REAL ANALYTICITY FOR RANDOM DYNAMICS OF TRANSCENDENTAL FUNCTIONS

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Abstract. Analyticity results of expected pressure and invariant densities in the context of random dynamics of transcendental functions are established. These are obtained by a refinement of work by Rugh [15] leading to a simple approach to analyticity. We work under very mild dynamical assumptions. Just the iterates of the Perron-Frobenius operator are assumed to converge.

We also provide Bowen's formula expressing the almost sure Hausdorff dimension of the radial fiberwise Julia sets in terms of the zero of an expected pressure function. Our main application establishes real analyticity for the variation of this dimension for suitable hyperbolic random systems of entire or meromorphic functions.

1. Introduction

Answering a conjecture of Sullivan, Ruelle [14] showed for hyperbolic rational functions that the Hausdorff dimension of the Julia sets does depend analytically on the map and gave a local formula for perturbations of the map $z \mapsto z^2$. Since then, there where several results of this type in various contexts and also different methods of proof. The monograph [22] treats the local formula and analyticity has been obtained, for example, in [21] for complex Henon mappings of $\mathbb{C}^2$, in [11] for basic sets of surface diffeomorphisms. In the context of entire and meromorphic functions, the first result was obtained in [20], further development appeared in [6, 7] and [17].

Whereas the latter papers use holomorphic motions, Rugh [15] introduces the method of positive cones and complex cones which allowed him to extend analyticity results to random dynamics of repellers. The present paper refines Rugh's approach, avoids complex cones, and allows us to get analyticity results for random dynamics of transcendental entire and meromorphic functions. The following is a particular case of our general result Theorem 9.11.

Theorem 1.1. *Let $f_\eta(z) = \eta e^z$ and let $a \in (\frac{1}{3e}, \frac{2}{3e})$ and $0 < r < r_{max}$, $r_{max} > 0$. Suppose that $\eta_1, \eta_2, ..$ are i.i.d. random variables uniformly distributed in $\mathbb{D}(a, r)$. Let*

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$J_{\eta_1,\eta_2,\dots}$ denote the Julia set of the sequence of compositions $\big(f_{\eta_n}\circ f_{\eta_{n-1}}\circ\ldots\circ f_{\eta_2}\circ f_{\eta_1}:\mathbb{C}\to\mathbb{C}\big)_{n=1}^\infty$ and let

$$\mathcal{J}_r(\eta_1,\eta_2,\ldots)=\{z\in J_{\eta_1,\eta_2,\ldots}:\ \liminf_{n\to\infty}|f_{\eta_n}\circ\ldots\circ f_{\eta_1}(z)|<+\infty\}$$

be the radial Julia set of $(f_{\eta_n}\circ\ldots\circ f_{\eta_1})_{n=1}^\infty$. Then, the Hausdorff dimension of $\mathcal{J}_r(\eta_1,\eta_2,\ldots)$ is almost surely constant and depends real-analytically on the parameters (a,r) provided that r_{max} is sufficiently small.

The common point in all papers on this topic is the fact that the Hausdorff dimension of Julia sets can be expressed in terms of the zero of a pressure function. This fact goes back to [1] and is now called Bowen's Formula. This formula also has been generalized in many contexts and we also provide one (Theorem 9.10). We would like to mention that the zero of the involved (expected in the random case) pressure does not really detect the dimension of the whole Julia set but the dimension of its subset consisting of all radial points. In fact, in the case of hyperbolic rational functions the radial Julia set and the Julia set itself coincide. However, for transcendental functions, especially for entire functions, there is a definite difference between these sets. McMullen [9] showed that the Julia set of sine or exponential functions is always maximal equal to two whereas for such hyperbolic functions the dimension of the radial Julia set, which is often called hyperbolic dimension, is never equal to two [18, 19].

The formulation of Theorem 1.1 has been chosen deliberately in analogy with Example 1.2 in [15] since our present work stems from Rugh's papers [15, 16]. However, we were not able to apply directly his machinery. Instead we worked out a refinement of Rugh's elegant approach to analyticity. In particular, we avoid any use of Hilbert's distance in positive cones and complex cones. Instead we provide a quite simple and direct calculation (see Proposition 6.2). The outcome, besides the results concerning random transcendental dynamics, provides an elementary and general tool. In short, it says that if the thermodynamic formalism holds and if the normalized iterated transfer operator converges with a uniform speed, then real analyticity holds. Let us explain this now in more detail.

We consider arbitrary analytic families of holomorphic functions $f_{j,\lambda}$, $j\in\mathbb{Z}$, having the following properties. There exists an open set $U\subset\hat{\mathbb{C}}$ and $\delta_0\in(0,1/4)$ such that, for all $w\in U$, $j\in\mathbb{Z}$ and $n\geq 1$, every inverse branch g of the non-autonomous composition

$$f^n_{j,\lambda}:=f_{j+n-1,\lambda}\circ\ldots\circ f_{j+1,\lambda}\circ f_{j,\lambda}$$

exists on $\mathbb{D}(w,2\delta_0)$, maps $\mathbb{D}(w,\delta_0)$ inside U and satisfies $|g'|\leq\gamma_n^{-1}$ on this disk. Here $(\gamma_n)_n$ is any sequence with $\lim_{n\to\infty}\gamma_n=\infty$. As for specific examples, the reader my have in mind rational functions, functions associated to finite or infinite iterated function systems or transcendental functions. In such a setting the thermodynamical formalism including a Ruelle-Perron-Frobenius Theorem usually holds (see for example [13], [22], [12], [4], [3], [5], [7] and [8]):

$$\mathcal{L}_{j,\lambda,t} g(w) = \sum_{f_{j,\lambda}(z)=w} |f'_{j,\lambda}(z)|^{-t} \left(\frac{1+|z|^2}{1+|w|^2} \right)^{-\tau \frac{t}{2}} g(z) \quad , \quad g \in C_b^0(U) \ ,$$

defines a bounded operator on the space of bounded continuous functions $C_b^0(U)$ equipped with the sup-norm such that, for every $j \in \mathbb{Z}$,

- there exists probability measures $\nu_{j,\lambda,t}$ and reals $P_j(\lambda, t)$ such that

$$\mathcal{L}^*_{j,\lambda,t} \nu_{j+1,\lambda,t} = e^{P_j(\lambda,t)} \nu_{j,\lambda,t} \tag{1.1}$$

- and that there exist functions $\hat{\rho}_{j,\lambda,t} \in C_b^0(U)$ such that $\hat{\mathcal{L}}_{j,\lambda,t} \hat{\rho}_{j,\lambda,t} = \hat{\rho}_{j+1,\lambda,t}$ where $\hat{\mathcal{L}}_{j,\lambda,t} = e^{-P_j(\lambda,t)} \mathcal{L}_{j,\lambda,t}$ is the normalized operator.

The functions $\hat{\rho}_{j,\lambda,t}$, called invariat densities give rise to an invariant family of measures $\mu_{j,\lambda,t}$, defined as $d\mu_{j,t,\lambda} = \hat{\rho}_{j,\lambda,t} d\nu_{j,\lambda,t}$. This family is invariant in a sense that $(f_{j,\lambda})_*(\mu_{j,\lambda,t}) = \mu_{j+1,\lambda,t}$.

In here, t belongs to an interval I of positive reals and $\tau \geq 0$. When $\tau = 0$ then the above operators are just the usual geometric transfer operators used, for example, for polynomials or iterated function systems. For the infinite to one transcendental functions we have to use the additional coboundary factor with some well chosen $\tau > 0$.

In such a setting the iterated normalized operators are uniformly bounded, i.e. there exists $M < \infty$ such that

$$\|\hat{\mathcal{L}}^n_{j,\lambda,t}\|_\infty \leq M \quad \text{for all} \quad j \in \mathbb{Z}\,, \lambda \in \Lambda \text{ and } t \in I \tag{1.2}$$

where $\hat{\mathcal{L}}^n_{j,\lambda,t} = \hat{\mathcal{L}}_{j+n-1,\lambda,t} \circ ... \circ \hat{\mathcal{L}}_{j,\lambda,t}$ (see [6] and [8] for the case of transcendental functions). Also, the densities satisfy the following positivity condition as soon as the dynamical system is mixing (see for example Lemma 5.5 in [8]): there exists $z_0 \in U$ and $a > 0$ such that

$$\hat{\rho}_{j,\lambda,t}(z_0) \geq a \quad \text{for all} \quad j \in \mathbb{Z}\,, \lambda \in \Lambda \text{ and } t \in I\,. \tag{1.3}$$

We use a bounded deformation property. It is formulated in Definition 3.3 and gives a uniform control of the variation of local inverse branches. Finally, to $\eta > 0$ we associate the space of Lipschitz functions $Lip(U, \eta)$ which is the space of bounded functions $g : U \to \mathbb{R}$ such that

$$Lip(g, \eta) = \sup \left\{ \frac{|g(z_1) - g(z_2)|}{|z_1 - z_2|} \ ; \ z_1, z_2 \in U \ , \ 0 < |z_1 - z_2| < \eta \right\} < \infty\,. \tag{1.4}$$

This space is equipped with the norm $\|g\|_{Lip,\eta} = \|g\|_\infty + Lip(g, \eta)$.

Theorem 1.2. *Suppose that $f_{j,\lambda}$ are of bounded deformation and that the above thermodynamical formalism holds, in particular with* (1.2) *and* (1.3)*. Suppose that the iterated normalized operators have uniform speed: for every $\eta > 0$ there exists $\omega_n \to 0$ such that*

$$\|\hat{\mathcal{L}}^n_{j,\lambda,t} g - \nu_{j,\lambda,t}(g) \hat{\rho}_{j+n,\lambda,t}\|_\infty \leq \omega_n \|g\|_{Lip,\eta} \quad \textit{for every} \quad n \geq 1 \ , \ j \in \mathbb{Z} \ \textit{and} \tag{1.5}$$

every $g \in Lip(U,\eta)$. Let $z_0 \in U$. Then, for every $j \in \mathbb{Z}$, the function

$$(\lambda, t, z) \mapsto \rho_{j,\lambda,t}(z) = \frac{\hat{\rho}_{j,\lambda,t}(z)}{\hat{\rho}_{j,\lambda,t}(z_0)}$$

is real analytic.

Theorem 1.2 will be a consequence of Theorem 7.3, Theorem 8.1 is its random analogue. All these results concern real analyticity of invariant densities. In fact Theorem 7.3 proves a stronger version of real analyticity than the one in Theorem1.2; namely that the mapping

$$(\lambda, t) \mapsto \rho_{j,\lambda,t}$$

is real-analytic, where $\rho_{j,\lambda,t}$ is considered as a member of an appropriate natural Banach space.

As it is explained in Remark 8.3, Theorem 8.1 could also include real analyticity of expected pressure. We worked this out in detail in the case of random transcendental dynamics and the cumulating result including real analyticity of the hyperbolic dimension is Theorem 9.11.

2. General setting

We already outlined the setting in the Introduction and present now details. They will be formulated for the non-autonomous setting since all the sections to follow including Section 7 are devoted to non-autonomous dynamics. Random dynamics are the object of Section 8 and 9. We denote by $D_z = \mathbb{D}(z, \delta)$ the Euclidean disk of radius δ centered at $z \in \mathbb{C}$. Suppose given

$$\text{an open set } \; U \subset \mathbb{C}\;, \;\; 0 < \delta < \delta_0 < \frac{1}{4} \;\text{ and a sequence }\; \gamma_n \to \infty.$$

For $j \in \mathbb{Z}$, we suppose that f_j is a holomorphic function defined on some open set $V_{f_j} \subset \mathbb{C}$ with range in $\hat{\mathbb{C}}$ such that the following holds for every $j \in \mathbb{Z}$ and $n \geq 1$: the composition $f_j^n = f_{j+n-1} \circ ... \circ f_j$ is defined on some domain such that the range contains the euclidean $2\delta_0$–neighborhood of U and such that for every $w \in U$ every inverse branch g of f_j^n is well defined on $\mathbb{D}(w, 2\delta_0)$ and satisfies

$$g(\mathbb{D}(w,\delta_0)) \subset U \quad \text{and} \quad |g'|_{|\mathbb{D}(w,\delta_0)} \leq \gamma_n^{-1}\,. \tag{2.1}$$

As often, replacing the functions by some of their iterates, we can assume that $\gamma_n > 1$ for all $n \geq 1$.

Example 2.1. *The reader may have in mind the following examples:*

- $f_{j,\lambda}(z) = z^2 + \lambda c_j$ *where* $\lambda \in \mathbb{D}(0,1)$ *and* $|c_j| < \frac{1}{8}$ *or other suitable perturbations of hyperbolic rational functions.*
- *Functions arising from (finite or infinite alphabet) conformal iterated functions systems.*
- *Families of transcendental functions such as the exponential family in Theorem 1.1 and all the examples treated in* [7, 8].

From the above definition follows that every function f_j has the set U in its range V_{f_j} and that $f_j^{-1}(\overline{U}) \subset U$. As a motivation for our non-autonomous setting, note that the radial Julia set of a hyperbolic meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ (see [6] and [7] for a precises definition of this concept) is

$$\mathcal{J}_r(f) = \left\{ z \in \bigcap_{n>0} f^{-n}(U) \ : \ \liminf_{n\to\infty} |f^n(z)| < +\infty \right\} ,$$

where U here is a sufficiently small neighborhood of the Julia set $J(f)$. The straightforward adaption of this definition to the non-autonomous case is the following:

$$\mathcal{J}_r(f_j, f_{j+1}, ...) = \left\{ z \in \bigcap_{n>0} (f_j^n)^{-1}(U) \ : \ \liminf_{n\to\infty} |f_j^n(z)| < +\infty \right\} , \tag{2.2}$$

where U is now as above, for ex. in (2.1). Notice that these radial Julia sets coincide with the usual Julia sets (of the same sequence) as soon as the open set U is bounded. This is the case for rational functions (after an appropriate change of coordinates) and for iterated function systems. Unbounded sets U and radial Julia sets are necessary for transcendental dynamics.

Our results concern holomorphic families of functions. Let Λ be the corresponding parameter space. Without loss of generality, we may assume that Λ is one-dimensional and, the results being of local nature, we can restrict to the case where $\Lambda = \mathbb{D}(\lambda_0, r)$ is an open disk in $\mathbb{C}$ having arbitrarily small radius $r > 0$.

Definition 2.2. *$\mathcal{F}_\Lambda = \{f_{j,\lambda} \ , \ j \in \mathbb{Z}$ and $\lambda \in \Lambda\}$ is called a non-autonomous holomorphic family if, for every $j \in \mathbb{Z}$, $f_{j,\lambda}$ depends holomorphically on $\lambda \in \Lambda$. This precisely means the following for every $j \in \mathbb{Z}$: if $V_{f_{j,\lambda}}$ is the domain of $f_{j,\lambda}$ then $\Gamma_j := \bigcup_{\lambda\in\Lambda} \{\lambda\} \times V_{f_{j,\lambda}}$ is an open subset of $\mathbb{C}^2$ and the map $(\lambda, z) \mapsto f_{j,\lambda}(z)$ is holomorphic on Γ_j.*

3. Pairings and bounded deformation

Let $\mathcal{F}_\Lambda = \{f_{j,\lambda} \ ; \ j \in \mathbb{Z} \ , \ \lambda \in \Lambda\}$ be a non-autonomous holomorphic family. We are interested in the solutions of the equation $f_{j,\lambda}^n(z) = w$. A direct application of the implicit function theorem along with analytic continuation and (2.1) gives the following observation.

