Open Conformal Systems and Perturbations of Transfer Operators

OPEN CONFORMAL SYSTEMS
AND
PERTURBATIONS OF TRANSFER OPERATORS

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Abstract. The escape rates for a ball in a dynamical systems has been much studied. Understanding the asymptotic behavior of the escape rate as the radius of the ball tends to zero is an especially subtle problem. In the case of hyperbolic conformal systems this has been addressed by various authors in several papers and these results apply in the case of real one dimensional expanding maps and conformal expanding repellers, particularly hyperbolic rational maps.

In this manuscript we consider a far more general realm of conformal maps where the analysis is correspondingly more involved. We prove the asymptotic existence of escape rates and calculate them in the context of countable alphabets, either finite or infinite, uniformly contracting conformal graph directed Markov systems with their special case of conformal countable alphabet iterated function systems. The reference measures are the projections of Gibbs/equilibrium states of Hölder continuous summable potentials from a countable alphabet subshifts of finite type to the limit set of the graph directed Markov system under consideration.

This goal is achieved firstly by developing the appropriate theory of singular perturbations of Perron-Frobenius (transfer) operators associated with countable alphabet subshifts of finite type and Hölder continuous summable potentials. This is done on the purely symbolic level and leads us also to provide a fairly full account of the structure of the corresponding open dynamical systems and, associated to them, surviving sets for the shift map with the holes used for singular perturbations. In particular, we prove the existence of escape rates for those open systems. Furthermore, we determine the corresponding conditionally invariant probability measures that are absolutely continuous with respect to the reference Gibbs/equilibrium/state. We also prove the existence and uniqueness of equilibrium measures on the symbol surviving sets. We equate the corresponding topological pressures with the negatives of escape rates. Eventually we show that these equilibria exhibit strong stochastic properties, namely the Almost Sure Invariance Principle, and therefore, also an exponential decay of correlations, the Central Limit Theorem and the Law of Iterated Logarithm.

In particular, this includes, as a second ingredient in its own right, the asymptotic behavior of leading eigenvalues of perturbed operators and their first and second derivatives.

Our third ingredient is to relate the geometry and dynamics, roughly speaking to relate the case of avoiding cylinder sets and that of avoiding Euclidean geometric balls. Towards this end, in particular, we investigate in detail thin boundary properties relating the measures of thin annuli to the measures of the balls they enclose. In particular we clarify the results in the case of expanding repellers and conformal graph directed Markov systems with finite alphabet.

The setting of conformal graph directed Markov systems is interesting in its own and moreover, in our approach, it forms the key ingredient for further results about other conformal systems. These include topological Collet-Eckmann multimodal interval maps and topological Collet-Eckmann rational maps of the Riemann sphere (an equivalent formulation is to be uniformly hyperbolic on periodic points), and also a large class of transcendental meromorphic functions.

Our approach here,in particular in relation to the applications mentioned in the previous paragraph, is firstly to note that all of these systems yield some sets, commonly referred to as nice ones, the first return (induced) map to which is isomorphic to a conformal countable alphabet iterated function system with some additional properties. Secondly, with the help of appropriate large deviation results, to relate escape rates of the original system with the induced one and then to apply the results of graph directed Markov systems. The reference measures are again Gibbs/equilibrium states of some large classes of Hölder continuous potentials.

Contents

1. Introduction

The escape rate for a dynamical system is a natural concept which describes the speed at which orbits of points first enter a small region of the space. The size of these sets is usually measured with respect to an appropriate probability. More precisely, given a metric space (X, d), we can consider a (usually) continuous transformation $T : X \to X$ and a ball

$$B(z, \epsilon) = \{x \in X : d(x, z) < \varepsilon\}$$

of radius $\epsilon > 0$ about a given point z. We then obtain an open system by removing $B(z, \epsilon)$ and considering the new space $X \setminus B(z, \varepsilon)$ and truncating those orbits that land in the ball $B(z, \epsilon)$, which can be thought of informally as a "hole" in the system. This is the reason that many authors speak of *escape rates* for the system, whereas it might be a more suitable nomenclature to call them *avoidable or survivor sets*.

We can then consider for each $n > 0$ the set $R_n(z, \varepsilon)$ of points $x \in X$ for which all the first n terms in the orbit omit the ball, i.e.,

$$x, T(x), \dots, T^{n-1}(x) \notin B(z, \varepsilon).$$

It is evident that these sets are nested in both parameters ε and n, i.e.,

$$R_{n+1}(z, \varepsilon) \subset R_n(z, \varepsilon)$$

for all $n \geq 1$ and that

$$R_n(z, \varepsilon) \subset R_n(z, \varepsilon')$$

for $\varepsilon > \varepsilon'$. We can first ask about the behavior of the size of the sets $R_n(z, \varepsilon)$ as $n \to +\infty$.

If we assume that μ is a T-invariant probability measure, say, then we can consider the measures $\mu(R_n(z, \varepsilon))$ of the sets $R_n(z, \varepsilon)$ as $n \to +\infty$. In particular, we can define the *lower and upper escape rates* respectively as

$$\underline{R}_\mu(B(z, \varepsilon)) = -\overline{\lim_{n\to+\infty}} \frac{1}{n} \log \mu(R_n(z, \varepsilon)) \quad \text{and} \quad \overline{R}_\mu(B(z, \varepsilon)) = -\underline{\lim_{n\to+\infty}} \frac{1}{n} \log \mu(R_n(z, \varepsilon)).$$

We say that the escape rates exist if

$$\underline{R}_\mu(B(z, \varepsilon)) = \overline{R}_\mu(B(z, \varepsilon)),$$

we denote this common number by

$$R_\mu(B(z,\varepsilon))$$

and refer to is as the *escape rate* from $B(z,\varepsilon)$. One can further consider how the escape rates behave as the radii of the balls tend to zero. This, in the context of appropriate conformal dynamics, is one of the three primary goals of the present work.

Our second goal is to understand the geometry of the avoiding/survivor sets

$$K_z(\varepsilon) := \{x \in X : T^n(x) \notin B(z,\varepsilon)\ \forall n \geq 0\}.$$

Such sets are T-invariant, meaning that

$$T(K_z(\varepsilon)) \subseteq K_z(\varepsilon),$$

and are usually of measure μ zero, but there is another natural quantity to measure their size and complexity, namely their Hausdorff dimension. Our goal is to put our hands on the asymptotic values of $\mathrm{HD}(K_z(\varepsilon))$ when $\varepsilon \searrow 0$; again in the context of appropriate conformal dynamics.

Escape rates and asymptotics of $\mathrm{HD}(K_z(\varepsilon))$ are indeed natural and well–motivated quantities to study in the context of open systems, they quantitatively measure the way points avoid the holes under iteration. Gerhard Keller in [24] has unraveled some connections of escape rates with non-singular perturbations of Perron–Frobenius operators. But these, i.e. escape rates and asymptotics of $\mathrm{HD}(K_z(\varepsilon))$, are not the only foci of our manuscript. Our approach to survivor sets is more comprehensive. There are more concepts, notions and results finely describing what is going on with the survivor sets. One should mention here first of all the pioneering works [38], [37], and [9] respectively of G. Pianigiani, G. G. Pianigiani and J. A. Yorke, and of P. Collet, S. Martínez and B. Schmitt. Furthermore [8], [11], [26], [24], [13],[14], [10], [28], [27], [12], [15], the references therein and many more; we are far from pretending for this list of references to be complete. The concepts worked out throughout these works include the following.

First one should notice that in all the above the ball $B(z,\varepsilon)$ can be replaced by any open set, and the concepts of survivor sets and escape rates remain unchanged. Let us denote such arbitrary open set by U. We write then $K(U)$ for the corresponding survivor set.

We call a Borel probability measure ν on $X \setminus U$ conditionally invariant if there exists $\alpha \in (0,+\infty)$ such that

$$\nu((X \setminus U) \cap T^{-1}(A)) = \alpha\nu(A) \tag{1.1}$$

for every Borel set $A \subset X \setminus U$. In slightly different terms, a Borel probability measure ν on X is conditionally invariant with respect to $X \setminus U$ if $\nu(X \setminus U) = 1$ and

$$\nu \circ T^{-1} = \alpha\nu. \tag{1.2}$$

If ν is absolutely continuous with respect to μ_φ, an equilibrium state of a "sufficiently good" continuous potential $\varphi : X \to \mathbb{R}$, then we call a T-invariant Borel probability measure

μ on $K(U)$ a surviving equilibrium for φ if it is “an ordinary” equilibrium state on $K(U)$ for the map $T : K(U) \to K(U)$, i.e. if

$$\sup\left\{ \mathrm{h}_\nu(\sigma) + \int_{K(U_n)} \varphi \, d\nu \right\} = \mathrm{h}_\mu(\sigma) + \int_{K(U_n)} \varphi \, d\mu,$$

where the supremum is taken over all Borel probability T–invariant measures ν on $K(U)$ for which $\int_{K(U)}(-\varphi)\, d\nu < +\infty$, and if in addition the above quantity is equal to $\log \alpha$.

The most natural and intriguing questions are about the existence and uniqueness of conditionally invariant measures absolutely continuous with respect the reference measure μ, the existence and uniqueness of surviving equlibria, and the value of the supremum above; the one being commonly expected to be equal to the negative escape rate. We address all these questions in full on the symbolic level resulting from complex dynamics considerations.

We begin by describing some of the background for this area. An early influential result in the direction of understanding asymptotic escape rates was [51]. Perhaps the simplest case is that of the doubling map $E_2 : [0, 1) \to [0, 1)$ defined by

$$E_2(x) = 2x (\mathrm{mod}\, 1)$$

and the usual Lebesgue measure λ. For this example it was Bunimovitch and Yurchenko [4] (see also [26]) who proved the following, perhaps surprising, result showing that

$$\lim_{\varepsilon\to 0} \frac{\underline{R}_\lambda(B(z,\varepsilon))}{\lambda(B(x_0,\varepsilon))} = \lim_{\varepsilon\to 0} \frac{\overline{R}_\lambda(B(z,\varepsilon))}{\lambda(B(x_0,\varepsilon))} = \tag{1.3}$$
$$= \begin{cases} 1 & \text{if } z \text{ is not periodic} \\ 1 - 2^{-p} & \text{if } E_2^p(z) = z \text{ is periodic (with minimal period } p). \end{cases}$$

In particular, the asymptotic escape rate can only take a certain set of values which are determined by the periods of periodic points. More results in this direction followed, particularly in [26], [24], [20], [12], [11], [14].

We will return to generalizations of these ideas after discussing the problem of the Hausdorff dimension of surviving sets $K_z(\varepsilon)$. The second named author already addressed this question in the early 80s by showing in [49] and [50] that in the case of the the doubling map E_2, or more generally, of any map $E_q(x) = qx\,(\mathrm{mod}\, 1)$, q being an integer greater than 1 in absolute value, or even more generally, in the case of any $C^{1+\eta}$ expanding map of the unit circle, the map

$$\varepsilon \mapsto \mathrm{HD}(K_z(\varepsilon))$$

is continuous. Moreover, he also showed that this function is almost everywhere locally constant, in fact, the set of points where it fails to be locally constant is a closed set of Hausdorff dimension 1 and Lebesgue measure zero. Rather curiously, the local Hausdorff dimension at each point ε of this set is equal to $\mathrm{HD}(K_z(\varepsilon))$. More about the function $e \mapsto \mathrm{HD}(K_z(\varepsilon))$ and related questions can be found for example in [2] and [6]. All of this suggests that it is interesting to study the asymptotic properties of $\mathrm{HD}(K_z(\varepsilon))$ when

$\varepsilon \searrow 0$. Andrew Ferguson and the first named author of this paper took up the challenge by proving in [20] that

(1.4)

$$\lim_{\varepsilon\to 0}\frac{\mathrm{HD}(J)-\mathrm{HD}(K_z(r))}{\mu_b(B(z,r))}=\begin{cases}1/\chi_{\mu_b} & \text{if } z \text{ is not a periodic point of } T\\ \frac{1-|(T^p)'(z)|^{-1}}{\chi_{\mu_b}} & \text{if } z \text{ is a periodic point of } T \text{ with prime period}\\ & p\geq 1.\end{cases}$$

in the case of any conformal expanding repeller $T : J \to J$; where b here is just the Hausdorff dimension $\mathrm{HD}(J)$ and μ_b is the equilibrium state of the Hölder continuous potential $J \ni x \mapsto -b\log|T'(x)|$. They have also established the analogue of (1.3) for such systems.

The approach of [20] was based on the method of singular perturbations, pioneered in the context of open systems by Véronique Maume–Deschamps and Carlangelo Liverani in [27], of the Perron–Frobenius operators determined by the open sets $B(z,\varepsilon)$. They first did this for neighborhoods of z formed by finite unions of cylinders of nth refinements of a Markov partition and then used appropriate approximation. This required leaving the realm of the familiar Banach space of Hölder continuous functions, to work with a more refined space, and they applied the seminal results of Keller and Liverani from [25] to control the spectral properties of perturbed operators.

