) *the number* $e^{\mathrm{P}(s)}$ *is a simple (isolated in the case of* $\mathrm{H}_\alpha(A)$*) eigenvalue of* $\mathcal{L}_s$ *and the Radon–Nikodym derivative* $\psi_s \in \mathrm{H}_\alpha(A)$ *generates its eigenspace.*

(i) *The reminder of the spectrum of the operator* $\mathcal{L}_s : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ *is contained in a union of finitely many eigenvalues (different from* $e^{\mathrm{P}(s)}$*) of modulus* $e^{\mathrm{P}(s)}$ *and a closed disk centered at* 0 *with radius strictly smaller than* $e^{\mathrm{P}(s)}$ *(if* A *is finitely primitive, then these eigenvalues of modulus smaller than* $e^{\mathrm{P}(s)}$ *disappear). In particular, the operator* $\mathcal{L}_s : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ *is quasi-compact.*

Given $s \in \Gamma_{\mathcal{S}}$ it immediately follows from this theorem and the definition of Gibbs states that

$$C_s^{-1} e^{-\mathrm{P}(s)|\omega|} \|\varphi_\omega'\|_\infty^s \le m_s([\omega]) \asymp \mu_s([\omega]) \le C_s e^{-\mathrm{P}(s)|\omega|} \|\varphi_\omega'\|_\infty^s \tag{2.3}$$

for all $\omega \in E_A^*$, where $C_s \in [1, +\infty)$ is some constant. We put

$$\widetilde{m}_s := m_s \circ \pi_{\mathcal{S}}^{-1} \quad \text{and} \quad \widetilde{\mu}_s := \mu_s \circ \pi_{\mathcal{S}}^{-1}. \tag{2.4}$$

The measure $\widetilde{m}_s$ is characterized (see [**47**]) by the following two properties:

$$\widetilde{m}_s(\varphi_e(F)) = e^{-\mathrm{P}(s)} \int_F |\varphi_e'| \, d\widetilde{m}_s \tag{2.5}$$

for every $e \in E$ and every Borel set $F \subseteq X_{t(e)}$, and

$$\widetilde{m}_s\big(\varphi_a(X_{t(a)}) \cap \varphi_b(X_{t(b)})\big) = 0 \tag{2.6}$$

whenever $a, b \in E$ and $a \ne b$. By a straightforward induction these extend to

$$\widetilde{m}_s(\varphi_\omega(F)) = e^{-\mathrm{P}(s)|\omega|} \int_F |\varphi_\omega'| \, d\widetilde{m}_s \tag{2.7}$$

for every $\omega \in E_A^*$ and every Borel set $F \subseteq X_{t(\omega)}$, and

$$\widetilde{m}_s\big(\varphi_\alpha(X_{t(\alpha)}) \cap \varphi_\beta(X_{t(\beta)})\big) = 0 \tag{2.8}$$

whenever $\alpha, \beta \in E_A^*$ and are incomparable.

The following theorem, providing a geometrical interpretation of the parameter $\delta_{\mathcal{S}}$, has been proved in [**47**] ([**42**] in the case of IFSs).

THEOREM 2.2.10. *If* $\mathcal{S}$ *is an finitely irreducible conformal GDMS, then*

$$\delta = \delta_{\mathcal{S}} := \mathrm{HD}(J_{\mathcal{S}}) = \inf\{s \ge 0 : \mathrm{P}(s) \le 0\} \ge \gamma_{\mathcal{S}}.$$

Following [**42**] and [**47**] we call the system $\mathcal{S}$ regular if there exists $s \in (0, +\infty)$ such that

$$\mathrm{P}(s) = 0.$$

Then by Theorems 2.2.10 and 2.2.8, such zero is unique and is equal to $\delta_{\mathcal{S}}$. So,

$$\mathrm{P}(\delta_{\mathcal{S}}) = 0. \tag{2.9}$$

Formula (2.3) then takes the following form:

$$C_{\delta_{\mathcal{S}}}^{-1} \|\varphi_\omega'\|_\infty^{\delta_{\mathcal{S}}} \le m_{\delta_{\mathcal{S}}}([\omega]) \asymp \mu_{\delta_{\mathcal{S}}}([\omega]) \le C_{\delta_{\mathcal{S}}} \|\varphi_\omega'\|_\infty^{\delta_{\mathcal{S}}} \tag{2.10}$$

for all $\omega \in E_A^*$. The measure $\widetilde{m}_{\delta_{\mathcal{S}}}$ is then referred to as the $\delta_{\mathcal{S}}$–conformal measure of the system $\mathcal{S}$.

Also following [**42**] and [**47**], we call the system $\mathcal{S}$ strongly regular if there exists $s \in [0, +\infty)$ (in fact in $(\gamma_{\mathcal{S}}, +\infty)$) such that

$$0 < \mathrm{P}(s) < +\infty.$$

Because of Theorem 2.2.8 each strongly regular conformal GDMS is regular. Furthermore, we record the following two immediate observations.

Observation 2.2.11. If $s \in \mathrm{Int}(\Gamma_{\mathcal{S}})$, then $\chi_{\mu_s} < +\infty$.

Observation 2.2.12. A finitely irreducible conformal GDMS $\mathcal{S}$ is strongly regular if and only if

$$\gamma_{\mathcal{S}} < \delta_{\mathcal{S}}.$$

In particular, if the system $\mathcal{S}$ is a strongly regular, then $\delta_{\mathcal{S}} \in \mathrm{Int}(\Gamma_{\mathcal{S}})$.

These two observations yield the following.

Corollary 2.2.13. *If a finitely irreducible conformal GDMS $\mathcal{S}$ is strongly regular, then $\chi_{\mu_\delta} < +\infty$.*

We will also need the following fact, well-known in the case of finite alphabets E, and proved for all countable alphabets in [**47**].

Theorem 2.2.14. *If $s \in \mathrm{Int}(\Gamma_{\mathcal{S}})$, then*

$$\mathrm{P}'(s) = -\chi_{\mu_s}.$$

In particular this formula holds if the system $\mathcal{S}$ is strongly regular and $s = \delta_{\mathcal{S}}$.

We end this section by noting that each finite irreducible system is strongly regular.

2.3. Complex Ruelle–Perron–Frobenius Operators; Spectrum and D–Genericity

A key ingredient when analyzing the Poincaré series $\eta_\xi(s)$ and $\eta_p(s)$, mentioned in the introduction, is to use complex Ruelle-Perron-Frobenius or transfer operators. These are closely related to the RPF operators already introduced, except that we now allow the weighting function to take complex values. More precisely, we extend the definition of operators $\mathcal{L}_s$, $s \in \Gamma_{\mathcal{S}}$, to the complex half-plane

$$\Gamma_{\mathcal{S}}^+ := \{s \in \mathbb{C} : \mathrm{Re}s > \gamma_{\mathcal{S}}\},$$

in a most natural way; namely, for every $s \in \Gamma_{\mathcal{S}}^+$, we set

$$\mathcal{L}_s(g)(\omega) = \sum_{e \in E : A_{e\omega_1} = 1} |\varphi_e'(\pi(\omega))|^s g(e\omega). \tag{2.1}$$

Clearly these linear operators $\mathcal{L}_s$ act on both Banach spaces $C_b(E_A^\infty)$ and $\mathrm{H}_\alpha(A)$, are bounded, and we have the following.

Observation 2.3.1. The function

$$\Gamma_{\mathcal{S}}^+ \ni s \mapsto \mathcal{L}_s \in L(\mathrm{H}_\alpha(A))$$

is holomorphic, where $L(\mathrm{H}_\alpha(A))$ is the Banach space of all bounded linear operators on $\mathrm{H}_a(A)$ endowed with the operator norm.

PROPOSITION 2.3.2. *Let $\mathcal{S}$ be a finitely irreducible conformal GDMS. Then for every $s = \sigma + it \in \Gamma_{\mathcal{S}}^+$*

(1) *the spectral radius $r(\mathcal{L}_s)$ of the operator $\mathcal{L}_s : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ is not larger than $e^{\mathrm{P}(\sigma)}$ and*
(2) *the essential spectral radius $r_{\mathrm{ess}}(\mathcal{L}_s)$ of the operator $\mathcal{L}_s : \mathrm{H}_\alpha(A) \to \mathrm{H}_a(A)$ is not larger than $e^{-\alpha}e^{\mathrm{P}(\sigma)}$.*

PROOF. Assume without loss of generality that $E = \mathbb{N}$. For every $\omega \in E_A^*$ choose arbitrarily $\hat{\omega} \in [\omega]$. Now for every integer $n \geq 1$ define the linear operator

$$E_n : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$$

by the formula

$$E_n(g) := \sum_{\omega \in E_A^n} g(\hat{\omega}) 1\!\!1_{[\omega]}. \tag{2.2}$$

Equivalently

$$E_n(g) = g(\hat{\omega}),\ \omega \in E_A^\infty.$$

Of course $||E_n(g)||_\alpha \leq ||g||_\alpha$ and E_n is a bounded operator with $||E_n||_\alpha \leq 1$. However, the series (2.2) is not uniformly convergent, i.e. it is not convergent in the supremum norm $||\cdot||_\infty$, thus not in the Hölder norm $||\cdot||_\alpha$ either. For all integers $N \geq 1$ and $n \geq 1$ denote

$$E_A^n(N) := \{\omega \in E_A^n : \forall_{j \leq n}\, \omega_j \leq N\}$$

and

$$E_A^n(N+) := \{\omega \in E_A^n : \exists_{j \leq n}\, \omega_j > N\}.$$

Let us further write

$$E_{n,N}g := \sum_{\omega \in E_A^n(N)} g(\hat{\omega}) 1\!\!1_{[\omega]}$$

and

$$E_{n,N}^+ g := \sum_{\omega \in E_A^n(N+)} g(\hat{\omega}) 1\!\!1_{[\omega]}.$$

Of course $E_{n,N} : \mathrm{H}_\alpha(A) \to \mathrm{H}_a(A)$ is a finite–rank operator, thus compact. Therefore, the composite operator $\mathcal{L}_s E_{n,N} : \mathrm{H}_\alpha(A) \to \mathrm{H}_a(A)$ is also compact. We know that

$$\begin{aligned} \|\mathcal{L}_s^n - \mathcal{L}_s^n E_{n,N}\|_\alpha &= \|(\mathcal{L}_s^n - \mathcal{L}_s^n E_n) + \mathcal{L}_s^n(E_n - E_{n,N})\|_\alpha = \|\mathcal{L}_s^n(I - E_n) + \mathcal{L}_s^n E_{n,N}^+\|_\alpha \\ &\leq \|\mathcal{L}_s^n(I - E_n)\|_\alpha + \|\mathcal{L}_s^n E_{n,N}^+\|_\alpha. \end{aligned} \tag{2.3}$$

We will estimate from above each of the last two terms separately. We begin first with the first of these two terms. In the same way as for real parameters s, which is done in [**47**], one proves for all operators $\mathcal{L}_s : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ the following form of the Ionescu–Tulcea–Marinescu inequality:

$$\|\mathcal{L}_s^n g\|_\alpha \leq Ce^{\mathrm{P}(\sigma)n}\big(\|g\|_\infty + e^{-\alpha n}\|g\|_\alpha\big) \tag{2.4}$$

with some constant $C > 0$. This establishes item (1) of our theorem. Since a straightforward calculation shows that $\|g - E_n g\|_\alpha \leq 2\|g\|_\alpha$ and $\|g - E_n g\|_\infty \leq \|g\|_\alpha e^{-\alpha n}$, we therefore get that

$$\|\mathcal{L}_s^n(I - E_n)g\|_\alpha \leq Ce^{\mathrm{P}(\sigma)n}\big(\|g\|_\alpha e^{-\alpha n} + 2e^{-\alpha n}\|g\|_\alpha\big) = 3Ce^{\mathrm{P}(\sigma)n}e^{-\alpha n}\|g\|_\alpha.$$

Thus,

$$||\mathcal{L}_s^n(I - E_n)||_\alpha \leq 3Ce^{\mathrm{P}(\sigma)n}e^{-\alpha n}. \tag{2.5}$$

Passing to the estimate of the second term, we have

$$\mathcal{L}_s^n E_{n,N}^+ g(\omega) = \sum_{\substack{\tau \in E_A^n(N+) \\ \tau\omega \in E_A^\infty}} g(\hat{\tau})|\varphi_\tau'(\pi(\sigma(\omega)))|^s.$$

Therefore,

$$\begin{aligned}\|\mathcal{L}_s^n E_{n,N}^+ g\|_\alpha &\le \sum_{\tau\in E_A^n(N+)} |g(\hat\tau)| \Big\| \big|\varphi_\tau'\circ\pi\circ\sigma\big|^s\Big\|_\alpha \\ &\le \|g\|_\infty \sum_{\tau\in E_A^n(N+)} \Big\| \big|\varphi_\tau'\circ\pi\circ\sigma\big|^s\Big\|_\alpha \\ &\le \|g\|_\infty \sum_{\tau\in E_A^n(N+)} \Big\| \big|\varphi_\tau'\circ\pi\circ\sigma\big|^s\Big\|_\alpha .\end{aligned}$$

Hence,

$$\|\mathcal{L}_s^n E_{n,N}^+\|_\alpha \le \sum_{\tau\in E_A^n(N+)} \Big\| \big|\varphi_\tau'\circ\pi\circ\sigma\big|^s\Big\|_\alpha . \tag{2.6}$$

But

$$\Big\| \big|\varphi_\tau'\circ\pi\circ\sigma\big|^s\Big\|_\alpha \le C\|\varphi_\tau'\|_\infty^\sigma .$$

for all $\tau\in E_A^*$ with some constant $C>0$. Since the matrix $A:E\times E\to\{0,1\}$ is finitely irreducible, there exists a finite set $\Lambda_\infty\subseteq E_A^\infty$ such that for every $e\in E$ there exists (at least one) $\hat e\in\Lambda_\infty$ such that $e\hat e\in E_A^\infty$. We further set for every $\tau\in E_A^*$,

$$\hat\tau := \widehat{\tau_{|\tau|}} .$$

For every $k\in E=\mathbb{N}$ let

$$\xi_k := \sup\{\|\varphi_n'\|_\infty : n\ge k\} \longrightarrow 0 \quad\text{as}\quad k\to\infty. \tag{2.7}$$

Fix an arbitrary $\varepsilon>0$ so small that $\sigma-\varepsilon>\gamma_{\mathcal{S}}$. By the Bounded Distortion Property and (2.7), we then have

$$\begin{aligned}\sum_{\tau\in E_A^n(N+)} \|\varphi_\tau'\|_\infty^\sigma &\le K^\sigma \sum_{\tau\in E_A^n(N+)} |\varphi_\tau'(\pi(\hat\tau))|^\sigma \le K^\sigma \sum_{\omega\in\Lambda_\infty} \sum_{\substack{\tau\in E_A^n(N+)\\ \tau\omega\in E_A^\infty}} |\varphi_\tau'(\pi(\omega))|^\sigma \\ &= K^\sigma \sum_{\omega\in\Lambda_\infty} \sum_{\substack{\tau\in E_A^n(N+)\\ \tau\omega\in E_A^\infty}} |\varphi_\tau'(\pi(\omega))|^\varepsilon |\varphi_\tau'(\pi(\omega))|^{\sigma-\varepsilon} \\ &\le K^\sigma \xi_N^\varepsilon \sum_{\omega\in\Lambda_\infty} \sum_{\substack{\tau\in E_A^n(N+)\\ \tau\omega\in E_A^\infty}} |\varphi_\tau'(\pi(\omega))|^{\sigma-\varepsilon} \\ &\le K^\sigma \#\Lambda_\infty \xi_N^\varepsilon \mathcal{L}_{\sigma-\varepsilon}^n \mathbb{1}(\omega) \le K^\sigma \#\Lambda_\infty \xi_N^\varepsilon \|\mathcal{L}_{\sigma-\varepsilon}^n\|_\infty \\ &\le K^\sigma \#\Lambda_\infty \xi_N^\varepsilon \|\mathcal{L}_{\sigma-\varepsilon}^n\|_\alpha \\ &\le CK^\sigma \#\Lambda_\infty \xi_N^\varepsilon e^{\mathrm{P}(\sigma-\varepsilon)n},\end{aligned} \tag{2.8}$$

where the last inequality was written due to (2.4) applied with $s=\sigma-1$ and $g=\mathbb{1}$. Inserting this to (2.7) and (2.8), we thus get that

$$\|\mathcal{L}_s^n E_{n,N}^+\|_\alpha \le CK^\sigma \#\Lambda_\infty \xi_N^\varepsilon e^{\mathrm{P}(\sigma-\varepsilon)n} .$$

Now, take an integer $N_n\ge 1$ so large that $\xi_N^\varepsilon \le (K^\sigma\#\Lambda_\infty)^{-1}e^{-\alpha n}$. Inserting this to the above display, we get that

$$\|\mathcal{L}_s^n E_{n,N_n}^+\|_\alpha \le Ce^{\mathrm{P}(\sigma-\varepsilon)n}e^{-\alpha n} .$$

Along with (2.5), (2.3), and the fact that $\mathrm{P}(\sigma)\le\mathrm{P}(\sigma-\varepsilon)$, this gives that

$$\|\mathcal{L}_s^n - \mathcal{L}_s^n E_{n,N_n}\|_\alpha \le 4Ce^{\mathrm{P}(\sigma-\varepsilon)n}e^{-\alpha n} .$$

Therefore,

$$r_{\mathrm{ess}}(\mathcal{L}_s) \leq \varlimsup_{n\to\infty} \|\mathcal{L}_s^n - \mathcal{L}_s^n \circ E_{n,N_n}\|_\alpha^{1/n} \leq e^{\mathrm{P}(\sigma-\varepsilon)}e^{-\alpha}.$$

Letting $\varepsilon \to 0$ and using continuity of the pressure function $\Gamma_{\mathcal{S}}^+ \ni t \mapsto \mathrm{P}(t) \in \mathbb{R}$, we thus get that

$$r_{\mathrm{ess}}(\mathcal{L}_s) \leq e^{-\alpha}e^{\mathrm{P}(\sigma)}.$$

The proof of item (2) is thus complete, and we are done. □

We recall that if λ_0 is an isolated point of the spectrum of a bounded linear operator L acting on a Banach space B, then the Riesz projector $P_{\lambda_0} : B \to B$ of λ_0 (with respect to L) is defined as

$$\frac{1}{2\pi i}\int_\gamma (\lambda I - L)^{-1} d\lambda$$

where, γ is any simple closed rectifiable Jordan curve enclosing λ_0 and enclosing no other point of the spectrum of L. We recall that λ_0 is called simple if the range $P_{\lambda_0}(B)$ of the projector P_{λ_0} is 1-dimensional. Then λ_0 is necessarily an eigenvalue of L. We recall the following well-known fact.

THEOREM 2.3.3. *Let λ_0 be an eigenvalue of a bounded linear operator L acting on a Banach space B. Assume that the Riesz projector P_{λ_0} of λ_0 (and L) is of finite rank. If there exists a constant $C \in [0, +\infty)$ such that*

$$\|L^n\| \leq C|\lambda_0|^n$$

for all integers $n \geq 0$, then (of course) $r(L) = |\lambda_0|$, and moreover

$$P_{\lambda_0}(B) = \mathrm{Ker}(\lambda_0 I - L).$$

What we will really need in conjunction with Proposition 2.3.2 is the following.

LEMMA 2.3.4. *If $\mathcal{S}$ is a finitely irreducible conformal GDMS and if $s = \sigma + it \in \Gamma_{\mathcal{S}}^+$, then every eigenvalue of $\mathcal{L}_s : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ with modulus equal to $e^{\mathrm{P}(\sigma)}$ is simple.*

PROOF. Since $\|\mathcal{L}_s^n\|_\alpha \leq 3||\mathcal{L}_\sigma^n||_\alpha \leq Ce^{\mathrm{P}(\sigma)n}$ for every $n \geq 0$ and some constant $C > 0$ independent of n, and since the Riesz projector of every eigenvalue of modulus $e^{\mathrm{P}(\sigma)}$ of $\mathcal{L}_s$ is of finite rank (as by Proposition 2.3.2 such an eigenvalue does not belong to the essential spectrum of $\mathcal{L}_s$), we conclude from Theorem 2.3.3 that in order to prove our lemma it suffices to show that

$$\dim\big(\mathrm{Ker}(\lambda I - \mathcal{L}_s)\big) = 1$$

for any such eigenvalue λ. Consider two operators $\hat{\mathcal{L}}_\sigma, \hat{\mathcal{L}}_s : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ given by the formulae

$$\hat{\mathcal{L}}_\sigma g(\omega) := e^{-\mathrm{P}(\sigma)}\frac{1}{\psi_\sigma(\omega)}\mathcal{L}_\sigma(g\psi_\sigma)(\omega) \tag{2.9}$$

and

$$\hat{\mathcal{L}}_s g(\omega) := e^{-\mathrm{P}(\sigma)}\frac{1}{\psi_\sigma(\omega)}\mathcal{L}_s(g\psi_\sigma)(\omega) \tag{2.10}$$

Both these operators are conjugate respectively to the operators $e^{-\mathrm{P}(\sigma)}\mathcal{L}_\sigma$ and $e^{-\mathrm{P}(\sigma)}\mathcal{L}_s$, $r(\hat{\mathcal{L}}_\sigma) = 1$,

$$\hat{\mathcal{L}}_\sigma 1\!\!1 = 1\!\!1 \ \ (\text{so } \ \hat{\mathcal{L}}_\sigma^n 1\!\!1 = 1\!\!1 \ \text{ for all } \ n \geq 0), \tag{2.11}$$

and in order to prove our lemma it is enough to show that

$$\dim\big(\mathrm{Ker}(\lambda I - \hat{\mathcal{L}}_s)\big) = 1$$

for every eigenvalue λ of $\hat{\mathcal{L}}_s$ with modulus equal to 1. We shall prove the following.

Claim 1^0*:* If $u \in \mathrm{H}_\alpha(A)$, then the sequence

$$\left(\frac{1}{n}\sum_{j=0}^{n-1}\hat{\mathcal{L}}_\sigma^j u\right)_{n=1}^{\infty}$$

converges uniformly on compact subsets of E_A^∞ to the constant function equal to $\int_{E_A^\infty} u\,d\mu_\sigma$.

Proof. The same proof as that of Theorem 4.3 in [**47**] asserts that any subsequence of the sequence $\left(\frac{1}{n}\sum_{j=0}^{n-1}\hat{\mathcal{L}}_\sigma^j u\right)_{n=1}^{\infty}$ has a subsequence converging uniformly on compact subsets of E_A^∞ to a function which is a fixed point of $\hat{\mathcal{L}}_\sigma$. By (2.11) and Corollary 7.5 in [**47**] each such function is a constant. Since the operator $\hat{\mathcal{L}}_\sigma$ preserves integrals ($\hat{\mathcal{L}}_\sigma^*\mu_\sigma = \mu_\sigma$) against Gibbs/equilibrium measure μ_σ, it follows that all these constants must be equal to $\int_{E_A^\infty} u\,d\mu_\sigma$. The proof of Claim 1^0 is thus complete. $\square$

Now, fix $\lambda \in \mathrm{Ker}(\lambda I - \hat{\mathcal{L}}_s)$ arbitrary and let $g \neq 0 \in \mathrm{Ker}(\lambda I - \hat{\mathcal{L}}_s)$ be arbitrary.

Claim 2^0*:* The function $E_A^\infty \ni \omega \mapsto |g(\omega)| \in \mathbb{R}$ is constant.

Proof. For every $\omega \in E_A^\infty$ and every integer $n \geq 0$ we have $|g(\omega)| = |\hat{\mathcal{L}}_s^n g(\omega)| \leq \hat{\mathcal{L}}_\sigma^n |g|(\omega)$, and therefore

$$|g(\omega)| \leq \frac{1}{n}\sum_{j=0}^{n-1}\hat{\mathcal{L}}_\sigma^j |g|(\omega).$$

So, invoking Claim 1^0, we get that

$$|g(\omega)| \leq \int_{E_A^\infty} |g|\,d\mu_\sigma.$$

Since g is continuous and $\mathrm{supp}(\mu_\sigma) = E_A^\infty$, this implies that

$$|g(\omega)| = \int_{E_A^\infty} |g|\,d\mu_\sigma$$

for all $\omega \in E_A^\infty$. The proof of Claim 2^0 is thus complete. $\square$

Formulae (2.9)–(2.11) give for every $\tau \in E_A^\infty$ that

$$\hat{\mathcal{L}}_\sigma^n g(\tau) = \sum_{\substack{\omega \in E_A^n \\ A_{\omega_n\tau_1}=1}} \exp\big(S_n h(\omega\tau)\big) g(\omega\tau)$$

and

$$\lambda^n g(\tau) = \hat{\mathcal{L}}_s^n g(\tau) = \sum_{\substack{\omega \in E_A^n \\ A_{\omega_n\tau_1}=1}} \exp\big(S_n h(\omega\tau)\big)|\varphi_\omega'(\pi(\tau))|^{it} g(\omega\tau),$$

where $h : E_A^\infty \to (-\infty, 0)$ is some Hölder continuous function resulting from (2.11) and

$$\sum_{\substack{\omega \in E_A^n \\ A_{\omega_n\tau_1}=1}} \exp\big(S_n h(\omega\tau)\big) = 1.$$

Since $\lambda^n = 1$, it follows from the last two formulas and Claim 1^0 that

$$|\varphi_\omega'(\pi(\tau))|^{it} g(\omega\tau) = \lambda^n g(\tau)$$

for all $\omega \in E_A^n$ with $A_{\omega_n\tau_1} = 1$. Equivalently:

$$g(\omega\tau) = \lambda^n |\varphi_\omega'(\pi(\tau))|^{-it} g(\tau).$$

This implies that if g_1, g_2 are two arbitrary functions in $\mathrm{Ker}(\lambda I - \mathcal{L}_s)$ such that

$$g_1(\tau) = g_2(\tau),$$

then g_1 coincides with g_2 on the set $\{\omega\tau : \omega \in E_A^* \text{ and } A_{\omega_{|om|}\tau_1} = 1\}$. But since this set is dense in E_A^∞ and both g_1 and g_2 are continuous, it follows that

$$g_1 = g_2.$$

Thus the vector space $\mathrm{Ker}(\lambda I - \mathcal{L}_s)$ is 1-dimensional and the proof is complete. □

Now we define

$$E_p^* := \{\omega \in E_A^* : A_{\omega_{|\omega|}\omega_1} = 1\}.$$

This set will be treated in greater detail in the forthcoming sections and will play an important role throughout the monograph, primarily in regard to periodic points of GDMSs.

For all $t, a \in \mathbb{R}$ we denote by $G_a(t)$ and $G_a^i(t)$ the multiplicative subgroups respectively of positive reals $(0, +\infty)$ and of the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ that are respectively generated by the sets

$$\left\{e^{-a|\omega|}|\varphi_\omega'(x_\omega)|^t : \omega \in E_p^*\right\} \subseteq (0, +\infty) \text{ and } \left\{e^{-ia|\omega|}|\varphi_\omega'(x_\omega)|^{it} : \omega \in E_p^*\right\} \subseteq S^1,$$

where x_ω is the only fixed point for $\varphi_\omega : X_{i(\omega_1)} \to X_{i(\omega_1)}$. The following proposition has been proved in [**68**] in the context of finite alphabets E, but the proof carries through without any change to the case of countable infinite alphabets as well.

Proposition 2.3.5. *Let $\mathcal{S} = \{\varphi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS. If $t \in \mathbb{R}$ and $a \in \mathbb{R}$, then the following conditions are equivalent.*

(a) *$G_a(t)$ is generated by $e^{2\pi k}$ with some $k \in \mathbb{N}_0$.*

(b) *$\exp(ia + \mathrm{P}(\sigma))$ is an eigenvalue for $\mathcal{L}_{\sigma+it} : C_b(E_A^{\mathbb{N}}) \to C_b(E_A^{\mathbb{N}})$ for some $\sigma \in \Gamma_{\mathcal{S}}$.*

(c) *$\exp(ia + \mathrm{P}(\sigma))$ is an eigenvalue for $\mathcal{L}_{\sigma+it} : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ for all $\sigma \in \Gamma_{\mathcal{S}}$.*

(d) *There exists $u \in C_b(E_A^\infty)\,(\mathrm{H}_\alpha(A))$ such that the function*

$$E_A^\infty \ni \omega \mapsto t\zeta(\omega) - a + u(\omega) - u \circ \sigma(\omega)$$

belongs to $C_b(E_A^\infty, 2\pi\mathbb{Z})\,(\mathrm{H}_\alpha(E_A^\infty, 2\pi\mathbb{Z}))$.

(e) *$G_a^i(t) = \{1\}$.*

As a matter of fact [**68**] establishes equivalence (in the case of finite alphabet) of conditions (a)–(d) but the equivalence of (a) and (e) is obvious.

We call a parameter $t \in \mathbb{R}$ $\mathcal{S}$-generic if the above condition (a) fails for $a = 0$ and we call it strongly $\mathcal{S}$–generic if it fails for all $a \in \mathbb{R}$. We call the system $\mathcal{S}$ D–generic if each parameter $t \in \mathbb{R} \setminus \{0\}$ is $\mathcal{S}$–generic and we call it strongly D-generic if each parameter $t \in \mathbb{R} \setminus \{0\}$ is strongly $\mathcal{S}$-generic.

Remark 2.3.6. We would like to remark that if the GDMS $\mathcal{S}$ is D-generic, then no function $t\zeta : E_A^\infty \to \mathbb{R}$, $t \in \mathbb{R} \setminus \{0\}$, is cohomologous to a constant. Precisely, there is no function $u \in C_b(E_A^\infty)$ such that

$$t\zeta(\omega) + u(\omega) - u \circ \sigma(\omega)$$

is a constant real-valued function.

The concept of D–genericity will play a pivotal role throughout our whole article. We start dealing with it by proving the following.

Proposition 2.3.7. *If $\mathcal{S}$ is a finitely irreducible strongly D-generic conformal GDMS and if $s = \sigma + it \in \Gamma_{\mathcal{S}}^+$ with $t \in \mathbb{R} \setminus \{0\}$, then $r(\mathcal{L}_s) < e^{\mathrm{P}(\sigma)}$.*

Proof. By Proposition 2.3.2 the set

$$\sigma(\mathcal{L}_s) \cap \big(\mathbb{C} \setminus \overline{B}(0, e^{-\alpha/2} e^{\mathrm{P}(\sigma)})\big)$$

is finite and consists only of eigenvalues of $\mathcal{L}_s$. So, by Proposition 2.3.5,

$$\sigma(\mathcal{L}_s) \cap \big(\mathbb{C} \setminus \overline{B}(0, e^{-\alpha/2} e^{\mathrm{P}(\sigma)})\big) \cap \{\lambda \in \mathbb{C} : |\lambda| = e^{\mathrm{P}(\sigma)}\} = \emptyset.$$

Therefore, using also Theorem 2.2.9 (g), we get that

$$r(\mathcal{L}_s) \leq \max\Big\{e^{-\alpha/2} e^{\mathrm{P}(\sigma)}, \max\big\{|\lambda| : \lambda \in \sigma(\mathcal{L}_s) \cap \big(\mathbb{C} \setminus \overline{B}(0, e^{-\alpha/2} e^{\mathrm{P}(\sigma)})\big)\big\}\Big\} < e^{\mathrm{P}(\sigma)}.$$

The proof is complete. □

We now shall provide a useful characterization of D-generic and strongly D-generic systems.

Proposition 2.3.8. *A finitely irreducible conformal GDMS $\mathcal{S} = \{\varphi_e\}_{e \in E}$ is D–generic if and only if the additive group generated by the set*

$$\big\{ \log |\varphi_\omega'(x_\omega)| : \omega \in E_p^*\big\}$$

is not cyclic.

Proof. Suppose first that the system $\mathcal{S} = \{\varphi_e\}_{e \in E}$ is not D–generic. This means that there exists $t \in \mathbb{R} \setminus \{0\}$ which is not $\mathcal{S}$-generic. This in turn means that the group $G_0(t)$ is generated by some non-negative integral power of $e^{2\pi}$, say by $e^{2q\pi}$, $q \in \mathbb{N}_0$. And this means that for every $\omega \in E_p^*$,

$$|\varphi_\omega'(x_\omega)|^t = \exp\big(2\pi q k_\omega\big)$$

with some (unique) $k_\omega \in \mathbb{Z}$. But then $t \log |\varphi_\omega'(x_\omega)| = 2\pi q k_\omega$ or equivalently

$$\log |\varphi_\omega'(x_\omega)| = \frac{2\pi q}{t} k_\omega.$$

This implies that the additive group generated by the set

$$\big\{ \log |\varphi_\omega'(x_\omega)| : \omega \in E_p^*\big\} \subseteq \mathbb{R}$$

is a subgroup of $\langle \frac{2\pi q}{t} \rangle$, the cyclic group generated by $\frac{2\pi q}{t}$, and is therefore itself cyclic.

For the converse implication suppose that the additive group generated by the set

$$\big\{ \log |\varphi_\omega'(x_\omega)| : \omega \in E_p^*\big\}$$

is cyclic. This means that there exists $\gamma \in (0, +\infty)$ such that

$$\log |\varphi_\omega'(x_\omega)| = 2\pi\gamma l_\omega$$

for all $\omega \in E_p^*$ and some $l_\omega \in -\mathbb{N}_0$. There then exists $t \in \mathbb{R} \setminus \{0\}$ such that $t\gamma \in \mathbb{N}$. But then

$$|\varphi_\omega'(x_\omega)|^t = \exp\big((2\pi t\gamma) l_\omega\big),$$

implying that the multiplicative group generated by the set

$$\{|\varphi_\omega'(x_\omega)|^t : \omega \in E_p^*\}$$

is a subgroup of $< e^{2\pi t\gamma} >$, the cyclic group generated by $e^{2\pi t\gamma}$, and is therefore itself cyclic. This means that $t \in \mathbb{R} \setminus \{0\}$ is not $\mathcal{S}$-generic, and this finally means that the system $\mathcal{S}$ is not D-generic. We are done. □

Remark 2.3.9. The D–genericity assumption is fairly generic. For example, it holds if there are two values $i, j \in E$ (or the weaker condition $i, j \in E_A^*$) such that $\frac{\log |\varphi_i'(x_i)|}{\log |\varphi_j'(x_j)|}$ is irrational; where we recall that x_i and x_j are the unique fixed points, respectively, of φ_i and φ_j. On the other hand, it is easy to construct specific conformal GDMSs for which it fails. For example, we can consider maps $\varphi_i(x) = \frac{x+1}{2^i}$ for $i \geq 1$ and than we can deduce that $\log |\varphi_i'(x)| \in (\log 2)\mathbb{Z}$.

Proposition 2.3.10. *A finitely irreducible conformal GDMS $\mathcal{S} = \{\varphi_e\}_{e\in E}$ is strongly D–generic if and only if the additive group generated by the set*

$$\big\{ \log|\varphi'_\omega(x_\omega)| - \beta|\omega| : \omega \in E_p^* \big\}$$

is not cyclic for any $\beta \in \mathbb{R}$.

Proof. Suppose first that the system $\mathcal{S} = \{\varphi_e\}_{e\in E}$ is not strongly D–generic. This means that there exists $t \in \mathbb{R}\setminus\{0\}$ which is not $\mathcal{S}$-generic. This in turn means that for some $a \in \mathbb{R}$ the group $G_a(t)$ is generated by some non-negative integral power of $e^{2\pi}$, say by $e^{2q\pi}$, $q \in \mathbb{N}_0$. And this means that for every $\omega \in E_p^*$,

$$e^{-a|\omega|}|\varphi'_\omega(x_\omega)|^t = \exp\big(2\pi q k_\omega\big)$$

with some (unique) $k_\omega \in \mathbb{Z}$. But then $t\log|\varphi'_\omega(x_\omega)| - a|\omega| = 2\pi q k_\omega$ or equivalently

$$\log|\varphi'_\omega(x_\omega)| - \frac{a}{t}|\omega| = \frac{2\pi q}{t}k_\omega.$$

This implies that the additive group generated by the set

$$\big\{ \log|\varphi'_\omega(x_\omega)| - \frac{a}{t}|\omega| : \omega \in E_p^* \big\}$$

is a subgroup of $< \frac{2\pi q}{t} >$, the cyclic groups generated by $\frac{2\pi q}{t}$, and is therefore itself cyclic.

For the converse implication suppose that the additive group generated by the set

$$\big\{ \log|\varphi'_\omega(x_\omega)| - \beta|\omega| : \omega \in E_p^* \big\}$$

is cyclic for some $\beta \in \mathbb{R}$. This means that there exists $\gamma \in (0, +\infty)$ such that

$$\log|\varphi'_\omega(x_\omega)| - \beta|\omega| = 2\pi\gamma l_\omega$$

for all $\omega \in E_p^*$ and some $l_\omega \in \mathbb{Z}$. There then exists $t \in \mathbb{R}\setminus\{0\}$ such that $t\gamma \in \mathbb{N}$. But then

$$e^{-t\beta|\omega|}|\varphi'_\omega(x_\omega)|^t = \exp\big((2\pi t\gamma)l_\omega\big),$$

implying that the multiplicative group generated by the set

$$\big\{e^{-t\beta|\omega|}|\varphi'_\omega(x_\omega)|^t : \omega \in E_p^*\big\}$$

is a subgroup of $< e^{2\pi t\gamma} >$, the cyclic group generated by $e^{2\pi t\gamma}$, and is therefore itself cyclic. This means that $t \in \mathbb{R}\setminus\{0\}$ is not strongly $\mathcal{S}$-generic, and this finally means that the system $\mathcal{S}$ is not strongly D-generic. We are done. $\square$

2.4. Asymptotic Results for Multipliers; Statements and First Preparations

In this section we keep the setting of the previous one. In this framework we can formulate our main asymptotic result, which has the dual virtues of being relatively easy to prove in this setting and also having many interesting applications, as illustrated in the introduction. In a later section we will also formulate the general result for C^2 multidimensional conformal contractions, although the basic statements will be exactly the same. We can now define two natural counting functions in the present context corresponding to “preimages” and “periodic points” respectively.

Definition 2.4.1. We can naturally order the countable family of the compositions of contractions $\varphi \in E_A^*$ in two different ways. Fix $\rho \in E_A^\infty$ arbitrary and set $\xi := \pi_{\mathcal{S}}(\rho) \in J_{\mathcal{S}}$. Let

$$E_\rho^* := \{\omega \in E_A^* : \omega\rho \in E_A^\infty\},$$

and, as we have already defined, for all integers $n \geq 1$ let

$$E_\rho^n := \{\omega \in E_A^n : \omega\rho \in E_A^\infty\}.$$

We recall from the previous section the set

$$E_p^* = \{\omega \in E_A^* : A_{\omega_{|\omega|}\omega_1} = 1\},$$

and for all integers $n \geq 1$ we put

$$E_p^n := \{\omega \in E_A^n : A_{\omega_n\omega_1} = 1\},$$

i.e., the words ω in E_A^* such that the words $\omega^\infty \in E_A^\infty$, the infinite concatenations of ωs, are periodic points of the shift map $\sigma : E_A^\infty \to E_A^\infty$ with period n.

(1) Firstly, we can associate the weights

$$\lambda_\rho(\omega) := -\log|\varphi'_\omega(\xi)| > 0, \quad \omega \in E_\rho^*,$$

and

(2) Secondly, we can use the weights

$$\lambda_p(\omega) := -\log|\varphi'_\omega(x_\omega)| > 0, \quad \omega \in E_p^*,$$

where we recall that $x_\omega(= \varphi_\omega(x_\omega))$ is the unique fixed point for the contraction $\varphi_\omega : X_{i(\omega_1)} \to X_{i(\omega_1)}$; we note that $t(\omega) = i(\omega_1)$.

We can associate appropriate counting functions to each of these weights, defined by

$$\pi_\rho(T) := \left\{\omega \in E_\rho^* : \lambda_\rho(\omega) \leq T\right\} \quad \text{and} \quad \pi_p(T) := \left\{\omega \in E_p^* : \lambda_p(\omega) \leq T\right\},$$

respectively, and their cardinalities

$$N_\rho(T) := \#\pi_\rho(T) \quad \text{and} \quad N_p(T) := \#\pi_p(T),$$

respectively, for each $T > 0$, i.e. the number of words $\omega \in E_i^*$ for which the corresponding weight $\lambda_i(\omega)$ doesn't exceed T for $i = \rho, p$.

The functions $\pi_\rho(T)$ and $\pi_p(T)$ are clearly both monotone increasing in T.
We first prove the following basic result, showing that the rates of growth of these two functions are both equal to the Hausdorff Dimension of the limit set $J_{\mathcal{S}}$.

PROPOSITION 2.4.2. *If the (finitely irreducible) conformal GDMS $\mathcal{S}$ is strongly regular, then*

$$\delta_{\mathcal{S}} = \lim_{T\to+\infty} \frac{1}{T}\log N_\rho(T) = \lim_{T\to+\infty} \frac{1}{T}\log N_p(T).$$

PROOF. Fix $i \in \{\rho, p\}$. Write $\delta := \delta_{\mathcal{S}}$. Assume for a contradiction that

$$\overline{\lim_{T\to+\infty}} \frac{1}{T}\log N_i(T) > \delta.$$

There then exists $\varepsilon > 0$ and an increasing unbounded sequence $T_n \to +\infty$ such that

$$N_i(T_n) \geq e^{(\delta+\varepsilon)T_n}.$$

We recall from the definition of a conformal GDMS that $\|\varphi'_e\|_\infty \leq \kappa \in (0,1)$ for all $e \in E$, and then $\|\varphi'_\omega\|_\infty \leq \kappa^{|\omega|}$ for all $\omega \in E_A^*$. Since

$$\lambda_i(\omega) + \log\|\varphi'_\omega\|_\infty \geq 0 \tag{2.1}$$

for all $\omega \in E_A^*$. we conclude that whenever $\omega \in \pi_i(T_n)$, i.e. whenever $\lambda_i(\omega) \leq T_n$, then

$$|\omega| \leq \frac{T_n}{|\log\kappa|} \leq k_n := \left[\frac{T_n}{|\log\kappa|}\right] + 1,$$

where $[\cdot]$ denotes the integer part. Therefore, we can also bound

$$\sum_{j=1}^{k_n} \sum_{\omega\in E_A^j} \|\varphi'_\omega\|_\infty^\delta \geq \sum_{\omega\in\pi_i(T_n)} \|\varphi'_\omega\|_\infty^\delta \geq N_i(T_n)e^{-\delta T_n} \geq e^{\varepsilon T_n}.$$

Hence, there exists $1 \le j_n \le k_n$ such that

$$\sum_{\omega \in E_A^{j_n}} \|\varphi_\omega'\|_\infty^\delta \ge \frac{1}{k_n} e^{\varepsilon T_n}.$$

In particular, $\lim_{n\to\infty} j_n = +\infty$. Recalling that each strongly regular system is regular and invoking (2.9), we finally get

$$\begin{aligned}
0 = \mathrm{P}(\delta) &= \lim_{n\to+\infty} \frac{1}{j_n} \log \sum_{\omega \in E_A^{j_n}} \|\varphi_\omega'\|_\infty^\delta \ge \varlimsup_{n\to+\infty} \frac{1}{j_n} \log\left(\frac{e^{\varepsilon T_n}}{k_n}\right) \\
&\ge \varlimsup_{n\to+\infty} \frac{1}{k_n} \log\left(\frac{e^{\varepsilon T_n}}{k_n}\right) = \varlimsup_{n\to+\infty} \frac{1}{k_n}(\varepsilon T_n - \log k_n) \\
&= \varepsilon \varlimsup_{n\to+\infty} \frac{T_n}{k_n} = \varepsilon |\log \kappa| > 0.
\end{aligned}$$

This contradiction shows that

$$\varlimsup_{T\to+\infty} \frac{1}{T} \log N_i(T) \le \delta. \tag{2.2}$$

For the lower bound recall that

$$\chi_\delta = -\int_{E_A^\infty} \log |\varphi_{\omega_1}'(\pi(\sigma(\omega)))| \, d\mu_\delta > 0$$

is the Lyapunov exponent of the measure μ_δ with respect to the shift map $\sigma : E_A^\infty \to E_A^\infty$. Since the system $\mathcal{S}$ is strongly regular, it follows from Observations 2.2.12 and 2.2.11 that χ_δ is finite. It then further follows from Theorem 2.2.9 (e) that h_{μ_δ} is finite and

$$\frac{\mathrm{h}_{\mu_\delta}}{\chi_\delta} = \delta.$$

Recall that along with (2.1) the Bounded Distortion Property, yields

$$0 \le \lambda_i(\omega) + \log \|\varphi_\omega'\|_\infty \le \log C \tag{2.3}$$

for all $\omega \in E_A^*$ and some constant $C > 1$. Using this and (2.10) we then get for every $\varepsilon > 0$ and all integers $n \ge 1$ large enough that

$$\begin{aligned}
\big\{\omega \in E_A^n : \lambda_i(\omega) \le (\chi_{\mu_\delta} + \varepsilon)n\big\} &= \\
&= \left\{\omega \in E_A^n : \lambda_i(\omega) \le \left(\frac{\mathrm{h}_{\mu_\delta}}{\delta} + \varepsilon\right) n\right\} \\
&\supseteq \left\{\omega \in E_A^n : -\frac{1}{\delta} \log \mu_\delta([\omega]) \le \left(\frac{\mathrm{h}_{\mu_\delta}}{\delta} + \varepsilon\right) n + \frac{\log C_\delta}{\delta} - \log C\right\} \\
&\supseteq \left\{\omega \in E_A^n : -\frac{1}{\delta} \log \mu_\delta([\omega]) \le \left(\frac{\mathrm{h}_{\mu_\delta}}{\delta} + 2\varepsilon\right) n\right\} \\
&= \big\{\omega \in E_A^n : \log \mu_\delta([\omega]) \ge -\big(\mathrm{h}_{\mu_\delta} + 2\varepsilon\delta\big)n\big\}.
\end{aligned}$$

Having this, it follows from Breiman-McMillan-Shannon Theorem that

$$\#\big\{\omega \in E_A^n : \lambda_i(\omega) \le (\chi_{\mu_\delta} + \varepsilon)n\big\} \ge \exp\big((\mathrm{h}_{\mu_\delta} - 3\varepsilon\delta)n\big)$$

for all integers $n \ge 1$ large enough. Since we also obviously have

$$\pi_i\big((\chi_{\mu_\delta} + \varepsilon)n\big) \supseteq \{\omega \in E_A^n : \lambda_i(\omega) \le (\chi_{\mu_\delta} + \varepsilon)n\},$$

we therefore get for every $T > 0$ large enough,

$$\begin{aligned}\log N_i(T) &= \log N_i\left((\chi_{\mu_\delta}+\varepsilon)\frac{T}{(\chi_{\mu_\delta}+\varepsilon)}\right) \geq \log N_i\left((\chi_{\mu_\delta}+\varepsilon)\left[\frac{T}{(\chi_{\mu_\delta}+\varepsilon)}\right]\right)\\ &\geq (\mathrm{h}_{\mu_\delta}-3\varepsilon\delta)\left[\frac{T}{(\chi_{\mu_\delta}+\varepsilon)}\right].\end{aligned}$$

Therefore,

$$\varliminf_{T\to+\infty}\frac{1}{T}\log N_i(T) \geq \frac{\mathrm{h}_{\mu_\delta}-3\varepsilon\delta}{\chi_{\mu_\delta}+\varepsilon}.$$

So, letting $\varepsilon \searrow 0$ yields

$$\varliminf_{T\to+\infty}\frac{1}{T}\log N_i(T) \geq \frac{\mathrm{h}_{\mu_\delta}}{\chi_{\mu_\delta}} = \delta.$$

Along with (2.2) this completes the proof. □

In particular, this proposition gives one more characterization of the value of δ.

One of our main objectives in this monograph is to provide a wide ranging substantial improvement of Proposition 2.4.2. This is the asymptotic formula below, formulated at level of conformal graph directed Markov systems, along with its further strengthenings, extensions, and generalizations, both for conformal graph directed Markov systems and beyond. Our first main result is the following.

THEOREM 2.4.3 (Asymptotic Formula). *If $\mathcal{S}$ is a strongly regular finitely irreducible D-generic conformal GDMS and $\rho \in E_A^\infty$, then with $\delta = \delta_{\mathcal{S}} = \mathrm{HD}(J_{\mathcal{S}})$, we have that*

$$\lim_{T\to+\infty}\frac{N_\rho(T)}{e^{\delta T}} = \frac{\psi_\delta(\rho)}{\delta\chi_{\mu_\delta}}$$

and

$$\lim_{T\to+\infty}\frac{N_p(T)}{e^{\delta T}} = \frac{1}{\delta\chi_{\mu_\delta}}.$$

The proof of this theorem will be completed as a special case of Theorem 2.4.8 (which is proved in Section 2.6).

REMARK 2.4.4. If the generic D-genericity hypothesis fails, then we may still have an asymptotic formulae, but of a different type, e.g., there exists $N_i(T) \sim C\exp(\delta a[(\log T)/a])$ as $T \to +\infty$. This is illustrated by the example in Remark 2.3.9 with $a = \log 2$.

As a preparation for the proof of Theorem 2.4.3 we now introduce a version of the main tool that will be used in the sequel. Our strategy, stemming from number theoretical considerations of distributions of prime numbers, is to use an appropriate complex function defined in terms of all of the weights $\lambda_\rho(\omega)$ and then to apply a Tauberian Theorem to convert properties of the function into the required asymptotic formula of $N_\rho(T)$, i.e. the first formula of Theorem 2.4.3. The asymptotic formula for $N_p(T)$, i.e. the second formula of Theorem 2.4.3 will be derived from the former, i.e. that of $N_\rho(T)$. The basic complex function in the symbolic context is the following.