Fact 3.1. *If $\lambda' \in \Lambda$, $w' \in U$ and $z' \in f_{j,\lambda'}^{-n}(w')$ are given, then there exists a unique holomorphic function*

$$\begin{array}{ccc} \Lambda \times \mathbb{D}(w', \delta_0) & \to & U \\ (\lambda, w) & \mapsto & z(\lambda, w) \end{array}$$

such that $z(\lambda', w') = z'$ and $f_{j,\lambda}^n(z(\lambda, w)) = w$ for every $\lambda \in \Lambda$ and $w \in \mathbb{D}(w', \delta_0)$.

This is simply the proper way of defining an inverse branch $f_{j,\lambda,*}^{-n}$ of $f_{j,\lambda}^n$. We will use the inverse branch notation rather than the function $z(\lambda, w)$. This precisely means that $f_{j,\lambda,*}^{-n}$ is a choice of inverse branch defined by a function z given by Fact 3.1:

$$f_{j,\lambda,*}^{-n}(w) = z(\lambda, w) \quad , \quad \lambda \in \Lambda \, , \; w \in \mathbb{D}(w', \delta_0) \, .$$

We can now introduce the notion of pairings used in the sequel. Let us recall that $0 < \delta \le \delta_0$. The number δ will be specified later on in (6.1).

Definition 3.2. *(w_1, w_2) is a 0–pairing if $w_1 \in U$ or $w_2 \in U$ and if $|w_1 - w_2| < \delta$. For $n \ge 1$, (z_1, z_2) is called n–pairing if there exists a 0–pairing (w_1, w_2), $j \in \mathbb{Z}$, parameters $\lambda_1, \lambda_2 \in \Lambda$ and a choice of inverse branch $f_{j,\lambda,*}^{-n}$ such that*

$$z_1 = f_{j,\lambda_1,*}^{-n}(w_1) \quad \textit{and} \quad z_2 = f_{j,\lambda_2,*}^{-n}(w_2) \, .$$

The following concept of bounded deformation has already been used in [7] but without the condition (3.2). This was so since for dynamically regular transcendental functions this second condition automatically is satisfied (see Lemma 9.4). It is also possible to relax this second condition in the setting of conformal infinite iterated function systems as it has been done [17].

Definition 3.3. *The family $\mathcal{F}_\Lambda$ is of bounded deformation if there exists $A, D < \infty$ such that for every $j \in \mathbb{Z}$ and for every choice of inverse branch $f_{j,\lambda,*}^{-1}$ we have*

$$\left| \frac{\partial f_{j,\lambda,*}^{-1}}{\partial \lambda} \right| \le D \, , \quad \lambda \in \Lambda \quad \textit{and} \tag{3.1}$$

$$\left| \frac{f'_{j,\lambda_1}(z_1)}{f'_{j,\lambda_2}(z_2)} \right| = \left| \frac{f'_{j,\lambda_1}(f_{j,\lambda_1,*}^{-1}(w_1))}{f'_{j,\lambda_2}(f_{j,\lambda_2,*}^{-1}(w_2))} \right| \le A \, , \quad \lambda_1, \lambda_2 \in \Lambda \, , \; w_1, w_2 \in \mathbb{D}(w, \delta_0) \, . \tag{3.2}$$

Bounded deformation holds for many transcendental families and especially for $f_\lambda(z) = \lambda e^z$ (see [7]). Notice that (3.1) is equivalent to the fact that $\left|\frac{\partial f_{j,\lambda}}{\partial \lambda}\right| \le D \left| f'_{j,\lambda} \right|$. This condition is automatically satisfied for all rational functions and for functions associated to finite iterated function systems subject to possible shrinking of the parameter space. Also, for all systems with compact phase space such as infinite iterated function systems one can use the theory of holomorphic motions in order to show that (3.1) holds for free. So, the bounded deformation condition is mainly instrumental in the case of transcendental, and especially entire, functions.

Remember that the expanding constant $\gamma_1 > 1$. This allows us to fix a constant

$$\kappa \in (\gamma_1^{-1}, 1) \, . \tag{3.3}$$

Lemma 3.4. *If $(f_{j,\lambda})$ satisfies (3.1) then there exists a (sufficiently small) choice of diam(Λ) (depending on δ) such that every 1–pairing (z_1, z_2) satisfies $|z_1 - z_2| < \kappa\delta$.*

Remark 3.5. *Lemma 3.4 implies that every 1–pairing is a 0–pairing and, inductively, that every n–pairing is a k–pairing for all $0 \le k < n$.*

Proof. Let a 1–pairing be given by $z_i = f^{-1}_{j,\lambda_i,*}(w_i)$, $i = 1,2$, and denote $z_2' = f^{-1}_{j,\lambda_2,*}(w_1)$. The condition (3.1) implies that

$$\left|z_1 - z_2'\right| = \left|f^{-1}_{j,\lambda_1,*}(w_1) - f^{-1}_{j,\lambda_2,*}(w_1)\right| \le D\, diam(\Lambda)\,.$$

On the other hand, $|z_2' - z_2| < \gamma_1^{-1}\delta$. Therefore, $|z_1 - z_2| < \delta\gamma_1^{-1} + Ddiam(\Lambda)$ and it suffices to take $diam(\Lambda) < \delta(\kappa - \gamma_1^{-1})/D$. □

A further consequence of bounded deformation, this time of condition (3.2), is the following.

Lemma 3.6. *There exists a constant $\tilde{A} < \infty$ independent of $\delta \in (0, \delta_0)$ such that, for every $j \in \mathbb{Z}$ and every 1–pairing $(z_1, z_2) = (f^{-1}_{j,\lambda_1,*}(w_1), f^{-1}_{j,\lambda_2,*}(w_2))$,*

$$\left|arg\left(\frac{f'_{j,\lambda_1}(z_1)}{f'_{j,\lambda_2}(z_2)}\right)\right| = \left|arg\left(\frac{f'_{j,\lambda_1}(f^{-1}_{j,\lambda_1,*}(w_1))}{f'_{j,\lambda_2}(f^{-1}_{j,\lambda_2,*}(w_2))}\right)\right| \le \tilde{A} \tag{3.4}$$

provided the parameters $\lambda_1, \lambda_2 \in \mathbb{D}(\lambda_0, r/2)$. In here, the argument is well defined and understood to be the principal choice, i.e. $arg(1) = 0$.

Proof. By Koebe's distortion theorem (see for ex. Theorem 2.7 in [10]) it suffices to consider pairings for which $f_{j,\lambda_1}(z_1) = f_{j,\lambda_2}(z_2) = w$ or, in terms of inverse branches, that $z_i = f^{-1}_{j,\lambda_i,*}(w)$, $i = 1,2$. Consider then the function

$$\varphi(\lambda) = \frac{f'_{j,\lambda}(f^{-1}_{j,\lambda,*}(w))}{f'_{j,\lambda_0}(f^{-1}_{j,\lambda_0,*}(w))} \quad , \quad \lambda \in \Lambda = \mathbb{D}(\lambda_0, r)\,.$$

It has the properties $\varphi(\lambda_0) = 1$ and $A^{-1} \le |\varphi| \le A$ by (3.2). The set of all holomorphic functions having these properties is compact which implies the estimate (3.4). □

In the rest of this paper we suppose that $r = diam\Lambda/2$ is chosen such that the conclusion of Lemma 3.4 holds as well as (3.4) for every 1–pairing, i.e. (3.4) holds for all parameters $\lambda_1, \lambda_2 \in \mathbb{D}(\lambda_0, r)$.

4. Mirror extension

One step towards real analyticity is complexification of the transfer operator and its potential. There are several possibilities for this but the elegant mirror extension of Rugh is now most appropriate for us. We use mainly the notation he used in his papers [15, 16]. The mirror of the parameter space Λ and the domain U is the set

$$\Upsilon = \left\{(\lambda_1, \overline{\lambda}_2, w_1, \overline{w}_2) \;:\; \lambda_1, \lambda_2 \in \Lambda\,,\ (w_1, w_2) \text{ is a 0–pairing}\right\}. \tag{4.1}$$

Consider also the w–mirror

$$\Upsilon_w = \left\{(w_1, \overline{w}_2) \;:\; (w_1, w_2) \text{ is a 0–pairing}\right\}.$$

The initial sets $\Lambda \times U$ and U naturally identify respectively with the diagonals

$$\Delta = \{(\lambda, \overline{\lambda}, w, \overline{w}) \;:\; \lambda \in \Lambda\,,\ w \in U\} \subset \Upsilon \quad \text{and} \quad \Delta_w = \{(\omega, \overline{\omega}) \;:\; \omega \in U\} \subset \Upsilon_w\,.$$

Let $\mathcal{A} = C_b^\omega(\Upsilon_w)$ be the space of functions that are holomorphic and bounded on Υ_w. This space will be equipped with the sup-norm defined by

$$\|h\|_\infty = \sup_{(w_1,\overline{w}_2)\in\Upsilon_w} |h(w_1,\overline{w}_2)]$$

and it makes it a Banach space. We also need the following notion of Lipschitz variation on n–pairings of a function $h : \Upsilon_w \to \mathbb{C}$:

$$Lip_n(h) = \sup\left\{ \frac{|h(z_1,\overline{z}_2) - h(z_1,\overline{z}_1)|}{|z_1 - z_2|} \;,\;\; (z_1,z_2) \; n\text{–pairing with } z_1 \neq z_2 \right\}. \tag{4.2}$$

Lemma 4.1. *For every $n \geq 1$ and $h \in \mathcal{A}$ we have $Lip_n(h) \leq \|h\|_\infty/((\kappa - \gamma_n^{-1})\delta)$, i.e. for every $h \in \mathcal{A}$ and every n–pairing (z_1,z_2)*

$$|h(z_1,\overline{z}_2) - h(z_1,\overline{z}_1)| \leq \frac{\|h\|_\infty}{(\kappa - \gamma_n^{-1})\delta}|z_1 - z_2|\,. \tag{4.3}$$

with κ the constant from (3.3).

Proof. Let $\sigma = \partial\mathbb{D}(z_1,\kappa\delta)$. Cauchy's Integral Formula implies

$$|h(z_1,\overline{z}_2) - h(z_1,\overline{z}_1)| \leq$$
$$\leq \frac{1}{(2\pi)^2}\int_\sigma\int_\sigma \left| \frac{h(\xi_1,\overline{\xi}_2)}{(\xi_1 - z_1)(\overline{\xi}_2 - \overline{z}_2)} - \frac{h(\xi_1,\overline{\xi}_2)}{(\xi_1 - z_1)(\overline{\xi}_2 - \overline{z}_1)} \right| |d\xi_1||d\overline{\xi}_2|.$$

Elementary estimations give $|\xi_i - z_1| = \kappa\delta$ and $|\xi_i - z_2| \geq \delta(\kappa - \gamma_n^{-1})$, $i = 1,2$. The required estimation follows now easily. □

The space $\mathcal{A}$ contains the relevant subspace

$$\mathcal{A}_\mathbb{R} = \left\{ h \in \mathcal{A} \;:\;\; h_{|\Delta_w} \in \mathbb{R} \right\}.$$

Functions from $\mathcal{A}_\mathbb{R}$ are real on the diagonal and can therefore be identified with a subclass of real functions defined on U. Up to identification, they belong to the space of Lipschitz functions $Lip(U,\eta)$ (see Introduction) provided $\eta < \kappa\delta$.

Lemma 4.2. *If $h \in \mathcal{A}_\mathbb{R}$ then $z \mapsto g(z) := h(z,\overline{z})$ belongs to $Lip(U,\kappa\delta)$ and*

$$\|g\|_{Lip,\kappa\delta} \leq C\|h\|_\infty$$

where $C = 1 + 2/((\sqrt{\kappa} - \kappa)\delta)$.

Proof. Let $h \in \mathcal{A}_\mathbb{R}$ and let $z_1,z_2 \in U$ with $0 < |z_1 - z_2| < \kappa\delta$. Consider $\sigma = \partial\mathbb{D}(z_1,\sqrt{\kappa}\delta)$ and use exactly the same argument then in the proof of Lemma 4.1 based on Cauchy's Integral Formula in order to obtain the estimates

$$|h(z_1,\overline{z}_2) - h(z_1,\overline{z}_1)| \leq \frac{1}{(\sqrt{\kappa} - \kappa)\delta}\|h\|_\infty|z_1 - z_2|\,.$$

The same argument also gives the following symmetric version of this estimaties:

$$|h(z_1,\overline{z}_2) - h(z_2,\overline{z}_2)| \leq \frac{1}{(\sqrt{\kappa} - \kappa)\delta}\|h\|_\infty|z_1 - z_2|\,.$$

It suffices now to combine these two estimations in order to complete this proof. □

4.1. **Potentials and extended operator.** The potentials under consideration must have two properties: they must admit holomorphic mirror extensions and have good distortion properties. We do not treat the most general setting but focus in the following on the most important class of potentials and will see that they have the required properties. So, suppose that $\tau \geq 0$ is fixed, that I is an open interval compactly contained in $(0, \infty)$, consider

$$\varphi_{j,\lambda,t}(z) = -t \log |f'_{j,\lambda}(z)| - t\frac{\tau}{2} \log \left(\frac{1+|z|^2}{1+|f_{j,\lambda}(z)|^2} \right) \tag{4.4}$$

and observe that $|f'_{j,\lambda}|_\tau^{-t} = e^{\varphi_{j,\lambda,t}}$, $\lambda \in \Lambda$ and $t \in I$, where $|f'_{j,\lambda}|_\tau$ denotes the derivative with respect to the Riemannian conformal metric $|dz|/(1+|z|^2)^{\frac{\tau}{2}}$. The transfer operator $\mathcal{L}_j = \mathcal{L}_{j,\lambda,t}$ of the function $f_{j,\lambda}$ and the potential $\varphi_{j,\lambda,t}$ is defined by

$$\mathcal{L}_j g(w) = \sum_{f_{j,\lambda}(z)=w} e^{\varphi_{j,\lambda,t}(z)} g(z) = \sum_{f_{j,\lambda}(z)=w} |f'_{j,\lambda}(z)|_\tau^{-t} g(z) \quad , \quad w \in U\,, \tag{4.5}$$

where $g \in C_b^0(U)$ is a continuous bounded function on U. The classical case, particularly when one deals with polynomials or iterated function systems, is when $\tau = 0$. For transcendental functions $\tau > 0$, i.e. the additional coboundary term $\log(1+|z|^2) - \log(1+|f_{j,\lambda}(z)|^2)$, is needed since otherwise the transfer operator is not well defined the series defining it being divergent.

