Skew product smale endomorphisms over countable shifts of finite type

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Abstract. We introduce and study skew product Smale endomorphisms over finitely irreducible topological Markov shifts with countable alphabets. We prove that almost all conditional measures of equilibrium states of summable Hölder continuous potentials are dimensionally exact, and their dimension is equal to the ratio of (global) entropy and Lyapunov exponent. We show that the exact dimensionality of conditional measures on fibers implies global exact dimensionality of the original measure. We then study equilibrium states and dimension for skew products over expanding Markov-Rényi transformations, and settle the question of exact dimensionality of such measures. In particular, we obtain the exact dimensionality of such measures with respect to skew products over the continued fractions transformation. We then prove two results related to Diophantine approximation, which extend and improve the Doeblin-Lenstra Conjecture on Diophantine approximation coefficients for a larger class of measures.

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1. Introduction

We introduce and explore skew product Smale endomorphisms modeled on countable alphabet subshifts of finite type. We study the thermodynamic formalism for skew product Smale endomorphisms over countable-to-1 maps, in particular natural extensions of countable-to-1 endomorphisms (such as EMR-expanding Markov-Rényi maps, Gauss map, etc). Our notion of Smale space is different, although inspired by the respective notion from [20]. One of our objectives is to develop the thermodynamic formalism of such dynamical systems. In order to do this, we first recall in Section 2 the foundations of thermodynamics formalism of one-sided subsifts of finite type modeled on a countable (either finite or infinite) alphabet, from [8], [7]. Passing on to two-sided shifts in Section 3, we provide a thermodynamic formalism of Hölder continuous potentials with respect to two-sided subshifts of finite type. It also includes a characterization of Gibbs states in terms of conditional measures; this has no counterpart for one sided shifts.

We then define in Section 4 skew product Smale endomorphisms, modeled on countable alphabet subshifts of finite type, and we specify several significant subclasses. Of particular interest is the projection from the symbol space to the Smale space. If a skew product Smale

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endomorphism is continuous and of compact type, then this projection gives a bijection between invariant measures for the symbol dynamics and for the Smale endomorphism.

A goal is to deal with conformal Hölder continuous Smale endomorphisms modeled on countable alphabet subshifts of finite type. We define them in Section 5 and in Section 6 we prove two theorems. In Theorem 6.1 we show that projections of a.e conditional measures of equilibrium states of summable Hölder continuous potentials are dimensionally exact, and their dimension is the ratio of the (global) entropy and the Lyapunov exponent. We prove in Theorem 7.3 a version of Bowen's formula giving the Hausdorff dimension of each fiber as the zero of a pressure function; we deal also with the case when pressure function has no zero. Exact dimensionality of measures has a long history and was studied in various cases, by Young ([25]), Barreira, Pesin and Schmeling ([1]), and other authors.

We then pass in Section 8 to general skew products over countable-to-1 endomorphisms. For endomorphisms, the study of Hausdorff dimension is in general different than for invertible systems and specific phenomena appear (for eg [21], [10], [11]). We prove, under a condition of μ-injectivity for the coding of the base map, the exact dimensionality of conditional measures of equilibrium measures in fibers, building on [12]. We consider general skew product endomorphisms $F : X \times Y \to X \times Y$, $F(x, y) = (f(x), g(x, y))$, over countable-to-1 endomorphisms $f : X \to X$ in the base X, where X is a general metric space (not only E_A^+), and $Y \subset \mathbb{R}^d$. Then f is coded by a shift space with countably many symbols, and we prove in Theorem 8.4 a result about the pointwise dimensions of conditional measures in fibers of F. Then, in Theorem 8.6 we prove that, if the conditional measures of an equilibrium measures μ_ϕ on fibers are exact dimensional, and if the projection of μ_ϕ in the base is also exact dimensional, then μ_ϕ is exact dimensional *globally*.

We then study several main classes of skew product endomorphisms over countable-to-1 maps, in particular natural extensions (inverse limits). In Section 9 we study EMR (expanding Markov-Rényi) maps $f : I \to I$ (see [17]), and conformal Smale skew product endomorphisms $F : I \times Y \to I \times Y$ over f. In Theorem 9.3 we prove exact dimensionality of conditional measures on fibers for F, for conditional measures of the equilibrium measures. In particular, we consider the continued fraction transformation $f_1(x) = \{\frac{1}{x}\}, x \in (0, 1]$ coded by a countable alphabet; and the Manneville-Pomeau maps $f_2(x) = x + x^{1+\alpha} \bmod 1$, $x \in [0, 1]$, $\alpha > 0$. In Theorem 9.5 we show that a class of equilibrium measures are exact dimensional globally on $I \times Y$.

In Section 10, we apply our results to Diophantine approximation of irrational numbers x, and we generalize the *Doeblin-Lenstra conjecture* about the approximation coefficients $\Theta_n(x)$ in continued fractions representation, to equilibrium measures μ_s of potentials $-s \log |T'|$, $s > \frac{1}{2}$ (where T is the Gauss map). If the continued fraction representation of an irrational number $x \in [0, 1)$ is $x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} = [a_1, a_2, \dots]$, with $a_i \geq 1$ integers, $i \geq 1$, and if $\frac{p_n(x)}{q_n(x)} = [a_1, \dots, a_n] \in \mathbb{Q}, n \geq 1$, then the approximation coefficients (see [6]) are:

$$\Theta_n(x) := q_n(x)^2 \cdot \big|x - \frac{p_n(x)}{q_n(x)}\big|, \; n \geq 1$$

The original Doeblin-Lenstra Conjecture (for eg [2], [6]) gives information about the frequency of having consecutive $\Theta_k(x), \Theta_{k-1}(x)$ in some set, and involves the lift of the Gauss measure μ_G to the natural extension space $[0,1)^2$ of the continued fraction transformation; thus, it is valid for Lebesgue-a.e $x \in [0,1)$. By contrast, in our case we take the numbers x from the complement of this set. The natural extension $([0,1)^2, \mathcal{T})$ of T is a skew product which falls in our class. Hence we can apply the results obtained in previous Sections. Using the exact dimensionality of the lift measure $\hat{\mu}_s$ of μ_s on the natural extension, we also make the Doeblin-Lenstra Conjecture more precise. Namely, for irrational x from $\Lambda_s \subset [0,1)$ with $\mu_s(\Lambda_s) = 1$ and $HD(\Lambda_s) > 0$ (but with $Leb(\Lambda_s) = 0$), we estimate the asymptotic frequency of having $(\Theta_k(x), \Theta_{k-1}(x))$ r-close to (z, z'), for $1 \le k \le n$ and n large.

Several authors studied various related aspects in thermodynamic formalism and dimension theory, for eg [1], [5], [8], [9], [10], [11], [13], [14], [16], [17], [20], [22], [23], etc.

2. One-Sided Thermodynamic Formalism

In this section we collect some fundamental ergodic (thermodynamic formalism) results concerning one-sided symbolic dynamics. All of them can be found with proofs in [8], [7]. Let E be a countable set and let $A : E \times E \to \{0,1\}$ be a matrix. A finite or countable infinite tuple ω of elements of E is called A-admissible if and only if $A_{ab} = 1$ for any two consecutive elements a, b of E. The matrix A is said to be *finitely irreducible* if there exists a finite set F of finite A-admissible words so that for any two elements a, b of E there exists $\gamma \in F$ such that the word $a\gamma b$ is A-admissible. In the sequel, the incidence matrix A is assumed to be finitely irreducible. Given $\beta > 0$, define the metric d_β on $E^{\mathbb{N}}$ by

$$d_\beta\big((\omega_n)_0^\infty, (\tau_n)_0^\infty\big) = \exp\big(-\beta \max\{n \ge 0 : (0 \le k \le n) \Rightarrow \omega_k = \tau_k\}\big)$$

with the standard convention that $e^{-\infty} = 0$. Note that all the metrics d_β, $\beta > 0$, on $E^{\mathbb{N}}$ are Hölder continuously equivalent and they induce the product topology on $E^{\mathbb{N}}$. Let

$$E_A^+ = \{(\omega_n)_0^\infty : \forall_{n\in\mathbb{N}}\ A_{\omega_n\omega_{n+1}} = 1\}$$

E_A^+ is a closed subset of $E^{\mathbb{N}}$ and we endow it with the topology and metrics d_β inherited from $E^{\mathbb{N}}$. The shift map $\sigma : E^{\mathbb{Z}} \to E^{Z}$ is defined by the formula $\sigma\big((\omega_n)_0^\infty\big) = \big((\omega_{n+1})_{n=0}^\infty\big)$, and $\sigma(E_A^+) \subset E_A^+$ and $\sigma : E_A^+ \to E_A^+$ is continuous. For every finite word $\omega = \omega_0\omega_1 \ldots \omega_{n-1}$, put $|\omega| = n$ the length of ω, and $[\omega] = \{\tau \in E_A^+ : \forall_{(0\le j\le n-1)} : \tau_j = \omega_j\}$ is the *cylinder* generated by ω. Let $\psi : E_A^+ \to \mathbb{R}$ continuous, then the topological pressure $\mathrm{P}(\psi)$ is

$$\mathrm{P}(\psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\big(\sup\big(S_n\psi|_{[\omega]}\big)\big)$$

and the limit exists, as the sequence $\log \sum_{|\omega|=n} \exp\big(\sup\big(S_n\psi|_{[\omega]}\big), n \in \mathbb{N}$, is sub-additive. The following theorem, a weaker version of the Variational Principle, was proved in [8].

Theorem 2.1. *If $\psi : E_A^+ \to \mathbb{R}$ is a continuous function and μ is a σ-invariant Borel probability measure on E_A^+ such that $\int \psi \, d\mu > -\infty$, then $\mathrm{h}_{\tilde{\mu}}(\sigma) + \int_{\mathrm{E}_A^+} \psi \, d\mu \le \mathrm{P}(\psi)$.*

We say that the function $\psi : E_A^+ \to \mathbb{R}$ is *summable* if and only if $\sum_{e\in E} \exp\big(\sup\big(\psi|_{[e]}\big)\big) < \infty$. A shift-invariant Borel probability measure μ on E_A^+ is called a Gibbs state of ψ provided that there are a constant $C \ge 1$ and $P \in \mathbb{R}$ such that

$$C^{-1} \le \frac{\mu([\omega])}{\exp(S_n\psi(\tau) - Pn)} \le C \tag{2.1}$$

for all $n \geq 1$, all admissible words ω of length n and all $\tau \in [\omega]$. It clearly follows from (2.1) that if ψ admits a Gibbs state, then $P = \mathrm{P}(\psi)$.

Definition 2.2. *A function $g : E_A^+ \to \mathbb{C}$ is called Hölder continuous if it is Hölder continuous with respect to one, equivalently all, metrics d_β. Then $\exists \beta > 0$ s.t g is Lipschitz continuous with respect to d_β. The corresponding Lipschitz constant is $L_\beta(g)$.*

The proofs of the following three results come from [8] and [7].

Theorem 2.3. *For every Hölder continuous summable potential $\psi : E_A^+ \to \mathbb{R}$ there exists a unique Gibbs state μ_ψ on E_A^+. The measure μ_ψ is ergodic.*

Theorem 2.4. *Suppose $\psi : E_A^+ \to \mathbb{R}$ is a Hölder continuous potential. Then, denoting by $\mathrm{P}_F(\psi)$ the topological pressure of $\psi|_{F_A^+}$ with respect to the shift map $\sigma : F_A^+ \to F_A^+$, we have $\mathrm{P}(\psi) = \sup\{\mathrm{P}_F(\psi)\}$, where the supremum is taken over all finite subsets F of E; equivalently over all finite subsets F of E such that the matrix $A|_{F\times F}$ is irreducible.*

Theorem 2.5 (Variational Principle for One-Sided Shifts)**.** *Suppose that $\psi : E_A^+ \to \mathbb{R}$ is a Hölder continuous summable potential. Then*

$$\sup\left\{\mathrm{h}_\mu(\sigma) + \int_{\mathrm{E}_\mathrm{A}^+} \psi \mathrm{d}\mu,\ \mu \circ \sigma^{-1} = \mu \text{ and} \int \psi \mathrm{d}\mu > -\infty\right\} = \mathrm{P}(\psi) = \mathrm{h}_{\mu_\psi}(\sigma) + \int_{E_A^+} \psi d\mu_\psi,$$

and μ_ψ is the only measure at which this supremum is attained.

Any measure that realizes the supremum in the above Variational Principle is called an equilibrium state for ψ. Then Theorem 2.5 can be reformulated as follows.

Theorem 2.6. *If $\psi : E_A^+ \to \mathbb{R}$ is a Hölder continuous summable potential, then the Gibbs state μ_ψ is a unique equilibrium state for ψ.*

Also due to the irreducibility of the incidence matrix A, we have:

Proposition 2.7. *A Hölder continuous $\psi : E_A^+ \to \mathbb{R}$ is summable if and only if $\mathrm{P}(\psi) < \infty$.*

3. Two-Sided Thermodynamic Formalism

As in the previous section let E be a countable set and let $A : E \times E \to \{0, 1\}$ be a finitely irreducible matrix. Given $\beta > 0$ we define the metric d_β on $E^{\mathbb{Z}}$ by setting

$$d_\beta\big((\omega_n)_{-\infty}^{\infty}, (\tau_n)_{-\infty}^{\infty}\big) = \exp\big(-\beta \max\{n \geq 0 : \forall_{k\in\mathbb{Z}} |k| \leq n \Rightarrow \omega_k = \tau_k\}\big)$$

with the standard convention that $e^{-\infty} = 0$. Note that all the metrics d_β, $\beta > 0$, on $E^{\mathbb{Z}}$ are Hölder continuously equivalent and they induce the product topology on $E^{\mathbb{Z}}$. We set

$$E_A = \{(\omega_n)_{-\infty}^{\infty} : \forall_{n\in\mathbb{Z}}\ A_{\omega_n\omega_{n+1}} = 1\}.$$

Obviously E_A is a closed subset of E^Z and we endow it with the topology and metrics d_β inherited from $E^{\mathbb{Z}}$. The two-sided shift map $\sigma : E^{\mathbb{Z}} \to E^Z$ is defined as $\sigma\big((\omega_n)_{-\infty}^{\infty}\big) = \big((\omega_{n+1})_{-\infty}^{\infty}\big)$. Clearly $\sigma(E_A) = E_A$ and $\sigma : E_A \to E_A$ is a homeomorphism.

Definition 3.1. *A function $g : E_A \to \mathbb{C}$ is said to be Hölder continuous provided that it is Hölder continuous with respect to one, equivalently all, metrics d_β. Then there exists at least one (in fact an open segment) parameter $\beta > 0$ such that g is Lipschitz continuous with respect to d_β. The corresponding Lipschitz constant is denoted by $L_\beta(g)$.*

For every $\omega \in E_A$ and all $-\infty \le m \le n \le \infty$, we set $\omega|_m^n = \omega_m\omega_{m+1}\dots\omega_n$.
Let E_A^* be the set of all A-admissible finite words. For $\tau \in E^*$, $\tau = \tau_m\tau_{m+1}\dots\tau_n$, we set

$$[\tau]_m^n = \{\omega \in E_A : \omega|_m^n = \tau\}$$

and call $[\tau]_m^n$ the cylinder generated by τ of size from m to n. The family of all cylinders of size from m to n will be denoted by C_m^n. If $m = 0$ we simply write $[\tau]$ for $[\tau]_m^n$.

Let $\psi : E_A \to \mathbb{R}$ be a continuous function. The topological pressure $\mathrm{P}(\psi)$ is defined by

$$\mathrm{P}(\psi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in C_0^{n-1}} \exp\bigl(\sup\bigl(S_n\psi|_{[\omega]}\bigr)\bigr), \tag{3.1}$$

and the limit exists due to the same subadditivity argument. Similarly we obtain:

Theorem 3.2. *If $\psi : E_A \to \mathbb{R}$ is a continuous function and μ is a σ-invariant Borel probability measure on E_A such that $\int \psi \, d\mu > -\infty$, then $\mathrm{h}_\mu(\sigma) + \int_{\mathrm{E_A}} \psi \, d\mu \le \mathrm{P}(\psi)$.*

A shift-invariant Borel probability measure μ on E_A is called a *Gibbs state* of ψ provided that there are a constant $C \ge 1$ and $P \in \mathbb{R}$ such that

$$C^{-1} \le \frac{\mu([\omega|_0^{n-1}])}{\exp(S_n\psi(\omega) - Pn)} \le C \tag{3.2}$$

for all $n \ge 1$ and all $\omega \in E_A$. It clearly follows from (3.2) that if ψ admits a Gibbs state, then $P = \mathrm{P}(\psi)$. Two functions ψ_1 and ψ_2 are called *cohomologous* in a class G of real-valued functions defined on E_A if and only if there exists $u \in G$ such that

$$\psi_2 - \psi_1 = u - u \circ \sigma.$$

Any function of the form $u - u \circ \sigma$ is called a *coboundary* in G. A function $\psi : E_A \to \mathbb{R}$ is called cohomologous to a constant, say $b \in \mathbb{R}$ provided that $\psi - b$ is a coboundary. Notice that any two functions on E_A, cohomologous in $C(E_A)$, the class of all real-valued bounded functions on B_A, have the same topological pressure and the same set of Gibbs measures.

A function $\psi : E_A \to \mathbb{R}$ is called *past-independent* if for every $\tau \in C_0^\infty$ and for all $\omega, \rho \in [\tau]$, we have $\phi(\omega) = \phi(\tau)$. To apply the previous Section, we need the following:

Lemma 3.3. *Every Hölder continuous function $\psi : E_A \to \mathbb{R}$ is cohomologous to a past-independent Hölder continuous function $\psi^+ : E_A \to \mathbb{R}$ in the class H_B of all bounded Hölder continuous functions.*

Proof. The proof is essentially the same as in [3], Lemma 1.6, page 11. For every $e \in E$ fix an arbitrary $\overline{e} \in E_A(-\infty, -1)$ such that $A_{\overline{e}_{-1}e} = 1$. Then, for every $\omega \in E_A$ put $\overline{\omega} = \overline{\omega_0}\omega|_0^\infty$, note that the mapping $\omega \mapsto \overline{\omega}$ is continuous and set

$$u(\omega) = \sum_{j=0}^{\infty} (\psi(\sigma^j(\omega)) - \psi(\sigma^j(\overline{\omega})).$$

We check first that u is well-defined and continuous. Fix $\beta > 0$ so that ψ is Lipschitz continuous with respect to the metric d_β. For every $j \geq 0$, $[\sigma^j(\omega)|_{-j}^{\infty}] = [\sigma^j(\overline{\omega})|_{-j}^{\infty}]$. Therefore $d_\beta(\sigma^j(\omega), \sigma^j(\overline{\omega})) \leq e^{-\beta j}$, and consequently

$$|\psi(\sigma^j(\omega)) - \psi(\sigma^j(\overline{\omega}))| \leq L_\beta(\psi) e^{-\beta j}. \tag{3.3}$$

Hence, by the Weierstrass M-test, $u : E_A \to \mathbb{R}$ is well-defined and continuous. If now $d_\beta(\omega, \tau) = e^{-\beta n}$, then $[\omega|_{-n}^{n}] = [\tau|_{-n}^{n}]$. Thus, for every $0 \leq j \leq n$,

$$|\psi(\sigma^j(\omega)) - \psi(\sigma^j(\tau))| \leq L_\beta(\psi) d_\beta(\sigma^j(\omega), \sigma^j(\tau)) \leq L_\beta(\psi) e^{-\beta(n-j)}$$

and

$$|\psi(\sigma^j(\overline{\tau})) - \psi(\sigma^j(\overline{\omega}))| \leq L_\beta(\psi) d_\beta(\sigma^j(\overline{\tau}), \sigma^j(\overline{\omega})) \leq L_\beta(\psi) e^{-\beta(n-j)}.$$

Thus using also (3.3), we get $|u(\omega) - u(\tau)| \leq 2L_\beta(\psi) \sum_{j=0}^{E(n/2)} e^{-\beta(n-j)} + 2L_\beta(\psi) \sum_{j>E(n/2)} e^{-\beta j} \leq 4L_\beta(\psi)(1 - e^{-\beta})^{-1} e^{-\beta \frac{n}{2}}$. So $u : E_A \to \mathbb{R}$ is Lipschitz continuous with respect to the metric $d_{\beta/2}$, and by (3.3) it is bounded. So $u \in \mathrm{H}_{\beta/2}$. Hence $\psi^+ = \psi - u + u \circ \sigma$ is Lipschitz continuous with respect to the metric $d_{\beta/2}$. Let us show that ψ^+ is past-independent. Let $\omega|_0^\infty = \tau_0^\infty$. Then $\overline{\omega} = \overline{\tau}$ and $\psi^+(\omega) = \psi(\omega) - \sum_{j=0}^{\infty}(\psi(\sigma^j(\omega)) - \psi(\sigma^j(\overline{\omega})) + \sum_{j=0}^{\infty}(\psi(\sigma^{j+1}(\omega)) - \psi(\sigma^{j+1}(\overline{\omega})) = \psi(\overline{\omega}) = \psi^+(\tau)$. □

In the setting of the above lemma, let $\overline{\psi}^+$ be the factorization of ψ^+ on E_A^+, i.e. $\psi^+ = \overline{\psi}^+ \circ \pi$. As an immediate consequence of this lemma we get,

Lemma 3.4. *If $\psi : E_A \to \mathbb{R}$ is a Hölder continuous potential, then $\mathrm{P}(\psi) = \mathrm{P}(\overline{\psi}^+)$, where, we remind, the former pressure is taken with respect to the two-sided shift $\sigma : E_A \to E_A$ while the latter one is taken with respect to the one-sided shift $\sigma : E_A^+ \to E_A^+$*

Then from this lemma and Theorem 2.4, we get

Theorem 3.5. *Suppose that $\psi : E_A \to \mathbb{R}$ is a Hölder continuous potential. Then, denoting by $\mathrm{P}_F(\psi)$ the topological pressure of $\psi|_{F_A^{+-}}$ with respect to $\sigma : F_A^{+-} \to F_A^{+-}$, we have that $\mathrm{P}(\psi) = \sup\{\mathrm{P}_F(\psi)\}$, where the supremum is taken over all finite subsets F of E; equivalently over all finite subsets F of E such that the matrix $A|_{F\times F}$ is irreducible.*

We call the function $\psi : E_A \to \mathbb{R}$ is *summable* if and only if

$$\sum_{e\in E} \exp\big(\sup(\psi|_{[e]})\big) < \infty.$$

As in the case of one-sided shift, we have the following.