DEFINITION 2.4.5. Given $s \in \mathbb{C}$ we define the (formal) *Poincaré series* by:

$$\eta_\rho(s) := \sum_{\omega\in E_\rho^*} e^{-s\lambda_\rho(\omega)} = \sum_{n=1}^{\infty}\sum_{\omega\in E_\rho^n} e^{-s\lambda_\rho(\omega)}.$$

In fact we will need a localized version of this function, which will be introduced and analyzed in Section 2.5. For the present, we observe that since

$$\sum_{\omega\in E_\rho^n} |e^{-s\lambda_\rho(\omega)}| = \sum_{\omega\in E_\rho^n} e^{-\mathrm{Re}(s\lambda_\rho(\omega))} \asymp \sum_{\omega\in E_\rho^n} \|\varphi_\omega'\|_\infty^{\mathrm{Re}s} \le \sum_{\omega\in E_A^n} \|\varphi_\omega'\|_\infty^{\mathrm{Re}s}$$

and since

$$\lim_{n\to\infty} \frac{1}{n}\log\left(\sum_{\omega\in E_\rho^n} \|\varphi_\omega'\|_\infty^{\mathrm{Re}s}\right) = \mathrm{P}(\mathrm{Re}s) < 0$$

whenever $\mathrm{Re}s > \delta_{\mathcal{S}}$, we get the following preliminary result.

Observation 2.4.6. The Poincaré series

$$\eta_\rho(s) = \sum_{n=1}^\infty \sum_{\omega\in E_\rho^n} e^{-s\lambda_\rho(\omega)}$$

converges absolutely uniformly on each set $\{s\in\mathbb{C} : \mathrm{Re}s > t\}$, for $t > \delta_{\mathcal{S}}$.

For notational convenience to follow we introduce the following set

$$\Delta_{\mathcal{S}}^+ := \{s\in\mathbb{C} : \mathrm{Re}s > \delta_{\mathcal{S}}\}.$$

As have said, the series $\eta_\rho(s)$ will be our main tool to acquire the asymptotic formula for the cardinalities of the sets $\pi_\rho(T)$, i.e. of the numbers $N_\rho(T)$. An appropriate knowledge of the behavior of the series $\eta_\rho(s)$ on the imaginary line $\mathrm{Re}(s) = \delta_{\mathcal{S}}$ is required for this end. Indeed, in fact one needs to know that the function $\eta_\rho(s)$ has a meromorphic extension to some open neighborhoods of $\overline{\Delta_{\mathcal{S}}^+} = \{s\in\mathbb{C} : \mathrm{Re}s \ge \delta_{\mathcal{S}}\}$ with the only pole at $s = \delta_{\mathcal{S}}$, that this pole is simple and the corresponding residue is to be calculated. This extension of $\eta_\rho(s)$ functions will come from an understanding of the spectral properties of the associated complex RPF operators.

With some additional work, we can actually get finer asymptotic results than those of Theorem 2.4.3. These count words subject to their weights being less than T and, additionally, their images being located in some, fairly arbitrarily prescribed, parts of the limit set.

Definition 2.4.7. Let $\rho\in E_A^\infty$ and let $\tau\in E_A^*$. Fix any Borel set $B\subset X$. Having $T>0$ we define:

$$\pi_\rho(B,T) := \left\{\omega\in E_\rho^* : \varphi_\omega(\pi_{\mathcal{S}}(\rho))\in B \text{ and } \lambda_\rho(\omega)\le T\right\}$$

and

$$\pi_p(B,T) := \left\{\omega\in E_p^* : x_\omega\in B \text{ and } \lambda_p(\omega)\le T\right\}.$$

We also define

$$\pi_\rho(\tau,T) := \left\{\omega\in E_\rho^* : \lambda_\rho(\tau\omega)\le T\right\} \quad \text{and} \quad \pi_p(\tau,T) := \left\{\omega\in E_p^* : \lambda_p(\tau\omega)\le T\right\}.$$

The corresponding cardinalities of these sets are denoted by:

$$N_\rho(B,T) := \#\pi_\rho(B,T) \text{ and } N_p(B,T) = \#\pi_p(B,T),$$

and

$$N_\rho(\tau,T) := \#\pi_\rho(\tau,T) \text{ and } N_p(\tau,T) = \#\pi_p(\tau,T),$$

i.e. the first pair count the number of words $\omega\in E_i^*$ for which the weight $\lambda_i(\omega)$ does not exceed T and, additionally, the image $\varphi_\omega(\pi_{\mathcal{S}}(\rho))$ is in B if $i=\rho$, or the fixed point x_ω, of φ_ω, is in B if $i=p$, while the second pair count the number of words $\omega\in E_i^*$ for which the weight $\lambda_i(\tau\omega)$ does not exceed T (for $i=p,\rho$) and an initial block of ω coincides with τ.

The following are refinements of the asymptotic results presented in Theorem 2.4.3, whose proof will be completed in Section 2.6.

Theorem 2.4.8 (Asymptotic Equidistribution Formula for Multipliers I). *Suppose that $\mathcal{S}$ is a strongly regular finitely irreducible D-generic conformal GDMS. Let $\delta = \delta_{\mathcal{S}} = \mathrm{HD}(J_{\mathcal{S}})$. Fix $\rho \in E_A^\infty$.*

If $\tau \in E_A^*$ then,

$$\lim_{T\to+\infty} \frac{N_\rho(\tau,T)}{e^{\delta T}} = \frac{\psi_\delta(\rho)}{\delta\chi_{\mu_\delta}} m_\delta([\tau]), \tag{2.4}$$

and

$$\lim_{T\to+\infty} \frac{N_p(\tau,T)}{e^{\delta T}} = \frac{1}{\delta\chi_{\mu_\delta}} \mu_\delta([\tau]). \tag{2.5}$$

Theorem 2.4.9 (Asymptotic Equidistribution Formula for Multipliers II). *Suppose that $\mathcal{S}$ is a strongly regular finitely irreducible D-generic conformal GDMS. Let $\delta = \delta_{\mathcal{S}} = \mathrm{HD}(J_{\mathcal{S}})$. Fix $\rho \in E_A^\infty$.*

If $B \subset X$ is a Borel set such that $\widetilde{m}_\delta(\partial B) = 0$ (equivalently $\widetilde{\mu}_\delta(\partial B) = 0$) then,

$$\lim_{T\to+\infty} \frac{N_\rho(B,T)}{e^{\delta T}} = \frac{\psi_\delta(\rho)}{\delta\chi_{\mu_\delta}} \widetilde{m}_\delta(B) \tag{2.6}$$

and

$$\lim_{T\to+\infty} \frac{N_p(B,T)}{e^{\delta T}} = \frac{1}{\delta\chi_{\mu_\delta}} \widetilde{\mu}_\delta(B). \tag{2.7}$$

After establishing the results of the next section (2.5), we will first prove in Section 2.6 formula (2.4). Then, in the same section, we will deduce from it formula (2.5). Finally, still within Section 2.6, we will deduce Theorem 2.4.9 as a consequence of Theorem 2.4.8. The asymptotic estimates for $N_\rho(B,T)$ given in this theorem, will turn out to have wider applications than the basic asymptotic results in Theorem 2.4.3. This will be apparent, particularly in Section 2.7 and Section 3.4 where we apply these results to deduce asymptotics of the diameters of circles.

Remark 2.4.10. Theorem 2.4.8 is formulated for a countable state symbolic system. In fact it could be formulated and proved with no real additional difficulty for ergodic sums of all summable Hölder continuous potentials rather than merely the functions $\lambda_\rho(\omega)$. In the particular case of a finite state symbolic system this would recover the corresponding results of Lalley [**37**].

2.5. Complex Localized Poincaré Series η_ρ

In order to prove the asymptotic statements of Theorem 2.4.8 we want to consider a localized Poincaré series, which in turn generalizes the Poincaré series introduced in the previous section. Again we denote by $\rho \in E_A^\infty$ our reference point and set $\xi := \pi_{\mathcal{S}}(\rho) \in J_{\mathcal{S}}$.

Definition 2.5.1. Given $s \in \mathbb{C}$ we define the following *localized (formal) Poincaré series.* Fixing $\tau \in E_A^*$ and denoting $q := |\tau|$, we formally write

$$\eta_\rho(\tau, s) := \sum_{\substack{\omega\in E_\rho^* \\ A_{\tau_q\omega_1}=1}} e^{-s\lambda_\rho(\tau\omega)}.$$

We formally expand the series $\eta_\rho(\tau, s)$ as follows.

$$\begin{aligned}
\eta_\rho(\tau, s) :&= \sum_{\substack{\omega \in E_\rho^* \\ A_{\tau_q \omega_1}=1}} e^{-s\lambda_\rho(\tau\omega)} = \sum_{\substack{\omega \in E_\rho^* \\ A_{\tau_q \omega_1}=1}} |\varphi_{\tau\omega}'|^s(\pi(\rho)) = \sum_{\substack{\omega \in E_\rho^* \\ A_{\tau_q \omega_1}=1}} |\varphi_\tau'|^s(\pi(\omega\rho))|\varphi_\omega'|^s(\pi(\rho)) \\
&= \sum_{n=1}^\infty \sum_{\substack{\omega \in E_\rho^n \\ A_{\tau_q \omega_1}=1}} |\varphi_\tau'|^s \circ \pi(\omega\rho)|\varphi_\omega'|^s(\pi(\rho)) \\
&= \sum_{n=1}^\infty \mathcal{L}_s^n\big(|\varphi_\tau'|^s \circ \pi\big)(\rho).
\end{aligned}$$

Defining the operator $\mathcal{L}_{s,\tau}^{(n)}$ from $\mathrm{H}_\alpha(A)$ to $\mathrm{H}_a(A)$ by

$$\mathrm{H}_\alpha(A) \ni g \longmapsto \mathcal{L}_{s,\tau}^{(n)} g := \mathcal{L}_s^n\big(g \cdot (|\varphi_\tau'|^s \circ \pi)\big) \in \mathrm{H}_\alpha(A),$$

we then formally write

$$\eta_\rho(\tau, s) = \sum_{n=1}^\infty \mathcal{L}_{s,\tau}^{(n)} 1\!\!1(\rho).$$

The same argument as that leading to Observation 2.4.6 leads to the following corresponding result.

Observation 2.5.2. For every $\tau \in E_\rho^*$ the localized Poincaré series $\eta_\rho(\tau, s)$ converges absolutely uniformly on each set

$$\{s \in \mathbb{C} : \mathrm{Re} s > t\} \, (\subseteq \Delta_\mathcal{S}^+),$$

$t > \delta_\mathcal{S}$, thus defining a holomorphic function on $\Delta_\mathcal{S}^+$.

Our main result about localized Poincaré series, which is crucial to us for obtaining the asymptotic behavior of $N_\rho(\tau, T)$, is the following.

Theorem 2.5.3. *Assume that the finitely irreducible strongly regular conformal GDMS $\mathcal{S}$ is D-generic. If $\tau \in E_A^*$ then*

(a) *the function $\Delta_\mathcal{S}^+ \ni s \longmapsto \eta_\rho(\tau, s) \in \mathbb{C}$ has a meromorphic extension to some neighborhood of the vertical line $\mathrm{Re}(s) = \delta_\mathcal{S}$,*

(b) *this extension has a single pole $s = \delta_\mathcal{S}$, and*

(c) *the pole $s = \delta = \delta_\mathcal{S}$ is simple and its residue is equal to $\frac{\psi_\delta(\rho)}{\chi_{\mu_\delta}} m_\delta([\tau])$.*

Proof. By Observation 2.5.2 and by the Identity Theorem for meromorphic functions, in order to prove the theorem it suffices to do the following.

(1) Show that for every $s_0 = \delta_\mathcal{S} + it_0 \in \Gamma_\mathcal{S}^+$ with $t_0 \neq 0$ the function $\eta_\rho(\tau, \cdot)$ has a holomorphic extension to some open neighborhood of s_0 in $\mathbb{C}$.

(2) Show that the function $\eta_\rho(\tau, \cdot)$ has a meromorphic extension to some open neighborhood of $\delta_\mathcal{S}$ in $\mathbb{C}$ with a simple pole at $\delta_\mathcal{S}$.

(3) Calculate the residue of this extension at the point $s = \delta_\mathcal{S}$ to show that it is equal to $\frac{\psi_\delta(\rho)}{\chi_{\mu_\delta}} m_\delta([\tau])$.

We first deal with item (1). Let $\Lambda \subseteq \mathbb{C}$ be the set of all eigenvalues of the operator $\mathcal{L}_{s_0} : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ whose moduli are equal to 1. By Proposition 2.3.2 this set is finite, and, by Lemma 2.3.4, it consists of only simple eigenvalues. Write

$$\Lambda = \{\lambda_j\}_{j=1}^q,$$

where $q := \#\Lambda$. Then, invoking Observation 2.2.6, Observation 2.3.1, and Proposition 2.3.2 (along with the fact that $\mathrm{P}(\delta_{\mathcal{S}}) = 0$), we see that the Kato–Rellich Perturbation Theorem applies and it produces holomorphic functions

$$U \ni s \mapsto \lambda_j(s) \in \mathbb{C}, \quad j = 1, 2, \ldots, q$$

defined on some sufficiently small neighborhood $U \subseteq \Gamma_{\mathcal{S}}^+$ of s_0 with the following properties for all $j = 1, 2, \ldots, q$:

- $\lambda_j(s_0) = \lambda_j$,
- $\lambda_j(s)$ is a simple isolated eigenvalue of the operator $\mathcal{L}_s : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$

Invoking Proposition 2.3.2 for the third time, we can further write, perhaps with a smaller neighborhood U of s_0, that

$$\mathcal{L}_s = \sum_{j=1}^{q} \lambda_j(s) P_{s,j} + \Delta_s,$$

where

- $P_{s,j} : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ are projections onto respective 1-dimensional spaces $\mathrm{Ker}\big(\lambda_j(s)I - \mathcal{L}_s\big)$,
- all functions $U \ni s \mapsto \Delta_s, P_{s,j}, \ j = 1, 2, \ldots, q$, are holomorphic,
- $r(\Delta_s) \le e^{-\alpha/2}$ for every $s \in U$, and
- $P_{s,i} P_{s,j} = 0$ whenever $i \neq j$ and $\Delta_s P_{s,j} = P_{s,j} \Delta_s = 0$ for all $s \in U$.

In consequence

$$\mathcal{L}_s^n = \sum_{j=1}^{q} \lambda_j^n(s) P_{s,j} + \Delta_s^n \tag{2.1}$$

for all integers $n \ge 0$. Shrinking U again if necessary, we will have that

$$||\Delta_s^n||_\alpha \le C e^{-\frac{\alpha}{3} n}$$

for all integers $n \ge 0$ and some constant $C \in (0, +\infty)$ independent of n. Since the system $\mathcal{S}$ is D-generic, it follows from Proposition 2.3.5 that $\lambda_j(s) \neq 1$ for all $s \in U$ and all $j = 1, 2, \ldots, q$. Denoting by $S_\infty(s)$ the holomorphic function

$$U \ni s \longmapsto \Delta_\infty(s) := \sum_{n=1}^{\infty} \Delta_s^n(|\varphi_\tau'|^s \circ \pi)(\rho)$$

and summing equation (2.1) over all $n \ge 1$, we obtain

$$\eta_\rho(\tau, s) = \sum_{n=1}^{\infty} \mathcal{L}_s^n\big(|\varphi_\tau'|^s \circ \pi\big)(\rho) = \sum_{j=1}^{q} \lambda_j(s)(1 - \lambda_j(s))^{-1} P_{s,j}\big(|\varphi_\tau'|^s \circ \pi\big)(\rho) + \Delta_\infty(s)$$

for all $s \in U \cap \{s \in \mathbb{C} : \mathrm{Re}(s) > \delta_{\mathcal{S}}\}$. But (remembering that $\lambda_j(s) \neq 1$) since, all the terms of the right-hand side of this equation are holomorphic functions from U to $\mathbb{C}$, the formula

$$U \ni s \mapsto \sum_{j=1}^{q} \lambda_j(s)(1 - \lambda_j(s))^{-1} P_{s,j}\big(|\varphi_\tau'|^s \circ \pi\big) + \Delta_\infty(s) \in \mathbb{C}$$

provides the required holomorphic extension of the function $\eta_\rho(\tau, s)$ to a neighborhood of s_0.

Now we shall deal will items (2) and (3). It follows from Theorem 2.2.9 (h) and (i), and the Kato–Rellich Perturbation Theorem that

$$\mathcal{L}_s^n = \lambda_s^n Q_s + S_s^n, \quad n \ge 0, \tag{2.2}$$

for all $s \in U \subseteq \Gamma_{\mathcal{S}}^+$, a sufficiently small neighborhood of δ, where

(4) λ_s is a simple isolated eigenvalues of $\mathcal{L}_s : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ and the function $U \ni s \mapsto \lambda_s \in \mathbb{C}$ is holomorphic,

(5) $Q_s : \mathrm{H}_\alpha(A) \to \mathrm{H}_\alpha(A)$ is a projector onto the 1-dimensional eigenspace of λ_s, and the map $U \ni s \mapsto Q_s \in L(\mathrm{H}_\alpha(A))$ is holomorphic,

(6) $\exists_{\kappa\in(0,1)} \exists_{C>0} \forall_{s\in U} \forall_{n\geq 0}$

$$\|S_s^n\|_\alpha \leq C\kappa^n,$$

and the map $U \ni s \mapsto S_s \in L(\mathrm{H}_\alpha(A))$ is holomorphic, and

(7) All three operators $\mathcal{L}_s$, Q_s, and S_s mutually commute and $Q_s S_s = 0$.

Let us write

$$H_{\tau,s} := Q_s\big(|\varphi_\tau'|^s \circ \pi\big).$$

It follows from (5) that the function $U \ni s \mapsto H_{\tau,s} \in \mathrm{H}_\alpha(A)$ is holomorphic, whence the function valued map $U \ni s \mapsto H_s(\rho) \in \mathbb{C}$ is holomorphic too. It follows from (6) that the series

$$S_\infty(s) := \sum_{n=1}^\infty S_s^n$$

converges absolutely uniformly to a holomorphic function, whence the function $U \ni s \mapsto \Sigma_\infty(s) \in \mathrm{H}_\alpha(A)$ is holomorphic too. Since, by Theorem 2.2.8, the function $s \mapsto \lambda_s$ is not constant on any neighborhood of δ, it follows from (4) that shrinking U if necessary, we will have that

$$\lambda_s \neq 1$$

for all $s \in U \setminus \{\delta\}$. It follows from Theorem 2.2.8, the definition of δ, and Proposition 2.3.2 (1) that

$$|\lambda_s| < 1$$

for all $s \in U \cap \{s \in \mathbb{C} : \mathrm{Re}(s) > \delta_{\mathcal{S}}\}$. It therefore follows from (2.2) that

$$\eta_\rho(s) = \lambda_s(1-\lambda_s)^{-1} H_{\tau,s}(\rho) + S_\infty(s)$$

for all $s \in U \cap \{s \in \mathbb{C} : \mathrm{Re}(s) > \delta_{\mathcal{S}}\}$, and consequently, the map

$$U \ni s \mapsto \lambda_s(1-\lambda_s)^{-1} H_{\tau,s}(\rho) + S_\infty(s) \tag{2.3}$$

is a meromorphic extension of $\eta_\rho(\tau,\cdot)$ to U. We keep the same symbol $\eta_\rho(\tau,s)$ for this extension. Now, using Theorem 2.2.14, we get

$$\begin{aligned}
\lim_{s\searrow\delta} \frac{s-\delta}{1-\lambda_s} &= -\left(\lim_{s\searrow\delta} \frac{\lambda_s - 1}{s-\delta}\right)^{-1} = -\left(\lim_{s\searrow\delta} \frac{\lambda_s - \lambda_\delta}{s-\delta}\right)^{-1} = -\big(\lambda_\delta'\big)^{-1} \\
&= -\left(\frac{d}{ds}\Big|_{s=\delta} e^{\mathrm{P}(s)}\right)^{-1} = -\big(\mathrm{P}'(\delta)e^{\mathrm{P}(\delta)}\big)^{-1} = -(\mathrm{P}'(\delta))^{-1} \\
&= \frac{1}{\chi_{\mu_\delta}}.
\end{aligned}$$

Since $\lambda_\delta = 1$ and

$$H_{\delta,\tau}(\rho) = Q_\delta\big(|\varphi_\tau'|^\delta \circ \pi\big)(\rho) = \left(\int_{E_A^\infty} |\varphi_\tau'|^\delta \circ \pi \, dm_\delta\right) \psi_\delta(\rho) = \psi_\delta(\rho) m_\delta([\tau]),$$

we therefore conclude that

$$\mathrm{res}_\delta\big(\eta_\rho(\tau,\cdot)\big) = \frac{\psi_\delta(\rho)}{\chi_{\mu_\delta}} m_\delta([\tau]).$$

The proof is thus complete. ∎

We can take τ to be the neutral (empty) word and deduce the corresponding results for the original Poincaré series

COROLLARY 2.5.4. *Assume that the finitely irreducible strongly regular conformal GDMS $\mathcal{S}$ is D-generic. Then*

(a) *the function $\eta_\rho(s)$ has a meromorphic extension to some neighborhood of the vertical line* $\mathrm{Re}(s) = \delta_{\mathcal{S}}$*,*

(b) *this extension has a single pole $s = \delta_{\mathcal{S}} = \mathrm{HD}(J_{\mathcal{S}})$, and*

(c) *the pole $s = \delta = \delta_{\mathcal{S}}$ is simple and its residue is equal to $\frac{\psi_\delta(\rho)}{\chi_{\mu_\delta}}$.*

2.6. Asymptotic Results for Multipliers; Concluding of Proofs

We are now in position to complete the proof of Theorem 2.4.8 and then, as its consequence, of Theorem 2.4.9. We aim to apply the Ikehara-Wiener Tauberian Theorem [**93**], which is a familiar ingredient in the classical analytic proof of the Prime Number Theorem in Number Theory.

THEOREM 2.6.1 (Ikehara-Wiener Tauberian Theorem, [**93**]). *Let M and θ be positive real numbers. Assume that $\alpha : [M, +\infty) \to (0, +\infty)$ is monotone increasing and continuous from the left, and also that there exists a (real) number $D > 0$ such that the function*

$$s \longmapsto \int_M^{+\infty} x^{-s} d\alpha(x) - \frac{D}{s-\theta} \in \mathbb{C}$$

is analytic in a neighborhood of $\mathrm{Re}(s) \geq \theta$*. Then*

$$\lim_{x\to+\infty} \frac{\alpha(x)}{x^\theta} = \frac{D}{\theta}.$$

We can now apply this general result in the present setting to prove the asymptotic equidistribution results. We begin with the proof of formula (2.4) in Theorem 2.4.8.

PROOF OF FORMULA (2.4) IN THEOREM 2.4.8. Let $\tau \in E_A^*$ be an arbitrary. We define the function $M_\rho(\tau, \cdot) : [1, +\infty) \to \mathbb{N}_0$ by the formula

$$M_\rho(\tau, T) := N_\rho(\tau, \log T) = \{\tau\omega \in E_\rho^* : |\varphi'_{\tau\omega}(\xi)|^{-1} \leq T\}.$$

We then have for every $s > \delta$ that

$$\eta_\rho(\tau, s) = \int_1^\infty T^{-s} dM_\rho(\tau, T).$$

Now Theorem 2.5.3 tells us that Theorem 2.6.1 applies with the function α being equal to $M_\rho(\tau, \cdot)$ and with $\theta := \delta_{\mathcal{S}}$, to give

$$\lim_{T\to+\infty} \frac{M_\rho(\tau, T)}{T^\delta} = \frac{\psi_\delta(\rho)}{\delta\chi_{\mu_\delta}} m_\delta([\tau]).$$

Consequently

$$\lim_{T\to+\infty} \frac{N_\rho(\tau, T)}{e^{\delta T}} = \lim_{T\to+\infty} \frac{M_\rho(\tau, e^T)}{e^{\delta T}} = \frac{\psi_\delta(\rho)}{\delta\chi_{\mu_\delta}} m_\delta([\tau]). \tag{2.1}$$

This means that (2.4) is proved. $\square$

Now we move onto the proof of (2.5). However the first step to do this is of quite general character and will be also used in Section 2.7. We therefore present it as a separate independent procedure. Fix an integer $q \geq 0$. Let $H \subseteq E_A^\infty$ be a set representable as a (disjoint) union of cylinders of length q. Let

$$\mathcal{R}_{q,H,\rho}(T) := \left\{\omega \in \pi_\rho(T) : |\omega| > q \text{ and } \omega|_{|\omega|-q+1}^{|\omega|} \in H\right\}$$

and the corresponding counting numbers

$$R_{q,H,\rho}(T) := \#\mathcal{R}_{q,H,\rho}(T).$$

We shall prove the following.

LEMMA 2.6.2. *If $q \geq 0$ is an integer and $H \subseteq E_A^\infty$ is a (disjoint) union of cylinders of length q, then the limit below exists and*

$$\lim_{T\to\infty} \frac{R_{q,H,\rho}(T)}{e^{\delta T}} \leq K^{2\delta}(\delta\chi_{\mu_\delta})^{-1} m_\delta(H). \tag{2.2}$$

PROOF. As in the proof of formula (2.4) in Theorem 2.4.8, the Poincaré series corresponding to the counting scheme $\#\mathcal{R}_{q,H,\rho}(T)$ is the function $\hat{\eta}_{H,\rho}(s)$, where for any $\gamma \in E_A^\infty$,

$$\begin{aligned}
\hat{\eta}_{H,\gamma}(s) := \sum_{\substack{\omega\in E_\gamma^*,\ |\omega|\geq q+1 \\ \omega|_{|\omega|-q+1}^{|\omega|}\in H}} |\varphi'_\omega(\pi_\mathcal{S}(\gamma))|^s &= \sum_{n=q+1}^\infty \sum_{\substack{\omega\in E_\gamma^n \\ \omega|_{n-q+1}^n\in H}} |\varphi'_\omega(\pi_\mathcal{S}(\gamma))|^s \\
&= \sum_{n=q+1}^\infty \sum_{\substack{\omega\in E_\gamma^n \\ \sigma^{n-q}(\omega\gamma)\in H}} |\varphi'_\omega(\pi_\mathcal{S}(\gamma))|^s = \sum_{n=q+1}^\infty \sum_{\substack{\omega\in E_\gamma^n \\ \omega\gamma\in\sigma^{-(n-q)}(H}} |\varphi'_\omega(\pi_\mathcal{S}(\gamma))|^s \\
&= \sum_{n=q+1}^\infty \sum_{\omega\in E_\gamma^n} 1\!\!1_H\circ\sigma^{n-q}(\omega\gamma)\cdot|\varphi'_\omega(\pi_\mathcal{S}(\gamma))|^s \\
&= \sum_{n=q+1}^\infty \mathcal{L}_s^n\big(1\!\!1_H\circ\sigma^{n-q}\big)(\gamma) = \sum_{n=q+1}^\infty \mathcal{L}_s^q\big(\mathcal{L}_s^{n-q}\big(1\!\!1_H\circ\sigma^{n-q}\big)\big)(\gamma) \\
&= \sum_{n=q+1}^\infty \mathcal{L}_s^q\big(1\!\!1_{[F_{q,\rho}^c]}\mathcal{L}_s^{n-q}1\!\!1\big)(\gamma) = \mathcal{L}_s^q\left(1\!\!1_H\sum_{n=q+1}^\infty \mathcal{L}_s^{n-q}1\!\!1\right)(\gamma).
\end{aligned}$$

Now, the same reasoning as in the proof of Theorem 2.5.3 shows that the function

$$s \longmapsto \eta_q(s) := \sum_{n=q+1}^\infty \mathcal{L}_s^{n-q}1\!\!1(\gamma)$$

has a meromorphic extension, denoted by the same symbol $\eta_q(s)$, to some neighborhood, call it G, of the vertical line $\mathrm{Re}(s) = \delta_\mathcal{S}$ with only pole at $s = \delta_\mathcal{S}$. This is again a simple pole with residue equal to $\chi_{\mu_\delta}\psi_\delta(\gamma)$. Since the operators $\mathcal{L}_s^q$ are locally uniformly bounded at all points of G, the function

$$s \longmapsto \mathcal{L}_s^q\left(1\!\!1_H\sum_{n=q+1}^\infty \mathcal{L}_s^{n-q}1\!\!1\right)(\gamma)$$

has holomorphic extension, which we will still call $\hat{\eta}_{H,\gamma}(s)$, to $G\setminus\{\delta\}$. In addition

$$\begin{aligned}
\lim_{s\to\delta}(\delta-s)\hat{\eta}_{H,\gamma}(s) &= \mathcal{L}_\delta^q\left(1\!\!1_H\lim_{s\to\delta}(\delta-s)\eta_q(s)\right)(g) = \mathcal{L}_\delta^q\left(1\!\!1_H\chi_{\mu_\delta}^{-1}\psi_\delta\right)(\gamma) \\
&= \chi_{\mu_\delta}^{-1}\mathcal{L}_\delta^q\left(1\!\!1_H\psi_\delta\right)(\gamma) \leq \chi_{\mu_\delta}^{-1}\|\psi_\delta\|_\infty\mathcal{L}_\delta^q\left(1\!\!1_H\right)(\gamma) \\
&\leq K^\delta\chi_{\mu_\delta}^{-1}\mathcal{L}_\delta^q\left(1\!\!1_H\right)(\gamma) \\
&\leq K^{2\delta}\chi_{\mu_\delta}^{-1}m_\delta(H).
\end{aligned}$$

Therefore, we can apply the Ikehara-Wiener Tauberian Theorem (Theorem 2.6.1) in exactly the same way as in the proof of (2.4), to conclude that

$$\lim_{T\to\infty}\frac{R_{q,H,\rho}(T)}{e^{\delta T}}=\frac{\mathrm{res}_\delta(\hat{\eta}_{q,H,\rho})}{\delta}\le K^{2\delta}(\delta\chi_{\mu_\delta})^{-1}m_\delta(H).$$

The proof is complete. □

Proof of formula (2.5) in Theorem 2.4.8. For every $\gamma\in E_A^*$ fix exactly one $\gamma^+\in E_A^\infty$ such that

$$\gamma\gamma^+\in E_A^\infty.$$

Observe that for every integer $q\ge 1$, every $\gamma\in E_A^q$, and every $\omega\in E_A^*$ such that $\gamma\omega\in E_p^*$, we have

$$K_q^{-1}|\varphi'_{\gamma\omega}(\pi(\gamma\gamma^+))|\le|\varphi'_{\gamma\omega}(x_{\gamma\omega})|\le K_q|\varphi'_{\gamma\omega}(\pi(\gamma\gamma^+))|. \tag{2.3}$$

It then follows from (2.3) that

$$\pi_p(\gamma,T)\subseteq\pi_{\gamma\gamma^+}(\gamma,T+\log K_q) \tag{2.4}$$

and

$$\pi_{\gamma\gamma^+}(\gamma,T)\subseteq\pi_p(\gamma,T+\log K_q). \tag{2.5}$$

Let

$$k:=|\tau|.$$

Using (2.5) and applying formula (2.4) of Theorem 2.4.9, we obtain that

$$\begin{aligned}\varliminf_{T\to\infty}\frac{N_p(\tau,T)}{e^{\delta T}}&\ge\varliminf_{T\to\infty}\sum_{\substack{\gamma\in E_A^q\\ \tau\gamma\in E_A^{q+k}}}\frac{N_{\tau\gamma(\tau\gamma)^+}\big(\tau\gamma,T-\log K_{q+k}\big)}{\exp\big(\delta(T-\log K_{q+k})\big)}K_{q+k}^{-\delta}\\
&\ge K_{q+k}^{-\delta}\sum_{\substack{\gamma\in E_A^q\\ \tau\gamma\in E_A^{q+k}}}\varliminf_{T\to\infty}\frac{N_{\tau\gamma(\tau\gamma)^+}\big(\tau\gamma,T-\log K_{q+k}\big)}{\exp\big(\delta(T-\log K_{q+k})\big)}\\
&=K_{q+k}^{-\delta}\frac{1}{\delta\chi_\delta}\sum_{\substack{\gamma\in E_A^q\\ \tau\gamma\in E_A^{q+k}}}\psi_\delta(\tau\gamma(\tau\gamma)^+)m_\delta([\tau\gamma])\\
&\ge K_{q+k}^{-2\delta}\frac{1}{\delta\chi_\delta}\sum_{\substack{\gamma\in E_A^q\\ \tau\gamma\in E_A^{q+k}}}\mu_\delta([\tau\gamma])\\
&=K_{q+k}^{-2\delta}\frac{1}{\delta\chi_\delta}\mu_\delta([\tau]).\end{aligned}$$

Therefore, taking the limit with $q\to\infty$, we obtain

$$\varliminf_{T\to\infty}\frac{N_p(\tau,T)}{e^{\delta T}}\ge\frac{1}{\delta\chi_\delta}\mu_\delta([\tau]). \tag{2.6}$$

Passing to the proof of the upper bound of the limit supremum, we split E_A^q, in a way that will be specified later, into two disjoint sets F_q and its complement $F_q^c:=E_A^q\setminus F_q$ (each of which naturally consists of words of length q) with F_q being finite. In particular,

$$E_A^q=F_q\cup F_q^c.$$

So far we have not imposed any additional hypotheses on the sets F_q and F_q^c. This will be done later in the course of the proof. We set

$$\mathcal{R}_{q,\rho}(T):=\mathcal{R}_{q,F_q^c}(T)$$

and

$$R_{q,\rho}(T) := \#\mathcal{R}_{q,\rho}(T),$$

and note that because of (2.4), we have

$$\begin{aligned}
\pi_p(\tau,T) &= \bigcup_{\substack{\gamma\in F_q\\ \gamma\tau\in E_A^*}} \pi_{\tau\gamma(\tau\gamma)^+}\big(\tau\gamma, T+\log K_{q+k}\big) \cup \bigcup_{\substack{\gamma\in F_q^c\\ \gamma\tau\in E_A^*}} \pi_{\tau\gamma(\tau\gamma)^+}\big(\tau\gamma, T+\log K_{q+k}\big)\\
&\subseteq \bigcup_{\substack{\gamma\in F_q\\ \gamma\tau\in E_A^*}} \pi_{\tau\gamma(\tau\gamma)^+}\big(\tau\gamma, T+\log K_{q+k}\big) \cup \bigcup_{\gamma\in F_q^c} \pi_{\tau\tau^+}\big(\gamma, T+\log K_{q+k}+\log K\big)\\
&= \bigcup_{\substack{\gamma\in F_q\\ \gamma\tau\in E_A^*}} \pi_{\tau\gamma(\tau\gamma)^+}\big(\tau\gamma, T+\log K_{q+k}\big) \cup \mathcal{R}_{q,\tau\tau^+}(T+\log K_{q+k}+\log K).
\end{aligned}$$

Therefore, using finiteness of the set F_q, Theorem 2.4.9, and (2.2), we further obtain

$$\begin{aligned}
\overline{\lim_{T\to\infty}}\frac{N_p(\tau,T)}{e^{\delta T}} &\le \sum_{\substack{\gamma\in F_q\\ \gamma\tau\in E_A^*}} \frac{N_{\tau\gamma(\tau\gamma)^+}\big(\tau\gamma, T+\log K_{q+k}\big)}{\exp\big(\delta(T+\log K_{q+k})\big)} K_{q+k}^\delta + \overline{\lim_{T\to\infty}}\frac{R_{q,\tau\tau^+}(T+\log K_{q+k}+\log K)}{e^{\delta T}}\\
&\le K_{q+k}^\delta\frac{1}{\delta\chi_\delta}\sum_{\substack{\gamma\in F_q\\ \gamma\tau\in E_A^*}} \psi_\delta(\tau\gamma(\tau\gamma)^+)m_\delta([\tau\gamma]) + K^{3\delta}K_{q+k}^\delta\frac{1}{\delta\chi_\delta}m_\delta\big([F_q^c]\big)\\
&\le K_{q+k}^{2\delta}\frac{1}{\delta\chi_\delta}\sum_{\substack{\gamma\in E_A^q\\ \tau\gamma\in E_A^{q+k}}} \mu_\delta([\tau\gamma]) + K^{3\delta}K_{q+k}^\delta\frac{1}{\delta\chi_\delta}m_\delta\big([F_q^c]\big)\\
&\le K_{q+k}^{2\delta}\frac{1}{\delta\chi_\delta}\mu_\delta([\tau]) + K^{3\delta}K_{q+k}^\delta\frac{1}{\delta\chi_\delta}m_\delta\big([F_q^c]\big).
\end{aligned}$$

Hence, taking finite sets $F_{q,\rho}$ with $m_\delta\big([F_{q,\rho}]\big)$ converging to one, so that $m_\delta\big([F_{q,\rho}^c]\big)$ converges to zero, we obtain

$$\overline{\lim_{T\to\infty}}\frac{N_p(\tau,T)}{e^{\delta T}} \le K_{q+k}^{2\delta}\frac{1}{\delta\chi_\delta}\mu_\delta([\tau]).$$

Therefore, taking the limit with $q\to\infty$, we obtain

$$\overline{\lim_{T\to\infty}}\frac{N_p(\tau,T)}{e^{\delta T}} \le \frac{1}{\delta\chi_\delta}([\tau]).$$

Along with (2.6) this yields

$$\lim_{T\to\infty}\frac{N_p(\tau,T)}{e^{\delta T}} = \frac{1}{\delta\chi_\delta}\mu_\delta([\tau]). \tag{2.7}$$

The proof of formula (2.5) in Theorem 2.4.8 is thus complete. This simultaneously finishes the proof of all of Theorem 2.4.8 $\square$

Proof of Theorem 2.4.9. The same proof, as a consequence of Theorem 2.4.8 goes through for $i=\rho$ and $i=p$. We therefore denote

$$C_i := \begin{cases} \frac{1}{\delta\chi_\delta}\psi_\delta(\rho) & \text{if } \ i=\rho\\ \frac{1}{\delta\chi_\delta} & \text{if } \ i=p,\end{cases}$$

$$\nu_i := \begin{cases} m_\delta & \text{if } \ i=\rho\\ \mu_d & \text{if } \ i=p\end{cases} \quad\text{and}\quad \widetilde{\nu}_i := \begin{cases} \widetilde{m}_\delta & \text{if } \ i=\rho\\ \widetilde{\mu}_\delta & \text{if } \ i=p.\end{cases}$$

We shall first prove both formulae (2.6) and (2.7) for all sets B that are open. To emphasize this, let us denote an arbitrary open subset of X by V. We assume that $\widetilde{\nu}_i(\partial V)=0$. Then for every $s\in(0,1)$ there exists a finite set $\Gamma_s(V)$ consisting of mutually incomparable elements of E_A^* such that

$$\bigcup_{\tau\in\Gamma_s(V)}\varphi_\tau\big(X_{t(\tau)}\big)\subseteq V \quad\text{ and }\quad \nu_i\left(\bigcup_{\tau\in\Gamma_s(V)}[\tau]\right)=\widetilde{\nu}_i\left(\bigcup_{\tau\in\Gamma_s(V)}\varphi_\tau\big(X_{t(\tau)}\big)\right)\ge s\widetilde{\nu}_i(V)$$

where the "=" sign in this formula is due to (2.8). So, for both $i=\rho,p$, using (2.1), we get that

$$\begin{aligned}\varliminf_{T\to+\infty}\frac{N_i(V,T)}{e^{\delta T}}&\ge\sum_{\tau\in\Gamma_\kappa(V)}\varliminf_{T\to+\infty}\frac{N_i(\tau,T)}{e^{\delta T}}=\sum_{\tau\in\Gamma_\kappa(V)}C_i\nu_i([\tau])\\&=C_i\nu_i\left(\bigcup_{\tau\in\Gamma_\kappa(V)}[\tau]\right)\\&\ge sC_i\widetilde{\nu}_i(V).\end{aligned}$$

Letting $s\nearrow 1$, we thus obtain

$$\varliminf_{T\to+\infty}\frac{N_i(V,T)}{e^{\delta T}}\ge C_i\widetilde{\nu}_i(V). \tag{2.8}$$

Therefore, we also have

$$\varliminf_{T\to+\infty}\frac{N_i(\overline{V}^c,T)}{e^{\delta T}}\ge C_i\widetilde{\nu}_i(\overline{V}^c). \tag{2.9}$$

But since $\nu_i(\partial V)=0$, we have $\nu_i(V)+\nu_i(\overline{V}^c)=1$, whence

$$\varliminf_{T\to+\infty}\frac{N_i(\overline{V}^c,T)}{e^{\delta T}}\ge C_i(1-\widetilde{\nu}_i(V)). \tag{2.10}$$

Therefore, using (2.1) and (2.7), both with τ replaced by $E_A^{\mathbb{N}}$, we get

$$\begin{aligned}C_i=\lim_{T\to+\infty}\frac{N_i(T)}{e^{\delta T}}&\ge\varlimsup_{T\to+\infty}\frac{N_i(V,T)+N_i(\overline{V}^c,T)}{e^{\delta T}}\\&\ge\varlimsup_{T\to+\infty}\frac{N_i(V,T)}{e^{\delta T}}+\varliminf_{T\to+\infty}\frac{N_i(\overline{V}^c,T)}{e^{\delta T}}\\&\ge\varlimsup_{T\to+\infty}\frac{N_i(V,T)}{e^{\delta T}}+C_i\big(1-\widetilde{\nu}_i(V)\big).\end{aligned} \tag{2.11}$$

Thus,

$$\varlimsup_{T\to+\infty}\frac{N_i(V,T)}{e^{\delta T}}\le C_i\widetilde{\nu}_i(V).$$

Along with (2.8) this implies

$$\lim_{T\to+\infty}\frac{N_i(V,T)}{e^{\delta T}}=C_i\widetilde{\nu}_i(V). \tag{2.12}$$

Finally, let B be an arbitrary Borel subset of X such that $\widetilde{\nu}_i(\partial B)=0$. Then $\overline{B}=B\cup\partial B$ and

$$\widetilde{\nu}_i(\overline{B})=\widetilde{\nu}_i(B).$$

Since the measure ν_i is outer regular, given $\varepsilon>0$ there exists an open set $G\subseteq X$ such that $B\subseteq G$ and

$$\widetilde{\nu}_i(G)\le\widetilde{\nu}_i(B)+\varepsilon. \tag{2.13}$$

Now, for every $x \in \overline{B}$ there exists an open set $V_x \subseteq G$, in fact an open ball centered at x, such that $x \in V_x$ and

$$\widetilde{\nu}_i(\partial V_x) = 0.$$

In particular, $\{V_x\}_{x\in\overline{B}}$ is a open cover of $\overline{B}$. Since $\overline{B}$ is compact, there thus exists a finite set $F \subseteq \overline{B}$ such that

$$\overline{B} \subseteq V := \bigcup_{x\in F} V_x \subseteq G.$$

Since F is finite, $\partial V \subseteq \bigcup_{x\in F} \partial V_x$, whence $\nu_i(\partial V) = 0$. Therefore, (2.12) applies to V to give

$$\begin{aligned}\overline{\lim_{T\to+\infty}} \frac{N_i(B,T)}{e^{\delta T}} &\le \overline{\lim_{T\to+\infty}} \frac{N_i(\overline{B},T)}{e^{\delta T}} \le \lim_{T\to+\infty} \frac{N_i(V,T)}{e^{\delta T}} = C_i\nu_i(V) \\ &\le C_i\widetilde{\nu}_i(G) \\ &\le C_i(\widetilde{\nu}_i(B) + \varepsilon).\end{aligned}$$

Letting $\varepsilon \searrow 0$, we therefore get

$$\overline{\lim_{T\to+\infty}} \frac{N_i(B,T)}{e^{\delta T}} \le C_i\widetilde{\nu}_i(B). \tag{2.14}$$

Now, we can finish the argument in the same way as in the case of open sets. Since $\partial B^c = \partial B$, we have $m_\delta(\partial B^c) = 0$. In particular, (2.14) also yields

$$\overline{\lim_{T\to+\infty}} \frac{N_i(B^c,T)}{e^{\delta T}} \le C_i\widetilde{\nu}_i(B^c) = C_i(1 - \widetilde{\nu}_i(B)).$$

Therefore, using Theorem 2.4.3 we can write

$$\begin{aligned}C_i = \lim_{T\to+\infty} \frac{N_i(T)}{e^{\delta T}} &= \lim_{T\to+\infty} \frac{N_i(B,T) + N_i(B^c,T)}{e^{\delta T}} \\ &\le \underline{\lim_{T\to+\infty}} \frac{N_i(B,T)}{e^{\delta T}} + \overline{\lim_{T\to+\infty}} \frac{N_i(B^c,T)}{e^{\delta T}} \\ &\le \underline{\lim_{T\to+\infty}} \frac{N_i(B,T)}{e^{\delta T}} + C_i\big(1 - \widetilde{\nu}_i(B)\big).\end{aligned}$$

Thus,

$$\underline{\lim_{T\to+\infty}} \frac{N_i(B,T)}{e^{\delta T}} \ge C_i\widetilde{\nu}_i(B).$$

Along with (2.14) this gives

$$\lim_{T\to+\infty} \frac{N_i(B,T)}{e^{\delta T}} = C_i\widetilde{\nu}_i(B),$$

and the proof of the theorem is complete. □

2.7. Asymptotic Results for Diameters

In this section we obtain asymptotic counting properties corresponding to the functions

$$-\log \operatorname{diam}\big(\varphi_\omega(X_{t(\omega)})\big), \quad \omega \in E_A^*.$$

These are relatively simple consequences of Theorem 2.4.9, but not quite so simple as one would expect. The subtle difficulty is due to the fact that the functions $N_i(B,T)$, $i = \rho, p$ are very sensitive to additive changes. In fact it follows from Theorem 2.4.9 that for every $u > 0$,

$$\lim_{T\to\infty} \frac{N_i(B,T+u)}{N_i(B,T)} = e^{\delta u} > 0.$$

In fact we will do something more general, namely for every $v \in V$ we fix an arbitrary set $Y_v \subseteq X_v$, having at least two points, and we look at asymptotic counting properties corresponding to the functions

$$-\log \operatorname{diam}\big(\varphi_\omega(Y_{t(\omega)})\big), \quad \omega \in E_A^*.$$

Such a generalization is interesting in its own right, but will turn out to be particularly useful when dealing with asymptotic counting properties for diameters in the context of parabolic GDMSs, see Section 3.4.

So, again $\mathcal{S}$ is a finitely irreducible conformal GDMS, we fix $\rho \in E_A^\infty$ and put $\xi = \pi_{\mathcal{S}}(\rho)$. We furthermore fix

$$Y \subseteq X_{i(\rho_1)}.$$

We denote

$$\Delta(\omega) = \Delta_Y(\omega) := -\log \operatorname{diam}\big(\varphi_\omega(Y)\big), \quad \omega \in E_\rho^*,$$

with the natural convention that for $\omega = \varepsilon$, being the empty (neutral) word:

$$\Delta_Y(\varepsilon) = -\log \operatorname{diam}(Y),$$

and further, for any $B \subseteq X$ and $T > 0$,

$$\mathcal{D}_Y^\rho(B,T) := \mathcal{D}^\rho(B,T) := \{\omega \in E_\rho^* : \Delta_Y(\omega) \le T \text{ and } \varphi_\omega(\xi) \in B\},$$

$$D_Y^\rho(B,T) := \#\mathcal{D}_Y^\rho(B,T).$$

The main result of this section is the following.

THEOREM 2.7.1. *Suppose that $\mathcal{S}$ is a strongly regular finitely irreducible conformal D-generic GDMS. Let $\delta = \delta_{\mathcal{S}} = \mathrm{HD}(J_{\mathcal{S}})$. Fix $\rho \in E_A^\infty$ and $Y \subseteq X_{i(\rho)}$ having at least two points.*

If $B \subset X$ is a Borel set such that $\widetilde{m}_\delta(\partial B) = 0$ (equivalently $\widetilde{\mu}_\delta(\partial B) = 0$) then,

$$\lim_{T \to +\infty} \frac{D_Y^\rho(B,T)}{e^{\delta T}} = C_\rho(Y)\widetilde{m}_\delta(B), \tag{2.1}$$

where $C_\rho(Y) \in (0,+\infty)$ is a constant depending only on the system $\mathcal{S}$, the word ρ (but see Remark 2.7.5), and the set Y. In addition

$$K^{-2\delta}(\delta\chi_\delta)^{-1}\operatorname{diam}^\delta(Y) \le C_\rho(Y) \le K^{2\delta}(\delta\chi_\delta)^{-1}\operatorname{diam}^\delta(Y). \tag{2.2}$$

We first shall prove the following auxiliary result. It is trivial in the case of finite alphabet E but requires an argument in the infinite case.

LEMMA 2.7.2. *With the hypotheses of Theorem 2.7.1, for every integer $q \ge 1$ let*

$$\pi_i^{(q)}(B,T) := \pi_i(B,T) \cap E_A^q, \quad i = \rho, p,$$

and

$$N_i^{(q)}(B,T) := \#\pi_i^{(q)}(B,T).$$

Then

$$\lim_{T \to \infty} \frac{N_i^{(q)}(B,T)}{e^{\delta T}} = 0.$$

PROOF. Since $N_i^{(q)}(B,T) \le N_i^{(q)}(T) := N_i^{(q)}(X,T)$, it suffices to prove that

$$\lim_{T \to \infty} \frac{N_i^{(q)}(T)}{e^{\delta T}} = 0.$$

By considering the iterate $\mathcal{S}^q$ of $\mathcal{S}$ it is further evident that it suffices to show that

$$\lim_{T \to \infty} \frac{N_i^{(1)}(T)}{e^{\delta T}} = 0.$$

To see this consider the Poincaré series

$$s \longmapsto \eta_\rho^{(1)}(s) := \mathcal{L}_s 1\!\!1(\rho),$$

notice that it is holomorphic throughout $\{s \in \mathbb{C} : \mathrm{Re}(s) > \gamma_{\mathcal{S}}\} \supseteq \overline{\Delta_{\mathcal{S}}^+}$, and conclude the proof with the help of the Ikehara-Wiener Tauberian Theorem (Theorem 2.6.1), in the same way as in the proof of Theorem 2.4.3. □

Denote also

$$\mathcal{D}^{(\rho,q)}(B,T) := \mathcal{D}^\rho(B,T) \cap E_\rho^q = \mathcal{D}^\rho(B,T) \cap E_A^q.$$

By (BDP)

$$N_i^{(\rho,q)}(B, T - \log K) \leq D^{(\rho,q)}(B,T) \leq N_i^{(\rho,q)}(B, T + \log K).$$

Therefore, as an immediate consequence of Lemma 2.7.2, we get the following.