The n-th composition of these operators is

$$\mathcal{L}_j^n = \mathcal{L}_{j+n-1} \circ ... \circ \mathcal{L}_j\,. \tag{4.6}$$

A standard calculation shows that $\mathcal{L}_j^n$ is the transfer operator as defined in (4.5) of the potential

$$S_n\varphi_j = \sum_{k=0}^{n-1} \varphi_{j+k} \circ f_j^k = \sum_{k=0}^{n-1} \varphi_{j+k,\lambda,t} \circ f_{j,\lambda}^k\,.$$

The potentials defined in (4.4), often called geometric, admit mirror extensions as we explain now. In the following, $\mathcal{I}$ is a complex neighborhood of $I \subset \mathbb{R}$. For $w \in U$, define $Z_w = \Lambda \times \overline{\Lambda} \times D_w \times \overline{D_w}$ and notice that $\Upsilon \subset \bigcup_{w\in U} Z_w$. From Fact 3.1 applied with $n = 1$ follows that, to every choice of $\lambda' \in \Lambda$ and $z' \in f_{j,\lambda'}^{-1}(w)$, there corresponds a choice of inverse branches $f_{j,\lambda,*}^{-1}$ defined on $\Lambda \times D_w$. Consider then on Z_w the map

$$(\lambda_1, \overline{\lambda}_2, w_1, \overline{w}_2) \mapsto (\lambda_1, \overline{\lambda}_2, f_{j,\lambda_1,*}^{-1}(w_1), \overline{f_{j,\lambda_2,*}^{-1}(w_2)}) \tag{4.7}$$

and denote its range by $Z_{j,w,*}^{-1}$. Notice that Lemma 3.4 and (2.1) imply

$$Z_{j,w,*}^{-1} \subset Z_{w'} \cap (\Lambda \times \overline{\Lambda} \times U \times \overline{U}) \quad \text{ for some } w' \in U.$$

Given the definition of the transfer operator in (4.5), it suffices to extend the potentials to

$$\Upsilon^{-1} \times \mathcal{I} := \bigcup_{w,*} Z_{j,w,*}^{-1} \times \mathcal{I} \subset \Upsilon \times \mathcal{I}\,. \tag{4.8}$$

The extension of $\varphi_{j,\lambda,t}$ to one of the sets $Z^{-1}_{j,w,*} \times \mathcal{I}$ is straightforward. Indeed, let

$$\Phi_{j,\lambda_1,\overline{\lambda}_2,t}(z_1,\overline{z}_2) = -\frac{t}{2}\log\left(f'_{j,\lambda_1}(z_1)\overline{f'_{j,\lambda_2}(z_2)}\right) - t\frac{\tau}{2}\log\left(\frac{1+z_1\overline{z}_2}{1+f_{j,\lambda_1}(z_1)\overline{f_{j,\lambda_2}(z_2)}}\right) \tag{4.9}$$

where $(\lambda_1,\overline{\lambda}_2,z_1,\overline{z}_2,t) \in Z^{-1}_{j,w,*} \times \mathcal{I}$. Notice that the expression in the first logarithm never equals zero. Also, the expression in the second logarithm is well defined and never equal to zero since (z_1,z_2) as well as $(w_1,w_2) = (f_{j,\lambda_1}(z_1), f_{j,\lambda_2}(z_2))$ are pairings and thus their respective distance is at most $\delta_0 \leq \frac{1}{4}$. Since, moreover, the set Λ is simply connected, both logarithms in (4.9) are well defined and we can and will take the principle branch since for $(\lambda_1,\overline{\lambda}_2,z_1,\overline{z}_2) = (\lambda,\overline{\lambda},z,\overline{z}) \in \Delta \cap Z^{-1}_{w,*}$ both expressions in the arguments of the logarithms are real positives. We thus have a properly defined map Φ_j on every set $Z^{-1}_{j,w,*}$.

The map Φ_j is in fact a global well defined map on the union $\bigcup_{w,*} Z^{-1}_{j,w,*} \times \mathcal{I}$. In order to see this, consider two sets $Z^{-1}_{j,w,*}$ and $Z^{-1}_{j,w',*'}$ having nonempty intersection. Then $\Delta \cap Z^{-1}_{j,w,*} \cap Z^{-1}_{j,w',*'}$ is a non-empty non-analytic subset of $Z^{-1}_{j,w,*} \cap Z^{-1}_{j,w',*'}$ and Φ_j restricted to $(\Delta \cap Z^{-1}_{j,w,*} \cap Z^{-1}_{j,w',*'}) \times I$ is real and coincides with the given potential φ_j. The map Φ_j is thus the desired extension of φ_j to $\Upsilon^{-1} \times \mathcal{I}$.

Given this extended potential and using the inclusion in (4.8), we can now consider the extended operator $L_{j,\lambda_1,\overline{\lambda}_2,t}$ acting on functions $g \in \mathcal{A}$ by

$$L_{j,\lambda_1,\overline{\lambda}_2,t}g(w_1,\overline{w}_2) = \sum_{z_1,z_2} \exp\left(\Phi_{j,\lambda_1,\overline{\lambda}_2,t}(z_1,\overline{z}_2)\right) g(z_1,\overline{z}_2) \tag{4.10}$$

where the summation is taken over all 1–pairings (z_1,z_2) such that $f_{j,\lambda_i}(z_i) = w_i$, $i = 1,2$. As for the initial real operator $\mathcal{L}_j$ it is convenient to write simply L_j instead of $L_{j,\lambda_1,\overline{\lambda}_2,t}$ when it is clear that the parameters λ_1,λ_2,t are fixed.

In the next proposition we will see that the image function $L_{\lambda_1,\overline{\lambda}_2,t}g \in \mathcal{A}$ provided the initial real operator $\mathcal{L}_{\lambda,t}$ is bounded. This will allow us to iterate the operator and this will be done again in a non-autonomous way: (in (4.11) we use the abbreviated notation $L_k := L_{k,\lambda_1,\overline{\lambda}_2,t}$)

$$L^n_j = L_{j+n-1} \circ ... \circ L_j \tag{4.11}$$

is the extension of $\mathcal{L}^n_j$ defined in (4.6). Notice that the cocyle properties of inverse branches along with Lemma 3.4 show that L^n_j can also be defined by formula (4.10) if one replaces the potential $\Phi_{j,\lambda_1,\overline{\lambda}_2,t}$ by

$$S_n\Phi_{j,\lambda_1,\overline{\lambda}_2,t}(z_1,\overline{z}_2) = \sum_{k=0}^{n-1} \Phi_{j+k,\lambda_1,\overline{\lambda}_2,t}(f^k_{\lambda_1}(z_1), \overline{f^k_{\lambda_2}(z_2)})$$

and where the summation is taken over all n–pairings (z_1,z_2) such that $f^n_{j,\lambda_i}(z_i) = w_i$, $i = 1,2$.

Proposition 4.3. *Suppose that the real operator $\mathcal{L}_{j,\lambda,t}$ is uniformly bounded for $j \in \mathbb{Z}$, $\lambda \in \Lambda$ and $t \in I$ and that $r = diam(\Lambda)/2$ is sufficiently small such that* (3.4) *holds for all* 1*–pairings. Then there exist $a > 0$ such that, with*

$$\mathcal{I} = \{x + iy \in \mathbb{C} \; ; \; x \in I \; , \; y \in]-a, a[\},$$

the extended operator $L_{j,\lambda_1,\overline{\lambda}_2,t}$ is a, uniformly for $j \in \mathbb{Z}$, $(\lambda_1, \overline{\lambda}_2, t) \in \Lambda \times \overline{\Lambda} \times \mathcal{I}$, bounded operator of $\mathcal{A}$. Moreover, if $\lambda_1 = \lambda_2 =: \lambda$ and if $t \in I$ is real, then each operator $L_k := L_{k,\lambda,\overline{\lambda},t}$ preserves $\mathcal{A}_{\mathbb{R}}$ and there exists $K < \infty$ such that, for every function $h \in \mathcal{A}$,

$$\left| L_j^n h(w_1, \overline{w}_2) - L_j^n h(w_1, \overline{w}_1) \right| \le \mathcal{L}^n_{j,\lambda,t} 1\!\!1(w_1) \left(K + \frac{\gamma_n^{-1}}{\delta(1 - \gamma_n^{-1})} \right) \|h\|_\infty |w_1 - w_2| \tag{4.12}$$

where $(w_1, \overline{w}_2) \in \Upsilon_w$ and $n \ge 1$, and, as in (4.11) *$L_j^n = L_{j+n-1} \circ \cdots \circ L_j$.*

Proof. Let $j \in \mathbb{Z}$, $(\lambda_1, \overline{\lambda}_2, w_1, \overline{w}_2) \in \Upsilon$ and let $t \in \mathcal{I}$ be complex. For every $g \in \mathcal{A}$ we have

$$\left| L_{j,\lambda_1,\overline{\lambda}_2,t} g(w_1, \overline{w}_2) \right| \le \|g\|_\infty \sum_{z_1, z_2} \left| \exp\left(\Phi_{j,\lambda_1,\overline{\lambda}_2,t}(z_1, \overline{z}_2) \right) \right|$$

where the summation is again over all corresponding 1–pairings like in (4.10). Therefore, it suffices to estimate the series on the right hand side of this inequality in order to get a bound of the norm of the operator $L_{j,\lambda_1,\overline{\lambda}_2,t}$ on $\mathcal{A}$.

Now, if $(z_1, \overline{z}_2)$ be a 1–pairing such that $f_{j,\lambda_i}(z_i) = w_i$, $i = 1, 2$, then

$$\left| \exp(\Phi_{j,\lambda_1,\overline{\lambda}_2,t}(z_1, \overline{z}_2)) \right| = \left| f'_{j,\lambda_1}(z_1) f'_{j,\lambda_2}(z_2) \right|^{-\frac{\Re t}{2}} \exp\left\{ \frac{\Im t}{2} \arg\left(f'_{j,\lambda_1}(z_1) \overline{f'_{\lambda_2}(z_2)} \right) \right\}$$
$$\times \left| \left(\frac{1 + z_1 \overline{z}_2}{1 + w_1 \overline{w}_2} \right)^{-t\frac{\tau}{2}} \right| .$$

The choice of $r > 0$ and (3.4) shows that $\left| \arg\left(f'_{j,\lambda_1}(z_1) \overline{f'_{j,\lambda_2}(z_2)} \right) \right| \le \tilde{A}$. Since $|\Im t| \le a$ it follows that

$$\exp\left\{ \frac{\Im t}{2} \arg\left(f'_{j,\lambda_1}(z_1) \overline{f'_{j,\lambda_2}(z_2)} \right) \right\} \le \exp\left\{ \frac{a}{2} \tilde{A} \right\} .$$

Clearly, $\left| \arg\left(\frac{1 + z_1 \overline{z}_2}{1 + w_1 \overline{w}_2} \right) \right|$ is bounded above uniformly with respect to z_i, w_i, $i = 1, 2$. Denote this bound again by $\tilde{A}$. Setting $B = \exp\left\{ a\tilde{A} \frac{1+\tau}{2} \right\}$ it follows that

$$\left| \exp(\Phi_{j,\lambda_1,\overline{\lambda}_2,t}(z_1, \overline{z}_2)) \right| \le B \left| f'_{j,\lambda_1}(z_1) f'_{j,\lambda_2}(z_2) \right|^{-\frac{\Re t}{2}} \left| \frac{1 + z_1 \overline{z}_2}{1 + w_1 \overline{w}_2} \right|^{-\frac{\tau}{2} \Re t} .$$

An elementary calculation shows that there exists a constant $C < \infty$ independent of z_i, w_i, $i = 1, 2$, and $t \in I$, such that

$$\left| \frac{1 + z_1 \overline{z}_2}{1 + w_1 \overline{w}_2} \right|^{-\frac{\tau}{2} \Re t} \le C \sqrt{ \frac{1 + |z_1|^2}{1 + |w_1|^2} \frac{1 + |z_2|^2}{1 + |w_2|^2} }^{\,-\frac{\tau}{2} \Re t} .$$

Therefore,

$$\left|\exp(\Phi_{j,\lambda_1,\overline{\lambda}_2,t}(z_1,\overline{z}_2))\right| \leq BC|f'_{j,\lambda_1}(z_1)|^{-\frac{\Re t}{2}}\left(\frac{1+|z_1|^2}{1+|w_1|^2}\right)^{-\frac{\tau\Re t}{4}} \times$$
$$\times |f'_{j,\lambda_2}(z_2)|^{-\frac{\Re t}{2}}\left(\frac{1+|z_2|^2}{1+|w_2|^2}\right)^{-\frac{\tau\Re t}{4}},$$

and thus the Cauchy-Schwarz inequality implies that

$$\sum_{z_1,z_2}\left|\exp\left(\Phi_{j,\lambda_1,\overline{\lambda}_2,t}(z_1,\overline{z}_2)\right)\right| \leq BC\sqrt{\mathcal{L}_{j,\lambda_1,\Re t}1\!\!1(w_1)}\sqrt{\mathcal{L}_{j,\lambda_2,\Re t}1\!\!1(w_2)}\,.$$

By our assumptions there exists $M<\infty$ such that $\|\mathcal{L}_{j,\lambda,t_0}1\!\!1\|_\infty \leq M$ for every $j\in\mathbb{Z}$, $\lambda\in\Lambda$ and $t_0\in I$. This shows that

$$\|L_{j,\lambda_1,\overline{\lambda}_2,t}\|_\infty \leq BCM\,. \tag{4.13}$$

Suppose now that $\lambda_1=\lambda_2=:\lambda$ and that $t\in I$ is real. In this case each operator $L_{k,\lambda,\overline{\lambda},t}$ clearly preserves $\mathcal{A}_\mathbb{R}$.

It remains to establish the distortion property (4.12). We have

$$\left|L_j^n h(w_1,\overline{w}_2) - L_j^n h(w_1,\overline{w}_1)\right| \leq I + II$$

where

$$I=\left|\sum \exp S_n\Phi_{j,\lambda,\overline{\lambda},t}(z_1,\overline{z}_1)\left(h(z_1,\overline{z}_2)-h(z_1,\overline{z}_1)\right)\right| \leq \mathcal{L}^n_{j,\lambda,t}1\!\!1(w_1)\,Lip_n(h)\gamma_n^{-1}|w_1-w_2|\,.$$

Lemma 4.1 gives an appropriate estimation for $Lip_n(h)$ and thus

$$I\leq \mathcal{L}^n_{j,\lambda,t}1\!\!1(w_1)\,\frac{\|h\|_\infty}{\delta(1-\gamma_n^{-1})}\gamma_n^{-1}|w_1-w_2|\,.$$

The second term is equal to

$$II=\left|\sum\left(\exp S_n\Phi_{j,\lambda,\overline{\lambda},t}(z_1,\overline{z}_2)-\exp S_n\Phi_{j,\lambda,\overline{\lambda},t}(z_1,\overline{z}_1)\right)h(z_1,\overline{z}_2)\right|\,.$$

The following distortion estimate directly results from the complex version of Koebe's distortion theorem in the case $\tau=0$ and from Lemma 4.7 in [7] if $\tau>0$:

$$\left|\frac{\exp S_n\Phi_{j,\lambda,\overline{\lambda},t}(f^{-n}_{j,\lambda,*}(w_1),\overline{f^{-n}_{j,\lambda,*}(w_2)})}{\exp S_n\Phi_{j,\lambda,\overline{\lambda},t}(f^{-n}_{j,\lambda,*}(w_1),\overline{f^{-n}_{j,\lambda,*}(w_1)})}-1\right| \leq K|w_1-w_2|\;,\;\; w_1,w_2\in\mathbb{D}(w,\delta).$$

Consequently,

$$II\leq \mathcal{L}_{j,\lambda,t}1\!\!1(w_1)\|h\|_\infty K|w_1-w_2|$$

and, combining this estimate with the one of I, the desired Lipschitz property follows. □

5. Complexification of the invariant density

We have to consider appropriate rescaled versions of the operators defined in the previous section. This section deals with the case where $\lambda_1 = \lambda_2 =: \lambda$ and $t \in I$ is real. Moreover, here and in the next section both parameters λ, t are fixed and so we will frequently surpress them:

$$\hat{\mathcal{L}}_j = e^{-P_j(t)}\mathcal{L}_j \quad \text{and} \quad \hat{L}_j = e^{-P_j(t)}L_j \quad , \quad j \in \mathbb{Z}\,. \tag{5.1}$$

The number $P_j(t)$ is usually called the topological pressure. Assume that for these rescaled operators there exist strictly positive functions $\hat{\rho}_j \in C_b^0(U)$ such that, for some $M < \infty$ and for every $j \in \mathbb{Z}$ and $n \geq 1$,

$$\|\hat{\mathcal{L}}_j^n\|_\infty \leq M \quad \text{and} \quad \hat{\mathcal{L}}_{-n+j}^n 1\!\!1 \to \hat{\rho}_j\,. \tag{5.2}$$

where the limit is with respect to the sup-norm as $n \to \infty$. Then clearly

$$\hat{\mathcal{L}}_j\hat{\rho}_j = \hat{\rho}_{j+1}, \quad j \in \mathbb{Z}\,,$$

and, for this reason, these functions are called invariant densities. The aim now is to extend the invariant densities to holomorphic functions of $\mathcal{A}_\mathbb{R}$ such that (5.2) still holds.