Proposition 3.6. *A Hölder continuous $\psi : E_A \to \mathbb{R}$ is summable if and only if $\mathrm{P}(\psi) < \infty$.*

From Lemma 3.3 (the coboundary appearing there is bounded), we get the following.

Lemma 3.7. *Every Hölder continuous summable function $\psi : E_A \to \mathbb{R}$ is cohomologous to a past-independent Hölder continuous summable function $\psi^+ : E_A \to \mathbb{R}$ in the class H_B of all bounded Hölder continuous functions.*

Theorem 3.8. *For every Hölder continuous summable potential $\psi : E_A \to \mathbb{R}$ there exists a unique Gibbs state μ_ψ on E_A. The measure μ_ψ is ergodic.*

Proof. Let ψ^+ be the past-independent Hölder continuous summable potential ascribed to ψ according to Lemma 3.7. Treating ψ^+ as defined on the one-sided symbol space E_A^+, it follows from Theorem 2.3 that there exists a unique Borel probability shift-invariant measure μ_ψ^+ on E_A^+ for which the formula (3.2) is satisfied. In addition μ_ψ^+ is ergodic. Since the measure μ_ψ^+ is shift-invariant, we conclude that the formula

$$\mu_\psi([\omega|_m^n]) = \mu_\psi^+(\sigma^m([\omega|_m^n])) = \mu_\psi^+((([|_0^{n-m}]),\ |\omega| = n - m + 1,$$

gives rise to a Borel probability shift-invariant measure μ_ψ on E_A, for which the formula (3.2) holds. Thus μ_ψ is a Gibbs state for ψ. It is easy to verify that μ_ψ is ergodic (remember that μ_ψ^+ was). Passing to the uniqueness, if μ is an arbitrary Gibbs state for ψ, then from its shift-invariance and (3.2), for all $n \geq 0$ and all $\omega \in E_A$, $C^{-1} \leq \frac{\mu([\omega|_{-n}^n])}{\exp\left(S_{2n+1}\psi(\sigma^{-n}(\omega)) - \mathrm{P}(\psi)n\right)} \leq C$. Any two Gibbs states of ψ are equivalent and, since one of them is ergodic, uniqueness follows. □

Let us now provide a variational characterization of Gibbs states.

Theorem 3.9 (Variational Principle for Two-Sided Shifts)**.** *Suppose that $\psi : E_A \to \mathbb{R}$ is a Hölder continuous summable potential. Then*

$$\sup\left\{\mathrm{h}_\mu(\sigma) + \int_{\mathrm{E_A}} \psi \mathrm{d}\mu : \mu \circ \sigma^{-1} = \mu \quad and \quad \int \psi \mathrm{d}\mu > -\infty\right\} = \mathrm{P}(\psi) = \mathrm{h}_{\mu_\psi}(\sigma) + \int_{E_A} \psi d\mu_\psi,$$

and μ_ψ is the only measure at which this supremum is taken on.

Proof. We replace ψ by the past-independent Hölder continuous summable potential ψ^+ resulting from Lemma 3.7. Since the dynamical system (σ, E_A), is canonically isomorphic to the natural extension of (σ, E_A^+), the map $\mu \mapsto \mu \circ \pi^{-1}$ gives a bijection between M_σ^{+-} and M_σ^+ which preserves entropies. Since $\mathrm{P}(\psi) = \mathrm{P}(\overline{\psi}^+)$ by Lemma 3.4, and since for every $\mu \in M_\sigma^{+-}$, $\int_{E_A^+} \overline{\psi}^+ d\mu\pi^{-1} = \int_{E_A} \overline{\psi}^+ \circ \pi d\mu = \int_{E_A} \psi^+ d\mu$, we are done due to Theorem 2.5. □

Any measure that realizes the supremum value in the above Variational Principle is called an *equilibrium state* for ψ. Then Theorem 3.9 can be reformulated as follows.

Theorem 3.10. *If $\psi : E_A \to \mathbb{R}$ is a Hölder continuous summable potential, then the Gibbs state μ_ψ is a unique equilibrium state for ψ.*

We will need however more characterizations of Gibbs states. Let the partition

$$\mathcal{P}_- = \{[\omega|_0^\infty] : \omega \in E_A\} = \{[\omega] : \omega \in E_A^+\}.$$

$\mathcal{P}_-$ is a measurable partition of E_A and two elements $\alpha, \beta \in E_A$ belong to the same element of this partition if and only if $\alpha|_0^\infty = \beta|_0^\infty$. If μ is a Borel probability measure on E_A, we let

$$\{\overline{\mu}^\tau : \tau \in E_A\}$$

be a *canonical system of conditional measures* induced by partition $\mathcal{P}_-$ and measure μ (see Rokhlin [19]). Each $\overline{\mu}^\tau$ is a Borel probability measure on $[\tau|_0^\infty]$ and we will frequently write $\overline{\mu}^\omega$, $\omega \in E_A^+$, to denote the corresponding conditional measure on $[\omega]$. Denote by

$$\pi_0 : E_A \to E_A^+,\ \pi_0(\tau) = \tau|_0^\infty, \tau \in E_A,$$

the canonical projection to E_A^+. The system $\{\overline{\mu}^\omega : \omega \in E_A^+\}$ is determined by the fact that:

$$\int_{E_A} g\, d\mu = \int_{E_A^+} \int_{[\omega]} g\, d\overline{\mu}^\omega\, d(\mu \circ \pi_0^{-1})(\omega)$$

for every measurable function $g \in L^1(\mu)$ ([19]). It is evident from this characterization that if we change such a system on a set of zero $\mu \circ \pi_0^{-1}$-measure, then we also obtain a system of conditional measures. The canonical system of conditional measures induced by μ is uniquely defined up to a set of zero $\mu \circ \pi_0^{-1}$-measure. We say that a collection

$$\{\overline{\mu}^\omega : \omega \in E_A^+\}$$

defines a *global system of conditional measures* of μ if this is indeed a system of conditional measures of μ and a measure $\overline{\mu}^\omega$ is defined for every $\omega \in E_A^+$, rather than only on a set of full $\mu \circ \pi_0^{-1}$-measure. The first characterization of Gibbs states is the following.

Theorem 3.11. *Suppose that $\psi : E_A \to \mathbb{R}$ is a Hölder continuous summable potential. Let μ be a Borel probability shift-invariant measure on E_A. Then $\mu = \mu_\psi$, the unique Gibbs state for ψ if and only if there exists $D \geq 1$ such that*

$$D^{-1} \leq \frac{\overline{\mu}^\omega([\tau\omega|_{-n}^\infty])}{\exp\big(S_n\psi(\rho) - \mathrm{P}(\psi)n\big)} \leq D \tag{3.4}$$

for every $n \geq 1$, $\mu \circ \pi_0^{-1}$-a.e. $\omega \in E_A^+$, $\overline{\mu}^\omega$-a.e. $\tau\omega \in E_A(-n, \infty)$ with $A_{\tau_{-1}\omega_0} = 1$, and $\rho \in [\tau\omega|_0^\infty]$. Also there exists a global system of conditional measures of μ_ψ s.t,

$$D^{-1} \leq \frac{\overline{\mu}_\psi^\omega([\tau\omega|_{-n}^\infty])}{\exp\big(S_n\psi(\rho) - \mathrm{P}(\psi)n\big)} \leq D \tag{3.5}$$

for every $\omega \in E_A^+$, $n \geq 1$, $\tau \in E_A(-n, -1)$ with $A_{\tau_{-1}\omega_0} = 1$, and every $\rho \in [\tau\omega|_0^\infty]$.

Proof. Suppose (3.4) holds. Then for every $\omega \in E_A$ (note that here indeed "for every", although (3.4) is assumed to hold only for $\mu \circ \pi_0^{-1}$-a.e. $\omega \in E_A^+$) and every $n \geq 1$, we get
(3.6)

$$\mu([\omega|_0^{n-1}]) = \mu(\sigma^n([\omega|_0^{n-1}])) = \mu([\omega|_0^{n-1}|_{-n}^{-1}]) = \int_{E_A^+} \overline{\mu}^\tau\big([\omega|_0^{n-1}|_{-n}^{-1}\tau]\big) d\mu \circ \pi^{-1}(\tau)$$

$$= \int_{E_A^+ : A_{\omega_{n-1}\tau_0}=1} \overline{\mu}^\tau\big([\omega|_0^{n-1}|_{-n}^{-1}\tau]\big) d\mu \circ \pi^{-1}(\tau) \asymp \exp\big(S_n\psi(\omega) - \mathrm{P}(\psi)n\big) \sum_{e \in E : A_{\omega_{n-1}e}=1} \mu([e])$$

Consequently,

$$\mu([\omega|_0^{n-1}]) \preceq \exp\big(S_n\psi(\omega) - \mathrm{P}(\psi)n\big). \tag{3.7}$$

In order to prove the opposite inequality notice that because of finite irreducibility of the matrix A there exists a finite set $F \subset E$ such that for every $a \in E$ there exists $b \in F$ such that $A_{ab} = 1$. Since μ is a non-zero measure, there exists $c \in E$ such that $\mu([c]) > 0$. Invoking finite irreducibility of the matrix A again, we see that for every $e \in E$ there exists a finite word α such that $e\alpha c$ is A-admissible. Put $k = |e\alpha|$. It then follows from (3.6) that

$$\mu([e]) \geq \mu([e\alpha]) \succeq \exp\big(S_k\psi(\rho) - \mathrm{P}(\psi)k\big)\mu([c]) > 0$$

for every $\rho \in [e\alpha]$. Hence $T = \min\{\mu([\varepsilon]) : e \in F\} > 0$. Continuing (3.6), we see that $\mu([\omega|_0^{n-1}]) \succeq T\exp\big(S_n\psi(\omega) - \mathrm{P}(\psi)n\big)$. Combining this with (3.7) we see that μ is a Gibbs state for ψ, and the first assertion of the theorem is established.

Now, to complete the proof, we need to define a global system of conditional measures of μ_ψ such that (3.5) holds for every $\omega \in E_A^+$, $n \geq 1$, $\tau \in E_A(-n,-1)$ with $A_{\tau_{-1}\omega_0} = 1$, and every $\rho \in \sigma^{-n}([\tau\omega|_{-n}^\infty]) = [\tau\omega|_0^\infty]$. Indeed, let $L : \ell_\infty \to \ell_\infty$ be a Banach limit. Note that:

(3.8)

$$\frac{\mu_\psi\big([\tau\omega|_{-n}^{k-1}]\big)}{\mu_\psi\big(\omega|_0^{k-1}]\big)} = \frac{\mu_\psi\big([\tau\omega|_0^{n+k-1}]\big)}{\mu_\psi\big(\omega|_0^{k-1}]\big)} \asymp \frac{\exp\big(S_{n+k}\psi(\rho) - \mathrm{P}(\psi)(n+k)\big)}{\exp\big(S_k\psi(\sigma^n(\rho) - \mathrm{P}(\psi)k\big)} = e^{S_n\psi(\rho)-\mathrm{P}(\psi)n} \asymp \mu_\psi([\tau]_0^{n-1}),$$

belongs to ℓ_∞ (comparability constants from Gibbs property of μ_ψ). So the sequence $\left(\frac{\mu_\psi\left([\tau\omega|_{-n}^{k-1}]\right)}{\mu_\psi\left(\omega|_0^{k-1}]\right)}\right)_{k=1}^\infty$ belongs to ℓ_∞. We can then define

$$\overline{\mu}_\psi^\omega\big([\tau\omega|_{-n}^\infty]\big) := L\left(\left(\frac{\mu_\psi\big([\tau\omega|_{-n}^{k-1}]\big)}{\mu_\psi\big(\omega|_0^{k-1}]\big)}\right)_{k=1}^\infty\right).$$

For every $g : [\omega] \to \mathbb{R}$, and a linear combination $\sum_{j=1}^s a_j \mathbb{1}_{[\tau^{(j)}\omega|_{-n_j}^\infty]}$, the sequence

$$\frac{\mu_\psi\left(\sum_{j=1}^s a_j \mathbb{1}_{[\tau^{(j)}\omega|_{-n_j}^{k-1}]}\right)}{\mu_\psi\big(\omega|_0^{k-1}]\big)} \asymp \mu_\psi\left(\sum_{j=1}^s a_j \mathbb{1}_{[\tau^{(j)}]_0^{n_j-1}}\right), \tag{3.9}$$

with the same comparability constants as above, belongs to ℓ_∞. We can then define

$$\overline{\mu}_\psi^\omega\left(\sum_{j=1}^s a_j \mathbb{1}_{[\tau^{(j)}\omega|_{-n_j}^\infty]}\right) := L\left(\left(\frac{\mu_\psi\left(\sum_{j=1}^s a_j \mathbb{1}_{[\tau^{(j)}\omega|_{-n_j}^{k-1}]}\right)}{\mu_\psi\big(\omega|_0^{k-1}]\big)}\right)_{k=1}^\infty\right).$$

So, we have defined a function $\overline{\mu}_\psi^\omega$ from the vector space $\mathcal{V}$ of all linear combinations as above the the set of real numbers. Since the Banach limit is a positive linear operator, so is the function $\overline{\mu}_\psi^\omega : \mathcal{V} \to \mathbb{R}$. Furthermore, because of monotonicity of Banach limits, and because of (3.9), $\overline{\mu}_\psi^\omega(g_n) \searrow 0$ whenever $(g_n)_{n=1}^\infty$ is a monotone decreasing sequence of functions in $\mathcal{V}$ converging pointwise to 0. Therefore, Daniell-Stone Theorem gives a unique Borel probability measure on $[\omega]$, whose restriction to $\mathcal{V}$ coincides with $\overline{\mu}_\psi^\omega$. We keep the same symbol $\overline{\mu}_\psi^\omega$ for this extension. Now, it follows from Martingale's Theorem that for $\mu_\psi \circ \pi_0^{-1}$-a.e. $\omega \in E_A^+$ and every $\tau \in E_A(-n,-1)$ with $A_{\tau_{-1}\omega_0} = 1$ the limit

$$\lim_{k\to\infty} \frac{\mu_\psi\big([\tau\omega|_{-n}^{k-1}]\big)}{\mu_\psi\big(\omega|_0^{k-1}]\big)}$$

exists and equals the conditional measure of μ_ψ on $[\omega]$. By properties of Banach limits, $\frac{\mu_\psi\left([\tau\omega|_{-n}^{k-1}]\right)}{\mu_\psi\left(\omega|_0^{k-1}]\right)} = \lim_{k\to\infty} \frac{\mu_\psi\left([\tau\omega|_{-n}^{k-1}]\right)}{\mu_\psi\left(\omega|_0^{k-1}]\right)}$, and thus the collection $\{\overline{\mu}_\psi^\omega : \omega \in E_A^+\}$ is indeed a global system of conditional measures of μ_ψ. Using also (3.8) this completes the proof.

□

Similarly, let

$$\mathcal{P}_+ = \{[\omega|_{-\infty}^{-1}] : \omega \in E_A\},$$

and given a Borel probability measure μ on E_A, let $\{\mu^{+\omega} : \omega \in E_A\}$ the corresponding canonical system of conditional measures. As in Theorem 3.11, we prove the following.

Theorem 3.12. *Suppose $\psi : E_A \to \mathbb{R}$ is a Hölder continuous summable potential. Let μ be a Borel probability shift-invariant measure on E_A. Then $\mu = \mu_\psi$, the unique Gibbs state for ψ if and only if there exists $D \geq 1$ s.t for all $\omega \in E_A(-\infty, -1)$, $n \geq 1$, $\tau \in E_A(0, n-1)$ with $A_{\omega_{-1}\tau_0} = 1$, and $\rho \in [\omega\tau|_{-\infty}^{n-1}]$, we have*

$$D^{-1} \leq \frac{\mu^{+\omega}([\omega\tau|_{-\infty}^{n-1}])}{\exp\big(S_n\psi(\rho) - \mathrm{P}(\psi)n\big)} \leq D \tag{3.10}$$

4. Skew product Smale Spaces of Countable Type

Keep notation from the previous two sections.

Definition 4.1. *Let (Y, d) be a complete bounded metric space, and take for every $\omega \in E_A^+$ an arbitrary set $Y_\omega \subset Y$ and a continuous injective map $T_\omega : Y_\omega \to Y_{\sigma(\omega)}$. Define*

$$\hat{Y} := \bigcup_{\omega \in E_A^+} \{\omega\} \times Y_\omega \subset E_A^+ \times Y.$$

Define the map $T : \hat{Y} \to \hat{Y}$ by $T(\omega, y) = (\sigma(\omega), T_\omega(y))$. The pair $(\hat{Y}, T : \hat{Y} \to \hat{Y})$ is called a skew product Smale endomorphism if there exists $\lambda > 1$ such that T is fiberwise uniformly contracting, i.e for all $\omega \in E_A^+$ and all $y_1, y_2 \in Y_\omega$,

$$d(T_\omega(y_2), T_\omega(y_1)) \leq \lambda^{-1} d(y_2, y_1) \tag{4.1}$$

Note that for every $\tau \in E_A(-n, \infty)$ the composition $T_\tau^n = T_{\tau|_{-1}^\infty} \circ T_{\tau|_{-2}^\infty} \circ \ldots \circ T_{\tau|_{-n}^\infty} : Y_\tau \to Y_{\tau|_0^\infty}$ is well-defined. Therefore for every $\tau \in E_A$ we can define the map

$$T_\tau^n := T_{\tau|_{-n}^\infty}^n := T_{\tau|_{-1}^\infty} \circ T_{\tau|_{-2}^\infty} \circ \ldots \circ T_{\tau|_{-n}^\infty} : Y_{\tau|_{-n}^\infty} \to Y_{\tau|_0^\infty}$$

Then the sequence $\big(T_\tau^n\big(Y_{\tau|_{-n}^\infty}\big)\big)_{n=0}^\infty$ consists of descending sets, and

$$\mathrm{diam}\big(T_\tau^n\big(Y_{\tau|_{-n}^\infty}\big)\big) \leq \lambda^{-n}\mathrm{diam}(Y). \tag{4.2}$$

The same is then true for the closures of these sets, i.e. we have that the sequence $\big(\overline{T_\tau^n\big(Y_{\tau|_{-n}^\infty}\big)}\big)_{n=0}^\infty$ consists of closed descending sets, and $\mathrm{diam}\big(\overline{T_\tau^n\big(Y_{\tau|_{-n}^\infty}\big)}\big) \leq \lambda^{-n}\mathrm{diam}(Y)$. Since the metric space (Y, d) is complete, we conclude that its intersection $\bigcap_{n=1}^\infty \overline{T_\tau^n\big(Y_{\tau|_{-n}^\infty}\big)}$ is a singleton. Denote its only element by $\hat{\pi}_2(\tau)$. So, we have defined the map

$$\hat{\pi}_2 : E_A \to Y,$$

and next define the map $\hat{\pi} : E_A \to E_A^+ \times Y$ by the formula

$$\hat{\pi}(\tau) = (\tau|_0^\infty, \hat{\pi}_2(\tau)), \tag{4.3}$$

and the truncation to the elements of non-negative indices by

$$\pi_0 : E_A \to E_A^+, \ \ \pi_0(\tau) = \tau|_0^\infty$$

In the notation for π_0 we drop the hat symbol, as this projection is in fact independent of the skew product on $\hat{Y}$. For all $\omega \in E_A^+$ define the $\hat{\pi}_2$-projection of the cylinder $[\omega] \subset E_A$,

$$J_\omega := \hat{\pi}_2([\omega]) \in Y,$$

and call these sets the *stable Smale fibers* of the system T. The global invariant set is:

$$J := \hat{\pi}(E_A) = \bigcup_{\omega \in E_A^+} \{\omega\} \times J_\omega \subset E_A^+ \times Y,$$

called the *Smale space* (or the *fibered limit set*) induced by the Smale pre-system T.