For completeness, we recall that Keller and Liverani introduced a different framework in [26], whereby one considers a family $P_\epsilon : V \to V$ $(0 \le \epsilon \le \epsilon_0)$ of bounded linear operators with a spectral gap, i.e.,

$$P_\epsilon = \lambda_\epsilon\nu_\epsilon(\cdot)\varphi_\epsilon + U_\epsilon$$

with $\lambda_\epsilon > 0$ and normalization $\nu_\epsilon(\varphi_\epsilon) = 1$, where $P_\epsilon U_\epsilon = 0$ and U_ϵ has spectral radius strictly smaller than λ_ϵ. The approach of Keller-Liverani requires a series of functional analytic assumptions. Firstly, they assume that

$$C_1 := \sum_{n=0}^{\infty}\sup_{\epsilon\in[0,\epsilon_0]}\|U_\epsilon^n\| < +\infty \text{ and } C_2 := \sup_{\epsilon\in[0,\epsilon_0]}\|\varphi_\epsilon\| < +\infty \text{ where } \nu_0(\varphi_\epsilon) = 1.$$

Secondly, they denote $\Delta_\epsilon := \nu_0\left((P_0 - P_\epsilon)(\varphi_0)\right)$ and require that there exists $C_3 > 0$ such that

$$\eta_\epsilon\|(P_0 - P_\epsilon)(\varphi_0)\| \le C_3|\Delta_\epsilon| \text{ where } \eta_\epsilon := \|\nu_0(P - P_\epsilon)\| \to 0 \text{ as } \epsilon \to 0.$$

In particular, they can first write

$$\lambda_0 - \lambda_\epsilon = \lambda_0\nu_0(\varphi_\epsilon) - \nu_0(\lambda_\epsilon(\varphi_\epsilon)) = \nu_0((P_0 - P_\epsilon)(\varphi_\epsilon)) \le C_2\eta_\epsilon.$$

If $\Delta_\epsilon \neq 0$ for sufficiently small $\epsilon > 0$, then for $k \ge 0$ they denote

$$q_k := \lim_{\epsilon\to 0}\frac{\nu_0((P_0 - P_\epsilon)P_\epsilon^k(P_0 - P_\epsilon))(\varphi_0)}{\Delta_\epsilon}$$

and then they have an expression

$$\lim_{\epsilon\to 0}\frac{\lambda_0 - \lambda_\epsilon}{\Delta_\epsilon} = 1 - \sum_{k=0}^{\infty} q_k.$$

Finally, to relate this formula to the escape rate problem, they apply these results to the special choice of operator $P_\epsilon := \mathcal{L}(I - 1\!\!1_{U_\epsilon})$ where $\mathcal{L}$ is the usual Perron-Frobenius operator for piecewise monotone expansive maps on the Banach space V of functions of bounded variation and where U_ϵ are a nested sequence of intervals shrinking to a point.

For our purposes it was more natural and more suitable to advance the approach in [20] rather than to deal with the one in [26].

We now turn to describing and discussing our results in relation to the three general goals and questions described above. In the current manuscript we want to understand the escape rates, in the sense of equations (1.3) and (1.4), of essentially all conformal dynamical systems with an appropriate type of expanding dynamics. By this we primarily mean a large class of topologically exact smooth maps of the interval $[0, 1]$, many rational functions of the Riemann sphere $\widehat{\mathbb{C}}$ with degree ≥ 2, a vast class of transcendental meromorphic functions from $\mathbb{C}$ to $\widehat{\mathbb{C}}$, and last, but not least, the class of all countable alphabet conformal iterated function systems, and somewhat more generally, the class of all countable alphabet conformal graph directed Markov systems. This last class, i.e the collection of all countable alphabet conformal iterated function systems (IFSs), has a special status for us. The reasons for this are two-folded. Firstly, this class is interesting by itself, and secondly, by means of appropriate inducing schemes (involving the first return map), it is our indispensable tool for understanding the escape rates of all other systems mentioned above.

In order to deal with escape rates for countable alphabet conformal IFSs and conformal graph directed Markov systems (GDMSs), motivated by the work [20] of Andrew Ferguson and the first named author of this paper, we first develop the singular perturbation theory for Perron-Frobenius operators associated to Hölder continuous summable potentials on countable alphabet shift of finite type symbol space.

A comprehensive account of the thermodynamic formalism in the symbolic context can be found in [31], cf. also [29] and [30]. The general approach to control the above mentioned singular perturbations is again based on the spectral results of Keller and Liverani from [25]. Because of its critical importance to us, for the convenience of the reader and for our convenience of reference, we bring up the setting of [25] in Appendix at the end of our manuscript. We formulate there Theorem 1 of [25] and all its consequences that we shall need.

The perturbations in the case of a countable infinite alphabet require further refinement of the Banach space on which the original and perturbed Perron–Frobenius operators act. This space, $\mathcal{B}_\theta$, with its (relatively) strong norm $||\cdot||_\theta$ is defined already in the beginning of Section 3. Its weak norm $||\cdot||_*$ is defined in Section 4 and plays a crucial role in Section 5 in showing the smallness of perturbations of Perron-Frobenius operators as acting from $\mathcal{B}_\theta$ endowed with the strong norm $||\cdot||_\theta$ to $\mathcal{B}_\theta$ endowed with the weak norm $||\cdot||_*$. As already said, this method of perturbing the Perron–Frobenius operators from a strong norm to a week one, was pioneered in the conext of open systems by Véronique Maume–Deschamps and Carlangelo Liverani in [27] and later was applied many times, for example in [24], [15], and [20].

The culminating technical result, a kind of source of all that follows, of our investigations of singular perturbations of Perron-Frobenius operators (Part 1) is Proposition 5.2 establishing spectral gaps for perturbed operators. Its further versions (such as singular perturbations of already perturbed operators) are needed and provided for example in Section 10; see Lemma 10.4 therein. We prove Proposition 5.2 by applying Theorem 1 of [25] and its consequences derived therein. As we have already said, for the convenience of the reader and convenience of referring to, we bring up the setting of [25] in Appendix at the end of our manuscript. We formulate there Theorem 1 of [25] and all its consequences we need.

Already the definition of the Banach space $\mathcal{B}_\theta$ is non-standard and non–canonical. Through the definition of the norm $||\cdot||_\theta$, it involves the corresponding Gibbs/equilibrium measures. These measures play a further prominent role when investigating singular perturbations. Qualitatively new difficulties here, caused by an infinite alphabet, are many fold and a great deal of them are related to the facts that the symbol space E_A^∞ need not longer be compact, that there are infinitely many cylinders of given finite length, and that summable (particular geometric) potentials are unbounded in the supremum norm. Some remedy to this unboundedness issue is our repetitive use of Hölder inequalities rather than estimating by the supremum norms.

Part 2: Symbol Escape Rates and the Survivor Set $K(U_n)$, is still on the level of symbolic dynamics, no geometry involved. The holes U_n, being special unions of cylinders of length n, are tailor crafted for the needs of the next part, Part 3: Escape Rates for Conformal GDMSs and IFSs. However, these holes are of fairly general form, are of their own interest, and become particularly simple if the alpabet E is finite. We present a full account of ergodic theory and thermodynamic formalism for the open dynamical systems they generate.

Let E be a countable alphabet and let $A : E \times E \to \{0,1\}$ be a finitely primitive incidence matrix. Let $\varphi : E_A^\infty \to \mathbb{R}$ be a Hölder continuous summable potential with equilibrium/Gibbs state μ_φ. Specifically, in Part 2, we prove the following results.

Theorem 1.1. *If an integer $n \ge 0$ is big enough as required in Proposition 5.2 then $\widehat{\mu}_n$, a probability multiple of $\mu_\varphi|_{U_n^c}$ is a unique conditionally invariant measure on U_n^c absolutely continuous with respect to $\mu_\varphi|_{U_n^c}$ whose Radon-Nikodyn derivative $d\widehat{\mu}_n/d\mu_\varphi$ belongs to $\mathcal{B}_\theta$. In addition, the coefficient α of (1.1) and (1.2) is equal to $\lambda_n (= \widehat{\mu}_n(\sigma^{-1}(U_n^c)))$ and for every Borel set $B \subset U_n^c$ we have that*

$$\lim_{k\to+\infty} \frac{\mu_\varphi(\sigma^{-k}(B) \cap U_n^c)}{\mu_\varphi(\sigma^{-k}(U_n^c) \cap U_n^c)} = \widehat{\mu}_n(B).$$

Theorem 1.2. *Assume that $\int(-\varphi)\, d\mu_\varphi < +\infty$. If an integer $n \ge 0$ is big enough as required in Proposition 5.2 then the escape rate $R_{\mu_\varphi}(U_n)$ exists and is equal to $-\log \lambda_n$, where (see Proposition 5.2) λ_n is the spectral radius of the perturbed operator $\mathcal{L}_n$ generated by the hole U_n.*

Theorem 1.3. *If an integer $n \geq 0$ is big enough as required in Proposition 5.2 then*

$$\sup\left\{\mathrm{h}_\nu(\sigma)+\int_{K(U_n)}\varphi\,d\nu : \nu\in\mathcal{M}_n^+(\sigma)\right\}=\sup\left\{\mathrm{h}_\nu(\sigma)+\int_{K(U_n)}\varphi\,d\nu : \nu\in\mathcal{M}_n^{+e}(\sigma)\right\}=\log\lambda_n,$$

where, as above, λ_n is the spectral radius of the perturbed operator $\mathcal{L}_n$ generated by the hole U_n.

Theorem 1.4. *Assume that $\int(-\varphi)\,d\mu_\varphi < +\infty$. If an integer $n \geq 0$ is big enough as required in Proposition 5.2 then there exists a unique (ergodic) surviving equilibrium state on the survivor set $K(U_n)$.*

Precisely, the bounded positive linear functional, defined in (10.1), restricted to the Banach space of bounded Hölder continuous functions on E_A^∞, extends uniquely to a Borel probability measure μ_n on E_A^∞ which is supported on $K(U_n)$, shift-invariant and ergodic. This is the unique surviving equilibrium mentioned above. Being an equilibrium means that

$$\mathrm{h}_{\mu_n}(\sigma)+\int_{K(U_n)}\varphi\,d\mu_n=\log\lambda_n.$$

Theorem 1.5. *If an integer $n \geq 0$ is big enough as required in Proposition 5.2 and μ_n is the unique (ergodic) surviving equilibrium state on the survivor set $K(U_n)$, described in the previous theorem, then the measure–preserving dynamical system $(\sigma : K(U_n) \to K(U_n), \mu_n)$ satisfies an Almost Sure Invariance Principle. In particular, the Central Limit Theorem and the Law of Iterated Logarithm hold:*

Let $d \geq 1$ be an integer. Fix an integer $n \geq 0$ so large as required in Proposition 5.2. Let $g : K(U_n) \to \mathbb{R}^d$ be a bounded Hölder continuous function. Then there exists a matrix $\Sigma^2 : \{1, 2, \dots, d\}^2 \to \mathbb{R}^d$ such that the process

$$\left(g\circ\sigma^k-\int_{K(U_n)}g\,d\mu_n\right)_{k=1}^\infty$$

satisfies an almost sure invariance principle with the limiting covariance Σ^2. In particular, the sequence

$$\left(\sum_{j=0}^{k-1}g\circ\sigma^j-k\int_{K(U_n)}g\,d\mu_n\right)_{k=1}^\infty$$

converges in distribution to the Gaussian (normal) distribution $\mathcal{N}(0, \sigma^2)$. In addition, if $d = 1$ then the Law of Iterated Logarithm holds in the form that for μ_n–a.e. $\omega \in K(U_n)$, we have that

$$\limsup_{k\to+\infty}\frac{\sum_{j=0}^{k-1}g\circ\sigma^j(\omega)-k\int_{K(U_n)}g\,d\mu_n}{\sqrt{k\log\log k}}=\sqrt{2\pi}\sigma,$$

where $\sigma^2 := \Sigma^2$ is a non-negative number. It is positive if an only if the function $g : K(U_n) \to \mathbb{R}$ is not cohomologous to a constant in $L^2(\mu_n)$.

Theorem 1.6 (Exponential Decay of Correlations)**.** *Suppose that* $(U_n)_{n=0}^\infty$ *is a sequence of open subsets of* E_A^∞ *satisfying conditions (U0)–(U5). Fix an integer* $n \geq 0$ *so large as required in Proposition* 5.2*. Then there exist* $\kappa \in (0,1)$ *and* $C \in (0,+\infty)$ *such that if* $g : K(U_n) \to \mathbb{R}$ *is a bounded Hölder continuous function and* $h \in L^1(\mu_n)$*, then*

$$\Big|\int_{K(U_n)} (g \circ \sigma^k \cdot h)\, d\mu_n - \int_{K(U_n)} g\, d\mu_n \int_{K(U_n)} h\, d\mu_n\Big| \leq C\kappa^n \|g\|_\theta \int_{K(U_n)} |f|\, d\mu_n$$

for every integer $k \geq 0$*.*

We cannot really do much better with the uniqueness part of Theorem 1.1; the hypothesis that the Radon-Nikodyn derivative $d\widehat{\mu}_n/d\mu_\varphi$ belongs to $\mathcal{B}_\theta$ is important. Indeed, it follows from Theorem 3.1 in [14], that for every $\alpha \in (0,1)$ there are uncountably (a continuum) of many conditionally invariant measures absolutely continuous with respect to μ_φ. Moreover, if $\alpha \in (0,1)$ is sufficiently small, then the Radon-Nikodym derivatives of all these measures with respect to μ_φ are bounded.

We would like to add that some of the results listed above were obtained in [11] in the context of open systems generated by holes in an infinite alphabet finitely primitive symbolic subshift of finite type. Their holes were unions of cylinders of length 1 and their methods were different, i.e. not perturbative. For the case of finite alphabet the reader is encouraged to consult the book [8].

We would also like to note that Parts 1, 3, and 4 are quite independent of Part 2. Except that the results of Section 7 in Part 2 are absolutely indispensable for all the rest the manuscript and the proof of (16.31) presented in Section 16 of Part3 is so simple because we were able to show in Part 2 that the functionals μ_n appearing there are in fact measures.