Proposition 5.1. *Suppose* (5.2) *does hold. Then, for every $j \in \mathbb{Z}$, the sequence $\hat{L}_{-n+j}^n 1\!\!1$ converges uniformly on compact sets to some function of $\mathcal{A}_\mathbb{R}$. These limit functions are extensions of $\hat{\rho}_j$ and they will be denoted by the same symbol. Moreover,*

$$|\hat{\rho}(w_1, \overline{w}_2) - \hat{\rho}(w_1, \overline{w}_1)| \leq M(K+1)|w_1 - w_2|\,, \quad (w_1, w_2) \in \Upsilon_w\,,$$

and the invariance property

$$\hat{L}_j\hat{\rho}_j = \hat{\rho}_{j+1} \quad \textit{holds on } \Upsilon_w \textit{ for every } j \in \mathbb{Z}. \tag{5.3}$$

Proof. Let $(w_1, \overline{w}_2) \in \Upsilon_w$. The distortion property (4.12) implies that there exists $n_0 \geq 0$ such that for every $n \geq n_0$

$$\left|\hat{L}_{-n+j}^n 1\!\!1(w_1, \overline{w}_2) - \hat{L}_{-n+j}^n 1\!\!1(w_1, \overline{w}_1)\right| \leq \hat{\mathcal{L}}_{-n+j}^n 1\!\!1(w_1)(K+1)|w_1 - w_2|\,. \tag{5.4}$$

Since $\hat{L}_{-n+j}^n 1\!\!1(w_1, \overline{w}_1) = \hat{\mathcal{L}}_{-n+j}^n 1\!\!1(w_1)$ it follows that

$$\begin{aligned}\left|\hat{L}_{-n+j}^n 1\!\!1(w_1, \overline{w}_2)\right| &\leq \hat{\mathcal{L}}_{-n+j}^n 1\!\!1(w_1)\Big(1 + (K+1)|w_1 - w_2|\Big) \\ &\leq M\Big(1 + (K+1)\delta\Big) \leq M(K+2) \quad \text{for every} \quad n \geq n_0\,.\end{aligned}$$

Therefore, the sequence $\left(\left|\hat{L}_{-n+j}^n 1\!\!1(w_1, \overline{w}_2)\right|\right)_{n=0}^\infty$ is uniformly bounded above. Montel's Theorem thus applies and yields normality of the family $\left(\hat{L}_{-n+j}^n 1\!\!1\right)_{n=0}^\infty$. Since the limit of every converging subsequence coincides with $\hat{\rho}_j$ on the non-analytic set Δ_w the whole sequence $\left(\hat{L}_{-n+j}^n 1\!\!1\right)_{n=0}^\infty$ converges to one and the same limit and this limit belongs to $\mathcal{A}_\mathbb{R}$.

The invariance property (5.3) holds since it holds on the non-analytic set Δ_w. Finally, the limit functions have the required Lipschitz property because of (5.2) and (5.4). □

The obvious modification of this proof, where $1\!\!1$ is replaced by an arbitrary element of $\mathcal{A}_{\mathbb{R}}$, also shows that the normalized extended operators and invariant densities are uniformly bounded above. Whenever (5.2) holds we may assume, increasing M if necessary, that

$$\|\hat{L}_j^n\|_\infty \le M \quad \text{and} \quad \|\hat{\rho}_j\|_\infty \le M \ \text{ for every } \ j\in\mathbb{Z}\,,\ \ n\ge 0\,. \tag{5.5}$$

In the condition (5.5), $\hat{\rho}_j$ is the extended density and the sup-norms are taken on the whole mirror Υ_w.

In the sequel we will need a different normalization. Let $l : C_b^0(U)\to\mathbb{R}$ be a bounded functional. It naturally acts on functions of $\mathcal{A}_{\mathbb{R}}$: if $h\in\mathcal{A}_{\mathbb{R}}$ then $g(z) = h(z,\overline{z})$, $z\in U$, defines a function $g\in C_b^0(U)$ and thus we can define

$$l(h) := l(g)\,.$$

In particular, $l(\hat{\rho}_j)$ is well defined regardless of whether $\hat{\rho}_j$ is understood as the initial function of $C_b^0(U)$ or the extended function that belong to $\mathcal{A}_{\mathbb{R}}$.

The functional l is assumed to be uniformly positive on the density functions meaning that there exists $a>0$ such that

$$l(\hat{\rho}_j)\ge a \quad \text{for every } j\in\mathbb{Z}. \tag{5.6}$$

Example 5.2. *Fix any point $\xi\in U$ and consider the functional l defined by $l(g) = g(\xi)$. Such a functional is uniformly positive on the functions $\hat{\rho}_j$ in the sense of* (5.6) *as soon as the system is mixing. This holds in particular for the transcendental random systems considered in* [8]*. Lemma 5.5 of that paper shows that there exists $n_0\ge 1$ and $a>0$ such that*

$$\hat{\mathcal{L}}_j^n 1\!\!1(\xi)\ge a \quad \textit{for every} \ \ n\ge n_0\,. \tag{5.7}$$

Consider then

$$\rho_j = \frac{\hat{\rho}_j}{l(\hat{\rho}_j)} \quad , \quad j\in\mathbb{Z}\,. \tag{5.8}$$

Clearly,

$$\lim_{n\to\infty}\frac{\mathcal{L}_{-n+j}^n 1\!\!1}{l(\mathcal{L}_{-n+j}^n 1\!\!1)} = \lim_{n\to\infty}\frac{\hat{\mathcal{L}}_{-n+j}^n 1\!\!1}{l(\hat{\mathcal{L}}_{-n+j}^n 1\!\!1)} = \rho_j$$

and, because of (5.3), the extended invariant densities satisfy

$$\frac{L_j^n(\rho_j)}{l(L_j^n(\rho_j))} = \rho_{j+n} \quad \text{for every } j\in\mathbb{Z} \text{ and } n\ge 1. \tag{5.9}$$

It is henceforth natural to consider maps $\Psi_{n,j}$ defined by

$$\Psi_{n,j}(g) = \frac{L_j^n(g)}{l(L_j^n(g))} = \frac{\hat{L}_j^n(g)}{l(\hat{L}_j^n(g))} \quad \text{for every } j \in \mathbb{Z} \text{ and } n \geq 1. \tag{5.10}$$

Lemma 5.3. *For every $j \in \mathbb{Z}$ and $n \geq 1$, the map $\Psi_{n,j}$ is well defined on the following neighborhood of ρ_j in $\mathcal{A}$:*

$$U_j := \left\{ g \in \mathcal{A} : \quad \|g - \rho_j\|_\infty < \frac{a}{2(\|l\|_\infty M)^2} \right\} .$$

Proof. For $g \in U_j$ we have to check that

$$l(\hat{L}_j^n(g)) = l(\hat{L}_j^n(\rho_j)) + l(\hat{L}_j^n(g - \rho_j)) \neq 0 .$$

Since

$$l(\hat{L}_j^n(\rho_j)) = \frac{l(\hat{L}_j^n(\hat{\rho}_j))}{l(\hat{\rho}_j)} = \frac{l(\hat{\rho}_{n+j})}{l(\hat{\rho}_j)} ,$$

since $l(\hat{\rho}_{n+j}) \geq a$ by (5.6) and, since $l(\hat{\rho}_j) \leq \|l\|_\infty M$ by (5.5), we have

$$l\big(\hat{L}^n(\rho_j)\big) \geq \frac{a}{\|l\|_\infty M} . \tag{5.11}$$

On the other hand, if $g \in U_j$ then $\|l(\hat{L}_j^n(g - \rho_j))\|_\infty \leq \|l\|_\infty M \|g - \rho_j\|_\infty < \frac{a}{2\|l\|_\infty M}$. Altogether we get $l(\hat{L}_j^n(g)) > \frac{a}{\|l\|_\infty M} - \frac{a}{2\|l\|_\infty M} = \frac{a}{2\|l\|_\infty M} > 0$. □

6. Contraction

We shall exploit in detail the convergence of the normalized iterated operators under the assumption that there is a uniform speed of the convergence in (5.2). Let us make this precise now (see also the condition (1.5) in Theorem 1.2). We keep in this section the setting and notation of Section 5 and assume again that (5.2) and (5.6) hold. We also recall that (5.2) implies (5.5).

We now fix $\delta > 0$ sufficiently small such that

$$\frac{M}{a} \|l\|_\infty \left(M \left(K + 1 \right) + Q \|l\|_\infty M \right) \delta \leq \frac{1}{4} , \tag{6.1}$$

where $Q = \frac{M}{a}(K + 1)$. Notice that diminishing δ does not influence the involved constants since M does not depend on δ and the distortion constant K becomes even better if δ is replaced by a smaller constant.

We shall formulate now the precise condition which we shall need in the sequel:

Uniform speed. *There exist bounded linear functionals $\nu_j \in \mathcal{A}'_{\mathbb{R}}$ and there exists a sequence $\omega_n \to 0$ such that*

$$\|\hat{\mathcal{L}}_j^n(h_{|\Delta_w}) - \nu_j(h)\hat{\rho}_{j+n}\|_{\infty,\Delta_w} \leq \omega_n \|h\|_{\infty,\Upsilon_w} \quad \textit{for every} \quad h \in \mathcal{A}_{\mathbb{R}} \,, \;\; n \geq 1 . \tag{6.2}$$

In order to avoid any confusion we indicated here the domain on which the sup-norm is taken. So on the left hand side of the inequality the supremum is taken over all points of the diagonal Δ_w, which is identified with U, whereas on the right-hand side one takes into account the whole mirror Υ_w.

We have chosen the notation ν_j since typical examples of these functionals are the measures of (1.1) that often are called conformal measures.

Lemma 6.1. *Assume that* (5.2), (5.6) *and* (6.2) *hold. Then*

$$\nu_j(\hat{\rho}_j) = 1 \quad , \quad j \in \mathbb{Z}. \tag{6.3}$$

Proof. Apply (6.2) with $h = \hat{\rho}_j$ and use the invariance property (5.3) in order to get

$$\|\hat{\mathcal{L}}_j^n \hat{\rho}_j - \nu_j(\hat{\rho}_j)\hat{\rho}_{j+n}\|_{\infty,\Delta_w} = |1 - \nu_j(\hat{\rho}_j)| \|\hat{\rho}_{j+n}\|_{\infty,\Delta_w} \leq \omega_n \|\hat{\rho}_j\|_{\infty,\Upsilon_w}$$

By (5.5), $\|\hat{\rho}_j\|_{\infty,\Upsilon_w} \leq M$. On the other hand, (5.6) implies that $\|\hat{\rho}_{j+n}\|_{\infty,\Delta_w} \geq a/\|l\|_\infty$. Since $\omega_n \to 0$ as $n \to \infty$ we thus must have $\nu_j(\hat{\rho}_j) = 1$. □

Let us now focus on $\mathcal{L}_0^n$, $n \geq 1$, and use the simplified notations

$$\mathcal{L}^n = \mathcal{L}_0^n,\ L^n = L_0^n,\ \nu = \nu_0,\ \rho = \rho_0,\ \hat{\rho} = \hat{\rho}_0,\ \Psi_n = \Psi_{n,0}.$$

Concerning the functional l, we already have explained the action of this functional on $\mathcal{A}_\mathbb{R}$. It also can be extended to $\mathcal{A}$ by first extending it to complex functions in the usual way and then to functions $h \in \mathcal{A}$ by $l(h) := l(h_{|\Delta_w})$. Remember also the map Ψ_n given by $\Psi_n(g) = \frac{L^n(g)}{l(L^n(g))}$ is, for every $n \geq 1$, well defined on the neighborhood U_0 of $\rho = \rho_0$ (see Lemma 5.3).

Proposition 6.2. *Suppose that* (5.2), (5.6) *and the uniform speed condition hold. Then, for every $\delta \in]0, \delta_0]$ sufficiently small there exists $n \geq 1$ such that the differential of Ψ_n at ρ satisfies*

$$\|D_\rho \Psi_n\|_\infty \leq \frac{\sqrt{2}}{2} < 1\,.$$

Remark 6.3. *The proof will show that the integer n does not depend on the operators $\mathcal{L}_j$ hence not on the functions f_j, $j \in Z$, but only on the involved constants such as a, M, ω_n. In other words, n is uniform for all families of operators as long as they satisfy the conditions* (5.5), (5.6) *and the uniform speed with the same constants. This is in particular the case for all $\Psi_{n,j}$, $j \in \mathbb{Z}$.*

Proof. Let $h \in \mathcal{A}$. From (5.9) we get $\Psi_n(\rho) = \rho_n$ and

$$\begin{aligned}\Psi_n(\rho + h) &= \frac{L^n(\rho) + L^n(h)}{l(L^n(\rho)) + l(L^n(h))} = \frac{\rho_n + L^n(h)/l(L^n(\rho))}{1 + l(L^n(h))/l(L^n(\rho))} \\ &= \rho_n + \frac{L^n(h)}{l(L^n(\rho))} - \rho_n \frac{l(L^n(h))}{l(L^n(\rho))} + o(\|h\|)\,.\end{aligned}$$

Hence,

$$D_\rho \Psi_n(h) = \frac{L^n(h)}{l(L^n(\rho))} - \rho_n \frac{l(L^n(h))}{l(L^n(\rho))}\,.$$

Consider first the case where $h \in \mathcal{A}_{\mathbb{R}}$. It suffices to consider functions h for which $\|h\|_\infty \le 1$. If we evaluate the above expression at points $(w,\overline{w}) \in \Delta_w$ of the diagonal then we can use (6.2) and it follows that there are functions ξ_n such that $\|\xi_n\|_\infty \le \omega_n$ and such that

$$\hat{L}^n(h)(w,\overline{w}) = \nu(h)\hat{\rho}_n(w) + \xi_n(w)\,.$$

Consequently,

$$\frac{L^n(h)}{l(L^n(\rho))} = \frac{\hat{L}^n(h)}{l(\hat{L}^n(\rho))} = \frac{\nu(h)\hat{\rho}_n + \xi_n}{l(\hat{L}^n(\rho))} \quad \text{on} \quad \Delta_w\,.$$

Thus

$$D_\rho\Psi_n(h)_{|\Delta_w} = \frac{\nu(h)\hat{\rho}_n + \xi_n}{l(\hat{L}^n(\rho))} - \rho_n\frac{\nu(h)l(\hat{\rho}_n) + l(\xi_n)}{l(\hat{L}^n(\rho))} = \frac{\xi_n - \rho_n l(\xi_n)}{l(\hat{L}^n(\rho))}\,.$$

This expression can be estimated as follows. From (5.11) we have $l\big(\hat{L}^n(\rho)\big) \ge \frac{a}{\|l\|_\infty M}$. For the same reasons, i.e. from (5.5) and (5.6), we also have that $\|\rho_n\|_\infty = \frac{\|\hat{\rho}_n\|_\infty}{l(\hat{\rho}_n)} \le \frac{M}{a}$. Altogether it follows that

(6.4)
$$\|D_\rho\Psi_n(h)_{|\Delta_w}\|_\infty \le \frac{\|\xi_n\|_\infty\,(1 + \|\rho_n\|_\infty\|l\|_\infty)}{a/M\|l\|_\infty} \le \omega_n\frac{M\|l\|_\infty}{a}\left(1 + \frac{M}{a}\|l\|_\infty\right) \le \frac{1}{4}\,.$$

for all $n \ge n_0$ and some sufficiently large n_0.