For each $\tau \in E_A$ we have $\hat{\pi}_2(\tau) \in \overline{Y}_{\tau|_0^\infty}$; therefore $J_\omega \subset \overline{Y}_\omega$, for every $\omega \in E_A^+$. Since all the maps $T_\omega : Y_\omega \to Y_{\sigma(\omega)}$ are Lipschitz continuous with a Lipschitz constant λ^{-1}, all of them extend uniquely to continuous maps from $\overline{Y}_\omega$ to $\overline{Y}_{\sigma(\omega)}$ and these extensions are Lipschitz continuous with a Lipschitz constant λ^{-1}.

Proposition 4.2. *For every $\omega \in E_A^+$ we have that*

$$T_\omega(J_\omega) \subset J_{\sigma(\omega)}, \tag{4.4}$$

$$\bigcup_{e \in E, A_{e\omega_0}=1} T_{e\omega}(J_{e\omega}) = J_\omega, \text{ and} \tag{4.5}$$

$$T \circ \hat{\pi} = \hat{\pi} \circ \sigma \tag{4.6}$$

Proof. Let $y \in J_\omega$; then $\exists \tau \in E_A(-\infty,-1)$ s.t $A_{\tau_{-1}\omega_0} = 1$ and $y = \hat{\pi}_2(\tau\omega)$. Then

(4.7)
$$\{T_\omega(y)\} = T_\omega\big(\bigcap_{n=1}^\infty \overline{T^n_{\tau\omega}(Y_{\tau|_{-n}^{-1}\omega})}\big) \subset \bigcap_{n=1}^\infty T_\omega\big(\overline{T^n_{\tau\omega}(Y_{\tau|_{-n}^{-1}\omega})}\big) \subset \bigcap_{n=1}^\infty \overline{T_\omega\big(T^n_{\tau\omega}(Y_{\tau|_{-n}^{-1}\omega})\big)}$$
$$= \bigcap_{n=1}^\infty \overline{T^{n+1}_{\tau\omega}(Y_{\tau|_{-n}^{-1}\omega})\big)} = \bigcap_{n=1}^\infty \overline{T^{n+1}_{\tau|_{-\infty}^{-1}\omega_0(\sigma(\omega))}(Y_{\tau|_{-\infty}^{-1}\omega_0(\sigma(\omega))}\big)} = \hat{\pi}_2\big(\tau|_{-\infty}^{-1}\omega_0(\sigma(\omega))\big) \subset J_{\sigma(\omega)}$$

Thus $T_\omega(J_\omega) \subset J_{\sigma(\omega)}$ meaning that (4.4) holds, and, as $\{T_\omega(y)\}$ and $\big\{\hat{\pi}_2\big(\tau|_{-\infty}^{-1}\omega_0(\sigma(\omega))\big)\big\}$, the respective sides of (4.7), are singletons, we therefore get

$$T_\omega \hat{\pi}_2(\tau\omega) = \hat{\pi}_2 \circ \sigma(\tau\omega), \tag{4.8}$$

meaning that (4.6) holds. The inclusion $\bigcup_{\substack{e \in E \\ A_{e\omega_0}=1}} T_{e\omega}(J_{e\omega}) \subset J_\omega$ holds because of (4.4). In order to prove the opposite one, let $z \in J_\omega$. Then $z = \hat{\pi}(\gamma\omega)$ with some $\gamma \in E_A(-\infty,-1)$, where $A_{\gamma_{-1}\omega_0} = 1$. Formula (4.8) then yields $z = \hat{\pi}_2 \circ \sigma\big(\gamma|_{-\infty}^{-2}\gamma_{-1}|_{-\infty}^0\omega\big) = T_{\gamma_{-1}\omega} \circ \hat{\pi}_2\big(\gamma|_{-\infty}^{-2}\gamma_{-1}|_{-\infty}^0\omega\big) \in T_{\gamma_{-1}\omega}\big(J_{\gamma_{-1}\omega}\big)$. So $J_\omega \subset \bigcup_{\substack{e \in E \\ A_{e\omega_0}=1}} T_{e\omega}(J_{e\omega})$, and (4.5) is proved. $\square$

Similarly we obtain $J_\omega = \bigcup_{\substack{\tau \in E_A^n \\ A_{\tau_n\omega_0}=1}} T_{\tau\omega}\big(J_{\tau\omega}\big)$, for all $\omega \in E_A^+$, and $n > 0$. By formula (4.4) we have $T(J) \subset J$, so consider the system

$$T : J \to J$$

which we call the *skew product Smale endomorphism* generated by the Smale system $T : \hat{Y} \to \hat{Y}$. By formula (4.5) we have the following.

Observation 4.3. *The map $T : J \to J$ is surjective.*

Observation 4.4. *If $T : \hat{Y} \to \hat{Y}$ is a skew product Smale system, then the following statements are equivalent:*

(a) *For every $\xi \in J$, the fiber $\hat{\pi}^{-1}(\xi) \subset E_A$ is compact.*
(b) *For every $y \in Y$, the fiber $\hat{\pi}_2^{-1}(y) \subset E_A$ is compact.*
(c) *For every $\xi = (\omega, y) \in J$, the set $\{e \in E : A_{e\omega_0} = 1 \text{ and } y \in T_{e\omega}(J_{e\omega})\}$ is finite.*

If either of these three above conditions is satisfied, we call the skew product Smale system $T : J \to J$ of *compact type*.

Remark 4.5. *In item (a) of Observation 4.4 one can replace J by $\hat{Y}$.*

Observation 4.6. *If for every $y \in Y$ the set $\{e \in E : A_{e\omega_0} = 1 \text{ and } y \in T_{e\omega}(J_{e\omega})\}$ is finite for every $\omega \in E_A^+$, then $T : J \to J$ is of compact type.*

From now on we assume $T : \hat{Y} \to \hat{Y}$ is a skew product Smale system of compact type.

If for every $\xi \in \hat{Y}$ (or in J), the fiber $\hat{\pi}^{-1}(\xi) \subset E_A$ is finite, we call the skew product Smale system T of *finite type*.

Observation 4.7. *If the skew product Smale system $T : \hat{Y} \to \hat{Y}$ is of finite type, then it is also of compact type.*

The Smale system $T : \hat{Y} \to \hat{Y}$ is called of *bijective type* if, for every $\xi \in J$ the fiber $\hat{\pi}^{-1}(\xi)$ is a singleton. Equivalently, the map $\hat{\pi} : E_A \to J$ is injective; then also $T : J \to J$ is bijective. A Smale skew product of bijective type is clearly of finite type, and thus of compact type.

Definition 4.8. *We call a Smale endomorphism continuous if the global map $T : J \to J$ is continuous with respect to the relative topology inherited from $E_A^+ \times Y$.*

Later in this section, we will provide a construction scheme giving rise to continuous Smale endomorphisms. In fact all of them will be Hölder continuous.

Lemma 4.9. *For every $n \geq 1$ and every $\tau \in E_A(-n, \infty)$, we have that*

$$\hat{\pi}_2\big([\tau]\big) = T_\tau^n\big(J_\tau\big), \text{ and equivalently for every } \tau \in E_A,\ \hat{\pi}_2\big([\tau|_{-n}^{\infty}]\big) = T_\tau^n\big(J_{\tau|_{-n}^{\infty}}\big)$$

Proof. From (4.6) we get $T_\tau^n\big(J_{\tau|_{-n}^{\infty}}\big) = T_\tau^n \circ \hat{\pi}_2\big([\tau|_{-n}^{\infty}]|_0^{\infty}\big) = \hat{\pi}_2 \circ \sigma^n\big([\tau|_{-n}^{\infty}]|_0^{\infty}\big) = \hat{\pi}_2\big([\tau|_{-n}^{\infty}]\big)$ □

As an immediate consequence of (4.2), we get the following

Observation 4.10. *For every $\omega \in E_A$, the map $[\omega]_0^{\infty} \ni \tau \mapsto \hat{\pi}_2(\tau) \in J_{\omega|_0^{\infty}} \subset Y$ is Lipschitz continuous if E_A is endowed with the metric $d_{\lambda^{-1}}$.*

Note that for every $\tau \in E_A^n$, $n \geq 1$, we have $\hat{\pi}([\tau]) = \bigcup_{\omega \in [\tau]} \{\omega\} \times J_\omega$.

Let $M(E_A)$ be the topological space of Borel probability measures on E_A with the topology of weak convergence, and $M_\sigma(E_A)$ be its closed subspace consisting of σ-invariant measures. Likewise, let $M(J)$ be the space of Borel probability measures on J with the topology of weak convergence, and let $M_T(J)$ be its closed subspace of T-invariant measures. The following fact is well known; we include its simple proof for completeness.

Lemma 4.11. *Let W and Z be Polish spaces. Let μ be a Borel probability measure on Z, let $\hat{\mu}$ be its completion, and denote by $\hat{\mathcal{B}}_\mu$ the complete σ-algebra of all $\hat{\mu}$-measurable subsets of Z. Let $f : W \to Z$ be a Borel measurable surjection and let $g : W \to \overline{\mathbb{R}}$ be a Borel measurable function. Define the functions $g_*, g^* : Z \to \overline{\mathbb{R}}$ respectively by*

$$g_*(z) := \inf\{g(w) : w \in f^{-1}(z)\} \quad \textit{and} \quad g_*(z) := \sup\{g(w) : w \in f^{-1}(z)\}.$$

Then these two functions are measurable with respect to the σ-algebra $\hat{\mathcal{B}}_\mu$. If in addition the map $f : W \to Z$ is locally 1-to-1, then both g_* and $g^* : Z \to \overline{\mathbb{R}}$ are Borel measurable.

Proof. Replacing g by $-g$ suffices to prove our lemma for the function $g^* : Z \to \overline{\mathbb{R}}$ only. Fix $t \in \mathbb{R}$. Then for any $z \in Z$ we have that $g^*(z) \in (t, \infty)$ if and only if $g(w) \in (t, \infty)$ for some $w \in f^{-1}(z)$. Thus $(g^*)^{-1}((t, \infty)) = f(g^{-1}((t, \infty)))$. Hence $(g^*)^{-1}((t, \infty))$ is an analytic set since $g^{-1}((t, \infty))$ is a Borel set, $f : W \to Z$ is a Borel map, and both spaces W and Z are Polish. The first assertion now follows from the fact that all analytic subsets of Z belong to $\hat{\mathcal{B}}_\mu$. If in addition the map $f : W \to Z$ is locally 1-to-1, then the f-images of all Borel subsets of W are Borel in Z, so $f(g^{-1}((t, \infty))) \subset Z$ is Borel. □

Now we prove the following.

Theorem 4.12. *If $T : J \to J$ is a continuous skew product Smales endomorphism of compact type, then the map $M_\sigma(E_A) \ni \mu \longmapsto \mu \circ \hat{\pi}^{-1} \in M_T(J)$ is surjective.*

Proof. Fix $\mu \in M_T(J)$. Let $\mathcal{B}_b(E_A)$ and $\mathcal{B}_b(J)$ be the vector spaces of all bounded Borel measurable real-valued functions defined respectively on E_A and on J. Let Let $\mathcal{B}_b^+(E_A)$ and $\mathcal{B}_b^+(J)$ be the respective convex cones consisting of non-negative functions. Let

$$\hat{\mathcal{B}}_b(E_A) := \{g \circ \hat{\pi} : g \in \mathcal{B}_b(J)\}.$$

Clearly $\hat{\mathcal{B}}_b(E_A)$ is a vector subspace of $\mathcal{B}_b(E_A)$ and, as $\hat{\pi} : E_A \to J$ is a surjection, for each $h \in \hat{\mathcal{B}}_b(E_A)$ there exists a unique $g \in \mathcal{B}_b(J)$ such that $h = g \circ \hat{\pi}$. Thus, treating, via integration, μ as a linear functional from $\mathcal{B}_b(J)$ to $\mathbb{R}$, the formula

$$\hat{\mathcal{B}}_b(E_A) \ni g \circ \hat{\pi} \longmapsto \hat{\mu}(g \circ \hat{\pi}) := \mu(g) \in \mathbb{R},$$

defines a positive linear functional from $\hat{\mathcal{B}}_b(E_A)$ to $\mathbb{R}$. Since, by Lemma 4.11 applied to the map f being equal to $\hat{\pi} : E_A \to \mathbb{R}$, for every $h \in \hat{\mathcal{B}}_b(E_A)$, the function $h_* \circ \hat{\pi} : E_A \to \mathbb{R}$ belongs to $\hat{\mathcal{B}}_b(E_A)$, and since $h - h_* \circ \hat{\pi} \geq 0$, meaning that $h - h_* \circ \hat{\pi} \in \mathcal{B}_b^+(E_A)$, Riesz Extension Theorem produces a positive linear functional $\mu^* : \mathcal{B}_b(E_A) \to \mathbb{R}$ s.t $\mu^*(h) = \hat{\mu}(h)$, for every $h \in \hat{\mathcal{B}}_b(E_A)$. But μ^* restricted to the vector space $C_b(E_A)$ of bounded continuous real-valued functions on E_A, remains linear and positive.

Claim 1^0: If $(g_n)_{n=1}^\infty$ is a monotone decreasing sequence of non-negative functions in $C_b(E_A)$ converging pointwise in E_A to the function identically equal to zero, then $\lim_{n\to\infty} \mu^*(g_n)$ exists and is equal to zero.

Proof. Clearly, $(g_n^*)_{n=1}^\infty$ is a monotone decreasing sequence of non-negative bounded functions that, by Lemma 4.11, all belong to $\mathcal{B}(J)$, thus to $\mathcal{B}_b^+(J)$. Fix $y \in J$. Since our map $T : J \to J$ is of compact type, the set $\hat{\pi}^{-1}(y) \subset E_A$ is compact. Therefore Dini's Theorem applies to let us conclude that the sequence $\left(g_n|_{\hat{\pi}^{-1}(y)}\right)_{n=1}^\infty$ converges uniformly to zero. Since all these functions are non-negative, this just means that the sequence $(g_n^*)_{n=1}^\infty$ converges to zero. In conclusion $(g_n^*)_{n=1}^\infty$ is a monotone decreasing sequence of functions in $\mathcal{B}_b^+(J)$ converging pointwise to zero. Therefore, as also $g_n \le g_n^* \circ \hat\pi$, we get

$$0 \le \varlimsup_{n\to\infty} \mu^*(g_n) \le \varlimsup_{n\to\infty} \mu^*(g_n^* \circ \hat\pi) = \varlimsup_{n\to\infty} \hat\mu(g_n^* \circ \hat\pi) = \varlimsup_{n\to\infty} \mu(g_n^*) = 0.$$

So, $\lim_{n\to\infty} \mu^*(g_n)$ exists and is equal to zero. The proof of Claim 1^0 is complete. □

Having Claim 1^0, Daniell-Stone Representation Theorem applies to tell us that μ^* extends uniquely from $C_b(E_A)$ to an element of $M(E_A)$. We denote it also by μ^*.

Claim 2^0**:** For every $\varepsilon > 0$ there exists K_ε, a compact subset of E_A such that $\hat\pi^{-1}(\hat\pi(K_\varepsilon)) = K_\varepsilon$ and $\mu(\hat\pi(K_\varepsilon)) \ge 1 - \frac{\varepsilon}{2}$.

Proof. Fix $k \in \mathbb{Z}$ and let $p_k : E^{+-} \to E$ the canonical projection on the kth coordinate, i.e. $p_k\big((\gamma_n)_{n=-\infty}^\infty\big) = \gamma_k$. Fix $\varepsilon > 0$. In the sequel we will assume without loss of generality that $E = \{1, 2, \dots\}$. Since the map $T : J \to J$ is of compact type, each set $\hat\pi^{-1}(y) \subset E_A$, $y \in J$, is compact, and consequently, the function $p_k^* : J \to \overline{\mathbb{R}}$, defined in Lemma 4.11, takes values in $\mathbb{R}$. So $p_k^* : J \to E$ is Borel measurable by Lemma 4.11; thus $p_k^* \circ \hat\pi : E_A \to \mathbb{N}$ is also Borel measurable. So there exists $n_k \ge 1$ such that

$$\mu\big((p_k^*)^{-1}([n_k+1, \infty))\big) < 2^{-|k|-4}\varepsilon. \tag{4.9}$$

Since μ is inner regular, by Lusin's Theorem, Borel measurability of the function $p_k^* : J \to \mathbb{N}$ yields the existence of closed subsets $J_k \subset J$ such that $\mu(J_k) \ge 1 - \varepsilon 2^{-|k|-4}$ and $p_k^*|_{J_k} : J_k \to \mathbb{N}$ is continuous. Define $J_\infty := \bigcap_{k\in\mathbb{Z}} J_k$. Then J_∞ is a closed subset of J, and

$$\mu(J_\infty) \ge 1 - \frac{\varepsilon}{4}, \tag{4.10}$$

and each map $p_k^*|_{J_\infty} : J_\infty \to \mathbb{N}$ is continuous. Define also

$$K_\varepsilon := \bigcap_{k\in\mathbb{Z}} \big(p_k^*|_{J_\infty} \circ \hat\pi|_{\hat\pi^{-1}(J_\infty)}\big)^{-1}([1, n_k])$$

By the definition of the maps p_k^* we have that

$$\hat\pi^{-1}(\hat\pi(K_\varepsilon)) = K_\varepsilon, \text{ and } \hat\pi(K_\varepsilon) = J_\infty \cap \bigcap_{k\in\mathbb{Z}} (p_k^*)^{-1}([1, n_k]) \tag{4.11}$$

Therefore, utilizing (4.10) and (4.9), we get

$$\mu(J \setminus \hat\pi(K_\varepsilon)) \le \mu(J \setminus J_\infty) + \sum_{k\in\mathbb{Z}} \mu\big((p_k^*)^{-1}([n_k+1, \infty))\big) \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \tag{4.12}$$

Since all the maps $p_k^*|_{J_\infty}$, $k \in \mathbb{Z}$, are continuous, K_ε is a closed subset of E_A. Since also $K_\varepsilon \subset \prod_{k\in\mathbb{Z}} [1, n_k]$ and this Cartesian product is compact, we conclude that K_ε is compact. Along with (4.11) and (4.12) this completes the proof of Claim 2^0. □

Using that μ is T-invariant, and Urysohn's Approximation Method, we prove,

Claim 3^0**:** If $\varepsilon > 0$ and $K_\varepsilon \subset E_A$ is the compact set produced in Claim 2^0, then $\mu^* \circ \sigma^{-j}(K_\varepsilon) \ge 1 - \varepsilon$, for all integers $j \ge 0$.