Having analyzed the symbolic part of the problem, we turn to escape rates for conformal GDMSs. With regard to formula (1.3), we consider the, already mentioned, measures on the limit set of the given conformal GDMS, that are projections of Gibbs/equilibrium states of Hölder continuous potentials from the symbol space. With respect to formula (1.4), we must consider geometric potentials, i.e. those of the form

$$E_A^\infty \ni \omega \longmapsto t \log \big|\varphi'_{\omega_0}(\pi_\mathcal{S}(\sigma(\omega)))\big| \in \mathbb{R}$$

where $\pi : E_A^\infty \to X$ is the canonical map for modeling the dynamics on X. Of particular interest are those for which t is close to $b_\mathcal{S}$, the Bowen parameter of the system conformal GDMS$\mathcal{S}$, which is defined as the only solution to the pressure equation

$$\mathrm{P}\big(\sigma, t \log \big|\varphi'_{\omega_0}(\pi_\mathcal{S}(\sigma(\omega)))\big|\big) = 0,$$

provided that such a solution exists. We can then consider the projection of the Gibbs/equilibrium state μ_t for the potential $t \log \big|\varphi'_{\omega_0}(\pi_\mathcal{S}(\sigma(\omega)))\big|$ on the limit set $J_\mathcal{S}$. This leads to the particularly technically involved task of calculating the asymptotic behavior of derivatives $\lambda_n'(t)$ and $\lambda_n''(t)$ of leading eigenvalues of perturbed operators when the integer $n \geq 0$ diverges to infinity and the parameter t approaches $b_\mathcal{S}$. This is again partially due to unboundedness of the function $E_A^\infty \ni \omega \longmapsto t \log \big|\varphi'_{\omega_0}(\pi_\mathcal{S}(\sigma(\omega)))\big| \in \mathbb{R}$ in the supremum norm and partially due to lack of uniform topological mixing on the sets $K_z(\varepsilon)$.

We say that a set $J \subseteq \mathbb{R}^d$, $d \geq 1$, is geometrically irreducible if it is not contained in any countable union of conformal images of hyperplanes or spheres of dimension $\leq d-1$ (see Definition 14.4). Our most general results about escape rates for conformal GDMSs can now be formulated in the following four theorems. We postpone detailed definitions of the hypotheses until later. We however want to mention that we adopt a simplified notation for Birkhoff's sums. Given a dynamical system $T : X \to X$ and a function $\varphi : X \to \mathbb{C}$ we set for every integer $n \geq 1$:

$$\varphi_n := \sum_{j=0}^{n-1} \varphi \circ T^j.$$

This notation does not encode the dynamical system T under considerations, but it will be virtually always clear from the context which dynamical system is meant. For example, in the realm of the symbol spaces E_A^∞, it will be always the shift map or its induced maps. However, we will be sometimes more traditional and will also use the notation

$$S_n\varphi$$

for φ_n; particularly in contexts where lots of functions with indeces appear.

Theorem 1.7. *Let $\mathcal{S} = \{\varphi_e\}_{e\in E}$ be a finitely primitive Conformal GDMS with limit set $J_{\mathcal{S}}$. Let $\varphi : E_A^\infty \to \mathbb{R}$ be a Hölder continuous summable potential with equilibrium/Gibbs state μ_φ. Assume that the measure $\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}$ is weakly boundary thin* (WBT) *at a point $z \in J_{\mathcal{S}}$. If z is either*

(a) *not pseudo-periodic,*

or

(b) *uniquely periodic, it belongs to* $\mathrm{Int}X$ *(and $z = \pi(\xi^\infty)$ for a (unique) irreducible word $\xi \in E_A^*$ with $\xi^\infty \in E_A^\infty$ being the infinite concatenation of ξ), and φ is the amalgamated function of a summable Hölder continuous system of functions,*

then, with $\underline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon)) := \underline{R}_{\mu_\varphi}\big(\pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))\big)$ and $\overline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon)) := \overline{R}_{\mu_\varphi}\big(\pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))\big)$, we have that

$$\begin{aligned}(1.5)\qquad &\lim_{\varepsilon\to 0}\frac{\underline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon))}{\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))} = \lim_{\varepsilon\to 0}\frac{\overline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon))}{\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))} = \\ &= d_\varphi(z) := \begin{cases} 1 & \text{if (a) holds} \\ 1-\exp\big(\varphi_p(\xi)-p\mathrm{P}(\varphi)\big) & \text{if (b) holds,}\end{cases}\end{aligned}$$

where in (b), *$\{\xi\} = \pi_{\mathcal{S}}^{-1}(z)$ and $p \geq 1$ is the prime period of ξ under the shift map.*

Theorem 1.8. *Assume that $\mathcal{S}$ is a finitely primitive conformal* GDMS *whose limit set $J_{\mathcal{S}}$ is geometrically irreducible. Let $\varphi : E_A^\infty \to \mathbb{R}$ be a Hölder continuous strongly summable potential. As usual, denote its equilibrium/Gibbs state by μ_φ. Then, with $R_{\mathcal{S},\varphi}(B(z,\varepsilon)) := R_{\mu_\varphi}\big(\pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))\big)$, we have that*

$$(1.6)\qquad \lim_{\varepsilon\to 0}\frac{\underline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon))}{\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))} = \lim_{\varepsilon\to 0}\frac{\overline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon))}{\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))} = 1$$

for $\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}$*–a.e. point* z *of* $\mathcal{J}_{\mathcal{S}}$*.*

These two theorems address the issue of (1.3). We would like to note that we do not know whether the actual escape rates $R_{\mathcal{S},\varphi}(B(z,\varepsilon))$ exist. As we have already explained it above we know this, see Section 7 in Part 2, on the symbolic level for the open systems generated bt the holes U_n, $n \geq 1$. However, all what our approach gives is that the balls $B(z,\varepsilon)$, in fact their inverse-images $\pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))$, are, in an appropriate sense, better and better approximated by the sets U_n as $\varepsilon \searrow 0$ and n depends on ε, but are actually never equal. This gives the asymptotic equality in (1.5) and (1.6) but no more. The symbol structure of the sets $\pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))$ themselves seems to be too complex (for example these sets usually cannot be represented as unions of cylinders of the same length) for the strong norm versus weak norm smallness of perturbations to hold. We are thus content with the asymptotic results of (1.5) and (1.6).

We would like to bring to the reader's attention a preprint [3] by H. Bruin, M.F. Demers and M. Todd, with results related to the above three, which we have recently received. In regard to (1.4), we have proved for conformal GDMSs the following two theorems. In regard to (1.4), we have proved the following two theorems for conformal GDMSs.

Theorem 1.9. *Let* $\mathcal{S}$ *be a finitely primitive strongly regular conformal* GDMS*. Assume both that* $\mathcal{S}$ *is* (WBT) *and the parameter* $b_{\mathcal{S}}$ *is powering at some point* $z \in J_{\mathcal{S}}$ *which is either*

(a) *not pseudo-periodic or else*
(b) *uniquely periodic and belongs to* Int$X$ *(and* $z = \pi(\xi^\infty)$ *for a (unique) irreducible word* $\xi \in E_A^*$*).*

Then

$$\lim_{r\to 0} \frac{\mathrm{HD}(J_{\mathcal{S}}) - \mathrm{HD}(K_z(r))}{\mu_b\big(\pi^{-1}(B(z,r))\big)} = \begin{cases} 1/\chi_{\mu_b} & \text{if (a) holds} \\ \big(1 - |\varphi_\xi'(z)|\big)/\chi_{\mu_b} & \text{if (b) holds}. \end{cases} \tag{1.7}$$

Corollary 1.10. *If* $\mathcal{S}$ *be a finitely primitive strongly regular conformal* GDMS *whose limit set* $J_{\mathcal{S}}$ *is a geometrically irreducible, then*

$$\lim_{r\to 0} \frac{\mathrm{HD}(J_{\mathcal{S}}) - \mathrm{HD}(K_z(r))}{\mu_b\big(\pi^{-1}(B(z,r))\big)} = \frac{1}{\chi_{\mu_b}} \tag{1.8}$$

at $\mu_{b_{\mathcal{S}}} \circ \pi^{-1}$*–a.e. point* z *of* $J_{\mathcal{S}}$*.*

As we have previously remarked, these four results are of independent interest, but they also provide a gateway to all other results on escape rates in this paper. There are necessarily several technical terms involved in formulations of these theorems. However, we hope that they do not obscure the overall meaning of the four theorems and all terms are carefully introduced and explained in appropriate sections dealing with them.

We would however like to comment on one of these terms, namely on (WBT). Its meaning can be understood as follows. Let

$$A(z;r,R) := B(z,R) \setminus \overline{B(z,r)}$$

be the annulus centered at z with the inner radius r and the outer radius R. We say that a finite Borel measure μ is weakly boundary thin (WBT) (with exponent $\beta > 0$) at the point x if

$$\lim_{r\to 0} \frac{\mu\big(A_\mu^\beta(x,r)\big)}{\mu(B(x,r))} = 0,$$

where we denote

$$A_\mu^\beta(x,r) := A\big(x; r - \mu(B(x,r))^\beta, r + \mu(B(x,r))^\beta\big).$$

This is a version of the problem of thin annuli, one that is notoriously challenging in dealing with the issue of relating dynamical and geometric properties, and which is particularly acute in the contexts of escape rates and return rates. Due to the breakthrough of [36], where some strong versions of the thin annuli properties are proved, we have been able in the current paper to prove (WBT) for almost all points, which is reflected in both Theorem 15.11 and Corollary 17.3. The (WBT) property allows us in turn to approximate sufficiently well the symbolic sets $\pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))$ by the sets U_n, the significance of which we discussed few paragraphs above.

In the case of finite alphabets E we have the following two results.

Theorem 1.11. *Let $\mathcal{S} = \{\varphi_e\}_{e\in E}$ be a primitive conformal GDMS with a finite alphabet E acting in the space $\mathbb{R}^d$, $d \geq 1$. Assume that either $d = 1$ or that the system $\mathcal{S}$ is geometrically irreducible. Let $\varphi : E_A^\infty \to \mathbb{R}$ be a Hölder continuous potential. As usual, denote its equilibrium/Gibbs state by μ_φ. Let $z \in J_{\mathcal{S}}$ be arbitrary. If either z is*

(a) *not pseudo-periodic,*
or
(b) *uniquely periodic, it belongs to* $\mathrm{Int}X$ *(and $z = \pi(\xi^\infty)$ for a (unique) irreducible word $\xi \in E_A^*$), and φ is the amalgamated function of a summable Hölder continuous system of functions,*

then,

$$\lim_{\varepsilon\to 0} \frac{\underline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))} = \lim_{\varepsilon\to 0} \frac{\overline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))} = = d_\varphi(z) := \begin{cases} 1 & \text{if (a) holds} \\ 1 - \exp\big(\varphi_p(\xi) - p\mathrm{P}(\varphi)\big) & \text{if (b) holds,} \end{cases} \tag{1.9}$$

where in (b), $\{\xi\} = \pi_{\mathcal{S}}^{-1}(z)$ *and $p \geq 1$ is the prime period of ξ under the shift map.*

Theorem 1.12. *Let $\mathcal{S} = \{\varphi_e\}_{e\in E}$ be a primitive conformal GDMS with a finite alphabet E acting in the space $\mathbb{R}^d$, $d \geq 1$. Assume that either $d = 1$ or that the system $\mathcal{S}$ is geometrically irreducible. Let $z \in J_{\mathcal{S}}$ be arbitrary. If either z is*

(a) *not pseudo-periodic or else*
(b) *uniquely periodic and belongs to* Int$X$ *(and* $z = \pi(\xi^\infty)$ *for a (unique) irreducible word* $\xi \in E_A^*$*).*

Then

$$\lim_{r\to 0} \frac{\mathrm{HD}(J_{\mathcal{S}}) - \mathrm{HD}(K_z(r))}{\mu_b\big(\pi^{-1}(B(z,r))\big)} = \begin{cases} 1/\chi_{\mu_b} & \text{if (a) holds} \\ \big(1-|\varphi_\xi'(z)|\big)/\chi_{\mu_b} & \text{if (b) holds}. \end{cases} \tag{1.10}$$

For these two theorems the two Thin Annuli Properties, Theorem 14.9 and Theorem 14.10, were also instrumental. With having both Theorem 15.12 and Theorem 17.5 proved we have fully recovered the results of [20].

As we have already explained, our next goal in this paper is to get the existence of escape rates in the sense of (1.3) and (1.4) for a a large class of topologically exact smooth maps of the interval $[0,1]$, many rational functions of the Riemann sphere $\widehat{\mathbb{C}}$ with degree ≥ 2, and a vast class of transcendental meromorphic functions from $\mathbb{C}$ to $\widehat{\mathbb{C}}$. In order to do this we employ two principle tools. The first is formed by the escape rates results, described above in detail, for the class of all countable alphabet conformal graph directed Markov systems. The second is a method based on the first return (induced) map developed in Section 19, Section 20, and Section 21. This method closely relates the escape rates of the original map and the induced map. It turns out that for the above mentioned classes of systems one can find a set of positive measure which gives rise to a first return map which is isomorphic to a countable alphabet conformal IFS or full shift map; the task being highly non-trivial and technically involved. But this allows us to conclude, for suitable systems, the existence of escape rates in the sense of (1.3) and (1.4). However, in order to reach this conclusion we need to know some non-trivial properties of the original systems. Firstly, that the tails of the first return time and the first entrance time decay exponentially fast, and secondly that the Large Deviation Property (LDP) of Section 20 holds. This in turn leads to Theorem 21.6, a kind of Large Deviation Theorem.

We shall now describe in some detail the above mentioned applications to (quite) general conformal systems. We start with one-dimensional systems. We consider the class of topologically exact piecewise C^3–smooth multimodal maps T of the interval $I=[0,1]$ with non-flat critical points and uniformly expanding periodic points, the property commonly referred to as Topological Collet–Eckmann. Topological exactness means that for every non-empty subset U of I there exists an integer $n \geq 0$ such that $T^n(U) = I$. Furthermore, our multimodal map $T : I \to I$ is assumed to be tame, meaning that

$$\overline{\mathrm{PC}(T)} \neq I,$$

where

$$\mathrm{Crit}(T) := \{c \in I : T'(c) = 0\}$$

is the critical set for T and

$$\mathrm{PC}(T) := \bigcup_{n=1}^{\infty} T^n(\mathrm{Crit}(T)),$$

is the postcritical set of T. A familiar example would be the famous unimodal map $x \mapsto \lambda x(1-x)$ with $0 < \lambda < 4$ for which the critical point $1/2$ is not in its own omega limit set, for example where λ is a Misiurewicz point.