Corollary 2.7.3. *With the hypotheses of Theorem 2.7.1, for every integer $q \geq 1$, we have*

$$\lim_{T\to\infty} \frac{D^{(\rho,q)}(B,T)}{e^{\delta T}} = 0.$$

Now we can turn to the actual proof of Theorem 2.7.1.

Proof of Theorem 2.7.1. Fix an integer $q \geq 0$ and define:

$$K_q := \sup\left\{ \frac{|\varphi_\omega'(y)|}{|\varphi_\omega'(x)|} : \tau \in E_A^q,\ x, y \in \mathrm{Conv}\big(\varphi_\tau(X_{t(\tau)})\big),\ \omega \in E_\tau^* \right\} \geq 1,$$

where $\mathrm{Conv}(F)$ is the convex hull of a set $F \subseteq \mathbb{R}^d$. In particular $K_0 = K$, the distortion constant of the system $\mathcal{S}$. (BDP) yields

$$\lim_{q\to\infty} K_q = 1. \tag{2.3}$$

(BDP) again, along with the Mean Value Theorem, imply that for all $\tau \in E_\rho^*$ and all $\omega \in E_\tau^*$, we have that

$$\mathrm{diam}\big(\varphi_{\omega\tau}(Y)\big) = \mathrm{diam}\big(\varphi_\omega\big(\varphi_\tau(Y)\big)\big) \leq K_q |\varphi_\omega'(\varphi_\tau(\xi))| \mathrm{diam}(\varphi_\tau(Y))$$

and

$$\mathrm{diam}\big(\varphi_{\omega\tau}(Y)\big) \geq K_q^{-1} |\varphi_\omega'(\varphi_\tau(\xi))| \mathrm{diam}(\varphi_\tau(Y)).$$

Equivalently

$$\lambda_{\tau\rho}(\omega) + \Delta_Y(\tau) - \log K_q \leq \Delta_Y(\omega\tau) \leq \lambda_{\tau\rho}(\omega) + \Delta_Y(\tau) + \log K_q. \tag{2.4}$$

Denote

$$\mathcal{D}_\tau^\rho(B,T) := \{\omega \in E_\tau^* : \omega\tau \in \mathcal{D}^\rho(B,T)\}$$

and

$$D_\tau^\rho(B,T) := \#\mathcal{D}_\tau^\rho(B,T).$$

Formula (2.4) then gives

$$\pi_{\tau\rho}(B,T) \subseteq \mathcal{D}_\tau^\rho(B, T + \Delta_Y(\tau) + \log K_q) \tag{2.5}$$

and

$$\mathcal{D}_\tau^\rho(B,T) \subseteq \pi_{\tau\rho}(B, T - \Delta_Y(\tau) + \log K_q). \tag{2.6}$$

The former equation is equivalent to

$$\mathcal{D}_\tau^\rho(B,T) \supseteq \pi_{\tau\rho}(B, T - \Delta_Y(\tau) - \log K_q).$$

This formula and (2.6) yield

$$N_{\tau\rho}\big(B, T - \Delta_Y(\tau) - \log K_q\big) \leq D_\tau^\rho(B,T) \leq N_{\tau\rho}\big(B, T - \Delta_Y(\tau) + \log K_q\big). \tag{2.7}$$

since

$$\mathcal{D}^\rho(B,T) = \bigcup_{\tau\in E_\rho^q} \mathcal{D}_\tau^\rho(B,T)\tau \cup \bigcup_{j=0}^{q} \mathcal{D}^{(\rho,j)}(B,T) \tag{2.8}$$

and since all the terms in this union are mutually disjoint, formula (2.8) yields

$$D^\rho(B,T) \ge \sum_{\tau\in E_\rho^q} D_\tau^\rho(B,T).$$

By inserting it into formula (2.7), we get

$$D^\rho(B,T) \ge \sum_{\tau\in E_\rho^q} N_{\tau\rho}\big(B, T-\Delta_Y(\tau)-\log K_q\big).$$

Therefore,

$$\begin{aligned}
\frac{D^\rho(B,T)}{e^\delta T} &\ge \sum_{\tau\in E_\rho^q} \frac{N_{\tau\rho}\big(B,T-\Delta_Y(\tau)-\log K_q\big)}{\exp\big(\delta(T-\Delta_Y(\tau)-\log K_q)\big)} \cdot \frac{\exp\big(\delta(T-\Delta_Y(\tau)-\log K_q)\big)}{e^\delta T} \\
&= \sum_{\tau\in E_\rho^q} \frac{N_{\tau\rho}\big(B,T-\Delta_Y(\tau)-\log K_q\big)}{\exp\big(\delta(T-\Delta_Y(\tau)-\log K_q)\big)} K_q^{-\delta} e^{-\delta\Delta_Y(\tau)} \\
&= K_q^{-\delta} \sum_{\tau\in E_\rho^q} \frac{N_{\tau\rho}\big(B,T-\Delta_Y(\tau)-\log K_q\big)}{\exp\big(\delta(T-\Delta_Y(\tau)-\log K_q)\big)} e^{-\delta\Delta_Y(\tau)}.
\end{aligned}$$

Hence, applying Theorem 2.4.9, we get

$$\begin{aligned}
\varliminf_{T\to\infty} \frac{D^\rho(B,T)}{e^{\delta T}} &\ge K_q^{-\delta} \sum_{\tau\in E_\rho^q} e^{-\delta\Delta_Y(\tau)} \varliminf_{T\to\infty} \frac{N_{\tau\rho}\big(B,T-\Delta_Y(\tau)-\log K_q\big)}{\exp\big(\delta(T-\Delta_Y(\tau)-\log K_q)\big)} \\
&\ge K_q^{-\delta} \sum_{\tau\in E_\rho^q} e^{-\delta\Delta_Y(\tau)} (\chi_\delta\delta)^{-1}\psi_\delta(\tau\rho) m_\delta(B) \\
&= (\chi_\delta\delta)^{-1} m_\delta(B) K_q^{-\delta} \sum_{\tau\in E_\rho^q} e^{-\delta\Delta_Y(\tau)} \psi_\delta(\tau\rho).
\end{aligned} \tag{2.9}$$

This is a good enough lower bound for us but getting a sufficiently good upper bound is more subtle. As in the proof of formula (2.5) in Theorem 2.4.8, we split E_A^q, at the moment arbitrarily, into two disjoint sets F_q and its complement $F_q^c := E_A^q \setminus F_q$ (each of which naturally consists of words of length q) with F_q being finite. In particular,

$$E_A^q = F_q \cup F_q^c.$$

So far we do not require anything more from the sets F_q and F_q^c. We will make specific choices later in the course of the proof. We are now primarily interested in the sets

$$\mathcal{R}_{q,\rho}(T) := \mathcal{R}_{q,[F_q^c],\rho}(T) = \big\{\omega\in\pi_\rho(T) : |\omega| > q \text{ and } \omega|_{|\omega|-q+1}^{|\omega|} \in F_{q,\rho}^c\big\}$$

and the corresponding counting numbers

$$R_{q,\rho}(T) := \#\mathcal{R}_{q,\rho}(T).$$

We are interested in estimating from above, the upper limit

$$\varlimsup_{T\to\infty} \frac{D_Y^\rho(B,T)}{e^{\delta T}}.$$

First of all, Lemma 2.6.2 yields

$$\lim_{T\to\infty}\frac{R_{q,\rho}(T)}{e^{\delta T}} \le K^{2\delta}\delta^{-1}\chi_{\mu_\delta} m_\delta\big([F_q^c]\big). \tag{2.10}$$

Denote now

$$\mathcal{R}^*_{q,\rho}(T) := \big\{\omega\in\mathcal{D}^\rho(T) : |\omega|>q \text{ and } \omega|_{|\omega|-q+1}^{|\omega|}\in F_q^c\big\}$$

and the corresponding counting numbers

$$R^*_{q,\rho}(T) := \#\mathcal{R}^*_{q,\rho}(T).$$

It follows from (2.4), applied with τ being empty (neutral) word, that

$$\mathcal{R}^*_{q,\rho}(T)\subseteq\mathcal{R}_{q,\rho}\big(T+\log\Delta_Y(\varepsilon)+\log K\big).$$

Along with (2.2) this yields

$$\varlimsup_{T\to\infty}\frac{R^*_{q,\rho}(T)}{e^{\delta T}} \le K^{3\delta}\delta^{-1}\chi_{\mu_\delta} m_\delta\Delta_Y(\varepsilon)\big([F_q^c]\big).$$

Now we write

$$\bigcup_{\tau\in E^q_\rho}\mathcal{D}^\rho_\tau(B,T)\tau = \bigcup_{\tau\in F_q\cap E^q_\rho}\mathcal{D}^\rho_\tau(B,T)\tau\cup\mathcal{R}^*_{q,\rho}(T).$$

Together with (2.8) and (2.7) this yields

$$\begin{aligned}
D^\rho_Y(B,T) &\le \sum_{\tau\in F_{q,\rho}\cap E^q_\rho} D^\rho_\tau(B,T)\tau + R^*_{q,\rho}(T) + \sum_{j=0}^q D^{(\rho,j)}(B,T)\\
&\le \sum_{\tau\in F_{q,\rho}\cap E^q_\rho} N_{\tau\rho}(B,T-\Delta_Y(\tau)+\log K_q) + R^*_{q,\rho}(T) + \sum_{j=0}^q D^{(\rho,j)}(B,T).
\end{aligned}$$

Hence, invoking also Corollary‘2.7.3 and finiteness of the set $F_{q,\rho}$, we get

$$\begin{aligned}
\varlimsup_{T\to\infty}\frac{D^\rho_Y(B,T)}{e^{\delta T}} &\le K_q^\delta\sum_{\tau\in F_{q,\rho}\cap E^q_\rho} e^{-\Delta_Y(\tau)}\varlimsup_{T\to\infty}\frac{N_{\tau\rho}(B,T-\Delta_Y(\tau)+\log K_q)}{\exp\big(\delta(T-\Delta_Y(\tau)+\log K_q)\big)} + \varlimsup_{T\to\infty}\frac{R^*_{q,\rho}(T)}{e^{\delta T}}\\
&\le (\chi_\delta\delta)^{-1}m_\delta(B)K_q^\delta\sum_{\tau\in E^q_\rho}e^{-\delta\Delta_Y(\tau)}\psi_\delta(\tau\rho) + K^{3\delta}(\delta\chi_\delta)^{-1}\Delta_Y(\varepsilon)m_\delta\big([F_q]\big).
\end{aligned} \tag{2.11}$$

Hence, taking finite sets $F_{q,\rho}$ with $m_\delta\big([F_{q,\rho}]\big)$ converging to one, with $m_\delta\big([F^c_{q,\rho}]\big)$ converging to zero, we obtain

$$\varlimsup_{T\to\infty}\frac{D^\rho_Y(B,T)}{e^{\delta T}} \le K_q^\delta(\chi_\delta\delta)^{-1}m_\delta(B)\sum_{\tau\in E^q_\rho}e^{-\delta\Delta_Y(\tau)}\psi_\delta(\tau\rho). \tag{2.12}$$

Since

$$\psi_\delta(\rho)=\mathcal{L}^q_\delta\psi_\delta(\rho)\preceq\sum_{\tau\in E^q_\rho}e^{-\delta\Delta_Y(\tau)}\psi_\delta(\tau\rho)\preceq\mathcal{L}^q_\delta\psi_\delta(\rho)=\psi_\delta(\rho),$$

we conclude from (2.9) and (2.12) that both $\varliminf_{T\to\infty}\frac{D^\rho_Y(B,T)}{e^{\delta T}}$ and $\varlimsup_{T\to\infty}\frac{D^\rho_Y(B,T)}{e^{\delta T}}$ are finite and positive numbers. Furthermore, we conclude from these same two formulae that for every $q\ge 1$,

$$1\le\frac{\varlimsup_{T\to\infty}\frac{D^\rho_Y(B,T)}{e^{\delta T}}}{\varliminf_{T\to\infty}\frac{D^\rho_Y(B,T)}{e^{\delta T}}}\le K_q^{2\delta}.$$

Formula (2.3) then yields that the limit $\lim_{T\to\infty}\frac{D_Y^\rho(B,T)}{e^{\delta T}}$ exists and is finite and positive. Invoking (2.9) and (2.12) again along with (2.3), we thus deduce the limit

$$\lim_{q\to\infty}\sum_{\tau\in E_\rho^q} e^{-\delta\Delta_Y(\tau)}\psi_\delta(\tau\rho)$$

also exists, is finite and positive. Denoting this limit by $C'_{\mathcal{S}}$, we thus conclude that

$$\lim_{T\to\infty}\frac{D_Y^\rho(B,T)}{e^{\delta T}}=\frac{1}{\delta\chi_\delta}C'_{\mathcal{S}}m_\delta(B),$$

and so, in order to complete the proof of Theorem 2.7.1, we only need to estimate $C'_{\mathcal{S}}$. Indeed,

$$\begin{aligned}\sum_{\tau\in E_\rho^q} e^{-\delta\Delta_Y(\tau)}\psi_\delta(\tau\rho)&=\sum_{\tau\in E_\rho^q}\mathrm{diam}^\delta\big(\varphi_\tau(Y)\big)\psi_\delta(\tau\rho)\le\sum_{\tau\in E_\rho^q}\|\varphi'_\tau\|_\infty^\delta\mathrm{diam}^\delta(Y)\psi_\delta(\tau\rho)\\&\le K^\delta\mathrm{diam}^\delta(Y)\sum_{\tau\in E_\rho^q}|\varphi'_\tau(\pi_{\mathcal{S}}(\rho))|^\delta\psi_\delta(\tau\rho)\\&=K^\delta\psi_\delta(\rho)\mathrm{diam}^\delta(Y)\\&\le K^{2\delta}\mathrm{diam}^\delta(Y),\end{aligned}$$

and similarly,

$$\sum_{\tau\in E_\rho^q} e^{-\delta\Delta_Y(\tau)}\psi_\delta(\tau\rho)\ge K^{-2\delta}\mathrm{diam}^\delta(Y).$$

The proof is complete. □

We can now consider a slightly different approach to counting diameters. Still keeping $\rho\in E_A^\infty$, $Y\subseteq X_{i(\rho_1)}$, a set $B\subseteq X$, and $T>0$, we define:

$$\mathcal{E}_Y^\rho(B,T):=\{\omega\in E_\rho^*:\ \Delta_Y(\omega)\le T\ \text{ and }\ \varphi_\omega(Y)\cap B\ne\emptyset\}$$

and

$$E_Y^\rho(B,T):=\#\mathcal{E}_Y^\rho(B,T).$$

THEOREM 2.7.4. *Suppose that $\mathcal{S}$ is a strongly regular finitely irreducible conformal D-generic GDMS. Let $\delta=\delta_{\mathcal{S}}=\mathrm{HD}(J_{\mathcal{S}})$. Fix $\rho\in E_A^\infty$ and $Y\subseteq X_{i(\rho)}$ having at least two points and such that $\pi_{\mathcal{S}}(\rho)\in Y$.*

If $B\subset X$ is a Borel set such that $\widetilde{m}_\delta(\partial B)=0$ (equivalently $\widetilde{\mu}_\delta(\partial B)=0$) then,

$$\lim_{T\to+\infty}\frac{E_Y^\rho(B,T)}{e^{\delta T}}=C_\rho(Y)\widetilde{m}_\delta(B),\tag{2.13}$$

where $C_\rho(Y)\in(0,+\infty)$ is a constant, in fact the one produced in Theorem 2.7.1, depending only on the system $\mathcal{S}$, the word ρ (but see Remark 2.7.5), and the set Y. In addition

$$K^{-2\delta}(\delta\chi_\delta)^{-1}\mathrm{diam}^\delta(Y)\le C_\rho(Y)\le K^{2\delta}(\delta\chi_\delta)^{-1}\mathrm{diam}^\delta(Y).\tag{2.14}$$

PROOF. Since $\pi_{\mathcal{S}}(\rho)\in Y$ we have that

$$D_Y^\rho(B,T)\le E_Y^\rho(B,T).$$

It therefore follows from Theorem 2.7.1 that

$$\liminf_{T\to+\infty}\frac{E_Y^\rho(B,T)}{e^{\delta T}}\ge C_\rho(Y)\widetilde{m}_\delta(B).\tag{2.15}$$

Since $\mathcal{E}_Y^\rho(T)=\mathcal{E}_Y^\rho(X,T)=\mathcal{D}_Y^\rho(T)$, Theorem 2.7.1, also yields

$$\lim_{T\to+\infty}\frac{E_Y^\rho(T)}{e^{\delta T}}=C_S(Y).\tag{2.16}$$

Now fix $(\epsilon_n)_{n=1}^{\infty}$, a sequence of positive numbers converging to zero such that for all $n \geq 1$

$$\widetilde{m}_\delta(\partial B(B,\epsilon_n)) = 0.$$

Then $\widetilde{m}_\delta(\partial B^c(B,\epsilon_n)) = 0$ and $\varphi_\omega(Y)$ intersects at most one of the sets B or $B^c(B;\epsilon_k) \cap B^c$ if $\Delta_Y(\omega) \geq \log(1/\epsilon_n)$. Hence applying formula (2.15) to the sets $B^c(B,\epsilon_n) \cap B^c$ and using (2.16) we get for every $n \geq 1$ that

$$\begin{aligned}
C_\rho(Y) &\geq \limsup_{T\to+\infty} \frac{E_Y^\rho(B,T) + E_Y^\rho(B^c(B,\epsilon_n),T)}{e^{\delta T}} \\
&\geq \limsup_{T\to+\infty} \frac{E_Y^\rho(B,T)}{e^{\delta T}} + \liminf_{T\to+\infty} \frac{E_Y^\rho(B^c(B,\epsilon_n),T)}{e^{\delta T}} \\
&\geq \limsup_{T\to+\infty} \frac{E_Y^\rho(B,T)}{e^{\delta T}} + C_\rho(Y)\widetilde{m}_\delta(B^c(B,\epsilon_n)).
\end{aligned}$$

But $\lim_{n\to+\infty} \widetilde{m}_\delta(B^c(B,\epsilon_n)) = \widetilde{m}_\delta(B^c) = 1 - m_\delta(B)$, (remembering that $\widetilde{m}_\delta(\partial B) = 0$), and therefore

$$C_\rho(Y) \geq \limsup_{T\to+\infty} \frac{E_Y^\rho(B,T)}{e^{\delta T}} + C_\rho(Y)(1 - m_\delta(B)).$$

Hence

$$\limsup_{T\to+\infty} \frac{E_Y^\rho(B,T)}{e^{\delta T}} \leq C_\rho(Y) m_\delta(B).$$

Along with (2.15) this finishes the proof of the first part of the theorem. The second part, i.e. (2.14), is just formula (2.2). $\square$

Remark 2.7.5. Since the left-hand side of (2.13) depends only on ρ_1, i.e. the first coordinate of ρ, we obtain that the constant $C_Y(\rho)$ of Theorem 2.7.4 and Theorem 2.7.1, also depends in fact only on ρ_1. We could have provided a direct argument of this already when proving Theorem 2.7.1 and this would not affect the proof of Theorem 2.7.4. However, our approach seems to be most economical.

We say that a graph directed Markov system $\mathcal{S}$ has the property (A) if for every vertex $v \in V$ there exists $a_v \in E$ such that

$$i(a_v) = v$$

and

$$A_{ea_v} = 1$$

whenever $t(e) = v$. As an immediate consequence of Theorem 2.7.1, Theorem 2.7.4 and Remark 2.7.5, we get the following.

Theorem 2.7.6. *Suppose that $\mathcal{S}$ is a strongly regular finitely irreducible D-generic conformal GDMS with property (A). Let $\delta = \delta_{\mathcal{S}} = \mathrm{HD}(J_{\mathcal{S}})$. For any $v \in V$ let $Y_v \subseteq X_v$ having at least two points fixed.*

If $B \subset X$ is a Borel set such that $\widetilde{m}_\delta(\partial B) = 0$ (equivalently $\widetilde{\mu}_\delta(\partial B) = 0$) and $\rho \in E_A^\infty$ is with $\rho_1 = a_v$, then,

$$\lim_{T\to+\infty} \frac{D_{Y_v}^\rho(B,T)}{e^{\delta T}} = \lim_{T\to+\infty} \frac{E_{Y_v}^\rho(B,T)}{e^{\delta T}} = C_v(Y_v)\widetilde{m}_\delta(B), \tag{2.17}$$

where $C_v(Y_v) \in (0,+\infty)$ is a constant depending only on the vertex $v \in V$ and the set Y_v. In particular, this holds for $Y_v := X_v$, $v \in V$.

Recall, see [**8**] for example, that a GDMS $\mathcal{S}$ is maximal if $A_{ab} = 1$ whenever $t(a) = i(b)$. Since every iterated function system is maximal and finitely irreducible and since each maximal GDMS has property (A), as an immediate consequence of Theorem 2.7.6, and Remark 2.7.5 (improved to claim that now $C_\rho(Y)$ depends only on $i(\rho_1)$ and Y) we get the following two corollaries.

Corollary 2.7.7. *Suppose that $\mathcal{S}$ is a strongly regular finitely irreducible D-generic maximal conformal GDMS. Let $\delta = \delta_{\mathcal{S}} = \mathrm{HD}(J_{\mathcal{S}})$. For any $v \in V$ let $Y_v \subseteq X_v$ having at least two points be fixed.*

If $B \subset X$ is a Borel set such that $\widetilde{m}_\delta(\partial B) = 0$ (equivalently $\widetilde{\mu}_\delta(\partial B) = 0$) and $\rho \in E_A^\infty$ is with $i(\rho_1) = v$, then,

$$\lim_{T\to+\infty} \frac{D_{Y_v}^\rho(B,T)}{e^{\delta T}} = \lim_{T\to+\infty} \frac{E_{Y_v}^\rho(B,T)}{e^{\delta T}} = C_v(Y_v)\widetilde{m}_\delta(B), \tag{2.18}$$

where $C_v(Y_v) \in (0, +\infty)$ is a constant depending only on the vertex $v \in V$ and the set Y_v. In particular, this holds for $Y_v := X_v$, $v \in V$.

Corollary 2.7.8. *Suppose that $\mathcal{S}$ is a strongly regular D-generic conformal IFS acting on a phase space X. Let $\delta = \delta_{\mathcal{S}} = \mathrm{HD}(J_{\mathcal{S}})$. Fix $Y \subseteq X$ having at least two points.*

If $B \subset X$ is a Borel set such that $\widetilde{m}_\delta(\partial B) = 0$ (equivalently $\widetilde{\mu}_\delta(\partial B) = 0$) and $\rho \in E_A^\infty$, then,

$$\lim_{T\to+\infty} \frac{D_{Y}^\rho(B,T)}{e^{\delta T}} = \lim_{T\to+\infty} \frac{E_{Y}^\rho(B,T)}{e^{\delta T}} = C(Y)\widetilde{m}_\delta(B), \tag{2.19}$$

where $C(Y) \in (0, +\infty)$ is a constant depending only on the set Y. In particular, this holds for $Y := X$.

CHAPTER 3

Parabolic Conformal Graph Directed Markov Systems

We want to apply the previous results (Theorem 2.4.8, Theorem 2.4.9, Theorem 2.7.1, Theorem 2.7.4) to prove counting theorems for a variety of dynamical and geometric examples. In particular, these theorems can be applied to prove geometric counting results for Apollonian packings and many other systems naturally arising in the realm of Kleinian groups and one-dimensional conformal, holomorphic and real, dynamical systems. But such systems do not really fit into the framework of previous sections. These however fit into the framework of conformal parabolic iterated function systems, and more generally of parabolic graph directed Markov systems. Therefore, and because parabolic systems are interesting on their own, following [**45**] and [**47**], we recall the definition of parabolic systems, bring up their basic properties, and, based on mentioned above results from previous sections, i.e attracting GDMSs, we prove appropriate counting results for them. This primarily means Theorem 3.3.1 and Theorem 3.3.2 for multipliers i. e. analogues of Theorem 2.4.8 and Theorem 2.4.9 in this setting, along with several of its quite involved, corollaries, primarily about counting diameters.

3.1. Parabolic GDMS; Preliminaries

In present section, following [**45**] and [**47**], we describe the suitable parabolic setting, we canonically associate to a parabolic system an ordinary (uniformly contracting) conformal graph directed Markov system (a kind of inducing), and we prove Theorem 3.1.7, which is a somewhat surprising and remarkable result about parabolic systems.

Similarly as in Section 2.2 we assume that we are given a directed multigraph (V, E, i, t) with V finite and E also finite (though in Section 2.2 E was merely assumed to be countable), an incidence matrix $A : E \times E \to \{0, 1\}$, and two functions $i, t : E \to V$ such that $A_{ab} = 1$ implies $t(b) = i(a)$. Also, we have nonempty compact metric spaces $\{X_v\}_{v\in V}$. Suppose further that we have a collection of conformal maps $\varphi_e : X_{t(e)} \to X_{i(e)}$, $e \in E$, satisfying the following conditions (which are more general than in Section 2.2 in that we do not necessarily assume the maps φ_e to be uniform contractions).

(1) (Open Set Condition) $\varphi_a(\mathrm{Int}(X)) \cap \varphi_b(\mathrm{Int}(X)) = \emptyset$ for all $a, b \in E$ with $a \neq b$.

(2) $|\varphi_e'(x)| < 1$ everywhere except for finitely many pairs (e, x_e), $e \in E$, for which x_e is the unique fixed point of φ_e and $|\varphi_e'(x_e)| = 1$. Such pairs and indices i will be called parabolic and the set of parabolic indices will be denoted by Ω. All other indices will be called hyperbolic. We assume that $A_{ee} = 1$ for all $e \in \Omega$.

(3) $\forall n \geq 1$ $\forall \omega = (\omega_1\omega_2...\omega_n) \in E_A^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then φ_ω extends conformally to an open connected set $W_{t(\omega_n)} \subseteq \mathbb{R}^d$ and maps $W_{t(\omega_n)}$ into $W_{i(\omega_n)}$.

(4) If $e \in E$ is a parabolic index, then

$$\bigcap_{n\geq 0} \varphi_{e^n}(X) = \{x_e\}$$

and the diameters of the sets $\varphi_{e^n}(X)$ converge to 0.

(5) (Bounded Distortion Property) $\exists K \geq 1\ \forall n \geq 1\ \forall \omega \in E_A^n\ \forall x, y \in W_{t(\omega_n)}$, if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then
$$\frac{|\varphi_\omega'(y)|}{|\varphi_\omega'(x)|} \leq K.$$
(6) $\exists \kappa < 1\ \forall n \geq 1\ \forall \omega \in E_A^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then $\|\varphi_\omega'\| \leq \kappa$.

(7) (Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subseteq \mathbb{R}^d$ there exists an open cone $\mathrm{Con}(x, \alpha, l) \subseteq \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure α, and altitude l.

(8) There exists a constant $L \geq 1$ such that
$$\left| \frac{|\varphi_e'(y)|}{|\varphi_e'(x)|} - 1 \right| \leq L\|y - x\|^\alpha,$$
for every $e \in E$ and every pair of points $x, y \in V$.

We call such a system of maps
$$\mathcal{S} = \{\varphi_e : e \in E\}$$
a *subparabolic conformal graph directed Markov system.*

Let us note that conditions (1), (3), (5)–(7) are modeled on similar conditions which were used to examine hyperbolic conformal systems.

DEFINITION 3.1.1. If $\Omega \neq \emptyset$, we call the system $\mathcal{S} = \{\varphi_i : i \in E\}$ *parabolic.*

As stated in (2) the elements of the set $E \setminus \Omega$ are called hyperbolic. We extend this name to all the words appearing in (5) and (6). It follows from (3) that for every hyperbolic word ω,
$$\varphi_\omega(W_{t(\omega)}) \subseteq W_{t(\omega)}.$$
Note that our conditions ensure that $\varphi_e'(x) \neq 0$ for all $e \in E$ and all $x \in X_{t(i)}$. It was proved (although only for IFSs nevertheless the case of GDMSs can be treated completely similarly) in [**45**] (comp. [**47**]) that

$$\lim_{n\to\infty} \sup_{\omega \in E_A^n} \big\{\mathrm{diam}(\varphi_\omega(X_{t(\omega)}))\big\} = 0. \tag{3.1}$$

As its immediate consequence, we record the following.

COROLLARY 3.1.2. *The map* $\pi = \pi_{\mathcal{S}} : E_A^\infty \to X := \bigoplus_{v\in V} X_v$,
$$\{\pi(\omega)\} := \bigcap_{n\geq 0} \varphi_{\omega|_n}(X),$$
is well defined, i.e. this intersection is always a singleton, and the map π *is uniformly continuous.*

As for hyperbolic (attracting) systems the limit set $J = J_{\mathcal{S}}$ of the system $\mathcal{S} = \{\varphi_e\}_{e\in e}$ is defined to be
$$J_{\mathcal{S}} := \pi(E_A^\infty)$$
and it enjoys the following self-reproducing property:
$$J = \bigcup_{e\in E} \varphi_e(J).$$
We now, still following [**45**] and [**47**], want to associate to the parabolic system $\mathcal{S}$ a canonical hyperbolic system $\mathcal{S}^*$. We will then be able to apply the ideas from the previous section to $\mathcal{S}^*$. The set of edges is defined as follows:
$$E_* := \big\{i^n j : n \geq 1,\ i \in \Omega,\ i \neq j \in E,\ A_{ij} = 1\big\} \cup (E \setminus \Omega) \subseteq E_A^*.$$
We set
$$V_* = t(E_*) \cup i(E_*)$$

and keep the functions t and i on E_* as the restrictions of t and i from E_A^*. The incidence matrix $A^* : E_* \times E_* \to \{0,1\}$ is defined in the natural (and the only reasonable) way by declaring that $A^*_{ab} = 1$ if and only if $ab \in E_A^*$. Finally

$$\mathcal{S}^* = \{\varphi_e : X_{t(e)} \to X_{t(e)} | \, e \in E^*\}.$$

It immediately follows from our assumptions (see [**45**] and [**47**] for more details) that the following is true.

Theorem 3.1.3. *The system S^* is a hyperbolic (contracting) conformal GDMS and the limit sets $J_{\mathcal{S}}$ and $J_{\mathcal{S}^*}$ differ only by a countable set. If the system $\mathcal{S}$ is finitely irreducible, then so is the system $\mathcal{S}^*$.*

The price we pay by replacing the non-uniform "contractions" in $\mathcal{S}$ with the uniform contractions in $\mathcal{S}^*$ is that even if the alphabet E is finite, the alphabet E^* of $\mathcal{S}^*$ is always infinite. In particular, already at the first level (just the maps φ_ω, $\omega \in E^*$,), we get more scaling factors to deal with. In order to understand them, we will need the following quantitative result, whose complete proof can be found in [**88**].

Proposition 3.1.4. *Let $\mathcal{S}$ be a conformal parabolic GDMS. Then there exists a constant $C \in (0,+\infty)$ and for every $i \in \Omega$ there exists some constant $p_i \in (0,+\infty)$ such that for all $n \geq 1$ and for all $z \in X_i := \bigcup_{j \in I\setminus\{i\}} \varphi_j(X)$,*

$$C^{-1} n^{-\frac{p_i+1}{p_i}} \leq |\varphi'_{i^n}(z)| \leq C n^{-\frac{p_i+1}{p_i}}.$$

Furthermore, if $d = 2$ then all constants p_i are integers ≥ 1 and if $d \geq 3$ then all constants p_i are equal to 1.

Let us also introduce the following auxiliary system:

$$\mathcal{S}^- := \{\varphi_e : e \in E \setminus \Omega\}.$$

As an immediate consequence of Proposition 3.1.4 we get the following.

Proposition 3.1.5. *If $\mathcal{S}$ is a conformal parabolic GDMS, then*

$$\Gamma_{\mathcal{S}^*} = \left(\frac{p_{\mathcal{S}}}{p_{\mathcal{S}}+1}, +\infty\right), \quad \gamma_{\mathcal{S}^*} = \frac{p_{\mathcal{S}}}{p_{\mathcal{S}}+1},$$

where

$$p_{\mathcal{S}} := \max\{p_i : i \in \Omega\}.$$

and the system $\mathcal{S}^*$ is, in the terminology of [**47**], *hereditarily (co-finitely) regular, in particular, strongly regular.*

We set

$$\delta_{\mathcal{S}} := \delta^*_{\mathcal{S}},$$

$$m_{\delta_{\mathcal{S}}} := m^*_{\delta_{\mathcal{S}^*}} \quad \text{and} \quad \widetilde{m}_{\delta_{\mathcal{S}}} := \widetilde{m}^*_{\delta_{\mathcal{S}^*}}.$$

Given $e \in E$, we set

$$\Omega_e := \{a \in \Omega : A_{ae} = 1\}$$

and

$$\Omega_\rho := \Omega_{\rho_1}$$

for every $\rho \in E_A^\infty$. We will need the following facts proved in [**45**], comp. [**47**].

Theorem 3.1.6. *If $\mathcal{S}$ is an irreducible conformal parabolic GDMS, then*

(1) $\delta_{\mathcal{S}} = \mathrm{HD}(J_{\mathcal{S}})$,

(2) *The measure* $\widetilde{m}_{\delta_{\mathcal{S}}}$ *is* δ*–conformal for the original system* $\mathcal{S}$ *in the sense that*

$$\widetilde{m}_{\delta_{\mathcal{S}}}(\varphi_\omega(F)) = \int_F |\varphi'_\omega|^{\delta_{\mathcal{S}}}\, d\widetilde{m}_{\delta_{\mathcal{S}}}$$

for every $\omega \in E_A$ *and every Borel set* $F \subseteq X_{t(\omega)}$, *and*

$$\widetilde{m}_{\delta_{\mathcal{S}}}\big(\varphi_\alpha(X_{t(\alpha)}) \cap \varphi_\beta(X_{t(\beta)})\big) = 0$$

whenever $\alpha, \beta \in E_A^*$ *and are incomparable.*

(3) *There exists a, unique up to multiplicative constant,* σ*–finite shift–invariant measure* $\mu_{\delta_{\mathcal{S}}}$ *on* E_A^∞, *absolutely continuous with respect to* $m_{\delta_{\mathcal{S}}}$. *The measure* $\mu_{\delta_{\mathcal{S}}}$ *is equivalent to* $m_{\delta_{\mathcal{S}}}$ *and*

(a) *The Radon–Nikodym derivative of* $\mu_{\delta_{\mathcal{S}}}$ *with respect to* m_δ *is given by the following formula:*

$$\psi_{\delta_{\mathcal{S}}}(\rho) := \frac{d\mu_{\delta_{\mathcal{S}}}}{dm_{\delta_{\mathcal{S}}}}(\rho) = \psi^*_{\delta_{\mathcal{S}}}(\rho) + \sum_{a\in\Omega_\rho}\sum_{k=1}^{\infty} |\varphi'_{a^k}(\pi(\rho))|^\delta \psi^*_{\delta_{\mathcal{S}}}(a^k\rho).$$

(b) *The measure* $\mu_{\delta_{\mathcal{S}}}$ *(and* $\widetilde{\mu}_{\delta_{\mathcal{S}}} := \mu_{\delta_{\mathcal{S}}} \circ \pi_{\mathcal{S}}^{-1}$*) is finite (we then always treat it as normalized so that it is a probability measure) if and only if*

$$\delta_{\mathcal{S}} > \frac{2p_{\mathcal{S}}}{p_{\mathcal{S}}+1}.$$

More precisely, the following conditions are equivalent:

(b1) $\delta_{\mathcal{S}} > \frac{2p_a}{p_a+1}$,

(b2) *There exists an integer* $l \geq 1$ *such that* $\mu_{\delta_{\mathcal{S}}}([a^l]) < +\infty$, *and*

(b3) *For every integer* $l \geq 1$, $\mu_{\delta_{\mathcal{S}}}([a^l]) < +\infty$.

(4) *Furthermore, we have that*

$$\chi_{\delta_{\mathcal{S}}} := -\int_{E_A^\infty} \log\big|\varphi'_{\omega_1}(\pi_{\mathcal{S}}(\omega)\big|d\mu_\delta = \chi^*_{\delta_{\mathcal{S}}} \in (0, +\infty)$$

and, as for attracting GDMSs, we call $\chi_{\delta_{\mathcal{S}}}$ *the Lyapunov exponent of the system* $\mathcal{S}$ *with respect to measure* $\mu_{\delta_{\mathcal{S}}}$.

For future use we denote

$$\Omega_\infty = \Omega_\infty(\mathcal{S}) := \left\{a \in \Omega : \frac{2p_a}{p_a+1} \geq \delta_{\mathcal{S}}\right\}.$$

A crucial feature of the hyperbolic systems arising from parabolic systems is that they are automatically D-hyperbolic. We have already seen that this is not necessarily true for hyperbolic systems.

THEOREM 3.1.7. *If* $\mathcal{S}$ *is an irreducible conformal parabolic GDMS, then* $\mathcal{S}^*$, *the associated contracting (hyperbolic) GDMS, is D-generic.*

PROOF. Assume for a contradiction that $\mathcal{S}^*$ is not D-generic. According to Proposition 2.3.8 this means that the additive group generated by the set

$$\big\{-\log|\varphi'_\omega(x_\omega)| : \omega \in E^*_{*A^*}\big\} \subseteq \mathbb{R}$$

is cyclic. Denote its generator by $M > 0$. Fix $b \in \Omega$ and then take $\alpha \in E_A^*$ such that $\alpha_1 \neq b$ and $\alpha b^2 \alpha_1 \in E_A^*$. Note that then $\alpha b^2\alpha_1 \in E^*_{*A^*}$ and moreover $\alpha b^n \alpha_1 \in E^*_{*A^*}$ for all integers $n \geq 2$. For every integer $n \geq 2$ denote by $x_n \in J_{\mathcal{S}^*}$ the only fixed point of the map $\varphi_{\alpha b^n\alpha_1} : X_{t(\alpha_1)} \to X_{t(\alpha_1)}$. We know from the above that for every $n \geq 2$ there exists an integer $k_n \geq 1$ such that

$$-\log\big|\varphi'_{\alpha b^n\alpha_1}(x_n)\big| = Mk_n. \tag{3.2}$$

By Proposition 3.1.4 we have that

$$\left|\varphi'_{\alpha b^n\alpha_1}(x_n)\right| = |\varphi'_{\alpha_1}(x_n)| \cdot \left|\varphi'_{b^n}(\varphi_{\alpha_1}(x_n))\right| \cdot \left|\varphi'_{\alpha}(\varphi_{b^n\alpha_1}(x_n))\right| = C_n n^{-\frac{p_b+1}{p_b}} \tag{3.3}$$

with some $C_n \in (C^{-1}, C)$, where C is the constant coming from Proposition 3.1.4. Combining this with (3.2) yields

$$k_n = -\frac{1}{M}\log C_n + \frac{p_b+1}{Mp_b}\log n.$$

On the other hand

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \varphi_{\alpha b^n\alpha_1}(x_n) = \varphi_\alpha\big(\lim_{n\to\infty}\varphi_b^n(\varphi_{\alpha_1}(x_n))\big) = \varphi_\alpha(x_b)$$

and

$$\lim_{n\to\infty} \varphi_{b^n\alpha_1}(x_n) = x_b.$$

Keeping in mind that $\varphi_b(x_b) = x_b$ and $|\varphi'_b(x_b)| = 1$ and using the Bounded Distortion Property, we therefore get

$$\begin{aligned}
\lim_{n\to\infty}\frac{\left|\varphi'_{\alpha b^{n+1}\alpha_1}(x_{n+1})\right|}{\left|\varphi'_{\alpha b^n\alpha_1}(x_n)\right|} &= \lim_{n\to\infty}\frac{\left|\varphi'_{\alpha b^{n+1}\alpha_1}(\varphi_\alpha(x_b))\right|}{\left|\varphi'_{\alpha b^n\alpha_1}(\varphi_\alpha(x_b))\right|} \\
&= \lim_{n\to\infty}\frac{\left|\varphi'_\alpha\big(\varphi_b^{n+1}(\varphi_{\alpha_1\alpha}(x_b))\big)\right|\cdot\left|\varphi'_b\big(\varphi_b^n(\varphi_{\alpha_1\alpha}(x_b))\big)\right|}{\left|\varphi'_\alpha\big(\varphi_b^n(\varphi_{\alpha_1\alpha}(x_b))\big)\right|} \\
&= \lim_{n\to\infty}\frac{|\varphi'_\alpha(x_b)|\cdot|\varphi'_b(x_b)|}{|\varphi'_\alpha(x_b)|} = |\varphi'_b(x_b)| = 1.
\end{aligned}$$

Equivalently:

$$\lim_{n\to\infty}\big(-\log|\varphi'_{\alpha b^{n+1}\alpha_1}(x_{n+1})| - \ (-\log|\varphi'_{\alpha b^n\alpha_1}(x_n)|\big) = 0.$$

Using (3.2) this gives that $\lim_{n\to\infty}(k_{n+1}-k_n) = 0$. Since all k_n, $n \geq 1$, are integers, this implies that the sequence $(k_n)_{n=1}^\infty$ is eventually constant. However, it follows from (3.2) that $\lim_{n\to\infty} k_n = +\infty$, and the contradiction we obtain finishes the proof. □

REMARK 3.1.8. We could generalize slightly the concepts of subparabolic and parabolic systems by requiring in item (2) of their definition that not merely some elements φ_e, $e \in E$, have parabolic fixed points but some finitely many elements φ_ω, $\omega \in E_A^*$, have such points. In other words it would suffice to assume that some iterate of the system $\mathcal{S}$ in the sense of Remark 2.2.2 is parabolic. Indeed, this would not really affect any considerations of this and any forthcoming section involving parabolic GDMSs, and such generalization will turn out to be needed in Subsection 5.2.1 for the Farey map, Subsections 6.2 and 6.2.2 when we deal respectively with Schottky groups with tangencies and Apollonian circle packings.

3.2. Poincaré's Series for $\mathcal{S}^*$, the Associated Countable Alphabet Attracting GDMS

In this section we again let $\mathcal{S}$ be an irreducible conformal parabolic GDMS. Our goal is to describe the Poincaré series and the associated asymptotic (equidistribution) results for the system $\mathcal{S}$. This is achieved by means of the transfer operator associated to the associated hyperbolic system $\mathcal{S}^*$.

We begin by formulating the required notation. Fix first $\rho \in E_{A^*}^\infty$ arbitrary. Denote $\xi := \pi_{\mathcal{S}^*}(\rho)$. Treating ρ in an obvious way as an element of E_A^∞, we can also write $\xi = \pi_{\mathcal{S}}(\rho)$. Fix next an arbitrary $\tau \in E_{*A^*}^*$.

Let $\eta_i^*(\tau, s)$, $i = \rho, p$, be the corresponding Poincaré series for the contracting system $\mathcal{S}^*$, and we continue to use

$$\eta_i(\tau, s), \quad i = \rho, p,$$

to denote the Poincaré series for the original (now parabolic) system $\mathcal{S}$. This allows to deduce the analytical properties of η_i from those for the η_i^*, to which we can apply the results already established in Proposition 2.5.3.

We show that the Poincaré series $\eta_i(\tau,s)$ for the parabolic system $\mathcal{S}$ can be expressed in terms of the Poincaré series for $\eta_i^*(\tau,s)$ for the hyperbolic system $\mathcal{S}^*$. In particular, we can deduce properties for $\eta_i^*(\tau,s)$ which are the analogue of those for $\eta_i(\tau,s)$, already established in Proposition 6.3. We can formally write

$$\begin{aligned}
\eta_\rho(\tau,s) &= \sum_{\omega\in E_\rho^*:\tau\omega\in E_A^*} |\varphi'_{\tau\omega}(\pi(\rho))|^s \\
&= \sum_{\substack{\omega\in E^*_{*,\rho}\\ \tau\omega\in E^*_{*A^*}}} |\varphi'_{\tau\omega}(\pi(\rho))|^s + \sum_{a\in\Omega_\rho}\sum_{k=1}^\infty \sum_{\substack{\omega\in E^*_{*A^*}\\ \tau\omega a\in E_A^*}} |\varphi'_{\tau\omega a^k}(\pi(\rho))|^s \\
&= \sum_{\substack{\omega\in E^*_{*\rho}\\ \tau\omega\in E^*_{*A^*}}} |\varphi'_{\tau\omega}(\pi(\rho))|^s + \sum_{a\in\Omega_\rho}\sum_{k=1}^\infty \sum_{\substack{\omega\in E^*_{*A^*}\\ \tau\omega a\in E_A^*}} |\varphi'_{\tau\omega}(\pi(a^k\rho))|^s|\varphi'_{a^k}(\pi(\rho))|^s \qquad (3.1)\\
&= \sum_{\substack{\omega\in E^*_{*\rho}\\ \tau\omega\in E^*_{*A^*}}} |\varphi'_{\tau\omega}(\pi(\rho))|^s + \sum_{a\in\Omega_\rho}\sum_{k=1}^\infty |\varphi'_{a^k}(\pi(\rho))|^s \sum_{\substack{\omega\in E^*_{*A^*}\\ \tau\omega a\in E_A^*}} |\varphi'_{\tau\omega}(\pi(a^k\rho))|^s \\
&= \eta_\rho^*(\tau,s) + \sum_{a\in\Omega_\rho}\sum_{k=1}^\infty |\varphi'_{a^k}(\pi(\rho))|^s \eta^*_{a^k\rho}(\tau,s).
\end{aligned}$$

Since by Theorem 3.1.7 we have that $\mathcal{S}^*$ is D-generic it follows from the proof of Theorem 2.5.3 that for every $s_0=\delta_{\mathcal{S}}+it_0\in\Gamma^+_{\mathcal{S}}$ with $t_0\neq 0$ all functions $\eta^*_{a^k\rho}(\tau,\cdot)$ have holomorphic extensions on a common neighborhood, denoted by U, of $s_0\in\Gamma^+_{\mathcal{S}^*}$ of the form

$$U\ni s\longmapsto \sum_{j=1}^q \lambda_j^*(s)(1-\lambda_j^*(s))^{-1}P^*_{s,j}(|\varphi'_\tau|^s\circ\pi)(a^k\rho)+S^*_\infty(s)\in\mathbb{C},$$

where all the symbols "$*$" indicate that the appropriate objects pertain to the system $\mathcal{S}^*$. Since

$$|P^*_{s,j}(|\varphi'_\tau|^s\circ\pi^*)(a^k\rho)|\le \|P^*_{s,j}(|\varphi'_\tau|^s\circ\pi^*)\|_\infty\le\|P^*_{s,j}(|\varphi'_\tau|^s\circ\pi^*)\|_\alpha<+\infty,$$

it follows that all the functions $\eta^*_{a^k\rho}(\tau,\cdot)$ are uniformly bounded on U. Since also $\delta_{\mathcal{S}}>\frac{p_a}{p_a+1}$ and since

$$\big||\varphi'_{a^k}(\pi(\rho))|^s\big|\le|\varphi'_{a^k}(\pi(\rho))|^{\delta_{\mathcal{S}}}\preceq(k+1)^{-\frac{p_a+1}{p_a}\delta_{\mathcal{S}}}, \qquad (3.2)$$

we eventually conclude that the series in (3.1) converges absolutely uniformly on U, thus representing a holomorphic function. We are therefore left to consider the case of $s_0=\delta_{\mathcal{S}}$. By virtue of (2.3) we then have for every $k\ge 0$ that

$$\eta^*_{a^k\rho}(\tau,s)=\lambda_s^*(1-\lambda_s^*)^{-1}H^*_{\tau,s}(a^k\rho)+\Sigma_\infty(s).$$

Substituting this into (3.1), we therefore get

$$\eta_\rho(\tau,s)=\eta_\rho^*(\tau,s)+\lambda_s^*(1-\lambda_s^*)^{-1}\sum_{a\in\Omega_\rho}\sum_{k=1}^\infty|\varphi'_{a^k}(\pi(\rho))|^sH^*_{\tau,s}(a^k\rho)+\sum_{a\in\Omega_\rho}\sum_{k=1}^\infty|\varphi'_{a^k}(\pi(\rho))|^s\Sigma_\infty(s),$$

and by (3.2) both series involved in the above formula converge absolutely uniformly on U. Looking up now at the calculations from the end of the proof of Theorem 2.5.3 and invoking Theorem 3.1.6 (3) and

(4), we conclude that the function $U \ni s \mapsto \eta_\rho(\tau, s)$ is meromorphic with a simple pole at $s = \delta_{\mathcal{S}}$ whose residue is equal to

$$\begin{aligned}
\frac{\psi^*_{\delta_{\mathcal{S}}}(\rho)}{\chi^*_{\delta_{\mathcal{S}}}} m^*_{\delta_{\mathcal{S}}}([\tau]) + \sum_{a\in\Omega_\rho}\sum_{k=1}^{\infty} |\varphi'_{a^k}(\pi(\rho))|^{\delta_{\mathcal{S}}} \psi^*_\delta(a^k\rho) m^*_{\delta_{\mathcal{S}}}([\tau]) = \\
= \frac{1}{\chi_{\delta_{\mathcal{S}}}} \Big(\psi^*_{\delta_{\mathcal{S}}}(\rho) + \sum_{a\in\Omega_\rho}\sum_{k=1}^{\infty} |\varphi'_{a^k}(\pi(\rho))|^{\delta} \psi^*_{\delta_{\mathcal{S}}}(a^k\rho)\Big) m_{\delta_{\mathcal{S}}}([\tau]) \\
= \frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{\chi_{\delta_{\mathcal{S}}}} m_{\delta_{\mathcal{S}}}([\tau]).
\end{aligned}$$

We have thus proved the following.