Proof. Fix $\varepsilon > 0$ arbitrary. Fix an integer $j \geq 0$. Since measure $\mu^* \circ \sigma^{-j} \circ \hat{\pi}^{-1}$ is outer regular and $\hat{\pi}(K_\varepsilon)$ is a Borel (since compact) set, there exists an open set $U \subset J$ such that

$$\hat{\pi}(K_\varepsilon) \subset U \quad \text{and} \quad \mu^* \circ \sigma^{-j} \circ \hat{\pi}^{-1}\big(U \setminus \hat{\pi}(K_\varepsilon)\big) \leq \varepsilon/2$$

Now, Urysohn's Lemma produces a continuous function $u : J \to [0,1]$ such that $u(\hat{\pi}(K_\varepsilon)) = \{1\}$ and $u(E_A \setminus U) \subset \{0\}$. Then, by our construction of μ^* and by Claim 2^0,

$$\begin{aligned}\mu^* \circ \sigma^{-j}(K_\varepsilon) &= \mu^* \circ \sigma^{-j} \circ \hat{\pi}^{-1}\big(\hat{\pi}(K_\varepsilon)\big) \geq \mu^* \circ \sigma^{-j} \circ \hat{\pi}^{-1}(U) - \frac{\varepsilon}{2} = \mu^*(1\!\!1_U \circ \hat{\pi} \circ \sigma^j) - \frac{\varepsilon}{2} \\ &= \mu^*(1\!\!1_U \circ T^j) - \frac{\varepsilon}{2} \geq \mu^*(u \circ T^j \circ \hat{\pi}) - \frac{\varepsilon}{2} = \mu(u) - \frac{\varepsilon}{2} \geq \mu(\hat{\pi}(K_\varepsilon)) - \frac{\varepsilon}{2} \geq 1 - \varepsilon\end{aligned}$$

$\square$

Now, for every $n \geq 1$ set

$$\mu_n^* := \frac{1}{n} \sum_{j=0}^{n-1} \mu^* \circ \sigma^{-j}.$$

It directly follows from Claim 3^0 that $\mu_n^*(K_\varepsilon) \geq 1 - \varepsilon$, for every $\varepsilon > 0$ and all $n \geq 1$. Also, since, by Claim 2^0, each set K_ε is compact, the sequence of measures $(\mu_n^*)_{n=1}^\infty$ is tight with respect to the weak topology on $M_\sigma(E_A)$. There thus exists $(n_k)_{k=1}^\infty$, an increasing sequence of positive integers such that $(\mu_{n_k}^*)_{k=1}^\infty$ converges weakly, and denote its limit by $\nu \in M(E_A)$. A standard argument shows that $\nu \in M_\sigma(E_A)$. By the definitions of $\hat{\mu}$ and μ^*, we get for every $g \in C_b(E_A)$, and every $n \geq 1$, that

$$\begin{aligned}\mu_n^* \circ \hat{\pi}^{-1}(g) &= \mu_n^*(g \circ \hat{\pi}) = \frac{1}{n} \sum_{j=0}^{n-1} \mu^* \circ \sigma^{-j}(g \circ \hat{\pi}) = \frac{1}{n} \sum_{j=0}^{n-1} \mu^*(g \circ \hat{\pi} \circ \sigma^j) = \frac{1}{n} \sum_{j=0}^{n-1} \mu^*\big((g \circ T^j) \circ \hat{\pi}\big) \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \hat{\mu}\big((g \circ T^j) \circ \hat{\pi}\big) = \frac{1}{n} \sum_{j=0}^{n-1} \mu(g \circ T^j) = \frac{1}{n} \sum_{j=0}^{n-1} \mu(g) = \mu(g)\end{aligned}$$

So $\mu_n^* \circ \hat{\pi}^{-1} = \mu$ for every $n \geq 1$, hence, $\nu \circ \hat{\pi}^{-1} = \lim_{k\to\infty} \mu_{n_k}^* \circ \hat{\pi}^{-1} = \lim_{k\to\infty} \mu_{n_k}^* \circ \hat{\pi}^{-1} = \mu$, which finishes the proof of the Theorem. $\square$

Observation 4.13. *If T is a Smale endomorphism (no additional hypotheses) and $\mu \in M_\sigma(E_A)$, then* $\mathrm{h}_{\mu\circ\hat{\pi}^{-1}}(\mathrm{T}) = \mathrm{h}_\mu(\sigma)$.

Proof. We have two standard inequalities $\mathrm{h}_{\mu\circ\hat{\pi}^{-1}}(\mathrm{T}) \leq \mathrm{h}_\mu(\sigma)$, and $\mathrm{h}_{\mu\circ\hat{\pi}^{-1}\circ\pi_0^{-1}}(\sigma) \leq \mathrm{h}_{\mu\circ\hat{\pi}^{-1}}(\mathrm{T})$. But $\pi_0 : E_A \to E_A^+$, $\pi_0(\tau) = \tau|_0^\infty$ is the canonical projection from E_A to E_A^+. So, the measure $\mu \in M_\sigma(E_A)$ is the Rokhlin's natural extension of the measure $\mu \circ \hat{\pi}^{-1} \circ \pi_0^{-1} \in M_\sigma(E_A^+)$. Hence, $\mathrm{h}_{\mu\circ\hat{\pi}^{-1}\circ\pi_0^{-1}}(\sigma) = \mathrm{h}_\mu(\sigma)$. So from the above inequalities, $\mathrm{h}_{\mu\circ\hat{\pi}^{-1}}(\mathrm{T}) = \mathrm{h}_\mu(\sigma)$.

$\square$

Now, we define the topological pressure of continuous real-valued functions on J with respect to the dynamical system $T : J \to J$. Since the space J is *not compact*, there is no canonical candidate for such definition and we choose the one which will turn out to behave well on the theoretical level (variational principle) and serves well for practical purposes (Bowen's formula). For every finite admissible word $\omega \in E_A^{+*}$ let

$$[\omega]_T =: \hat{\pi}_2([\omega]) \subset J.$$

If $\psi : J \to \mathbb{R}$ is a continuous function, we define

$$\mathrm{P}(\psi) = \mathrm{P}_T(\psi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in C^{n-1}} \exp\bigl(\sup(S_n\psi|_{[\omega]_T}\bigr),$$

where $S_n\psi = \sum_{j=0}^{n-1} \psi \circ T^j, \quad n \ge 1$. The limit above exists since the sequence $\mathbb{N} \ni n \longmapsto \log \sum_{\omega\in C^{n-1}} \exp\bigl(\sup(S_n\psi|_{[\omega]_T}\bigr)$ is subadditive. We call $\mathrm{P}_T(\psi)$ the *topological pressure* of the potential $\psi : J \to \mathbb{R}$ with respect to the dynamical system $T : J \to J$. As an immediate consequence of this definition and Definition 3.1, we get the following.

Observation 4.14. *If $\psi : J \to \mathbb{R}$ is a continuous function, then*

$$\mathrm{P}_T(\psi) = \mathrm{P}_\sigma(\psi \circ \hat{\pi}).$$

The following theorem follows immediately from Theorem 3.9, Observation 4.14, Theorem 4.12, and Observation 4.13, and we will provide such proof.

Theorem 4.15. *If $\psi : J \to \mathbb{R}$ is a continuous function, and $\mu \in M_T(J)$ is such that $\psi \in L^1(J,\mu)$ and $\int \psi\, d\mu > -\infty$, then $\mathrm{h}_\mu(\mathrm{T}) + \int_\mathrm{J} \psi\, \mathrm{d}\mu \le \mathrm{P}_\mathrm{T}(\psi)$.*

Proof. By Theorem 4.12 there exists $\nu \in M_\sigma(E_A)$ such that $\nu \circ \hat{\pi}^{-1} = \mu$. The other theorems listed immediately above give: $\mathrm{h}_\mu(\mathrm{T}) + \int_\mathrm{J} \psi\, \mathrm{d}\mu = \mathrm{h}_{\nu\circ\hat{\pi}^{-1}}(\mathrm{T}) + \int_\mathrm{J} \psi\, \mathrm{d}(\nu \circ \hat{\pi}^{-1}) = \mathrm{h}_\nu(\sigma) + \int_{\mathrm{E_A}} \psi \circ \hat{\pi}\, \mathrm{d}\hat{\nu} \le \mathrm{P}_\sigma(\psi \circ \hat{\pi}) = \mathrm{P}_\mathrm{T}(\psi)$. □

We have the following two definitions.

Definition 4.16. *The measure $\mu \in M_T(J)$ is called an equilibrium state of the continuous potential $\psi : \hat{Y} \to \mathbb{R}$, if $\int \psi\, d\mu > -\infty$ and $\mathrm{h}_\mu(\mathrm{T}) + \int_\mathrm{J} \psi\, \mathrm{d}\mu = \mathrm{P}_\mathrm{T}(\psi)$.*

Definition 4.17. *The potential $\psi : J \to \mathbb{R}$ is called summable if*

$$\sum_{e\in E} \exp\bigl(\sup(\psi|_{[e]_T})\bigr) < \infty.$$

Observation 4.18. *$\psi : J \to \mathbb{R}$ is summable if and only if $\psi \circ \hat{\pi} : E_A \to \mathbb{R}$ is summable.*

Definition 4.19. *We call a continuous skew product Smale endomorphism $T : \hat{Y} \to \hat{Y}$ Hölder, if the projection $\hat{\pi} : E_A \to J$ is Hölder continuous.*

We now establish an important property of Hölder skew product Smale endomorphisms of compact type, and then will describe a general construction of such endomorphisms.

Theorem 4.20. *If $T : J \to J$ is Hölder skew product Smale endomorphism of compact type and $\psi : J \to \mathbb{R}$ is a Hölder summable potential, then ψ admits a unique equilibrium state, denoted by μ_ψ. In addition $\mu_\psi = \mu_{\psi\circ\hat{\pi}} \circ \hat{\pi}^{-1}$, where $\mu_{\psi\circ\hat{\pi}}$ is the unique equilibrium state of $\psi \circ \hat{\pi} : E_A \to \mathbb{R}$ with respect to $\sigma : E_A \to E_A$.*

Proof. $\psi \circ \hat{\pi} : E_A \to \mathbb{R}$ is a summable Hölder continuous potential, so it has a unique equilibrium state $\mu_{\psi\circ\hat{\pi}}$ by Theorem 2.6. By Observation 4.14 and Observation 4.4 we have

$$\mathrm{h}_\mathrm{T}\bigl(\mu_{\psi\circ\hat{\pi}} \circ \hat{\pi}^{-1}\bigr) + \int_\mathrm{J} \psi\, \mathrm{d}\bigl(\mu_{\psi\circ\hat{\pi}} \circ \hat{\pi}^{-1}\bigr) = \mathrm{h}_\sigma(\mu_{\psi\circ\hat{\pi}}) + \int_{\mathrm{E_A}} \psi \circ \hat{\pi}\, \mathrm{d}(\mu_{\psi\circ\hat{\pi}}) = \mathrm{P}_\sigma(\psi \circ \hat{\pi}). = \mathrm{P}_\mathrm{T}(\psi)$$

So we have to show that if μ is an equilibrium state of ψ, then $\mu = \mu_{\psi\circ\hat{\pi}} \circ \hat{\pi}^{-1}$. Assume that μ is such equilibrium. It then follows from Theorem 4.12 that $\mu = \nu \circ \hat{\pi}^{-1}$ for some $\nu \in M_\sigma(E_A)$. But then by Observation 4.14,

$$\mathrm{h}_\nu(\sigma) + \int_{\mathrm{E_A}} \psi\circ\hat{\pi}\,\mathrm{d}\nu \ge \mathrm{h}_{\nu\circ\hat{\pi}^{-1}}(\mathrm{T}) + \int_{\mathrm{J}} \psi\,\mathrm{d}\big(\nu\circ\hat{\pi}^{-1}\big) = \mathrm{h}_\mu(\mathrm{T}) + \int_{\mathrm{J}} \psi\,\mathrm{d}\mu = \mathrm{P_T}(\psi) = \mathrm{P}_\sigma(\psi\circ\hat{\pi}).$$

Hence, ν is an equilibrium state of the potential $\psi\circ\hat{\pi} : E_A \to \mathbb{R}$ and the dynamical system $\sigma : E_A \to E_A$. Thus $\nu = \mu_{\psi\circ\hat{\pi}}$ (see Theorem 2.6).

□

Now we provide the promised construction of Hölder Smale skew product endomorphisms. Start with (Y, d), a complete bounded metric space, and assume given for every $\omega \in E_A^+$ a continuous closed injective map $T_\omega : Y \to Y$, satisfying the following conditions

$$d(T_\omega(y_2), T_\omega(y_1)) \le \lambda^{-1} d(y_2, y_1), \tag{4.13}$$

for all $y_1, y_2 \in Y$ and some $\lambda > 1$ independent of ω,

$$d_\infty(T_\beta, T_\alpha) := \sup\big\{d(T_\beta(\xi), T_\alpha(\xi)) : \xi \in Y\big\} \le C d_\kappa(\beta, \alpha) \tag{4.14}$$

with some constants $C \in (0, \infty)$, $\kappa > 0$, and all $\alpha, \beta \in E_A^+$. Then

$$\hat{Y} = E_A^+ \times Y,$$

and call $T : \hat{Y} \to \hat{Y}$ a skew product Smale system of *global character*. We may assume without loss of generality that

$$\kappa \le \frac{1}{2}\log\lambda. \tag{4.15}$$

Theorem 4.21. *Each skew product Smale system of global character is Hölder.*

Proof. Let $T : E_A^+ \times Y \to E_A^+ \times Y$ be such skew product Smale system. We first show that $T : E_A^+ \times Y \to E_A^+ \times Y$ is continuous. Enough to show that $p_2 \circ T : E_A^+ \times Y \to Y$ is continuous, with p_2 the projection to second coordinate. For all $\alpha, \beta \in E_\omega^+$ and $z, w \in Y$,

$$\begin{aligned} d(p_2\circ T(\alpha, z), p_2\circ T(\beta, w)) &= d(T_\alpha(z), T_\beta(w)) \le d(T_\alpha(z), T_\beta(z)) + d(T_\beta(z), T_\beta(w)) \\ &\le d_\infty(T_\alpha, T_\beta) + \lambda^{-1} d(z, w) \le C d_\kappa(\alpha, \beta) + \lambda^{-1} d(z, w), \end{aligned}$$

and continuity of the map $p_2\circ T : E_A^+ \times Y \to Y$ is proved. So the continuity of $T : E_A^+ \times Y \to E_A^+ \times Y$ is proved, and thus $T : J \to J$ is continuous too. We now show that $T : J \to J$ is Hölder. So, fix an integer $n \ge 0$, two words $\alpha, \beta \in E_A$ and $\xi \in Y$. We then have

$$\begin{aligned} &d\big(T_\alpha^{n+1}(\xi), T_\beta^{n+1}(\xi)\big) = d\left(T_\alpha^n\big(T_{\alpha|_{-(n+1)}^\infty}(\xi)\big), T_\beta^n\big(T_{\beta|_{-(n+1)}^\infty}(\xi)\big)\right) \\ &\le d\left(T_\alpha^n\big(T_{\alpha|_{-(n+1)}^\infty}(\xi)\big), T_\alpha^n\big(T_{\beta|_{-(n+1)}^\infty}(\xi)\big)\right) + d\left(T_\alpha^n\big(T_{\beta|_{-(n+1)}^\infty}(\xi)\big), T_\beta^n\big(T_{\beta|_{-(n+1)}^\infty}(\xi)\big)\right) \\ &\le \lambda^{-n} d\big(T_{\alpha|_{-(n+1)}^\infty}(\xi)), T_{\beta|_{-(n+1)}^\infty}(\xi)\big) + d_\infty\big(T_\alpha^n, T_\beta^n\big) \le \lambda^{-n} C d_\kappa\big(\alpha|_{-(n+1)}^\infty, \beta|_{-(n+1)}^\infty\big) + d_\infty\big(T_\alpha^n, T_\beta^n\big) \end{aligned} \tag{4.16}$$

Let $p \ge -1$ be uniquely determined by the property that

$$d_\kappa(\alpha, \beta) = e^{-\kappa p}. \tag{4.17}$$

Consider two cases. First assume that $d_\kappa(\alpha,\beta)\ge e^{-\kappa n}$. Then using also (4.15), we get

$$\lambda^{-n}d_\kappa\big(\alpha|_{-(n+1)}^\infty,\beta|_{-(n+1)}^\infty\big)\le e^{-2\kappa n}\le e^{-\kappa n}d_\kappa(\alpha,\beta). \tag{4.18}$$

So, assume that

$$d_\kappa(\alpha,\beta)<e^{-\kappa n}. \tag{4.19}$$

Then $n<p$, so $n+1\le p$, whence $d_\kappa\big(\alpha|_{-(n+1)}^\infty,\beta|_{-(n+1)}^\infty\big)=\exp\big(-\kappa((n+1)+1+p)\big)=e^{-\kappa(n+2)}e^{-\kappa p}=e^{-\kappa(n+2)}d_\kappa(\alpha,\beta)\le e^{-\kappa n}d_\kappa(\alpha,\beta)$. Hence, $\lambda^{-n}d_\kappa\big(\alpha|_{-(n+1)}^\infty,\beta|_{-(n+1)}^\infty\big)\le e^{-\kappa n}d_\kappa(\alpha,\beta)$. Inserting this and (4.18) to (4.16) in either case yields $d\big(T_\alpha^{n+1}(\xi),T_\beta^{n+1}(\xi)\big)\le d_\infty\big(T_\alpha^n,T_\beta^n\big)+Ce^{-\kappa n}d_\kappa(\alpha,\beta)$. Taking supremum over all $\xi\in Y$, we get $d_\infty\big(T_\alpha^{n+1},T_\beta^{n+1}\big)\le d_\infty\big(T_\alpha^n,T_\beta^n\big)+Ce^{-\kappa n}d_\kappa(\alpha,\beta)$. Thus, by induction

$$d_\infty\big(T_\alpha^n,T_\beta^n\big)\le Cd_\kappa(\alpha,\beta)\sum_{j=0}^{n-1}e^{-\kappa n}\le Cd_\kappa(\alpha,\beta)\sum_{j=0}^{\infty}e^{-\kappa n}=C(1-e^{-\kappa})^{-1}d_\kappa(\alpha,\beta) \tag{4.20}$$

for all $\alpha,\beta\in E_A$ and all integers $n\ge 0$. Recall that the integer $p\ge -1$ is determined by (4.17). Assume that $p\ge 0$. Then using (4.20), (4.19), and (4.2), we get

$$\begin{aligned}&d(\hat\pi_2(\alpha),(\hat\pi_2(\alpha))\le \operatorname{diam}\big(T_\alpha^p(Y)\big)+\operatorname{diam}\big(T_\beta^p(Y)\big)+d_\infty\big(T_\alpha^p,T_\beta^p\big)\\ &\le\lambda^{-p}\operatorname{diam}(Y)+\lambda^{-p}\operatorname{diam}(Y)+\frac{C}{1-e^{-\kappa}}d_\kappa(\alpha,\beta)\le 2\operatorname{diam}(Y)d_\kappa^{\frac{\log\lambda}{\kappa}}(\alpha,\beta)+\frac{C}{1-e^{-\kappa}}d_\kappa(\alpha,\beta)\end{aligned}$$

As d is a bounded metric and $d_\kappa(\alpha,\beta)=e^\kappa$ if $p=-1$, we get that $\hat\pi_2:E_A\to Y$ is Hölder continuous, so $\hat\pi:E_A\to Y$ is Hölder continuous. □

5. Conformal Skew Product Smale Endomorphisms

In this section we keep the setting of skew product Smale endomorphisms. However we assume more about the spaces Y_ω, $\omega\in E_A^+$, and the fiber maps $T_\omega:Y_\omega\to Y_{\sigma(\omega)}$, namely:

(a) Y_ω is a closed bounded subset of $\mathbb{R}^d$, with some $d\ge 1$ such that $\overline{\operatorname{Int}(Y_\omega)}=Y_\omega$.

(b) Each map $T_\omega:Y_\omega\to Y_{\sigma(\omega)}$ extends to a C^1 conformal embedding from Y_ω^* to $Y_{\sigma(\omega)}^*$, where Y_ω^* is a bounded connected open subset of $\mathbb{R}^d$ containing Y_ω. The same symbol T_ω denotes this extension and we assume that $T_\omega:Y_\omega^*\to Y_{\sigma(\omega)}^*$ satisfy:

(c) Formula (4.1) holds for all $y_1,y_2\in Y_\omega^*$, perhaps with some smaller constant $\lambda>1$.

(d) (Bounded Distortion Property 1) There exist constants $\alpha>0$ and $H>0$ such that for all $y,z\in Y_\omega^*$ we have that: $\big|\log|T_\omega'(y)|-\log|T_\omega'(z)|\big|\le H||y-z||^\alpha$.

(e) The function $E_A\ni\tau\longmapsto\log|T_\tau'(\hat\pi_2(\omega))|\in\mathbb{R}$ is Hölder continuous.

(f) (Open Set Condition) For every $\omega\in E_A^+$ and for all $a,b\in E$ with $A_{a\omega_0}=A_{b\omega_0}=1$ and $a\ne b$, we have $T_{a\omega}(\operatorname{Int}(Y_{a\omega}))\cap T_{b\omega}(\operatorname{Int}(Y_{b\omega}))=\emptyset$.

(g) (Strong Open Set Condition) There exists a measurable function $\delta:E_A^+\to(0,\infty)$ such that for every $\omega\in E_A^+$, $J_\omega\cap\big(Y_\omega\setminus\overline{B}(Y_\omega^c,\delta(\omega)\big)\ne\emptyset$.

Any skew product Smale endomorphism satisfying conditions (a)–(g) will be called in the sequel a *conformal skew product Smale endomorphism*.

Remark 5.1. *The Bounded Distortion Property 1, i.e (d), is always automatically satisfied if $d \geq 2$. If $d = 2$, this is so because of Koebe's Distortion Theorem and because each conformal map in $\mathbb{C}$ is either holomorphic or antiholomorphic. If $d \geq 3$ this follows from Liouville's Representation Theorem asserting that each conformal map in $\mathbb{R}^d$, $d \geq 3$, is either a Möbius transformation or similarity, see [8] for details.*

A standard calculation based on (c), (d), and (e), yields in fact the following.

(BDP2) (Bounded Distortion Property 2) For some constant H, we have that

$$\Big|\log\big|\big(T_\tau^n\big)'(y)\big| - \log\big|\big(T_\tau^n\big)'(z)\big|\Big| \leq H||y-z||^\alpha.$$

for all $\tau \in E_A$, $y, z \in Y^*_{\tau|_{-n}^{\infty}}$, and all $n > 0$.

An immediate consequence of (BDP2) is the following version.