The class of potentials, called acceptable in the sequel, is provided by all Lipschitz continuous functions $\psi : I \to \mathbb{R}$ for which

$$\sup(\psi) - \inf(\psi) < \mathrm{h_{top}}(T).$$

The first escape rates theorem in this setting is this.

Theorem 1.13. *Let $T : I \to I$ be a tame topologically exact Topological Collet–Eckmann map. Let $\psi : I \to \mathbb{R}$ be an acceptable potential. Let $z \in I \setminus \overline{\mathrm{PC}(T)}$ be a recurrent point. Assume that the equilibrium state μ_ψ is* (WBT) *at z. Then*

$$\lim_{\varepsilon \to 0} \frac{\underline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} = \lim_{\varepsilon \to 0} \frac{\overline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} =$$
$$= \begin{cases} & \text{if } z \text{ is not any periodic point of } T, \\ 1 - \exp\big(\psi_p(z) - p\mathrm{P}(f,\psi)\big) & \text{if } z \text{ is a periodic point of } T. \end{cases}$$

We have also the following.

Theorem 1.14. *Let $T : I \to I$ be a tame topologically exact Topological Collet–Eckmann map map. Let $\psi : I \to \mathbb{R}$ be an acceptable potential. Then*

$$\lim_{\varepsilon \to 0} \frac{\underline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} = \lim_{\varepsilon \to 0} \frac{\overline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} = 1$$

for μ_ψ–a.e. point $z \in I$.

In order to address formula (1.4) in this context we need a stronger assumption on the map $T : I \to I$. Our multimodal map $T : I \to I$ is said to be subexpanding if

$$\mathrm{Crit}(T) \cap \overline{\mathrm{PC}(T)} = \emptyset.$$

It is evident that each subexpanding map is tame and it is not hard to see that the subexpanding property entails being Topological Collet–Eckmann. It is well known that in this case there exists a unique Borel probability T-invariant measure μ absolutely continuous with respect to Lebesgue measure λ. In fact, μ is equivalent to λ and (therefore) has full topological support. It is ergodic, even K-mixing, has Rokhlin's natural extension metrically isomorphic to some two sided Bernoulli shift. The Radon–Nikodym derivative $\frac{d\mu}{d\lambda}$ is uniformly bounded above and separated from zero on the complement of every fixed neighborhood of $\overline{\mathrm{PC}(T)}$. We prove in this setting the following.

Theorem 1.15. *Let $T : I \to I$ be a topologically exact multimodal subexpanding map. Fix $\xi \in I \backslash \overline{\mathrm{PC}(T)}$. Assume that the parameter 1 is powering at ξ with respect to the conformal* GDS $\mathcal{S}_T$ *defined in Section 22. Then the following limit exists, is finite, and positive:*

$$\lim_{r\to 0} \frac{1 - \mathrm{HD}(K_\xi(r))}{\mu(B(\xi, r))}.$$

Theorem 1.16. *If $T : I \to I$ is a topologically exact multimodal subexpanding map, then for Lebesgue–a.e. point $\xi \in I \setminus \overline{\mathrm{PC}(T)}$ the following limit exists, is finite and positive:*

$$\lim_{r\to 0} \frac{1 - \mathrm{HD}(K_\xi(r))}{\mu(B(\xi, r))}.$$

We now turn to complex one-dimensional maps. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of the Riemann sphere with degree $\deg(f) \geq 2$. The sets $\mathrm{Crit}(f)$ and $\mathrm{PC}(f)$ have the same meaning as for the multimodal maps of the interval I. Let $\psi : \widehat{\mathbb{C}} \to \mathbb{R}$ be a Hölder continuous function. Following [18] we say that $\psi : \widehat{\mathbb{C}} \to \mathbb{R}$ has a pressure gap if

$$n\mathrm{P}(f, \psi) - \sup\left(\psi_n\right) > 0 \tag{1.11}$$

for some integer $n \geq 1$. It was proved in [18] that there exists a unique equilibrium state μ_ψ for such ψ. Some more ergodic properties of μ_ψ were established there, and a fairly extensive account of them was provided in [48]. For example, if $\psi = 0$ then $\mathrm{P}(f, 0) = \log\deg(f) > 0$ is the topological entropy of f and the condition automatically holds. More generally, it always holds whenever

$$\sup(\psi) - \inf(\psi) < \mathrm{h_{top}}(f)\,(= \log\deg(f)).$$

We would like to also add that (1.11) always holds (with all $n \geq 0$ sufficiently large) if the function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ restricted to its Julia set is expanding (also frequently referred to as hyperbolic). This is the best understood and the easiest to deal with class of rational functions. The rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is said to be expanding if the restriction $f|_{J(f)} : J(f) \to J(f)$ satisfies

$$\inf\{|f'(z)| : z \in J(f)\} > 1 \tag{1.12}$$

or, equivalently,

$$|f'(z)| > 1 \tag{1.13}$$

for all $z \in J(f)$. Another, topological, characterization of the expanding property is the following.

Fact 1.17. A rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is expanding if and only if

$$J(f) \cap \overline{\mathrm{PC}(f)} = \emptyset.$$

It is immediate from this characterization that all the polynomials $z \mapsto z^d$, $d \geq 2$, are expanding along with their small perturbations $z \mapsto z^d + \varepsilon$; in fact expanding rational functions are commonly believed to form the vast majority amongst all rational functions.

Being a tame rational function and Topological Collet–Eckmann both mean the same as in the setting of multimodal interval maps. Nowadays this property is somewhat more frequently used in its equivalent form of exponential shrinking (see (23.4)) (ESP), and we this follow tradition. All expanding functions are tame and (ESP). Finally, as in the context of interval maps, we have the following.

Theorem 1.18. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a tame rational function having the exponential shrinking property* (ESP)*. Let $\psi : \widehat{\mathbb{C}} \to \mathbb{R}$ be a Hölder continuous potential with pressure gap. Let $z \in J(f) \setminus \overline{\mathrm{PC}(f)}$ be recurrent. Assume that the equilibrium state μ_ψ is* (WBT) *at z. Then*

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\underline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} &= \lim_{\varepsilon\to 0}\frac{\overline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))}\\ &= \begin{cases}1 & \text{if } z \text{ is not a periodic point for } f,\\ 1-\exp\big(S_p\psi(z)-p\mathrm{P}(f,\psi)\big) & \text{if } z \text{ is a periodic point of } f.\end{cases}\end{aligned}$$

Corollary 1.19. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a tame rational function having the exponential shrinking property* (ESP) *whose Julia set $J(f)$ is geometrically irreducible. If $\psi : \widehat{\mathbb{C}} \to \mathbb{R}$ is a Hölder continuous potential with pressure gap, then*

$$\lim_{\varepsilon\to 0}\frac{\underline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} = \lim_{\varepsilon\to 0}\frac{\overline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} = 1$$

for μ_ψ–a.e. $z \in J(f)$.

As for the case of maps of an interval, in order to establish formula (1.4) in this context we need a stronger assumption on the rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Because the Julia set need not be equal to $\widehat{\mathbb{C}}$ (and usually it is not) the definition of subexpanding rational functions is somewhat more involved, see Definition 23.13. It is evident that each subexpanding map is tame and it is not hard to see that being subexpanding entails also being Topological Collet–Eckmann. All expanding functions are necessarily subexpanding.

Theorem 1.20. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a subexpanding rational function of degree $d \geq 2$. Fix $\xi \in J(f) \setminus \overline{\mathrm{PC}(f)}$. Assume that the measure μ_h is* (WBT) *at ξ and the parameter $h := \mathrm{HD}(J(f))$ is powering at ξ with respect to the conformal* GDS *$\mathcal{S}_f$ defined in Section* 23*. Then the following limit exists, is finite and positive:*

$$\lim_{r\to 0}\frac{\mathrm{HD}(J(f)) - \mathrm{HD}(K_\xi(r))}{\mu_h(B(\xi,r))}.$$

Theorem 1.21. *If $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a subexpanding rational function of degree $d\geq 2$ whose Julia set $J(f)$ is geometrically irreducible, then for μ_h–a.e. point $\xi\in J(f)\setminus\overline{\mathrm{PC}(f)}$ the following limit exists, is finite and positive:*

$$\lim_{r\to 0}\frac{\mathrm{HD}(J(f))-\mathrm{HD}(K_\xi(r))}{\mu_h(B(\xi,r))}.$$

Remark 1.22. We would like to note that if the rational function $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is expanding (or hyperbolic as such functions are frequently called), then it is subexpanding and each Hölder continuous potential has a pressure gap. In particular all four theorems above pertaining to rational functions hold for it.

In both theorems μ_h is a unique (ergodic) Borel probability f–invariant measure on $J(f)$ equivalent to m_h, a unique h-conformal measure m_h on $J(f)$ for f. Th was proved studied in [53], comp. also [52].

The last applications are in the realm of transcendental meromorphic functions. There is a large class of such systems, introduced in [33] and [34] for which it is possible to build (see these two papers) a fairly rich and complete account of thermodynamic formalism. Applying again our escape rates theorems for conformal graph directed Markov systems, one prove in this setting four main theorems which are analogous of those for the multimodal maps of an interval and rational functions of the Riemann sphere. These can be found with complete proofs in Section 24, the last section of our manuscript.

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Part 1. Singular Perturbations of Countable Alphabet Symbol Space Classical Perron–Frobenius Operators

2. The classical original Perron-Frobenius Operator $\mathcal{L}_\varphi$, Gibbs and Equilibrium States μ_φ, Thermodynamic Formalism; Preliminaries

In this section we present some notation and basic results on Thermodynamic Formalism as developed in [31], see also [30] and [7]. It will be the base for our subsequent work.

Let E be a countable, either finite or infinite, set, called in the sequel the alphabet. Let $A:E\times E\to\{0,1\}$ an arbitrary matrix. For every integer $n\geq 0$ let

$$E_A^n:=\{\omega\in E^n: A_{\omega_j\omega_{j+1}}=1\ \forall\, 0\leq j\leq n-1\},$$

denote the finite words of length n, let

$$E_A^\infty:=\{\omega\in E^{\mathbb{N}}: A_{\omega_j\omega_{j+1}}=1\ \forall\, j\geq 0\},$$

denote the space of one-sided infinite sequences, and let

$$E^* := \bigcup_{n=0}^{\infty} E^n, \quad \text{and} \quad E_A^* := \bigcup_{n=0}^{\infty} E_A^n.$$

be set of all finite strings of words, the former being without restrictions and the latter being called A-admissible.

We call elements of E_A^* and E_A^∞ A-admissible. The matrix A is called finitely primitive (or aperiodic) if there exist an integer $p \geq 0$ and a finite set $\Lambda \subseteq E^p$ such that for all $i, j \in E$ there exists $\omega \in \Lambda$ such that $i\omega j \in E_A^*$. Denote by $\sigma : E_A^\infty \to E_A^\infty$ the shift map, i. e. the map uniquely defined by the property that

$$\sigma(\omega)_n := \omega_{n+1}$$

for every $n \geq 0$. Fixing $\theta \in (0,1)$ endow E_A^∞ with the standard metric

$$d_\theta(\omega, \tau) := \theta^{|\omega \wedge \tau|},$$

where for every $\gamma \in E^* \cup E^{\mathbb{N}}$, $|\gamma|$ denotes the length of γ, i. e. the unique $n \in \mathbb{N} \cup \{\infty\}$ such that $\gamma \in E^n$. Given $0 \leq k \leq |\gamma|$, we set

$$\gamma|_k := \gamma_0\gamma_1\gamma_2 \dots \gamma_k.$$

We then also define

$$[\gamma] := \{\omega \in E_A^\infty : \omega|_n = \gamma\},$$

and call $[\gamma]$ the (initial) cylinder generated by γ.

Given an element $\xi \in E_A^*$ and $\xi \in \{0, 1, 2, \dots, +\infty\}$ we denote by $\xi^k \in E_A^*$ the concatenation of k copies of ξ; in particular $\xi^0 = \emptyset$, the empty word, $\xi^1 = \xi$, and $\xi^\infty = \xi\xi\xi \dots$ is the infinite concatenation of the word ξ. We frequently refer to ξ^k as the kth power of ξ.