THEOREM 3.2.1. *If $\mathcal{S}$ is an irreducible parabolic conformal GDMS, $\rho \in E^\infty_{A^*}$, and $\tau \in E^*_{*A^*}$, then*

(a) *The function $\Delta^+_{\mathcal{S}} \ni s \longmapsto \eta_\rho(\tau, s) \in \mathbb{C}$ has a meromorphic extension to some neighborhood of the vertical line $\mathrm{Re}(s) = \delta_{\mathcal{S}}$,*

(b) *This extension has a single pole $s = \delta_{\mathcal{S}}$, and*

(c) *The pole $s = \delta_{\mathcal{S}}$ is simple and its residue is equal to $\frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{\chi_{\delta_{\mathcal{S}}}} m_{\delta_{\mathcal{S}}}([\tau])$.*

3.3. Asymptotic Results for Multipliers

Now that we have established Theorem 3.2.1, we are ready to prove the following theorem which, along with its two corollaries below, constitutes the main results of this section.

THEOREM 3.3.1 (Asymptotic Equidistribution of Multipliers for Parabolic Systems I). *Suppose that $\mathcal{S}$ is an irreducible parabolic conformal GDMS. Fix $\rho \in E^\infty_A$. If $\tau \in E^*_A$ then,*

$$\lim_{T\to+\infty} \frac{N_\rho(\tau, T)}{e^{\delta_{\mathcal{S}} T}} = \frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{\delta_{\mathcal{S}} \chi_{\mu_\delta}} m_{\delta_{\mathcal{S}}}([\tau]), \tag{3.1}$$

and

$$\lim_{T\to+\infty} \frac{N_p(\tau, T)}{e^{\delta_{\mathcal{S}} T}} = \frac{1}{\delta_{\mathcal{S}} \chi_{\mu_{\delta_{\mathcal{S}}}}} \mu_{\delta_{\mathcal{S}}}([\tau]). \tag{3.2}$$

PROOF. We first prove formula (3.1). If $\rho \in E^\infty_{*A^*}$ and $\tau \in E^*_{*A^*}$, this formula follows from Theorem 3.2.1 in exactly the same way as formula (2.5) in Theorem 2.4.8 follows from Theorem 2.5.3.

Now keep $\tau \in E^*_{*A^*}$ and let $\rho \in E^\infty_A$ be arbitrary. Then for every $q \geq 1$ large enough there exists $\rho_q \in E^\infty_{*A^*}$ such that

$$\rho|_q = \rho_q|_q.$$

Since $\lim_{q\to\infty} d(\rho, \rho_q) = 0$, the Bounded Distortion Property (BDP) for the attracting system $\mathcal{S}^*$ yields a function $q \mapsto \widehat{K}_q \in [1, +\infty)$ such that

$$\lim_{q\to\infty} \widehat{K}_q = 1 \tag{3.3}$$

and

$$\widehat{K}_q^{-1} \leq \frac{|\varphi'_\tau(\pi_{\mathcal{S}}(\rho))|}{|\varphi'_\tau(\pi_{\mathcal{S}}(\rho_q))|} \leq \widehat{K}_q$$

for all $q \geq 1$ large enough as indicated above. Hence

$$N_{\rho_q}(\tau, T - \log \widehat{K}_q) \leq N_\rho(\tau, T) \leq N_{\rho_q}(\tau, T + \log \widehat{K}_q).$$

Therefore, dividing by $e^{\delta_{\mathcal{S}}T}$ we get that

$$\frac{N_{\rho_q}(\tau, T-\log\widehat{K}_q)}{\exp(\delta_{\mathcal{S}}(T-\log\widetilde{K}_q))}\widehat{K}_q^{-\delta_{\mathcal{S}}} \le \frac{N_\rho(\tau,T)}{e^{\delta_{\mathcal{S}}T}} \le \frac{N_{\rho_q}(\tau, T+\log\widehat{K}_q)}{\exp(\delta_{\mathcal{S}}(T+\log\widetilde{K}_q))}\widehat{K}_q^{\delta_{\mathcal{S}}}.$$

Since $\rho_q \in E^\infty_{*A^*}$ and $\tau \in E^*_{*A^*}$ we thus obtain

$$\widehat{K}_q^{-\delta_{\mathcal{S}}}\frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}}m_\delta([\tau]) \le \liminf_{T\to+\infty}\frac{N_\rho(\tau,T)}{e^{\delta_{\mathcal{S}}T}} \le \limsup_{T\to+\infty}\frac{N_\rho(\tau,T)}{e^{\delta_{\mathcal{S}}T}} \le \widehat{K}_q^{\delta_{\mathcal{S}}}\frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}}m_{\delta_{\mathcal{S}}}([\tau]).$$

Invoking (3.3) we now conclude that

$$\lim_{T\to+\infty}\frac{N_\rho(\tau,T)}{e^{\delta_{\mathcal{S}}T}} = \frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}}m_{\delta_{\mathcal{S}}}([\tau]). \tag{3.4}$$

Working in full generality, we now assume that $\rho \in E^\infty_A$ and $\tau \in E^*_A$. Then there exists $\mathcal{F}_\tau$, a countable collection of mutually incomparable elements of $E^*_{*A^*}$, each of which extends τ, such that

$$m_{\delta_{\mathcal{S}}}\left([\tau] \setminus \bigcup_{\omega\in\mathcal{F}_\tau}[\omega]\right) = 0.$$

Noting that then the family $\{[\omega] : \omega \in \mathcal{F}_\tau\}$ consists of mutually disjoint sets, we thus get that from (3.4) that

$$\begin{aligned}\liminf_{T\to+\infty}\frac{N_\rho(\tau,T)}{e^{\delta_{\mathcal{S}}T}} &\ge \liminf_{T\to+\infty}\frac{\sum_{\omega\in\mathcal{F}_\tau}N_\rho(\omega,T)}{e^{\delta_{\mathcal{S}}T}} \ge \sum_{\omega\in\mathcal{F}_\tau}\liminf_{T\to+\infty}\frac{N_\rho(\omega,T)}{e^{\delta_{\mathcal{S}}T}}\\ &= \sum_{\omega\in\mathcal{F}_\tau}\frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{\delta_{\mathcal{S}}\chi_{\mu_{\delta_{\mathcal{S}}}}}m_{\delta_{\mathcal{S}}}([\omega]) = \frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{\delta_{\mathcal{S}}\chi_{\mu_{\delta_{\mathcal{S}}}}}m_{\delta_{\mathcal{S}}}([\tau]).\end{aligned}$$

Having this (and already knowing that the neutral word $\emptyset$ belongs to $E^*_{*A^*}$) then (3.4) gives that

$$\lim_{T\to+\infty}\frac{N_\rho(T)}{e^{\delta_{\mathcal{S}}T}} = \frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}}m_{\delta_{\mathcal{S}}}([\emptyset]) = \frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}}m_{\delta_{\mathcal{S}}}(E^\infty_{*A^*}) = \frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}},$$

we deduce that

$$\lim_{T\to+\infty}\frac{N_\rho(\tau,T)}{e^{\delta_{\mathcal{S}}T}} = \frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{e^{\delta_{\mathcal{S}}T}}m_{\delta_{\mathcal{S}}}([\tau])$$

in the say way (although it is now in fact simpler) as formula (2.12) is deduced from (2.8) and (2.1), the latter applied with $\tau = \emptyset$ (i.e., the empty word). The proof of formula (3.1) is then complete.

Now we prove formula (3.2). First assume that τ is not a power of an element from Ω. This means that either

$$\tau = a^j\beta$$

where $a \in \Omega$, $j \ge 1$, and $\beta_1 \ne a$ or

$$\tau = \beta$$

where $\beta_1 \notin \Omega$. In either case,

$$\tau = a^j\beta,$$

with $j \ge 0$. As in the proof of formula (2.5) in Theorem 2.4.8, for every $\gamma \in E^*_A$ fix $\gamma^+ \in E^\infty_A$ (which in fact can be selected to depend only on $\gamma_{|\gamma|}$) such that

$$\gamma\gamma^+ \in E^\infty_A.$$

Fix $q \ge 1$ and $\gamma \in E^q_A$ arbitrarily. Consider an arbitrary element $\omega b^k \in E^*_A$, $\omega \in E^*_{*,A^*}$, $b \in \Omega$ such that $a^j\beta\gamma\omega b^k \in E^*_p$. Consider two cases:

Case 1^0. Assume $b \ne a$ if $j \ge 1$. Then

$$\left|\varphi'_{a^j\beta\gamma\omega b^k}(x_{a^j\beta\gamma\omega b^k})\right| = \left|\varphi'_{a^j\beta\gamma\omega}\big(\pi_{\mathcal{S}}((b^ka^j\beta\gamma\omega)^\infty)\big)\right| \cdot \left|\varphi'_{b^k}\big(\pi_{\mathcal{S}}((a^j\beta\gamma\omega b^k)^\infty)\big)\right|$$

and

$$\left|\varphi'_{a^j\beta\omega b^k}\big(\pi_{\mathcal S}(a^j\beta\gamma\gamma^+)\big)\right| = \left|\varphi'_{a^j\beta\gamma\omega}\big(\pi_{\mathcal S}(b^k a^j\beta\gamma\gamma^+)\big)\right|\cdot\left|\varphi'_{b^k}\big(\pi_{\mathcal S}(a^j\beta\gamma\gamma^+)\big)\right|.$$

Since $\omega\in E^*_{*A^*}$ and since either $b\neq a$ if $j\geq 1$ or $\beta_1\notin\Omega$ if $j=0$, by the (BDP) we get that

$$\widetilde K_q^{-1}\leq\frac{\left|\varphi'_{a^j\beta\gamma\omega}\big(\pi_{\mathcal S}\big((b^k a^j\beta\gamma\omega)^\infty\big)\big)\right|}{\left|\varphi'_{a^j\beta\gamma\omega}\big(\pi_{\mathcal S}(b^k a^j\beta\gamma\gamma^+)\big)\right|}\leq\widetilde K_q$$

and

$$\widetilde K_q^{-1}\leq\frac{\left|\varphi'_{b^k}\big(\pi_{\mathcal S}((a^j\beta\gamma\omega b^k)^\infty)\big)\right|}{\left|\varphi'_{b^k}\big(\pi_{\mathcal S}((a^j\beta\gamma\gamma^+)\big)\right|}\leq\widetilde K_q$$

with some "distortion" function $q\mapsto\widetilde K_q\in[1,+\infty)$ such that $\lim_{q\to\infty}\widehat K_q=1$. Consequently,

$$\widetilde K_q^{-2}\leq\frac{\left|\varphi'_{a^j\beta\gamma\omega b^k}(x_{a^j\beta\gamma\omega b^k})\right|}{\left|\varphi'_{a^j\beta\gamma\omega b^k}\big(\pi_{\mathcal S}((a^j\beta\gamma\gamma^+)\big)\right|}\leq\widetilde K_q^2. \tag{3.5}$$

Case 2^0. Assume $j\geq 1$ and $b=a$. Then

$$\left|\varphi'_{a^j\beta\gamma\omega b^k}(x_{a^j\beta\gamma\omega b^k})\right| = \left|\varphi'_{a^j\beta\gamma\omega}\big(\pi_{\mathcal S}((a^{j+k}\beta\gamma\omega)^\infty)\big)\right|\cdot\left|\varphi'_{a^k}(\pi_{\mathcal S}\big((a^j\beta\gamma\omega b^k)^\infty)\big)\right| \tag{3.6}$$

and

$$\left|\varphi'_{a^j\beta\gamma\omega b^k}\big(\pi_{\mathcal S}(a^j\beta\gamma\gamma^+)\big)\right| = \left|\varphi'_{a^j\beta\gamma\omega}\big(\pi_{\mathcal S}(a^{j+k}\beta\gamma\gamma^+)\big)\right|.\left|\varphi'_{a^k}\big(\pi_{\mathcal S}(a^j\beta\gamma\gamma^+)\big)\right|. \tag{3.7}$$

Again by (BDP) we have that

$$\widetilde K_q^{-1}\leq\frac{\left|\varphi'_{a^j\beta\gamma\omega}\big(\pi_{\mathcal S}((a^{j+k}\beta\gamma\omega)^\infty))\big)\right|}{\left|\varphi'_{a^j\beta\gamma\omega}\big(\pi_{\mathcal S}(a^{j+k}\beta\gamma\gamma^+)\big)\right|}\leq\widetilde K_q. \tag{3.8}$$

By the Chain Rule

$$\left|\varphi'_{a^k}(\pi_{\mathcal S}(a^j\beta\gamma\kappa))\right| = \left|\varphi'_{a^{k+j}}(\pi_{\mathcal S}(\beta\gamma\kappa))\right|\cdot\left|\varphi'_{a^j}\big(\pi_{\mathcal S}(\beta\gamma\kappa)\big)\right|^{-1} \tag{3.9}$$

for every $\kappa\in E_A^*$ such that $\gamma\kappa\in E_A^*$. Since $\beta_1\neq a$ we have that

$$\widetilde K_q^{-1}\leq\frac{\left|\varphi'_{a^{j+k}}\big(\pi_{\mathcal S}((\beta\gamma\omega a^k)^\infty)\big)\right|}{\left|\varphi'_{a^{j+k}}\big(\pi_{\mathcal S}(\beta\gamma\gamma^+)\big)\right|}\leq\widetilde K_q$$

and

$$\widetilde K_q^{-1}\leq\frac{\left|\varphi'_{a^j}(\pi_{\mathcal S}\big((\beta\gamma\omega a^k)^\infty)\big)\right|}{\left|\varphi'_{a^j}\big(\pi_{\mathcal S}(\beta\gamma\gamma^+)\big)\right|}\leq\widetilde K_q.$$

Hence, invoking (3.9) we get that

$$\widetilde K_q^{-2}\leq\frac{\left|\varphi'_{a^k}\big(\pi_{\mathcal S}((a^j\beta\gamma\omega a^k)^\infty)\big)\right|}{\left|\varphi'_{a^k}(\pi_{\mathcal S}(a^j\beta\gamma\gamma^+)\big)\right|}\leq\widetilde K_q^2.$$

Along with (3.8), (3.6) and (3.7) this yields

$$\widetilde K_q^{-3}\leq\frac{\left|\varphi'_{a^j\beta\gamma\omega b^k}(x_{a^j\beta\gamma\omega b^k})\right|}{\left|\varphi'_{a^j\beta\gamma\omega b^k}\big(\pi_{\mathcal S}(a^j\beta\gamma\gamma^+)\big)\right|}\leq\widetilde K_q^3. \tag{3.10}$$

Now it follows from (3.5) and (3.10) that

$$\pi_{a^j\beta\gamma\gamma^+}(a^j\beta\gamma,T-3\log\widetilde K_q)\subseteq\pi_p(\alpha^j\beta\gamma,T)\subseteq\pi_{a^j\beta\gamma\gamma^+}(a^j\beta\gamma,T+3\log\widetilde K_q). \tag{3.11}$$

Therefore, applying (3.1) we get that

$$
\begin{aligned}
\liminf_{T\to+\infty}\frac{N_p(a^j\beta,T)}{e^{\delta_{\mathcal S}T}} &\ge \liminf_{T\to+\infty}\sum_{\substack{\gamma\in E_A^q\\ a^j\beta\gamma\in E_A^*}}\frac{N_{\alpha^j\beta\gamma\gamma^+}(a^j\beta\gamma,T-3\log\widetilde K_q)}{e^{\delta_{\mathcal S}T}}\\
&\ge \sum_{\substack{\gamma\in E_A^q\\ a^j\beta\gamma\in E_A^*}}\liminf_{T\to+\infty}\frac{N_{\alpha^j\beta\gamma\gamma^+}(a^j\beta\gamma,T-3\log\widetilde K_q)}{e^{\delta_{\mathcal S}T}}\\
&= \sum_{\substack{\gamma\in E_A^q\\ a^j\beta\gamma\in E_A^*}}\liminf_{T\to+\infty}\frac{N_{\alpha^j\beta\gamma\gamma^+}(a^j\beta\gamma,T-3\log\widetilde K_q)}{e^{\delta_{\mathcal S}(T-3\log\widetilde K_q)}}\widetilde K_q^{-3\delta_{\mathcal S}}\\
&= \widetilde K_q^{-3\delta_{\mathcal S}}\sum_{\substack{\gamma\in E_A^\infty\\ a^j\beta\gamma\in E_A^*}}\frac{\psi_{\delta_{\mathcal S}}(a^j\beta\gamma\gamma^+)}{\delta_{\mathcal S}\chi_{\delta_{\mathcal S}}}m_{\delta_{\mathcal S}}([a^j\beta\gamma])\\
&\ge K_q^{-1}(a^j\beta)\frac{1}{\delta_{\mathcal S}\chi_{\delta_{\mathcal S}}}\mu_{\delta_{\mathcal S}}([a^j\beta]),
\end{aligned}
\tag{3.12}
$$

with some function $q\mapsto K_q(a^j\beta)\in[1,+\infty)$ for which $\lim_{q\to+\infty}K_q(a^j\beta)=1$ and which exists because $a^j\beta$ is not a power of an element from Ω. Taking the limit in (3.12) as $q\to+\infty$, we thus get that

$$
\liminf_{T\to+\infty}\frac{N_p(a^j\beta,T)}{e^{\delta_{\mathcal S}T}}\ge\frac{1}{\delta_{\mathcal S}\chi_\delta}\mu_{\delta_{\mathcal S}}([a^j\beta]). \tag{3.13}
$$

In the general case, i.e., making no assumptions on $\tau\in E_A^*$ we proceed in the same way as in the proof of formula (3.1). We can fix $\mathcal F_\tau$, a countable collection of mutually incomparable words extending τ, not being powers (concatenations) of elements from Ω, and such that

$$
\mu_{\delta_{\mathcal S}}\left([\tau]\setminus\bigcup_{\omega\in\mathcal F_\tau}\right)=0.
$$

Noting that then the family $\{[\omega]:\omega\in\mathcal F_\tau\}$ consists of mutually disjoint sets, we thus get that from (3.13) that

$$
\begin{aligned}
\liminf_{T\to+\infty}\frac{N_p(\tau,T)}{e^{\delta_{\mathcal S}T}} &\ge\liminf_{T\to+\infty}\frac{\sum_{\omega\in\mathcal F_\tau}N_p(\omega,T)}{e^{\delta_{\mathcal S}T}}\ge\sum_{\omega\in\mathcal F_\tau}\liminf_{T\to+\infty}\frac{N_p(\omega,T)}{e^{\delta_{\mathcal S}T}}\\
&=\frac{1}{\delta_{\mathcal S}\chi_{\delta_{\mathcal S}}}\sum_{\omega\in\mathcal F_\tau}\mu_{\delta_{\mathcal S}}([\omega])=\frac{1}{\delta_{\mathcal S}\chi_{\delta_{\mathcal S}}}\mu_{\delta_{\mathcal S}}([\tau]).
\end{aligned}
\tag{3.14}
$$

For the upper bound we again deal first with words $a^j\beta$, i.e., the same as those leading to (3.13). Since the alphabet E is finite it follows from the left hand side of (3.11) and from (3.1) that

$$
\begin{aligned}
\limsup_{T\to+\infty}\frac{N_p(a^j\beta,T)}{e^{\delta_{\mathcal S}T}} &\le \limsup_{T\to+\infty}\sum_{\substack{\gamma\in E_A^\infty\\ a^j\beta\gamma\in E_A^*}}\frac{N_{\alpha^j\beta\gamma\gamma^+}(a^j\beta\gamma,T+3\log\widetilde{K}_q)}{e^{\delta_{\mathcal S}T}}\\
&\le \sum_{\substack{\gamma\in E_A^\infty\\ a^j\beta\gamma\in E_A^*}}\limsup_{T\to+\infty}\frac{N_{\alpha^j\beta\gamma\gamma^+}(a^j\beta\gamma,T+3\log\widetilde{K}_q)}{e^{\delta_{\mathcal S}T}}\\
&= \sum_{\substack{\gamma\in E_A^\infty\\ a^j\beta\gamma\in E_A^*}}\limsup_{T\to+\infty}\frac{N_{\alpha^j\beta\gamma\gamma^+}(a^j\beta\gamma,T+3\log\widetilde{K}_q)}{e^{\delta_{\mathcal S}(T+3\log\widetilde{K}_q)}}\widetilde{K}_q^{3\delta_{\mathcal S}} \qquad (3.15)\\
&= \widetilde{K}_q^{3\delta_{\mathcal S}}\sum_{\substack{\gamma\in E_A^\infty\\ a^j\beta\gamma\in E_A^*}}\frac{\psi_{\delta_{\mathcal S}}(a^j\beta\gamma\gamma^+)}{\delta_{\mathcal S}\chi_{\delta_{\mathcal S}}}m_{\delta_{\mathcal S}}([a^j\beta\gamma])\\
&\le K_q(a^j\beta)\frac{1}{\delta_{\mathcal S}\chi_{\delta_{\mathcal S}}}\mu_{\delta_{\mathcal S}}([a^j\beta]).
\end{aligned}
$$

Taking the limit as $q\to+\infty$ in (3.15) we thus get that

$$\liminf_{T\to+\infty}\frac{N_p(a^j\beta,T)}{e^{\delta_{\mathcal S}T}}\le\frac{1}{\delta_{\mathcal S}\chi_{\delta_{\mathcal S}}}\mu_{\delta_{\mathcal S}}([a^j\beta]).$$

Along with (3.13) this gives

$$\lim_{T\to+\infty}\frac{N_p(a^j\beta,T)}{e^{\delta_{\mathcal S}T}}=\frac{1}{\delta_{\mathcal S}\chi_{\delta_{\mathcal S}}}\mu_{\delta_{\mathcal S}}([a^j\beta]). \qquad (3.16)$$

Passing to the upper bound in the general case, we only need to deal with powers of parabolic elements. Because of (3.14) and Theorem 3.1.6 (b1)–(b3), formula (3.2) holds for all words $\tau=a^l$, $l\ge1$, where $a\in\Omega$ is such that $\delta_{\mathcal S}\le\frac{2p_a}{p_a+1}$. In what follows, we can thus assume that

$$\delta_{\mathcal S}>\frac{2p_a}{p_a+1}.$$

Then for every integer $j\ge-1$, we have

$$[a^{j+1}]\setminus[a^{j+2}]=\bigcup\big\{[a^{j+1}e]:e\in E\setminus\{a\}\text{ and}A_{ae}=1\big\}. \qquad (3.17)$$

Since the set $E\setminus\{a\}$ is finite it thus follows from (3.16) that

$$\frac{N_p([a^{j+1}]\setminus[a^{j+2}],T)}{e^{\delta_{\mathcal S}T}}=\frac{1}{\delta_{\mathcal S}\chi_{\delta_{\mathcal S}}}\mu_{\delta_{\mathcal S}}([a^{j+1}]\setminus[a^{j+2}])=\frac{1}{\delta_{\mathcal S}\chi_{\delta_{\mathcal S}}}(\mu_{\delta_{\mathcal S}}([a^{j+1}])-\mu_{\delta_{\mathcal S}}([a^{j+2}])). \qquad (3.18)$$

Now if $\omega\in[a^{j+1}]\setminus[a^{j+2}]$ then $\omega=a^j(a\kappa a^l)$ with $\kappa_1,\kappa_{|\kappa|}\in E\setminus\{a\}$, $A_{a\kappa_1}=1$, $A_{\kappa_{|\kappa|}a}=1$, and $l\ge0$. Then

$$e^{-T}\le\big|\varphi'_{a^j(a\kappa a^l)}(x_{a^j(a\kappa a^l)})\big|=\big|\varphi'_{a\kappa a^l}(x_{a^j(a\kappa a^l)})\big|\cdot\big|\varphi'_{a^j}(x_{a\kappa a^{j+l}})\big|\asymp(j+2)^{-(p_a+1)/p_a}\big|\varphi'_{a\kappa a^l}(x_{a^j(a\kappa a^l)})\big|.$$

Denoting by $Q\ge1$ the multiplicative constant corresponding to the "$\asymp$" sign above, we thus get

$$\big|\varphi'_{a\kappa a^l}(x_{a^j(a\kappa a^l)})\big|\ge Q^{-1}(j+2)^{-\frac{p_a+1}{p_a}}e^{-T}. \qquad (3.19)$$

Now fix a word $\beta\in E_A^\infty$ with $\beta_1=a$ and $\beta_2\ne a$. Then

$$\big|\varphi'_{a\kappa a^l}(x_{a^j(a\kappa a^l)})\big|=\big|\varphi'_{a^l}(x_{a^j(a\kappa a^l)})\big|\cdot|\varphi'_{a\kappa}(x_{a^{j+l}\kappa})|\asymp(j+l+2)^{-\frac{p_a+1}{p_a}}\cdot j^{\frac{p_a+1}{p_a}}\big|\varphi'_{a\kappa}(\pi_{\mathcal S}(\beta))\big|.$$

It therefore follows from (3.19) that

$$\big|\varphi'_{ax}(\pi_{\mathcal S}(\beta))\big|\ge Q^{-2}(j+l+2)^{\frac{p_a+1}{p_a}}e^{-T}.$$

Equivalently,

$$-\log\left|\varphi'_{a\kappa}(\pi_{\mathcal{S}}(\beta))\right| \le 2\log Q - \frac{(p_a+1)}{p_a}\log(j+l+2)+T.$$

Hence

$$a\kappa \in \pi_\beta\left([a\kappa_1], 2\log Q - \frac{(p_a+1)}{p_a}\log(j+l+2)+T\right).$$

Therefore,

$$N_p([\alpha^{j+1}]\setminus[a^{j+2}]) \le \sum_{\substack{b\in E\setminus\{a\}\\ A_{ab}=1}}\sum_{l=0}^{\infty} N_\beta\left([ab], 2\log Q - \frac{p_a+1}{p_a}\log(j+l+2)+T\right). \tag{3.20}$$

By formula (3.1), and since the alphabet E is finite, there exists $T_1>0$ such that

$$e^{-\delta_{\mathcal{S}}S}N_\beta([ab],S) \le \frac{\psi_{\delta_{\mathcal{S}}}(\beta)}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}}m_{\delta_{\mathcal{S}}}([ab]) \le \frac{\psi_{\delta_{\mathcal{S}}}(\beta)}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}} \tag{3.21}$$

for every $b\in E\setminus\{a\}$ with $A_{ab}=1$ and every $S\ge T_1$. Now

$$2\log Q - \frac{p_a+1}{p_a}\log(j+l+2)+T \ge T_1$$

if and only if

$$j+l+2 \le s_T := Q^{\frac{2p_a}{p_a+1}}\exp\left(\frac{p_a}{p_a+1}(T-T_1)\right). \tag{3.22}$$

In addition, if

$$2\log Q - \frac{p_a+1}{p_a}\log(j+l+2)+T \le -1, \tag{3.23}$$

then

$$N_\beta\left([ab], 2\log Q - \frac{p_a+1}{p_a}\log(j+l+2)+T\right) = 0. \tag{3.24}$$

Formula (3.23) just means that

$$j+l+2 \ge u_T := eQ^{\frac{2p_a}{p_a+1}}e^{\frac{p_a}{p_a+1}T}. \tag{3.25}$$

Therefore, returning to formula (3.20), for every $q \geq 1$ we get that

(3.26)

$$
\begin{aligned}
&\sum_{j=q+1}^{\infty} e^{-\delta_{\mathcal{S}} T} N_p([\alpha^{j+1}] \setminus [a^{j+2}]) \leq \\
&\quad \leq \sum_{\substack{b \in E\setminus\{a\} \\ A_{ab}=1}} \sum_{j=q}^{\infty} \sum_{l : j+l+2 \leq s_T} \frac{N_\beta\left([ab], 2\log Q - \frac{p_a+1}{p_a}\log(j+l+2)+T\right)}{\exp\left(\delta_{\mathcal{S}}(2\log Q - \frac{p_a+1}{p_a}\log(j+l+2)+T)\right)} Q^{2\delta_{\mathcal{S}}}(j+l+2)^{-\frac{p_a+1}{p_a}\delta_{\mathcal{S}}} + \\
&\qquad\qquad + \sum_{s_T+1 \leq j+l+2 \leq u_T} e^{-\delta_{\mathcal{S}} T} N_\beta([ab], T_1) \\
&\quad \leq Q^{2\delta_{\mathcal{S}}} \# E \frac{\psi_{\delta_{\mathcal{S}}}(\beta)}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}} \sum_{j=q}^{\infty}\sum_{k=j}^{\infty} k^{-\frac{p_a+1}{p_a}\delta_{\mathcal{S}}} + N_a e^{-\delta_{\mathcal{S}} T} u_T^2 \\
&\quad \leq \widehat{Q}_1 \sum_{j=q}^{\infty} j^{1-\frac{p_a+1}{p_a}\delta_{\mathcal{S}}} + \widehat{Q}_2 \exp\left(\left(\frac{2p_a}{p_a+1} - \delta_{\mathcal{S}}\right)T\right) \\
&\quad \leq \widehat{Q}_3 q^{2-\frac{p_a+1}{p_a}\delta_{\mathcal{S}}} + \widehat{Q}_2 \exp\left(\left(\frac{2p_a}{p_a+1} - \delta_{\mathcal{S}}\right)T\right),
\end{aligned}
$$

where $N_a := \max\left\{N_\beta([ab], T_1) : b \in E\setminus\{a\},\, A_{ab} = 1\right\}$, $\widehat{Q}_1, \widehat{Q}_2, \widehat{Q}_3 \geq 1$ are universal constants, and the last inequality holds because $\delta_{\mathcal{S}} > \frac{2p_a}{p_a+1}$.

Applying (3.18) and (3.26) we obtain for all integers $q \geq k+2$ the following estimate

$$
\begin{aligned}
&\varlimsup_{T\to+\infty} \left| \frac{N_p([a^k], T)}{e^{\delta_{\mathcal{S}} T}} - \frac{1}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}}\mu_{\delta_{\mathcal{S}}}([a^k]) \right| = \\
&= \varlimsup_{T\to+\infty} \left| \sum_{j=k-1}^{q} \frac{N_p([a^{j+1}] \setminus [a^{j+2}], T)}{e^{\delta_{\mathcal{S}} T}} + \sum_{j=q+1}^{\infty} \frac{N_p([a^{j+1}] \setminus [a^{j+2}], T)}{e^{\delta_{\mathcal{S}} T}} - \frac{1}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}}\mu_{\delta_{\mathcal{S}}}([a^k]) \right| \\
&\leq \varlimsup_{T\to+\infty} \left| \sum_{j=k-1}^{q} \frac{N_p([a^{j+1}] \setminus [a^{j+2}], T)}{e^{\delta_{\mathcal{S}} T}} - \frac{1}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}}\mu_{\delta_{\mathcal{S}}}([a^k]) \right| + \varlimsup_{T\to+\infty} \left| \sum_{j=q+1}^{\infty} \frac{N_p([a^{j+1}] \setminus [a^{j+2}], T)}{e^{\delta_{\mathcal{S}} T}} \right| \\
&\leq \frac{1}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}} \left| \sum_{j=k-1}^{q} \mu_{\delta_{\mathcal{S}}}([a^{j+1}] \setminus [a^{j+2}]) - \mu([a^k]) \right| + \varlimsup_{T\to+\infty} \widehat{Q}_2 \sum_{a\in\Omega} \exp\left(\left(\frac{2p_a}{p_a+1} - \delta_{\mathcal{S}}\right)T\right) \\
&= \frac{1}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}} \left| \mu_{\delta_{\mathcal{S}}}([a^k] \setminus [a^{q+2}]) - \mu([a^k]) \right| \\
&= \frac{1}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}} \mu_{\delta_{\mathcal{S}}}([a^{q+2}]).
\end{aligned}
$$

But since $\delta_{\mathcal{S}} > \frac{2p_a}{p_a+1}$ we have that $\lim_{q\to\infty} \mu_{\delta_{\mathcal{S}}}([a^{q+2}]) = 0$ and therefore

$$
\varlimsup_{T\to+\infty} \left| \frac{N_p([a^k], T)}{e^{\delta_{\mathcal{S}} T}} - \frac{1}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}}\mu_{\delta_{\mathcal{S}}}([a^k]) \right| = 0.
$$

This just means that

$$
\lim_{T\to\infty} \frac{N_p([a^k], T)}{e^{\delta_{\mathcal{S}} T}} = \frac{1}{\delta_{\mathcal{S}}\chi_{\delta_{\mathcal{S}}}}\mu_{\delta_{\mathcal{S}}}([a^k]).
$$

The proof of our theorem is thus complete. $\square$

The proof of the following theorem, based on Theorem 3.3.1, is exactly the same as the proof of Theorem 2.4.9 based on Theorem 2.4.8.

THEOREM 3.3.2 (Asymptotic Equidistribution of Multipliers for Parabolic Systems II). *Suppose that $\mathcal{S}$ is an irreducible parabolic conformal GDMS. Fix $\rho \in E_A^\infty$. If $B \subset X$ is a Borel set such that $\widetilde{m}_{\delta_{\mathcal{S}}}(\partial B) = 0$ (equivalently $\widetilde{\mu}_{\delta_{\mathcal{S}}}(\partial B) = 0$) then,*

$$\lim_{T \to +\infty} \frac{N_\rho(B,T)}{e^{\delta_{\mathcal{S}} T}} = \frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{\delta_{\mathcal{S}} \chi_{\mu_{\delta_{\mathcal{S}}}}} \widetilde{m}_{\delta_{\mathcal{S}}}(B) \tag{3.27}$$

and

$$\lim_{T \to +\infty} \frac{N_p(B,T)}{e^{\delta_{\mathcal{S}} T}} = \frac{1}{\delta_{\mathcal{S}} \chi_{\mu_{\delta_{\mathcal{S}}}}} \widetilde{\mu}_{\delta_{\mathcal{S}}}(B). \tag{3.28}$$

We have as an immediate corollary the following:

THEOREM 3.3.3 (Asymptotic Equidistribution of Multipliers for Parabolic Systems). *Suppose that $\mathcal{S}$ is an irreducible parabolic conformal GDMS. Fix $\rho \in E_A^\infty$. Then*

$$\lim_{T \to +\infty} \frac{N_\rho(T)}{e^{\delta_{\mathcal{S}} T}} = \frac{\psi_{\delta_{\mathcal{S}}}(\rho)}{\delta_{\mathcal{S}} \chi_{\mu_{\delta_{\mathcal{S}}}}} \tag{3.29}$$

and

$$\lim_{T \to +\infty} \frac{N_p(T)}{e^{\delta_{\mathcal{S}} T}} = \frac{1}{\delta_{\mathcal{S}} \chi_{\mu_{\delta_{\mathcal{S}}}}} \widetilde{\mu}_{\delta_{\mathcal{S}}}(J_{\mathcal{S}}). \tag{3.30}$$

3.4. Asymptotic Results for Diameters

We now want to use the asymptotic results established in the previous section to show the asymptotic formulae for diameters of images of a set.

In this section, as in the previous one, we assume that $\mathcal{S}$ is an irreducible conformal parabolic GDMS. Our task here is, for parabolic systems, the same as the one in Section 2.7 for attracting systems, i.e. to obtain asymptotic counting properties corresponding to the function $-\log \operatorname{diam}(\varphi_\omega(Y))$, $\omega \in E_A^*$. The notation here is the same as in Section 2.7. Our strategy now is to use the full generality of Theorem 2.7.1 and to deduce from it the first main result of the current section, which is the following.

THEOREM 3.4.1 (Asymptotic Equidistribution Formula of Diameters for Parabolic Systems, I). *Suppose that $\mathcal{S}$ is an irreducible parabolic conformal GDMS. Fix $\rho \in E_A^\infty$ and $Y \subseteq X_{i(\rho)}$ having at least two points. If $B \subset X$ is a Borel set such that $\widetilde{m}_{\delta_{\mathcal{S}}}(\partial B) = 0$ (equivalently $\widetilde{\mu}_{\delta_{\mathcal{S}}}(\partial B) = 0$) then,*

$$\lim_{T \to +\infty} \frac{D_Y^\rho(B,T)}{e^{\delta_{\mathcal{S}} T}} = C_\rho(Y) \widetilde{m}_{\delta_{\mathcal{S}}}(B), \tag{3.1}$$

where $C_\rho(Y) \in (0, +\infty]$ is a constant depending only on the system $\mathcal{S}$, the word ρ (but see Remark 3.4.3), and the set Y. In addition $C_\rho(Y)$ is finite if and only if either

(1)
$$\overline{Y} \cap \Omega_\infty = (\overline{Y} \cap \Omega_\infty \cap \Omega_{\rho_1}) = \emptyset$$

or

(2)
$$\delta_{\mathcal{S}} > \max\{p(a) : a \in \Omega_{\rho_1} \text{ and } x_a \in \overline{Y}\}.$$

Then the function $[\rho_1] \ni \omega \longmapsto C_\omega(Y)$ is uniformly separated away from zero and bounded above.

PROOF. Recall that

$$\Omega_{\rho_1} = \{a \in \Omega : A_{a\rho_1} = 1\}.$$

We know that

$$E_\rho^* = E_{*\rho}^* \cup \bigcup_{a\in\Omega_\rho} \bigcup_{k=1}^\infty E_{*a}^* a^k$$

and this union consists of mutually incomparable terms. Therefore,

$$\mathcal{D}_Y^\rho(B,T) = \mathcal{D}_{Y,\mathcal{S}^*}^\rho(B,T) \cup \bigcup_{a\in\Omega_\rho} \bigcup_{k=1}^\infty \mathcal{D}_{\varphi_{a^k}(Y),\mathcal{S}^*}^{a^k\rho}(B,T),$$

and this union consists of mutually disjoint terms. Therefore,

$$\frac{D_Y^\rho(B,T)}{e^{\delta_{\mathcal{S}}T}} \ge \frac{D_{Y,\mathcal{S}^*}^\rho(B,T)}{e^{\delta_{\mathcal{S}}T}} + \sum_{a\in\Omega_\rho} \sum_{k=1}^\infty \frac{D_{\varphi_{a^k}(Y),\mathcal{S}^*}^{a^k\rho}(B,T)}{e^{\delta_{\mathcal{S}}T}}, \tag{3.2}$$

and for every $q \ge 1$:

$$\frac{D_Y^\rho(B,T)}{e^{\delta_{\mathcal{S}}T}} \le \frac{D_{Y,\mathcal{S}^*}^\rho(B,T)}{e^{\delta_{\mathcal{S}}T}} + \sum_{a\in\Omega_\rho} \sum_{k=1}^q \frac{D_{\varphi_{a^k}(Y),\mathcal{S}^*}^{a^k\rho}(B,T)}{e^{\delta_{\mathcal{S}}T}} + \sum_{a\in\Omega_\rho} \sum_{k=q+1}^\infty \frac{D_{\varphi_{a^k}(Y),\mathcal{S}^*}^{a^k\rho}(B,T)}{e^{\delta_{\mathcal{S}}T}}. \tag{3.3}$$

Assume first that $\rho \in E_{*A^*}^{\mathbb{N}}$. Then, $a^k\rho \in E_{*A^*}^{\mathbb{N}}$ for every $a \in \Omega_\rho$ and for all integers $k \ge 0$, whence we can invoke Theorem 2.7.1 and 3.2, to conclude that

$$\begin{aligned}
\varliminf_{T\to\infty} \frac{D_Y^\rho(B,T)}{e^{\delta_{\mathcal{S}}T}} &\ge \varliminf_{T\to\infty} \frac{D_{Y,\mathcal{S}^*}^\rho(B,T)}{e^{\delta_{\mathcal{S}}T}} + \sum_{a\in\Omega_\rho} \sum_{k=1}^\infty \varliminf_{T\to\infty} \frac{D_{\varphi_{a^k}(Y),\mathcal{S}^*}^{a^k\rho}(B,T)}{e^{\delta_{\mathcal{S}}T}} \\
&= \Big(C_\rho^{\mathcal{S}^*}(Y) + \sum_{a\in\Omega_\rho} \sum_{k=1}^\infty C_{a^k\rho}^{\mathcal{S}^*}(\varphi_{a^k}(Y))\Big) m_{\delta_{\mathcal{S}}}^*(B) \\
&= \Big(C_\rho^{\mathcal{S}^*}(Y) + \sum_{a\in\Omega_\rho} \sum_{k=1}^\infty C_{a^k\rho}^{\mathcal{S}^*}(\varphi_{a^k}(Y))\Big) m_{\delta_{\mathcal{S}}}(B).
\end{aligned} \tag{3.4}$$

Since for every $a \in \Omega_\rho$ and for all integers $k \ge 0$

$$\operatorname{diam}\big(\varphi_{a^k}(Y)\big) \asymp \begin{cases} (k+1)^{-\frac{1}{p_a}} & \text{if} \quad \overline{Y} \cap \Omega_\infty \cap \Omega_\rho \ne \emptyset, \\ (k+1)^{-\frac{p_a+1}{p_a}} & \text{if} \quad \overline{Y} \cap \Omega_\infty \cap \Omega_\rho = \emptyset, \end{cases}$$

formula (3.4) along with (2.2), complete the proof of Theorem 3.4.1 if neither (1) nor (2) hold. So, for the rest of the proof of the present case of $\rho \in E_{*A^*}^{\mathbb{N}}$, we assume that at least one of (1) or (2) holds. Then

$$C_\rho^{\mathcal{S}^*}(Y) + \sum_{a\in\Omega_\rho} \sum_{k=1}^\infty C_{a^k\rho}^{\mathcal{S}^*}(\varphi_{a^k}(Y)) < +\infty, \tag{3.5}$$

and in addition, this number is bounded away from zero and bounded above independently of $\rho \in E_{*A}^{\mathbb{N}}$ because of (2.2).

Now fix $a \in \Omega_\rho$. If $\omega \in \mathcal{D}_{\varphi_{a^k}(Y),\mathcal{S}^*}^{a^k\rho}(B,T)$, then

$$\operatorname{diam}\big(\varphi_\omega(\varphi_{a^k}(Y))\big) \ge e^{-T},$$

and, as

$$\begin{aligned}
\operatorname{diam}\big(\varphi_\omega(\varphi_{a^k}(Y))\big) &\le \|\varphi_\omega'\|_\infty \operatorname{diam}\big(\varphi_{a^k}(Y)\big) \le Q_1 \operatorname{diam}(\varphi_\omega(X_{t(\omega)}))\operatorname{diam}\big(\varphi_{a^k}(Y)\big) \\
&= Q_1 \operatorname{diam}(\varphi_\omega(X_{i(a)}))\operatorname{diam}\big(\varphi_{a^k}(Y)\big),
\end{aligned}$$

with some constant $Q_1 > 0$, we thus conclude that

$$\operatorname{diam}(\varphi_\omega(X_{i(\omega)})) \geq Q_1^{-1} e^{-T} \operatorname{diam}^{-1}\big(\varphi_{a^k}(Y)\big).$$

Equivalently,

$$\Delta_{X_{i(a)}}(\omega) \leq \log Q_1 + \log \operatorname{diam}\big(\varphi_{a^k}(Y)\big) + T.$$

Thus

$$\omega \in \mathcal{D}^{a\rho}_{X_{i(a)},\mathcal{S}^*}\big(\log Q_1 + \log \operatorname{diam}\big(\varphi_{a^k}(Y)\big) + T\big).$$

In conclusion,

$$\mathcal{D}^{a^k\rho}_{\varphi_{a^k}(Y),\mathcal{S}^*}(B,T) \subseteq \mathcal{D}^{a\rho}_{X_{i(a)},\mathcal{S}^*}\big(\log Q_1 + \log \operatorname{diam}\big(\varphi_{a^k}(Y)\big) + T\big). \tag{3.6}$$

By virtue of Theorem 2.7.1 there exists $T_1 > 0$ such that

$$\frac{D^{a\rho}_{X_{i(a)},\mathcal{S}^*}(B,S)}{e^{\delta_{\mathcal{S}} S}} \leq C^{\mathcal{S}^*}_{a\rho}(X_{i(a)}) + 1 \tag{3.7}$$

for all $S \geq T_1$. Now, let $k_2(T)$ be the least integer such that

$$\log Q_2 + \log \operatorname{diam}\big(\varphi_{a^k}(Y)\big) + T < 0.$$

Then

$$\mathcal{D}^{a\rho}_{X_{i(a)},\mathcal{S}^*}\big(\log Q_2 + \log \operatorname{diam}\big(\varphi_{a^k}(Y)\big) + T\big) = \emptyset \tag{3.8}$$

for all $k \geq k_2(T)$ and

$$k_2(T) \leq \begin{cases} Q_2^{p_a} e^{p_a(T-T_1)} & \text{if } (2) \text{ holds} \\ Q_2^{\frac{p_a}{p_a+1}} e^{\frac{p_a}{p_a+1}(T-T_1)} & \text{if } (1) \text{ holds} \end{cases}$$

with some constant $Q_2 \in (0, +\infty)$, which in general depends on Y if (1) holds. Furthermore, let $k_1(T)$ be least integer such that

$$\log Q_2 + \log \operatorname{diam}\big(\varphi_{a^k}(Y)\big) + T < T_1.$$

Then, on the one hand,

$$\log Q_2 + \log \operatorname{diam}\big(\varphi_{a^k}(Y)\big) + T < T_1,$$

for all $k \geq k_1(T)$ and (so) it follows from (3.6) that

$$\mathcal{D}^{a^k\rho}_{\varphi_{a^k}(Y),\mathcal{S}^*}(B,T) \subseteq \mathcal{D}^{a\rho}_{X_{i(a)},\mathcal{S}^*}(T_1).$$

On the other hand,

$$\log Q_2 + \log \operatorname{diam}\big(\varphi_{a^k}(Y)\big) + T \geq T_1$$

for all $0 \le k \le k_1(T)$. All of this, together with (3.6)–(3.8), yield

$$
\begin{aligned}
\sum_{k=q+1}^{\infty} \frac{D^{a^k\rho}_{\varphi_{a^k}(Y),\mathcal{S}^*}(B,T)}{e^{\delta_{\mathcal{S}}T}} &= \\
&= \sum_{k=q+1}^{k_1(T)} \frac{D^{a^k\rho}_{\varphi_{a^k}(Y),\mathcal{S}^*}(B,T)}{e^{\delta_{\mathcal{S}}T}} + \sum_{k=k_1(T)+1}^{k_2(T)} \frac{D^{a^k\rho}_{\varphi_{a^k}(Y),\mathcal{S}^*}(B,T)}{e^{\delta_{\mathcal{S}}T}} \\
&\le \sum_{k=q+1}^{[Q_2e^{p_a(T-T_1)}]} \frac{D^{a\rho}_{X_{i(a)},\mathcal{S}^*}\big(\log Q_2 + \log\operatorname{diam}\big(\varphi_{a^k}(Y)\big)+T\big)}{\exp\big(\delta_{\mathcal{S}}\big(\log Q_2+\log\operatorname{diam}\big(\varphi_{a^k}(Y)\big)+T\big)\big)} Q_2^{\delta_{\mathcal{S}}}\operatorname{diam}^{\delta_{\mathcal{S}}}\big(\varphi_{a^k}(Y)\big)+ \\
&\qquad + \sum_{k=k_1(T)+1}^{k_2(T)} \frac{D^{a\rho}_{X_{i(a)},\mathcal{S}^*}(T_1)}{e^{\delta_{\mathcal{S}}T_1}} e^{\delta_{\mathcal{S}}(T_1-T)} \\
&\le Q_2^{\delta_{\mathcal{S}}} \sum_{k=q+1}^{k_1(T)} \big(C^{\mathcal{S}^*}_{a\rho}(X_{i(a)})+1\big)\operatorname{diam}^{\delta_{\mathcal{S}}}\big(\varphi_{a^k}(Y)\big) + \big(C^{\mathcal{S}^*}_{a\rho}(X_{i(a)})+1\big)e^{\delta_{\mathcal{S}}(T_1-T)}k_2(T) \\
&\le Q_2^{\delta_{\mathcal{S}}} \sum_{k=q+1}^{\infty} \big(C^{\mathcal{S}^*}_{a\rho}(X_{i(a)})+1\big)\operatorname{diam}^{\delta_{\mathcal{S}}}\big(\varphi_{a^k}(Y)\big) + \big(C^{\mathcal{S}^*}_{a\rho}(X_{i(a)})+1\big)e^{\delta_{\mathcal{S}}(T_1-T)}k_2(T).
\end{aligned}
\tag{3.9}
$$

Denote by $\Sigma_1(q,T)$ the maximum over all $a \in \Omega_\rho$ of the first term in the last line of the above formula and by $\Sigma_2(T)$ the second term. Because we are assuming either (1) or (2) from our current theorem, we have that in either case

$$
\lim_{q\to\infty} \Sigma_1(q,T) = 0 \quad\text{and}\quad \lim_{T\to\infty}\Sigma_2(T) = 0. \tag{3.10}
$$

Keeping $q \ge 1$ fixed, inserting (3.9) to (3.3), and applying Theorem 2.7.1, we obtain

$$
\begin{aligned}
\varlimsup_{T\to\infty} \frac{D^\rho_Y(B,T)}{e^{\delta_{\mathcal{S}}T}} &\le \\
&\le \varlimsup_{T\to\infty} \frac{D^\rho_{Y,\mathcal{S}^*}(B,T)}{e^{\delta_{\mathcal{S}}T}} + \sum_{a\in\Omega_\rho}\sum_{k=1}^{q} \varlimsup_{T\to\infty} \frac{D^{a^k\rho}_{\varphi_{a^k}(Y),\mathcal{S}^*}(B,T)}{e^{\delta_{\mathcal{S}}T}} + \#\Omega_\rho\big(\Sigma_1(q,T)+\Sigma_2(T)\big) \\
&\le \left(C^{\mathcal{S}^*}_\rho(Y) + \sum_{a\in\Omega_\rho}\sum_{k=1}^{q} C^{\mathcal{S}^*}_{a^k\rho}(\varphi_{a^k}(Y))\right)\widetilde{m}_{\delta_{\mathcal{S}}}(B) + \#\Omega\big(\Sigma_1(q,T)+\Sigma_2(T)\big).
\end{aligned}
$$

Therefore, invoking (3.10), we obtain by letting $q \to \infty$, that

$$
\varlimsup_{T\to\infty} \frac{D^\rho_Y(B,T)}{e^{\delta_{\mathcal{S}}T}} \le \left(C^{\mathcal{S}^*}_\rho(Y) + \sum_{a\in\Omega_\rho}\sum_{k=1}^{\infty} C^{\mathcal{S}^*}_{a^k\rho}(\varphi_{a^k}(Y))\right)\widetilde{m}_{\delta_{\mathcal{S}}}(B).
$$

Along with (3.4) this shows that formula (3.1) holds. The number

$$
C^{\mathcal{S}^*}_\rho(Y) + \sum_{a\in\Omega_\rho}\sum_{k=1}^{\infty} C^{\mathcal{S}^*}_{a^k\rho}(\varphi_{a^k}(Y))
$$

is finite because of (3.5). Invoking also the sentence following this formula, we conclude the proof in the case of words $\rho \in E^{\mathbb{N}}_{*A^*}$.