For general points $(w_1,\overline{w}_2) \in \Upsilon$ we can proceed as follows. First of all we have

$$D_\rho\Psi_n(h)(w_1,\overline{w}_2) = \frac{1}{l(\hat{L}^n(\rho))}\Big(\hat{L}^n(h)(w_1,\overline{w}_2) - \rho_n(w_1,\overline{w}_2)l(\hat{L}^n(h))\Big)\,.$$

We already have an appropriated estimate for the first factor. From the Lipschitz property of $\hat{\rho}$ (Proposition 5.1) follows that

$$|\rho(w_1,\overline{w}_2) - \rho(w_1,\overline{w}_1)| \le \frac{M(K+1)}{l(\hat{\rho})}|w_1 - w_2| \le Q|w_1 - w_2|\,, \quad (w_1,\overline{w}_2) \in \Upsilon_w$$

where, we remember, $Q = \frac{M}{a}(K+1)$. If we combine this with the Lipschitz behavior of $L^n h$ given in (4.12) and use $|w_1 - w_2| < \delta$, we finally get for large n

$$\Big|D_\rho\Psi_n(h)(w_1,\overline{w}_2) - D_\rho\Psi_n(h)(w_1,\overline{w}_1)\Big| \le \frac{M}{a}\|l\|_\infty\left(M\left(K + \frac{8}{\delta\gamma_n}\right) + Q\|l\|_\infty M\right)\delta\,.$$

Remember now that $\delta > 0$ has been fixed small enough such that (6.1) holds. This constant δ being chosen, we can choose n sufficiently large such that $\frac{8}{\delta\gamma_n} \le 1$. Then

$$\Big|D_\rho\Psi_n(h)(w_1,\overline{w}_2) - D_\rho\Psi_n(h)(w_1,\overline{w}_1)\Big| \le \frac{1}{4}\,.$$

Combing this with (6.4) implies that for real h such that $\|h\|_\infty \le 1$ we have, for this choice of n,

$$\|D_\rho\Psi_n(h)\|_\infty \le \frac{1}{2}\,.$$

If $h \in \mathcal{A}$ is arbitrary with $\|h\|_\infty = 1$, then h can be expressed as $h = h_1 + ih_2$ where both h_1, h_2 are in $\mathcal{A}_{\mathbb{R}}$ and such that $\max\{\|h_1\|_\infty, \|h_2\|_\infty\} \le \|h\|_\infty = 1$. It suffices then to use the case of functions in $\mathcal{A}_\mathbb{R}$ of norm at most one in order to conclude this proof. $\square$

7. Analyticity: the non-autonomous case

We now come to the final part where we investigate analytic dependence on the parameter λ. In this section we still continue with the non-autonomous case and thus with the notations introduced in the previous sections 3 to 6. The assumptions are also unchanged: (5.2), thus (5.5), (5.6) and the uniform speed assumption (6.2) are kept throughout this section.

The first observation concerns the extended operators introduced in (4.10).

Proposition 7.1. *For every $j \in \mathbb{Z}$ and every $g \in \mathcal{A}$, the map*

$$(t, \lambda_1, \overline{\lambda_2}) \mapsto L_{j,\lambda_1,\overline{\lambda}_2,t}\, g \in \mathcal{A}$$

is holomorphic on $\mathcal{I} \times \Lambda \times \overline{\Lambda}$.

Proof. Let $t_0 \in \mathcal{I}$ and $\varepsilon > 0$ such that $\mathbb{D}(t_0, \varepsilon) \subset \mathcal{I}$. We have to show that there are functions $h_{k_1,k_2,k_3} \in \mathcal{A}$ such that for every $(t, \lambda_1, \overline{\lambda_2}) \in \mathbb{D}(t_0, \varepsilon) \times \Lambda \times \overline{\Lambda}$, we have a power series representation:

$$L_{j,\lambda_1,\overline{\lambda}_2,t}\, g = \sum_{k_1,k_2,k_3 \geq 0} h_{k_1,k_2,k_3} (\lambda_1 - \lambda_0)^{k_1} (\overline{\lambda}_2 - \overline{\lambda}_0)^{k_2} (t - t_0)^{k_3}. \tag{7.1}$$

Every point $(w_1, \overline{w}_2) \in \Upsilon_w$ belongs to a disk D_w for some $w \in U$. By Formula (4.7) we have well defined holomorphic functions

$$\Lambda \times \overline{\Lambda} \ni (\lambda_1, \overline{\lambda}_2) \mapsto F_{j,\lambda_1,\overline{\lambda}_2,*}\big|_{D_w \times \overline{D}_w} \in \mathcal{A}\big|_{(D_w \times \overline{D}_w) \cap \Upsilon_w}$$

ascribing to every $(w_1, \overline{w}_2) \in (D_w \times \overline{D}_w) \cap \Upsilon_w$ a 1–pairing $(z_1, \overline{z}_2)$:

$$F_{j,\lambda_1,\overline{\lambda}_2,*}(w_1, w_2) := (z_1, \overline{z}_2) = \big(f^{-1}_{j,\lambda_1,*}(w_1), \overline{f^{-1}_{j,\lambda_2,*}(w_2)}\big).$$

In consequence, the function

$$\begin{aligned}\mathbb{D}(t_0, \varepsilon) \times \Lambda \times \overline{\Lambda} \ni (\lambda_1, \overline{\lambda}_2) &\longmapsto L_{j,\lambda_1,\overline{\lambda}_2,t}\, g\big|_{(D_w \times \overline{D}_w) \cap \Upsilon_w} = \\ &= \sum_{*} \exp \Phi_{j,\lambda_1,\overline{\lambda}_2,t} \circ F_{j,\lambda_1,\overline{\lambda}_2,*}\, g \circ F_{j,\lambda_1,\overline{\lambda}_2,*} \in \mathcal{A}\big|_{(D_w \times \overline{D}_w) \cap \Upsilon_w}\end{aligned}$$

is also holomorphic as the sum of an absolutely uniformly convergent series of holomorphic functions. Hence we have the representation,

$$L_{j,\lambda_1,\overline{\lambda}_2,t}\, g|_{(D_w \times \overline{D}_w) \cap \Upsilon_w} = \sum_{k_1,k_2,k_3 \geq 0} h_{k_1,k_2,k_3;w} (\lambda_1 - \lambda_0)^{k_1} (\overline{\lambda}_2 - \overline{\lambda}_0)^{k_2} (t - t_0)^{k_3},$$

where all the functions $h_{k_1,k_2,k_3;w}$ belong to $\mathcal{A}|_{(D_w \times \overline{D}_w) \cap \Upsilon_w}$. From the uniqueness theorem for holomorphic functions, all these functions $h_{k_1,k_2,k_3;w}$, $w \in U$, glue to one element h_{k_1,k_2,k_3} of $\mathcal{A}$ giving rise to the representation (7.1). The proof is complete. $\square$

Consider now a new Banach space $\mathcal{A}_{\mathbb{Z}}$ of all bounded sections $g = (g_j)_{j\in\mathbb{Z}}$ where $g_j \in \mathcal{A}$ for every $j \in \mathbb{Z}$ and such that

$$|g| = \sup_{j\in\mathbb{Z}} ||g_j||_\infty .$$

The space $\mathcal{A}_{\mathbb{Z}}$ equipped with this norm $|\cdot|$ is a Banach space. One then considers the global operator $L_{\lambda_1,\overline{\lambda}_2,t}$ mapping $g = (g_j)_{j\in\mathbb{Z}} \in \mathcal{A}_{\mathbb{Z}}$ to the function $L_{\lambda_1,\overline{\lambda}_2,t}g \in \mathcal{A}_{\mathbb{Z}}$ which is defined by

$$\Big(L_{\lambda_1,\overline{\lambda}_2,t}g\Big)_{j+1} = L_{j,\lambda_1,\overline{\lambda}_2,t}g_j \quad , \quad j \in \mathbb{Z} .$$

In the same way, the map $\Psi_{n,j,\lambda_1,\overline{\lambda_2},t}$ introduced in (5.10) gives rise to a global map $g \mapsto \Psi_{n,\lambda_1,\overline{\lambda}_2,t}(g)$ defined by

$$(\Psi_{n,\lambda_1,\overline{\lambda}_2,t}(g))_{j+1} = \frac{L^n_{j,\lambda_1,\overline{\lambda}_2,t}(g_j)}{l(L^n_{j,\lambda_1,\overline{\lambda}_2,t}(g_j))} = \frac{\hat{L}^n_{j,\lambda_1,\overline{\lambda}_2,t}(g_j)}{l(\hat{L}^n_{j,\lambda_1,\overline{\lambda}_2,t}(g_j))} \quad , \quad j \in \mathbb{Z}. \tag{7.2}$$

The integer $n \geq 1$ will be fixed such that the conclusion of Proposition 6.2 holds.

Remember also that for $t \in I$ real and for $\lambda = \lambda_1 = \lambda_2$ the function

$$\rho_{\lambda,\overline{\lambda},t} = (\rho_{j,\lambda,\overline{\lambda},t})_{j\in\mathbb{Z}}$$

is a fixed point of $\Psi_{n,\lambda,\overline{\lambda},t}$ (see (5.9)).

Lemma 7.2. *Let $\lambda_0 \in \Lambda$, let $t_0 \in I$ be real and let $n \geq 1$. Then there exist U_{λ_0,t_0}, an open neighborhood of $\rho_{\lambda_0,\overline{\lambda}_0,t_0}$ in $\mathcal{A}_{\mathbb{Z}}$ and an open neighborhood W_{λ_0,t_0} of the point $(\lambda_0, \overline{\lambda}_0, t_0)$ in $\Lambda\times\overline{\Lambda}\times\mathcal{I}$ such that $\Psi_{n,\lambda_1,\overline{\lambda_2},t}$ is well defined on U_{λ_0,t_0} for every $(\lambda_1, \overline{\lambda}_2, t) \in W_{\lambda_0,t_0}$. Moreover, the map*

$$U_{\lambda_0,t_0} \times W_{\lambda_0,t_0} \ni (h, \lambda_1, \overline{\lambda}_2, t) \mapsto \Psi_{n,\lambda_1,\overline{\lambda}_2,t}(h) \in \mathcal{A}_{\mathbb{Z}}$$

is holomorphic.

Proof. First of all note that for every $j \in \mathbb{Z}$ and $n \geq 1$ the function

$$\mathcal{A}_{\mathbb{Z}} \times \Lambda \times \overline{\Lambda} \times \mathcal{I} \ni (h, \lambda_1, \overline{\lambda_2}, t) \mapsto L^n_{j,\lambda_1,\overline{\lambda}_2,t}(h_j) \in \mathcal{A}$$

is holomorphic since it is linear with respect to the first variable, holomorphic with respect to all three other variables (Proposition 7.1), and one applies Hartogs' Theorem. Hence, also the function

$$\mathcal{A}_{\mathbb{Z}} \times \Lambda \times \overline{\Lambda} \times \mathcal{I} \ni (h, \lambda_1, \overline{\lambda_2}, t) \mapsto l\big(L^n_{j,\lambda_1,\overline{\lambda}_2,t}(h_j)\big) \in \mathbb{C} \tag{7.3}$$

is holomorphic. Now, in order to conclude the proof, we shall find U_{λ_0,t_0}, an open neighborhood of $\rho_{\lambda_0,\overline{\lambda}_0,t_0}$ in $\mathcal{A}_{\mathbb{Z}}$ and an open neighborhood W_{λ_0,t_0} of the point $(\lambda_0, \overline{\lambda}_0, t_0)$ in $\Lambda\times\overline{\Lambda}\times\mathcal{I}$ such that $|l\big(L^n_{j,\lambda_1,\overline{\lambda}_2,t}(h_j)\big)|$ is uniformly bounded below for every $h \in U_{\lambda_0,t_0}$ and for every $(\lambda_1, \overline{\lambda_2}, t) \in W_{\lambda_0,t_0}$. This will tell us that all coordinates of the function $\Psi_{(\cdot,\cdot,\cdot)}(\cdot)$ are continuous and uniformly bounded, and ultimately the function $\Psi_{\cdot,\cdot,\cdot}(\cdot)$ is holomorphic.

Let $n \geq 1$ be fixed. In order to find these neighborhoods we deduce from (5.5) that $\|L^n_{j,\lambda_1,\overline{\lambda}_2,t}1\|_\infty$ is uniformly bounded above with respect to $j \in \mathbb{Z}$ and $(\lambda_1, \overline{\lambda}_2, t) \in \Lambda\times\overline{\Lambda}\times\mathcal{I}$. Cauchy's Integral Formula thus implies that the map $(\lambda_1, \overline{\lambda}_2, t) \mapsto L^n_{j,\lambda_1,\overline{\lambda}_2,t}\mathbb{1}$ is uniformly Lipschitz with respect to $j \in \mathbb{Z}$. Consequently, for every $\varepsilon > 0$ there exists a neighborhood W_{λ_0,t_0} of $(\lambda_0, \overline{\lambda}_0, t_0)$ such that for every $h \in \mathcal{A}_\mathbb{Z}$, we have that

$$|L^n_{\lambda_1,\overline{\lambda}_2,t}(h) - L^n_{\lambda_0,\overline{\lambda}_0,t_0}(h)| = \sup_{j\in\mathbb{Z}} \|L^n_{j,\lambda_1,\overline{\lambda}_2,t}(h_j) - L^n_{j,\lambda_0,\overline{\lambda}_0,t_0}(h_j)\|_\infty \leq \varepsilon|h| \,. \tag{7.4}$$

The existence of U_{λ_0,t_0} easily follows now from the above Lipschitz property (7.4) along with the estimate (5.11) of the proof of Lemma 5.3. □

We are now in a position to extend the invariant density $\rho_{\lambda_0,\overline{\lambda}_0,t_0}$ (i.e., to extend the function assigning the density $\rho_{\lambda_0,\overline{\lambda}_0,t_0}$ to parameters $(\lambda_0, \overline{\lambda}_0, t_0)$) analytically to a neighbourhood of $(\lambda_0, \overline{\lambda}_0, t_0)$ by making use of the Implicit Function Theorem. Indeed, $\rho_{\lambda_0,\overline{\lambda}_0,t_0}$ is a fixed point of $\Psi_{n,\lambda_0,\overline{\lambda}_0,t_0}$, the map $(h, \lambda_1, \overline{\lambda}_2, t) \mapsto \Psi_{n,\lambda_1,\overline{\lambda}_2,t}(h)$ is analytic (Lemma 7.2) and Proposition 6.2 along with the Remark 6.3 imply that

$$|D_{\rho_{\lambda_0,\overline{\lambda}_0,t_0}}\Psi_{n,\lambda_0,\overline{\lambda}_0,t_0}| = \sup_{j\in\mathbb{Z}} \|D_{\rho_{j,\lambda_0,\overline{\lambda}_0,t_0}}\Psi_{n,j,\lambda_0,\overline{\lambda}_0,t_0}\|_\infty \leq \frac{\sqrt{2}}{2} < 1$$

provided n has been chosen sufficiently large. In conclusion we get the following.