Let $\varphi : E_A^\infty \to \mathbb{R}$ be a Hölder continuous function, called in the sequel potential. We assume that φ is summable, meaning that

$$\sum_{e \in E} \exp\left(\sup(\varphi|_{[e]}\right) < +\infty.$$

It is well known (see [31] or [29]) that the following limit

$$\mathrm{P}(\varphi) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in E_A^n} \exp\left(\sup(\varphi|_{[\omega]}\right)$$

exists. It is called the topological pressure of φ. It was proved in [29] (compare [31]) that there exists a unique shift-invariant Gibbs/equilibrium measure μ_φ for the potential φ. The Gibbs property means that

$$C_\varphi^{-1} \leq \frac{\mu_\varphi([\omega|_n])}{\exp\left(\varphi_n(\omega) - \mathrm{P}(\varphi)n\right)} \leq C_\varphi$$

with some constant $C_\varphi \geq 1$ for every $\omega \in E_A^\infty$ and every integer $n \geq 1$, where here and in the sequel

$$g_n(\omega) := \sum_{j=0}^{n-1} g \circ \sigma^j$$

for every function $g : E_A^\infty \to \mathbb{C}$. For the measure μ_φ being an equilibrium state for the potential φ means that

$$\mathrm{h}_{\mu_\varphi}(\sigma) + \int_{E_A^\infty} \varphi d\mu_\varphi = \mathrm{P}(\varphi).$$

It has been proved in [31] that

$$\mathrm{h}_\mu(\sigma) + \int_{E_A^\infty} \varphi d\mu < \mathrm{P}(\varphi)$$

for any other Borel probability σ-invariant measure μ such that $\int \varphi d\mu > -\infty$. For every bounded function $g : E_A^\infty \to \mathbb{R}$ define $\mathcal{L}_\varphi(g) : E_A^\infty \to \mathbb{R}$ as follows

$$\mathcal{L}_\varphi(g)(\omega) := \sum_{e \in E : A_{e\omega_0}=1} g(e\omega) \exp(\varphi(e\omega)).$$

Then $\mathcal{L}_\varphi(g)$ is bounded again, and we get by induction that

$$\mathcal{L}_\varphi^k(g)(\omega) := \sum_{\tau \in E_A^k : A_{\tau_{k-1}\omega_0}=1} g(\tau\omega) \exp(\varphi_k(\tau\omega)).$$

Let $C_b(A)$ be the Banach space of all complex-valued bounded continuous functions defined on E_A^∞ endowed with the supremum norm $||\cdot||_\infty$. Let $\mathrm{H}_\theta^b(A)$ be its vector subspace consisting of all Lipschitz continuous functions with respect to the metric d_θ. Equipped with the norm

$$\mathrm{H}_\theta(g) := ||g||_\infty + v_\theta(g), \tag{2.1}$$

where $v_\theta(g)$ is the least constant $C \geq 0$ such that

$$|g(\omega) - g(\tau)| \leq C d_\theta(\omega, \tau), \tag{2.2}$$

whenever $d_\theta(\omega, \tau) \leq \theta$ (i. e. $\omega_0 = \tau_0$), the vector space $\mathrm{H}_\theta^b(A)$ becomes a Banach space. It is easy to see that the operator $\mathcal{L}_\varphi$ preserves both Banach spaces $C_b(A)$ (as we have observed some half-page ago) and $\mathrm{H}_\theta^b(A)$ and also acts continuously on each of them. The adjective "original" indicates that we do not deal with its perturbations while "classical" refers to standard Banach spaces $C_b(A)$ and $\mathrm{H}_\theta^b(A)$. The following theorem, describing fully the spectral properties of $\mathcal{L}_\varphi$, has been proved in [31] and [29].

Theorem 2.1. *If $A : E \times E \to \{0, 1\}$ is finitely primitive and $\varphi \in \mathrm{H}_\theta^b(A)$, then*

(a) *The spectral radius of the operator $\mathcal{L}_\varphi$ considered as acting either on $C_b(A)$ or $\mathrm{H}_\theta^b(A)$ is in both cases equal to $e^{\mathrm{P}(\varphi)}$.*

(b) *In both cases of (a) the number $e^{\mathrm{P}(\varphi)}$ is a simple eigenvalue of $\mathcal{L}_\varphi$ and there exists corresponding to it an everywhere positive eigenfunction $\rho_\varphi \in \mathrm{H}_\theta^b(A)$ such that $\log \rho_\varphi$ is a bounded function.*

(c) *The reminder of the spectrum of the operator* $\mathcal{L}_\varphi : \mathrm{H}^b_\theta(A) \to \mathrm{H}^b_\theta(A)$ *is contained in a closed disk centered at* 0 *with radius strictly smaller than* $e^{\mathrm{P}(\varphi)}$. *In particular, the operator* $\mathcal{L}_\varphi : \mathrm{H}^b_\theta(A) \to \mathrm{H}^b_\theta(A)$ *is quasi-compact.*

(d) *There exists a unique Borel probability measure* m_φ *on* E_A^∞ *such that*

$$\mathcal{L}_\varphi^* m_\varphi = e^{\mathrm{P}(\varphi)} m_\varphi,$$

where $\mathcal{L}_\varphi^* : C_b^*(A) \to C_b^*(A)$, *is the operator dual to* $\mathcal{L}_\varphi$ *acting on the space of all bounded linear functionals from* $C_b(A)$ *to* $\mathbb{C}$.

(e) *If* $\rho_\varphi : E_A^\infty \to (0, \infty)$ *is normalized so that* $m_\varphi(\rho_\varphi) = 1$, *then* $\rho_\varphi m_\varphi = \mu_\varphi$, *where, we recall, the latter is the unique shift-invariant Gibbs/equilibrium measure for the potential* φ.

(e) *The Riesz projector* $Q_\varphi : \mathrm{H}^b_\theta(A) \to \mathrm{H}^b_\theta(A)$, *corresponding to the eigenvalue* $e^{\mathrm{P}(\varphi)}$, *is given by the formula*

$$Q_\varphi(g) = e^{\mathrm{P}(\varphi)} m_\varphi(g) \rho_\varphi.$$

If we multiply the operator $\mathcal{L}_\varphi : \mathrm{H}^b_\theta(A) \to \mathrm{H}^b_\theta(A)$ by $e^{-\mathrm{P}(\varphi)}$ and conjugate it via the linear homeomorphism

$$g \mapsto \rho_\varphi^{-1} g,$$

then the resulting operator $T : \mathrm{H}^b_\theta(A) \to \mathrm{H}^b_\theta(A)$ has the same properties, described above, as the operator $\mathcal{L}_\varphi$, with $e^{\mathrm{P}(\varphi)}$ replaced by 1, ρ_φ by the function $\mathbb{1}$ which is identically equal to 1, and m_φ replaced by μ_φ. Since in addition it is equal to $\mathcal{L}_{\tilde{\varphi}} : \mathrm{H}^b_\theta(A) \to \mathrm{H}^b_\theta(A)$ with

$$\tilde{\varphi} := \varphi - \mathrm{P}(\varphi) + \log \rho_\varphi,$$

we will frequently deal with the operator $\mathcal{L}_{\tilde{\varphi}}$ instead of $\mathcal{L}_\varphi$, exploiting its useful property

$$\mathcal{L}_{\tilde{\varphi}} \mathbb{1} = \mathbb{1}.$$

We will occasionally refer to $\mathcal{L}_{\tilde{\varphi}}$ as fully normalized. Sometimes, we will only need the semi-normalized operator $\mathcal{L}_\varphi$ given by the formula

$$\hat{\mathcal{L}}_\varphi := e^{-\mathrm{P}(\varphi)} \mathcal{L}_\varphi.$$

It essentially differs from only by having $e^{\mathrm{P}(\varphi)}$ replaced by 1. Now we bring up two standard well-known technical facts about the above concepts. These can be found for example in [31].

Lemma 2.2. *There exists a constant* $M_\varphi \in (0, +\infty)$ *such that*

$$|\varphi_k(\omega) - \varphi_k(\tau)| \le M_\varphi \theta^m$$

for all integers $k, m \ge 1$, *and all words* $\omega, \tau \in E_A^\infty$ *such that* $\omega|_{k+m} = \tau|_{k+m}$.

Lemma 2.3. *With the hypotheses of Lemma* 2.2 *and increasing the constant* M_φ *if necessary, we have that*

$$\left|1 - \exp\big(\varphi_k(\gamma\omega) - \varphi_k(\gamma\tau)\big)\right| \le M_\varphi |\varphi_k(\omega) - \varphi_k(\tau)|.$$

3. Non-standard original Perron-Frobenius Operator $\mathcal{L}_\varphi$; Definition and first technical Results

We keep the setting of the previous section. We still deal with the original operator $\mathcal{L}_\varphi$ but we let it act on a different non-standard Banach space $\mathcal{B}_\theta$ defined below. This space is more suitable for consideration of perturbations of $\mathcal{L}_\varphi$.

Given a function $g \in L^1(\mu_\varphi)$ and an integer $m \geq 0$, we define the function $\mathrm{osc}_m(g) : E_A^\infty \to [0, \infty)$ by the following formula:

$$\mathrm{osc}_m(g)(\omega) := \operatorname{ess\,sup}\{|g(\alpha) - g(\beta)| : \alpha, \beta \in [\omega|_m]\} \tag{3.1}$$

and

$$\mathrm{osc}_0(g) := \mathrm{esssup}(g) - \mathrm{essinf}(g).$$

We further define:

$$|g|_\theta := \sup_{m \geq 0}\{\theta^{-m}||\mathrm{osc}_m(g)||_1\}, \tag{3.2}$$

where $|\cdot|$ denotes the L^1-norm with respect to the measure μ_φ. Note the subtle difference between this definition and the analogous one, which motivated us, from [20]. Therein in the analogue of formula (3.2) the supremum is taken over integers $m \geq 1$ only. Including $m = 0$ causes some technical difficulties, particularly the (tedious) part of the proof of Lemma 3.2 for the integer $m = 0$. However, without the case $m = 0$ we would not be able to prove Lemma 3.1, in contrast to the finite alphabet case of [20], which is indispensable for our entire approach. The, previously announced, non-standard (it even depends on the dynamics – via μ_φ) Banach space is defined as follows:

$$\mathcal{B}_\theta := \{g \in L^1(\mu_\varphi) : |g|_\theta < +\infty\}$$

and we denote

$$||g||_\theta := ||g||_1 + |g|_\theta. \tag{3.3}$$

Of course $\mathcal{B}_\theta$ is a vector space and the function

$$\mathcal{B}_\theta \ni g \mapsto ||g||_\theta \tag{3.4}$$

is a norm on $\mathcal{B}_\theta$. This is the non-standard Banach space we will be working with throughout the whole manuscript. We shall prove the following.

Lemma 3.1. *If $g \in \mathcal{B}_\theta$, then g is essentially bounded and*

$$||g||_\infty \leq ||g||_\theta.$$

Proof. For all $\omega \in E_A^\infty$, we have

$$\begin{aligned} |g(\omega)| &\leq \left| \int_{E_A^\infty} g\, d\mu_\varphi + \mathrm{osc}_0(g)(\omega) \right| = \left| \int_{E_A^\infty} g\, d\mu_\varphi + \int_{E_A^\infty} \mathrm{osc}_0(g)\, d\mu_\varphi \right| \\ &\leq \int_{E_A^\infty} |g|\, d\mu_\varphi + ||\mathrm{osc}_0(g)||_1 \\ &\leq ||g||_\theta. \end{aligned}$$

The proof is complete. □

From now on, unless otherwise stated, we assume that the potential $\varphi : E_A^\infty \to \mathbb{R}$ is normalized (by adding a constant and a coboundary) so that

$$\mathcal{L}_\varphi 1\!\!1 = 1\!\!1.$$

For ease of notation we also abbreviate $\mathcal{L}_\varphi$ to $\mathcal{L}$. We shall prove the following.

Lemma 3.2. *There exists a constant $C > 0$ for every integer $k \geq 0$ and every $g \in \mathcal{B}_\theta$, we have*

$$|\mathcal{L}^k g|_\theta \leq C(\theta^k |g|_\theta + ||g||_1).$$

Proof. For every $e \in E$ let

$$E_A^k(e) := \{\gamma \in E_A^k : A_{\gamma_k e} = 1\}.$$

Fix first an integer $m \geq 1$ and then $\omega, \tau \in E_A^\infty$ such that $\omega|_m = \tau|_m$. Using Lemmas 2.2 and 2.3, we then get

$$\begin{aligned}
|\mathcal{L}^k g(\omega) - \mathcal{L}^k g(\tau)| &\leq \sum_{\gamma \in E_A^k(\omega_1)} e^{\varphi_k(\gamma\omega)} |g(\gamma\omega) - g(\gamma\tau)| + \sum_{\gamma \in E_A^k(\omega_1)} |g(\gamma\tau)| \left| e^{\varphi_k(\gamma\omega)} - e^{\varphi_k(\gamma\tau)} \right| \\
&\leq \sum_{\gamma \in E_A^k(\omega_1)} \mathrm{osc}_{k+m}(g)(\gamma\omega) e^{\varphi_k(\gamma\omega)} + \\
&\qquad + \sum_{\gamma \in E_A^k(\omega_1)} |g(\gamma\tau)| e^{\varphi_k(\gamma\tau)} \left| 1 - \exp\big(\varphi_k(\gamma\omega) - \varphi_k(\gamma\tau)\big) \right| \\
&\leq \sum_{\gamma \in E_A^k(\omega_1)} \mathrm{osc}_{k+m}(g)(\gamma\omega) e^{\varphi_k(\gamma\omega)} + \\
&\qquad + \sum_{\gamma \in E_A^k(\omega_1)} |g(\gamma\tau)| e^{\varphi_k(\gamma\tau)} M_\varphi |\varphi_k(\gamma\omega) - \varphi_k(\gamma\tau)| \\
&\leq \mathcal{L}^k(\mathrm{osc}_{k+m}(g))(\omega) + M_\varphi^2 \theta^m \sum_{\gamma \in E_A^k(\omega_1)} \big(|g(\gamma\omega)| + \mathrm{osc}_{k+m}(g)(\gamma\omega) \big) e^{\varphi_k(\gamma\omega)} \\
&\leq \mathcal{L}^k(\mathrm{osc}_{k+m}(g))(\omega) + M_\varphi^2 \theta^m \mathcal{L}^k(|g|)(\omega) + M_\varphi^2 \theta^m \mathcal{L}^k(\mathrm{osc}_{k+m}(g))(\omega) \\
&\leq (1 + M_\varphi^2) \mathcal{L}^k(\mathrm{osc}_{k+m}(g))(\omega) + M_\varphi^2 \theta^m \mathcal{L}^k(|g|)(\omega)
\end{aligned}$$

Hence,

$$\mathrm{osc}_m(\mathcal{L}^k g)(\omega) \leq (1 + M_\varphi^2) \mathcal{L}^k(\mathrm{osc}_{k+m}(g))(\omega) + M_\varphi^2 \theta^m \mathcal{L}^k(|g|)(\omega)$$