(BDP3) (Bounded Distortion Property 3) For all $\tau \in E_A$, all $n \geq 0$, and all $y, z \in Y^*_{\tau|_{-n}^{\infty}}$, if $K := \exp\big(H\mathrm{diam}^\alpha(Y)\big)$, then we have that

$$K^{-1} \leq \frac{\big|\big(T_\tau^n\big)'(y)\big|}{\big|\big(T_\tau^n\big)'(z)\big|} \leq K$$

Recall also that we say that a conformal skew product Smale endomorphism is *Hölder*, if the condition of Hölder continuity for $\hat{\pi} : E_A \to J$ is satisfied, see Definition 4.19.

Remark 5.2. *Note that condition (e) is satisfied for instance if $T : \hat{Y} \to \hat{Y}$ is of global character (then by Theorem 4.21, it is Hölder) and if in addition*

$$||T'_\alpha - T'_\beta||_\infty \leq Cd_\kappa(\alpha, \beta) \tag{5.1}$$

for all $\alpha, \beta \in E_A^+$. Actually if the conformal endomorphism $T : \hat{Y} \to \hat{Y}$ is of global character, then (5.1) also automatically follows in all dimensions $d \geq 2$. For $d = 2$ this is just Cauchy's Formula for holomorphic functions, and for $d \geq 3$ it would follow from the Liouville's Representation Theorem, although in this case the proof is not straightforward.

As an immediate consequence of the Open Set Condition (f) we get the following.

Lemma 5.3. *Let $T : \hat{Y} \to \hat{Y}$ a conformal skew product Smale endomorphism. If $n \geq 1$, $\alpha, \beta \in E_A(-n, \infty)$, $\alpha|_0^\infty = \beta|_0^\infty$, and $\alpha|_{-n}^{-1} \neq \beta|_{-n}^{-1}$, then*

$$T_\alpha^n(\mathrm{Int}(Y_\alpha)) \cap T_\beta^n(\mathrm{Int}(Y_\beta)) = \emptyset.$$

In fact we have more: $T_\alpha^n(\mathrm{Int}(Y_\alpha)) \cap T_\beta^n(Y_\beta) = \emptyset = T_\alpha^n(Y_\alpha) \cap T_\beta^n(\mathrm{Int}(Y_\beta))$.

Lemma 5.4. *Let $T : \hat{Y} \to \hat{Y}$ be a conformal skew product Smale endomorphism. If $n \geq 1$ and $\tau \in E_A(-n, \infty)$, then $\hat{\pi}_2^{-1}\big(T_\tau^n(\mathrm{Int}(Y_\tau))\big) \subset [\tau]$.*

Proof. Let $\gamma \in \hat{\pi}_2^{-1}\big(T_\tau^n(\mathrm{Int}(Y_\tau))\big)$, hence $\gamma|_0^\infty = \tau|_0^\infty$ and $\hat{\pi}_2(\gamma) \in T_\tau^n(\mathrm{Int}(Y_\tau)) \subset Y_{\tau|_0^\infty}$. Also, $\hat{\pi}_2(\gamma) \in T^n_{\gamma|_{-n}^\infty}\big(Y_{\gamma|_{-n}^\infty}\big)$. From Lemma 5.3 it follows that $\gamma|_{-n}^0 = \tau$, so $\gamma \in [\tau]$. □

We will also use the following:

(h) (Uniform Geometry Condition) $\exists(R > 0)\ \forall(\omega \in E_A^+)\ \exists(\xi_\omega \in Y_\omega)\ B(\xi_\omega, R) \subset Y_\omega$.
The primary significance of Uniform Geometry Condition (h) lies in:

Lemma 5.5. *If $T : \hat{Y} \to \hat{Y}$ is a Hölder conformal skew product Smale endomorphism satisfying Uniform Geometry Condition (h), then for every $\gamma \geq 1$, $\exists\, \Gamma_\gamma > 0$ such that:*

If $\mathcal{F} \subset E_A^*(-\infty, -1)$ is a collection of mutually incomparable (finite) words, so that $A_{\tau_{-1}\omega_0} = 1$ for some $\omega \in E_A^*$ and all $\tau \in \mathcal{F}$, and so that for some $\xi \in Y_\omega$,

$$T_{\tau\omega}^{|\tau|}(Y_{\tau\omega}) \cap B(\xi, r) \neq \emptyset \quad \text{with } \gamma^{-1} r \leq \operatorname{diam}\big(T_{\tau\omega}^{|\tau|}(Y_{\tau\omega})\big) \leq \gamma r,$$

then the cardinality of $\mathcal{F}$ is bounded above by Γ_γ.

Proof. The family $\{T_{\tau\omega}^{|\tau|}(\mathrm{Int}(Y_{\omega\tau})) : \tau \in \mathcal{F}\}$ consists of mutually disjoint sets in Y_ω. We get $T_{\tau\omega}^{|\tau|}(\mathrm{Int}(Y_{\tau\omega})) \supset T_{\tau\omega}^{|\tau|}(B(\xi_{\tau\omega}, R)) \supset B\big(T_{\tau\omega}^{|\tau|}(\xi_{\tau\omega}, K^{-1}R\big|\big(T_{\tau\omega}^{|\tau|}\big)'(\xi_{\tau\omega})\big|\big) \supset B\big(T_{\tau\omega}(\xi_{\tau\omega}), K^{-2}R\gamma^{-1}r\big)$, from the Uniform Geometry condition. Also $T_{\tau\omega}^{|\tau|}(\mathrm{Int}(Y_{\omega\tau})) \subset B(\xi, (1+\gamma)r)$. □

6. Volume Lemmas

We keep the setting of Section 5, with $T : \hat{Y} \to \hat{Y}$ a conformal skew product Smale endomorphism, i.e. satisfying conditions (a)–(g) of Section 5. We emphasize that Uniform Geometry Condition (h) is not required in this section; it will be used in the next one.

If μ is a Borel probability σ-invariant measure on E_A, then by $\chi_\mu(\sigma)$ we denote its *Lyapunov exponent*, defined by the formula

$$\chi_\mu(\sigma) := -\int_{E_A} \log\Big|T'_{\tau|_0^\infty}\big(\hat{\pi}_2(\tau)\big)\Big|\, d\mu(\tau) = -\int_{E_A^+}\int_{[\omega]} \log\big|T'_\omega\big(\hat{\pi}_2(\tau)\big)\big|\, d\overline{\mu}^\omega(\tau)\, dm(\omega),$$

where $m = \mu \circ \pi_0^{-1} = \pi_{1*}\mu$ is the canonical projection of μ onto E_A^+. We shall prove:

Theorem 6.1. *Let $T : \hat{Y} \to \hat{Y}$ be a Hölder conformal skew product Smale endomorphism, and let $\psi : E_A \to \mathbb{R}$ be a Hölder continuous summable potential. Then for the projection $\hat{\pi}_{2*}\overline{\mu}_\psi^\omega = \overline{\mu}_\psi^\omega \circ \hat{\pi}_2^{-1}$, of the conditional measure onto the fiber J_ω, we have that*

$$\mathrm{HD}\big(\overline{\mu}_\psi^\omega \circ \hat{\pi}_2^{-1}\big) = \frac{\mathrm{h}_{\mu_\psi}(\sigma)}{\chi_{\mu_\psi}(\sigma)} = \frac{\mathrm{P}_\sigma(\psi) - \int \psi\, d\mu_\psi}{\chi_{\mu_\psi}(\sigma)}$$

for m_ψ-a.e $\omega \in E_A^+$, where $m_\psi = \mu_\psi \circ \pi_0^{-1}$. Moreover for m_ψ-a.e $\omega \in E_A^+$ the measure $\overline{\mu}_\psi^\omega \circ \hat{\pi}_2^{-1}$ is dimensional exact, and its pointwise dimension is given by:

$$\lim_{r\to 0} \frac{\log \overline{\mu}_\psi^\omega \circ \hat{\pi}_2^{-1}(B, r))}{\log r} = \frac{\mathrm{h}_{\mu_\psi}(\sigma)}{\chi_{\mu_\psi}(\sigma)}, \tag{6.1}$$

for m_ψ-a.e. $\omega \in E_A^+$ and $\overline{\mu}_\psi^\omega \circ \hat{\pi}_2^{-1}$-a.e. $z \in J_\omega$ (and equivalently for $\mu_\psi \circ \hat{\pi}^{-1}$-a.e. $(\omega, z) \in J$).

Proof. We only need to show that (6.1) holds. Since μ_ψ is ergodic, Birkhoff's Ergodic Theorem applied to $\sigma^{-1} : E_A \to E_A$ gives a measurable set $E_{A,\psi} \subset E_A$ s.t $\mu_\psi(E_{A,\psi}) = 1$,

$$\lim_{n\to\infty} \frac{1}{n} \log\big|\big(T_\tau^n\big)'(\hat{\pi}_2(\sigma^{-n}(\tau)))\big| = -\chi_{\mu_\psi}(\sigma), \text{ and } \lim_{n\to\infty} \frac{1}{n} S_n\psi(\sigma^{-n}(\tau)) = \int_{E_A} \psi\, d\mu_\psi \tag{6.2}$$

for every $\tau \in E_{A,\psi}$. For arbitrary $\omega \in E_A^+$ denote now:

$$\nu_\omega := \overline{\mu}_\psi^\omega \circ \hat{\pi}_2^{-1},$$

which is a Borel probability measure on J_ω. Fix $\tau \in E_{A,\psi}$. Fix a radius $r \in \big(0, \operatorname{diam}\big(Y_{p_2(\tau)}\big)/2\big)$. Let $z = \hat{\pi}_2(\tau)$, and consider the least integer $n = n(z,r) \geq 0$ so that

$$T_\tau^n\big(Y_{\tau|_{-n}^\infty}\big) \subset B(z,r). \tag{6.3}$$

If $r > 0$ is small enough (depending on τ), then $n \geq 1$ and $T_\tau^{n-1}\big(Y_{\tau|_{-(n-1)}^\infty}\big) \not\subset B(z,r)$. Since $z \in T_\tau^{n-1}\big(Y_{\tau|_{-(n-1)}^\infty}\big)$, this implies that

$$\operatorname{diam}\left(T_\tau^{n-1}\big(Y_{\tau|_{-(n-1)}^\infty}\big)\right) \geq r. \tag{6.4}$$

Write $\omega := \tau|_0^\infty$. It follows from (6.3), Lemma 4.9, and Theorem 3.11 that

$$\begin{aligned}\nu_\omega(B(z,r)) &\geq \nu_\omega\big(\hat{\pi}_2\big([\tau|_{-n}^\infty]\big)\big) = \overline{\mu}_\psi^\omega \circ \hat{\pi}_2^{-1}\big(\hat{\pi}_2\big([\tau|_{-n}^\infty]\big)\big) \geq \overline{\mu}_\psi^\omega\big([\tau|_{-n}^\infty]\big) \\ &\geq D^{-1}\exp\big(S_n\psi(\sigma^{-n}(\tau)) - \mathrm{P}_\sigma(\psi)n\big).\end{aligned} \tag{6.5}$$

By taking logarithms and using (6.4), this gives that

$$\frac{\log \nu_\omega(B(z,r))}{\log r} \leq \frac{-\log D + S_n\psi(\sigma^{-n}(\tau)) - \mathrm{P}_\sigma(\psi)n}{\log\left(\operatorname{diam}\left(T_\tau^{n-1}\big(Y_{\tau|_{-(n-1)}^\infty}\big)\right)\right)}$$

So applying (BDP3), we get that

$$\frac{\log \nu_\omega(B(z,r))}{\log r} \leq \frac{-\log D + S_n\psi(\sigma^{-n}(\tau)) - \mathrm{P}_\sigma(\psi)n}{\log K + \log\left(\operatorname{diam}\left(Y_{\tau|_{-(n-1)}^\infty}\right)\right) + \log\left|\left(T_\tau^{n-1}\right)'\left(\hat{\pi}_2\big(\sigma^{-n}(\tau)\big)\right)\right|}$$

so by dividing both numerator and denominator by n, and using that $\operatorname{diam}\left(Y_{\tau|_{-(n-1)}^\infty}\right) = \operatorname{diam}(Y)$ and (6.2), this yields,

$$\varlimsup_{r\to 0}\frac{\log \nu_\omega(B(z,r))}{\log r} \leq \frac{\lim_{n\to\infty}\frac{1}{n}S_n\psi(\sigma^{-n}(\tau)) - \mathrm{P}_\sigma(\psi)}{\lim_{n\to\infty}\frac{1}{n}\log\left|\left(T_\tau^{n-1}\right)'\left(\hat{\pi}_2\big(\sigma^{-n}(\tau)\big)\right)\right|} = \frac{\mathrm{P}_\sigma(\psi) - \int\psi\, d\mu_\psi}{\chi_{\mu_\psi}(\sigma)}. \tag{6.6}$$

To prove the opposite inequality, note that the set $\hat{\pi}_2^{-1}\big(J_\omega \backslash \overline{B}(Y_\omega^c, \delta(\omega))\big)$ is open in $[\omega] \subset E_A$, it is not empty by (g), and thus

$$\overline{\mu}_\psi^\omega\big(\hat{\pi}_2^{-1}\big(J_\omega \setminus \overline{B}(Y_\omega^c, \delta(\omega))\big)\big) > 0$$

for every $\omega \in E_A^+$. Consequently, $\mu_\psi(Z) > 0$, where $Z := \bigcup_{\omega\in E_A^+}\hat{\pi}_2^{-1}\big(J_\omega \setminus \overline{B}(Y_\omega^c, \delta(\omega))\big)$. Since $\delta : E_A^+ \to (0,\infty)$ is measurable, there exists $R > 0$ s.t $\mu_\psi(Z_R) > 0$, where

$$Z_R := \bigcup_{\omega\in E_A^+}\hat{\pi}_2^{-1}\big(J_\omega \setminus \overline{B}(Y_\omega^c, R)\big)$$

Consider the set $N(\tau) := \{k \geq 0 : \sigma^{-k}(\tau) \in Z_R\}$. Represent this set $N(\tau)$ as a strictly increasing sequence $(k_n(\tau))_{n=1}^\infty$. By Birkhoff's Ergodic Theorem there is a measurable set $\tilde{E}_{A,\psi} \subset E_{A,\psi}$ with $\mu_\psi(\tilde{E}_{A,\psi}) = 1$ and for every $\tau' \in \tilde{E}_{A,\psi}$,

$$\lim_{n\to\infty}\frac{Card\{0 \leq i \leq n,\ \sigma^{-i}(\tau') \in Z_R\}}{n} = \mu_\psi(Z_R)$$

Now we put $k_n(\tau) \geq n$, instead of n above, and notice that $Card\{0 \leq i \leq k_n(\tau),\ \sigma^{-i}(\tau') \in Z_R\} = n$. Therefore as $\mu_\psi(Z_R) > 0$, we obtain for every $\tau \in \tilde{E}_{A,\psi}$ and any n large, that:

$$\lim_{n\to\infty} \frac{k_n(\tau)}{n} = \frac{1}{\mu_\psi(Z_R)}$$

Hence for every $\tau \in \tilde{E}_{A,\psi}$,

$$\lim_{n\to\infty} \frac{k_{n+1}(\tau)}{k_n(\tau)} = 1 \tag{6.7}$$

Fix $\tau \in \tilde{E}_{A,\psi}$, $\omega = \tau|_0^\infty$, and let the largest $n = n(\tau, r) \geq 1$ s.t with $k_j := k_j(\tau)$, $j \geq 1$,

$$K^{-1} \left| \left(T_\tau^{k_n}\right)' \left(\hat{\pi}_2(\sigma^{-k_n}(\tau))\right) \right| R \geq r. \tag{6.8}$$

Then

$$K^{-1} \left| \left(T_\tau^{k_{n+1}}\right)' \left(\hat{\pi}_2(\sigma^{-k_{n+1}}(\tau))\right) \right| R < r. \tag{6.9}$$

It follows from (6.8) and (BDP3) that $B(z,r) \subset T_\tau^{k_n}\left(B(\hat{\pi}_2(\sigma^{-k_n}(\tau)), R)\right) \subset T_\tau^{k_n}\left(\text{Int}\left(Y_{\tau|_{-k_n}^\infty}\right)\right)$. Hence, invoking also Lemma 5.4 and Theorem 3.11, we infer that

$$\nu_\omega(B(z,r)) \leq \overline{\mu}_\psi^\omega\left([\tau|_{-k_n}^\infty]\right) \leq D \exp\left(S_{k_n}\psi(\sigma^{-k_n}(\tau)) - \mathrm{P}_\sigma(\psi)k_n\right).$$

By taking logarithms and using (6.9), this gives

$$\frac{\log \nu_\omega(B(z,r))}{\log r} \geq \frac{\log D + S_{k_n}\psi(\sigma^{-k_n}(\tau)) - \mathrm{P}_\sigma(\psi)k_n}{-\log K + \log\left|\left(T_\tau^{k_{n+1}}\right)'\left(\hat{\pi}_2\left(\sigma^{-k_{n+1}}(\tau)\right)\right)\right|}$$

Dividing both numerator and denominator above by k_n, and using (6.2), (6.7), it yields

$$\varliminf_{r\to 0} \frac{\log \nu_\omega(B(z,r))}{\log r} \geq \frac{\lim_{n\to\infty} \frac{1}{k_n} S_{k_n}\psi(\sigma^{-k_n}(\tau)) - \mathrm{P}_\sigma(\psi)}{\lim_{n\to\infty} \frac{1}{k_n}\log\left|\left(T_\tau^{k_{n+1}}\right)'\left(\hat{\pi}_2\left(\sigma^{-k_{n+1}}(\tau)\right)\right)\right|} = \frac{\mathrm{P}_\sigma(\psi) - \int \psi\, d\mu_\psi}{\chi_{\mu_\psi}(\sigma)}.$$

From (6.6), it follows that (6.1) holds for all $\tau \in \tilde{E}_{A,\psi}$.

□

If μ is now a Borel probability T-invariant measure on the fibered limit set J, then by $\chi_\mu(T)$ we denote its *Lyapunov exponent*, which is defined by the formula

$$\chi_\mu(T) := -\int_J \log\left|T_\omega'(z)\right| d\mu(\omega,z) = -\int_{E_A^+}\int_{J_\omega} \log\left|T_\omega'(z)\right| d\overline{\mu}^\omega(z)\, dm(\omega),$$

where $m = \mu \circ \pi_0^{-1}$ is the projection of μ onto E_A^+, and $\left(\overline{\mu}^\omega\right)_{\omega\in E_A^+}$ is the canonical system of conditional measures of μ for the measurable partition $\{\{\omega\}\times J_\omega\}_{\omega\in E_A^+}$. Now we prove

Corollary 6.2. *Let $T : \hat{Y} \to \hat{Y}$ be a Hölder conformal Smale endomorphism of compact type. Let $\psi : J \to \mathbb{R}$ be a Hölder continuous summable potential. Then*

$$\mathrm{HD}\left(\overline{\mu}_\psi^\omega\right) = \frac{\mathrm{h}_{\mu_\psi}(\mathrm{T})}{\chi_{\mu_\psi}(T)} = \frac{\mathrm{P}_T(\psi) - \int \psi\, d\mu_\psi}{\chi_{\mu_\psi}(T)}$$

for m_ψ-a.e. $\omega \in E_A^+$, where $m_\psi = \mu_\psi \circ p_1^{-1}$. Moreover, for m_ψ-a.e. $\omega \in E_A^+$ the measure $\overline{\mu}_\psi^\omega$ is dimensional exact, and for m_ψ-a.e. $\omega \in E_A^+$ and $\overline{\mu}_\psi^\omega$-a.e. $z \in J_\omega$,

$$\lim_{r\to 0} \frac{\log \overline{\mu}_\psi^\omega(B(z,r))}{\log r} = \frac{\mathrm{h}_{\mu_\psi}(\mathrm{T})}{\chi_{\mu_\psi}(T)} \tag{6.10}$$

Proof. Let $\hat{\psi} := \psi \circ \hat{\pi} : E_A \to \mathbb{R}$. By Theorem 4.20 $\mu_\psi = \mu_{\hat{\psi}} \circ \hat{\pi}^{-1}$ is the unique equilibrium state of the potential ψ and the shift map $\sigma : E_A \to E_A$. By Observation 4.14, $\mathrm{P}_T(\psi) = \mathrm{P}_\sigma(\hat{\psi})$, and by Observation 4.13, $\mathrm{h}_{\mu_\psi}(\mathrm{T}) = \mathrm{h}_{\mu_{\hat{\psi}}}(\sigma)$. Since in addition $\chi_{\mu_\psi}(T) = \chi_{\mu_{\hat{\psi}}}(\sigma)$, the proof follows immediately from Theorem 6.1 applied to $\hat{\psi} : E_A \to \mathbb{R}$. □

7. Bowen's Formula

We keep the setting of Sections 5 and Section 6, so $T : \hat{Y} \to \hat{Y}$ is a conformal skew product Smale endomorphism, i.e. satisfies conditions (a)–(g) of Section 5. We however emphasize that in Section 7, Condition (h) i.e. the Uniform Geometry Condition, is assumed.