Now, we pass to the general case, i.e., all we assume is that $\rho \in E_A^{\mathbb{N}}$. For every $k \geq 1$ choose $\rho^{(k)} \in E_{*A^*}^{\mathbb{N}}$ such that

$$\rho^{(k)}|_k = \rho|_k.$$

We already know that there exists a constant $M \geq 1$ such that

$$M^{-1} \leq C_Y(\rho^{(k)}) \leq M$$

for all integers $k \geq 1$. So, passing to a subsequence, we may assume without loss of generality that the limit

$$\lim_{k\to+\infty} C_Y(\rho^{(k)})$$

exists and belongs to the interval $[M^{-1}, M]$. We denote this limit by $C_Y(\rho)$.

Assume first that $B \subseteq X$ is an open set. In order to emphasize the openness of the set B and in order to clearly separate the present setup from the next one, we now denote B by V. Fixing $\varepsilon > 0$, there then exist F_ε, a compact subset of V and a number $r(\varepsilon) > 0$ such that

$$\widetilde{m}_{\delta_{\mathcal{S}}}(V \backslash F_\varepsilon) < \varepsilon \quad \text{and} \quad \widetilde{m}_{\delta_{\mathcal{S}}}(B(V, r(\varepsilon)) \backslash V) < \varepsilon \tag{3.11}$$

and

$$\widetilde{m}_{\delta_{\mathcal{S}}}(\partial F_\varepsilon) = 0 \quad \text{and} \quad \widetilde{m}_{\delta_{\mathcal{S}}}(\partial B(V, r(\varepsilon))) = 0, \tag{3.12}$$

where in writing the latter of these four requirements we used the fact that $\widetilde{m}_{\delta_{\mathcal{S}}}(\partial V) = 0$. Hence there exists $k \geq 1$ so large that for every $\omega \in E_\rho$ (simultaneously meaning that $\omega \in E_{\rho_k}$, we have that

$$\varphi_\omega\big(\pi_{\mathcal{S}}(\rho^{(k)})\big) \in F_\varepsilon \quad \Longrightarrow \quad \varphi_\omega(\pi_{\mathcal{S}}(\rho) \in V)$$

and

$$\varphi_\omega\big(\pi_{\mathcal{S}}(\rho)\big) \in V \quad \Longrightarrow \quad \varphi_\omega\big(\pi_{\mathcal{S}}(\rho^{(k)})\big) \in B(V, r(\varepsilon))).$$

Therefore, for every $T > 0$,

$$\mathcal{D}_Y^{\rho^{(k)}}(F_\varepsilon, T) \subseteq \mathcal{D}_Y^{\rho}(V, T) \subseteq \mathcal{D}_Y^{\rho^{(k)}}(B(V, r(\varepsilon)), T)$$

so,

$$D_Y^{\rho^{(k)}}(F_\varepsilon, T) \leq D_Y^{\rho}(V, T) \leq D_Y^{\rho^{(k)}}(B(V, r(\epsilon)), T).$$

Hence, applying the already proven assertion for words in $E_{*A^*}^\infty$ one gets

$$\begin{aligned} C_{\rho^{(k)}}(Y)\widetilde{m}_{\delta_{\mathcal{S}}}(F_\epsilon) &= \lim_{T\to+\infty} \frac{D_Y^{(\rho^{(k)})}(F_\epsilon, T)}{e^{\delta_{\mathcal{S}} T}} \leq \liminf_{T\to+\infty} \frac{D_Y^{(\rho)}(V, T)}{e^{\delta_{\mathcal{S}} T}} \leq \limsup_{T\to+\infty} \frac{D_Y^{(\rho)}(V, T)}{e^{\delta_{\mathcal{S}} T}} \\ &\leq \lim_{T\to+\infty} \frac{D_Y^{\rho^{(k)}}(B(V, r(\epsilon), T))}{e^{\delta_{\mathcal{S}} T}} = C_{\rho^{(k)}}(Y)\widetilde{m}_{\delta_{\mathcal{S}}}(B(V, r(\epsilon))). \end{aligned}$$

So, letting $k \to +\infty$ and invoking (3.12) we obtain that

$$C_\rho(Y)\widetilde{m}_{\delta_{\mathcal{S}}}(F_\epsilon) \leq \liminf_{T\to+\infty} \frac{D_Y^\rho(V, T)}{e^{\delta_{\mathcal{S}} T}} \leq \limsup_{T\to+\infty} \frac{D_Y^\rho(V, T)}{e^{\delta_{\mathcal{S}} T}} \leq C_\rho(Y)\widetilde{m}_{\delta_{\mathcal{S}}}(B(V, r(\epsilon))).$$

Hence, letting $\epsilon \to 0$ and invoking 3.11 we get that

$$C_\rho(Y)\widetilde{m}_{\delta_{\mathcal{S}}}(V) \leq \liminf_{T\to+\infty} \frac{D_Y^\rho(V, T)}{e^{\delta_{\mathcal{S}} T}} \leq \limsup_{T\to+\infty} \frac{D_Y^\rho(V, T)}{e^{\delta_{\mathcal{S}} T}} \leq C_\rho(Y)\widetilde{m}_{\delta_{\mathcal{S}}}(V),$$

and the theorem is fully proved for all open sets B. Having shown this, the general case can be taken care of in exactly the same way as the part of the proof of Theorem 2.4.9, starting right after formula (2.12). This completes the proof. □

Having established Theorem 3.4.1, by proceeding in a similar way to the way Theorem 2.7.4 was based on Theorem 2.7.1, we derive from Theorem 3.4.1, the following second main result of the current section.

THEOREM 3.4.2 (Asymptotic Equidistribution Formula of Diameters for Parabolic Systems, II). *Suppose that $\mathcal{S}$ is an irreducible parabolic conformal GDMS. Fix $\rho \in E_A^\infty$ and $Y \subseteq X_{i(\rho)}$ having at least two points and such that $\pi_{\mathcal{S}}(\rho) \in Y$. If $B \subset X$ is a Borel set such that $\widetilde{m}_{\delta_{\mathcal{S}}}(\partial B) = 0$ (equivalently $\widetilde{\mu}_{\delta_{\mathcal{S}}}(\partial B) = 0$) then,*

$$\lim_{T \to +\infty} \frac{E_Y^\rho(B,T)}{e^{\delta_{\mathcal{S}} T}} = C_\rho(Y) \widetilde{m}_{\delta_{\mathcal{S}}}(B), \tag{3.13}$$

where $C_\rho(Y) \in (0, +\infty]$ is a constant (the same as that of Theorem 3.4.1) depending only on the system $\mathcal{S}$, the word ρ (but see Remark 3.4.3), and the set Y. In addition $C_\rho(Y)$ is finite if and only if either

(1)
$$\overline{Y} \cap \Omega_\infty = (\overline{Y} \cap \Omega_\infty \cap \Omega_\rho) = \emptyset$$

or

(2)
$$\delta_{\mathcal{S}} > \max\big\{p(a) : a \in \Omega_\rho \ \text{ and } \ x_a \in \overline{Y}\big\}.$$

Then the function $[\rho_1] \ni \omega \longmapsto C_\omega(Y)$ is uniformly bounded away from zero and bounded above.

REMARK 3.4.3. We now can essentially repeat Remark 2.7.5 verbatim with the only change being the replacement of Theorem 2.7.4 and Theorem 2.7.1, respectively, by Theorem 3.4.2 and Theorem 3.4.1. For the sake of completeness, convenience of the reader, and ease of referencing we summarize:

Since the left-hand side of (3.13) depends only on ρ_1, i.e. the first coordinate of ρ, we obtain that the constant $C_Y(\rho)$ of Theorem 3.4.2 and Theorem 3.4.1, depends in fact only on ρ_1. Again, we could have provided a direct argument for this already when proving Theorem 3.4.1 and this would not affect the proof of Theorem 3.4.2. Thus our approach seems most economical.

The last three results of this section are derived from the, already established, results, in the same way as the last three results of Section 2.7 were derived from the earlier results of that section.

THEOREM 3.4.4. *Suppose that $\mathcal{S}$ is an irreducible parabolic conformal GDMS with property (A). For any $v \in V$ let $Y_v \subseteq X_v$ having at least two points. If $B \subset X$ is a Borel set such that $\widetilde{m}_{\delta_{\mathcal{S}}}(\partial B) = 0$ (equivalently $\widetilde{\mu}_{\delta_{\mathcal{S}}}(\partial B) = 0$) and $\rho \in E_A^\infty$ is with $\rho_1 = a_v$, then,*

$$\lim_{T \to +\infty} \frac{D_Y^\rho(B,T)}{e^{\delta_{\mathcal{S}} T}} = \lim_{T \to +\infty} \frac{E_Y^\rho(B,T)}{e^{\delta_{\mathcal{S}} T}} = C_v(Y_v) \widetilde{m}_{\delta_{\mathcal{S}}}(B), \tag{3.14}$$

where $C_v(Y_v) \in (0, +\infty]$ is a constant depending only on the vertex $v \in V$ and the set Y_v. In particular, this holds for $Y_v := X_v$, $v \in V$. In addition $C_v(Y)$ is finite if and only if either

(1)
$$\overline{Y} \cap \Omega_\infty = (\overline{Y} \cap \Omega_\infty \cap \Omega_{a_v}) = \emptyset$$

or

(2)
$$\delta_{\mathcal{S}} > \max\big\{p(a) : a \in \Omega_{a_v} \ \text{ and } \ x_a \in \overline{Y}\big\}.$$

COROLLARY 3.4.5. *Suppose that $\mathcal{S}$ is an irreducible maximal parabolic conformal GDMS. For any $v \in V$ let $Y_v \subseteq X_v$ having at least two points be fixed. If $B \subset X$ is a Borel set such that $\widetilde{m}_{\delta_{\mathcal{S}}}(\partial B) = 0$ (equivalently $\widetilde{\mu}_{\delta_{\mathcal{S}}}(\partial B) = 0$) and $\rho \in E_A^\infty$ is with $i(\rho_1) = v$, then,*

$$\lim_{T \to +\infty} \frac{D_Y^\rho(B,T)}{e^{\delta_{\mathcal{S}} T}} = \lim_{T \to +\infty} \frac{E_Y^\rho(B,T)}{e^{\delta_{\mathcal{S}} T}} = C_v(Y_v) \widetilde{m}_{\delta_{\mathcal{S}}}(B), \tag{3.15}$$

where $C_v(Y_v) \in (0, +\infty]$ is a constant depending only on the vertex $v \in V$ and the set Y_v. In particular, this holds for $Y_v := X_v$, $v \in V$. In addition $C_v(Y)$ is finite if and only if either

(1)
$$\overline{Y} \cap \Omega_\infty = (\overline{Y} \cap \Omega_\infty \cap \Omega_v) = \emptyset$$

or

(2)
$$\delta_{\mathcal{S}} > \max\big\{p(a) : a \in \Omega_v \quad \text{and} \quad x_a \in \overline{Y}\big\}.$$

Corollary 3.4.6. *Suppose that $\mathcal{S}$ is a conformal parabolic IFS acting on a phase space X. Fix $Y \subseteq X$ having at least two points. If $B \subset X$ is a Borel set such that $\widetilde{m}_{\delta_{\mathcal{S}}}(\partial B) = 0$ (equivalently $\widetilde{\mu}_{\delta_{\mathcal{S}}}(\partial B) = 0$) and $\rho \in E_A^\infty$, then,*

$$\lim_{T\to+\infty} \frac{D_Y^\rho(B,T)}{e^{\delta_{\mathcal{S}} T}} = \lim_{T\to+\infty} \frac{E_Y^\rho(B,T)}{e^{\delta_{\mathcal{S}} T}} = C(Y)\widetilde{m}_{\delta_{\mathcal{S}}}(B), \tag{3.16}$$

where $C(Y) \in (0,+\infty]$ is a constant depending only on the set Y. In particular, this holds for $Y := X$. In addition $C(Y)$ is finite if and only if either

(1)
$$\overline{Y} \cap \Omega_\infty = \emptyset$$

or

(2)
$$\delta_{\mathcal{S}} > \max\big\{p(a) : a \in \Omega \quad \text{and} \quad x_a \in \overline{Y}\big\}.$$

CHAPTER 4

Central Limit Theorems

We now consider the distribution of weights and the Central Limit Theorems. In this section we will formulate the results in full generality and provide their applications in subsequent sections.

Let us consider a conformal, either attracting or parabolic, GDMS. As we did in previous sections, we can associate to finite words $\omega \in E_A^*$ both the weights $\lambda_i(\omega)$ $(i = p, \rho)$ and the word length $|\omega|$. We would like to understand how these quantities are related for typical orbits, which leads naturally to the study of Central Limit Theorems. The most familiar and natural formulation of Central Limit Theorems (CLT) is with respect to invariant measures. However, in the present context it is equally natural to give versions for preimages and periodic points.

4.1. Central Limit Theorems for Multipliers and Diameters: Attracting GDMSs with Invariant Measure $\mu_{\delta_{\mathcal{S}}}$

As an immediate consequence of Theorem 2.5.4 (which easiliy follows from Theorem 7.1 in [**65**]), Lemma 2.5.6, Lemma 4.8.8 from [**47**], and Remark 2.3.6 from our present monograph, we get the following version of the Central Limit Theorem for attracting systems and Gibbs/equilibrium states.

Theorem 4.1.1. *If $\mathcal{S}$ is a strongly regular finitely irreducible D–generic conformal GDMS* [1]*, then there exists $\sigma^2 > 0$ (in fact $\sigma^2 = \mathrm{P}''(0) \neq 0$ because of Remark 2.3.6 and since the system $\mathcal{S}$ is D–generic) such that if $G \subset \mathbb{R}$ is a Lebesgue measurable set with* $\mathrm{Leb}(\partial G) = 0$*, then*

$$\lim_{n\to+\infty} \mu_{\delta_{\mathcal{S}}}\left(\left\{\omega \in E_A^\infty : \frac{-\log\left|\varphi'_{\omega|_n}(\pi_{\mathcal{S}}(\sigma^n(\omega)))\right| - \chi_{\mu_{\delta_{\mathcal{S}}}} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty} \mu_{\delta_{\mathcal{S}}}\left(\left\{\omega \in E_A^\infty : \alpha \le \frac{-\log\left|\varphi'_{\omega|_n}(\pi_{\mathcal{S}}(\sigma^n(\omega)))\right| - \chi_{\mu_{\delta_{\mathcal{S}}}} n}{\sqrt{n}} \le \beta\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

Since by the Bounded Distortion Property (BDP) of the definition of attracting GDMSs, the numbers

$$\left|\log\operatorname{diam}\bigl(\varphi|_{\omega|_n}(Y_{t(\omega)})\bigr) - \log|\varphi'_{\omega|_n}(\pi_{\mathcal{S}}(\sigma^n(\omega))|\right|$$

are uniformly bounded above and since $\lim_{n\to+\infty}\sqrt{n} = +\infty$ we immediately obtain from Theorem 4.1.1 its version with $-\log\left|\varphi'_{\omega|_n}(\pi_{\mathcal{S}}(\sigma^n(\omega)))\right|$ replaced by $-\log\operatorname{diam}\bigl(\varphi|_{\omega|_n}(Y_{t(\omega)})\bigr)$. This gives the following.

Theorem 4.1.2. *Suppose that $\mathcal{S}$ is a strongly regular finitely irreducible D–generic conformal GDMS*[2]*. Let $\sigma^2 := \mathrm{P}''(0)(\neq 0)$. For every $v \in V$ let $Y_v \subset X_v$ be a set with at least two points. If $G \subset \mathbb{R}$ is a Lebesgue measurable set with* $\mathrm{Leb}(\partial G) = 0$*, then*

$$\lim_{n\to+\infty} \mu_{\delta_{\mathcal{S}}}\left(\left\{\omega \in E_A^\infty : \frac{-\log\operatorname{diam}(\varphi_{\omega|_n}(Y_{t(\omega_n)})) - \chi_{\mu_{\delta_{\mathcal{S}}}} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

---

[1] In fact $\mu_{\delta_{\mathcal{S}}}$ below can be replaced by the (unique) Gibbs/equilibrium state of any Hölder continuous summable potential $f : E_A^\infty \to \mathbb{R}$.

[2] In fact $\mu_{\delta_{\mathcal{S}}}$ below can be replaced by the (unique) Gibbs/equilibrium state of any Hölder continuous summable potential $f : E_A^\infty \to \mathbb{R}$.

In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty}\mu_{\delta_{\mathcal S}}\left(\left\{\omega\in E_A^\infty\,:\,\alpha\le\frac{-\log\mathrm{diam}(\varphi_{\omega|_n}(Y_{t(\omega_n)}))-\chi_{\mu_{\delta_{\mathcal S}}}n}{\sqrt{n}}\le\beta\right\}\right)=\frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

Also, as an immediate consequence of the appropriate results from [**47**] and Remark 2.3.6 from our present monograph, we get the following Law of Iterated Logarithm.

THEOREM 4.1.3. *Suppose that there* $\mathcal S$ *is a strongly regular finitely irreducible D–generic conformal GDMS*[3]. *Let* $\sigma^2 := \mathrm{P}''(0) > 0$. *For every* $v \in V$ *let* $Y_v \subset X_v$ *be a set with at least two points. Then for* $\mu_{\delta_{\mathcal S}}$*–a.e.* $\omega \in E_A^\infty$*, we have that*

$$\limsup_{n\to+\infty}\frac{-\log\left|(\varphi'_{\omega|_n}(\pi_{\mathcal S}(\sigma^n(\omega)))\right|-\chi_{\mu_{\delta_{\mathcal S}}}n}{\sqrt{n\log\log n}}=\sqrt{2\pi}\sigma$$

and

$$\limsup_{n\to+\infty}\frac{-\log\mathrm{diam}\big(\varphi_{\omega|_n}(Y_{t(\omega_n)})\big)-\chi_{\mu_{\delta_{\mathcal S}}}n}{\sqrt{n\log\log n}}=\sqrt{2\pi}\sigma.$$

REMARK 4.1.4. It is possible to reverse the roles of the word length and the weights. More precisely, given $\omega \in E_A$ and $t \ge 0$ we can define $n = n(t,\omega)$ to be the only integer for which

$$\lambda(\omega|_n) \le t < \lambda(\omega|_{n+1}).$$

Ergodicity of measure $\mu_{\delta_{\mathcal S}}$ and Birkhoff's Ergodic Theorem then yield

$$\lim_{t\to+\infty}\frac{t}{n(t,\omega)}=\chi_{\mu_{\delta_{\mathcal S}}}$$

for $\mu_{\delta_{\mathcal S}}$–a.e. $\omega \in E_A^\infty$. We claim that there exists $\sigma_0^2 > 0$ such that for any $\alpha < \beta$

$$\lim_{t\to+\infty}\mu_{\delta_{\mathcal S}}\left(\left\{\omega\in E_A^\infty\,:\,\alpha\le\frac{\lambda(\omega|_{n(t,\omega)})-\chi_{\mu_{\delta_{\mathcal S}}}t}{\sqrt{t}}\le\beta\right\}\right)=\frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-u^2/2\sigma_0^2}\,du.$$

This is obtained by reinterpreting an approach of Melbourne and Török, originally applied in the case of suspended flow [**49**]. In particular, they showed that if a discrete system satisfies a central limit theorem with variance σ^2, then a suitable suspension flows also satisfy the CLT. [4] In the present case one takes $\sigma : E_A \to E_A$ as the discrete transformation and a roof function $r : E_A \to \mathbb{R}$ defined by $r = -\log\left|\varphi'_{\omega_1}(\pi_{\mathcal S}(\sigma(\omega)))\right|$. For the suspension space $E_A^r = \{(\omega,u) : 0 \le u \le r(\omega)\}$ with the identifications $(\omega, r(\omega)) \sim (\sigma\omega, 0)$ one can consider the suspension flow $\sigma_t^r : E_A^r \to E_A^r$ defined by $\sigma_t^r(\omega,u) = (\omega, u+t)$, up to the identifications. We can associate to the σ-invariant probability measure a φ-invariant probability measure $\widehat{\mu}_\sigma$ defined by $d\widehat{\mu}_\sigma = d\mu_\sigma \times dt/\int r d\mu_{\delta_{\mathcal S}}$. Given a function $F : E_A^r \to \mathbb{R}$ the CLT for the flow gives that

$$\lim_{t\to+\infty}\widehat{\mu}_{\delta_{\mathcal S}}\left(\left\{(\omega,u)\in E_A^r\,:\,\alpha\le\frac{\int_0^t F\circ\varphi_s(\omega,u)ds-t\int d\hat{\mu}_{\delta_{\mathcal S}}}{\sqrt{t}}\le\beta\right\}\right)=\frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-u^2/2\sigma_1^2}\,du,$$

where $\sigma_1^2 = \sigma_0^2/\chi_{\mu_{\delta_{\mathcal S}}}$ cf. [**49**], §3.We would like to choose F so that $\int_0^t F\circ\varphi_s(\omega,u)ds$ corresponds to $\lambda(\omega|_{n(t,\omega)})$. To this end one chooses a function F which integrates to unity on fibers, i.e., $\int_0^{r(\omega)} F(\omega,u)du = 1$ for all $\omega \in \Sigma_A$, and has support close to $E_A \times \{0\}$. Thus the Central Limit Theorem for the suspension flow corresponds to the Central Limit Theorem formulated above in t. The variances are related by a factor of $\int r d\mu_{\delta_{\mathcal S}}$.

We now turn the the parabolic setting.

---

[3] In fact $\mu_{\delta_{\mathcal S}}$ below can be replaced by the (unique) Gibbs/equilibrium state of any Hölder continuous summable potential $f : E_A^\infty \to \mathbb{R}$.

[4] There is a mild hypothesis on the roof function r which is satisfied if $r \in L^4$, say. This is the case in our present context.

4.2. Central Limit Theorems for Multipliers and Diameters: Parabolic GDMSs with Finite Invariant Measure $\mu_{\delta_\mathcal{S}}$

Through this whole section we assume that the invariant measure $\mu_{\delta_\mathcal{S}}$ is finite, so normalized to be probability one. We want to consider analogous comparison results in the context of parabolic GDMSs. Following the approach described in Section 3.1, given a parabolic conformal GDMS S we associate to it a conformal GDMS S^*. In this case the Central Limit Theorem for the measure $\mu^*_{\delta_\mathcal{S}}$ associated to $\mathcal{S}^*$ translates into a Central Limit Theorem for the parabolic system $\mathcal{S}$ and its measure $\mu_{\delta_\mathcal{S}}$. This leads to the following results, the first of which is the analogue of Theorem 4.1.1.

THEOREM 4.2.1. *If $\mathcal{S}$ is a finitely irreducible parabolic conformal GDMS with $\delta_\mathcal{S} > \frac{2p}{p+1}$*[5]*, then there exists $\sigma^2 > 0$ such that if $G \subset \mathbb{R}$ is a Lebesgue measurable set with* $\text{Leb}(\partial G) = 0$*, then*

$$\lim_{n\to+\infty} \mu_{\delta_\mathcal{S}}\left(\left\{\omega \in E_A^\infty : \frac{-\log\left|\varphi'_{\omega|_n}(\pi_\mathcal{S}(\sigma^n(\omega)))\right| - \chi_{\mu_{\delta_\mathcal{S}}} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty} \mu_{\delta_\mathcal{S}}\left(\left\{\omega \in E_A^\infty : \alpha \le \frac{-\log\left|\varphi'_{\omega|_n}(\pi_\mathcal{S}(\sigma^n(\omega)))\right| - \chi_{\mu_{\delta_\mathcal{S}}} n}{\sqrt{n}} \le \beta\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

PROOF. By Theorem 3.1.6, the hypothesis that $\delta_\mathcal{S} > \frac{2p}{p+1}$ precisely means that measure $\mu_{\delta_\mathcal{S}}$ is finite, and, as always, we normalize it to be a probability measure. Because of Theorem 3.1.7 and Remark 2.3.6 Theorem 4.2.1 then is a standard consequence of L. S. Young's tower approach [**94**], [**95**], comp. [**26**], [**26**], and [**26**]. $\square$

The second result is the parabolic analogue of Theorem 4.1.2.

THEOREM 4.2.2. *Let $\mathcal{S}$ be a finite alphabet irreducible parabolic GDMS with $\delta_\mathcal{S} > \frac{2p}{p+1}$*[6]*. Then there exists $\sigma^2 > 0$ such that if for every $v \in V$, a set $Y_v \subset X_v$ is given having at least two points and whose closure is disjoint from the set of parabolic fixed points Ω, then for every Lebesgue measurable set $G \subset \mathbb{R}$ with* $\text{Leb}(\partial G) = 0$*, we have that*

$$\lim_{n\to+\infty} \mu_{\delta_\mathcal{S}}\left(\left\{\omega \in E_A^\infty : \frac{-\log \operatorname{diam}(\varphi_{\omega|_n}(Y_{t(\omega_n)})) - \chi_{\mu_t} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-t^2/2\sigma}dt.$$

In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty} \mu_{\delta_\mathcal{S}}\left(\left\{\omega \in E_A^\infty : \alpha \le \frac{-\log \operatorname{diam}(\varphi_{\omega|_n}(Y_{t(\omega_n)})) - \chi_{\mu_t} n}{\sqrt{n}} \le \beta\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-t^2/2\sigma}dt.$$

PROOF. Because of Theorem 4.2.1, it suffices to show that

$$\lim_{n\to+\infty} \mu_{\delta_\mathcal{S}}\left(\{\omega \in E_A^\infty : \left|\log \operatorname{diam}(\varphi_{\omega|_n}(Y_{t(\omega_n)})\right| - \log\left|\varphi'_{\omega|_n}(\pi_\mathcal{S}(\sigma^n(\omega))\right| \ge n^{1/4}\}\right) = 0.$$

To show this, write

$$g_n(\omega) := \left|\log \operatorname{diam}(\varphi_{\omega|_n}(Y_{t(\omega_n)})\right| - \log\left|\varphi'_{\omega|_n}(\pi_\mathcal{S}(\sigma^n(\omega))\right|.$$

Since the set $E_A^\mathbb{N} \setminus E_{A^*}^\mathbb{N}$ is countable and the measure $\mu_{\delta_\mathcal{S}}$ is atomless, it suffices to deal with the elements of $E_A^\mathbb{N} \setminus E_{A^*}^\mathbb{N}$ only. Each such element ω has a unique representation in the form

$$\omega = \tau a^j \sigma^n(\omega),$$

[5] By Theorem 3.1.6, this precisely means that measure $\mu_{\delta_\mathcal{S}}$ is finite, and, as always, we normalize it to be a probability measure.

[6] As above

where $\tau \in E^*_{*,A^*}$, $a \in \Omega$ and $j = j(\omega) \in \{0, 1, \cdots, n - |\tau|\}$. Then for every $n \geq 0$ either

$$\operatorname{diam}(\varphi_{\omega|_n}(Y_{t(\omega_n)})) \asymp \|\varphi'_\tau\|(j+1)^{-1/p_a} \quad \text{or} \quad \operatorname{diam}(\varphi_{\omega|_n}(Y_{t(\omega_n)})) \asymp \|\varphi'_\tau\|(j+1)^{(p_a+1)/p_a},$$

respectively, depending on whether $a \in \overline{Y}_{t(\omega_n)}$ or not. In either case

$$\operatorname{diam}(\varphi_{\omega|n}(Y_{t(\omega_n)})) \asymp \|\varphi'_\tau\|(j+1)^{-\alpha}$$

where $\alpha \in \{1/p_a, (p_a+1)/p_a\}$. Since $\omega \in E_A^{\mathbb{N}} \setminus E_{A^*}^{\mathbb{N}}$, there exists a largest (finite) $k \geq 0$ such that

$$\omega \in [\tau a^{j+k}].$$

Then

$$\left|\varphi'_{\omega|_n}\big(\pi_{\mathcal{S}}(\sigma^n(\omega)\big)\right| \asymp \|\varphi'_\tau\|(j+k+1)^{-\frac{p_a+1}{p_a}}(k+1)^{\frac{p_a+1}{p_a}}.$$

Hence

$$g_n(\omega) \leq \frac{p_a+1}{p_a}\big(\log(k+1) + \log(j+k+1) + \alpha\log(j+1) + \Gamma_+\big) \leq \Gamma\log(j+k+1)$$

where $\Gamma_+ \in [0, +\infty)$ and $\Gamma \in [1, +\infty)$ are some universal constants independent of ω and n. Then

$$\int_{\omega \in E_{A^*}^\infty : j(\omega)=j} g_n(\omega)\, d\mu_{\delta_{\mathcal{S}}}(\omega) \leq \Sigma_j := \Gamma \sum_{\tau \in E_A^{n-j}(*)} \sum_{a \in \Omega} \sum_{\substack{b \neq a \\ A_{ab}=1}} \sum_{k=0}^{\infty} \log(j+k+1)\mu_{\delta_{\mathcal{S}}}([\tau a^{j+k} b]),$$

where $E_A^{n-j}(*)$ denotes the set of all finite words of "real" length $n-j$ that belong to $E^*_{*A^*}$. Now represent each element $\tau \in E_A^{n-j}(*)$ uniquely as $c^l d\gamma$, where $l \geq 0$, $c \in \Omega$, $d \neq c$. Then both $c^l d$ and γ belong to $E^*_{*A^*}$, and we can write

$$\Sigma_j = \Gamma \sum_{c \in \Omega} \sum_{\substack{d \neq c \\ A_{cd}=1}} \sum_{\gamma \in E^*_{*A^*}} \sum_{l=0}^{n-j-1} \sum_{a \in \Omega} \sum_{\substack{b \neq a \\ A_{ab}=1}} \sum_{k=0}^{\infty} \log(j+k+1)\mu_{\delta_{\mathcal{S}}}([c^l d\gamma a^{j+k} b]).$$

Now since the Radon-Nikodym derivative $\frac{d\mu_{\delta_{\mathcal{S}}}}{dm_{\delta_{\mathcal{S}}}}$ is comparable to $l+1$ on $c^l d$ and since the three words $c^l d$, γ and $a^{j+k}b$, belong to E_{*A^*}, we obtain

$$\begin{aligned}
\Sigma_j &\asymp \sum_{c \in \Omega} \sum_{\substack{d \neq c \\ A_{cd}=1}} \sum_{\substack{\gamma \in E^*_{*A^*} \\ d\gamma \in E_A^{n-k-l-1}}} \sum_{l=0}^{n-j-1} \sum_{a \in \Omega} \sum_{\substack{b \neq a \\ A_{ab}=1}} \sum_{k=0}^{\infty} \log(j+k+1)(l+1) m_{\delta_{\mathcal{S}}}([c^l d\gamma a^{j+k} b]) \\
&\asymp \sum_{c \in \Omega} \sum_{\substack{d \neq c \\ A_{cd}=1}} \sum_{\substack{\gamma \in E^*_{*A^*} \\ d\gamma \in E_A^{n-k-l-1}}} \sum_{l=0}^{n-j-1} \sum_{a \in \Omega} \sum_{\substack{b \neq a \\ A_{ab}=1}} \sum_{k=0}^{\infty} \log(j+k+1)(l+1) m_{\delta_{\mathcal{S}}}([c^l d]) m_{\delta_{\mathcal{S}}}([\gamma]) m_{\delta_{\mathcal{S}}}([a^{j+k} b]) \\
&\preceq \sum_{c \in \Omega} \sum_{a \in \Omega} \sum_{l=0}^{\infty} \sum_{k=1}^{\infty} \log(j+k+1)(l+1)^{1-\frac{p_a+1}{p_a}\delta_{\mathcal{S}}}(j+k+1)^{-\frac{p_a+1}{p_a}\delta_{\mathcal{S}}} \\
&\asymp \sum_{a \in \Omega} \sum_{k=0}^{\infty} \log(j+k+1)(j+k+1)^{-\frac{p_a+1}{p_a}\delta_{\mathcal{S}}}
\end{aligned}$$

where the last comparability sign we wrote because $1 - \frac{p_a+1}{p_a}\delta_{\mathcal{S}} < -1$ for all $c \in \Omega$. Therefore,

$$\begin{aligned}\int_{E_A^\infty} g_n d\mu_{\delta_{\mathcal{S}}} &= \sum_{j=0}^{n} \int_{\{\omega\in E_{A^*}^\infty : j(\omega)=j\}} g_n(\omega) d\mu_(\omega) \\ &\preceq \sum_{j=0}^{\infty} \sum_{a\in\Omega} \sum_{k=1}^{\infty} \log(j+k)(j+k)^{-\frac{p_a+1}{p_a}\delta_{\mathcal{S}}} \\ &\asymp D := \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \log(j+k)(j+k)^{-\frac{p+1}{p}\delta_{\mathcal{S}}} < +\infty,\end{aligned}$$

where, we recall, $p = \max\{p_a : a \in \Omega\}$ and the constant D is finite since $\frac{p+1}{p}\delta_{\mathcal{S}} > 2$. Therefore, Tchebyschev's Inequality tells us that

$$\mu_{\delta_{\mathcal{S}}}\big(\{\omega \in E_A^\infty : g_n(\omega) \ge n^{1/4}\}\big) \le \frac{\int_{E_A^\infty} g_n d\mu_{\delta_{\mathcal{S}}}}{n^{1/4}} \le Dn^{-1/4},$$

and the proof is complete. □

REMARK 4.2.3. There are a variety of even stronger results, e.g., Functional Central Limit Theorems and Invariance Principles, based on approximation by Brownian Motion, which should also hold with a little more work. Similarly, there are other complementary results such as large deviation results.

REMARK 4.2.4. There are possible stronger results of other kinds as well. For example, in both the hyperbolic and parabolic settings there is the possibility of estimating error terms and obtaining local limit theorems as in [**24**] and [**25**].

4.3. Central Limit Theorems: Asymptotic Counting Functions for Attracting GDMSs

In this subsection we work in the setting of attracting GDMSs. We again fix $\rho \in E_A^\infty$. For any $n \ge 1$ and $\omega \in E_\rho^n$ consider the weights

$$e^{-\delta_{\mathcal{S}}\lambda_\rho(\omega)} = |\varphi_\omega'(\pi(\rho))|^{\delta_{\mathcal{S}}}.$$

More precisely, for every set $H \subset E_\rho^n$, we define

$$\mu_n(H) := \frac{\sum_{\omega\in H} e^{-\delta_{\mathcal{S}}\lambda_\rho(\omega)}}{\sum_{\omega\in E_\rho^n} e^{-\delta_{\mathcal{S}}\lambda_\rho(\omega)}} = \frac{\mathcal{L}_{\delta_{\mathcal{S}}}^n 1\!\!1_{[H]}(\rho)}{\mathcal{L}_{\delta_{\mathcal{S}}}^n 1\!\!1(\rho)}. \tag{4.1}$$

Define the function $\lambda : E_A^\infty \to \mathbb{R}$ by the formula:

$$\lambda(\omega) = -\log|\varphi_{\omega_1}'(\sigma(\omega))|.$$

In particular, for every $\tau \in E_\rho^*$, say $\tau \in E_\rho^n$,

$$\lambda_\rho(\tau) = \sum_{j=0}^{n-1} \lambda\big(\sigma^j(\tau\rho)\big).$$

We first prove the following.

THEOREM 4.3.1. *If $\mathcal{S}$ is a finitely irreducible strongly regular conformal GDMS, then for every $\rho \in E_A^\infty$ we have that*

$$\lim_{n\to+\infty} \int_{E_\rho^n} \frac{\lambda_\rho}{n} d\mu_n = \chi_{\mu_{\delta_{\mathcal{S}}}} = \int_{E_\rho^\infty} \lambda d\mu_{\delta_{\mathcal{S}}}. \tag{4.2}$$

Proof. The idea of the proof is to represent the integral

$$\int_{E_\rho^n} \frac{\lambda_\rho}{n} d\mu_n$$

as the ratio of (sums of) Perron–Frobenius operators, and then to use the spectral properties of the operator $\mathcal{L}_{\delta_\mathcal{S}}$. However, there is a difficulty in such an approach which does not appear in the case of a finite alphabet. The character of this difficulty is that although the function $\lambda : E_A^\infty \to \mathbb{R}$ is always Hölder continuous, in the case of infinite alphabet it is unbounded. The remedy comes from the fact that $\mathcal{L}_{\delta_\mathcal{S}}(\mathbb{1})$ is a Hölder continuous bounded function. Beginning the proof, we have

$$\begin{aligned}\int_{E_\rho}^n \frac{\lambda_\rho}{n} d\mu_n &= \frac{\frac{1}{n}\mathcal{L}_{\delta_\mathcal{S}}^n\big(\sum_{j=0}^{n-1}\lambda\circ\sigma^j\big)(\rho)}{\mathcal{L}_{\delta_\mathcal{S}}^n(\mathbb{1})(\rho)} = \frac{\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_{\delta_\mathcal{S}}^n(\lambda\circ\sigma^j)(\rho)}{\mathcal{L}_{\delta_\mathcal{S}}^n(\mathbb{1})(\rho)} \\ &= \frac{\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_{\delta_\mathcal{S}}^{n-j}\big(\mathcal{L}_{\delta_\mathcal{S}}^j(\lambda\circ\sigma^j)\big)(\rho)}{\mathcal{L}_{\delta_\mathcal{S}}^n(\mathbb{1})(\rho)} \\ &= \frac{\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_{\delta_\mathcal{S}}^{n-j}\big(\lambda\mathcal{L}_{\delta_\mathcal{S}}^j\mathbb{1}\big)(\rho)}{\mathcal{L}_{\delta_\mathcal{S}}^n(\mathbb{1})(\rho)} \\ &= \frac{\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_{\delta_\mathcal{S}}^{n-(j+1)}\big(\mathcal{L}_{\delta_\mathcal{S}}(\lambda\mathcal{L}_{\delta_\mathcal{S}}^j\mathbb{1})\big)(\rho)}{\mathcal{L}_{\delta_\mathcal{S}}^n(\mathbb{1})(\rho)}.\end{aligned}$$

Now a straightforward calculation based on the strong regularity of the system $\mathcal{S}$ shows that the Hölder norms of the functions $\mathcal{L}_{\delta_\mathcal{S}}(\lambda\mathcal{L}_{\delta_\mathcal{S}}^i\mathbb{1})$, $i \ge 0$, are uniformly bounded above. With the fact that the sequence $(\mathcal{L}_{\delta_\mathcal{S}}^i g)_{i=0}^\infty$ converges uniformly (in fact exponentially fast) to $\int g dm_{\delta_\mathcal{S}}\psi_{\delta_\mathcal{S}}$ for every bounded Hölder continuous function $g : E_A^\infty \to \mathbb{R}$ we conclude that the sequence $(\mathcal{L}_{\delta_\mathcal{S}}(\lambda\mathcal{L}_{\delta_\mathcal{S}}^j\mathbb{1}))_{j=0}^\infty$ converges uniformly to $\mathcal{L}_{\delta_\mathcal{S}}(\lambda\psi_{\delta_\mathcal{S}})$. So, fixing $\epsilon > 0$, we can find $k_1 \ge 1$ such that

$$\|\mathcal{L}_{\delta_\mathcal{S}}(\lambda\mathcal{L}_{\delta_\mathcal{S}}^j\mathbb{1}) - \mathcal{L}_{\delta_\mathcal{S}}(\lambda\psi_{\delta_\mathcal{S}})\|_\alpha \le \epsilon$$

for all $j \ge k_1$. Furthermore, there exist $N \ge k_2 \ge k_1$ such that for all $n \ge N$ and all $j \le n - k_2$,

$$\Big\|\mathcal{L}_{\delta_\mathcal{S}}^{n-j}(\lambda\mathcal{L}_{\delta_\mathcal{S}}^j\mathbb{1}) - \int \mathcal{L}_{\delta_\mathcal{S}}(\lambda\psi_{\delta_\mathcal{S}})dm_{\delta_\mathcal{S}}\psi_{\delta_\mathcal{S}}\Big\|_\alpha \le \epsilon.$$

But $\int \mathcal{L}_{\delta_\mathcal{S}}(\lambda\psi_{\delta_\mathcal{S}})dm_{\delta_\mathcal{S}} = \int \lambda\psi_{\delta_\mathcal{S}}dm_{\delta_\mathcal{S}} = \int \lambda d\mu_{\delta_\mathcal{S}}$ and $M := \sup\{\|\mathcal{L}_{\delta_\mathcal{S}}^n\mathbb{1}\|_\alpha : n \ge 0\}$ is finite. So we can conclude that

$$\Big\|\mathcal{L}_{\delta_\mathcal{S}}^{n-(j+1)}\mathcal{L}_{\delta_\mathcal{S}}(\lambda\mathcal{L}_{\delta_\mathcal{S}}^j\mathbb{1}) - \int \lambda d\mu_{\delta_\mathcal{S}}\psi_{\delta_\mathcal{S}}\Big\|_\alpha \le (1+M)\epsilon$$

for all $n \ge N$ and all $k_1 \le j \le n - k_2$. Hence

$$\int \lambda d\mu_{\delta_\mathcal{S}} - (1+M)\epsilon \le \liminf_{n\to+\infty}\int_{E_\mathcal{S}^n}\frac{\lambda_{\delta_\mathcal{S}}}{n}d\mu \le \limsup_{n\to+\infty}\int_{E_\mathcal{S}^n}\frac{\lambda_{\delta_\mathcal{S}}}{n}d\mu \le \int \lambda d\mu_{\delta_\mathcal{S}} + (M+1)\epsilon.$$

Letting $\epsilon \to 0$, then concludes the proof. □

Now we are next going to prove versions of the Central Limit Theorem (CLT) that involve counting. This requires some preparatory steps.

We define the functions $\Delta_n : E_\rho^n \to \mathbb{R}$ by the formulae

$$\Delta_n(\omega) := \frac{\lambda_\rho(\omega) - \chi_{\delta_\mathcal{S}} n}{\sqrt{n}} \tag{4.3}$$

and consider the sequence $(\mu_n \circ \Delta_n^{-1})_{n=1}^\infty$ of probability distributions on $\mathbb{R}$. Observe that for every Borel set $F \subset \mathbb{R}$, we have that

$$
\begin{aligned}
\mu_n \circ \Delta_n^{-1}(F) = \frac{\mathcal{L}_{\delta_\mathcal{S}}^n \mathbb{1}_{[\Delta_n^{-1}(F)]}(\rho)}{\mathcal{L}_{\delta_\mathcal{S}}^n \mathbb{1}(\rho)} &= \frac{\mathcal{L}_{\delta_\mathcal{S}}^n (\mathbb{1}_F \circ \Delta_n)(\rho)}{\mathcal{L}_{\delta_\mathcal{S}}^n \mathbb{1}(\rho)} \\
&= \frac{\sum_{\omega \in E_\rho^n} e^{-\delta_\mathcal{S}\lambda_p(\omega)} \mathbb{1}_F(\Delta_n(\omega))}{\sum_{\omega \in E_\rho^n} e^{-\delta_\mathcal{S}\lambda_p(\omega)}} \\
&= \frac{\sum_{\omega \in E_\rho^n} e^{-\delta_\mathcal{S}\lambda_p(\omega)} \mathbb{1}_F\left(\frac{\lambda_\rho(\omega) - \chi_{\delta_\mathcal{S}} n}{\sqrt{n}}\right)}{\sum_{\omega \in E_\rho^n} e^{-\delta_\mathcal{S}\lambda_p(\omega)}}
\end{aligned} \tag{4.4}
$$

where in the third term the function Δ_n is considered as defined on $E_A^\mathbb{N}$ by the formula

$$\Delta_n(\omega) = \frac{\lambda_\rho(\omega|_n) - \chi_{\delta_\mathcal{S}} n}{\sqrt{n}}.$$

Our last counting result for attracting systems is the following.

THEOREM 4.3.2. *If $\mathcal{S}$ is a strongly regular finitely irreducible D–generic attracting conformal graph directed Markov system, then the sequence of random variables $(\Delta_n)_{n=1}^\infty$ converges in distribution to the normal (Gaussian) distribution $\mathcal{N}_0(\sigma)$ with mean value zero and the variance $\sigma^2 = \mathrm{P}''(\delta_\mathcal{S})$ (the latter being positive because of Remark 2.3.6 and since the system $\mathcal{S}$ is D–generic). Equivalently, the sequence $(\mu_n \circ \Delta_n^{-1})_{n=1}^\infty$ converges weakly to the normal distribution $\mathcal{N}_0(\sigma^2)$. This means that for every Borel set $F \subset \mathbb{R}$ with $\mathrm{Leb}(\partial F) = 0$, we have*

$$\lim_{n\to+\infty} \mu_n(\Delta_n^{-1}(F)) = \frac{1}{\sqrt{2\pi}\sigma}\int_F e^{-t^2/2\sigma^2}\,dt. \tag{4.5}$$

PROOF. This theorem is equivalent to showing that the characteristic functions (or Fourier transforms) of the measures $\mu_n \circ \Delta_n^{-1}$ converge to the characteristic function of $\mathcal{N}_0(\sigma^2)$, i.e., to the function $\mathbb{R} \ni t \longmapsto e^{-\sigma^2 t^2/2}$. By the formula (2.2) we have for every $t \in \mathbb{R}$ that

$$
\begin{aligned}
\int_\mathbb{R} e^{itx} d\mu_n \circ \Delta_n^{-1}(x) = \int_{E_\rho^n} e^{it\Delta_n(\omega)} d\mu_n(\omega) &= \frac{\mathcal{L}_{\delta_\mathcal{S}}^n (e^{it\Delta_n})(\rho)}{\mathcal{L}_{\delta_\mathcal{S}}^n \mathbb{1}(\rho)} \\
&= e^{-t\chi_{\delta_\mathcal{S}}\sqrt{n}} \frac{\mathcal{L}_{\delta_\mathcal{S} - \frac{it}{\sqrt{n}}}^n \mathbb{1}(\rho)}{\mathcal{L}_{\delta_\mathcal{S}}^n \mathbb{1}(\rho)} \\
&= e^{-t\chi_{\delta_\mathcal{S}}\sqrt{n}} \left(\frac{\lambda_{\delta_\mathcal{S} - \frac{it}{\sqrt{n}}}^n Q_{\delta_\mathcal{S} - \frac{it}{\sqrt{n}}}(\mathbb{1})(\rho) + S_{\delta_\mathcal{S} - \frac{it}{\sqrt{n}}}^n \mathbb{1}(\rho)}{\psi_{\delta_\mathcal{S}}(\rho) + S_{\delta_\mathcal{S}}^n \mathbb{1}(\rho)}\right).
\end{aligned}
$$

It therefore follows from items (4), (5) and (6) following formula 2.2 that

$$\lim_{n\to+\infty} \int_\mathbb{R} e^{itx} d\mu_n \circ \Delta_n^{-1}(x) = \lim_{n\to+\infty} e^{-it\chi_{\delta_\mathcal{S}}\sqrt{n}} \lambda_{\delta_\mathcal{S} - \frac{it}{\sqrt{n}}}^n.$$

Denote by $\log\lambda_s$, s belonging to some sufficiently small neighborhood of $\delta_\mathcal{S}$, the principle branch of the logarithm of λ_s, i.e., that determined by the requirement that $\log\lambda_{\delta_\mathcal{S}} = 0$. Since $\log\lambda_s = \mathrm{P}(s)$ for real $s > \gamma_s$ and since $\mathrm{P}'(0) = -\chi_{\delta_\mathcal{S}}$, we therefore get that

$$\lambda_s = \exp(\log\lambda_s) = \exp\left(-\chi_{\delta_\mathcal{S}}(s-\delta) + \frac{{\delta_\mathcal{S}}^2}{2}(s-\delta_\mathcal{S})^2 + O(|s-\delta_\mathcal{S}|^3)\right).$$

So for $s = \delta_{\mathcal{S}} - \frac{it}{\sqrt{n}}$ we get

$$\lambda_{\delta_{\mathcal{S}} - \frac{it}{\sqrt{n}}} = \exp\left(i\frac{t\chi_{\delta_{\mathcal{S}}}}{\sqrt{n}} - \frac{\sigma^2 t^2}{2n} + O(n^{-3/2})\right).$$

Therefore,

$$\begin{aligned} e^{-it\chi_{\delta_{\mathcal{S}}}\sqrt{n}}\lambda^n_{\delta_{\mathcal{S}} - \frac{it}{\sqrt{n}}} &= e^{-it\chi_{\delta_{\mathcal{S}}}\sqrt{n}} \exp\left(it\chi_{\delta_{\mathcal{S}}}\sqrt{n} - \frac{\sigma^2 t^2}{2} + O(n^{-1/2})\right) \\ &= \exp\left(-\frac{\sigma^2 t^2}{2} + O(n^{-1/2})\right). \end{aligned}$$

So finally

$$\lim_{n\to+\infty} \int_{\mathbb{R}} e^{itx} d\mu_n \circ \Delta_n^{-1}(x) = \exp\left(-\sigma^2 t^2/2\right).$$

Thus since $\mathbb{R} \ni t \longmapsto \exp\left(-\sigma^2 t^2/2\right)$ is the characteristic function of the Gaussian distribution $\mathcal{N}_0(\sigma^2)$, the proof is complete. □

4.4. Central Limit Theorems: Asymptotic Counting Functions for Parabolic GDMSs

We want to extend the Central Limit Theorem for counting functions from the previous (attracting GDNSs) subsection to the case of parabolic GDMSs. We are in the same setting as in Section 3.1 i.e., $\mathcal{S} = \{\varphi_e\}_{e\in E}$ is an irreducible conformal parabolic GDMS. Furthermore, the functions Δ_n and measures μ_n have formally the same definitions as their "attracting" counterparts given in Subsection 4.1 respectively by formulae (4.3) and (4.1). We start with the following analogue of Theorem 4.3.1.