Integrating against the measure μ_φ, this yields

$$
\begin{aligned}
\theta^{-m}||\mathrm{osc}_m(\mathcal{L}^k g)||_1 &\le (1+M_\varphi^2)\theta^{-m}\int_{E_A^\infty}\mathcal{L}^k(\mathrm{osc}_{k+m}(g))\,d\mu_\varphi + M_\varphi^2\int_{E_A^\infty}\mathcal{L}^k(|g|)\,d\mu_\varphi \\
&= (1+M_\varphi^2)\theta^{-m}\int_{E_A^\infty}\mathrm{osc}_{k+m}(g)\,d\mu_\varphi + M_\varphi^2\int_{E_A^\infty}|g|\,d\mu_\varphi. \qquad (3.5)\\
&\le (1+M_\varphi^2)\theta^k|g|_\theta + M_\varphi^2||g||_1 \\
&\le (1+M_\varphi^2)(\theta^k|g|_\theta + ||g||_1).
\end{aligned}
$$

Some separate considerations are needed if $m=0$. However, we note that it would require no special treatment in the case of a full shift, i. e. when the incidence matrix A consists of 1s only. Let $p\ge 1$ be the value in the definition of finite primitivity of the matrix A. Replacing p by a sufficiently large integral multiple, we will have that the set

$$E_A^p(a,b) := \{\alpha\in E_A^p : a\alpha b\in E_A^*\}$$

consisting of words of length p prefixed by a and suffixed by b is non-empty for all $a,b\in E$ and it is countable infinite if the alphabet E is infinite. For every function $h: E_A^\infty\to\mathbb{R}$ and every finite word $\gamma\in E_A^*$ with associated cylinder $[\gamma]$ consisting of all infinite sequences beginning with γ let $\hat{h}(\gamma)\in\mathbb{R}$ be a number with the following two properties:

(a) $\hat{h}(\gamma)\in\overline{h([\gamma])}$ and
(b) $|\hat{h}(\gamma)| = \inf\{|h(\rho)| : \rho\in[\gamma]\}$.

Let us introduce the following two functions:

$$\Delta_1\mathcal{L}^{k+p}(g)(\rho) := \sum_{|\gamma|=k}\sum_{\alpha\in E_A^p(\gamma_k,\rho_1)}\left(g(\gamma\alpha\rho)e^{\varphi_k(\gamma\alpha\rho)}e^{\varphi_p(\alpha\rho)} - \hat{g}(\gamma)e^{\hat{\varphi}_k(\gamma)}e^{\varphi_p(\alpha\rho)}\right)$$

and

$$\Delta_2\mathcal{L}^{k+p}(g)(\omega,\tau) := \sum_{|\gamma|=k}\hat{g}(\gamma)e^{\hat{\varphi}_k(\gamma)}\left(\sum_{\alpha\in E_A^p(\gamma_k,\omega_1)}e^{\varphi_p(\alpha\omega)} - \sum_{\beta\in E_A^p(\gamma_k,\tau_1)}e^{\varphi_p(\beta\tau)}\right).$$

We then have

$$\mathcal{L}^{k+p}(g)(\omega)-\mathcal{L}^{k+p}(g)(\tau) = \Delta_1\mathcal{L}^{k+p}(g)(\omega)+\Delta_2\mathcal{L}^{k+p}(g)(\omega,\tau)-\Delta_1\mathcal{L}^{k+p}(g)(\tau). \qquad (3.6)$$

We will estimate the absolute value of each of these three summands in terms of ω only (i. e. independently of τ) and then we will integrate against the measure μ_φ. First:

$$
\begin{aligned}
|\Delta_1\mathcal{L}^{k+p}(g)(\rho)| &\le \sum_{|\gamma|=k}\sum_{\alpha\in E_A^p(\gamma_k,\rho_1)} |g(\gamma\alpha\rho)e^{\varphi_k(\gamma\alpha\rho)}-\hat g(\gamma)e^{\hat\varphi_k(\gamma)}|e^{\varphi_p(\alpha\rho)} \\
&\le \sum_{|\gamma|=k}\sum_{\alpha\in E_A^p(\gamma_k,\rho_1)} \big(|g(\gamma\alpha\rho)-\hat g(\gamma)|e^{\varphi_{k+p}(\gamma\alpha\rho)}+|e^{\varphi_k(\gamma\alpha\rho)}-e^{\hat\varphi_k(\gamma)}|\cdot|\hat g(\gamma)|e^{\varphi_p(\alpha\rho)}\big) \\
&\le \sum_{|\gamma|=k}\sum_{\alpha\in E_A^p(\gamma_k,\rho_1)} \big(\mathrm{osc}_k\big(g|_{[\gamma]}\big)e^{\varphi_{k+p}(\gamma\alpha\rho)}+M_\varphi e^{\varphi_k(\gamma\alpha\rho)}e^{\varphi_p(\alpha\rho)}|\hat g(\gamma)|\big) \\
&\le \sum_{|\gamma|=k}\sum_{\alpha\in E_A^p(\gamma_k,\rho_1)} \mathrm{osc}_k\big(g|_{[\gamma]}\big)e^{\varphi_{k+p}(\gamma\alpha\rho)}+M_\varphi\sum_{|\gamma|=k}\sum_{\alpha\in E_A^p(\gamma_k,\rho_1)} |g(\gamma\alpha\rho)|e^{\varphi_{k+p}(\gamma\alpha\rho)} \\
&= \mathcal{L}^{k+p}(\mathrm{osc}_k(g))(\rho)+M_\varphi\mathcal{L}^{k+p}(|g|)(\rho),
\end{aligned}
\tag{3.7}
$$

with some appropriately large constant M_φ. Plugging into the above inequality, $\rho=\omega$, this gives

$$
|\Delta_1\mathcal{L}^{k+p}(g)(\omega)| \le \mathcal{L}^{k+p}(\mathrm{osc}_k(g))(\omega)+M_\varphi\mathcal{L}^{k+p}(|g|)(\omega). \tag{3.8}
$$

Now notice that because of our choice of $p\ge 1$ there exists a number $Q\ge 1$ and for every $e\in E$ there exists an at most Q-to-1 function $f_e: E_A^p(e,\tau_1)\to E_A^p(e,\omega_1)$ (can be chosen to be a bijection if the alphabet E is infinite). So, plugging in turn $\rho=\tau$ to (3.7), we get

$$
\begin{aligned}
&|\Delta_1\mathcal{L}^{k+p}(g)(\tau)| \le \\
&\le \sum_{|\gamma|=k}\sum_{\beta\in E_A^p(\gamma_k,\tau_1)} \mathrm{osc}_k\big(g|_{[\gamma]}\big)e^{\varphi_{k+p}(\gamma\beta\tau)}+M_\varphi\sum_{|\gamma|=k}\sum_{\beta\in E_A^p(\gamma_k,\tau_1)} |g(\gamma\beta\tau)|e^{\varphi_{k+p}(\gamma\beta\tau)} \\
&\le M_\varphi\sum_{|\gamma|=k}\sum_{\beta\in E_A^p(\gamma_k,\tau_1)} \big(\mathrm{osc}_k(g)(\gamma f_e(\beta)\omega)e^{\varphi_{k+p}(\gamma f_e(\beta)\omega)}+M_\varphi|\hat g(\gamma)|e^{\varphi_{k+p}(\gamma f_e(\beta)\omega)}\big) \\
&\le M_\varphi\sum_{|\gamma|=k}\sum_{\beta\in E_A^p(\gamma_k,\tau_1)} \mathrm{osc}_k(g)(\gamma f_e(\beta)\omega)e^{\varphi_{k+p}(\gamma f_e(\beta)\omega)}+M_\varphi\sum_{|\gamma|=k}\sum_{\beta\in E_A^p(\gamma_k,\tau_1)} |g(\gamma f_e(\beta)\omega)|e^{\varphi_{k+p}(\gamma f_e(\beta)\omega)} \\
&\le QM_\varphi\left(\sum_{|\gamma|=k}\sum_{\alpha\in E_A^p(\gamma_k,\omega_1)} \mathrm{osc}_k(g)(\gamma\alpha\omega)e^{\varphi_{k+p}(\gamma\alpha\omega)}+M_\varphi\sum_{|\gamma|=k}\sum_{\alpha\in E_A^p(\gamma_k,\omega_1)} |g(\gamma\alpha\omega)|e^{\varphi_{k+p}(\gamma\alpha\omega)}\right) \\
&= QM_\varphi\big(\mathcal{L}^{k+p}(\mathrm{osc}_k(g))(\omega)+M_\varphi\mathcal{L}^{k+p}(|g|)(\omega)\big)
\end{aligned}
\tag{3.9}
$$

with some appropriate constant $Q > 0$. Turning to $\Delta_2\mathcal{L}^{k+p}(g)$, we get

$$
\begin{aligned}
|\Delta_2\mathcal{L}^{k+p}(g)(\omega,\tau)| &\le \sum_{|\gamma|=k} |\hat{g}(\gamma)| e^{\hat{\varphi}_k(\gamma)} \left(\sum_{\alpha\in E_A^p(\gamma_k,\omega_1)} e^{\varphi_p(\alpha\omega)} + \sum_{\beta\in E_A^p(\gamma_k,\tau_1)} e^{\varphi_p(\beta\tau)} \right) \\
&\le \sum_{|\gamma|=k} |\hat{g}(\gamma)| e^{\hat{\varphi}_k(\gamma)} \big(\mathcal{L}^p 1\!\!1(\omega) + \mathcal{L}^p 1\!\!1(\tau)\big) \\
&= 2\sum_{|\gamma|=k} |\hat{g}(\gamma)| e^{\hat{\varphi}_k(\gamma)} \\
&\le 2M_\varphi \sum_{|\gamma|=k} |g(\gamma\alpha(\gamma_k,\omega_1)\omega)| e^{\varphi_{k+p}(\gamma\alpha(\gamma_k,\omega_1)\omega)} e^{-\varphi_p(\alpha(\gamma_k,\omega_1)\omega)} \\
&\le 2M_\varphi e^{-C_p} \sum_{|\gamma|=k} |g(\gamma\alpha(\gamma_k,\omega_1)\omega)| e^{\varphi_{k+p}(\gamma\alpha(\gamma_k,\omega_1)\omega)} \\
&\le 2M_\varphi e^{-C_p} \mathcal{L}^{k+p}(|g|)(\omega),
\end{aligned} \tag{3.10}
$$

where $\alpha(\gamma_k,\omega_1)$ is one, arbitrarily chosen, element from Λ, a finite set witnessing finite primitivity of A, such that $\gamma\alpha(\gamma_k,\omega_1) \in E_A^*$, and $C_p := \min\{\inf\{\varphi_p|_{[\alpha]} : \alpha \in \Lambda\} > 0$. Inserting now (3.10), (3.9), and (3.8) to (3.6), we get for all $\omega,\tau \in E_A^\infty$ that

$$\left|\mathcal{L}^{k+p}(g)(\omega) - \mathcal{L}^{k+p}(g)(\tau)\right| \le C(\mathcal{L}^{k+p}(\mathrm{osc}_k(g))(\omega) + \mathcal{L}^{k+p}(|g|)(\omega))$$

with some universal constant $C > 0$. Integrating against the measure μ_φ, this gives

$$
\begin{aligned}
\theta^{-0}||\mathrm{osc}_0(\mathcal{L}^{k+p}(g))||_1 &\le C\left(\int_{E_A^\infty} \mathcal{L}^{k+p}(\mathrm{osc}_k(g))\, d\mu_\varphi + \int_{E_A^\infty} \mathcal{L}^{k+p}(|g|)\, d\mu_\varphi \right) \\
&= C\left(\int_{E_A^\infty} \mathrm{osc}_k(g)\, d\mu_\varphi + \int_{E_A^\infty} |g|\, d\mu_\varphi \right) \\
&\le C(\theta^k|g|_\theta + ||g||_1) \\
&\le C\theta^{-p}(\theta^{k+p}|g|_\theta + ||g||_1).
\end{aligned} \tag{3.11}
$$

Along with (3.5) this gives that

$$|\mathcal{L}^k g|_\theta \le C(\theta^k|g|_\theta + ||g||_1) \tag{3.12}$$

for all $k \ge p$ with some suitable constant $C > 0$. Also, for every $0 \le k \le p$ we have

$$
\begin{aligned}
|\mathcal{L}^k g|_\theta \le ||\mathcal{L}^k g||_\theta &\le \max\{||\mathcal{L}||_\theta^j : 0 \le j \le p\}||g||_\theta \\
&\le \theta^{-p}\max\{||\mathcal{L}||_\theta^j : 0 \le j \le p\}||g||_\theta(\theta^k||g||_\theta) \\
&\le \theta^{-p}\max\{||\mathcal{L}||_\theta^j : 0 \le j \le p\}||g||_\theta(\theta^k|g|_\theta + ||g||_1).
\end{aligned}
$$

Along with (3.12) this finishes the proof. □

In conjunction with Theorem 2.1 this lemma gives the following.

Proposition 3.3. *The following hold:*

(1)
$$\mathrm{H}_\theta^b(A) \subseteq \mathcal{B}_\theta,$$
(2)
$$\mathcal{L}(\mathcal{B}_\theta) \subseteq \mathcal{B}_\theta,$$

In addition,

(3) *The operator $\mathcal{L} : \mathcal{B}_\theta \to \mathcal{B}_\theta$ is bounded, in fact all its iterates are uniformly bounded,*

(4) *The function $1\!\!1$ belongs $\mathcal{B}_\theta$ and is an eigenfunction of the operator $\mathcal{L} : \mathcal{B}_\theta \to \mathcal{B}_\theta$ with the eigenfunction equal to 1. Prior to any normalization of the operator $\mathcal{L}_\varphi$ the corresponding statement would read:*
The function ρ_φ belongs $\mathcal{B}_\theta$ and is an eigenfunction of the operator $\mathcal{L}_\varphi : \mathcal{B}_\theta \to \mathcal{B}_\theta$ with the eigenfunction equal to $e^{\mathrm{P}(\varphi)}$.