Theorem 7.3. *For every $(\lambda_0, t_0) \in \Lambda \times I$ there exists an open neighborhood W_{λ_0,t_0} in $\Lambda \times \overline{\Lambda} \times \mathcal{I}$ of $(\lambda_0, \overline{\lambda}_0, t_0)$, and U_{λ_0,t_0}, an open neighborhood of $\rho_{\lambda_0,\overline{\lambda}_0,t_0}$ in $\mathcal{A}_\mathbb{Z}$, along with an analytic map $(\lambda_1, \overline{\lambda}_2, t) \mapsto \rho_{\lambda_1,\overline{\lambda}_2,t} \in U_{\lambda_0,t_0}$ such that*

$$\Psi_{n,\lambda_1,\overline{\lambda}_2,t}(\rho_{\lambda_1,\overline{\lambda}_2,t}) = \rho_{\lambda_1,\overline{\lambda}_2,t} \quad \textit{for every} \quad (\lambda_1, \overline{\lambda}_2, t) \in W_{\lambda_0,t_0} \,.$$

Theorem 1.2 follows now easily.

Proof of Theorem 1.2. An assumption of Theorem 1.2 is that there exists $a > 0$ and $z_0 \in U$ such that $\hat{\rho}_{j,\lambda,t}(z_0) \geq a$ for all (j, λ, t). This enables us to consider the functional $l : C^0_b(U) \to \mathbb{R}$ defined by $l(g) := g(z_0)$. It clearly satisfies (5.6) and thus Theorem 7.3 implies Theorem 1.2 provided the uniform speed condition (6.2) holds. So, consider $h \in \mathcal{A}_\mathbb{R}$. By Lemma 4.2 the associated function $z \mapsto g(z) = h(z, \overline{z})$ belongs to $Lip(U, \kappa\delta)$ with $\|g\|_{Lip,\kappa\delta} \leq C\|h\|_\infty$. It follows from the assumption (1.5) that there exists $\omega_n \to 0$ such that

$$\|\hat{\mathcal{L}}^n_j g_{|\Delta_w} - \nu_j(g)\hat{\rho}_{j+n}\|_{\infty,\Delta_w} \leq \omega_n \|g\|_{Lip,\kappa\delta} \leq C\omega_n\|h\|_{\infty,\Upsilon_w}$$

for every $n \geq 1$ and $j \in \mathbb{Z}$. This implies (6.2) with ω_n replaced by $C\omega_n$. □

Remark 7.4. *Note that the uniqueness part of the Implicit Function Theorem guarantees the functions $\rho_{\lambda,\overline{\lambda},t}$, $t \in I$ being real, to coincide with the ones resulting from Proposition 5.1.*

8. Analyticity: the random case

The final part of this paper is devoted to random dynamics. So we now consider the following setting. Let X be an arbitrary set and $\mathcal{B}$ a σ–algebra on X. We consider a complete probability space $(X, \mathcal{B}, m)$. As usual, the randomness will be modeled by an invertible map $\theta : X \to X$ preserving the measure m. All objects like functions and operators do now depend on $x \in X$ instead of the integer dependence $j \in \mathbb{Z}$ in the non-autonomous case. In particular, we consider functions $f_{x,\lambda}$, $x \in X$ and $\lambda \in \Lambda$, that satisfy the conditions described in Section 2. In the random case one has to require in addition that these functions are measurable. This means that the map $(x, z) \mapsto f_{x,\lambda}(z)$ is measurable for every $\lambda \in \Lambda$. We are interested in the dynamics of the random compositions

$$f^n_{x,\lambda} = f_{\theta^{n-1}(x),\lambda} \circ ... \circ f_{x,\lambda}, \quad n \geq 1,$$

where $\lambda \in \Lambda$ and $x \in X$. The associated radial Julia set $\mathcal{J}_r(f_{x,\lambda})$ is defined by the formula (2.2) with functions $f_j, f_{j+1}, ...$ replaced by $f_{x,\lambda}, f_{\theta(x),\lambda}, ...$.

The space of analytic functions $\mathcal{A}_{\mathbb{Z}}$ is now replaced by $\mathcal{A}_X$. It has the same meaning as before except that the functions depend measurably on $x \in X$. Thus, $g \in \mathcal{A}_X$ if $(z_1, \overline{z}_2) \mapsto g_x(z_1, \overline{z}_2)$ is holomorphic on Υ_w for every $x \in X$, if $x \mapsto g_x(z_1, \overline{z}_2)$ is measurable for every $(z_1, \overline{z}_2) \in \Upsilon_w$ and if

$$|g| := \operatorname*{ess\,sup}_{x \in X} \|g_x\|_\infty < \infty.$$

The transfer operators $\mathcal{L}_{x,\lambda,t}$ must also have measurable dependence on $x \in X$ in the sense that each function

$$X \ni x \longmapsto \mathcal{L}_{x,\lambda,t} g(z_1, \overline{z}_2) \in \mathbb{C}$$

is measurable for all arguments $\lambda, t, g, (z_1, z_2)$ fixed in their appriopriate domains. Notice that one can show with the help of the Measurable Selection Theorem (see [2]) that this is indeed the case. In the case of transcendental functions this has been worked out in Lemma 3.6 of [8]. In this case, the invariant densities $\rho_{x,\lambda,t}$ as well as their extensions $\rho_{x,\lambda,\overline{\lambda},t}$ also depend measurably on $x \in X$ since they are obtained as a limit of measurable maps (see (5.2) and Proposition 5.1). Clearly, exactly as for the above composition of the functions $f_{x,\lambda}$, the iterated operators are of the form $\mathcal{L}^n_{x,\lambda,t} = \mathcal{L}_{\theta^{n-1}(x),\lambda,t} \circ ... \circ \mathcal{L}_{x,\lambda,t}$. In the same way, the definitions given in the part on non-autonomous dynamics have straightforward counterparts. For example, the invariance of the density is the relation $\hat{\mathcal{L}}_{x,\lambda,t} \hat{\rho}_{x,\lambda,t} = \hat{\rho}_{\theta(x),\lambda,t}$ and the uniform speed assumption (6.2) takes on the following form:

$$\|\hat{\mathcal{L}}^n_{x,\lambda,t} h - \nu_x(h) \hat{\rho}_{\theta^n(x),\lambda,t}\|_{\infty,\Delta_w} \leq \omega_n \|h\|_{\infty,\Upsilon_w} \quad \text{for every} \quad h \in \mathcal{A}_{\mathbb{R}}\ , \ \ n \geq 1\,. \tag{8.1}$$

Also, the definition of the global map $g \mapsto \Psi_{\lambda_1,\overline{\lambda}_2,t}(g)$, $g \in \mathcal{A}_X$, is

$$(\Psi_{\lambda_1,\overline{\lambda}_2,t}(g))_{\theta(x)} = \frac{L_{x,\lambda_1,\overline{\lambda}_2,t}(g_x)}{l(L_{x,\lambda_1,\overline{\lambda}_2,t}(g_x))} \quad , \quad x \in X,$$

where again l is a functional that satisfies (5.6). Proceeding now exactly as in the previous section and applying the Implicit Function Theorem in the Banach space $(\mathcal{A}_X, |.|)$ we see that Theorem 7.3 holds also in the present random setting.

The results can now be summarized as follows. Assume again that the expanding property (2.1) is satisfied with γ_n independent of λ, that this family is of bounded deformation (Definition 3.3) and the bounded distortion of the arguments of (3.4) holds. Finally, we assume that the, most natural in this context, thermodynamical formalism property (5.2) holds with some universal (i.e., independent of λ) constant M.

Theorem 8.1. *Suppose the following:*

(1) *There exists a bounded functional $l : C_b^0(U) \to \mathbb{R}$ that is uniformly positive on the invariant densities (see (5.6)).*
(2) *The uniform speed condition (8.1) holds with some constants ω_n independent of λ.*

Then, the map $(\lambda_1, \overline{\lambda}_2, t) \mapsto \rho_{\lambda_1,\overline{\lambda}_2,t} \in \mathcal{A}_X$ is analytic. In particular for a.e. $x \in X$ the map $(\lambda_1, \overline{\lambda}_2, t) \mapsto \rho_{x,\lambda_1,\overline{\lambda}_2,t} \in \mathcal{A}_X$ is analytic.

Remark 8.2. *Note that the uniqueness part of the Implicit Function Theorem guarantees the functions $\rho_{\lambda,\overline{\lambda},t}$, $t \in I$ being real, to coincide with the ones resulting from Proposition 5.1.*

Remark 8.3. *In fact, in this theorem we also could include real analyticity of the expected pressure as defined in the transcendental case in (9.6) and established in Lemma 9.5.*

9. Transcendental random systems

In this last part we apply the preceding results to the case of transcendental random systems. Such systems have been considered in [8] and the full thermodynamical formalism including spectral gap property has been established there. We here complete the picture in establishing analyticity in this general context. As a consequence we get a proof for the particular example in the Introduction (Theorem 1.1).

Assume now that the functions $f_{x,\lambda}$ are transcendental functions and that this family consists of *transcendental random systems* as defined in [8]. We use notation from that paper such as $\mathcal{J}_{x,\lambda}$ for the Julia set of $(f^n_{x,\lambda})_{n\geq 1}$. Clearly, the radial Julia set $\mathcal{J}_r(f_{x,\lambda}) \subset \mathcal{J}_{x,\lambda}$. Here are some other notions from [8] that are necessary for the present work. First of all, the following mild technical conditions are used in [8] with the same enumeration:

Condition 2. *There exists $T > 0$ such that*

$$\Big(\mathcal{J}_{x,\lambda} \cap \mathbb{D}_T\Big) \cap f^{-1}_{x,\lambda}\left(\mathcal{J}_{\theta(x),\lambda} \cap \mathbb{D}_T\right) \neq \emptyset \ \ , \ \ x \in X \ \textit{and}\ \lambda \in \Lambda\,.$$

Condition 4. *For every $R > 0$ and $N \geq 1$ there exists $C_{R,N}$ such that*

$$|\left(f^N_{x,\lambda}\right)'(z)| \leq C_{R,N} \quad \textit{for all} \quad z \in \mathbb{D}_R \cap f^{-N}_{x,\lambda}\left(\mathbb{D}_R\right)\,,\ x \in X \ \textit{and}\ \lambda \in \Lambda\,.$$

Then, there must be some common bound for the growth of the (spherical) characteristic functions $\mathring{T}_{x,\lambda}(r) = \mathring{T}(f_{x,\lambda}, r)$ of $f_{x,\lambda}$, $x \in X$ and $\lambda \in \Lambda$. We use here a stronger version of the Condition 1 in [8] and would like to mention that this is only used in order to show that the expected pressure function has a zero (see Proposition 9.7):

Condition 1'. *There exists $\rho > 0$ and $\iota > 0$ such that*

$$\iota r^\rho \leq \mathring{T}_{x,\lambda}(r) \leq \iota^{-1} r^\rho \quad \textit{for all } \ r \geq 1\ ,\ x \in X \textit{ and all } \ \lambda \in \Lambda\,. \tag{9.1}$$

Definition 9.1. *The transcendental random family $(f_{x,\lambda})_{x\in X,\lambda\in\Lambda}$ is called:*

(1) Topologically hyperbolic *if there exists $0 < \delta_0 \leq \frac{1}{4}$ such that for every $x \in X$, $\lambda \in \Lambda$, $n \geq 1$ and $w \in \mathcal{J}_{\theta^n(x),\lambda}$ all holomorphic inverse branches of $f^n_{x,\lambda}$ are well defined on $\mathbb{D}(w, 2\delta_0)$.*

(2) Expanding *if there exists $c > 0$ and $\gamma > 1$ such that*

$$|(f^n_{x,\lambda})'(z)| \geq c\gamma^n$$

for every $z \in \mathcal{J}_{x,\lambda} \setminus f^{-n}_{x,\lambda}(\infty)$ and every $x \in X$, $\lambda \in \Lambda$.

(3) Hyperbolic *if it is both topologically hyperbolic and expanding.*

Definition 9.2. *The transcendental random family $(f_{x,\lambda})_{x\in X,\lambda\in\Lambda}$ satisfies the balanced growth condition if there are $\alpha_2 > \max\{0, -\alpha_1\}$ and $\kappa \geq 1$ such that for every $(x, \lambda) \in X \times \Lambda$ and every $z \in f^{-1}_{x,\lambda}(U)$,*

$$\kappa^{-1} \leq \frac{|f'_{x,\lambda}(z)|}{(1 + |z|^2)^{\frac{\alpha_1}{2}}(1 + |f_{x,\lambda}(z)|^2)^{\frac{\alpha_2}{2}}} \leq \kappa\,. \tag{9.2}$$

In the following we always assume that the above conditions are satisfied.

Definition 9.3. *A transcendental holomorphic random family $(f_{x,\lambda})_{x\in X,\lambda\in\Lambda}$ will be called* admissible *if*

(1) *the base map $\theta : X \to X$ is ergodic with respect to the measure m,*
(2) *the system $(f_{x,\lambda})$ is hyperbolic,*
(3) *the balanced growth condition is satisfied,*
(4) *the Conditions 1', 2 and 4 hold.*

In this context, the right potential to work with is $\varphi_{\lambda,t}$ as defined in (4.4) but with $\tau = \alpha_1 + \tau'$ where $\tau' < \alpha_2$ is arbitrarily close to α_2 such that

$$t > \rho/\tau > \rho/\alpha \quad , \quad \alpha = \alpha_1 + \alpha_2\,. \tag{9.3}$$

With such a choice, the following has been shown in [8]:

- *The full thermodynamical formalism holds. In particular, there exist $\nu_{x,t}$, the Gibbs states, in fact generalized eigenmeasures of dual transfer operators, and unique equilibrium states*

$$\mu_{x,t} = \hat{\rho}_{x,t}\nu_{x,t}, \quad \nu_{x,t}(\hat{\rho}_{x,t}) = 1.$$

Moreover, for every $t > \rho/\alpha$, there are constants $A_t, C_t < \infty$ and $\varepsilon_t > 0$ such that

$$\hat{\rho}_{x,t}(z) \le C_t(1+|z|)^{-\varepsilon_t t} \quad \text{and} \quad \|\hat{\rho}_{x,t}\|_\infty \le A_t \quad \text{for all } z \in U \text{ and } x \in X\,. \tag{9.4}$$

- The normalized iterated transfer operator converge exponentially fast (Theorem 5.1 (2)).

For admissible transcendental random families one has the bounded deformation property. Indeed, the following uniform control is a complete analogue of Lemma 9.7 in [7] and can be shown with exactly the same normal family argument as in the proof given in [7]. Let us recall that $\Lambda = \mathbb{D}(\lambda_0, r)$.