Theorem 4.4.1. *If $\mathcal{S}$ is an irreducible parabolic conformal GDMS for which*

$$\delta_{\mathcal{S}} > \frac{2p_{\mathcal{S}}}{p_{\mathcal{S}} + 1},$$

i.e the invariant measure $\mu_{\delta_{\mathcal{S}}}$ is finite (so a probability after normalization), then for every $\rho \in E^\infty_{A^*}$

$$\lim_{n\to+\infty} \int \frac{\lambda_\rho}{n} d\mu_n = \int_{E_A^\infty} \lambda d\mu_{\delta_{\mathcal{S}}} = \chi_{\delta_{\mathcal{S}}}.$$

Proof. Since the behavior of iterates of the Perron–Frobenius operator $\mathcal{L}_{\delta_{\mathcal{S}}}$ is now (in the parabolic context) more complicated than in the attracting case, we need to provide a conceptually different proof than that of Theorem 4.3.1. We will make an essential use of Birkhoff's Ergodic Theorem instead.

Firstly, we fix $\epsilon > 0$. Then it follows from Birkhoff's Ergodic Theorem, along with both Lusin's Theorem and Egorov's Theorem, that there exists an integer $N_\epsilon \geq 1$ and a measurable set $F(\epsilon) \subset E_A^\infty$ such that $m_{\delta_{\mathcal{S}}}(F(\epsilon)) > 1 - \epsilon$ (remembering that $m_{\delta_{\mathcal{S}}}$ is equivalent to $\mu_{\delta_{\mathcal{S}}}$) for every $\tau \in F(\epsilon)$ and every integer $n \geq N_\epsilon$,

$$\left|\frac{\sum_{j=0}^{n-1} \lambda \circ \sigma^j(\tau)}{n} - \chi_{\delta_{\mathcal{S}}}\right| \leq \epsilon.$$

For all $n \geq N_2$ let

$$F_\rho(\epsilon, n) := \{\omega \in E_\rho^n : \omega\rho \in F(\epsilon)\} \quad \text{and} \quad F_\rho^c(\epsilon, n) := \{\omega \in E_\rho^n : \omega\rho \in F^c(\epsilon)\}$$

Then

$$
\begin{aligned}
&\left| \sum_{\omega \in F_\rho(\epsilon,n)} \frac{\lambda_\rho(\omega)}{n} \frac{|\varphi'_\omega(\pi(\rho))|^{\delta_{\mathcal{S}}}}{\mathcal{L}^n_{\delta_{\mathcal{S}}} 1\!\!1(\rho)} - \sum_{\omega \in F_\rho(\epsilon,n)} \chi_{\delta_{\mathcal{S}}} \frac{|\varphi'_\omega(\pi(\rho))|^{\delta_{\mathcal{S}}}}{\mathcal{L}^n_{\delta_{\mathcal{S}}} 1\!\!1(\rho)} \right| = \\
&= \left| \sum_{\omega \in F_\rho(\epsilon,n)} \left(\frac{\lambda_\rho(\omega)}{n} - \chi_{\delta_{\mathcal{S}}} \right) \frac{|\varphi'_\omega(\pi(\rho))|^{\delta_{\mathcal{S}}}}{\mathcal{L}^n_{\delta_{\mathcal{S}}} 1\!\!1(\rho)} \right| \\
&= \left| \frac{\lambda_\rho(\omega)}{n} - \chi_{\delta_{\mathcal{S}}} \right| \sum_{\omega \in F_\rho(\epsilon,n)} \frac{|\varphi'_\omega(\pi(\rho))|^{\delta_{\mathcal{S}}}}{\mathcal{L}^n_{\delta_{\mathcal{S}}} 1\!\!1(\rho)} \\
&= \left| \frac{\lambda_\rho(\omega)}{n} - \chi_{\delta_{\mathcal{S}}} \right| \le \epsilon.
\end{aligned} \tag{4.1}
$$

Now given a positive number M and an arbitrary function $g : E_A^\infty \to \mathbb{R}$ for which $|g| \le M$, we have that

$$
\begin{aligned}
\left| \sum_{\omega \in F^c_\rho(\epsilon,n)} g(\omega\rho) \frac{|\varphi'_\omega(\pi(\rho))|^{\delta_{\mathcal{S}}}}{\mathcal{L}^n_{\delta_{\mathcal{S}}} 1\!\!1(\rho)} \right| &\le \frac{M}{\mathcal{L}^n_{\delta_{\mathcal{S}}} 1\!\!1(\rho)} \sum_{\omega \in F^c_\rho(\epsilon,n)} |\varphi'_\omega(\pi(\rho))|^{\delta_{\mathcal{S}}} \le \frac{M'}{\mathcal{L}^n_{\delta_{\mathcal{S}}} 1\!\!1(\rho)} \sum_{\omega \in F^c_\rho(\epsilon,n)} m_{\delta_{\mathcal{S}}}([\omega]) \\
&= \frac{M'}{\mathcal{L}^n_{\delta_{\mathcal{S}}} 1\!\!1(\rho)} m_{\delta_{\mathcal{S}}}(F^c_\rho(\epsilon,n)) \\
&\le \frac{M'}{\mathcal{L}^n_{\delta_{\mathcal{S}}} 1\!\!1(\rho)} \epsilon,
\end{aligned}
$$

with some appropriate constant $M' > 0$. Now it follows from Theorem E of [**33**] that there exists a constant $Q_\rho \ge 1$, depending on ρ (in fact depending only on $\mathrm{dist}(\pi(\rho), \Omega)$) such that

$$
Q_\rho^{-1} \le \mathcal{L}^n_{\delta_{\mathcal{S}}}(\rho) \le Q_\rho
$$

for every integer $n \ge 0$. We therefore get

$$
\left| \sum_{\omega \in F^c_\rho(\epsilon,n)} g(\omega\rho) \frac{|\varphi'_\omega(\pi(\rho))|^{\delta_{\mathcal{S}}}}{\mathcal{L}^n_{\delta_{\mathcal{S}}} 1\!\!1(\rho)} \right| \le M' Q_\rho \epsilon. \tag{4.2}
$$

Since

$$
0 \le \frac{1}{n} \sum_{j=0}^{n-1} \lambda \circ \sigma^j \le M
$$

for every $n \ge 1$, applying (4.1) and also (4.2) for both

$$
g = \frac{1}{n} \sum_{j=0}^{n-1} \lambda \circ \sigma^j \quad \text{and} \quad g = \chi_{\delta_{\mathcal{S}}},
$$

we get the following bound:

$$
\begin{aligned}
&\left|\int_{E_\rho^n}\frac{\lambda_\rho}{n}d\mu_n-\chi_{\delta_{\mathcal S}}\right|\le\\
&\quad\le\left|\left(\int_{F_\rho(\epsilon,n)}\frac{\lambda_\rho}{n}d\mu_n-\int_{F_\rho(\epsilon,n)}\chi_{\delta_{\mathcal S}}d\mu_n\right)+\left(\int_{F_\rho^c(\epsilon,n)}\frac{\lambda_\rho}{n}d\mu_n-\int_{F_\rho^c(\epsilon,n)}\chi_{\delta_{\mathcal S}}d\mu_n\right)\right|\\
&\quad\le\left|\int_{F_\rho(\epsilon,n)}\frac{\lambda_\rho}{n}d\mu_n-\int_{F_\rho(\epsilon,n)}\chi_{\delta_{\mathcal S}}d\mu_n\right|+\left|\int_{F_\rho(\epsilon,n)^c}\frac{\lambda_\rho}{n}d\mu_n-\int_{F_\rho(\epsilon,n)^c}\chi_{\delta_{\mathcal S}}d\mu_n\right|\\
&\quad\le\left|\sum_{\omega\in F_s(\epsilon,n)}\frac{\lambda_\rho(\omega)}{n}\frac{|\varphi_\omega'(\pi(\rho))|^{\delta_{\mathcal S}}}{\mathcal L_{\delta_{\mathcal S}}^n1(\rho)}-\sum_{\omega\in F_s(\epsilon,n)}\chi_{\delta_{\mathcal S}}\frac{|\varphi_\omega'(\pi(\rho))|^{\delta_{\mathcal S}}}{\mathcal L_{\delta_{\mathcal S}}^n1(\rho)}\right|+\\
&\qquad\qquad+\left|\int_{F_\rho^c(\epsilon,n)}\frac{\lambda_\rho}{n}d\mu_u\right|+\left|\int_{F_\rho^c(\epsilon,n)}\chi_{\delta_{\mathcal S}}d\mu_u\right|\\
&\quad\le\epsilon+\left|\sum_{\omega\in F_s^c(\epsilon,n)}\frac{\lambda_\rho(\omega)}{n}\frac{|\varphi_\omega'(\pi(\rho))|^{\delta_{\mathcal S}}}{\mathcal L_{\delta_{\mathcal S}}^n\mathbb 1(\rho)}\right|+\left|\sum_{\omega\in F_s^c(\epsilon,n)}\chi_{\delta_{\mathcal S}}\frac{|\varphi_\omega'(\pi(\rho))|^{\delta_{\mathcal S}}}{\mathcal L_{\delta_{\mathcal S}}^n\mathbb 1(\rho)}\right|\\
&\quad\le\epsilon+M'Q_\rho\epsilon+M'Q_\rho\epsilon\\
&\quad\le(1+2M'Q_\rho)\epsilon.
\end{aligned}
$$

Hence, letting $\epsilon\to0$ we obtained

$$\int_{E_\rho^n}\frac{\lambda_\rho}{n}d\mu_n=\chi_{\delta_{\mathcal S}}$$

and the proof is complete. □

Our main theorem in this subsection is the following.

THEOREM 4.4.2. *If $\mathcal S$ is an irreducible parabolic conformal GDMS for which*

$$\delta_{\mathcal S}>\frac{2p_{\mathcal S}}{p_{\mathcal S}+1},$$

i.e the invariant measure $\mu_{\delta_{\mathcal S}}$ is finite (so a probability after normalization), then the sequence of random variables $(\Delta_n)_{n=1}^\infty$ converges in distribution to the normal (Gaussian) distribution $\mathcal N_0(\sigma^2)$ with mean value zero and the variance $\sigma^2=P_*''(\delta_{\mathcal S})>0$. Equivalently, the sequence $(\mu_n\circ\Delta_n^{-1})_{n=1}^\infty$ converges weakly to the normal distribution $\mathcal N_0(\sigma^2)$. This means that for every Borel set $F\subset\mathbb R$ with $\mathrm{Leb}(\partial F)=0$, we have

$$\lim_{n\to+\infty}\mu_n(\Delta_n^{-1}(F))=\frac{1}{\sqrt{2\pi}\sigma}\int_F e^{-t^2/2\sigma^2}dt. \tag{4.3}$$

PROOF. Using our previous notation recall that

$$\psi_{\delta_{\mathcal S}}=\frac{d\mu_{\delta_{\mathcal S}}}{dm_{\delta_{\mathcal S}}}.$$

Then

$$\mathcal L_{\delta_{\mathcal S}}\psi_{\delta_{\mathcal S}}=\psi_{\delta_{\mathcal S}},$$

and we can define the operator $\widehat{\mathcal L}_{\delta_{\mathcal S}}:L^1(\mu_{\delta_{\mathcal S}})\to L^1(\mu_{\delta_{\mathcal S}})$ by the formula

$$\widehat{\mathcal L}_{\delta_{\mathcal S}}(g)=\frac{1}{\psi_{\delta_{\mathcal S}}}\mathcal L_{\delta_{\mathcal S}}(g\psi_{\delta_{\mathcal S}}).$$

Then

$$\widehat{\mathcal L}_{\delta_{\mathcal S}}(\mathbb 1)=\mathbb 1$$

and $\widehat{\mathcal{L}}_{\delta_{\mathcal{S}}}$ is the Perron–Frobenius operator associated to the measure–preserving symbolic dynamical system $(\sigma, \mu_{\delta_{\mathcal{S}}})$. Following Gouëzel [**27**], for every integer $q \geq 1$ we consider the set

$$Z_q := \bigcup_{b\in\Omega} \bigcup_{\substack{e\in E\setminus\{b\} \\ A_{be}=1}} \{b^k e : 1 \leq k \leq q\} \cup (E \setminus \Omega)$$

and the first return map $\sigma_q : Z_q \to Z_q$. Still following [**27**], given an integer $n \geq 1$ we define an operator $\widehat{\mathcal{L}}^{(n)}_{\delta_{\mathcal{S}}} : L^1(\mu_{\delta_{\mathcal{S}}}) \to L^1(\mu_{\delta_{\mathcal{S}}})$ by the formula

$$\widehat{\mathcal{L}}^{(n)}_{\delta_{\mathcal{S}}}(g) := 1\!\!1_{Z_q} \widehat{\mathcal{L}}^n_{\delta_{\mathcal{S}}}(g 1\!\!1_{Z_q}).$$

Now our setting entirely fits into the hypothesis of section 2, 3 and 4 of Gouëzel's paper [**27**]. In particular, Theorem 2.1 (especially its formula (2)), Theorem 3.7 and Lemma 4.4 of [**27**] apply to give (compare the last formula of the proof of Proposition 4.6 in [**27**]) for any $\tau \in Z_q$ and any $t \in \mathbb{R}$ that

$$\lim_{n\to+\infty} \left| \widehat{\mathcal{L}}^{(n)}_{\delta_{\mathcal{S}}}(e^{it\Delta_n})(\rho) - \mu_{\delta_{\mathcal{S}}}(Z_q)^2 e^{-\sigma^2/2t^2} \right| = 0. \tag{4.4}$$

Now there exists $q_0 \geq 0$ such that $\rho \in Z_{q_0}$. Fix $\epsilon > 0$. Take $q \geq q_0$ sufficiently large, say, $q \geq q_1 \geq q_0$ that

$$1 - \mu_{\delta_{\mathcal{S}}}(Z_q)^2 < \epsilon. \tag{4.5}$$

Then by (4.4)

$$\limsup_{n\to+\infty} \left| \widehat{L}^{(n)}_{\delta_{\mathcal{S}}}(e^{it\Delta_n})(\rho) - e^{-\sigma^2/2t^2} \right| \leq \epsilon e^{-\sigma^2/2t^2}. \tag{4.6}$$

Now define μ_n' analogously to (4.1), i.e., for $H \subset E^n_{*,\rho}$:

$$\mu_n'(H) = \sum_{\omega\in H} e^{-\delta_{\mathcal{S}}\lambda_\rho(\omega)}.$$

Then the same calculation as (4.5) gives

$$\int_{\mathbb{R}} e^{itx} d\mu_n' \circ \Delta_n^{-1}(x) = \widehat{\mathcal{L}}^n_{\delta_{\mathcal{S}}}(e^{it\Delta_n})(\rho) = \widehat{\mathcal{L}}^{(n)}_{\delta_{\mathcal{S}}}(e^{it\Delta_n})(\rho) + \widehat{\mathcal{L}}_{\delta_{\mathcal{S}}}(1\!\!1_{Z_q^c} e^{it\Delta_n})(\rho).$$

But

$$\left| \widehat{\mathcal{L}}_{\delta_{\mathcal{S}}}(1\!\!1_{Z_q^c} e^{it\Delta_n})(\rho) \right| \leq \left| \widehat{\mathcal{L}}_{\delta_{\mathcal{S}}}(1\!\!1_{Z_q^c})(\rho) \right| = \widehat{\mathcal{L}}_{\delta_{\mathcal{S}}}(1\!\!1_{Z_q^c})(\rho), \tag{4.7}$$

and according to Theorem E in [**33**] we can write

$$\lim_{n\to+\infty} \widehat{\mathcal{L}}_{\delta_{\mathcal{S}}}(1\!\!1_{Z_q^c})(\rho) = \mu_{\delta_{\mathcal{S}}}(1\!\!1_{Z_q^c}) = 1 - \mu_{\delta_{\mathcal{S}}}(1\!\!1_{Z_q}).$$

Combining this along with (4.4), (4.6) and (4.7) gives

$$\limsup_{n\to+\infty} \left| \int_{\mathbb{R}} e^{itx} d\mu_n' \circ \Delta_n^{-1}(x) - e^{-\sigma^2/2t^2} \right| \leq \epsilon e^{-\sigma^2/2t^2} + 1 - \mu_{\delta_{\mathcal{S}}}(Z_q) \leq (1 + e^{-\sigma^2/2t^2})\epsilon.$$

Hence

$$\lim_{n\to+\infty} \int_{\mathbb{R}} e^{itx} d\mu_n' \circ \Delta_n^{-1}(x) = e^{-\sigma^2/2t^2}.$$

Therefore, formula (4.3) holds with μ_n replaced μ_n'. Because of this, because the measures μ_n and μ_n' are equivalent for all $n \geq 1$, and since, by Theorem E of [**33**] again, for the sequence $(\mu_n')_{n=1}^\infty$,

$$\lim_{n\to+\infty} \frac{d\mu_n}{d\mu_n'}(x) = 1$$

uniformly with respect to all $x \in \mathbb{R}$, we finally conclude that the formula (4.3) holds for measures μ_n, $n \geq 1$. Thus the proof of Theorem 4.4.2 is complete. $\square$

CHAPTER 5

Examples and Applications, I

5.1. Attracting/Expanding Conformal Dynamical Systems

In this section we deal with a class of conformal dynamical systems that are expanding and we show that their, appropriately organized, inverse holomorphic branches form conformal attracting GDMSs. We also examine in greater detail some special countable alphabet conformal attracting GDMSs.

5.1.1. Conformal Expanding Repellers. In this section we deal with conformal expanding repellers. We do it by applying the theory developed in the previous sections. In fact it suffices to work here with conformal GDMSs modeled on finite alphabets E. However, most of the results proved in this section are entirely new.

Let us start with the the definition of a conformal expanding repeller, the primary object of interest in this subsection.

Definition 5.1.1. Let U be an open subset of $\mathbb{R}^d$, $d \geq 1$. Let X be a compact subset of U. Let $f : U \to \mathbb{R}^d$ be a conformal map. The map f is called a conformal expanding repeller if the following conditions are satisfied:

(1) $f(X) = X$,

(2) $|f'|_X| > 1$,

(3) there exists an open set V such that $\overline{V} \subset U$ and

$$X = \bigcap_{k=0}^{\infty} f^{-n}(V),$$

and

(4) the map $f|_X : X \to X$ is topologically transitive.

Note that f is not required to be one-to-one; in fact usually it is not one-to-one. Abusing notation slightly we frequently refer also to the set X alone as a conformal expanding repeller. In order to use a uniform terminology we also call X the limit set of f.

Typical examples of conformal expanding repellers are provided by the following.

Proposition 5.1.2. *If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational function of degree $d \geq 2$, such that the map f restricted to its Julia set $J(f)$ is expanding, then $J(f)$ is a conformal expanding repeller.*

The basic concept associated with such repellers which will be used in this section is given by the following definition.

Definition 5.1.3. A finite cover $\mathcal{R} = \{R_e : e \in F\}$ of X is said to be a Markov partition of the space X for the mapping T if the following conditions are satisfied.

(a) $R_e = \overline{\text{Int} R_e}$ for all $e \in F$.

(b) $\text{Int} R_a \cap \text{Int} R_b = \emptyset$ for all $a \neq b$.

(c) $\mathrm{Int} R_b \cap f(\mathrm{Int} R_a) \neq \emptyset \implies R_b \subset f(R_a)$ whenever $a, b \in F$.

The elements of a Markov partition will be called cells in the sequel. The basic theorem about Markov partitions proved, for ex. in [**74**], is this.

THEOREM 5.1.4. *Any conformal expanding repeller $f : X \to X$ admits Markov partitions of arbitrarily small diameters.*

Fix $\beta > 0$ so small that for every $x \in X$ and every $n \geq 0$ there exists $f_x^{-n} : B(f^n(x), 4\beta) \to \mathbb{R}^d$, a unique continuous branch of f^{-n} sending $f^n(x)$ to x. Theorem 5.1.4 guarantees us the existence of $\mathcal{R} = \{R_j : j \in F\}$, a Markov partition of f with all cells of diameter smaller than β. Having such a Markov partition $\mathcal{R}$ we now associate to it a finite graph directed Markov system. The set of vertices is equal to $\mathcal{R}$ while the alphabet E is defined as follows.

$$E := \big\{(i,j) \in F \times F : \mathrm{Int} R_j \cap f(\mathrm{Int} R_i) \neq \emptyset\big\}.$$

Now, from the above for every $(i,j) \in E$ there exists a unique conformal map $f_{i,j}^{-1} : B(R_j, \beta) \to \mathbb{R}^d$ such that

$$f_{i,j}^{-1}(R_j) \subseteq R_i.$$

Define the incidence matrix $A : E \times E \to \{0,1\}$ by

$$A_{(i,j)(k,l)} = \begin{cases} 1 & \text{if } \ l = i \\ 0 & \text{if } \ l \neq i. \end{cases}$$

We further define:

$$t(i,j) = j \ \text{ and } \ i(i,j) = i.$$

Of course

$$\mathcal{S}_{\mathcal{R}} = \{f_{i,j}^{-1} : (i,j) \in E\} \tag{5.1}$$

forms a finite conformal directed Markov system, and $\mathcal{S}_{\mathcal{R}}$ is irreducible since the map $f : X \to X$ is transitive. Let

$$\pi_{\mathcal{R}} := \pi_{\mathcal{S}_{\mathcal{R}}} : E_A^\infty \to X$$

be the canonical projection onto the limit set $J_{\mathcal{S}}$ of the conformal GDMS $\mathcal{S}$ which is easily seen to be equal to X.

Returning to the actual topic of the paper, i.e., counting inverse images and periodic points, we fix a point $\xi \in X$, a Markov Partition

$$\mathcal{R} = \{R_e : e \in F\},$$

with

$$\xi \in \bigcup_{e \in F} \mathrm{Int}(R_e). \tag{5.2}$$

So, there exists a unique element $e(\xi) \in F$ such that $\xi \in \mathrm{Int}(R_{e(\xi)})$, and we fix a radius $\alpha > 0$ so small that

$$B(\xi, \alpha) \subset R_{e(\xi)}.$$

Furthermore, there exists a unique code of ξ, i.e. a unique infinite word $\rho \in E_A^\infty$ such that

$$\pi_{\mathcal{R}}(\rho) = \xi.$$

Using our usual notation we set

$$\lambda(z) = \log |(f^{n(z)})'(z)|, \tag{5.3}$$

where z is an inverse image of ξ under an iterate of f and the integer $n(z) \geq 0$ is uniquely determined by the following two conditions:

$$f^{n(z)}(z) = \xi \tag{5.4}$$

and

$$f^k(z) \neq \xi \quad \text{for every integer} \quad 0 \leq k < k(z). \tag{5.5}$$

We immediately note that if ξ is not periodic then condition (5.4) alone determines $n(z)$ uniquely. We further note that that if $\omega\rho$ is a (unique by (5.2)) coding of z ($\omega \in E_\rho^*$) then

$$\lambda(z) = \lambda_\rho(\omega).$$

We denote the set of all inverse images of ξ under iterates of f by $f^{-*}(\xi)$, i.e.

$$f^{-*}(\xi) := \bigcup_{n=0}^{\infty} f^{-n}(\xi).$$

We call $z := (x, n) \in X \times \mathbb{N}$, a periodic pair of f (of period n) if

$$f^n(x) = x.$$

We then denote x by $\hat{z}$ and n by $n(z)$. Of course x is a periodic point of f (of period n). We emphasize that we do not assume n to be a prime (least) period of x. We set

$$\lambda_p(z) := \log |(f^{n(z)})'(\hat{z})|.$$

We denote by $\widehat{\mathrm{Per}}(f)$ (respectively $\widehat{\mathrm{Per}}_n(f)$) the set of all periodic pairs (of period n) and by $\mathrm{Per}(f)$ (respectively $\mathrm{Per}_n(f)$) the set of all periodic points (of period n) of f.

Given $T \geq 0$ we set

$$\pi_\xi(f, T) := \{z \in f^{-*}(\xi) : \lambda(z) \leq T\}$$

and

$$\pi_p(f, T) = \{z \in \widehat{\mathrm{Per}}(f) : \lambda_p(z) \leq T\}.$$

Furthermore, given a set $B \subset X$, we denote

$$\pi_\xi(f, B, T) := B \cap \pi_\xi(f, T) \quad \text{and} \quad \pi_p(f; B, T) := B \cap \pi_p(f, T).$$

As in the case of graph directed Markov systems we denote

$$N_\xi(f, T) := \#\pi_\xi(f, T), \quad N_\xi(f; B, T) := \#\pi_\xi(f; B, T)$$

and

$$N_p(f, T) := \#\pi_p(f, T), \quad N_p(f, B, T) := \#\pi_p(f, B, T).$$

Given a set $Y \subset B(\xi, \alpha)$ we denote

$$\mathcal{D}_Y^\xi(f; B, T) := \{z \in f^{-*}(\xi) \cap B : \log \operatorname{diam}\big(f_{\hat{z}}^{-n(z)}(Y)\big) \leq T\},$$

$$\mathcal{E}_Y^\xi(f; B, T) := \big\{z \in f^{-*}(\xi) : \log \operatorname{diam}\big(f_{\hat{z}}^{-n(z)}(Y)\big) \leq T \quad \text{and} \quad f_{\hat{z}}^{-n(z)}(Y) \cap B \neq \emptyset\big\},$$

and then

$$D_Y^\xi(f; B, T) := \#\mathcal{D}_Y^\xi(f; B, T) \quad \text{and} \quad E_Y^\xi(f; B, T) := \#\mathcal{E}_Y^\xi(f; B, T).$$

Now we record a straightforward, but basic observation which links this section to the previous ones. It is the following.

Observation 5.1.5. If $f : X \to X$ is a conformal expanding repeller, then with the notation as above

$$N_\xi(f; B, T) = N_\rho(B, T), \quad D_Y^\xi(f; B, T) = D_Y^\rho \rho(B, T)$$

and

$$\Gamma N_p(B, T) \leq N_p(f; B, T) \leq N_\rho(B, T)$$

with some universal constant $\Gamma \in (0, +\infty)$. In addition,

$$N_p(f; B, T) = N_p(B, T)$$

whenever $B \subseteq \bigcup_{e \in F} \mathrm{Int}(R_e)$.

We call a conformal expanding repeller $f : X \to X$ D–generic if and only if the additive group generated by the set

$$\{\lambda_p(z) : z \in \widehat{\mathrm{Per}}(f)\}$$

is not cyclic. It is immediate from the definition of the graph directed Markov system $\mathcal{S}_{\mathcal{R}}$ and Proposition 2.3.8 that we have the following.

Proposition 5.1.6. *A conformal expanding repeller $f : X \to X$ is D–generic if and only if the conformal graph directed Markov system $\mathcal{S}_{\mathcal{R}}$ is D–generic.*

A concept of essentially non–linear conformal expanding repellers was introduced by Dennis Sullivan in [**87**], Section 3, although the terminology used there was "non-linear $\mathbb{C}$–analytic expanding systems". This was explored in detail in [**74**], where they were called "non–linear conformal expanding repellers". The additional adjective "essentially" is to indicate that the system is not merely non-linear but in fact is not even conformally conjugate to a linear system. One of many characterizations (see Chapeter 6 of [**74**] for these) of essentially non–linear conformal expanding repellers is that there is no conformal atlas covering X with respect to which the map f is affine, i.e. a similarity composed with a translation. Analogously, as for graph directed Markov systems, with the help of Chapter 10 from [**74**], we get the following proposition, which adds considerably to our knowledge that D–generic conformal expanding repellers abound.

Proposition 5.1.7. *An essentially non–linear conformal expanding repeller $f : X \to X$ is D–generic.*

As a fairly direct consequence of Theorem 2.4.9 and Theorem 2.7.1, we get the following.

Theorem 5.1.8. *Let $f : X \to X$ be a D–generic conformal expanding repeller and let $\delta := \mathrm{HD}(X)$.*

1. *Let m_δ be the unique δ-conformal measure for f on X, which coincides with the normalized δ–dimensional Hausdorff measure on X.*
2. *Let μ_δ be the unique f-invariant Borel probability measure on X absolutely continuous (in fact known to be equivalent) with respect to m_δ. It is also known to be the unique equilibrium state of the potential $X \ni x \mapsto -\delta \log |f'(x)| \in \mathbb{R}$.*
3. *Let $\psi_\delta := \frac{d\mu_\delta}{dm_\delta}$.*
4. *Fix $\xi \in X$ arbitrarily and $Y \subset B(\xi, \alpha)$, an arbitrary set consisting of at least two distinct points.*
5. *Let $B \subset X$ be an arbitrary Borel set such that $m_\delta(\partial B) = 0$ (equivalently that $\mu_\delta(\partial B) = 0$).*

Then

$$\lim_{T\to+\infty} \frac{N_\xi(f; B, T)}{e^{\delta T}} = \frac{\psi_\delta(\xi)}{\delta\chi_\delta} m_\delta(B), \tag{5.6}$$

$$\lim_{T\to+\infty} \frac{N_p(f; B, T)}{e^{\delta T}} = \frac{1}{\delta\chi_\delta} \mu_\delta(B), \tag{5.7}$$

and

$$\lim_{T\to+\infty} \frac{D_Y^\xi(f; B, T)}{e^{\delta T}} = \lim_{T\to+\infty} \frac{E_Y^\xi(f; B, T)}{e^{\delta T}} = C_\xi(Y) m_\delta(B), \tag{5.8}$$

where $C_\xi(Y) \in (0, +\infty)$ is a constant depending only on the repeller f, the point $\xi \in X$, and the set Y. In addition

$$K^{-2\delta}(\delta\chi_\delta)^{-1}\mathrm{diam}^\delta(Y) \le C_\xi(Y) \le K^{2\delta}(\delta\chi_\delta)^{-1}\mathrm{diam}^\delta(Y), \tag{5.9}$$

and the function

$$\xi \longmapsto C_\xi(Y) \in (0, +\infty)$$

is locally constant on some sufficiently small neighborhood of Y.

Proof. By making use of Observation 5.1.5, formulae (5.6) and (5.8) are immediate consequences of formula (2.6) of Theorem 2.4.9, along with Theorem 2.7.1 and Theorem 2.7.4, once we notice that the measures m_δ and μ_σ are respectively δ-conformal and invariant, equivalent to m_δ, for both the conformal expanding repeller $f : X \to X$ and the associated conformal GDMS $\mathcal{S}_\mathcal{R}$. In order to get formula (5.7) one uses formula (2.7) of Theorem 2.4.9, and also, in a straightforward way, the fact that $\mu_\mathcal{S}(\partial\mathcal{R}) = 0$. The fact the function $\xi \longmapsto C_\xi(Y)$ is locally constant follows from Remark 2.7.5. $\square$

From the results of Section 4, in particular the versions of the Central Limit Theorem, proved for attracting conformal GDMSs, we directly get the following consequences for expanding repellers.

Theorem 5.1.9. *Let $f : X \to X$ be a D–generic conformal expanding repeller. With notation of Theorem 5.1.8, there exists $\sigma^2 > 0$ (in fact $\sigma^2 = \mathrm{P}''(0) > 0$) such that if $G \subset \mathbb{R}$ is a Lebesgue measurable set with $\mathrm{Leb}(\partial G) = 0$, then*

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in X : \frac{\log\left|(f^n)'(z)\right| - \chi_{\mu_\delta} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in X : \alpha \le \frac{\log\left|(f^n)'(z)\right| - \chi_{\mu_\delta} n}{\sqrt{n}} \le \beta\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

For every point $z \in X$ and every integer $n \ge 0$ let $e(z,n) \in F$ be such that

$$f^n(z) \in R_e.$$

Theorem 5.1.10. *Let $f : X \to X$ be a D–generic conformal expanding repeller. With notation of Theorem 5.1.8, there exists $\sigma^2 > 0$ (in fact $\sigma^2 = \mathrm{P}''(0) > 0$) such that if $G \subset \mathbb{R}$ is a Lebesgue measurable set with $\mathrm{Leb}(\partial G) = 0$, then*

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in X : \frac{-\log\operatorname{diam}\big(f_x^{-n}(Y_{e(z,n)})\big) - \chi_{\mu_\delta} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in X : \alpha \le \frac{-\log\operatorname{diam}\big(f_x^{-n}(Y_{e(z,n)})\big) - \chi_{\mu_\delta} n}{\sqrt{n}} \le \beta\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

The next result is a law of the iterated logarithm.

Theorem 5.1.11. *Let $f : X \to X$ be a D–generic conformal expanding repeller. Assume the some notation as in Theorem 5.1.8, For every $e \in F$ let $Y_e \subset R_e$ be a set with at least two points. There exists $\sigma^2 > 0$ (in fact $\sigma^2 := \mathrm{P}''(0) > 0$) such that for μ_δ–a.e. $z \in X$, we have that*

$$\limsup_{n\to+\infty} \frac{\log\left|(f^n)'(z)\right| - \chi_{\mu_\delta} n}{\sqrt{n\log\log n}} = \sqrt{2\pi}\sigma$$

and

$$\limsup_{n\to+\infty} \frac{-\log\operatorname{diam}\big(f_x^{-n}(Y_{e(z,n)})\big) - \chi_{\mu_\delta} n}{\sqrt{n\log\log n}} = \sqrt{2\pi}\sigma.$$

Let $\xi \in X$ be fixed. For every set $H \subset f^{-n}(\xi)$, define

$$\mu_n(H) := \frac{\sum_{z\in H}\left|(f^n)'(z)\right|^{-\delta}}{\sum_{z\in f^{-n}(\xi)}\left|(f^n)'(z)\right|^{-\delta}}. \tag{5.10}$$

THEOREM 5.1.12. *If $f : X \to X$ is a conformal expanding repeller, then for every $\xi \in X$, we have that*

$$\lim_{n\to+\infty}\int_{f^{-n}(\xi)} \frac{\log|(f^n)'|}{n}d\mu_n = \chi_\delta. \tag{5.11}$$

Analogously to (4.3) we define the functions $\Delta_n : f^{-n}(\xi) \to \mathbb{R}$ by the formulae

$$\Delta_n(z) := \frac{\log|(f^n)'(z)| - \chi_{\mu_\delta} n}{\sqrt{n}} \tag{5.12}$$

and consider the sequence $(\mu_n \circ \Delta_n^{-1})_{n=1}^\infty$ of probability distributions on $\mathbb{R}$.
We have the following.

THEOREM 5.1.13. *If $f : X \to X$ is a D–generic conformal expanding repeller, then the sequence of random variables $(\Delta_n)_{n=1}^\infty$ converges in distribution to the normal (Gaussian) distribution $\mathcal{N}_0(\sigma)$ with mean value zero and the variance $\sigma^2 = P''(\delta) > 0$. Equivalently, the sequence $(\mu_n \circ \Delta_n^{-1})_{n=1}^\infty$ converges weakly to the normal distribution $\mathcal{N}_0(\sigma^2)$. This means that for every Borel set $F \subset \mathbb{R}$ with $\mathrm{Leb}(\partial F) = 0$, we have*

$$\lim_{n\to+\infty}\mu_n(\Delta_n^{-1}(F)) = \frac{1}{\sqrt{2\pi}\sigma}\int_F e^{-t^2/2\sigma^2}dt. \tag{5.13}$$

5.1.2. 1–Dimensional Attracting Conformal GDMSs and 1–Dimensional Conformal Expanding Repellers. In this subsection we briefly discuss 1–Dimensional systems. We start with the following.

EXAMPLE 5.1.14. Theorem 2.4.9, Theorem 2.7.1, and Theorem 2.7.4 hold in particular if a system $\mathcal{S}$ in one–dimensional, i.e., if X is a compact interval of $\mathbb{R}$. Perhaps the the best known and one of the most often considered, is the infinite IFS $\mathcal{G}$ formed by all continuous inverse branches of the Gauss map

$$G(x) = x - [x].$$

So $\mathcal{G}$ consists of the maps

$$[0,1] \ni x \longmapsto g_n(x) := \frac{1}{x+n}, \quad n \in \mathbb{N}.$$

and with $q = 2$ in the sense of Remark 2.2.2 it becomes a conformal IFS. The corresponding conformal measure m_1 is just Lebesgue measure Leb on $[0,1]$ (or somewhat more precisely on the set of irrational numbers of $[0,1]$ being $J_{\mathcal{G}}$, the limit set of the Gauss system $\mathcal{G}$. The corresponding invariant measure μ_1, is in this case the well-known Gauss measure defined by

$$\frac{d\mu_1}{dm_1}(x) = \frac{1}{\log 2}\cdot\frac{1}{1+x}.$$

Looking at the fixed points of g_1, g_2, and g_3 one immediately concludes that the Gauss system $\mathcal{G}$ is D–generic. It is also known (see ex. [**44**]) to be strongly regular, even more, in the terminology of [**47**], it is hereditarily regular. So, Theorem 2.4.9, 2.7.1 and 2.7.4 do indeed apply to this system. Because of importance of the Gauss map we formulate below all the above mentioned applications expressed in the language of the Gauss map itself rather than the associated IFS $\mathcal{G}$. We adopt the, naturally adjusted, notation of Subsection 5.1.1.

We begin with the growth estimates.

THEOREM 5.1.15. *If $G : [0,1] \to [0,1]$ is the Gauss map, then with notation of subsection 5.1.1 we have the following. Fix $\xi \in [0,1]$. If $B \subset [0,1]$ is a Borel set such that $\mathrm{Leb}(\partial B) = 0$ and $Y \subset [0,1]$ is any set having at least two elements, then*

$$\lim_{T\to+\infty}\frac{N_\xi(G;B,T)}{e^T} = \frac{\psi_1(\xi)}{\chi_1}\mathrm{Leb}(B),$$

$$\lim_{T\to+\infty}\frac{N_p(G;B,T)}{e^T}=\frac{1}{\chi_1}\mu_1(B),$$

and

$$\lim_{T\to+\infty}\frac{D_Y^\xi(G;B,T)}{e^T}=\lim_{T\to+\infty}\frac{E_Y^\xi(G;B,T)}{e^T}=C(Y)\mathrm{Leb}(B),$$

where $C(Y)\in(0,+\infty]$ is a constant depending only on the map G and the set Y.

We next formulate a Central Limit Theorem for diameters.

THEOREM 5.1.16. *Let $G:[0,1]\to[0,1]$ be the Gauss map. Let $\sigma^2:=\mathrm{P}''(0)>0$. With the notation of Theorem 5.1.8 we have the following. Let $Y\subset[0,1]$ be a set with at least two points. If $H\subset\mathbb{R}$ is a Lebesgue measurable set with $\mathrm{Leb}(\partial H)=0$, then*

$$\lim_{n\to+\infty}\mu_1\left(\left\{z\in[0,1]\,:\,\frac{-\log\mathrm{diam}\big(G_x^{-n}(Y)\big)-\chi_{\mu_1}n}{\sqrt{n}}\in H\right\}\right)=\frac{1}{\sqrt{2\pi}\sigma}\int_H e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

In particular, for any $\alpha<\beta$

$$\lim_{n\to+\infty}\mu_1\left(\left\{z\in[0,1]\,:\,\alpha\le\frac{-\log\mathrm{diam}\big(G_x^{-n}(Y)\big)-\chi_{\mu_1}n}{\sqrt{n}}\le\beta\right\}\right)=\frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

The law of the iterated logarithm takes the following form.

THEOREM 5.1.17. *Let $G:[0,1]\to[0,1]$ be the Gauss map. Let $\sigma^2:=\mathrm{P}''(0)>0$. With notation of Theorem 5.1.8 we have the following. Let $Y\subset[0,1]$ be a set with at least two points. Then for Leb–a.e. $z\in[0,1$, we have that*

$$\limsup_{n\to+\infty}\frac{\log\big|(G^n)'(z)\big|-\chi_{\mu_1}n}{\sqrt{n\log\log n}}=\sqrt{2\pi}\sigma$$

and

$$\limsup_{n\to+\infty}\frac{-\log\mathrm{diam}\big(G_x^{-n}(Y)\big)-\chi_{\mu_1}n}{\sqrt{n\log\log n}}=\sqrt{2\pi}\sigma.$$

Finally, we have a Central Limit Theorem for counting functions.

THEOREM 5.1.18. *If $G:[0,1]\to[0,1]$ is the Gauss map, then for every $\xi\in[0,1]$, we have that*

$$\lim_{n\to+\infty}\int_{G^{-n}(\xi)}\frac{\log\big|(G^n)'\big|}{n}d\mu_n=\chi_1.$$

THEOREM 5.1.19. *If $G:[0,1]\to[0,1]$ is the Gauss map, then the sequence of random variables $(\Delta_n)_{n=1}^\infty$ converges in distribution to the normal (Gaussian) distribution $\mathcal{N}_0(\sigma)$ with mean value zero and the variance $\sigma^2=P''(\delta)>0$. Equivalently, the sequence $(\mu_n\circ\Delta_n^{-1})_{n=1}^\infty$ converges weakly to the normal distribution $\mathcal{N}_0(\sigma^2)$. This means that for every Borel set $F\subset\mathbb{R}$ with $\mathrm{Leb}(\partial F)=0$, we have*

$$\lim_{n\to+\infty}\mu_n(\Delta_n^{-1}(F))=\frac{1}{\sqrt{2\pi}\sigma}\int_F e^{-t^2/2\sigma^2}dt. \tag{5.14}$$

REMARK 5.1.20. Theorem 5.1.8 holds in particular if $f:X\mapsto X$ is a conformal expanding repeller with X a compact subset (a topological Cantor set) of $\mathbb{R}$.

5.1.3. Hyperbolic (Expanding) Rational Functions of the Riemann Sphere $\widehat{\mathbb{C}}$. One of the most celebrated conformal expanding repellers are hyperbolic (expanding) rational functions of the Riemann sphere $\widehat{\mathbb{C}}$ restricted to the Julia sets and already mentioned in subsection 5.1.1. For the sake of completeness and convenience of the reader, let us briefly describe them. Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a rational function of degree $d\ge 2$. Let $J(f)$ denote the Julia sets of f and let

$$\mathrm{Crit}(f):=\{c\in\widehat{\mathbb{C}}:f'(c)=0\}$$

be the set of all critical (branching) points of f. Put

$$\mathrm{PC}(f):=\bigcup_{n=1}^{\infty}f^n(\mathrm{Crit}(f))$$

and call it the postcritical set of f. The rational map $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is said to be hyperbolic (expanding) if the restriction $f|_{J(f)}:J(f)\to J(f)$ satisfies

$$\inf\{|f'(z)|:z\in J(f)\}>1 \tag{5.15}$$

or, equivalently,

$$|f'(z)|>1 \tag{5.16}$$

for all $z\in J(f)$. Another, topological, characterization of expandingness is the following.

FACT 5.1.21. A rational function $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is expanding if and only if

$$J(f)\cap\overline{\mathrm{PC}(f)}=\emptyset.$$

It is immediate from this characterization that all the polynomials $z\mapsto z^d$, $d\ge 2$, are expanding along with their small perturbations $z\mapsto z^d+\varepsilon$; in fact expanding rational functions are commonly believed to form a vast majority amongst all rational functions. This is known at least for polynomials with real coefficients.

It is known from [**96**] (see also Section 3 of [**73**]) that the only essentially linear expanding rational functions are the maps of the form

$$\widehat{\mathbb{C}}\ni z\longmapsto f_d(z)=:z^d\in\widehat{\mathbb{C}},\quad |d|\ge 2.$$

In consequence the only non D-generic rational functions of the Riemann sphere $\widehat{\mathbb{C}}$ are these functions f_d. So, as an immediate consequence of Theorem 5.1.8, we get the following.

THEOREM 5.1.22. *Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a hyperbolic (expanding) rational function of the Riemann sphere $\widehat{\mathbb{C}}$ not of the form $\widehat{\mathbb{C}}\ni z\longmapsto z^d\in\widehat{\mathbb{C}}$, $|d|\ge 2$. Let $\delta:=\mathrm{HD}(J(f))$.*

(1) *Let m_δ be the unique δ-conformal measure for f on the Julia set $J(f)$, which coincides with the normalized δ–dimensional Hausdorff measure on $J(f)$.*
(2) *Let μ_δ be the unique f-invariant Borel probability measure on $J(f)$ absolutely continuous (in fact known to be equivalent) with respect to m_δ. It is also known to be the unique equilibrium state of the potential $J(f)\ni x\mapsto -\delta\log|f'(x)|\in\mathbb{R}$.*
(3) *Let $\psi_\delta:=\frac{d\mu_\delta}{dm_\delta}$.*
(4) *Fix $\xi\in J(f)$ arbitrary and $Y\subset B(\xi,\alpha)$ (where $\alpha>0$ is sufficiently small as described in subsection 5.1.1), an arbitrary set consisting of at least two distinct points.*
(5) *Let $B\subset J(f)$ be an arbitrary Borel set such that $m_\delta(\partial B)=0$ (equivalently that $\mu_\delta(\partial B)=0$).*

Then

$$\lim_{T\to+\infty}\frac{N_\xi(f;B,T)}{e^{\delta T}}=\frac{\psi_\delta(\xi)}{\delta\chi_\delta}m_\delta(B), \tag{5.17}$$

$$\lim_{T\to+\infty}\frac{N_p(f;B,T)}{e^{\delta T}}=\frac{1}{\delta\chi_\delta}\mu_\delta(B), \tag{5.18}$$

and

$$\lim_{T\to+\infty}\frac{D_Y^\xi(f;B,T)}{e^{\delta T}}=\lim_{T\to+\infty}\frac{E_Y^\xi(f;B,T)}{e^{\delta T}}=C_\xi(Y)m_\delta(B), \tag{5.19}$$

where $C_\xi(Y)\in(0,+\infty)$ is a constant depending only on the repeller f, the point $\xi\in J(f)$, and the set Y. In addition

$$K^{-2\delta}(\delta\chi_\delta)^{-1}\mathrm{diam}^\delta(Y)\le C_\xi(Y)\le K^{2\delta}(\delta\chi_\delta)^{-1}\mathrm{diam}^\delta(Y), \tag{5.20}$$

and the function

$$\xi\longmapsto C_\xi(Y)\in(0,+\infty)$$

is locally constant on some sufficiently small neighborhood of Y.

Fixing a Markov partition for the map $f:J(f)\to J(f)$, as immediate consequences of Theorems 5.1.9 – 5.1.13 we get the following stochastic laws, primarily Central Limit Theorems, for the dynamical system (f,μ_δ).

We begin with a Central Limit Theorem for the expansion on orbits.

Theorem 5.1.23. *Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a hyperbolic (expanding) rational function of the Riemann sphere $\widehat{\mathbb{C}}$ not of the form $\widehat{\mathbb{C}}\ni z\longmapsto z^d\in\widehat{\mathbb{C}}$, $|d|\ge 2$. With notation of Theorem 5.1.8 there exists $\sigma^2>0$ (in fact $\sigma^2=\mathrm{P}''(0)>0$) such that if $G\subset\mathbb{R}$ is a Lebesgue measurable set with $\mathrm{Leb}(\partial G)=0$, then*

$$\lim_{n\to+\infty}\mu_\delta\left(\left\{z\in J(f)\,:\,\frac{\log\left|(f^n)'(z)\right|-\chi_{\mu_\delta}n}{\sqrt{n}}\in G\right\}\right)=\frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

In particular, for any $\alpha<\beta$

$$\lim_{n\to+\infty}\mu_\delta\left(\left\{z\in J(f)\,:\,\alpha\le\frac{\log\left|(f^n)'(z)\right|-\chi_{\mu_\delta}n}{\sqrt{n}}\le\beta\right\}\right)=\frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

We next have a Central Limit Theorem for diameters.

Theorem 5.1.24. *Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a hyperbolic (expanding) rational function of the Riemann sphere $\widehat{\mathbb{C}}$ not of the form $\widehat{\mathbb{C}}\ni z\longmapsto z^d\in\widehat{\mathbb{C}}$, $|d|\ge 2$. Let $\sigma^2:=\mathrm{P}''(0)>0$. With the notation of Subsection 5.1.1 for every $e\in F$ let $Y_e\subset R_e$ be a set with at least two points and if $G\subset\mathbb{R}$ is a Lebesgue measurable set with $\mathrm{Leb}(\partial G)=0$, then*

$$\lim_{n\to+\infty}\mu_\delta\left(\left\{z\in J(f)\,:\,\frac{-\log\mathrm{diam}\big(f_x^{-n}(Y_{e(z,n)})\big)-\chi_{\mu_\delta}n}{\sqrt{n}}\in G\right\}\right)=\frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

In particular, for any $\alpha<\beta$

$$\lim_{n\to+\infty}\mu_\delta\left(\left\{z\in J(f)\,:\,\alpha\le\frac{-\log\mathrm{diam}\big(f_x^{-n}(Y_{e(z,n)})\big)-\chi_{\mu_\delta}n}{\sqrt{n}}\le\beta\right\}\right)=\frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

The following is a version of the law of the iterated function scheme.