(5) *The measure μ_φ can be viewed as an element of the dual Banach space $\mathcal{B}_\theta^*$ and it is an eigenmeasure of the dual operator $\mathcal{L}^* : \mathcal{B}_\theta^* \to \mathcal{B}_\theta^*$ with the eigenfunction equal to 1. Prior to any normalization of the operator $\mathcal{L}_\varphi$ the corresponding statement would read:*
The measure m_φ can be viewed as an element of the dual Banach space $\mathcal{B}_\theta^*$ and it is an eigenmeasure of the dual operator $\mathcal{L}_\varphi^* : \mathcal{B}_\theta^* \to \mathcal{B}_\theta^*$ with the eigenfunction equal to $e^{\mathrm{P}(\varphi)}$.

(6) *The operator $Q_\varphi : \mathrm{H}_\theta^b(A) \to \mathrm{H}_\theta^b(A)$ extends to the Banach space $\mathcal{B}_\theta$ by the same formula (e) of Theorem 2.1 ($Q_\varphi(g)\mu_\varphi(g)1\!\!1$ after normalizations) and the linear operator $Q_\varphi : \mathcal{B}_\theta \to \mathcal{B}_\theta$ is bounded.*

4. Singular Perturbations, generated by open holes U_n, of (original) Perron–Frobenius Operators $\mathcal{L}_\varphi$ I: Fundamental Inequalities

This is the first section in which we deal with singular perturbations of the operator $\mathcal{L}_\varphi$. We work in the quite general setting described below. We keep the same non-standard Banach space $\mathcal{B}_\theta$ but, motivated by [20], we introduce an even more exotic norm $||\cdot||_*$, which depends even more on dynamics than $||\cdot||_\theta$.

The ultimate goal of this and the next section is Proposition 5.2. We prove it by applying Theorem 1 of [25] and its consequences derived therein. For the convenience of the reader and convenience of our referring to, we bring up the setting of [25] in Appendix at the end of our manuscript. We formulate there Theorem 1 of [25] and all its consequences we need.

Passing to details, in this section we assume that $(U_n)_{n=0}^\infty$, a nested sequence of open subsets of E_A^∞ is given, with the following properties:

(U0) $U_0 = E_A^\infty$,

(U1) For every $n \geq 0$ the open set U_n is a (disjoint) union of cylinders all of which are of length n,

(U2) There exists $\rho \in (0,1)$ such that such that

$$\mu_\varphi(U_n) \leq \rho^n$$

for all $n \geq 0$.

Let $|\cdot|_*$, $||\cdot||_* : \mathcal{B}_\theta \to [0,+\infty]$ be the functions defined by respective formulas

$$|g|_* := \sup_{j\geq 0}\sup_{m\geq 0}\left\{\theta^{-m}\int_{\sigma^{-j}(U_m)} |g|\, d\mu_\varphi\right\}$$

and

$$||g||_* := ||g||_1 + |g|_*.$$

Without loss of generality assume from now on that $\theta \in (\rho, 1)$. We shall prove the following.

Lemma 4.1. *For all $g \in \mathcal{B}_\theta$, we have that*

$$||g||_* \leq 2||g||_\infty \leq 2||g||_\theta.$$

Proof. By virtue of (U2), we get

$$|g|_* \leq \sup_{m\geq 0}\left\{\theta^{-m}\mu_\varphi(U_m)||g||_\infty\right\} \leq \sup_{m\geq 0}\left\{\theta^{-m}\rho^m||g||_\infty\right\} = \sup_{m\geq 0}\{(\rho/\theta)^m||g||_\infty\} = ||g||_\infty.$$

Hence,

$$||g||_* = ||g||_1 + |g|_* \leq ||g||_\infty + ||g||_\infty = 2||g||_\infty.$$

Combining this with Lemma 3.1 completes the proof. □

In particular, this lemma assures us that $|\cdot|_*$ and $||\cdot||_*$, respectively, are a semi-norm and a norm on $\mathcal{B}_\theta$. It is straightforward to check that $\mathcal{B}_\theta$ endowed with the norm $||\cdot||_*$ becomes a Banach space. For all integers $k \geq 1$ and $n \geq 0$ let

$$\mathbb{1}_n^k := \prod_{j=0}^{k-1} \mathbb{1}_{\sigma^{-j}(U_n^c)} = \prod_{j=0}^{k-1} \mathbb{1}_{U_n^c}\circ\sigma^j. \tag{4.1}$$

We also abbreviate

$$\mathbb{1}_n := \mathbb{1}_n^1$$

and set

$$\mathbb{1}_n^c := \mathbb{1}_{U_n} = \mathbb{1} - \mathbb{1}_n.$$

Let $\mathcal{L}_n : \mathcal{B}_\theta \to \mathcal{B}_\theta$ be defined by the formula

$$\mathcal{L}_n(g) := \mathcal{L}(\mathbb{1}_n^1 g).$$

These, for $n \geq 0$, are our perturbations of the operator $\mathcal{L}$. The difference $\mathcal{L} - \mathcal{L}_n$ in the supremum, or even $||\cdot||_\theta$, norm can be quite large even for arbitrarily large n, however, as Lemma 5.1 shows, the incorporation of the $||\cdot||_*$ norm makes this difference kind of small. The main result of this section is Proposition 5.2, complemented by Proposition 5.3, which describes in detail how well the spectral properties of the operator $\mathcal{L}$ are preserved under perturbations $\mathcal{L}_n$. Note that for every $k \ge 1$, we then have

$$\mathcal{L}_n^k(g) := \mathcal{L}^k(\mathbb{1}_n^k g).$$

The results we now obtain, leading ultimately to Proposition 5.2 and Proposition 5.3, stem from Lemma 3.9 and Lemma 3.10 in [20]. We develop these and extend them to the case of infinite alphabets. Since the sets U_n may, and in applications, will, consist of infinitely many cylinders (of the same length), we are cannot take advantage of good mixing properties of the symbol dynamical system $(\sigma : E_A^\infty \to E_A^\infty, \mu_\varphi)$. We use instead the Hölder inequality, which also, as a by-product, simplifies some of the reasonings of [20]. In what follows, the last fragment, directly preceding Proposition 5.2, and leading to verifying the requirements (24.3), (24.4), (24.5) from Appendix, corresponding to Remark 3 in [25], is particularly delicate and entirely different from the one of [20].

Lemma 4.2. *For all integers $k \ge 1$ and $n \ge 0$, we have*

$$||\mathcal{L}_n^k||_* \le 1.$$

Proof. Let $g \in L^1(\mu_\varphi)$. Then,

$$||\mathcal{L}_n^k(g)||_1 = \int |\mathcal{L}^k(\mathbb{1}_n^k g))|\, d\mu_\varphi \le \int \mathcal{L}^k(|\mathbb{1}_n^k g|)\, d\mu_\varphi = \int |\mathbb{1}_n^k g|\, d\mu_\varphi \le ||g||_1. \tag{4.2}$$

Also, for all integers $j, m \ge 0$, we have

$$\begin{aligned} \theta^{-m} \int_{\sigma^{-j}(U_m)} |\mathcal{L}^k(\mathbb{1}_n^k g))|\, d\mu_\varphi &\le \theta^{-m} \int_{\sigma^{-j}(U_m)} \mathcal{L}^k(|\mathbb{1}_n^k g|)\, d\mu_\varphi = \theta^{-m} \int_{\sigma^{-(j+1)}(U_m)} |\mathbb{1}_n^k g|\, d\mu_\varphi \\ &\le \theta^{-m} \int_{\sigma^{-(j+1)}(U_m)} |g|\, d\mu_\varphi \\ &\le |g|_*. \end{aligned}$$

Taking the supremum over j and m yields

$$|\mathcal{L}_n^k(g)|_* \le |g|_*.$$

Combining this and (4.2) completes the proof. ☐

Lemma 4.3. *For all integers $j, n \ge 0$ and for $g \in \mathcal{B}_\theta$, we have that*

$$|g\mathbb{1}_{\sigma^{-j}(U_n^c)}|_\theta \le |g|_\theta + \theta^{-j}||g||_*$$

Proof. Fix an integer $m \ge 1$. We consider two cases. Namely: $j + n \le m$ and $m < j + n$. Suppose first that $j + n \le m$. Then, $\text{osc}_m\big(g\mathbb{1}_{\sigma^{-j}(U_n^c)}\big)(\omega) \le \text{osc}_m(g)(\omega)$ for all $\omega \in E_A^\infty$. Thus

$$\theta^{-m} \int \text{osc}_m\big(g\mathbb{1}_{\sigma^{-j}(U_n^c)}\big)\, d\mu_\varphi \le \theta^{-m} \int \text{osc}_m(g)\, d\mu_\varphi \le |g|_\theta. \tag{4.3}$$

On the other hand, if $m < j + n$, then it is easy to see that if $[\omega|_m] \subseteq \sigma^{-j}(U_n^c)$, then

$$\text{osc}_m\big(g\mathbb{1}_{\sigma^{-j}(U_n^c)}\big)(\omega) = \text{osc}_m(g)(\omega). \tag{4.4}$$

On the other hand, if $[\omega|_m] \cap \sigma^{-j}(U_n) \neq \emptyset$, then

$$\mathrm{osc}_m\big(g\mathbb{1}_{\sigma^{-j}(U_n^c)}\big)(\omega) = \max\{\mathrm{osc}_m(g)(\omega), ||g\mathbb{1}_{[\omega|m]}||_\infty\}.$$

In this latter case

$$\mathrm{osc}_m\big(g\mathbb{1}_{\sigma^{-j}(U_n^c)}\big) \leq \max\{\mathrm{osc}_m(g)(\omega), ||g\mathbb{1}_{[\omega|m]}||_\infty\} \leq \mathrm{osc}_m(g)(\omega) + \frac{1}{\mu_\varphi\big([\omega|_m]\big)}\int_{[\omega|_m]} |g|\, d\mu_\varphi.$$

Together with (4.4) this implies that

$$\theta^{-m}\int \mathrm{osc}_m\big(g\mathbb{1}_{\sigma^{-j}(U_n^c)}\big)\, d\mu_\varphi \leq |g|_\theta + \theta^{-m}\int_{\{\omega\in E_A^\infty:[\omega|_m]\cap\sigma^{-j}(U_n)\neq\emptyset\}} |g|\, d\mu_\varphi. \tag{4.5}$$

We now consider two further sub-cases. If $m \leq j$, then we see that

$$\theta^{-m}\int_{\{\omega\in E_A^\infty:[\omega|_m]\cap\sigma^{-j}(U_n)\neq\emptyset\}} |g|\, d\mu_\varphi \leq \theta^{-j}\int_{\{\omega\in E_A^\infty:[\omega|_m]\cap\sigma^{-j}(U_n)\neq\emptyset\}} |g|\, d\mu_\varphi \leq \theta^{-j}||g||_1. \tag{4.6}$$

If $j < m < j+n$, the descending property of the sequence $\big(U_n\big)_{n=0}^\infty$ yields

$$\{\omega \in E_A^\infty : [\omega|_m] \cap \sigma^{-j}(U_n) \neq \emptyset\} \subseteq \sigma^{-j}(U_{m-j}).$$

In this case

$$\theta^{-m}\int_{\{\omega\in E_A^\infty:[\omega|_m]\cap\sigma^{-j}(U_n)\neq\emptyset\}} |g|\, d\mu_\varphi \leq \theta^{-j}\theta^{-(m-j)}\int_{\sigma^{-j}(U_{m-j})} |g|\, d\mu_\varphi \leq \theta^{-j}|g|_*. \tag{4.7}$$

Combining (4.3), (4.5), and (4.7) yields the desired inequality, and completes the proof. □

As a fairly straightforward inductive argument using Lemma 4.3, we shall prove the following.

Lemma 4.4. *For all integers $k \geq 1$ and $n \geq 0$, and all functions $g \in \mathcal{B}_\theta$, we have that*

$$|\mathbb{1}_n^k g|_\theta \leq |g|_\theta + \theta(1-\theta)^{-1}\theta^{-k}||g||_*. \tag{4.8}$$

Proof. Keeping $n \geq 0$ fixed, we will proceed by induction with respect to the integer $k \geq 1$. The case of $k = 1$ follows directly from Lemma 4.3. Assuming for the inductive step that (4.8) for some integer $k \geq 1$ and applying again Lemma 4.3, we get

$$\begin{aligned}
|\mathbb{1}_n^{k+1} g|_\theta &= \big|\mathbb{1}_{\sigma^{-k}(U_n^c)}(\mathbb{1}_n^k g)\big|_\theta \leq |\mathbb{1}_n^k g|_\theta + \theta^{-k}||\mathbb{1}_n^k g||_* \\
&\leq |\mathbb{1}_n^k g|_\theta + \theta^{-k}||g||_* \\
&\leq |g|_\theta + \theta(1-\theta)^{-1}\theta^{-k}||g||_* + \theta^{-k}||g||_* \\
&= |g|_\theta + \theta(1-\theta)^{-1}\theta^{-(k+1)}||g||_*.
\end{aligned}$$

The proof is complete. □

As a fairly immediate consequence of Lemma 4.4 and Lemma 3.2, we get the following.

Corollary 4.5. *There exists a constant $c > 0$ such that*

$$||\mathcal{L}_n^k g||_\theta \leq c\big(\theta^k||g||_\theta + ||g||_*\big) \tag{4.9}$$

for all $g \in \mathcal{B}_\theta$ and all integers $k, n \geq 0$.