Lemma 9.4. *For every $\varepsilon > 0$ there exists $0 < r_\varepsilon < r$ such that*

$$\left| \frac{f'_\lambda(f^{-1}_{\lambda,*}(w))}{f'_{\lambda_0}(f^{-1}_{\lambda_0,*}(w))} - 1 \right| < \varepsilon$$

for every inverse branch $f^{-1}_{\lambda,*}$ defined on $\mathbb{D}(w,\delta_0)$, $w \in U$, and every $\lambda \in \mathbb{D}(\lambda_0, r_\varepsilon)$.

If we combine this with Koebe's Distortion Theorem (see for ex. Theorem 2.7 in [10]) then it follows that the condition (3.2) of the bounded deformation property always holds. The first property of the bounded deformation property (3.1) holds for many families (see again [7]) and clearly for the exponential family in Theorem 1.1.

9.1. **Expected pressure.** Fix $t > \rho/\alpha$ and let us first discuss the numbers $P_{x,\lambda}(t)$ of (5.1). They depend on the transfer operator $\mathcal{L}_{x,\lambda}$ which itself has been defined with the auxiliary parameter τ such that (9.3) holds. Let us indicate for a moment this dependence by a superscript τ: $\mathcal{L}^\tau_{x,\lambda}$, $P^\tau_{x,\lambda}(t)$ and let $\nu^\tau_{x,\lambda}$ denote the associated Gibbs states (conformal measures) such that (1.1) holds: $\mathcal{L}^{\tau\,*}_{x,\lambda}\nu^\tau_{\theta(x),\lambda} = e^{P^\tau_{x,\lambda}(t)}\nu^\tau_{x,\lambda}$.

If $\tau' \ne \tau$, for example if $\tau' = \tau + \Delta\tau > \tau$, then the potentials of the operators corresponding to τ, τ' respectively are related by

$$|f'_{x,\lambda}(z)|^{-t}_{\tau'} = |f'_{x,\lambda}(z)|^{-t}_{\tau} \frac{v(z)}{v(f_{x,\lambda}(z))} \quad \text{where} \quad v(z) = (1+|z|^2)^{-\frac{t\Delta\tau}{2}}\,.$$

Notice that this function v does not depend on the parameter $x \in X$ and thus

$$\mathcal{L}^{\tau'}_{x,\lambda} g = \frac{1}{v}\mathcal{L}^{\tau}_{x,\lambda}(vg) \quad , \quad g \in C^0_b(U)\,.$$

Then, $v\nu^\tau_{x,\lambda}$ is a finite measure and, with $\gamma_x^{-1} = v\nu^\tau_{x,\lambda}(1\!\!1) = \int v d\nu^\tau_{x,\lambda}$, $\upsilon_{x,\lambda} = \gamma_x v \nu^\tau_{x,\lambda}$ a probability measure such that, for $g \in C^0_b(U)$,

$$\begin{aligned}\mathcal{L}^{\tau'\,*}_{x,\lambda}\upsilon_{\theta(x),\lambda}(g) &= \gamma_{\theta(x)}\int \mathcal{L}^{\tau'}_{x,\lambda}(g) v d\nu^\tau_{\theta(x),\lambda} = \gamma_{\theta(x)}\int \mathcal{L}^{\tau}_{x,\lambda}(vg) d\nu^\tau_{\theta(x),\lambda} \\ &= \gamma_{\theta(x)} e^{P^\tau_{x,\lambda}(t)} \int vg d\nu^\tau_{x,\lambda} = \frac{\gamma_{\theta(x)}}{\gamma_x} e^{P^\tau_{x,\lambda}(t)} \upsilon_{x,\lambda}(g)\,.\end{aligned}$$

Thus, for τ' we have Gibbs states $\nu^{\tau'}_{x,\lambda} = \upsilon_{x,\lambda}$ with corresponding pressures

$$P^{\tau'}_{x,\lambda}(t) = P^\tau_{x,\lambda}(t) + \log\gamma_{\theta(x)} - \log\gamma_x \quad , \quad x \in X\,. \tag{9.5}$$

Theorem 3.1 in [8] states that $\sup_{x\in X}|P_{x,\lambda}(t)|<\infty$ for every $\lambda\in\Lambda$. This allow us now to introduce the expected pressure:

$$\mathcal{E}P_\lambda(t)=\int_X P_{x,\lambda}(t)dm(x)\,. \tag{9.6}$$

The cohomological equation (9.5) and invariance of m implies that $\mathcal{E}P_\lambda(t)$ does not depend on the auxiliary parameter τ. The function $t\mapsto \mathcal{E}P_\lambda(t)$ is well defined for $t>\rho/\alpha$. Real analyticity of this function is a consequence of the following result. Here and in the following l is again a functional that satisfies (5.6). Notice that the existence of such a functional is guaranteed thanks to Example 5.2.

Lemma 9.5. *For the expected pressure we have the following expression*

$$\mathcal{E}P_\lambda(t)=\int_X \log l\left(\mathcal{L}_{x,\lambda,t}\rho_{x,\lambda,t}\right)dm(x)$$

and the function $(\lambda,t)\mapsto\mathcal{E}P_\lambda(t)$ is real analytic in $\Lambda\times(\rho/\alpha,\infty)$.

Proof. On the one hand we know that $\mathcal{L}_{x,\lambda,t}\hat\rho_{x,\lambda,t}=e^{P_{x,\lambda}(t)}\hat\rho_{\theta(x),\lambda,t}$ and on the other hand $\mathcal{L}_{x,\lambda,t}\rho_{x,\lambda,t}=l\left(\mathcal{L}_{x,\lambda,t}\rho_{x,\lambda,t}\right)\rho_{\theta(x),\lambda,t}$. Since $\rho_{x,\lambda,t}=\frac{\hat\rho_{x,\lambda,t}}{l(\hat\rho_{x,\lambda,t})}$ it follows that

$$\log\left(l\left(\mathcal{L}_{x,\lambda,t}\rho_{x,\lambda,t}\right)\right)=P_{x,\lambda}(t)+\log\big(l(\hat\rho_{\theta(x),\lambda,t})\big)-\log\left(l(\hat\rho_{x,\lambda,t})\right)\,.$$

It suffices to integrate this expression with respect to m and to use that the measure m is θ–invariant. The statement on analyticity results from this expression and the fact (see (7.3) and Theorem 8.1) that the function $(\lambda_1,\overline\lambda_2,t)\mapsto l\big(L_{x,\lambda_1,\overline\lambda_2,t}\rho_{x,\lambda_1,\overline\lambda_2,t}\big)\in\mathbb{C}$ is holomorphic. ☐

9.2. **Bowen's Formula.** This formula concerns a fixed random system or, in other words, a fixed parameter $\lambda\in\Lambda$. We can therefore neglect this parameter throughout this subsection and consider a fixed random system $(f_x)_{x\in X}$. As our preparation for the proof of Bowen's Formula we are to deal with expected pressure in greater detail.

Lemma 9.6. *Let $t>\rho/\alpha$. Then for m-a.e. $x\in X$ and every $w\in\mathcal{J}_x$,*

$$\mathcal{E}\mathrm{P}(t)=\lim_{n\to\infty}\frac{1}{n}\log\mathcal{L}^n_{\theta^{-n}(x),t}1\!\!1(w).$$

Proof. Taking $g_x:=1\!\!1$, item (2) of Theorem 5.1 in [8] yields for every $n\geq 1$ that

$$\left|\hat{\mathcal{L}}^n_{\theta^{-n}(x),t}1\!\!1(w)-\hat\rho_{x,t}(w)\right|\leq B\vartheta^n$$

for some $B\in(0,+\infty)$ and some $\vartheta\in(0,1)$. Since $\rho_{x,t}(w)>0$ this yields

$$\left|\log\left(\frac{1}{\hat\rho_{x,t}(w)}\hat{\mathcal{L}}^n_{\theta^{-n}(x),t}1\!\!1(w)\right)\right|\leq\frac{B'}{\hat\rho_{x,t}(w)}\vartheta^n$$

for every $n\geq 1$ with some constant $B'>0$. Using the standard Birkhoff's sum notation $S_nP_y=P_y+P_{\theta(y)}+...+P_{\theta^{n-1}(y)}$, we have

$$\hat{\mathcal{L}}^n_{\theta^{-n}(x),t}1\!\!1(w)=e^{-S_nP_{\theta^{-n}(x)}(t)}\mathcal{L}^n_{\theta^{-n}(x),t}1\!\!1(w)\,.$$

With this notation, it follows that

$$\left|\frac{1}{n}\log \mathcal{L}^n_{\theta^{-n}(x),t}1\!\!1(w) - \frac{1}{n} S_n P_{\theta^{-n}(x)}(t)\right| \leq \frac{B'}{\hat\rho_{x,t}(w)}\frac{\vartheta^n}{n} + \frac{|\log(\hat\rho_{x,t}(w))|}{n} \longrightarrow 0$$

as $n\to\infty$. The lemma now follows by applying Birkhoff's Ergodic Theorem to the function $x\mapsto P_x(t)$. □

This characterization of expected pressure along with hyperbolicity of the system $(f_x)_{x\in X}$ and of Condition 1' allow us to establish the desired description of the behavior of the expected pressure.

Proposition 9.7. *The function $t\mapsto \mathcal{E}\mathrm{P}(t)$ is real-analytic (hence continuous) on $(\rho/\alpha,\infty)$, strictly decreasing with $\frac{d}{dt}\mathcal{E}\mathrm{P}(t)\leq -\log\gamma<0$ and satisfies*

$$\lim_{t\searrow\rho/\alpha}\mathcal{E}\mathrm{P}(t)=+\infty \quad \textit{and} \quad \lim_{t\to+\infty}\mathcal{E}\mathrm{P}(t)=-\infty\,.$$

Proof. Analyticity has been established in Lemma 9.5, while the strict monotonicity and the limit at $+\infty$ are straightforward and standard whith the use of Lemma 9.6. The estimate of the derivative is due to the expanding property and the formula in Lemma 9.6. Here are the details:

Condition 2 implies that there exists $w_x\in\mathcal{J}_{x,\lambda}\cap\mathbb{D}_T$. Using the expanding property in Definition 9.1 one can estimate as follows:

$$\begin{aligned}\mathcal{L}^n_{\theta^{-n}(x),t+s}1\!\!1(w_x) &= \sum_{f^n_{\theta^{-n}(x)}(z)=w_x} |f^n_{\theta^{-n}(x)}(z)|_\tau^{-t}\,|f^n_{\theta^{-n}(x)}(z)|_\tau^{-s}\\ &\leq (c\gamma^n)^{-s}(1+T^2)^{\frac{s\tau}{2}}\mathcal{L}^n_{\theta^{-n}(x),t}1\!\!1(w_x)\ ,\ s>0\,.\end{aligned}$$

Taking logarithms and dividing by n yields

$$\frac{1}{n}\log\mathcal{L}^n_{\theta^{-n}(x),t+s}1\!\!1(w_x) - \frac{1}{n}\log\mathcal{L}^n_{\theta^{-n}(x),t}1\!\!1(w_x) \leq \frac{s}{n}\log\left(\frac{(1+T^2)^{\frac{\tau}{2}}}{c}\right) - s\log\gamma\,.$$

The estimate of the derivative $\frac{d}{dt}\mathcal{E}\mathrm{P}(t)$ follows now from differentiability of the expected pressure along with the formula in Lemma 9.6.

It remains to analyze the behavior of $\mathcal{E}\mathrm{P}$ near ρ/α. In order to do so, we will use Condition 1' along with Nevanlinna Theory as explained in [8]. In the following we use the notations from that paper especially from the proof of Lemma 3.17. It is shown there that there exists $k>0$ and $\tilde R_0>0$ sufficiently large such that for every $R>\tilde R_0$ and every $w\in U\cap\mathbb{D}_R$

$$\mathcal{L}_x 1\!\!1_{\mathbb{D}_R}(w) \geq kR^{-(\alpha_2-\tau)t}\int_{r_R}^R \frac{\mathring{T}_x(r)}{r^{\hat\tau t+1}}dr$$

where $r_R=\omega^{-1}(8\log R)$ and where ω comes from Condition 1 in [8]. This condition being replaced here by Condition 1', we have $\omega(r)=\iota r^\rho$ and $\mathring{T}_x(r)\geq \iota r^\rho$. Therefore, still with $\hat{\tau} = \alpha_1 + \tau$ and with $\hat{k} = k\iota$, we get, uniformly in $w \in U \cap \mathbb{D}_R$ and $x \in X$, the lower bound

$$\begin{aligned}\mathcal{L}_x 1\!\!1_{\mathbb{D}_R}(w) \geq& \hat{k} R^{-(\alpha_2-\tau)t} \int_{r_R}^{R} \frac{dr}{r^{\hat{\tau}t-\rho+1}} \\ =& \hat{k} R^{-(\alpha_2-\tau)t} \big(\log R - \log r_R + O(\hat{\tau}t - \rho) \big) .\end{aligned}$$

The number $\tau \in (0, \alpha_2)$ is chosen in dependence of t arbitrarily close to α_2 such that $t > \rho/(\alpha_1+\tau) > \rho/\alpha$ (see Remark 1.2 in [8]). It is therefore clear that for every $H > 0$ one can choose $R = R_H > \tilde{R}_0$ and then $t_H > \rho/\alpha$ such that for every $t \in (\rho/\alpha, t_H)$

$$\mathcal{L}_x 1\!\!1_{\mathbb{D}_R}(w) \geq H \quad \text{for every} \quad w \in U \cap \mathbb{D}_R \ , \ x \in X \ .$$

Now, if $\mathcal{L}_x^{n-1} 1\!\!1_{\mathbb{D}_R} \geq H^{n-1}$ on $U \cap \mathbb{D}_R$ for some $n \geq 1$ then

$$\mathcal{L}_x^n 1\!\!1 \geq \mathcal{L}_x \left(1\!\!1_{\mathbb{D}_R} \mathcal{L}_{\theta(x)}^{n-1} (1\!\!1_{\mathbb{D}_R}) \right) \geq H^{n-1} \mathcal{L}_x 1\!\!1_{\mathbb{D}_R} \geq H^n \quad \text{on} \quad U \cap \mathbb{D}_R \, .$$

The formula $\lim_{t \searrow \rho/\alpha} \mathcal{E}\mathrm{P}(t) = +\infty$ follows now by induction and Lemma 9.6. □

Now, let $\mu_{x,t}$ be the invariant family of measures defined in Section 1, i.e., $d\mu_{x,t} = \hat{\rho}_{x,t} d\nu_{x,t}$.

Lemma 9.8. *For every $t > \rho/\alpha$, the function $(x,z) \mapsto \log |f_x'(z)|$ is $\mu_{x,t}$–integrable meaning that the integral*

$$\chi_t := \int_X \int_{J_x} \log |f_x'(z)| \, d\mu_{x,t}(z) \, dm(x)$$

is well-defined and finite. Moreover, $\chi_t > 0$.