For every $t \geq 0$ let $\psi_t : J \to \mathbb{R}$ be the function $\psi_t(\omega, y) = -t \log |T'_\omega(y)|$.
Define $\mathcal{F}(T)$ to be the set of parameters $t \geq 0$ for which the potential ψ_t is summable, i.e.

$$\sum_{e\in E} \exp\bigl(\sup(\psi_t|_{[e]_T})\bigr) < \infty.$$

This means that $\sum_{e\in E} \sup\bigl\{||T_{e\tau}||_\infty^t : \tau \in E_A(1,\infty),\ A_{e\tau_1} = 1\bigr\} < \infty$. For every $t \geq 0$, let

$$\mathrm{P}(t) := \mathrm{P}_T(\psi_t),$$

and call $\mathrm{P}(t)$ the topological pressure of the parameter t. From Proposition 3.6, we have $\mathcal{F}(T) = \{t \geq 0 : \mathrm{P}(t) < \infty\}$. We record the following basic properties of this pressure.

Proposition 7.1. *The pressure function $t \mapsto \mathrm{P}(t)$, $t \in [0, \infty)$ has the following properties:*

(a) P *is monotone decreasing*

(b) $\mathrm{P}|_{\mathcal{F}(T)}$ *is strictly decreasing.*

(c) $\mathrm{P}|_{\mathcal{F}(T)}$ *is convex, real-analytic, and Lipschitz continuous.*

Proof. All these statements except real analyticity follow easily from definitions, plus, due to Lemma 3.4 and Observation 4.14, from their one-sided shift counterparts. □

Now we can define two significant numbers associated with the Smale endomorphism T:

$$\theta_T := \inf\bigl\{t \geq 0 : \mathrm{P}(t) < \infty\bigr\} \text{ and } B_T := \inf\bigl\{t \geq 0 : \mathrm{P}(t) \leq 0\bigr\}.$$

The number B_T is called the *Bowen's parameter* of the system T. Of course $\theta_T \leq B_T$. The main result of this section is the following.

Theorem 7.2. *If $T : \hat{Y} \to \hat{Y}$ is a Hölder conformal skew product Smale endomorphism satisfying the Uniform Geometry Condition (h), then for every $\omega \in E_A^+$,*

$$\mathrm{HD}(J_\omega) = B_T$$

We first shall prove this theorem under the assumption that the alphabet E is finite. In this case we will actually prove more. Recall that if (Z,ρ) is a separable metric space, then a finite Borel measure ν on Z is called *Ahlfors regular* (or *geometric*) if and only if

$$C^{-1}r^h \le \nu(B(z,r)) \le Cr^h,$$

for all $r>0$, with some independent constants $h\ge 0$, $C\in(0,\infty)$. It is well known and easy to prove that there is at most one h with such property and all Ahlfors regular measures on Z are mutually equivalent, with bounded Radon-Nikodym derivatives. Moreover

$$h = \mathrm{HD}(Z) = \mathrm{PD}(Z) = \mathrm{BD}(Z),$$

the two latter dimensions being, respectively the packing and box-counting dimensions of Z. In addition, the h-dimensional Hausdorff measure H_h, and the h-dimensional packing measure P_h on Z, are Ahlfors regular, equivalent to each other and equivalent to ν.

Now, if the alphabet E is finite, then the Smale endomorphism $T:\hat{Y}\to\hat{Y}$ is of compact type, and in particular, for every $t\ge 0$ there exists μ_t, a unique equilibrium state for the potential $\psi_t : J\to\mathbb{R}$. Since $0\le \mathrm{P}(0)<\infty$ it follows from Proposition 7.1 that $\mathrm{P}(B_T)=0$.

Theorem 7.3. *If $T:\hat{Y}\to\hat{Y}$ is a Hölder conformal skew product Smale endomorphism satisfying the Uniform Geometry Condition (h) and the alphabet E is finite, then $\overline{\mu}^\omega_{B_T}$ is an Ahlfors regular measure on J_ω, for every $\omega\in E_A^+$. In particular, for every $\omega\in E_A^+$,*

$$\mathrm{HD}(J_\omega) = B_T$$

Proof. Put $h:=B_T$. Fix $\omega\in E_A^+$ and $z=\hat{\pi}_2(\tau)\in J_\omega$ arbitrary. Let $n=n(z,r)$ be given by (6.3), and denote $\nu_\omega := \overline{\mu}^\omega_h\circ\hat{\pi}_2^{-1}$. The formula (6.5) gives, for $\psi=\psi_h$,

$$\nu_\omega(B(z,r)) \ge D^{-1}\exp\big(S_n\psi(\sigma^{-n}(\tau))\big) = D^{-1}\big|\big(T_\tau^n\big)'(\hat{\pi}_2(\sigma^{-n}(\tau)))\big|^h. \tag{7.1}$$

Now, since E_A is compact (as E is finite) and since $E_A\ni\tau\mapsto|T_\tau'(\hat{\pi}_2(\tau))|\in(0,\infty)$ is continuous, we conclude that there exists a constant $M\in(0,\infty)$ such that

$$M^{-1}\le\inf\big\{|T_\tau'(\hat{\pi}_2(\tau))| : \tau\in E_A\big\}\le\sup\big\{|T_\tau'(\hat{\pi}_2(\tau))| : \tau\in E_A\big\}\le M. \tag{7.2}$$

Having this and inserting (6.4) to (7.1), we get

$$\nu_\omega(B(z,r)) \ge (DM^h)^{-1}r^h. \tag{7.3}$$

In order to prove an appropriate inequality in the opposite direction let

$$\mathcal{F}(z,r) := \Big\{\tau\in E_A^*(-\infty,-1) : T_{\tau\omega}^{|\tau|}(Y_{\tau\omega})\cap B(z,r/2)\ne\emptyset,$$
$$\mathrm{diam}\big(T_{\tau\omega}^{|\tau|}(Y_{\tau\omega})\big)\le r/2 \ \text{ and } \ \mathrm{diam}\Big(T^{|\tau|}_{\tau|^{-1}_{-(|\tau|-1)}\omega}\big(Y_{\tau|^{-1}_{-(|\tau|-1)}\omega}\big)\Big) > r/2\Big\}.$$

By its definition $\mathcal{F}(z,r)$ consists of mutually incomparable elements of $E_A^*(-\infty,-1)$, so using (7.2) along with (BDP3), we get for every $\tau\in\mathcal{F}(z,r)$, with $n:=|\tau|$, that

$$\begin{aligned}\operatorname{diam}\big(T^n_{\tau\omega}(Y_{\tau\omega})\big) &= \operatorname{diam}\Big(T^{n-1}_{\tau|_{-(n-1)}^{-1}\omega}\big(T_{\tau\omega}(Y_{\tau\omega})\big)\Big) \ge K^{-1}\Big\|\Big(T^{n-1}_{\tau|_{-(n-1)}^{-1}\omega}\Big)'\Big\|_\infty \operatorname{diam}\big(T_{\tau\omega}(Y_{\tau\omega})\big)\\ &\ge K^{-2}\Big\|\Big(T^{n-1}_{\tau|_{-(n-1)}^{-1}\omega}\Big)'\Big\|_\infty \|T'_{\tau\omega}\|_\infty \operatorname{diam}(Y_{\tau\omega}) \ge 2K^{-2}M^{-1}R\Big\|\Big(T^{n-1}_{\tau|_{-(n-1)}^{-1}\omega}\Big)'\Big\|_\infty\\ &\ge 2K^{-3}M^{-1}R\operatorname{diam}(Y)^{-1}\operatorname{diam}\Big(T^{n-1}_{\tau|_{-(n-1)}^{-1}\omega}\big(T_{\tau\omega}\big(Y_{\tau|_{-(n-1)}^{-1}\omega}\big)\big)\Big) \ge K^{-3}M^{-1}R\operatorname{diam}(Y)^{-1}r\end{aligned}$$

Thus Lemma 5.5 applies with the radius equal to $r/2$ given that $\#\mathcal{F}(z,r)\le\Gamma_\gamma$, where $\gamma:=\max\{1,2K^3MR^{-1}\operatorname{diam}(Y)\}$. Since also $\hat\pi_2^{-1}(B(z,r))\subset\bigcup_{\tau\in\mathcal{F}(z,r)}[\tau\omega]$, we therefore get

$$\nu_\omega(B(z,r))\le\overline{\mu}_h^\omega\circ\hat\pi_2^{-1}\Big(\bigcup_{\tau\in\mathcal{F}(z,r)}[\tau\omega]\Big)\le\sum_{\tau\in\mathcal{F}(z,r)}\overline{\mu}_h^\omega\circ\hat\pi_2^{-1}([\tau\omega])\le K^h\sum_{\tau\in\mathcal{F}(z,r)}\operatorname{diam}^h\big(T^{|\tau|}_{\tau\omega}(Y_{\tau\omega})\big)\le 2^hK^h\#_\Gamma r^h$$

along with (7.3) this shows that ν_ω is Ahlfors regular with exponent $h=B_T$. □

Proof of Theorem 7.2: Fix $t>B_T$ arbitrary; then $\mathrm{P}(t)<0$, so for every integer $n\ge1$ large and $\omega\in E_A^+$, we have $\sum_{\substack{\tau\in E_A^*(-n,-1)\\ A_{\tau_{-1}\omega_0}=1}}\big\|\big(T^n_{\tau\omega}\big)'\big\|_\infty^t\le\exp\big(\frac12\mathrm{P}(t)n\big)$. Thus by (BDP2),

$$\sum_{\substack{\tau\in E_A^*(-n,-1)\\ A_{\tau_{-1}\omega_0}=1}}\operatorname{diam}^t\big(T^n_{\tau\omega}(Y_{\tau\omega})\big)\le K^t\exp\left(\frac12\mathrm{P}(t)n\right). \tag{7.4}$$

Since $\mathrm{P}(t)<0$, since $\big\{T^n_{\tau\omega}(Y_{\tau\omega}):\tau\in E_A^*(-n,-1),\ A_{\tau_{-1}\omega_0}=1\big\}$ is a cover of J_ω and since the diameters of this cover converge to zero $\big(\operatorname{diam}\big(T^n_{\tau\omega}(Y_{\tau\omega})\big)\le\lambda^{-n}\operatorname{diam}(Y)\big)$, it follows from (7.4), by letting $n\to\infty$, that $\mathrm{H}_t(J_\omega)=0$. So $\mathrm{HD}(J_\omega)\le t$, and, thus $\mathrm{HD}(J_\omega)\le B_T$.

In order to prove the opposite inequality fix $0\le t<B_T$. Then $\mathrm{P}(t)>0$ and it thus follows from Theorem 3.5 that $\mathrm{P}_F(t)>0$ for some finite set $F\subset E$ such that the matrix $A|_{F\times F}$ is irreducible. It then further follows from Theorem 7.3 that $\mathrm{HD}(J_\omega(F))>t$ for all $\omega\in E_A^+$. Since $J_\omega(F)\subset J_\omega$, this yields $\mathrm{HD}(J_\omega)\ge t$. Thus, by arbitrariness of $t<B_T$, we get that $\mathrm{HD}(J_\omega)\ge B_T$. Hence this completes the proof of Theorem 7.2. □

8. General Skew Products over Countable-to-1 Endomorphisms.

We want to enlarge the class of endomorphisms for which we can prove exact dimensionality of conditional measures on fibers. For general thermodynamic formalism of endomorphisms related to our approach, one can see [20], [12], [11], [10], [13], etc. Our results on exact dimensionality of conditional measures in fibers extend a result on exact dimensionality of conditional measures on stable manifolds of hyperbolic endomorphisms (see [12]). We want to apply the results obtained in the previous sections to skew products over countable-to-1 endomorphisms. This includes EMR maps, continued fractions transformation, etc.

First, we prove a result about skew products whose base transformations are modeled by 1-sided shifts on a countable alphabet. Assume we have a skew product $F:X\times Y\to X\times Y$, where X and Y are complete bounded metric spaces, $Y \subset \mathbb{R}^d$ for some $d \geq 1$, and

$$F(x, y) = (f(x), g(x, y)),$$

where the map $Y \ni y \longmapsto g(x, y)$ is injective and continuous for every $y \in Y$. Denote the map $Y \ni y \mapsto g(x, y)$ also by $g_x(y)$. Assume $f : X \to X$ is at most countable-to-1, and its dynamics is modeled by a 1-sided Markov shift on a countable alphabet E with the matrix A finitely irreducible, i.e there exists a surjective Hölder continuous map, called *coding*,

$$p : E_A^+ \to X \;\text{ such that }\; p \circ \sigma = f \circ p$$

Assume conditions (a)–(g) from Section 5 are satisfied for $T_\omega : Y_\omega \to Y_{\sigma_\omega},\ \omega \in E_A^+$. Then we call $F : X \times Y \to X \times Y$ a *generalized conformal skew product Smale endomorphism.*

Given the skew product F as above, we can also form a skew product endomorphism in the following way: define for every $\omega \in E_A^+$, the fiber map $\hat{F}_\omega : Y \to Y$ by

$$\hat{F}_\omega(y) = g(p(\omega), y).$$

The system $(\hat{Y}, \hat{F})$ is called *the symbolic lift* of F. If $\hat{Y} = E_A^+ \times Y$, we obtain a conformal skew product Smale endomorphism $\hat{F} : \hat{Y} \to \hat{Y}$ given by

$$\hat{F}(\omega, y) = (\sigma(\omega), \hat{F}_\omega(y)) \tag{8.1}$$

As in the beginning of Section 4, we study the structure of fibers $J_\omega,\ \omega \in E_A^+$ and later of the sets J_x, $x \in X$. From definition, $J_\omega = \hat{\pi}_2([\omega])$ and it is the set of points of type

$$\bigcap_{n \geq 1} \overline{\hat{F}_{\tau_{-1}\omega} \circ \hat{F}_{\tau_{-2}\tau_{-1}}\omega \circ \ldots \circ \hat{F}_{\tau_{-n}\ldots\tau_{-1}\omega}(Y)}.$$

Let us call *n-prehistory* of the point x with respect to the system (f, X), any finite sequence of points in X: $(x, x_{-1}, x_{-2}, \ldots, x_{-n}) \in X^{n+1}$, where $f(x_{-1}) = x$, $f(x_{-2}) = x_{-1}, \ldots, f(x_{-n}) = x_{-n+1}$. Call a *complete prehistory* (or simply a *prehistory*) of x with respect to (f, X), any infinite sequence of consecutive preimages in X, i.e. $\hat{x} = (x, x_{-1}, x_{-2}, \ldots)$, where $f(x_{-i}) = x_{-i+1}$, $i \geq -1$. The space of complete prehistories is denoted by $\hat{X}$ and is called the *natural extension* (or *inverse limit*) of (f, X). We have a bijection $\hat{f} : \hat{X} \to \hat{X}$,

$$\hat{f}(\hat{x}) = (f(x), x, x_{-1}, \ldots).$$

In this paper, we use the terms inverse limit and natural extension interchangeably, without having necessarily a fixed invariant measure defined on the space X.
We consider on $\hat{X}$ the canonical metric, which induces the topology equivalent to the one inherited from the product topology on $X^{\mathbb{N}}$. Then $\hat{f}$ becomes a homeomorphism. For more on the dynamics of endomorphisms and their inverse limits, one can see [20], [11], [13], [10].

In the above notation, we have $f(p(\tau_{-1}\omega)) = p(\omega) = x$, and for all the prehistories of x, $\hat{x} = (x, x_{-1}, x_{-2}, \ldots) \in \hat{X}$, consider the set J_x of points of type

$$\bigcap_{n \geq 1} \overline{g_{x_{-1}} \circ g_{x_{-2}} \circ \ldots \circ g_{x_{-n}}(Y)}$$

Notice that, if $\hat{\eta} = (\eta_0, \eta_1, \ldots)$ is another sequence in E_A^+ such that $p(\hat{\eta}) = x$, then for any η_{-1} so that $\eta_{-1}\hat{\eta} \in E_A^+$, we have $p(\eta_{-1}\hat{\eta}) = x'_{-1}$ where x'_{-1} is some 1-preimage (i.e preimage of order 1) of x. Hence from the definitions and the discussion above, we see that

$$J_x = \bigcup_{\omega \in E_A^+, p(\omega)=x} J_\omega \tag{8.2}$$

Let us denote the respective fibered limit sets for T and F by:

$$J = \bigcup_{\omega \in E_A^+} \{\omega\} \times J_\omega \subset E_A^+ \times Y \quad \text{and} \quad J(X) := \bigcup_{x \in X} \{x\} \times J_x \subset X \times Y \tag{8.3}$$

So $\hat{F}(J) = J$ and $F(J(X)) = J(X)$. The Hölder continuous projection $p_J : J \to J(X)$ is

$$p_J(\omega, y) = (p(\omega), y),$$

we obtain $F \circ p_J = p_J \circ \hat{F}$. In the sequel, $\hat{\pi}_2 : E_A \to Y$ and $\hat{\pi} : E_A \to E_A^+ \times Y$ are the maps defined in Section 4 and,

$$\hat{\pi}(\tau) = (\tau|_0^\infty, \hat{\pi}_2(\tau)).$$

Now, it is important to know if enough points $x \in X$ have unique coding sequences in E_A^+.

Definition 8.1. *Let $F : X \times Y \to X \times Y$ be a generalized conformal skew product Smale endomorphism. Let μ be a Borel probability measure X. We then say that the coding $p : E_A^+ \to X$ is μ-injective, if there exists a μ-measurable set $G \subset X$ with $\mu(G) = 1$ such that for every point $x \in G$, the set $p^{-1}(x)$ is a singleton in E_A^+.*
Denote such a set G by G_μ and for $x \in G_\mu$ the only element of $p^{-1}(x)$ by $\omega(x)$.

Proposition 8.2. *If the coding $p : E_A^+ \to X$ is μ-injective, then for every $x \in G_\mu$, we have*

$$J_x = J_{\omega(x)}.$$

Proof. Take $x \in G_\mu$, and let $x_{-1} \in X$ be an f-preimage of x, i.e $f(x_{-1}) = x$. Since $p : E_A^+ \to X$ is surjective, there exists $\eta \in E_A^+$ such that $p(\eta) = x_{-1}$. But this implies that $f(x_{-1}) = f \circ p(\eta) = p \circ \sigma(\eta) = x$. Then, from the uniqueness of the coding sequence for x, it follows that $\sigma(\eta) = \omega(x)$, whence $x_{-1} = p(\omega_{-1}\omega(x))$, for some $\omega_{-1} \in E$. Since $J_x = \bigcap_{n \geq 1} \overline{g_{x_{-1}} \circ g_{x_{-2}} \circ \ldots \circ g_{x_{-n}}(Y)}$, it follows that $J_x = J_{\omega(x)}$. □

In the sequel we work only with μ-injective codings, and the measure μ will be clear from the context. Also given a metric space X with a coding $p : E_A^+ \to X$, and a potential $\phi : X \to \mathbb{R}$, we say that ϕ is *Hölder continuous* if $\phi \circ p$ is Hölder continuous.

Now consider a potential $\phi : J(X) \to \mathbb{R}$ such that the potential

$$\widehat{\phi} := \phi \circ p_J \circ \hat{\pi} : E_A \to \mathbb{R}$$

is Hölder continuous and summable. For example, $\widehat{\phi}$ is Hölder continuous if $\phi : J(X) \to \mathbb{R}$ is itself Hölder continuous. This case will be quite frequent in certain of our examples given later, when we will have Hölder continuous potentials ϕ on a set in $\mathbb{R}^2$ containing $J(X)$. Define now

$$\mu_\phi := \mu_{\widehat{\phi}} \circ (p_J \circ \hat{\pi})^{-1}, \tag{8.4}$$

and call it the equilibrium measure of ϕ on $J(X)$ with respect to the skew product F.

Now, let us consider the partition ξ' of $J(X)$ into the fiber sets $\{x\} \times J_x$, $x \in X$, and the conditional measures μ_ϕ^x associated to μ_ϕ with respect to the measurable partition ξ' (see [19]). Recall that for each $\omega \in E_A^+$, we have $\hat{\pi}_2([\omega]) = J_\omega$.