THEOREM 5.1.25. *Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a hyperbolic (expanding) rational function of the Riemann sphere $\widehat{\mathbb{C}}$ not of the form $\widehat{\mathbb{C}}\ni z\longmapsto z^d\in\widehat{\mathbb{C}}$, $|d|\geq 2$. Let $\sigma^2:=\mathrm{P}''(0)>0$. With the notation of Subsection 5.1.1 for every $e\in F$ let $Y_e\subset R_e$ be a set with at least two points and if $G\subset\mathbb{R}$ is a Lebesgue measurable set with $\mathrm{Leb}(\partial G)=0$, then for μ_δ–a.e. $z\in J(f)$, we have that*

$$\limsup_{n\to+\infty}\frac{\log\left|(f^n)'(z)\right|-\chi_{\mu_\delta}n}{\sqrt{n\log\log n}}=\sqrt{2\pi}\sigma$$

and

$$\limsup_{n\to+\infty}\frac{-\log\mathrm{diam}\big(f_x^{-n}(Y_{e(z,n)})\big)-\chi_{\mu_\delta}n}{\sqrt{n\log\log n}}=\sqrt{2\pi}\sigma.$$

THEOREM 5.1.26. *If $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is a hyperbolic (expanding) rational function of the Riemann sphere $\widehat{\mathbb{C}}$, then for every $\xi\in J(f)$, we have that*

$$\lim_{n\to+\infty}\int_{f^{-n}(\xi)}\frac{\log\left|(f^n)'\right|}{n}d\mu_n=\chi_\delta. \tag{5.21}$$

Finally, we have a Central Limit Theorem for counting.

THEOREM 5.1.27. *If $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is a hyperbolic (expanding) rational function of the Riemann sphere $\widehat{\mathbb{C}}$ not of the form $\widehat{\mathbb{C}}\ni z\longmapsto z^d\in\widehat{\mathbb{C}}$, $|d|\geq 2$, then the sequence of random variables $(\Delta_n)_{n=1}^\infty$ converges in distribution to the normal (Gaussian) distribution $\mathcal{N}_0(\sigma)$ with mean value zero and the variance $\sigma^2=P''(\delta)>0$. Equivalently, the sequence $(\mu_n\circ\Delta_n^{-1})_{n=1}^\infty$ converges weakly to the normal distribution $\mathcal{N}_0(\sigma^2)$. This means that for every Borel set $F\subset\mathbb{R}$ with $\mathrm{Leb}(\partial F)=0$, we have*

$$\lim_{n\to+\infty}\mu_n(\Delta_n^{-1}(F))=\frac{1}{\sqrt{2\pi}\sigma}\int_F e^{-t^2/2\sigma^2}dt. \tag{5.22}$$

5.2. Conformal Parabolic Dynamical Systems

Now we move onto dealing with parabolic systems. We consider first 1–dimensional examples.

5.2.1. 1–Dimensional Parabolic IFSs. Theorems 3.3.1, 3.4.1 and 3.4.2 hold in particular if a parabolic system $\mathcal{S}$ is 1–dimensional, i.e., if X is a compact interval of $\mathbb{R}$. Perhaps the best known, and one of the most often considered, are the 1-dimensional parabolic IFSs formed by (two) continuous inverse branches of Manneville–Pomeau maps $f_\alpha:[0,1]\to[0,1]$ defined by the

$$f_\alpha(x)=x+x^{1+\alpha}\ (\mathrm{mod}\ 1),$$

where $\alpha>0$ is a fixed number and by the (two) continuous inverse branches of the Farey map (for this one Remark 2.3.6 applies with $q=2$)

$$f(x)=\begin{cases}\frac{x}{1-x} & \text{if } 0\leq x\leq\frac12\\ \frac{1-x}{x} & \text{if } \frac{1-x}{x}\leq x\leq 1.\end{cases}$$

Observe that for parabolic points,

$$\Omega(f)=\Omega(f_\alpha)=\{0\}$$

for all $\alpha>0$. Furthermore,

$$p(f)=1\ \text{ and }\ p(f_\alpha)=\alpha$$

for all $\alpha>0$, and

$$\Omega_\infty(f_\alpha)=\begin{cases}\emptyset & \text{if }\ \alpha<1\\ \{0\} & \text{if }\ \alpha\geq 1,\end{cases}$$

while

$$\Omega_\infty(f) = \{0\}.$$

Of course for both systems, arising from f_α and f, the corresponding δ number is equal to 1 and m_δ is the Lebesgue measure Leb.

Another large class of 1-dimensional parabolic maps, actually comprising the above, whose continuous inverse branches form a 1-dimensional parabolic GDMS can be found in [**90**]. In conclusion, using also Corollary 3.4.6, we have the following results which apply to all of them.

THEOREM 5.2.1. *If $f : [0,1] \to [0,1]$ is the Farey map, then with notation of subsection 5.1.1 we have the following. Fix $\xi \in [0,1]$. If $B \subset [0,1]$ is a Borel set such that* $\mathrm{Leb}(\partial B) = 0$ *and $Y \subset [0,1]$ is any set having at least two elements, then*

$$\lim_{T\to+\infty} \frac{N_\xi(f; B, T)}{e^T} = \frac{\psi_1(\xi)}{\chi_1}\mathrm{Leb}(B), \tag{5.1}$$

$$\lim_{T\to+\infty} \frac{N_p(f; B, T)}{e^T} = \frac{1}{\chi_1}\mu_1(B), \tag{5.2}$$

and

$$\lim_{T\to+\infty} \frac{D_Y^\xi(f; B, T)}{e^T} = \lim_{T\to+\infty} \frac{E_Y^\xi(f; B, T)}{e^T} = C(Y)\mathrm{Leb}(B), \tag{5.3}$$

where $C(Y) \in (0, +\infty]$ is a constant depending only on the map f and the set Y. In addition $C(Y)$ is finite if and only if

$$0 \notin \overline{Y}.$$

Although this is not needed for our results in this monograph, it is interesting that a simple calculation reveals that the attracting "*" IFS of Section 3.1 associated with the Farey IFS is just the Gauss IFS $\mathcal{G}$ described in Remark 5.1.14.

As the next theorem shows, the counting situation is more complex in the case of Manneville-Pomeau maps.

THEOREM 5.2.2. *If $\alpha > 0$ and $f_\alpha : [0,1] \to [0,1]$ is the corresponding Manneville-Pomeau map, then with the notation of subsection 5.1.1 we have the following. Fix $\xi \in [0,1]$. If $B \subset [0,1]$ is a Borel set such that* $\mathrm{Leb}(\partial B) = 0$ *and $Y \subset [0,1]$ is any set having at least two elements, then*

$$\lim_{T\to+\infty} \frac{N_\xi(f_\alpha; B, T)}{e^T} = \frac{\psi_1(\xi)}{\chi_1}\mathrm{Leb}(B), \tag{5.4}$$

$$\lim_{T\to+\infty} \frac{N_p(f_\alpha; B, T)}{e^T} = \frac{1}{\chi_1}\mu_1(B), \tag{5.5}$$

and

$$\lim_{T\to+\infty} \frac{D_Y^\xi(f_\alpha; B, T)}{e^T} = \lim_{T\to+\infty} \frac{E_Y^\xi(f_\alpha; B, T)}{e^T} = C(Y)\mathrm{Leb}(B), \tag{5.6}$$

where $C(Y) \in (0, +\infty]$ is a constant depending only on the map f_α and the set Y. In addition $C(Y)$ is finite if and only if either

(1) $0 \notin \overline{Y}$ *or*
(2) $\alpha < 1$.

In general, we have the following.

THEOREM 5.2.3. *If f is generated by a parabolic Cantor set of* [**90**]*, then with notation of subsection 5.1.1, we have the following.*

Fix ξ belonging to the limit set of the iterated function system associated to f. If $B \subset X$ is a Borel set such that $m_\delta(\partial B) = 0$ and $Y \subset [0,1]$ is any set having at least two elements and contained in a sufficiently small ball centered at ξ, then

$$\lim_{T\to+\infty} \frac{N_\xi(f;B,T)}{e^{\delta T}} = \frac{\psi_\delta(\xi)}{\delta\chi_\delta} m_\delta(B), \tag{5.7}$$

$$\lim_{T\to+\infty} \frac{N_p(f;B,T)}{e^{\delta T}} = \frac{1}{\delta\chi_\delta}\mu_\delta(B), \tag{5.8}$$

and

$$\lim_{T\to+\infty} \frac{D_Y^\xi(f;B,T)}{e^{\delta T}} = \lim_{T\to+\infty} \frac{E_Y^\xi(f;B,T)}{e^{\delta T}} = C_\xi(Y) m_\delta(B), \tag{5.9}$$

where $C_\xi(Y) \in (0,+\infty]$ is a constant depending only on the map f, the point ξ, and the set Y. In addition $C_\xi(Y)$ is infinite if and only if

$$\xi \in \Omega_\infty(f) \cap \overline{Y} \quad \text{and} \quad p(\xi) \le \delta.$$

With respect to the stochastic laws, as an immediate consequence of the results in Subsections 4.2 and 4.4 we get that the following results hold for systems considered in the current subsection.

We begin with a Central Limit Theorem for the expansion along orbits.

THEOREM 5.2.4. *Let T be either a Manneville-Pomeau map f_α with $\alpha < 1$, or generally, the map generated by a parabolic Cantor set of* [**90**] *with $\Omega_\infty(T) = \emptyset$. Let J be either the interval $[0,1]$ (Manneville-Pomeau) or the parabolic Cantor set. Let $\sigma^2 = \mathrm{P}''(0) > 0$. With the notation of Subsection 5.1.1 if $G \subset \mathbb{R}$ is a Lebesgue measurable set with $\mathrm{Leb}(\partial G) = 0$, then*

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in J : \frac{\log\left|(T^n)'(z)\right| - \chi_{\mu_\delta} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in J : \alpha \le \frac{\log\left|(T^n)'(z)\right| - \chi_{\mu_\delta} n}{\sqrt{n}} \le \beta\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

We next have a Central Limit Theorems for diameters.

THEOREM 5.2.5. *Let T be either a Manneville-Pomeau map f_α with $\alpha < 1$, or generally, the map generated by a parabolic Cantor set of* [**90**] *with $\Omega_\infty(T) = \emptyset$. Let J be either the interval $[0,1]$ (Manneville-Pomeau) or the parabolic Cantor set. Let $\sigma^2 = \mathrm{P}''(0) > 0$. With the notation of Subsection 5.1.1, for every $e \in F$ let $Y_e \subset R_e$ be a set with at least two points, then if $G \subset \mathbb{R}$ is a Lebesgue measurable set with $\mathrm{Leb}(\partial G) = 0$ we have*

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in J : \frac{-\log\operatorname{diam}\big(T_x^{-n}(Y_{e(z,n)})\big) - \chi_{\mu_\delta} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{\omega \in J : \alpha \le \frac{-\log\operatorname{diam}\big(T_x^{-n}(Y_{e(z,n)})\big) - \chi_{\mu_\delta} n}{\sqrt{n}} \le \beta\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

Next, we have a Central Limit Theorem for preimages.

THEOREM 5.2.6. *Let T be either a Manneville-Pomeau map f_α with $\alpha < 1$, or generally, the map generated by a parabolic Cantor set of* [**90**] *with $\Omega_\infty(T) = \emptyset$. Let J be either the interval* $[0, 1$ *(Manneville-Pomeau) or the parabolic Cantor set. Then for every $\xi \in J$, we have that*

$$\lim_{n\to+\infty} \int_{T^{-n}(\xi)} \frac{\log |(T^n)'|}{n} d\mu_n = \chi_\delta.$$

Finally, we have a Central Limit Theorem for counting.

THEOREM 5.2.7. *Let T be either a Manneville-Pomeau map f_α with $\alpha < 1$, or generally, the map generated by a parabolic Cantor set of* [**90**] *with $\Omega_\infty(T) = \emptyset$. Let J be either the interval* $[0, 1$ *(Manneville-Pomeau) or the parabolic Cantor set. Then for every $\xi \in J$ the sequence of random variables $(\Delta_n)_{n=1}^\infty$ converges in distribution to the normal (Gaussian) distribution $\mathcal{N}_0(\sigma)$ with mean value zero and the variance $\sigma^2 = \mathrm{P}''(\delta) > 0$. Equivalently, the sequence $(\mu_n \circ \Delta_n^{-1})_{n=1}^\infty$ converges weakly to the normal distribution $\mathcal{N}_0(\sigma^2)$. This means that for every Borel set $F \subset \mathbb{R}$ with* $\mathrm{Leb}(\partial F) = 0$, *we have*

$$\lim_{n\to+\infty} \mu_n(\Delta_n^{-1}(F)) = \frac{1}{\sqrt{2\pi}\sigma} \int_F e^{-t^2/2\sigma^2} dt. \tag{5.10}$$

5.2.2. Parabolic Rational Functions. Now we pass to the counting applications for parabolic rational functions. We recall that if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational function then $\xi \in \widehat{\mathbb{C}}$ is called a rationally indifferent (or just parabolic) periodic point of f if $f^q(\xi) = \xi$ for some integer $q \geq 1$ and $(f^q)'(\xi) = 1$. It is well known and easy to to see that then $\xi \in J(f)$, the Julia set of f. The number $p(\xi) \geq 1$, closely related to the one of parabolic GDMSs, comes from the Taylor series expansion of f about ξ:

$$f^q(z) = z + a(z - \xi)^{p(\xi)+1} + \text{higher terms}$$

with $a \neq 0$. Another, more geometric, characterization of $p(\xi)$ is that it is equal to the number of Fatou petals for f^q coming out of ξ. Let

$$p_f := \max\{p(\xi)\},$$

where the maximum is taken over the (finite) set of all rationally indifferent periodic points of f .

The following theorem has been proved in [**16**].

THEOREM 5.2.8. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational function, then the following two conditions are equivalent.*

(1) *$f|_{\mathcal{J}(f)} : J(f) \to J(f)$ is expansive.*

(2) *$|f'(z)| > 0$ for all $z \in J(f)$, i.e. $J(f)$ contains no critical point of f.*

In addition, if (a) or (b) hold then the map $\widehat{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is not expanding iff $J(f)$ contains a parabolic periodic point. Following [**16**] and [**17**] we then call the map $\widehat{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ parabolic.

Probably, the best known example of a parabolic rational function is the polynomial

$$\widehat{\mathbb{C}} \ni z \longmapsto f_{1/4}(z) := z^2 + \frac{1}{4} \in \widehat{\mathbb{C}}.$$

It has only one parabolic point, namely $z = 1/2$. In fact this is a fixed point of $f_{1/4}$ and $f'_{1/4}(1/2) = 1$. It was independently proved in [**89**] and [**96**] that

$$\delta_{1/4} := \mathrm{HD}(\mathcal{J}_{1/4}) > 1. \tag{5.11}$$

The GDMS associated to f as in formula (5.1) is now parabolic. The measures m_δ and μ_δ (being inconsistent but these now denote the objects on the Julia sets rather than on the symbol space) come either from the theory of parabolic conformal GDMS of Subsection 3.1, particularly, Theorem 3.1.6, or can be traced back much earlier to [**16**], [**17**] and [**1**]. Either from these three papers or from Theorem 3.1.6, we have the following.

THEOREM 5.2.9. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a parabolic rational function then the invariant measure μ_δ is finite if and only of*

$$\delta = \delta_f = \mathrm{HD}(J(f)) > \frac{2p_f}{p_f + 1}.$$

With the arguments parallel to those in the proof of Theorem 5.1.8, as a consequence of Theorem 3.3.2, Theorem 3.4.1 and Theorem 3.4.2, we get the following.

COROLLARY 5.2.10. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a parabolic rational function then with notation of Subsection 5.1.1, we have the following.*

Fix $\xi \in J(f)$. If $B \subset \widehat{\mathbb{C}}$ is a Borel set such that $m_\delta(\partial B) = 0$ and $Y \subset \widehat{\mathbb{C}}$ is any set having at least two elements and contained in a sufficiently small ball centered at ξ, then

$$\lim_{T\to+\infty} \frac{N_\xi(f; B, T)}{e^{\delta T}} = \frac{\psi_\delta(\xi)}{\delta\chi_\delta} m_\delta(B), \tag{5.12}$$

$$\lim_{T\to+\infty} \frac{N_p(f; B, T)}{e^{\delta T}} = \frac{1}{\delta\chi_\delta} \mu_\delta(B), \tag{5.13}$$

and

$$\lim_{T\to+\infty} \frac{D_Y^\xi(f; B, T)}{e^{\delta T}} = \lim_{T\to+\infty} \frac{E_Y^\xi(f; B, T)}{e^{\delta T}} = C_\xi(Y) m_\delta(B), \tag{5.14}$$

where $C_\xi(Y) \in (0, +\infty]$ is a constant depending only on the map f, the point ξ, and the set Y. In addition $C_\xi(Y)$ is infinite if and only if

$$\xi \in \Omega_\infty(f) \cap \overline{Y} \quad \text{and} \quad p(\xi) \le \delta.$$

As in the previous subsection, the stochastic laws appear as immediate consequences of the results in Subsections 4.2 and 4.4.

We begin with a Central Limit Theorem for the expansion along orbits.

THEOREM 5.2.11. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a parabolic rational function of the Riemann sphere $\widehat{\mathbb{C}}$ with $\delta > \frac{2p_f}{p_f+1}$*[1]*. With notation of Theorem 5.1.8 we have the following.*

There exists $\sigma^2 > 0$ (in fact $\sigma^2 = \mathrm{P}''(0) > 0$) such that if $G \subset \mathbb{R}$ is a Lebesgue measurable set with $\mathrm{Leb}(\partial G) = 0$, then

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in J(f) \,:\, \frac{\log\left|(f^n)'(z)\right| - \chi_{\mu_\delta} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}}\, dt.$$

In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in J(f) \,:\, \alpha \le \frac{\log\left|(f^n)'(z)\right| - \chi_{\mu_\delta} n}{\sqrt{n}} \le \beta\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\, dt.$$

We next have a Central Limit Theorem for diameters.

THEOREM 5.2.12. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a parabolic rational function of the Riemann sphere $\widehat{\mathbb{C}}$ with $\delta > \frac{2p_f}{p_f+1}$*[2]*. Let $\sigma^2 := \mathrm{P}''(0) > 0$. With notation of Subsection 5.1.1 we have the following.*

For every $e \in F$ let $Y_e \subset R_e$ be a set with at least two points. If $G \subset \mathbb{R}$ is a Lebesgue measurable set with $\mathrm{Leb}(\partial G) = 0$, then

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in J(f) \,:\, \frac{-\log\mathrm{diam}\big(f_x^{-n}(Y_{e(z,n)})\big) - \chi_{\mu_\delta} n}{\sqrt{n}} \in G\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_G e^{-\frac{t^2}{2\sigma^2}}\, dt.$$

[1]This precisely means that the invariant measure μ_δ is finite, thus normalized to be probabilistic.

[2]The same as above

In particular, for any $\alpha < \beta$

$$\lim_{n\to+\infty} \mu_\delta\left(\left\{z \in J(f) \,:\, \alpha \le \frac{-\log \operatorname{diam}\big(f_x^{-n}(Y_{e(z,n)})\big) - \chi_{\mu_\delta} n}{\sqrt{n}} \le \beta\right\}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{t^2}{2\sigma^2}}\, dt.$$

Finally, we have a Central Limit Theorem for counting.

THEOREM 5.2.13. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a parabolic rational function of the Riemann sphere $\widehat{\mathbb{C}}$ with $\delta > \frac{2p_f}{p_f+1}$*[3]*, then for every $\xi \in J(f)$, we have that*

$$\lim_{n\to+\infty} \int_{f^{-n}(\xi)} \frac{\log\left|(f^n)'\right|}{n} d\mu_n = \chi_\delta. \tag{5.15}$$

THEOREM 5.2.14. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a parabolic rational function of the Riemann sphere $\widehat{\mathbb{C}}$ with $\delta > \frac{2p-f}{p_f+1}$*[4]*, then the sequence of random variables $(\Delta_n)_{n=1}^\infty$ converges in distribution to the normal (Gaussian) distribution $\mathcal{N}_0(\sigma)$ with mean value zero and the variance $\sigma^2 = P''(\delta) > 0$. Equivalently, the sequence $(\mu_n \circ \Delta_n^{-1})_{n=1}^\infty$ converges weakly to the normal distribution $\mathcal{N}_0(\sigma^2)$. This means that for every Borel set $F \subset \mathbb{R}$ with $\operatorname{Leb}(\partial F) = 0$, we have*

$$\lim_{n\to+\infty} \mu_n(\Delta_n^{-1}(F)) = \frac{1}{\sqrt{2\pi}\sigma}\int_F e^{-t^2/2\sigma^2} dt. \tag{5.16}$$

Note that for the map $f_{1/4} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$,

$$p\big(f_{1/4}\big) = p_{1/4} = \max\{p(a) \,:\, a \in \Omega\} = 1,$$

so by (5.11) we have that,

$$\delta > p_{1/4} = p\big(f_{1/4}\big) = \frac{2p_{1/4}}{2p_{1/4}+1}. \tag{5.17}$$

Thus, Theorem 5.2.9 gives the following.

THEOREM 5.2.15. *For the map $f_{1/4} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $\Omega_\infty = \emptyset$ and the invariant measure μ_δ is finite, so a probability after normalization.*

Thus, as a consequence of all in this subsection, we get the following.

COROLLARY 5.2.16. *If $f_{1/4} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is parabolic quadratic polynomial*

$$\widehat{\mathbb{C}} \ni z \longmapsto f_{1/4}(z) := z^2 + \frac{1}{4} \in \widehat{\mathbb{C}},$$

then with notation of Subsection 5.1.1, we have the following.

Fix $\xi \in J(f_{1/4})$. If $Y \subset \widehat{\mathbb{C}}$ is any set having at least two elements and contained in a sufficiently small ball centered at ξ, then there exists a constant $C_\xi(Y) \in (0, +\infty)$ such that if $B \subset \widehat{\mathbb{C}}$ is a Borel set with $m_\delta(\partial B) = 0$, then

$$\lim_{T\to+\infty} \frac{N_\xi(f_{1/4}; B, T)}{e^{\delta T}} = \frac{\psi_\delta(\xi)}{\delta\chi_\delta} m_\delta(B), \tag{5.18}$$

$$\lim_{T\to+\infty} \frac{N_p(f_{1/4}; B, T)}{e^{\delta T}} = \frac{1}{\delta\chi_\delta}\mu_\delta(B), \tag{5.19}$$

and

$$\lim_{T\to+\infty} \frac{D_Y^\xi(f_{1/4}; B, T)}{e^{\delta T}} = \lim_{T\to+\infty} \frac{E_Y^\xi(f_{1/4}; B, T)}{e^{\delta T}} = C_\xi(Y) m_\delta(B). \tag{5.20}$$

[3]The same as above

[4]The same as above

Remark 5.2.17. Because of (5.17) all the hypotheses of Theorems 5.2.11 – 5.2.14 are satisfied for the map $f_{1/4} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$; so, in particular, all these theorems hold for the map $f = f_{1/4}$.

On the other hand if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a parabolic rational function with $\mathrm{HD}(J(f)) \le 1$, which is the case for many maps, in particular those of the form $\widehat{\mathbb{C}} \ni z \mapsto 2 + 1/z + t$ where $t \in \mathbb{R}$ or parabolic Blaschke products, then

$$\delta \le 1 \le p_a$$

for every point $a \in \Omega(f)$. Thus also

$$\Omega_\infty(f) = \Omega(f)$$

and, as an immediate consequence of Corollary 5.2.10, we get the following.

Corollary 5.2.18. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a parabolic rational function with $\mathrm{HD}(J(f)) \le 1$, then with notation of Subsection 5.1.1, we have the following.*

Fix $\xi \in J(f)$. If $B \subset \widehat{\mathbb{C}}$ is a Borel set such that $m_\delta(\partial B) = 0$ and $Y \subset \widehat{\mathbb{C}}$ is any set having at least two elements and contained in a sufficiently small ball centered at ξ, then

$$\lim_{T\to+\infty} \frac{N_\xi(f; B, T)}{e^{\delta T}} = \frac{\psi_\delta(\xi)}{\delta\chi_\delta} m_\delta(B), \tag{5.21}$$

$$\lim_{T\to+\infty} \frac{N_p(f; B, T)}{e^{\delta T}} = \frac{1}{\delta\chi_\delta} \mu_\delta(B), \tag{5.22}$$

and

$$\lim_{T\to+\infty} \frac{D_Y^\xi(f; B, T)}{e^{\delta T}} = \lim_{T\to+\infty} \frac{E_Y^\xi(f; B, T)}{e^{\delta T}} = C_\xi(Y) m_\delta(B), \tag{5.23}$$

where $C_\xi(Y) \in (0, +\infty]$ is a constant depending only on the map f, the point ξ, and the set Y. In addition $C_\xi(Y)$ is finite if and only if

$$\xi \notin \Omega(f) \cap \overline{Y}.$$

CHAPTER 6

Examples and Applications, II: Kleinian Groups

In this part we apply our counting results to some large classes of Kleinian groups. These include all finitely generated classical Schottky groups and essentially all finitely generated Fuchsian groups. The applications described in this section would actually fit into two previous sections: Convex co-compact (hyperbolic) groups would fit to Section 5.1 while parabolic ones would fit to Section 5.2. However, because of their distinguished character and the specific methods used to deal with them, we collect all the applications to Kleinian groups in one separate part.

6.1. Finitely Generated Classical Schottky Groups with no Tangencies

In this section we first recall the definition of hyperbolic finitely generated classical Schottky groups. Next, we associate to them appropriate conformal graph directed Markov systems and then we express many concepts pertaining to such groups in the language of such GDMSs. This enables us to apply the counting results for graph directed Markov systems, obtained in previous parts, to such Schottky groups.

Doing this we also, on the way, associate to a finitely generated classical Schottky group an appropriate symbolic dynamics, precisely, a countable alphabet finitely irreducible subshift of finite type as defined in the first sections of our manuscript.

The use of symbolic dynamics to study Schottky groups can be viewed in the more general framework of convex cocompact Fuchsian and Kleinian groups, which can be traced back to the work of Hedlund. A specific instance of the coding for (non–classical) Schottky groups, and developing the corresponding thermodynamic formalism, occurs in Bowen's famous 1979 paper on the Hausdorff Dimension of quasi-circles [**5**]. A nice recent exposition of this construction is given in the book [**12**]. The coding in Bowen's influential paper was used, either implicitly or explicitly, in a number of subsequent works. These include both the paper of Lalley [**37**], and its generalization by Quint to higher rank Schottky settings [**75**]. Further development of these ideas covers the more general case of infinitely generated Schottky groups described, for example, in [**86**]. In a different direction Mark Pollicott in [**70**] and Dal'bo and Peigné [**13**] used symbolic dynamics (based on continued fractions) to count closed geodesics on the non-compact modular surface in the context of metrics of variable negative curvature.

Fix an integer $d \geq 1$. Fix also an integer $q \geq 2$. Let

$$B_j, \quad j = \pm 1, \pm 2, \cdots, \pm q,$$

be open balls in $\mathbb{R}^d$ with mutually disjoint closures. For every $j = 1, 2, \cdots, q$ let

$$g_j : \widehat{\mathbb{R}}^d \to \widehat{\mathbb{R}}^d$$

be a conformal self-map of the one point compactification of $\mathbb{R}^d$ (thus making $\widehat{\mathbb{R}}^d$ conformally equivalent to the unit sphere $S^d \subset \mathbb{R}^{d+1}$) such that

$$g_j(B_{-j}^c) = \overline{B}_j. \tag{6.1}$$

The group G generated by the maps g_j, $j = 1, \ldots, q$, is called a hyperbolic classical Schottky group; hyperbolic alluding to the lack of tangencies. If there is no danger of misunderstanding, we will frequenly skip in this section the adjective "hyperbolic", speaking simply about Schottky groups. Note that if we set

$$g_j := g_{-j}^{-1}$$

for all $j = -1, \dots, -q$ then (6.1) holds for all $j = \pm 1, \pm 2, \cdots, \pm q$.

Denote by $\mathbb{H}^{d+1}$ the space $\mathbb{R}^d \times (0, +\infty)$ endowed with the Poincaré metric. The Poincaré Extension Theorem ensures that all the maps g_j, $j = 1, \dots, q$, uniquely extend to conformal self-maps of

$$\overline{\mathbb{H}}^{d+1} := \widehat{\mathbb{R}}^d \times [0, +\infty),$$

also denoted by g_j, onto itself. Their restrictions to $\mathbb{H}^{d+1}$, which are again also denoted by g_j, are isometries with respect to the Poincaré metric ρ on $\mathbb{H}^d$. The group generated by these isometries in discrete, is also denoted by G, and is also called the Schottky group generated by the maps g_j, $j = 1, \dots, q$. For every $j = \pm 1, \pm 2, \cdots, \pm q$ denote by $\hat{B}_j$ the half-ball in $\mathbb{H}^{d+1}$ with the same center and radius as those of B_j. We recall the following well-known standard fact.

Fact 6.1.1. The region

$$\mathcal{R} := \mathbb{H}^{d+1} \setminus \bigcup_{j=1}^{q} (\hat{B}_j \cup \hat{B}_{-j})$$

is a fundamental domain for the action of G on $\mathbb{H}^{d+1}$ and

$$\hat{\mathbb{R}}^d \setminus \bigcup_{j=1}^{q} (B_j \cup B_{-j})$$

is a fundamental domain for the action of G on $\hat{\mathbb{R}}^d$.

For any $z \in \overline{\mathbb{H}}^{d+1}$ the set of cluster points of the set Gz is contained in

$$\bigcup_{j=1}^{q} \overline{B}_j \cup \overline{B}_{-j},$$

and is independent of z. We call it the limit set of G and denote it by $\Lambda(G)$. This set is compact, perfect, $G(\Lambda(G)) = \Lambda(G)$ and G acts minimally on $\Lambda(G)$. We denote

$$V := \{\pm 1, \pm 2, \dots, \pm q\}, \quad E := V \times V \setminus \{(i, -i) : i \in V\},$$

and introduce an incidence matrix $A : E \times E \to \{0, 1\}$ by declaring that

$$A_{(a,b),(c,d)} = \begin{cases} 1 & \text{if } b = c \\ 0 & \text{if } b \neq c \end{cases}$$

Furthermore, we set for all $(a, b) \in E$, $t(a, b) := b$ and $i(a, b) := a$, and

$$g_{(a,b)} := g_a|_{\overline{B}_b} : \overline{B}_b \to \overline{B}_a.$$

In this way we have associated to G the conformal graph directed Markov system

$$\mathcal{S}_G := \{g_e : e \in E\}.$$

By the very definition of this system, for every $\omega \in E_A^*$, say $\omega = (a_1, b_1)(a_2, b_2) \dots (a_n, b_n)$, we have that

$$g_\omega = g_{(a_1,b_1)} \circ g_{(a_2,b_2)} \circ \dots \circ g_{(a_n,b_n)}|_{\overline{B}_{b_n}} = g_{a_1} \circ g_{a_2} \circ \dots \circ g_{a_n}|_{\overline{B}_{b_n}} : \overline{B}_{b_n} \to \overline{B}_{a_1}.$$

Of course,

$$\Lambda(G) = J_{\mathcal{S}_G}$$

and we make the following observation:

Observation 6.1.2. The projection map

$$\pi = \pi_G := \pi_{\mathcal{S}_G} : E_A^{\mathbb{N}} \to \Lambda(G)$$

is a homeomorphism, in particular, a bijection.

We will now make some preparatory comments on our approach to counting problems for the group G by means of the conformal GDMS $\mathcal{S}_G$. For any element $\xi \in \Lambda(G)$ there exists a unique $k \in V$ such that $\xi \in \overline{B}_k$ and by Observation 6.1.2, a unique $\rho \in E_A^\infty$ such that

$$\xi = \pi_G(\rho);$$

of course $i(\rho) = k$. Set

$$G_\xi := \{g_\omega : \omega \in E_\rho^*\} = \{g_\omega : \omega \in E_A^*,\ t(\omega) = i(\rho) = k\} := G_k.$$

The next obvious observation is the following.

Observation 6.1.3. The maps

$$E_\rho^* \ni \omega \mapsto g_\omega \in G \quad \text{and} \quad E_\rho^* \ni \omega \longmapsto g_\omega(\xi) \in G(\xi)$$

are both 1-to-1.

For every $g = g_\omega \in G_\xi$, $\omega \in E_\rho^*$, we denote

$$\lambda_\xi(g) = -\log|g'(\xi)| = -\log|g'_\omega(\xi)| = \lambda_\rho(\omega).$$

Furthermore, for every set $Y \subset \overline{B}_k$ we denote

$$\Delta_Y(\omega) = -\log(\operatorname{diam}(g_\omega(Y)))$$

Now we move onto the discussion of periodic points of the system $\mathcal{S}_G$ along with periodic orbits of the geodesic flow and closed geodesics on the hyperbolic manifold $\mathbb{H}^{d+1}/G$.

Indeed, first of all we recall the following.

Observation 6.1.4. The map $E_p^* \ni \omega \longmapsto g_\omega \in G$ is 1-to-1.

Now, if $\omega \in E_p^*$ then

$$g_\omega(\overline{B}_{t(\omega)}) \subset \overline{B}_{t(\omega)}$$

and the map $g_\omega : \overline{B}_{t(\omega)} \to \overline{B}_{t(\omega)}$ has a unique fixed point. Call it x_ω. We know that the map $\overline{g}_\omega : \widehat{\mathbb{R}}^d \to \widehat{\mathbb{R}}^d$ has exactly one other fixed point. Call it y_ω. Denoting by $-\omega$ the word

$$(-\alpha_n, -\alpha_{n-1})(-\alpha_{n-1}, -\alpha_{n-2})(-\alpha_{n-2}, -\alpha_{n-3}) \cdots (-\alpha_2, -\alpha_1)(-\alpha_1, -\alpha_n)$$

and marking that $\omega = (\alpha_1, \beta_1)(\alpha_2, \beta_2) \cdots (\alpha_n, \beta_n)$ belongs to E_p^*, we see that $-\omega \in E_p^*$ and $g_{-\omega} = g_\omega^{-1}$ as elements of the group G. Then $x_{-\omega} \in \overline{B}_{-\alpha_n} \neq \overline{B}_{\beta_n}$. So as $g_\omega(x_{-\omega}) = x_{-\omega}$ we must have $y_\omega = x_{-\omega}$. Therefore, we have the following.

Proposition 6.1.5. *If $\omega \in E_p^*$ then γ_ω, the geodesic in $\mathbb{H}^{d+1}$ joining y_ω and x_ω (oriented from y_ω to x_ω), is fixed by g_ω, crosses the fundamental domain $\mathcal{R}$, γ_ω/G is a closed geodesic on $\mathbb{H}^{d+1}/G$ with length*

$$\lambda_p(\omega) = -\log|g'_\omega(x_\omega)|, \tag{6.2}$$

and simultaneously represents a periodic orbit of the geodesic flow on the unit tangent bundle of $\mathbb{H}^{d+1}/G$ with the period equal to $\lambda_p(\omega)$.

On the other hand, if γ is a closed oriented geodesic in $\mathbb{H}^{d+1}/G$ then its full lift $\widetilde{\gamma}$ in $\mathbb{H}^{d+1}$ consists of a countable union of mutually disjoint geodesics in $\mathbb{H}^{d+1}$. Then the set $\widetilde{\gamma} \cap \mathcal{R}$ is not empty and each of its connected components is an oriented geodesic joining two distinct faces of $\mathcal{R}$. Fix Δ, one of the such connected components. Let $\widehat{\Delta}$ be the full geodesic in $\mathbb{H}^{d+1}$ containing Δ and oriented in the direction of Δ. Fix $z \in \widehat{\Delta}$ arbitrarily. Denote by $l(\gamma)$ the length of γ. Let z^* be the unique point on $\widehat{\Delta}$ such that

$\rho(z^*, z) = l(\gamma)$ and the segment $[z, z^*]$ is oriented in the direction of $\widehat{\Delta}$. Since both points z and z^* project to the same element of $\mathbb{H}^{d+1}/G$, there exists a unique element $g_{\gamma,\Delta} \in G$ such that $g_{\gamma,\Delta}(z) = z^*$. Since γ has no self intersections it follows that

$$g_{\gamma,\Delta}(\widetilde{\Delta}) = \widetilde{\Delta}.$$

Denote be x_Δ and y_Δ the endpoints of $\widehat{\Delta}$ labeled so that the direction of $\widehat{\Delta}$ is from y_Δ to x_Δ. Let a, b be unique elements of V such that $x_\Delta \in \overline{B}_a$ and $y_\Delta \in \overline{B}_b$. Let $\widehat{\omega}_\Delta \in E_A^*$ and $k \in V$ be the unique elements respectively of E_A^* and V such that

$$g_{\gamma,\Delta} = g_{\widehat{\omega}_\Delta} \quad \text{and} \quad t(\widehat{\omega}_\Delta) = k,$$

the first equality meant in the group G. We will prove the following.

Claim 1. $k = -b$

PROOF. By our choice of the endpoints x_Δ and y_Δ, y_Δ is an attracting fixed point of $g_k^{-1}(g_{\widehat{\omega_\Delta}})^{-1} = (g_{\widehat{\omega}_\Delta} \circ g_k)^{-1}$. Since also $y_\Delta \in \overline{B}_b$, we thus conclude that

$$g_{-k} \circ (g_{\widehat{\omega}_\Delta})^{-1}(\overline{B}_b) \subseteq \overline{B}_b. \tag{6.3}$$

Consequently, $-k = b$, and Claim 1 is proved. $\square$

Since also $a \neq b$ as Δ intersects $\mathcal{R}$, we thus conclude that

$$\omega_{\gamma,\Delta} := \widehat{\omega}_\Delta(-b, a) \in E_A^* \quad \text{and} \quad g_{\gamma,\Delta} = g_{\omega_{\gamma,\Delta}}. \tag{6.4}$$

In addition, by the same token as (6.3) we get that $g_{\widehat{\omega}_\Delta} \circ g_k(\overline{B_a}) \subset \overline{B_a}$. Thus $i(\widehat{\omega}_\Delta) = a$. Consequently

$$\omega_\Delta \in E_p^*.$$

In addition,

$$\lambda_p(\omega_{\gamma,\Delta}) = \lambda(g_{\omega_{\gamma,\Delta}}) = \lambda(g_{\gamma,\Delta}) = \rho(z^*, z) = l(\gamma)$$

and

$$\gamma_{\omega_{\gamma,\Delta}}/G = \gamma.$$

Denote by $\mathcal{C}(\gamma)$ the set of all connected components of $\widetilde{\gamma} \cap \mathcal{R}$. Of course we have the following.

OBSERVATION 6.1.6. The function $\mathcal{C}(\gamma) \ni \Delta \longmapsto \omega_{\gamma,\Delta} \in E_p^*$ is one-to-one.

We shall prove the following.

PROPOSITION 6.1.7. *The map $E_p^* \longmapsto \gamma_\omega/G$ is a surjection from E_p^* onto $\mathcal{C}(G)$, the set of all closed oriented geodesics on $\mathcal{H}^{d+1}/G$. Furthermore, if γ is a closed oriented geodesic on $\mathbb{H}^{d+1}/G$ then*

$$\mathrm{Per}(\gamma) := \{\omega \in E_p^* : \gamma_\omega/G = \gamma\} = \{\omega_{\gamma,\Delta} \in E_p^* : \Delta \in \mathcal{C}(\gamma)\}$$

and $\mathrm{Per}(\gamma)$ *forms a full periodic cycle, i.e. the orbit of any element of $\omega \in \mathrm{Per}(\gamma)$ under the map σ^* : $\omega \longmapsto \sigma(\omega)\omega_1$.*

PROOF. The first part of this proposition has already been proved. More precisely, it is contained in Proposition 6.1.5 and formula (6.1). The inclusion

$$\{\omega_\Delta \in E_p^* : \Delta \in \mathcal{C}(\gamma)\} \subset \{\omega \in E_p^* : \gamma_\omega/G = \gamma\}$$

follows immediately from (6.1). The inclusion

$$\mathrm{Per}(\gamma) = \{\omega_\Delta \in E_p^* : \gamma_\omega/G = \gamma\} \subset \{\omega_{\gamma,\Delta} \in E_p^* : \Delta \in \mathcal{C}(\gamma)\}$$

follows from the fact that for each $\omega \in E_p^*$ the geodesic γ_ω crosses $\mathcal{R}$. So formula (6.1.7) is established. Now,

$$g_{\sigma(\omega)\omega_1}(g_{\omega_1}^{-1}(x_\omega)) = g_{\omega_1}^{-1} \circ g_{\omega_1} \circ g_{\sigma(\omega)}(x_\omega) = g_{\omega_1}^{-1} \circ g_\omega(x_\omega) = g_{\omega_1}^{-1}(x_\omega).$$

Similarly,

$$g_{\sigma(\omega))\omega_1}(g_{\omega_1}^{-1}(y_\omega)) = g_{\omega_1}^{-1}(y_\omega).$$

Also, by the Chain Rule,

$$l(g_{\sigma(\omega)\omega_1}) = \lambda_p(\sigma(\omega)\omega_1) = \lambda_p(\omega) = l(g_\omega).$$

Therefore, noting also that $g_{\omega_1}^{-1}(\gamma_\omega)$ crosses $\mathcal{R}$, we get

$$\gamma_{\sigma(\omega)\omega_1} = g_{\omega_1}^{-1}(\gamma_\omega) \quad \text{and} \quad \gamma_{\sigma(\omega)\omega_1}/G = \gamma_\omega/G = \gamma.$$

So, $\sigma(\omega)\omega_1 \in \mathrm{Per}(\gamma)$ and we have proved that $\mathrm{Per}(\gamma)$ is a union of full periodic cycles. Let $\omega \in \mathrm{Per}(\gamma)$ be arbitrary. Put $n := |\omega|$. Since

$$\sum_{j=0}^{n-1} l(\gamma_{\sigma^{*j}(\omega)} \cap \mathcal{R}) = l(\gamma) = \sum_{\Delta \in \mathcal{C}(\gamma)} |\Delta|,$$

since all elements $\gamma_{\sigma^{*j}(\omega)} \cap \mathcal{R}$ are mutually disjoint, and since $\{\gamma_{\sigma^{*j}(\omega)} \cap \mathcal{R} : 0 \le j \le n-1\} \subset \mathcal{C}(\gamma)$ we can conclude

$$\{\gamma_{\sigma^{*j}(\omega)} \cap \mathcal{R} : 0 \le j \le n-1\} = \mathcal{C}(\gamma).$$

Along with (6.1.7) and Observation 6.1.6 this yields the last assertion of Proposition 6.1.7 and the proof of this proposition is complete □

Denote by $\widehat{G} \subset G$ the set of those elements in G for which γ_g, the oriented geodesic in $\mathbb{H}^{d+1}$ from its repelling fixed point y_g to its attracting fixed point x_g crosses the fundamental domain $\mathcal{R}$. We can now complete Observation 6.1.4 by proving the following.

Proposition 6.1.8. *The map $E_p^* \ni \omega \longmapsto g_\omega \in G$ is a bijection from E_p^* onto $\widehat{G}$.*

Proof. Observation 6.1.4 tells us that this map is one-to-one and Proposition 6.1.5 tells us that its range is contained in $\widehat{G}$. Thus, in order to complete the proof we have to show that $\widehat{G}$ is contained in this range. So fix $g \in \widehat{G}$. Let α be the projection on $\mathbb{H}^{d+1}/G$ of the geodesic γ_g such that $l(\alpha) = \alpha(g)$. Then $g = g_{\omega_\alpha,\Delta}$ where $\Delta = \gamma_g \cap \mathcal{R}$. Since $\omega_{\alpha,\Delta} \in E_p^*$ we are done. □

Propositions 6.1.5 and 6.1.7 provide a full description of closed oriented geodesics and periodic orbits of the geodesic flow in terms of symbolic dynamics and graph directed Markov systems. For the picture to be complete we also describe all periodic points of the group G.

Proposition 6.1.9. *The map*

$$E_p^* \ni \omega \longmapsto \langle\omega\rangle = \{g \circ g_\omega \circ g^{-1} : g \in G\}$$

has the following properties:

(1) $\langle\omega\rangle = \langle\tau\rangle \iff \langle\omega\rangle \cap \langle\tau\rangle \ne \emptyset \iff \tau = \sigma^{*j}(\omega)$ *for some* $j \ge 0$.
(2) *Each element $g \circ g_\omega \circ g^{-1}$ has precisely two fixed points $g(x_\omega)$ and $g(y_\omega)$. In addition*
$$(g \circ g_\omega \circ g^{-1})'(g(x_\omega)) = g_\omega'(x_\omega) \quad \text{and} \quad (g \circ g_\omega \circ g^{-1})'(g(y_\omega)) = g_\omega'(y_\omega)$$
(3) *For each $h \in G \setminus \{\mathrm{Id}\}$ there exists a unique periodic cycle such that*
    (a) *there exists $\omega \in E_p^*$ in this periodic cycle and a unique $g \in G$, depending on ω, such that $h = g \circ g_\omega \circ g^{-1}$,*
    (b) *for each $\omega \in E_p^*$ in this periodic cycle there exists a unique $g \in G$, depending on ω, such that $h = g \circ g_\omega \circ g^{-1}$.*

The proof of this proposition is straightforward and we omit it.

Now we pass to the main goal of this monograph, i.e., counting estimates. We deal with these in the symbol space and on both $\mathbb{H}^{d+1}$ and $\mathbb{H}^{d+1}/G$. We start with appropriate definitions.

Let B denote a Borel subset of $\mathbb{R}^d$. Set

$$\pi_\xi(G; T, B) := \{g \in G_\xi : \lambda_\xi(g) \le T \text{ and } g(\xi) \in B\}$$

$$\pi_\xi(G;T) := \pi_\xi(G;T,\mathbb{R}^d) = \{g \in G_\xi : \lambda_\xi(g) \le T\}$$

$$\pi_p(G;T,B) := \{\omega \in E_p^* : \lambda_p(\omega) = l(\gamma_\omega) \le T \text{ and } x_\omega \in B\},$$

$$\pi_p(G;T) := \pi_p(G;T,\mathbb{R}^d) = \{\omega \in E_p^* : \lambda_p(\omega) = l(\gamma_\omega) \le T\},$$

$$\widehat{\pi}_p(G,T) := \{g \in \widehat{G} : l(\gamma_g) \le T\}$$

Having $k \in V = \{\pm j\}_{j=1}^q$ and $Y \subset \overline{B}_k$ put

$$\Delta_Y(g) := -\log\big(\mathrm{diam}(g(Y))\big).$$

We further denote

$$\mathcal{D}_\xi(G;T,B,Y) := \{g \in G_k : \Delta_Y(g) \le T \quad \text{and} \quad g(\xi) \in B\},$$

$$\mathcal{E}_k(G;T,B,Y) := \{g \in G_k : \Delta_Y(g) \le T \quad \text{and} \quad g(Y) \cap B \neq \emptyset\},$$

and

$$\mathcal{E}_k(G;T,Y) := \mathcal{E}_k(G;T,\mathbb{R}^d,Y) := \{g \in G_k : \Delta_Y(g) \le T\}.$$

We denote by $N_\xi(G;T,B)$, $N_\xi(G;T)$, $N_p(G;T,B)$, $N_p(G;T)$, $\widehat{N}_p(G;T)$, $D_\xi(G;T,B,Y)$, $E_k(G;T,B,Y)$ and $E_k(G;T,Y)$ the corresponding cardinalities.

As an immediate consequence of Theorem 2.4.9, Theorem 2.7.1, and Theorem 2.7.4 along with Observation 6.1.3, Proposition 6.1.5, Observation 6.1.4, Observation 6.1.6 and Proposition 6.1.8 we get the following.

THEOREM 6.1.10. *Let $G = \langle g_j \rangle_{j=1}^q$ be a hyperbolic finitely generated classical Schottky group acting on $\hat{\mathbb{R}}^d$, $d \ge 2$.*

- *Let δ_G be the Poincaré exponent of G; it is known to be equal to* $\mathrm{HD}(\Lambda(G))$.
- *Let m_{δ_G} be the Patterson-Sullivan conformal measure for G on $\Lambda(G)$.*
- *Let μ_{δ_G} be the $\mathcal{S}_G$-invariant measure on $\Lambda(G)$ equivalent to m_{δ_G}.*
- *Fix $k \in \{\pm 1, \pm 2, \cdots, \pm q\}$ and $\xi \in \Lambda(G) \cap \overline{B}_k$.*

Let $B \subseteq \mathbb{R}^d$ be a Borel set with $m_{\delta_G}(\partial B) = 0$ (equivalently $\mu_{\delta_G}(\partial B) = 0$) and let $Y \subseteq \overline{B}_k$ be a set having at least two distinct points. Then with some constant $C_k(Y) \in (0,+\infty)$, we have that

$$\lim_{T\to+\infty} \frac{N_\xi(G;T,B)}{e^{\delta_G T}} = \frac{\psi_{\delta_G}(\xi)}{\delta_G \chi_{\delta_G}} m_{\delta_G}(B), \qquad \lim_{T\to+\infty} \frac{N_\xi(G;T)}{e^{\delta_G T}} = \frac{\psi_{\delta_G}(\xi)}{\delta_G \chi_{\delta_G}},$$

$$\lim_{T\to+\infty} \frac{N_p(G;T,B)}{e^{\delta_G T}} = \frac{1}{\delta_G \chi_{\delta_G}} \mu_{\delta_G}(B), \qquad \lim_{T\to+\infty} \frac{N_p(G;T)}{e^{\delta_G T}} = \frac{1}{\delta_G \chi_{\delta_G}},$$

$$\lim_{T\to+\infty}\frac{\widehat{N}_p(G;T)}{e^{\delta_G T}}=\frac{1}{\delta_G\chi_{\delta_G}},$$

$$\lim_{T\to+\infty}\frac{D_\xi(G;T,B,Y)}{e^{\delta_G T}}=C_k(Y)m_{\delta_G}(B),$$

$$\lim_{T\to+\infty}\frac{E_k(G;T,B,Y)}{e^{\delta_G T}}=C_k(Y)m_{\delta_G}(B),$$

$$\lim_{T\to+\infty}\frac{E_k(G;T,Y)}{e^{\delta_G T}}=C_k(Y).$$

Theorem 4.1.1 – Theorem 4.1.3 for the conformal GDMS $\mathcal{S}_G$, associated to the group G, are valid without changes. Therefore, we do not repeat them here. However, we present the appropriate versions of Theorems 4.3.1 and 4.3.2 as their formulations are closer to the group G. In order to get appropriate expressions in the language of the group G itself, given $\xi\in\Lambda(G)$, and an integer $n\ge 1$, we set

$$G_\xi^n:=\{g_\omega:\omega\in E_\rho^n\}\subseteq G_\xi.$$

Furthermore, we define a probability measure μ_n on G_ξ^n by setting that

$$\mu_n(H):=\frac{\sum_{g\in H}e^{-\delta\lambda_\xi(g)}}{\sum_{\omega\in G_\xi^n}e^{-\delta\lambda_\xi(g)}} \tag{6.5}$$

for every set $H\subset G_\xi^n$. As an immediate consequence of Theorem 4.3.1 we get the following.