Remark 9.9. *The measures $(\mu_{x,t})_{x \in X}$ depend measurably on $x \in X$ and they are in fact disintegrations of a measure μ_t on the global space $\mathcal{J} = \bigcup_{x \in X} \{x\} \times \mathcal{J}_x$ having marginal m. Such a measure is often called random measure. Crauel's book* [2] *contains the general background related to random measures and* [8] *all the details concerning the present setting. Also, Theorem 5.1 in* [8] *tells us that μ_t is ergodic and invariant under the global skew product $(x,z) \mapsto (\theta(x), f_x(z))$.*

Proof of Lemma 9.8. Let $t > \rho/\alpha$. The expanding property implies $\chi_t > 0$. It remains to show that $\chi_t < \infty$. It follows from the estimate given in (9.4) that

$$\int_X \int_{\mathcal{J}_x} \log |z| \, d\mu_{x,t} \, dm(x) = \int_X \int_{\mathcal{J}_x} \log |z| \, \hat{\rho}_{x,t} \, d\nu_{x,t} \, dm(x) < \infty \ , \quad x \in X \, ,$$

and from invariance that

$$\int_X \int_{\mathcal{J}_x} \log |f_x(z)| \, d\mu_{x,t} \, dm(x) = \int_X \int_{\mathcal{J}_{\theta(x)}} \log |z| \, d\mu_{\theta(x),t} \, dm(x) < \infty \ , \quad x \in X \, .$$

Thus, both functions $(x,z) \mapsto \log(1 + |z|^2)$ and $(x,z) \mapsto \log(1 + |f_x(z)|^2)$ are μ_t–integrable. From the balanced growth condition follows now μ_t–integrability of the function $(x,z) \mapsto \log |f_x'(z)|$. □

Proposition 9.7 yields the existence of a unique zero $h > \rho/\alpha$ of the expected pressure function. It turns out that this number coincides almost everywhere with the Hausdorff dimension of the radial Julia set.

Theorem 9.10 (A version of Bowen's Formula)**.** *If $(f_x)_{x\in X}$ is an admissible random system, then*

$$\mathrm{HD}(\mathcal{J}_r(f_x)) = h \quad \textit{for } m\textit{-a.e. } x \in X.$$

Proof. Since μ_h is an ergodic measure, there is $M \in (0, +\infty)$ such that

$$\mu_{x,h}\big(J_r(x, M)\big) = 1 \quad \text{for all} \quad x \in X_1\,,$$

where $X_1 \subset X$ is some measurable set with $m(X_1) = 1$, and

$$J_r(x, M) := \big\{z \in J_r(x) : \varliminf_{n\to\infty} |(f_x^n(z)| < M\big\}.$$

First we shall prove that

$$\mathrm{HD}(J_r(x, M)) \geq h \tag{9.7}$$

or m-a.e. $x \in X_1$. Fix $x \in X_1$ and $z \in J_r(x, M)$. Set $y := (x, z)$ and denote by f_y^{-n} the inverse branch of f_x^n defined on $\mathbb{D}(f_x^n(z), \delta)$ mapping $f_x^n(z)$ back to z. For every $r \in (0, \delta)$ let $k := k(y, r)$ be the largest integer $n \geq 0$ such that

$$\mathbb{D}(z, r) \subset f_y^{-n}\big(\mathbb{D}(f_x^n(z), \delta)\big). \tag{9.8}$$

Since our system is expanding this inclusion holds for all $0 \leq n \leq k$ and

$$\lim_{r\to 0} k(y, r) = +\infty.$$

Fix $n = n_k \geq 0$ to be the largest integer in $\{0, 1, 2\dots, k\}$ such that $f_x^n(z) \in \mathbb{D}(0, M)$ and $s = s_k$ to be the least integer $\geq k+1$ such that $f_x^s(z) \in \mathbb{D}(0, M)$. It follows from Birkhoff's Ergodic Theorem that

$$\lim_{k\to\infty} \frac{s_k}{n_k} = 1 \tag{9.9}$$

for m-a.e. $x \in X_1$, say $x \in X_2 \subset X_1$ with $m(X_2) = 1$ and $\mu_{x,h}$-a.e. $z \in J_r(x, M)$, say $z \in J_r^1(x, M)$, with $\mu_{x,h}\big(J_r^1(x, M)\big) = 1$. Since the random measure ν_h is h-conformal, i.e., since $\nu_{\theta(x),h}(f_x(A)) = \exp(P_x(h)) \int_A |f_x'|_\tau^h d\nu_{x,h}$ for every Borel set A on which f_x is injective, we get from (9.8) and the definition of n that

$$\nu_{x,h}(\mathbb{D}(z, r)) \leq \nu_{x,h}\big(f_y^{-n}\big(\mathbb{D}(f_x^n(z), \delta)\big)\big) \leq K_{z,M}^h\big|\big(f_x^n)'(z)\big|^{-h} e^{-S_nP_x(h)}, \tag{9.10}$$

where the constant $K_{z,M}$ compensates the replacement of the τ-derivative $|\big(f_x^n)'(z)\big|_\tau$ by the Euclidean derivative $|\big(f_x^n)'(z)\big|$. On the other hand $\mathbb{D}(z, r) \not\subset f_y^{-s}\big(\mathbb{D}(f_x^s(z), \delta)\big)$. But since, by $\frac14$-Koebe's Distortion Theorem,

$$f_y^{-s}\big(\mathbb{D}(f_x^s(z), \delta)\big) \supset \mathbb{D}\big(z, \frac{1}{4}|(f_x^s)'(z)|^{-1}\delta\big),$$

we thus get that $r \geq \frac14 |(f_x^s)'(z)|^{-1}\delta$. Equivalently,

$$|(f_x^s)'(z)|^{-1} \leq 4\delta^{-1} r.$$

By inserting this into (9.10) and using also the Chain Rule, we obtain

$$\nu_{x,h}(\mathbb{D}(z,r)) \le (4K_{z,M}\delta^{-1})^h r^h e^{-S_nP_x(h)}\left|\left(f^{s-n}_{\theta^n(x)}\right)'\left(f^n_x(z)\right)\right|^h.$$

Equivalently:

$$\frac{\log \nu_{x,h}(\mathbb{D}(z,r))}{\log r} \ge h + \frac{h\log(4K_{z,M}\delta^{-1})}{\log r} - \frac{S_nP_x(h)}{\log r} + h\frac{\log\left|\left(f^{s-n}_{\theta^n(x)}\right)'\left(f^n_x(z)\right)\right|}{\log r}. \tag{9.11}$$

Now, Koebe's Distortion Theorem yields

$$f_y^{-n}\big(\mathbb{D}(f_x^n(z),\delta)\big) \subset \mathbb{D}\big(z, K\delta|(f_x^n)'(z)|^{-1}\big).$$

Along with (9.8) this yields $r \le K\delta|(f_x^n)'(z)|^{-1}$. Equivalently:

$$-\log r \ge -\log(K\delta) + \log|(f_x^n)'(z)|. \tag{9.12}$$

By Lemma 9.8 the function $(x,z)\mapsto \log|f_x'(z)|$ is μ_h–integrable with $\chi_h>0$. Therefore, there exists a measurable set $X_3\subset X_2$ with $m(X_3)=1$ and for every $x\in X_3$ there exists a measurable set $J_r^2(x,M)\subset J_r^1(x,M)$ such that $\mu_{x,h}\big(J_r^2(x,M)\big)=1$ and

$$\lim_{j\to\infty}\frac{1}{j}\log|(f_x^j)'(z)| = \chi_h \in (0,+\infty) \tag{9.13}$$

for every $x\in X_3$ and every $z\in J_r^2(x,M)$, the equality holding because of Birkhoff's Ergodic Theorem. This formula, along with (9.9) also yields

$$\lim_{n\to\infty}\frac{1}{n}\log\left|\left(f^{s-n}_{\theta^n(x)}\right)'\left(f^n_x(z)\right)\right| = 0 \tag{9.14}$$

for every $x\in X_3$ and every $z\in J_r^2(x,M)$. Since $\int_X P_x(h)\,dm(x)=0$, Birkhoff's Ergodic Theorem gives:

$$\lim_{j\to\infty}\frac{1}{j}S_jP_x(h)=0, \tag{9.15}$$

for all $x\in X_4\subset X_3$, where X_4 is some measurable set with $m(X_4)=1$. By combining this formula taken together with the three formulas (9.14), (9.13), and (9.12), and formula (9.11), we get

$$\varliminf_{r\to 0}\frac{\log\nu_{x,h}(\mathbb{D}(z,r))}{\log r}\ge h$$

for every $x\in X_4$ and every $z\in J_r^2(x,M)$. Since $\mu_{x,h}\big(J_r^2(x,M)\big)=1$, we thus obtain, using a version of Frostman's lemma (see, e.g., [12], Theorem 8.6.3):

$$\mathrm{HD}(J_r(x)) \ge \mathrm{HD}(\mu_{x,h}) \ge h \tag{9.16}$$

for every $x\in X_4$ (with $m(X_4)=1$).

We now shall establish the opposite inequality. We know from Lemma 3.19 in [8] that for any $n\ge 1$ large enough, say $n\ge q\ge 1$,

$$Q_n := \inf\big\{\nu_{x,h}(\mathbb{D}(w,\delta)) : x\in X,\ w\in J_x\cap\mathbb{D}(0,n)\big\} > 0.$$

By the very definition of $J_r(x)$ we have that

$$J_r(x) = \bigcup_{n=q}^{\infty} J_r(x,n). \tag{9.17}$$

Fix $n \ge q$. Keep both $x \in X_4$ and $z \in J_r(x,n)$ fixed (still $y := (x,z)$), and consider an arbitrary integer $l \ge 0$ such that

$$f_x^l(z) \in \mathbb{D}(0,n). \tag{9.18}$$

Let $r_l > 0$ be the least radius such that

$$f_y^{-l}\big(\mathbb{D}(f_x^l(z),\delta)\big) \subset \mathbb{D}(z,r_l). \tag{9.19}$$

But, by Koebe's Distortion Theorem, $f_y^{-l}\big(\mathbb{D}(f_x^l(z),\delta)\big) \subset \mathbb{D}\big(z, K\delta|(f_x^l)'(z)|^{-1}\big)$; hence

$$r_l \le K\delta|(f_x^l)'(z)|^{-1}. \tag{9.20}$$

Formula (9.19) along with Koebe's Distortion Theorem and (9.20), yield

$$\begin{aligned}
\nu_{x,h}(\mathbb{D}(z,r_l)) &\ge \nu_{x,h}\big(f_y^{-l}\big(\mathbb{D}(f_x^l(z),\delta)\big)\\
&\ge K_{z,M}^{-h}\big|\big(f_x^l\big)'(z)\big|^{-h} e^{-S_lP_x(h)}\nu_{h,\theta^l(x)}\big(\mathbb{D}(f_x^l(z),\delta)\big)\\
&\ge K_{z,M}^{-h}Q_n e^{-S_lP_x(h)}\big|\big(f_x^l\big)'(z)\big|^{-h}\\
&\ge (K\delta K_{z,M})^{-h}Q_n e^{-S_lP_x(h)} r_l^h.
\end{aligned} \tag{9.21}$$

where the constant $K_{z,M}$ again compensates the replacement of the τ-derivative $\big|\big(f_x^l\big)'(z)\big|_\tau$ by the Euclidean derivative $\big|\big(f_x^l\big)'(z)\big|$. Therefore,

$$\frac{\log \nu_{x,h}(\mathbb{D}(z,r_l))}{\log r_l} \le h - \frac{h\log(K\delta K_{z,M})}{\log r_l} - \frac{S_lP_x(h)}{\log r_l} - \frac{Q_n}{\log r_l}. \tag{9.22}$$

Formula (9.20) equivalently means that

$$-\log r_l \ge \log\big|\big(f_x^l\big)'(z)\big| - \log(K\delta) \ge \hat{\chi} l - \log(K\delta) \tag{9.23}$$

with some $\hat{\chi} > 0$ resulting from uniform expanding property of the system $(f_x)_{x\in X}$. Since the set of all integers $l \ge 1$ for which (9.18) holds is infinite (as $z \in J_r(x,n)$), taking the limit of the right-hand side of (9.22) over all such ls. and applying (9.23), (9.15), and also recalling that, by Birkhoff's Ergodic Theorem,

$$\lim_{j\to\infty} \frac{1}{j} S_jP_x(h) = 0,$$

we obtain

$$\underline{\lim}_{r\to 0} \frac{\log \nu_{x,h}(\mathbb{D}(z,r))}{\log r} \le \underline{\lim}_{l\to\infty} \frac{\log \nu_{x,h}(\mathbb{D}(z,r_l))}{\log r_l} \le h$$

Consequently, $\mathrm{HD}(J_r(x,n)) \le h$ for all $x \in X_4$. Together with (9.17) and σ-stability of Hausdorff dimension, we thus get that $\mathrm{HD}(J_r(x)) \le h$ for all $x \in X_4$. Along with (9.16) this finishes the proof. □

9.3. **Conclusion.** All in all we now get the following analyticity result for the dimension of the radial limit set.

Theorem 9.11. *Suppose that the transcendental holomorphic random family* $(f_{x,\lambda})_{x,\lambda}$ *is admissible and let* h_λ *be the fiberwise Hausdorff dimension of the radial limit set of* $(f_{x,\lambda})_{x\in X}$*,* $\lambda \in \Lambda$*. Then,* $\lambda \mapsto h_\lambda$ *is real-analytic.*

Proof. Bowen's Formula shows that h_λ is the unique zero of the expected pressure function. The later is analytic and $\frac{\partial}{\partial t}\mathcal{E}\mathrm{P}_\lambda(t) < 0$ (Proposition 9.7). Therefore the Implicit Function Theorem applies and yields analyticity of $\lambda \mapsto h_\lambda$. □

It remains to discuss the initial example given in the Introduction.

Proof of Theorem 1.1. Let $U = \{z \in \mathbb{C} : \Re z > 1\}$. It is well known that $f_\eta = \eta e^z$ is a hyperbolic exponential map if η is real and $\frac{1}{6e} < \eta < \frac{5}{6e}$. Moreover, the closure $\overline{f_\eta^{-1}(U)} \subset U$. An elementary calculation shows that there exists $b > 0$ such that $\overline{f_\eta^{-1}(U)} \subset U$ for every $\eta \in \Omega_b$ where

$$\Omega_b = \left\{ \eta \in \mathbb{C}\ ;\ \ \frac{1}{6e} < \Re(\eta) < \frac{5}{6e} \text{ and } |\Im(\eta)| < b \right\}.$$

It follows that $f_{\eta_n} \circ ... \circ f_{\eta_1}$, $n \geq 1$, defines an expanding non-autonomous sequence that satisfies (2.1) for any choice of $\eta_1, \eta_2, ... \in \Omega_b$. It is straightforward to see that we thus have for these parameters a admissible transcendental random family provided that we explain the random model.

In order to do so, let $X = \mathbb{D}(0,1)^{\mathbb{Z}}$, $\mathcal{B}$ the Borel σ-algebra, m the infinite product measure of the normalized Lebesgue measure of the unit disk and θ the left-shift map on X.

Consider now parameters (a, r) such that $\mathbb{D}(a, r) \subset \Omega_{b/2}$. Let $x \in X$ and x_0 the 0–coordinate of x. We associate to these parameters the function $\eta e^z = (a + rx_0)e^z$. In such a way we get for every $x \in X$ a family $(a, r) \mapsto f_\eta$. However, this family only depends real analytically on $(a, r) \in \mathbb{R}^2$. In order to turn this into a holomorphic family it suffices to replace these parameters by complex ones with small imaginary part such that $a + rx_0 \in \Omega_b$ for every $x_0 \in \mathbb{D}(0, 1)$. Theorem 9.11 applies to this family. □

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