Denote by $p_1 : X \times Y \to X$ the canonical projection onto the first coordinate, i.e.

$$p_1(x, y) = x.$$

Theorem 8.3. *Let $F : X \times Y \to X \times Y$ be a generalized conformal skew product Smale endomorphism. Let $\phi : J(X) \to \mathbb{R}$ be a potential such that $\widehat{\phi} = \phi \circ p_J \circ \hat{\pi} : E_A \to \mathbb{R}$ is a Hölder continuous summable potential on E_A. Assume that the coding $p : E_A^+ \to X$ is $\mu_\phi \circ p_1^{-1}$–injective, and denote the corresponding set $G_{\mu_\phi} \subset X$ by G_ϕ. Then:*

(1) *$J_x = J_{\omega(x)}$ for every $x \in G_\phi$.*

(2) *With $\bar{\mu}_\psi^\omega$, $\omega \in E_A^+$ the conditional measures of $\mu_{\widehat{\phi}}$, we have for $\mu_\phi \circ p_1^{-1}$–a.e. $x \in G_\phi$,*

$$\mu_\phi^x = \bar{\mu}_{\widehat{\phi}}^{\omega(x)} \circ (p_J \circ \hat{\pi})^{-1},$$

or equivalently, if μ_ϕ^x and $\bar{\mu}_{\widehat{\phi}}^{\omega(x)}$ are viewed on J_x and E_A^-, $\mu_\phi^x = \bar{\mu}_{\widehat{\phi}}^{\omega(x)} \circ \hat{\pi}_2^{-1}$.

Proof. Part (1) is just Proposition 8.2. We thus deal with part (2) only. By the definition of conditional measures, we have for every μ_ϕ–integrable function $H : J(X) \to \mathbb{R}$ that

$$\int_{J(X)} H d\mu_\phi = \int_{E_A} H \circ p_J \circ \hat{\pi} d\mu_{\widehat{\phi}} = \int_{E_A^+} \int_{[\omega]} H \circ p_J \circ \hat{\pi}\, d\bar{\mu}_{\widehat{\phi}}^\omega\, d\mu_{\widehat{\phi}} \circ \pi_1^{-1}(\omega) \tag{8.5}$$

and

$$\int_{J(X)} H d\mu_\phi = \int_X \int_{\{x\} \times J_x} H\, d\mu_\phi^x\, d\mu_\phi \circ p_1^{-1}(x) \tag{8.6}$$

But from the definitions of various projections:

$$\mu_\phi \circ p_1^{-1} = \mu_{\widehat{\phi}} \circ (p_J \circ \hat{\pi})^{-1} \circ p_1^{-1} = \mu_{\widehat{\phi}} \circ (p_1 \circ p_J \circ \hat{\pi})^{-1} = \mu_{\widehat{\phi}} \circ (p \circ \pi_1)^{-1} = \mu_{\widehat{\phi}} \circ \pi_1^{-1} \circ p^{-1}. \tag{8.7}$$

Therefore, remembering also that $\mu_\phi \circ p_1^{-1}(G_\phi) = 1$, we get that

(8.8)
$$\int_{E_A^+} \int_{[\omega]} H \circ p_J \circ \hat{\pi}\, d\bar{\mu}_{\widehat{\phi}}^\omega\, d\mu_{\widehat{\phi}} \circ \pi_1^{-1}(\omega) = \int_{E_A^+} \int_{\{p(\omega)\} \times J_{p(\omega)}} H d\bar{\mu}_{\widehat{\phi}}^\omega \circ (p_J \circ \hat{\pi})^{-1}\, d\mu_{\widehat{\phi}} \circ \pi_1^{-1}(\omega)$$
$$= \int_{G_\phi} \int_{\{x\} \times J_x} H d\bar{\mu}_{\widehat{\phi}}^{\omega(x)} \circ (p_J \circ \hat{\pi})^{-1}\, d\mu_{\widehat{\phi}} \circ \pi_1^{-1} \circ p^{-1}(x) = \int_{G_\phi} \int_{\{x\} \times J_x} H d\bar{\mu}_{\widehat{\phi}}^{\omega(x)} \circ (p_J \circ \hat{\pi})^{-1}\, d\mu_\phi \circ p_1^{-1}(x).$$

Hence this, together with (8.5) and (8.6), gives

$$\int_{G_\phi} \int_{\{x\} \times J_x} H\, d\mu_\phi^x\, d\mu_\phi \circ p_1^{-1}(x) = \int_{G_\phi} \int_{\{x\} \times J_x} H d\bar{\mu}_{\widehat{\phi}}^{\omega(x)} \circ (p_J \circ \hat{\pi})^{-1}\, d\mu_\phi \circ p_1^{-1}(x).$$

Thus, the uniqueness of the system of Rokhlin's canonical conditional measures yields $\mu_\phi^x = \bar{\mu}_{\widehat{\phi}}^{\omega(x)} \circ (p_J \circ \hat{\pi})^{-1}$ for $\mu_\phi \circ p_1^{-1}$–a.e. $x \in G_\phi$. This means that the first part of (2) is established. But $p_J \circ \hat{\pi} = (p \circ \pi_1) \times \hat{\pi}_2$, and thus $p_J \circ \hat{\pi}|_{[\omega(x)]} = \{x\} \times \hat{\pi}_2|_{[\omega(x)]}$.

□

As in the previous Section, define a *Lyapunov exponent* for an F-invariant measure μ on the fibered limit set $J(X) = \bigcup_{x\in X} \{x\} \times J_x$, by:

$$\chi_\mu(F) = -\int_{J(X)} \log |g_x'(y)| \, d\mu(x,y).$$

In conclusion, from Theorem 8.3, Theorem 6.1, and definition (8.4), we obtain the following result for skew product endomorphisms over countable-to-1 maps $f : X \to X$:

Theorem 8.4. *Let $F : X \times Y \to X \times Y$ a generalized conformal skew product Smale endomorphism. Let $\phi : J(X) \to \mathbb{R}$ be a potential such that*

$$\psi := \phi \circ p_J \circ \hat{\pi} : E_A \to \mathbb{R}$$

is Hölder continuous summable. Assume the coding $p : E_A^+ \to X$ is $\mu_\phi \circ p_1^{-1}$–injective. Then for $\mu_\phi \circ p_1^{-1}$-a.e $x \in X$, the conditional measure μ_ϕ^x is exact dimensional on J_x, and

$$\lim_{r\to 0} \frac{\log \mu_\phi^x(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_\phi}(\mathrm{F})}{\chi_{\mu_\phi}(F)} = \mathrm{HD}(\mu_\phi^x),$$

for μ_ϕ^x-a.e $y \in J_x$; hence, equivalently, for μ_ϕ-a.e $(x,y) \in J(X)$.

As an immediate consequence of this theorem, we get the following.

Corollary 8.5. *Let $F : X \times Y \to X \times Y$ a generalized conformal skew product Smale endomorphism. Let $\phi : J(X) \to \mathbb{R}$ be a Hölder continuous potential such that*

$$\sum_{e\in E} \exp(\sup(\phi|_{\pi([e])\times Y})) < \infty.$$

Assume that the coding $p : E_A^+ \to X$ is $\mu_\phi \circ p_1^{-1}$–injective. Then, for $\mu_\phi \circ p_1^{-1}$-a.e $x \in X$, the conditional measure μ_ϕ^x is exact dimensional on J_x, and for μ_ϕ^x-a.e $y \in J_x$,

$$\lim_{r\to 0} \frac{\log \mu_\phi^x(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_\phi}(\mathrm{F})}{\chi_{\mu_\phi}(F)} = \mathrm{HD}(\mu_\phi^x)$$

By using Theorem 8.4, we will prove exact dimensionality of conditional measures of equilibrium states on fibers for many types of skew products.

First, let us prove a general result about *global* exact dimensionality of measures on fibered limit sets $J(X)$.

Theorem 8.6. *Let $F : X \times Y \to X \times Y$ a generalized conformal skew product Smale endomorphism. Assume that $X \subset \mathbb{R}^d$ with some integer $d \geq 1$. Let μ be a Borel probability F–invariant measure on $J(X)$, and $(\mu^x)_{x\in X}$ be the Rokhlin's canonical sytem of conditional measures of μ, with respect to the partition $\big(\{x\} \times J_x\big)_{x\in X}$. Assume that:*

a) There exists $\alpha > 0$ such that for $\mu \circ p_1^{-1}$-a.e $x \in X$ the conditional measure μ^x is exact dimensional and $\mathrm{HD}(\mu_x) = \alpha$*,*

b) The measure $\mu \circ p_1^{-1}$ is exact dimensional on X.

Then, the measure μ is exact dimensional on $J(X)$, and for μ-a.e $(x,y) \in J(X)$,

$$\mathrm{HD}(\mu) = \lim_{r\to 0} \frac{\log \mu(B((x,y),r))}{\log r} = \alpha + \mathrm{HD}(\mu \circ p_1^{-1}).$$

Proof. Denote the canonical projection to first coordinate by $p_1 : X \times Y \to X$. Let then $\nu := \mu \circ p_1^{-1}$. Denote the Hausdorff dimension $\mathrm{HD}(\nu)$ by γ. From the exact dimensionality of the conditional measures of μ, we know that for ν–a.e $x \in X$ and for μ^x–a.e $y \in Y$,

$$\lim_{r\to 0} \frac{\log \mu^x(B(y,r))}{\log r} = \alpha.$$

Then for any $\varepsilon \in (0,\alpha)$ and any integer $n \geq 1$, consider the following Borel set in $X \times Y$:

$$A(n,\varepsilon) := \left\{ z = (x,y) \in X \times Y : \ \alpha - \varepsilon < \frac{\log \mu^x(B(y,r))}{\log r} < \alpha + \varepsilon \ \text{ for all } \ r \in (0,1/n) \right\}.$$

From definition it is clear that $A(n,\varepsilon) \subset A(n+1,\varepsilon)$ for all $n \geq 1$. Moreover, setting $X'_Y := \bigcap_{\varepsilon>0} \bigcup_{n=1}^{\infty} A(n,\varepsilon)$, it follows from the exact dimensionality of almost all the conditional measures of μ and from the equality of their pointwise dimensions, that $\mu(X'_Y) = 1$. For $\varepsilon > 0$ and $n \geq 1$, consider also the following Borel subset of X:

$$D(n,\varepsilon) := \left\{ x \in X : \ \gamma - \varepsilon < \frac{\log \nu(B(x,r))}{\log r} < \gamma + \varepsilon \ \text{ for all } \ r \in (0,1/n) \right\}.$$

We know that $D(n,\varepsilon) \subset D(n+1,\varepsilon)$ for all $n \geq 1$, and from the exact dimensionality of ν, we obtain that for every $\varepsilon > 0$, we have $\nu\left(\bigcup_{n=1}^{\infty} D(n,\varepsilon)\right) = 1$. For $\varepsilon > 0$ and an integer $n \geq 1$, let us denote now

$$E(n,\varepsilon) := A(n,\varepsilon) \cap p_1^{-1}(D(n,\varepsilon)).$$

Clearly from above, we have that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(E(n,\varepsilon)) = 1. \tag{8.9}$$

From the definition of conditional measures and the definition of $A(n,\varepsilon)$ and $D(n,\varepsilon)$, we have that, for any $z \in E(n,\varepsilon)$, $x = \pi_1(z)$ and any $n \geq 1, \varepsilon > 0, 0 < r < 1/n$,

(8.10)
$$\begin{aligned}
\mu(E(n,\varepsilon) \cap B(z,r)) &= \int_{D(n,\varepsilon)\cap B(x,r)} \mu^y\big(B(z,r) \cap (\{y\} \times Y) \cap A(n,\varepsilon)\big) \, d\nu(y) \\
&\leq \int_{D(n,\varepsilon)\cap B(x,r)} r^{\alpha-\varepsilon} \, d\nu(y) = r^{\alpha-\varepsilon}\nu(D(n,\varepsilon) \cap B(x,r)) \leq r^{\alpha+\gamma-2\varepsilon}
\end{aligned}$$

Since $\mu(E(n,\varepsilon)) > 0$ for all $n \geq 1$ large enough, it follows from Borel Density Lemma - Lebesgue Density Theorem that, for μ-a.e $z \in E(n,\varepsilon)$, we have that

$$\lim_{r\to 0} \frac{\mu(B(z,r) \cap E(n,\varepsilon))}{\mu(B(z,r))} = 1.$$

Thus for any $\theta > 1$ arbitrary, there exists a subset $E(n,\varepsilon,\theta)$ of $E(n,\varepsilon)$, such that

$$\mu(E(n,\varepsilon,\theta)) = \mu(E(n,\varepsilon)),$$

and for every $z \in E(n,\varepsilon,\theta)$ there exists $r(z,\theta) > 0$ so that for any $0 < r < \inf\{r(z,\theta), 1/n\}$, we have from 8.10:

$$\mu(B(z,r)) \leq \theta\mu(E(n,\varepsilon) \cap B(z,r)) \leq \theta \cdot r^{\alpha+\gamma-2\varepsilon}$$

Thus for $z \in E(n, \varepsilon, \theta)$, we obtain $\lim_{r\to 0} \frac{\log \mu(B(z,r))}{\log r} \geq \alpha+\gamma-2\varepsilon$. Now, since $\mu(E(n,\varepsilon,\theta)) = \mu(E(n,\varepsilon))$, it follows from (8.9) that $\mu(\bigcup_n E(n,\varepsilon,\theta)) = 1$. Hence

$$\mu\left(\bigcap_{\varepsilon>0}\bigcap_{\theta>1}\bigcup_{n=1}^{\infty} E(n,\varepsilon,\theta)\right) = 1,$$

and for $z \in \bigcap_{\varepsilon>0}\bigcap_{\theta>1}\bigcup_{n} E(n,\varepsilon,\theta)$, we have $\lim_{r\to 0} \frac{\log \mu(B(z,r))}{\log r} \geq \alpha + \gamma$. Conversely, from the exact dimensionality of ν and of the conditional measures of μ, and with $x = \pi_1(z)$, we have that for $r \in (0, 1/n)$,

(8.11)
$$\mu(B(z,r)\cap E(n,\varepsilon)) = \int_{D(n,\varepsilon)\cap B(x,r)} \mu^y\big(B(z,r)\cap A(n,\varepsilon)\cap\{y\}\times Y\big)\, d\nu(y) \geq r^{\alpha+\gamma+2\varepsilon}$$

Thus, $\mu(B(z,r)) \geq \mu(B(z,r)\cap E(n,\varepsilon)) \geq r^{\alpha+\gamma+2\varepsilon}$, for $z \in E(n,\varepsilon)$ and $r \in (0,1/n)$. Making use of (8.9) we deduce that μ is exact dimensional, and for μ-a.e $z \in X \times Y$ we obtain the conclusion $\lim\limits_{r\to 0} \frac{\log \mu(B(z,r))}{\log r} = \alpha + \gamma$. □

9. Skew products over EMR-endomorphisms.

We now consider EMR (expanding Markov-Rényi) maps on the interval, and we construct skew product endomorphisms over these maps which contract in fibers. This EMR class contains important examples of endomorphisms coded by a shift space with countable alphabet, like the continued fractions transformation, and the Manneville-Pomeau map. In particular, the Manneville-Pomeau transformation is an example of a non-uniformly hyperbolic system with an indifferent fixed point (*parabolic point*), but one can associate to it a countable uniformly hyperbolic system by inducing using the Schweiger jump transformation ([23], [8]). Let us first give the definition of EMR maps from [17].

Definition 9.1. *Let I be an interval in $\mathbb{R}$, and assume $I = \cup_{n\geq 0} I_n$, where $I_n, n \geq 0$ are closed intervals with mutually disjoint interiors. A map $f : I \to I$ is called EMR if:*

a) f is $\mathcal{C}^2$ on $\cup_{n\geq 0} int(I_n)$.

b) there exists an iterate of f which is uniformly expanding, i.e $\exists K > 1$ and m a positive integer, so that $|(f^m)'(x)| \geq K > 1, \forall x \in \cup_{n\geq 0} int(I_n)$.

c) the map f is Markov, i.e for any $n \geq 0$, $f|_{int(I_n)}$ is a homeomorphism from the interior of I_n to the interior of a union of some of the I_j's, $j \geq 0$.

d) f satisfies Rényi condition, i.e $\exists K' > 0$ such that $\sup\limits_n \sup\limits_{x,y,z\in I_n} \frac{|f''(x)|}{|f'(y)|\cdot|f'(z)|} \leq K' < \infty$.

For an EMR map f, there exists a coding with a shift space on countably many symbols,

$$\pi : \mathbb{N}^{\mathbb{N}} \to I,\ \pi((k_1,k_2,\ldots)) = \bigcap_{n\geq 0} f^{-n}(I_{k_n})$$

Every point x which never hits the boundary of some interval I_n under an iterate of f, has a unique such coding, i.e there exists a unique $(k_1,k_2,\ldots) \in \mathbb{N}^{\mathbb{N}}$ with $\pi((k_1,k_2,\ldots)) = x$. Thus, $\pi : E_A^+ \to X$ is injective outside a countable set.

Two important examples used in the sequel, are the continued fractions map and the Manneville-Pomeau maps. The continued fractions (Gauss) map is $f_1 : [0,1] \to [0,1]$,

$$f_1(x) = \frac{1}{x} - [\frac{1}{x}] = \{\frac{1}{x}\},\ x \neq 0,\ \text{and}\ f_1(0) = 0$$

The Manneville-Pomeau map $f_2 : [0,1] \to [0,1]$ is defined by:

$$f_2(x) = x + x^{1+\alpha} \text{ mod } 1,$$

for some $\alpha > 0$; we fix such an arbitrary $\alpha > 0$ and, for simplicity of notation, will not record it in the notation for f_2. Notice that f_2 has an indifferent fixed point at 0, so f_2 is not strictly EMR. It was shown that an induced map of f_2 is EMR. First, f_2 is injective on two maximal intervals $[0, a_0]$ and $[a_0, 1]$, where a_0 is given by $1 = a_0 + a_0^\alpha$. The induced map of f_2 on $[a_0, 1]$ is hyperbolic, since we are far from the indifferent point 0. Take a decreasing sequence $(a_n)_n$ s.t $f_2(a_{n+1}) = a_n, n \geq 0$, and let $I_n := [a_n, a_{n-1}], n \geq 1$, and $I_0 = [a_0, 1]$. Then $f_2^n(I_{n+1}) = I_0$, for all $n > 0$, so the induced map (first return time) to I_0 is:

$$f_{2,I_0}(x) = f_2^n(x),\ x \in I_{n+1}, n \geq 0 \tag{9.1}$$

Proposition 9.2. *a) From [23] it follows that the Gauss map f_1 is an EMR map.*
b) From [24] it follows that the induced map f_{2,I_0} of the Manneville-Pomeau map is EMR.

Consider now a general EMR map $f : I \to I$, and a skew product $F : I \times Y \to I \times Y$, where $Y \subset \mathbb{R}^d$ is a bounded open set, with $F(x, y) = (f(x), g(x, y))$. Recall that the symbolic lift of F is $\hat{F} : \mathbb{N}^{\mathbb{N}} \times Y \to \mathbb{N}^{\mathbb{N}} \times Y$,

$$\hat{F}(\omega, y) = (\sigma\omega, g(\pi(\omega), y),\ \forall (\omega, y) \in \mathbb{N}^{\mathbb{N}} \times Y$$

If the symbolic lift $\hat{F}$ is a Hölder conformal skew product Smale endomorphism, then we say by extension that F is a *Hölder conformal skew product endomorphism over f*.

Recall now the observation after Definition 9.1, that the coding π of an EMR map is injective outside a countable set; and, from (8.3) the fibered limit set of F is $J(I) = \bigcup\limits_{x \in I} \{x\} \times J_x$. Let ϕ be a Hölder continuous summable potential on $J(I)$, and μ_ϕ be its equilibrium measure. Then π is easily shown to be $\mu_\phi \circ p_1^{-1}$-injective (as μ_ϕ is invariant, ergodic and has full topological support). So, from Theorem 8.4 follows:

Theorem 9.3. *Let an EMR map $f : I \to I$, an open bounded set $Y \subset \mathbb{R}^d$, and a Hölder conformal skew product endomorphism over f, $F : I \times Y \to I \times Y$. Let $\phi : J(I) \to \mathbb{R}$ be a Hölder continuous potential, such that $\sum_{e \in \mathbb{N}} \exp(\sup(\phi|_{\pi([e]) \times Y})) < \infty$.*
Then, for $(\pi_{1*}\mu_\phi)$-a.e $x \in I$, the conditional measure μ_ϕ^x is exact dimensional on J_x and

$$\lim_{r \to 0} \frac{\log \mu_\phi^x(B(y, r))}{\log r} = \frac{h_{\mu_\phi}(F)}{\chi_{\mu_\phi}(F)},$$

for μ_ϕ^x-a.e $y \in J_x$; equivalently for μ_ϕ-a.e $(x, y) \in J(I)$.

For the continued fractions transformation f_1 and the induced map f_{2,I_0} of the Manneville-Pomeau map, we can use Theorem 9.3, and Theorem 8.6, to prove the exact dimensionality of certain equilibrium measures for skew products over f_1 or f_{2,I_0}. Here the intervals I_n from EMR definition are, respectively:

- for $f_1 : I \to I,\ \ I_n = [\frac{1}{n+1}, \frac{1}{n}], n \geq 1$.
- for $f_{2,I_0} : I_0 \to I_0,\ \ I_n = [a_n, a_{n-1}], n \geq 1$ as defined in (9.1).