THEOREM 6.1.11. *If $G=\langle g_j\rangle_{j=1}^q$ is a hyperbolic finitely generated classical Schottky group acting on $\hat{\mathbb{R}}^d$, $d\ge 2$, then for every $\xi\in\Lambda(G)$ we have that*

$$\lim_{n\to+\infty}\int_{G_\xi^n}\frac{\lambda_\xi}{n}d\mu_n=\chi_{\mu_\delta}.$$

Now define the functions $\Delta_n:G_\xi^n\to\mathbb{R}$ by the formulae

$$\Delta_n(g)=\frac{\lambda_\xi(g)-\chi n}{\sqrt{n}},$$

As an immediate consequence of Theorem 4.3.2 we get the following.

THEOREM 6.1.12. *If $G=\langle g_j\rangle_{j=1}^q$ is a hyperbolic finitely generated classical Schottky group acting on $\hat{\mathbb{R}}^d$, $d\ge 2$, then for every $\xi\in\Lambda(G)$ the sequence of random variables $(\Delta_n)_{n=1}^\infty$ converges in distribution to the normal (Gaussian) distribution $\mathcal{N}_0(\sigma)$ with mean value zero and the variance $\sigma^2=\mathrm{P}''_{\mathcal{S}_G}(\delta)>0$. Equivalently, the sequence $(\mu_n\circ\Delta_n^{-1})_{n=1}^\infty$ converges weakly to the normal distribution $\mathcal{N}_0(\sigma^2)$. This means that for every Borel set $F\subset\mathbb{R}$ with* $\mathrm{Leb}(\partial F)=0$, *we have*

$$\lim_{n\to+\infty}\mu_n(\Delta_n^{-1}(F))=\frac{1}{\sqrt{2\pi}\sigma}\int_F e^{-t^2/2\sigma^2}dt. \tag{6.6}$$

6.2. Generalized (allowing tangencies) Classical Schottky Groups

In this section we keep to the same setting and the same notation as in Subsection 6.1. except that we now do not assume that the closures $\overline{B}_j$, $j=\pm1,\cdots,\pm q$ to be disjoint but merely that the open balls B_j, $j=\pm1,\cdots,\pm q$ themselves are mutually disjoint.

6.2.1. General Schottky Groups. We also assume that if an element $g \in G \setminus \{\mathrm{Id}\}$ has a fixed point (call it z_q) in ∂B_j for some $j \in \{\pm 1, \cdots, \pm q\}$ then g is parabolic. Then z_g is a unique fixed point of g and there exists a unique $j^* \in \{\pm 1, \cdots, \pm q\} \setminus \{j\}$ such that

$$z_g \in \overline{B}_j \cap \overline{B}_{j^*}.$$

We refer to z_g as a parabolic fixed point of G (and of g). We denote by $p(g) \geq 1$ its rank. We further denote by $\Omega(G)$ the set of all parabolic fixed points of G. Any such group G is called a generalized Schottky group (GSG). If G has at least one parabolic element, it is called a parabolic Schottky group (PSG). We associate to the group G the conformal GDMS $\mathcal{S}_G$ in exactly the same way as for hyperbolic (i.e. without tangencies) Schottky groups in Section 6.1. Since any generalized Schottky group G is geometrically finite, the number of conjugacy classes of parabolic elements of G and the number of orbit classes of parabolic fixed points of G, i.e. $\Omega(G)/G$, are both finite. In consequence, we have the following.

Observation 6.2.1. The conformal GDMS $\mathcal{S}_G$ associated to G is attracting if G has no parabolic fixed points and it is (finite) parabolic (in the sense of Remark 2.3.6) if G has some parabolic fixed points.

and

Observation 6.2.2. We have that:

- Each parabolic fixed point of G has a representative in
$$\bigcup_{-q \leq j < k \leq q} \overline{B}_j \cap \overline{B}_k,$$
and
- $$\Omega(\mathcal{S}_G) = \Omega(G) \cap \bigcup_{-q \leq j < k \leq q} \overline{B}_j \cap \overline{B}_k.$$

We define

$$p_G := p(\mathcal{S}_G) := \sup\{p(g) : g \in \Omega(G)\}. \tag{6.1}$$

So, as an immediate consequence of Theorems 3.3.2, 3.4.1, and 3.4.2, in the same way as Theorem 6.1.10, i.e., along with Observation 6.1.3, Proposition 6.1.5, Observation 6.1.4, Observation 6.1.6 and Proposition 6.1.8, we get the following.

Theorem 6.2.3. *Let $G = \langle g_j \rangle_{j=1}^q$ be a parabolic classical Schottky group acting on $\mathbb{R}^d$, $d \geq 2$.*

- *Let δ_G be the Poincaré exponent of G; which is known to be equal to $\mathrm{HD}(\Lambda(G))$.*
- *Let m_{δ_G} be the Patterson-Sullivan conformal measure for G on $\Lambda(G)$.*
- *Let μ_{δ_G} be the $\mathcal{S}_G$-invariant measure on $\Lambda(G)$ equivalent to m_{δ_G}.*
- *Fix $k \in \{\pm 1, \pm 2, \cdots, \pm q\}$ and $\xi \in \Lambda(G) \cap \overline{B}_k$.*

Let $B \subseteq \mathbb{R}^d$ be a Borel set with $m_{\delta_G}(\partial B) = 0$ (equivalently $\mu_{\delta_G}(\partial B) = 0$) and let $Y \subseteq \overline{B}_k$ be a set having at least two distinct points. Then with some constant $C_k(Y) \in (0, +\infty)$, we have that

$$\lim_{T \to +\infty} \frac{N_\xi(G; T, B)}{e^{\delta_G T}} = \frac{\psi_{\delta_G}(\xi)}{\delta_G \chi_{\delta_G}} m_{\delta_G}(B), \qquad \lim_{T \to +\infty} \frac{N_\xi(G; T)}{e^{\delta_G T}} = \frac{\psi_{\delta_G}(\xi)}{\delta_G \chi_{\delta_G}},$$

$$\lim_{T \to +\infty} \frac{N_p(G; T, B)}{e^{\delta_G T}} = \frac{1}{\delta_G \chi_{\delta_G}} \mu_{\delta_G}(B), \qquad \lim_{T \to +\infty} \frac{N_p(G; T)}{e^{\delta_G T}} = \frac{1}{\delta_G \chi_{\delta_G}},$$

$$\lim_{T\to+\infty}\frac{\widehat{N}_p(G;T)}{e^{\delta_G T}}=\frac{1}{\delta_G\chi_{\delta_G}},$$

$$\lim_{T\to+\infty}\frac{D_\xi(G;T,B,Y)}{e^{\delta_G T}}=C_k(Y)m_{\delta_G}(B),$$

$$\lim_{T\to+\infty}\frac{E_k(G;T,B,Y)}{e^{\delta_G T}}=C_k(Y)m_{\delta_G}(B),$$

$$\lim_{T\to+\infty}\frac{E_k(G;T,Y)}{e^{\delta_G T}}=C_k(Y).$$

In addition, $C_k(Y)>0$ is finite if and only if

(1)
$$\overline{Y}\cap\Omega_\infty(\mathcal{S}_G)=(\overline{Y}\cap\Omega_\infty(\mathcal{S}_G)\cap\partial B_k)=\emptyset$$

or

(2)
$$\delta_G>\max\big\{p(g):z_g\in\partial B_k\big\}.$$

As in the case of hyperbolic Schottky groups, there are also Central Limit Theorems on the distribution of the preimages for parabolic Schottky groups. Theorem 4.2.1 and Theorem 4.2.2 for the parabolic conformal GDMS $\mathcal{S}_G$, associated to the group G, take the same form. Therefore, we do not repeat them here. However, we present the appropriate versions of Theorems 4.4.1 and 4.4.2 as their formulations are closer to the actual group G. As in the case of hyperbolic groups, in order to get appropriate expressions in the language of the group G itself, given $\xi\in\Lambda(G)$, and an integer $n\ge 1$, we set

$$G_\xi^n:=\{g_\omega:\omega\in E_\rho^n\}\subseteq G_\xi.$$

Furthermore, we define a probability measure μ_n on G_ξ^n by setting that

$$\mu_n(H):=\frac{\sum_{g\in H}e^{-\delta\lambda_\xi(g)}}{\sum_{\omega\in G_\xi^n}e^{-\delta\lambda_\xi(g)}} \tag{6.2}$$

for every set $H\subset G_\xi^n$. As an immediate consequence of Theorem 4.4.1 we get the following.

THEOREM 6.2.4. *If $G=\langle g_j\rangle_{j=1}^q$ is a parabolic finitely generated classical Schottky group acting on $\hat{\mathbb{R}}^d$, $d\ge 2$, and*

$$\delta_G>\frac{2p_G}{p_G+1},$$

i.e the invariant measure μ_δ is finite (so a probability after normalization), then for every $\xi\in\Lambda(G)$ we have that

$$\lim_{n\to+\infty}\int_{G_\xi^n}\frac{\lambda_\xi}{n}d\mu_n=\chi_{\mu_\delta}.$$

Again as in the hyperbolic (no tangencies) case, we define the functions $\Delta_n:G_\xi^n\to\mathbb{R}$, $n\in\mathbb{N}$, by the formulae

$$\Delta_n(g)=\frac{\lambda_\xi(g)-\chi n}{\sqrt{n}}.$$

As an immediate consequence of Theorem 4.4.2 we get the following.

THEOREM 6.2.5. *If $G = \langle g_j \rangle_{j=1}^q$ is a parabolic finitely generated classical Schottky group acting on $\hat{\mathbb{R}}^d$, $d \geq 2$, and*

$$\delta_G > \frac{2p_G}{p_G + 1},$$

i.e., the invariant measure μ_δ is finite (thus a probability measure after normalization), then for every $\xi \in \Lambda(G)$ the sequence of random variables $(\Delta_n)_{n=1}^\infty$ converges in distribution to the normal (Gaussian) distribution $\mathcal{N}_0(\sigma)$ with mean value zero and the variance $\sigma^2 = \mathrm{P}''_{\mathcal{S}_G^*}(\delta) > 0$. Equivalently, the sequence $(\mu_n \circ \Delta_n^{-1})_{n=1}^\infty$ converges weakly to the normal distribution $\mathcal{N}_0(\sigma^2)$. This means that for every Borel set $F \subset \mathbb{R}$ with $\mathrm{Leb}(\partial F) = 0$, we have

$$\lim_{n \to +\infty} \mu_n(\Delta_n^{-1}(F)) = \frac{1}{\sqrt{2\pi}\sigma} \int_F e^{-t^2/2\sigma^2} dt.$$

6.2.2. Apollonian Circle Packings. We now describe the application of Theorem 6.2.3 to Apollonian circle packings, as explained in the introduction. This can be formulated in the framework we described in the introduction to this section. Some additional information related to the subject of this section and the one following it can be found in works such as [**2**], [**7**], [**21**], [**23**], [**36**], [**28**], [**51**], [**56**]–[**58**], [**59**], and [**83**]. Of course we make no claims for this list to be even remotely complete.

Let C_1, C_2, C_3, C_4 be four distinct circles in the Euclidean (complex) plane, each of which shares a common tangency point with each of the others. We assume that the bounded component of the complement of one of these circles contains the bounded components of the complements of the remaining three circles. Without loss of generality C_4 is this circle enclosing the three other. We refer to such configuration of circles C_1, C_2, C_3, C_4 as bounded. This name will be justified in a moment. We can now choose the new four circles K_1, K_2, K_3, K_4 that are dual to the original four tangent circles, i.e., those circles that pass through the three of the four possible tangent points between the initial circles C_1, C_2, C_3, C_4. We label them (uniquely) so that

$$C_i \cap K_i = \emptyset$$

for all $i = 1, 2, 3, 4$. Figure 2 depicts this construction.

We associate to the dual circles K_1, K_2, K_3, K_4 the respective inversions g_1, g_2, g_3, g_4 in these four dual circles. More precisely, if K_i, $i = 1, 2, 3, 4$, is a circle with center $a_i \in \mathbb{C}$ and radius $r_i > 0$ then we define

$$g_i(z) = \frac{1}{r_i^2} \frac{z - a_i}{|z - a_i|^2} + a_i,$$

Denote by B_1, B_2, B_3 and B_4 the open balls (disks) enclosed, respectively, by the circles K_1, K_2, K_3, K_4. Let

$$G := \langle g_1, g_2, g_3, g_4 \rangle$$

be the group generated by the four inversions g_1, g_2, g_3, g_4. Let Γ be the subgroup of G consisting of its all orientation preserving elements. Observe that Γ is a free group generated by three elements, for example by

$$\gamma_1 := g_4 \circ g_1, \quad \gamma_2 := g_4 \circ g_2, \quad \gamma_3 := g_4 \circ g_3.$$

Now noting that the the balls

$$B_1, \quad B_2, \quad B_3; \quad B_{-1} := g_4(B_1), \quad B_{-2} := g_4(B_2), \quad B_{-3} := g_4(B_3),$$

are mutually disjoint (see Figure 3), and that for every $i = 1, 2, 3$:

$$\gamma_i(\overline{B}_i) = g_4 \circ g_i(\overline{B}_i) = g_4(B_i^c) = (g_4(B_i))^c = B_{-i}^c$$

we get the following.

OBSERVATION 6.2.6. $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ is a parabolic classical Schottky group.

In addition,

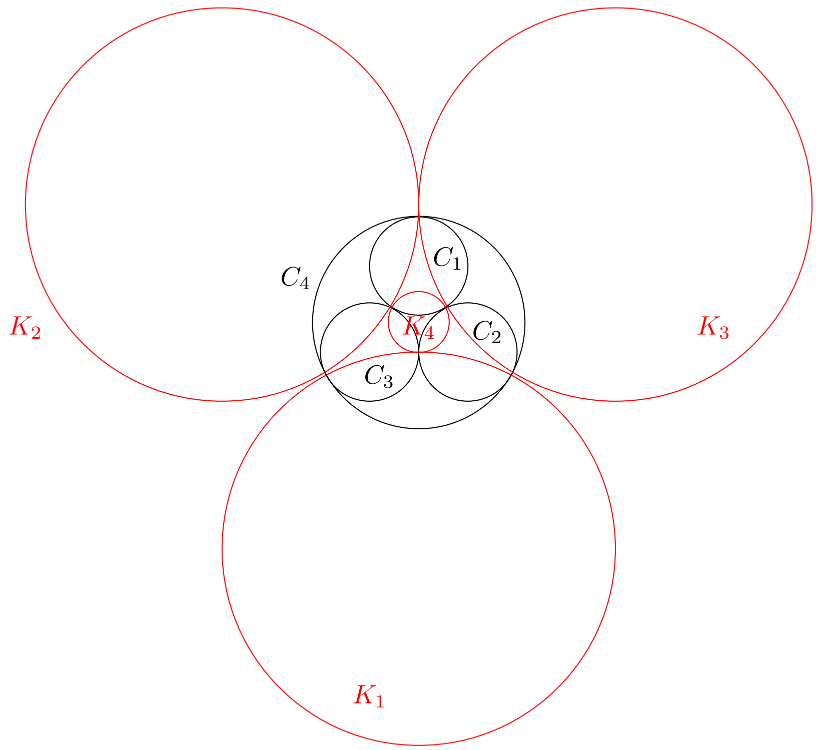

FIGURE 1. The Tangent Circles C_1, C_2, C_3, C_4 and Dual Circles K_1, K_2, K_3, K_4

OBSERVATION 6.2.7. The parabolic classical Schottky group Γ has six conjugacy classes of parabolic elements whose representatives are

$$\gamma_1, \quad \gamma_2, \quad \gamma_3, \quad \gamma_1\gamma_2^{-1}, \quad \gamma_1\gamma_3^{-1}, \quad \gamma_2\gamma_3^{-1},$$

with the corresponding parabolic fixed points being the only elements, respectively, of

$$\overline{B}_1 \cap \overline{B}_4, \quad \overline{B}_2 \cap \overline{B}_4, \quad \overline{B}_3 \cap \overline{B}_4, \quad \overline{B}_{-1} \cap \overline{B}_{-2}, \quad \overline{B}_{-1} \cap \overline{B}_{-3}, \quad \overline{B}_{-2} \cap \overline{B}_{-3}.$$

These objects are depicted in Figure 3. We have the following.

OBSERVATION 6.2.8. The limit set $\Lambda(\Gamma)$ coincides with the residual set of the Apollonian circle packing generated by the circles C_1, C_2, C_3, C_4. In addition (see [**6**], [**43**], and Theorem 3.1.6), we have the following.

(1) $\delta_\Gamma = \mathrm{HD}(\Lambda(\Gamma)) > 1$,

(2) $p(g) = 1$ for every parabolic element of Γ, and so

$$\delta_\Gamma > \sup\{p(g)\},$$

where $g \in \Gamma$ ranges of all parabolic elements of G,

(3) $\Omega_\infty(\mathcal{S}_\Gamma) = \emptyset$, and so μ_{δ_Γ}, the probability $\mathcal{S}_\Gamma$-invariant measure on $\Lambda(\Gamma)$, is finite, thus probability after normalization.

Hence, as an immediate consequence of Theorem 6.2.3, we get the following.

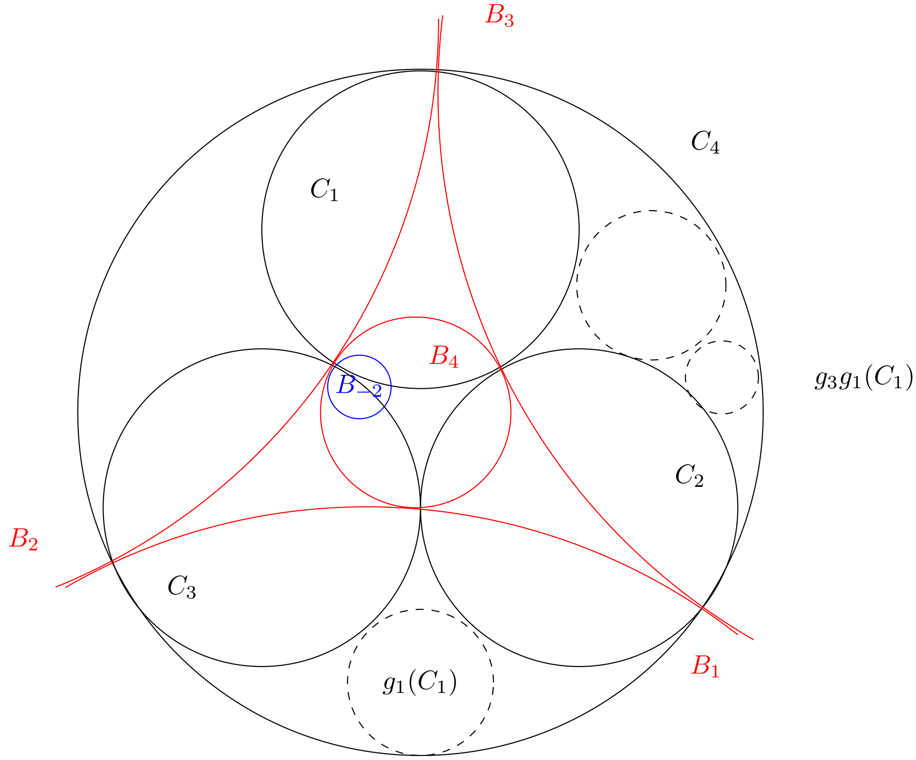

FIGURE 2. Circles, Disks, and Generators of G

COROLLARY 6.2.9. *Let C_1, C_2, C_3, C_4 be a bounded*[1] *configuration of four distinct circles in the plane, each of which shares a common tangency point with each of the others. Let Γ be the corresponding parabolic classical Schottky group.*

- *Let δ_Γ be the Poincaré exponent of Γ; it is known to be equal to $\mathrm{HD}(\Lambda(\Gamma))$.*
- *Let m_{δ_Γ} be the Patterson-Sullivan conformal measure for Γ on $\Lambda(\Gamma)$.*
- *Let μ_{δ_Γ} be the probability $\mathcal{S}_\Gamma$-invariant measure on $\Lambda(\Gamma)$ equivalent to m_{δ_Γ}.*
- *Fix $k \in \{\pm 1, \pm 2, \pm 3\}$ and $\xi \in \Lambda(\Gamma) \cap \overline{B}_k$.*

Then for every set $Y \subset \overline{B_k}$ having at least two distinct points there exists a constant $C_k(Y) \in (0, +\infty)$ such that for every Borel set $B \subset \mathbb{R}^d$ with $m_{\delta_\Gamma}(\partial B) = 0$ (equivalently $\mu_{\delta_\Gamma}(\partial B) = 0$), we have that

$$\lim_{T\to+\infty} \frac{N_\xi(\Gamma; T, B)}{e^{\delta_\Gamma T}} = \frac{\psi_{\delta_\Gamma}(\xi)}{\delta_\Gamma \chi_{\delta_\Gamma}} m_{\delta_\Gamma}(B), \qquad \lim_{T\to+\infty} \frac{N_\xi(\Gamma; T)}{e^{\delta_\Gamma T}} = \frac{\psi_{\delta_\Gamma}(\xi)}{\delta_\Gamma \chi_{\delta_\Gamma}},$$

$$\lim_{T\to+\infty} \frac{N_p(\Gamma; T, B)}{e^{\delta_\Gamma T}} = \frac{1}{\delta_\Gamma \chi_{\delta_\Gamma}} \mu_{\delta_\Gamma}(B), \qquad \lim_{T\to+\infty} \frac{N_p(\Gamma; T)}{e^{\delta_\Gamma T}} = \frac{1}{\delta_\Gamma \chi_{\delta_\Gamma}},$$

[1] Boundedness of the configuration C_1, C_2, C_3, C_4 guarantees us that the group Γ is Schottky in the sense of our previous section, and, in particular, all the numbers $N_\xi(\Gamma; T)$ and $N_p(\Gamma; T)$ are finite.

$$\lim_{T\to+\infty}\frac{\widehat{N}_p(\Gamma;T)}{e^{\delta_\Gamma T}}=\frac{1}{\delta_\Gamma\chi_{\delta_\Gamma}},$$

$$\lim_{T\to+\infty}\frac{D_\xi(\Gamma;T,B,Y)}{e^{\delta_\Gamma T}}=C(Y)m_{\delta_G}(B),$$

$$\lim_{T\to+\infty}\frac{E_k(\Gamma;T,B,Y)}{e^{\delta_\Gamma T}}=C_k(Y)m_{\delta_\Gamma}(B),$$

$$\lim_{T\to+\infty}\frac{E_k(\Gamma;T,Y)}{e^{\delta_\Gamma T}}=C_k(Y).$$

Making use of Observation 6.2.8, as an immediate consequence respectively of Theorem 6.2.4 and Theorem 6.2.5, we get the following two theorems.

Theorem 6.2.10. *Let C_1, C_2, C_3, C_4 be a bounded configuration of four distinct circles in the plane, each of which shares a common tangency point with each of the others. If Γ is the corresponding parabolic classical Schottky group, then for every $\xi \in \Lambda(\Gamma)$ we have that*

$$\lim_{n\to+\infty}\int_{\Gamma_\xi^n}\frac{\lambda_\xi}{n}d\mu_n=\chi_{\mu_\delta}.$$

The next theorem is a Central Limit Theorem for diameters of circles in the Apollonian Circle Packing.

Theorem 6.2.11. *Let C_1, C_2, C_3, C_4 be a bounded configuration of four distinct circles in the plane, each of which shares a common tangency point with each of the others. If Γ is the corresponding parabolic classical Schottky group, then for every $\xi \in \Lambda(\Gamma)$ the sequence of random variables $(\Delta_n)_{n=1}^\infty$ converges in distribution to the normal (Gaussian) distribution $\mathcal{N}_0(\sigma)$ with mean value zero and the variance $\sigma^2 = \mathrm{P}''_{\mathcal{S}_\Gamma^*}(\delta) > 0$. Equivalently, the sequence $(\mu_n \circ \Delta_n^{-1})_{n=1}^\infty$ converges weakly to the normal distribution $\mathcal{N}_0(\sigma^2)$. This means that for every Borel set $F \subset \mathbb{R}$ with $\mathrm{Leb}(\partial F) = 0$, we have*

$$\lim_{n\to+\infty}\mu_n(\Delta_n^{-1}(F))=\frac{1}{\sqrt{2\pi}\sigma}\int_F e^{-t^2/2\sigma^2}dt.$$

In Figure 2 we illustrate the Central Limit Theorem for the diameters in the standard Apollonian Circle Packing in Theorem 6.2.5.

Now, we consider the actual counting of the circles in the Apollonian circle packing generated by the bounded configuration of the circles C_1, C_2, C_3 and C_4. The following immediate observation is crucial to this goal.

Observation 6.2.12. The elements of $\mathcal{A}$, the Apollonian circle packing generated by the bounded configuration of the circles C_1, C_2, C_3, C_4, is bounded[2] and coincide with the following disjoint union

$$\{C_1,C_2,C_3,C_4\}\cup\bigcup_{j=1}^{3}(\Gamma_j\cup\{\mathrm{Id}\})(g_j(C_j))\cup\bigcup_{j=1}^{3}\bigcup_{i=1,i\neq j}^{4}(\Gamma_i\cup\{\mathrm{Id}\})(g_i\circ g_j(C_j))\cup$$
$$\cup\{g_4(C_4)\}\cup\cup_{j=1}^{3}(\Gamma_j\cup\{\mathrm{Id}\})(g_j\circ g_4)(C_4),$$

and for $j = 1, 2, 3$ and $i \in \{1, 2, 3\} \setminus \{j\}$ we have that

$$g_j(C_j)\subset\overline{B}_j,\quad g_i\circ g_j(C_j)\subset\overline{B}_i,\quad g_4\circ g_j(C_j)\subset\overline{B}_{-j},\quad g_j\circ g_4(C_4)\subset\overline{B}_j.$$

[2]This justifies the name "bounded" in regards to the configuration C_1, C_2, C_3, C_4.

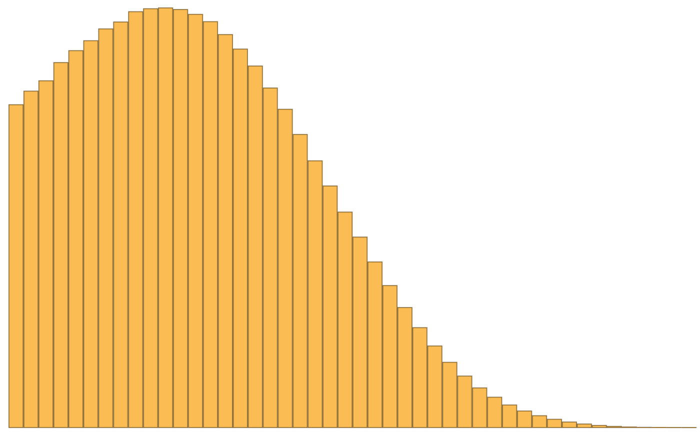

FIGURE 3. We plot a portion of the weighted histogram of the 6,377,292 values $-\log r$ where r is a circle of generation $n = 14$ for standard Apollonian circle packing. There are 46 bins with a weighting corresponding to r^δ.

For every $T > 0$ and every set $B \subset \mathbb{C}$, we denote

$$\mathcal{E}(T;B) := \{C \in \mathcal{A} : -\log \operatorname{diam}(C) \le T \text{ and } C \cap B \ne \emptyset\},$$
$$\mathcal{E}(T) := \mathcal{E}(T;\mathbb{C})$$
$$N_{\mathcal{A}}(T;B) := \#\mathcal{E}(T;B) \quad \text{and} \quad N_{\mathcal{A}}(T) := \#\mathcal{E}(T).$$

As an immediate consequence of the last two formulas of Corollary 6.2.9 and Observation 6.2.12 we get the following result proved in [**36**] (see also [**56**]–[**58**]) by entirely different methods.

THEOREM 6.2.13. *Let C_1, C_2, C_3, C_4 be a bounded configuration of four distinct circles in the plane, each of which shares a common tangency point with each of the others. Let $\mathcal{A}$ be the corresponding circle packing.*

Let $\delta = 1.30561\ldots$ be the Hausdorff dimension of the residual set of $\mathcal{A}$ and let m_δ be the Patterson-Sullivan measure of the corresponding parabolic classical Schottky group Γ.

Then the limit

$$\lim_{T\to+\infty} \frac{N_{\mathcal{A}}(T)}{e^{\delta T}}$$

exists, is positive, and finite. Moreover, there exists a constant $C \in (0,+\infty)$ such that

$$\lim_{T\to+\infty} \frac{N_{\mathcal{A}}(T;B)}{e^{\delta T}} = C m_\delta(B)$$

for every Borel set $B \subset \mathbb{C}$ with $m_\delta(\partial B) = 0$.

6.2.3. Apollonian Triangle. Now we consider the Apollonian triangle. Let C_1, C_2, C_3 be three mutually tangent circles in the plane having mutually disjoint interiors. Let C_4 be the circle tangent to all the circles C_1, C_2, C_3 and having all of them in its interior, i.e. the configuration C_1, C_2, C_3, C_4 is bounded.

We look at the curvilinear triangle $\mathcal{T}$ formed by the three edges joining the three tangency points of C_1, C_2, C_3 and lying on these circles. The bounded collection

$$\mathcal{G} := \{C \in \mathcal{A} : C \subset \mathcal{T}\}$$

is called the Apollonian gasket generated by the circles C_1, C_2, C_3. Since $\partial\mathcal{T} \cap \Lambda(\Gamma) = \partial\mathcal{T}$ has Hausdorff dimension 1, since $\delta > 1$ and since m_δ is a constant multiple of δ–dimensional Hausdorff measure restricted to $\Lambda(\Gamma)$, we have that $m_\delta(\partial T) = 0$. Another, a more general argument for this, would be to invoke Corollary 1.4 from [**20**]. Therefore, as an immediate consequence of Theorem 6.2.13 we get the following result, also proved by Kontorovich and Oh in [**36**] (see also [**56**]–[**58**]) with entirely different methods.

Corollary 6.2.14. *Let C_1, C_2, C_3 be three mutually tangent circles in the plane having mutually disjoint interiors. Let C_4 be the circle tangent to all the circles C_1, C_2, C_3 and having all of them in its interior, i.e. the configuration C_1, C_2, C_3, C_4 is bounded. Let $\mathcal{A}$ be the corresponding (bounded) circle packing.*

Let $\delta = 1.30561\ldots$ be the Hausdorff dimension of the residual set of $\mathcal{A}$ and let m_δ be the Patterson-Sullivan measure of the corresponding parabolic classical Schottky group Γ.

If $\mathcal{T}$ is the curvilinear triangle formed by C_1, C_2 and C_3, then the limit

$$\lim_{T\to+\infty} \frac{N_{\mathcal{A}}(T;\mathcal{T})}{e^{\delta T}}$$

exists, is positive, and finite; we just count the elements of $\mathcal{G}$. Moreover, there exists a constant $C \in (0, +\infty)$, in fact the one of Theorem 6.2.13, such that

$$\lim_{T\to+\infty} \frac{N_{\mathcal{A}}(T;B)}{e^{\delta T}} = C m_\delta(B)$$

for every Borel set $B \subset \mathcal{T}$ with $m_\delta(\partial B) = 0$.

Now we will provide a somewhat different proof of Corollary 6.2.14, by appealing directly to the theory of parabolic conformal IFSs and avoiding the intermediate step of parabolic Schottky groups. Indeed, let C_0 be the circle inscribed in $\mathcal{T}$ and tangent to the circles C_1, C_2 and C_3. Let x_1, x_2 and x_3 be the vertices of the curvilinear triangle $\mathcal{T}$, i.e., for $i = 1, 2, 3$, x_i is the only element of the intersection $K_i \cap K_4$. Let

$$\varphi_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

be the Möbius transformation fixing the point x_i and mapping the other vertices x_j and x_k, respectively, onto the only points of the intersections $C_0 \cap C_j$ and $C_0 \cap C_k$. Then

$$\mathcal{S} = \{\varphi_1, \varphi_2, \varphi_3\}$$

is a parabolic IFS defined on $\overline{B}_4$, x_i is a parabolic fixed point of φ_i, $i = 1, 2, 3$, and

$$\mathcal{G} = \{\varphi_\omega(C_0) : \omega \in \{1, 2, 3\}^*\},$$

see Figure 5. We therefore obtain Corollary 6.2.14 immediately from Theorem 3.4.6.

Remark 6.2.15. In the context of limit sets, such as circle packings, there is scope for finding error terms in the above asymptotic formulae, see ex. [**39**] and [**60**]. It could be also done using the techniques worked out in our present manuscript. However, in the general setting of conformal graph directed Markov systems quite delicate technical hypotheses might well be required.

Remark 6.2.16. For these analytic maps it would be equally possible to work with Banach spaces of analytic functions, rather than Hölder continuous functions. This would have the advantage that the transfer operator operator is compact (even trace class or nuclear) and might help to simplify some of the arguments as well as being useful in explicit numerical computations. On the other hand, working with Hölder functions allows the results to be applied to a far greater range of examples.

Remark 6.2.17. In higher dimensions, we can consider the packing of the sphere S^d by mutually tangent d-spheres. The same analysis gives a corresponding asymptotic for the diameters of spheres. In an overlapping setting and with entirely different methods this question has been addressed in Oh's paper [**55**].

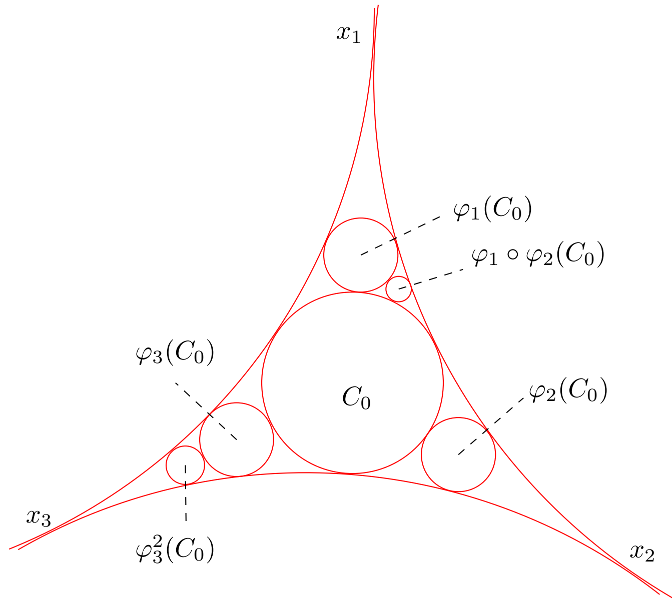

FIGURE 4. Apollonian Triangle

6.3. Fuchsian Groups

We recall that a Fuchsian group Γ is a discrete group of orientation preserving Poincaré isometries acting on the unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

in the complex plane. A Poincaré isometry means that the Poincaré metric

$$\frac{|dz|}{1-|z|^2}$$

is preserved, equivalently the map is a holomorphic homeomorphism of the disk $\mathbb{D}$ onto itself. The limit set $\Lambda(\Gamma)$ of Γ is a compact perfect subset of $S^1 = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. Assume that Γ is finitely generated and denote a minimal (in the sense of inclusion) set of its generators by $\{g_j\}_{j=\pm 1}^{\pm q}$ where $g_j = g_j^{-1}$. Assume that $q \geq 2$, so that Γ is non-elementary. Following [**84**] (see also [**85**]) we call Γ non-exceptional if at least one of the following conditions holds (corresponding to conditions (10.1)-(10.3) from [**37**]):

(1) $\mathbb{D}/\Gamma$ is not compact;
(2) The generating set has at least 5 elements (i.e., $q \geq 5$) and every non-trivial relation has length 5; and
(3) At least 3 of the generating relations have length at least 7.

In particular, every finitely generated parabolic Fuchsian group is non-exceptional as the condition (1) above is satisfied. In the language of conformal GDMSs, C. Series proved in [**84**] (see also [**66**], [**67**] for an alternative account and [**85**] where a more algebraic approach is employed) the following:

THEOREM 6.3.1. *If Γ is a non-exceptional Fuchsian group then there exists a finite irreducible pre-parabolic GDMS $\mathcal{S}_\Gamma$ with an incidence matrix A, a finite set of vertices V and a finite alphabet $E = \{\pm 1, \pm 2, \cdots, \pm q\}$ such that*

(1) *For every $j \in E$ the corresponding element of $\mathcal{S}_\Gamma$ is $g_j : X_{t(j)} \to X_{i(j)}$*

(2) *All sets* X_v, $v \in V$ *are closed subarcs of* S^1

(3) *The map* $E_A^* \ni \omega \longmapsto g_\omega \in \Gamma$ *is a bijection*

(4) $\Lambda(\Gamma) = \mathcal{J}_{\mathcal{S}_\Gamma}$

(5) *The map* $\pi_{\mathcal{S}_\Gamma} : E_A^\infty \longrightarrow \mathcal{J}_{\mathcal{S}_\Gamma} = \Lambda(\Gamma)$ *is a continuous surjection and it is* 1*-to-*1 *except at countably many points, where it is* 2*-to-*1.

Similarly to (but not exactly) as in Section 6.1, given $e \in E$ we define

$$\Gamma_e := \{\gamma\omega \in E_A^* \text{ and } \omega_1 = e\}.$$

Then having $\rho \in A_A^{\mathbb{N}}$ we set

$$\Gamma_\rho := \Gamma_{\rho_1}$$

Again, similarly as in Section 6.1, we denote

$$\lambda_\rho(\gamma) = -\log|\gamma'(\pi_\Gamma(\rho))| = -\log|\gamma_\omega'(\pi_\Gamma(\omega))| = \lambda_\rho(\omega)$$

for every $\omega \in E_\rho^*$ ($\gamma = \gamma_\omega \in \Gamma_{\rho_1} = \Gamma_\rho$) and

$$\Delta(Y) := -\log(\mathrm{diam}(\gamma_\omega(Y)))$$

if $Y \subset X_{t(\rho_1)}$. Also

$$\lambda_p(\omega) = -\log|\gamma_\omega'(x_\omega)|$$

if $\omega \in E_p^*$

Let B denote a Borel subset of the set S^1. Set

$$\pi_\rho(\Gamma; T, B) := \{\gamma \in \Gamma_\rho : \lambda_\rho(\gamma) \le T \text{ and } \gamma(\pi_\Gamma(\rho)) \in B\}$$

$$\pi_\rho(\Gamma; T) := \pi_\xi(\Gamma; T, S^1) = \{\gamma \in \Gamma_\rho : \lambda_\rho(\gamma) \le T\}$$

$$\pi_p(\Gamma; T, B) := \{\omega \in E_p^* : \lambda_p(\omega) = l(\gamma_\omega) \le T \text{ and } x_\omega \in B\},$$

$$\pi_p(\Gamma; T) := \pi_p(\Gamma; T, S^1) = \{\omega \in E_p^* : \lambda_p(\omega) = l(\gamma_\omega) \le T\},$$

$$\widehat{\pi}_p(\Gamma, T) := \{\gamma \in \widehat{\Gamma} : l(\gamma_\gamma) \le T\}$$

With $e := \rho_1$ we further denote

$$\mathcal{D}_\rho(\Gamma; T, B, Y) := \{\gamma \in \Gamma_e : \Delta_\gamma(Y) \le T \text{ and } \gamma(\pi_\Gamma(\rho)) \in B\},$$

$$\mathcal{E}_e(\Gamma; T, B, Y) = \{\gamma \in \Gamma_e : \Delta_\gamma(Y) \le T \text{ and } \gamma(Y) \cap B \neq \emptyset\},$$

and

$$\mathcal{E}_e(\Gamma; T, Y) := \mathcal{E}_e(\Gamma; T, S^1, Y) = \{\gamma \in \Gamma_e : \Delta_\gamma(Y) \le T\}.$$

We denote by $N_\xi(\Gamma; T, B)$, $N_\xi(\Gamma; T)$, $N_p(\Gamma; T, B)$, $N_p(\Gamma; T)$, $\widehat{N}_p(\Gamma; T)$, $D_\xi(\Gamma; T, B, Y)$, $E_e(\Gamma; T, B, Y)$ and $E_e(\Gamma; T, Y)$ the corresponding cardinalities.

As immediate consequences of Theorem 2.4.9, Theorem 2.7.1, Theorem 2.7.4, Theorems 3.3.2, 3.4.1, and 3.4.2, along with Theorem 6.3.1 and Fuchsian counterparts of Proposition 6.1.5, Observation 6.1.6 and Proposition 6.1.8, following from [**84**] and [**85**], we get the following.

THEOREM 6.3.2. *Let* $\Gamma = \langle\gamma_j\rangle_{j=1}^q$ *be a finitely generated non-exceptional Fuchsian group.*

- *Let* δ_Γ *be the Poincaré exponent of* Γ*; it is known to be equal to* $\mathrm{HD}(\Lambda(\Gamma))$.
- *Let* m_{δ_Γ} *be the Patterson-Sullivan conformal measure for* G *on* $\Lambda(\Gamma)$.

- *Let μ_{δ_Γ} be the $\mathcal{S}_\Gamma$-invariant measure on $\Lambda(\Gamma)$ equivalent to m_{δ_Γ}.*
- *Fix $e \in E = \{\pm 1, \pm 2, \cdots, \pm q\}$ and $\rho \in E_A^\infty$ with $\rho_1 = e$.*

Let $B \subseteq S^1$ be a Borel set with $m_{\delta_\Gamma}(\partial B) = 0$ (equivalently $\mu_{\delta_\Gamma}(\partial B) = 0$) and let $Y \subseteq X_{t(e)}$ be a set having at least two distinct points. Then with some constant $C_e(Y) \in (0, +\infty]$, we have that

$$\lim_{T\to+\infty} \frac{N_\xi(\Gamma; T, B)}{e^{\delta_\Gamma T}} = \frac{\psi_{\delta_\Gamma}(\xi)}{\delta_\Gamma \chi_{\delta_\Gamma}} m_{\delta_\Gamma}(B), \qquad \lim_{T\to+\infty} \frac{N_\xi(\Gamma; T)}{e^{\delta_\Gamma T}} = \frac{\psi_{\delta_\Gamma}(\xi)}{\delta_\Gamma \chi_{\delta_\Gamma}},$$

$$\lim_{T\to+\infty} \frac{N_p(\Gamma; T, B)}{e^{\delta_\Gamma T}} = \frac{1}{\delta_\Gamma \chi_{\delta_\Gamma}} \mu_{\delta_\Gamma}(B), \qquad \lim_{T\to+\infty} \frac{N_p(\Gamma; T)}{e^{\delta_\Gamma T}} = \frac{1}{\delta_\Gamma \chi_{\delta_\Gamma}},$$

$$\lim_{T\to+\infty} \frac{\widehat{N}_p(\Gamma; T)}{e^{\delta_\Gamma T}} = \frac{1}{\delta_\Gamma \chi_{\delta_\Gamma}},$$

$$\lim_{T\to+\infty} \frac{D_\xi(\Gamma; T, B, Y)}{e^{\delta_\Gamma T}} = C_e(Y) m_{\delta_\Gamma}(B),$$

$$\lim_{T\to+\infty} \frac{E_k(\Gamma; T, B, Y)}{e^{\delta_\Gamma T}} = C_e(Y) m_{\delta_\Gamma}(B),$$

$$\lim_{T\to+\infty} \frac{E_k(\Gamma; T, Y)}{e^{\delta_\Gamma T}} = C_e(Y).$$

In addition, $C_e(Y) > 0$ is finite if and only if

$$\overline{Y} \cap \Omega(\mathcal{S}_\Gamma) = \emptyset,$$

in particular if Γ has no parabolic points, i.e. if it is convex co-compact.

We would like to add that the geodesic flow of a non-compact surface was coded by a sususpension flow over countable shift in [**15**] and was, in particular, used to get appropriate counting results.

Theorem 4.1.1 – Theorem 4.1.3 hold for the conformal GDMS $\mathcal{S}_\Gamma$, associated to the group Γ, without changes. Therefore, we do not repeat them here. However, as in Section 6.1, we present the appropriate versions of Theorems 4.3.1 and 4.3.2 as their formulations are closer to the group Γ. In order to get appropriate expressions in the language of the group Γ itself, given $\rho \in E_A^\infty$, and an integer $n \geq 1$, we set

$$\Gamma_\rho^n := \{\gamma_\omega : \omega \in E_\rho^n\} \subseteq \Gamma_\rho.$$

Furthermore, we define a probability measure μ_n on Γ_ρ^n by setting that

$$\mu_n(H) := \frac{\sum_{\gamma\in H} e^{-\delta\lambda_\rho(\gamma)}}{\sum_{\gamma\in\Gamma_\rho^n} e^{-\delta\lambda_\rho(\gamma)}} \tag{6.1}$$

for every set $H \subset \Gamma_\rho^n$. As an immediate consequence of Theorem 4.3.1 we get the following.

THEOREM 6.3.3. *If $\Gamma = \langle\gamma_j\rangle_{j=1}^q$ is a finitely generated non-exceptional convex co-compact (i. e. without parabolic fixed points) Fuchsian group, then for every $\rho \in E_A^\infty$ we have that*

$$\lim_{n\to+\infty} \int_{\Gamma_\rho^n} \frac{\lambda_\rho}{n} d\mu_n = \chi_{\mu_\delta}.$$

Now define the functions $\Delta_n : \Gamma_\rho^n \to \mathbb{R}$ by the formulae

$$\Delta_n(\gamma) = \frac{\lambda_\xi(\gamma) - \chi n}{\sqrt{n}}.$$

As an immediate consequence of Theorem 4.3.2 we get the following.

THEOREM 6.3.4. *If $\Gamma = \langle \gamma_j \rangle_{j=1}^q$ is a finitely generated non-exceptional convex co-compact (i. e. without parabolic fixed points) Fuchsian group, then for every $\rho \in E_A^\infty$ the sequence of random variables $(\Delta_n)_{n=1}^\infty$ converges in distribution to the normal (Gaussian) distribution $\mathcal{N}_0(\sigma)$ with mean value zero and the variance $\sigma^2 = \mathrm{P}''_{\mathcal{S}_\Gamma}(\delta) > 0$. Equivalently, the sequence $(\mu_n \circ \Delta_n^{-1})_{n=1}^\infty$ converges weakly to the normal distribution $\mathcal{N}_0(\sigma^2)$. This means that for every Borel set $F \subset \mathbb{R}$ with $\mathrm{Leb}(\partial F) = 0$, we have*

$$\lim_{n \to +\infty} \mu_n(\Delta_n^{-1}(F)) = \frac{1}{\sqrt{2\pi}\sigma} \int_F e^{-t^2/2\sigma^2} \, dt. \tag{6.2}$$

6.3.1. Hecke Groups. A special class of Fuchsian parabolic (so non-exceptional) groups are Hecke groups. These are easiest to express in the Lobachevsky model of hyperbolic geometry and plane rather than in the Poincaré one. The 2-dimensional hyperbolic (Lobachevsky) plane is the set

$$\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$$

endowed with the Riemannian metric

$$\frac{|dz|}{\mathrm{Im} z}.$$

Given $\epsilon > 0$ the corresponding Hecke group is defined as follows

$$\Gamma_\epsilon := \big\langle z \mapsto -1/z, \, z \mapsto z + 1 + \epsilon \big\rangle.$$

This group has an elliptic element order 2 which is the map $z \longmapsto -1/z$ and one (conjugacy class) of parabolic elements which is the map $z \longmapsto z + 1 + \epsilon$. Its (parabolic) fixed point is ∞. In particular all the limit sets $\Lambda(\Gamma_\varepsilon)$ are unbounded, and therefore the Hecke groups Γ_ε do not really fit into the setting of our current manuscript. However, any Möbius transformation

$$H : \mathbb{D} \to \mathbb{H}$$

is an isometry with respect to corresponding Poincaré metrics and the map

$$\Gamma_\varepsilon \ni \gamma \longmapsto H^{-1} \circ \gamma \circ H$$

establishes an algebraic isomorphism between Γ_ε and the group

$$\hat{\Gamma}_\varepsilon := \{H^{-1} \circ \gamma \circ H : \gamma \in \Gamma_\varepsilon\}.$$

Of course, the conjugacy H between $\hat{\Gamma}_\varepsilon$ and Γ_ε congregates elements of $\hat{\Gamma}_\varepsilon$ and Γ_ε viewed as isometric actions. The groups $\hat{\Gamma}_\varepsilon$ are Fuchsian parabolic (so non-exceptional) groups acting on $\mathbb{D}$ and perfectly fit into the setting of Section 6.3. In particular, Theorem 6.3.2 holds for them.

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