Corollary 9.4. *a) Let f be either the continued fraction map f_1, or the induced map of the Manneville-Pomeau map f_{2,I_0}. Take an open bounded set $Y \subset \mathbb{R}^d$, and a Hölder conformal skew product endomorphism over f, $F : I \times Y \to I \times Y$. Let $\phi : I \to \mathbb{R}$ a Hölder continuous potential s.t $\sum_{e\in\mathbb{N}} \exp(\sup(\phi|_{I_n}) < \infty$, and $\psi := \phi \circ \pi : I \times Y \to I \times Y$. Then, for μ_ϕ-a.e $x \in I$, the conditional measure μ_ψ^x is exact dimensional on J_x, and*

$$\lim_{r\to 0} \frac{\log \mu_\psi^x(B(y,r))}{\log r} = \frac{h_{\mu_\psi}(F)}{\chi_{\mu_\psi}(F)},$$

for μ_ψ^x-a.e $y \in J_x$; hence, equivalently for μ_ψ-a.e $(x, y) \in J(I)$.

b) If μ_ϕ is exact dimensional on I, then μ_ψ is exact dimensional on $I \times Y$.

Proof. a) If $\phi : I \to \mathbb{R}$ is a potential on I and if $\psi = \phi \circ \pi$, then the projection of μ_ψ on the first coordinate is μ_ϕ. We then apply Theorem 9.3 for f_1 or the induced map f_{2,I_0}. Notice that if ϕ is Hölder continuous on I, then $\psi = \phi \circ \pi$ is Hölder continuous on $I \times Y$.

b) If μ_ϕ is exact dimensional, apply part a) and Theorem 8.6. □

In [17] Pollicott and Weiss studied multifractal analysis for a class of potentials. Given an EMR map $f : I \to I$ and $\phi : I \to \mathbb{R}$ with $\exp \phi$ continuous, ϕ belongs to the class $\mathcal{W}$ iff:

$\mathcal{W}1$) there exists a constant $C > 0$ with: $\sum\limits_{y, f(y)=x} \exp \phi(y) \leq C,\ \forall x \in I$.

$\mathcal{W}2$) the function $C_\phi(x, x') = \sum\limits_{n\geq 1} \sum\limits_{y\in f^{-n}x, y'\in f^{-n}x'} \sum\limits_{0\leq j\leq n-1} |\phi(f^j y) - \phi(f^j y')|$ is bounded above by a constant C_ϕ and $C_\phi(x, x') \to 0$ when $|x - x'| \to 0$.

For any $\phi \in \mathcal{W}$, from conditions $\mathcal{W}1$) and $\mathcal{W}2$) it follows that there exists a unique equilibrium measure μ_ϕ for ϕ with respect to f on I (see Prop 7 of [17], or Section 8 above). For all $n \geq 1$ and $x \in I$, μ_ϕ satisfies the usual estimates on the set $I_n(x)$ containing x of the partition $\bigvee\limits_{0\leq i\leq n-1} f^{-i}(\{I_m\}_{m\geq 0})$. Namely $\exists$ a constant $C > 0$ s.t for all $x \in I$, $y \in I_n(x)$,

$$\frac{1}{C} \exp\Big(\sum_{0\leq j\leq n-1} \phi(f^j(y)) - nP(\phi)\Big) \leq \mu(I_n(x)) \leq C \exp\Big(\sum_{0\leq j\leq n-1} \phi(f^j(y)) - nP(\phi)\Big) \tag{9.2}$$

Now, for a potential $\phi \in \mathcal{W}$ and real parameters q, t, one can form the family of potentials

$$\phi_{q,t} = -t \log |f'| + q(\phi - P(\phi)), \tag{9.3}$$

and define the number $t(q)$ by the condition $P(\phi_{q,t(q)}) = 0$. We see that $P(\phi_{1,0}) = 0$, so $t(1) = 0$. Let μ_q be the equilibrium measure of the potential $\phi_{q,t(q)} \in \mathcal{W}$.

Consider a skew product $F : I \times Y \to I \times Y$ over f_1 or over f_{2,I_0} as in Corollary 9.4, and let $\phi \in \mathcal{W}$. If π_1 is the projection on the first coordinate, define the potentials on $I \times Y$,

$$\psi_{q,t} := \phi_{q,t} \circ \pi_1, \ \text{ and } \ \psi_q := \psi_{q,t(q)}$$

As in Sections 4 and 5, from conditions $\mathcal{W}1$) and $\mathcal{W}2$), it follows that there exists a unique equilibrium measure μ_{ψ_q} for ψ_q, with respect to F on $I \times Y$. Consider skew product endomorphisms F as above. We prove that the equilibrium measures of certain potentials ψ_q with respect to F, are *exact dimensional on* $I \times Y$.

Theorem 9.5. *a) In the above setting, if $F : I \times Y \to I \times Y$ is a Hölder conformal skew product endomorphism over the continued fractions transformation f_1, and if $\phi \in \mathcal{W}$, then μ_{ψ_q} is exact dimensional on $I \times Y$, for all parameters q satisfying $t(q) > \frac{1}{2}$.*

b) In the above setting, if F is a Hölder conformal skew product endomorphism over the induced map f_{2,I_0} of the Manneville-Pomeau map, and if $\phi \in \mathcal{W}$, then μ_{ψ_q} is exact dimensional on $I \times Y$, for all parameters q satisfying $\frac{1}{\alpha} < t(q) < 1$.

Proof. We have that the estimates (9.2) for equilibrium measures on intervals I_n (and thus on cylinders), hold when $\phi \in \mathcal{W}$. Thus we have Theorem 6.1 and Corollary 9.4, and obtain the exact dimensionality of a.e conditional measure of μ_{ψ_q} on the fibers contained in Y. From Theorems 1 and 2 and Proposition 3 of [17], we obtain the exact dimensionality of measures μ_{ϕ_q} on I, for the respective ranges of parameters q for f_1, and f_{2,I_0}. But $\pi_{1*}(\mu_{\psi_q}) = \mu_{\phi_q}$ is thus exact dimensional. So, from Theorem 8.6 μ_{ψ_q} is exact dimensional on $I \times Y$. □

10. Diophantine approximants and the Doeblin-Lenstra conjecture

We want to apply the results about skew products to certain properties of diophantine approximants, making the conjecture of Doeblin and Lenstra more general and precise. Consider an irrational number $x \in [0, 1]$, whose continued fraction representation is:

$$x = [a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}},$$

where $a_i \geq 1, i \geq 1$. Recall also that the associated continued fraction transformation is

$$T : [0, 1] \to [0, 1],\ T(x) = \{\frac{1}{x}\},\ x \neq 0,\ \text{and } T(0) = 0$$

If we truncate the representation at n, then we obtain a rational number $\frac{p_n}{q_n}$ (called the n-th convergent of x), where $p_n, q_n \geq 1, n \geq 1,\ (p_n, q_n) = 1$, and

$$\frac{p_n}{q_n} = [a_1, \ldots, a_n]$$

When need be, we shall also denote a_n, p_n, q_n by $a_n(x), p_n(x), q_n(x)$, respectively, in order to emphasize their dependence on x. Let us now denote (see for eg [6]) by

$$\Theta_n := |x - \frac{p_n}{q_n}| \cdot q_n^2,\ n \geq 1$$

This number Θ_n depends on x, so we will also denote it by $\Theta_n(x)$.
Notice that the Gauss map T above can be coded by the shift on a symbolic space with a countable set of generators $E_{\mathbb{N}}^+$, and that $Tx = [a_2, a_3, \ldots]$. Denote by $T_n := T^n(x)$, hence

$$T_n = [a_{n+1}, a_{n+2}, \ldots],\ \text{and } V_n = [a_n, \ldots, a_1],\ n \geq 1$$

Hence T_n represents the future of x, and V_n represents the past of x. Now, for every $n \geq 1$,

$$V_n = \frac{q_{n-1}}{q_n},\ \text{and } \Theta_{n-1} = \frac{V_n}{1 + T_n V_n},\ \text{and } \Theta_n = \frac{T_n}{1 + T_n V_n}$$

The natural extension of $([0,1),T)$ is given by (see for eg [6]):

$$\mathcal{T}(x,y)=(Tx,\frac{1}{a_1(x)+y}),\ (x,y)\in[0,1)^2$$

From this, it follows that $\mathcal{T}(x,0)=(Tx,\frac{1}{a_1(x)})$, and $\mathcal{T}^2(x,0)=(T^2(x),\frac{1}{a_2(x)+\frac{1}{a_1(x)}})$. By induction, we obtain that in general, for every $n\geq 1$,

$$\mathcal{T}^n(x,0)=(T_n,[a_n,\dots,a_1])=(T_n,V_n)$$

The coefficients Θ_n were the object of an important Conjecture by Doeblin and reformulated by Lenstra (see [6]), that for Lebesgue-a.e $x\in[0,1)$ and all $z\in[0,1]$, the frequency of $\Theta_n(x)$ appearing in the interval $[0,z]$ is given by the function $F(z)$ defined on $[0,1]$ by

$$F(z)=\frac{z}{\log 2},z\in[0,\frac{1}{2}],\ \text{and}\ F(z)=\frac{1}{\log 2}(1-z+\log 2z),z\in[\frac{1}{2},1]$$

The **Doeblin-Lenstra Conjecture** says that for a.e $x\in[0,1]$ and all $z\in[0,1]$, the limit

$$\lim_{n\to\infty}\frac{Card\{1\leq k\leq n,\Theta_n(x)\leq z\}}{n}$$

exists, and equals the above distribution function $F(z)$. This conjecture was solved by Bosma, Jager and Wiedijk in the '80's ([2]), and they needed the *natural extension* $([0,1)^2,\mathcal{T},\tilde{\mu}_G)$, of the continued fraction dynamical system with the Gauss measure μ_G.

Let us see how we can apply our results on skew products for the natural extension $([0,1)^2,\mathcal{T})$ of the continued fraction transformation $([0,1),T)$, and to the lifts of certain invariant measures. First, notice that the natural extension map of T, namely

$$\mathcal{T}(x,y)=(Tx,\frac{1}{a_1(x)+y}),\ (x,y)\in[0,1)^2,$$

falls into our class of skew products. From the representation of real numbers in continued fraction, it follows that the endomorphism T is coded completely by the shift map on a symbolic space with infinite alphabet $E_{\mathbb{N}}^+$. Consider now the potentials

$$\phi_s(x)=-s\log|T'(x)|,x\in[0,1),$$

for $s>\frac{1}{2}$. As discussed above, the potential ϕ_s belongs to the class $\mathcal{W}$, and it has an equilibrium measure denoted by μ_s on $[0,1)$. Let us now denote by

$$\psi_s(x,y)=\phi_s(x),\ (x,y)\in[0,1)^2,$$

and let $\hat{\mu}_s$ be the equilibrium measure of ψ_s w.r.t $\mathcal{T}$ on $[0,1)^2$. From Theorem 9.8, we know that $\hat{\mu}_s$ is exact dimensional on $[0,1)\times[0,1)$. Our purpose is now to describe the asymptotic frequencies with which $\Theta_n(x)$ come close to arbitrary values, when x is μ_s-generic (instead of x in a set of full Lebesgue measure as in the original Doeblin-Lenstra conjecture).

Theorem 10.1. *Consider the measure $\hat{\mu}_s$ on $[0,1)^2$ and the measure μ_s on $[0,1)$. Then for μ_s-a.e $x\in[0,1)$ we have that for all $z,z'\in[0,1)$, and $r,r'>0$,*

$$\lim_{n\to\infty}\frac{Card\big\{k,0\leq k\leq n-1,(T_k,V_k)\in B(z,r)\times B(z',r')\big\}}{n}=\hat{\mu}_s(B(z,r)\times B(z',r'))$$

Proof. First recall that $\mathcal{T}^n(x,0) = (T_n, V_n), n \geq 1$. It is enough to prove the result for the set $A = (a,b) \times (c,d)$ for a dense set of c, d, instead of $B(z,r) \times B(z', r')$. Let us consider $\varepsilon > 0$ arbitrary and denote $A(\varepsilon) = (a,b) \times (c-\varepsilon, d+\varepsilon)$ and $A(-\varepsilon) = (a,b) \times (c+\varepsilon, d-\varepsilon)$. Thus $A(-\varepsilon) \subset A \subset A(\varepsilon)$.

If $x = [a_1, a_2, \ldots] \notin \mathbb{Q}$, then there exists $n_0(\varepsilon) \geq 1$ such that for any $y \in [0,1]$, $|[a_n, a_{n-1}, \ldots, a_1 + y] - [a_n, \ldots, a_1]| < \varepsilon$. But then, if $\mathcal{T}^n(x,y) = (T^n(x), [a_n, \ldots, a_1 + y]) \in A(-\varepsilon)$, it follows automatically that $(T_n, V_n) \in A$. And, if $(T_n, V_n) \in A$, then $\mathcal{T}^n(x,y) \in A(\varepsilon)$.

Thus, we compare the condition of existence of an iterate of (x,y) in a slightly modified rectangle with the existence of an iterate of $(x,0)$ in A. But then, from above, it follows:

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{0\leq k\leq n-1} 1_{A(-\varepsilon)}(\mathcal{T}^k(x,y)) \leq \liminf_{n\to\infty} \frac{1}{n} \sum_{0\leq k\leq n-1} 1_A(\mathcal{T}^k(x,0))$$
$$\leq \limsup_{n\to\infty} \frac{1}{n} \sum_{0\leq k\leq n-1} 1_A(\mathcal{T}^k(x,0)) \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{0\leq k\leq n-1} 1_{A(-\varepsilon)}(\mathcal{T}^k(x,y))$$

Now we know that the equilibrium measure $\hat{\mu}_s$ is ergodic on $[0,1)^2$ w.r.t $\mathcal{T}$, hence from Birkhoff Ergodic Theorem it follows that for μ_s-a.e $x \in [0,1)$,
(10.1)
$$\hat{\mu}_s(A(-\varepsilon)) \leq \liminf_{n\to\infty} \frac{1}{n} \sum_{0\leq k\leq n-1} 1_A(\mathcal{T}^k(x,0)) \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{0\leq k\leq n-1} 1_A(\mathcal{T}^k(x,0)) \leq \hat{\mu}_s(A(\varepsilon))$$

But outside a set of $\{c,d\}$'s which is at most countable, the measure $\hat{\mu}_s$ is zero on $(a,b) \times \{c,d\}$. Hence from (10.1), $\liminf\limits_{n\to\infty} \frac{1}{n} \sum\limits_{0\leq k\leq n-1} 1_A(T_k, V_k) = \hat{\mu}_s(A)$. □

Let us denote the Lyapunov exponent associated to the measure μ_s, by

$$\lambda(\mu_s) := \int_I \log|T'|(x)\ d\mu_s(x)$$

Denote also by λ_0 the Lyapunov exponent of the Gauss measure, i.e $\lambda_0 = \int \log|T'| d\mu_G = \frac{\pi^2}{6\log 2}$. Then, from Pollicott-Weiss, we have that for each $\lambda \in [\lambda_0, \infty)$, there exists $s = s(\lambda) > \frac{1}{2}$ and an uncountable dense set $\Lambda_s \subset [0,1)$, so that $\lambda(\mu_s) = \lambda$, and

$$\Lambda_s = \{x \in [0,1), \lim_{n\to\infty} \frac{1}{n} \log|x - \frac{p_n(x)}{q_n(x)}| = \lambda\}, \text{ and } HD(\Lambda_s) = \frac{h_{\mu_s}(T)}{\lambda} \tag{10.2}$$

We want to prove now that for $x \in \Lambda_s$, the approximation coefficients $\Theta_n(x), \Theta_{n-1}(x)$ behave *very erratically*, and we estimate the asymptotic frequency that $(\Theta_k(x), \Theta_{k-1}(x))$ is r-close to some (z, z'), *independently of* x.

Theorem 10.2. *In the above setting, for any $\lambda \in [\lambda_0, \infty)$, there exists $s > \frac{1}{2}$ and a set $\Lambda_s \subset [0,1)$ with $HD(\Lambda_s) = \frac{h_{\mu_s}(T)}{\lambda}$, such that for any $\varepsilon > 0$, $x \in \Lambda_s$, and $\hat{\mu}_s$-a.e $(z,z') \in [0,1)^2$, there exists $r(x,z,z') > 0$ so that for any $0 < r < r(x,z,z',\varepsilon)$, we have the following asymptotic estimates:*

$$r^{\delta(\hat{\mu}_s)-\varepsilon} \leq \lim_{n\to\infty} \frac{Card\{k, 0 \leq k \leq n-1,\ \big(\Theta_k(x), \Theta_{k-1}(x)\big) \in B(\frac{z}{1+zz'}, r) \times B(\frac{z'}{1+zz'}, r)\}}{n} \leq r^{\delta(\hat{\mu}_s)+\varepsilon},$$

where $\delta(\hat{\mu}_s)$ is the Hausdorff dimension of $\hat{\mu}_s$, and where for $\hat{\mu}_s$-a.e $(z, z') \in [0,1)^2$, we have

$$\delta(\hat{\mu}_s) = \frac{h_{\mu_s}(T)}{\lambda} + \frac{h_{\hat{\mu}_s}(\mathcal{T})}{2\int_{[0,1)^2} \log(a_1(x)+y)\ d\hat{\mu}_s(x,y)}$$

Proof. First, for any $\lambda \in [\lambda_0, \infty)$, there exists an $s > \frac{1}{2}$ and a set Λ_s is defined in (10.2), and by [17] we know that its Hausdorff dimension is given by the formula in (10.2).

We now want to use Theorem 9.5 and the formula for the Hausdorff dimension of the measure $\hat{\mu}_s$. The measure $\hat{\mu}_s$ is exact on $[0,1)^2$, since μ_s is exact for $s > \frac{1}{2}$ (from [17]) and since by Theorem 8.4 and Theorem 9.3 the conditional measures of $\hat{\mu}_s$ on fibers are also exact dimensional. Given an arbitrary point $x \in \Lambda_s$, we will work with the associated numbers $T_k(x), V_k(x), \Theta_k(x)$, but for simplicity of notation will denote them just by T_k, V_k, Θ_k respectively. We use Theorem 10.1 to show that the asymptotic frequencies of (T_k, V_k) being in certain set A is given by the measure $\hat{\mu}_s(A)$. In our case $A = B(z,r) \times B(z',r)$, and if $(T_k, V_k) \in B(z,r) \times B(z',r)$, then there exist constants $C, C' > 0$ so that $(\Theta_k, \Theta_{k-1}) \in B(\frac{z}{1+zz'}, Cr) \times B(\frac{z'}{1+zz'}, Cr)$, and vice-versa if $(\Theta_k, \Theta_{k-1}) \in B(\frac{z}{1+zz'}, r) \times B(\frac{z'}{1+zz'}, r)$, then $(T_k, V_k) \in B(z, C'r) \times B(z', C'r)$. Thus, there exists some constant $C_1 > 0$, such that from Theorem 10.1, the asymptotic frequency behaves as:

$$C_1^{-1}\hat{\mu}_s(B(z,r) \times B(z',r)) \leq$$

$$\leq \lim_{n\to\infty} \frac{Card\{k, k \leq n-1,\ \big(\Theta_k(x), \Theta_{k-1}(x)\big) \in B(\frac{z}{1+zz'}, r) \times B(\frac{z'}{1+zz'}, r)\}}{n} \leq C_1\hat{\mu}_s(B(z,r) \times B(z',r))$$

But we know from Theorem that the measure $\hat{\mu}_s$ is exact dimensional, hence $\hat{\mu}_s(B(z,r) \times B(z',r)) \approx r^{\delta(\hat{\mu}_s)}$, where $\delta(\hat{\mu}_s)$ is the pointwise dimension of $\hat{\mu}_s$,

$$\delta(\hat{\mu}_s) = \lim_{t\to 0} \frac{\log \hat{\mu}_s(B(z,r) \times B(z',r))}{\log r}$$

In our case, by Theorem 8.6, the pointwise dimension $\delta(\hat{\mu}_s)$ is given as the sum between the dimension of tμ_s and the dimension of the conditional measures on the vertical fibers. From the fact that $\hat{\mu}_s$ is exact dimensional, it follows also that for small $r > 0$,

$$r^{\delta(\hat{\mu}_s)-\varepsilon} \leq \hat{\mu}_s(B(z,r) \times B(z',r)) \leq r^{\delta(\hat{\mu}_s)+\varepsilon}$$

The final formula for $\delta(\hat{\mu}_s)$ follows then from Theorems 8.4 and 9.5, where we compute the Lyapunov exponent of the contraction $y \to \frac{1}{a_1(x)+y}$ in the fiber over x. Thus we obtain, $\delta(\hat{\mu}_s) = \frac{h_{\mu_s}(T)}{\lambda} + \frac{h_{\hat{\mu}_s}(\mathcal{T})}{2\int_{[0,1)^2} \log(a_1(x)+y)\ d\hat{\mu}_s(x,y)}$.

□

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