Proof. Substituting $\mathbb{1}_n^k g$ for g into the statement of Lemma 3.2 and then applying Lemma 4.3, we get

$$\begin{aligned}|\mathcal{L}_n^k g|_\theta = |\mathcal{L}^k(\mathbb{1}_n^k g)|_\theta &\le C\big(\theta^k|\mathbb{1}_n^k g|_\theta + ||g||_1\big)\\ &\le C\big(\theta^k(|g|_\theta + \theta(1-\theta)^{-1}\theta^{-k}||g||_*) + ||g||_1\big)\\ &\le C\big(\theta^k(|g|_\theta + \theta(1-\theta)^{-1}||g||_* + ||g||_1\big).\end{aligned}$$

Hence,

$$\begin{aligned}||\mathcal{L}_n^k g||_\theta = |\mathcal{L}_n^k g|_\theta + ||\mathcal{L}_n^k g||_1 &\le |\mathcal{L}_n^k g|_\theta + ||g||_1\\ &\le (C+1)\big(\theta^k|g|_\theta + \theta(1-\theta)^{-1}||g||_* + ||g||_1\big)\\ &\le \tilde{C}\big(\theta^k||g||_\theta + ||g||_*\big),\end{aligned}$$

for some sufficiently large $\tilde{C} > 0$ depending only on C and θ. The proof is complete. □

5. Singular Perturbations, generated by open holes U_n, of (original) Perron–Frobenius Operators $\mathcal{L}_\varphi$ II: Stability of the Spectrum

As noted in the previous section, the ultimate goal of this and the previous section is Proposition 5.2. We prove it by applying Theorem 1 of [25] and its consequences derived therein. For the convenience of the reader and convenience of referring to, we bring up the setting of [25] in Appendix at the end of our manuscript. We formulate there Theorem 1 of [25] and all its consequences we need.

For a linear operator $Q : \mathcal{B}_\theta \to \mathcal{B}_\theta$ define

$$|||Q||| := \sup\{||Qg||_* : ||g||_\theta \le 1\}.$$

From now on fix $p, q > 1$, $q \ge 2$ being an integer, such that $\frac{1}{p} + \frac{1}{q} = 1$ and, by taking $0 < \rho < 1$ coming from (U2), sufficiently close to 1, assume without loss of generality that

$$\theta \in (\rho^{1/p}, 1).$$

We shall prove the following.

Lemma 5.1. *For every $n \ge 0$ we have*

$$|||\mathcal{L} - \mathcal{L}_n||| \le 2(\rho^{1/q})^n.$$

Proof. Fix an arbitrary $g \in \mathcal{B}_\theta$ with $||g||_\theta \le 1$. Using Lemma 3.1 we then get

$$\begin{aligned}||(\mathcal{L} - \mathcal{L}_n)g||_1 = ||\mathcal{L}(\mathbb{1} - \mathbb{1}_n^1)g)||_1 = ||\mathcal{L}(\mathbb{1}_{U_n} g)||_1 = ||\mathbb{1}_{U_n} g||_1 &\le \mu_\varphi(U_n)||g||_\infty\\ &\le \mu_\varphi(U_n)||g||_\theta \le \mu_\varphi(U_n) \le \rho^n\\ &\le (\rho^{1/q})^n\end{aligned} \tag{5.1}$$

Now fix also two integers $m, j \geq 0$. Using the Hölder Inequality, we get

$$
\begin{aligned}
\theta^{-m} \int_{\sigma^{-j}(U_m)} |(\mathcal{L}-\mathcal{L}_n)g| \, d\mu_\varphi &\leq \theta^{-m} \mu_\varphi(\sigma^{-(j+1)}(U_m) \cap U_n)||g||_\theta = \\
&= \theta^{-m}||g||_\theta \int \mathbb{1}_{\sigma^{-(j+1)}(U_m)} \mathbb{1}_{U_n} \, d\mu_\varphi \\
&\leq ||g||_\theta \theta^{-m} \left(\int \mathbb{1}_{\sigma^{-(j+1)}(U_m)} \, d\mu_\varphi \right)^{1/p} \left(\int \mathbb{1}_{U_n} \, d\mu_\varphi \right)^{1/q} \\
&= ||g||_\theta \theta^{-m} \mu_\varphi(U_m)^{1/p} \mu_\varphi(U_n)^{1/q} \\
&\leq ||g||_\theta \big(\rho^{1/p}/\theta\big)^m \rho^{n/q} \leq (\rho^{1/q})^n ||g||_\theta \leq (\rho^{1/q})^n,
\end{aligned} \tag{5.2}
$$

where the second to the last inequality follows from the fact that $\theta \in (\rho^{1/p}, 1)$. Along with (5.1) this implies that $||\mathcal{L}-\mathcal{L}_n||_* \leq 2(\rho^{1/q})^n$. So, taking the supremum over all $g \in \mathcal{B}_\theta$ with $||g||_\theta \leq 1$, we get that $|||\mathcal{L}-\mathcal{L}_n||| \leq 2(\rho^{1/q})^n$. The proof is complete. □

With Lemma 4.2, Corollary 4.5, and Lemma 4.3, we have checked that the respective conditions (KL2), (KL3), and (KL5), from Appendix are satisfied. We shall now check that condition (KL4) from there also holds. We will do this by showing that the requirements (24.3)–(24.5) from Remark 24.9 in Appendix hold.

For every integer $k \geq 1$ let $\mathcal{A}^k$ be the partition of E_A^∞ into cylinders of length k. Let $\pi_k^* : L^1(\mu_\varphi) \to L^1(\mu_\varphi)$ be the operator of expected value with respect to the probability measure μ_φ and the σ-algebra $\sigma(\mathcal{A}^k)$ generated by the elements of $\mathcal{A}^k$; i. e.

$$
\pi_k^*(g) = E_{\mu_\varphi}(g|\sigma(\mathcal{A}^k)).
$$

If $g \in \mathcal{B}_\theta$ then $|\pi_k^*(g) - g| \leq \mathrm{osc}_k(g)$, and therefore

$$
||\pi_k^*(g) - g||_1 = \int_{E_A^\infty} |\pi_k^*(g) - g| \, d\mu_\varphi \leq \int_{E_A^\infty} \mathrm{osc}_k(g) \, d\mu_\varphi \leq \theta^k |g|_\theta. \tag{5.3}
$$

Let now $\mathcal{A}_0^k$ be a finite subset of $\mathcal{A}^k$ such that

$$
\mu_\varphi\big(A_c^k\big) \leq \theta^k, \tag{5.4}
$$

where

$$
A_c^k := \bigcup_{A \in \mathcal{A}^k \setminus \mathcal{A}_0^k} A.
$$

Let also

$$
A_0^k := \bigcup_{A \in \mathcal{A}_0^k} A.
$$

Let $\hat{\mathcal{A}}^k$ be the partition of E_A^∞ consisting of A_c^k and all elements of $\mathcal{A}_0^k$. Similarly as above, let $\pi_k : L^1(\mu_\varphi) \to L^1(\mu_\varphi)$ be defined by the formula

$$
\pi_k(g) = E_{\mu_\varphi}(g|\sigma(\hat{\mathcal{A}}^{qk})).
$$

We then have that

$$||\pi_k||_1 \le 1, \tag{5.5}$$

and for every $g \in \mathcal{B}_\theta$, because of (5.3) and Lemma 3.1, and (5.4):

$$\begin{aligned} ||\pi_k(g) - g||_1 &= \int_{E_A^\infty} |\pi_k(g) - g| \, d\mu_\varphi = \int_{A_0^{qk}} |\pi_k(g) - g| \, d\mu_\varphi + \int_{A_c^{qk}} |\pi_k(g) - g| \, d\mu_\varphi \\ &= \int_{A_0^{qk}} |\pi_k^*(g) - g| \, d\mu_\varphi + \int_{A_c^{qk}} |\pi_k(g) - g| \, d\mu_\varphi \\ &\le \int_{E_A^\infty} |\pi_k^*(g) - g| \, d\mu_\varphi + 2||g||_\infty \mu_\varphi(A_c^{qk}) \\ &\le \theta^{qk} |g|_\theta + 2||g||_\infty \theta^{qk} \\ &\le 3\theta^{qk} ||g||_\theta. \end{aligned} \tag{5.6}$$

Now, for all m and k we have that

$$\mathrm{osc}_m(\pi_k(g)) = \begin{cases} 0 & \text{if } m \ge qk \\ \le \mathrm{osc}_0(g) \le 2||g||_\infty \le 2||g||_\theta & \text{if } m < qk. \end{cases}$$

Moreover, if $\omega \in A_0^{qk}$ and $m < qk$, then

$$\mathrm{osc}_m(\pi_k(g))(\omega) = \mathrm{osc}_m(\pi_k^*(g))(\omega) \le \mathrm{osc}_m(g)(\omega).$$

Thus,

$$\begin{aligned} \theta^{-m} ||\mathrm{osc}_m(\pi_k(g))||_1 &= \theta^{-m} \int_{E_A^\infty} \mathrm{osc}_m(\pi_k(g) \, d\mu_\varphi \\ &= \theta^{-m} \int_{A_0^{qk}} \mathrm{osc}_m(\pi_k(g)) \, d\mu_\varphi + \theta^{-m} \int_{A_c^{qk}} \mathrm{osc}_m(\pi_k(g)) \, d\mu_\varphi \\ &\le \theta^{-m} \int_{A_0^{qk}} \mathrm{osc}_m(g) \, d\mu_\varphi + 2\theta^{-k} ||g||_\theta \mu_\varphi(A_c^{qk}) \\ &\le |g|_\theta + 2||g||_\theta \\ &\le 3||g||_\theta. \end{aligned}$$

Therefore $|\pi_k(g)|_\theta \le 3||g||_\theta$. Together with (5.5), this gives $||\pi_k||_\theta \le 4$. In other words:

$$\sup_{k \ge 1} \{||\pi_k||_\theta\} \le 4 < +\infty. \tag{5.7}$$

This means that condition (24.3) is satisfied. Now assume that $||g||_\theta \le 1$. Recall that we have fixed $p, q > 1$ such that $(1/p) + (1/q) = 1$. Using Hölder's Inequality and (5.6) we then get for all integers $k \geq 1$, $j \geq 0$, and $n \geq 0$, that

$$\begin{aligned}\int_{\sigma^{-j}(U_n)} |\pi_k(g) - g| \, d\mu_\varphi &= \int_{E_A^\infty} 1\!\!1_{\sigma^{-j}(U_n)} |\pi_k(g) - g| \, d\mu_\varphi \\ &\leq \left(\int_{E_A^\infty} 1\!\!1_{\sigma^{-j}(U_n)} \, d\mu_\varphi \right)^{1/p} \left(\int_{E_A^\infty} |\pi_k(g) - g|^q \, d\mu_\varphi \right)^{1/q} \\ &\leq \mu_\varphi(U_n)^{1/p} 2^{\frac{q-1}{q}} \left(\int_{E_A^\infty} |\pi_k(g) - g| \, d\mu_\varphi \right)^{1/q} \\ &\leq \mu_\varphi(U_n)^{1/p} \big(3\theta^{qk} ||g||_\theta\big)^{1/q} \\ &\leq 3\rho^{n/p}\theta^k.\end{aligned}$$

Recall that $\theta \in (0,1)$ was fixed so large that $\theta > \rho^{1/p}$. In other words $\rho^{1/p}/\theta < 1$, and we get

$$\theta^{-n} \int_{\sigma^{-j}(U_n)} |\pi_k(g) - g| \, d\mu_\varphi \leq 3\big(\rho^{1/p}/\theta\big)^n \theta^k \leq 3\theta^k.$$

In other words $|\pi_k(g) - g|_* \leq 3\theta^k$. Together with (5.6) this gives

$$\|\pi_k(g) - g\|_* \leq 3\theta^{qk} + 3\theta^k \leq 6\theta^k. \tag{5.8}$$

It therefore follows from formula (4.9) of Corollary 4.5 that formula (24.4) in Appendix is satisfied with

$$\alpha = \theta \quad \text{and} \quad M = 1. \tag{5.9}$$

Since all the operators $\pi_k : \mathcal{B}_\theta \to \mathcal{B}_\theta$ have finite–dimensional ranges, all the operators $\mathcal{L}_n \circ \pi_k : \mathcal{B}_\theta \to \mathcal{B}_\theta$ are compact. This establishes formula (24.5) in Appendix.

All the hypotheses of Theorem 24.8 in Appendix (i.e. Theorem 1 in [25]) have been thus verified. Note also that the number 1 is a simple eigenvalue of the operator $\mathcal{L} : \mathcal{B}_\theta \to \mathcal{B}_\theta$ as there exists exactly one Borel probability σ-invariant measure absolutely continuous with respect to the Gibbs measure μ_φ. Applying Theorem 24.8 in Appendix and all the corollaries listed therein, we get the following fundamental perturbative result which extends Propositions 3.17, 3.19, and 3.7 from [20] to the case of infinite alphabet.

Proposition 5.2 (Fundamental Perturbative Result)**.** *For all $n \geq 0$ sufficiently large there exist two bounded linear operators $Q_n, \Delta_n : \mathcal{B}_\theta \to \mathcal{B}_\theta$ and complex numbers $\lambda_n \neq 0$ with the following properties:*

(a) *λ_n is a simple eigenvalue of the operator $\mathcal{L}_n : \mathcal{B}_\theta \to \mathcal{B}_\theta$.*

(b) *$Q_n : \mathcal{B}_\theta \to \mathcal{B}_\theta$ is a projector ($Q_n^2 = Q_n$) onto the 1–dimensional eigenspace of λ_n.*

(c) *$\mathcal{L}_n = \lambda_n Q_n + \Delta_n$.*

(d) *$Q_n \circ \Delta_n = \Delta_n \circ Q_n = 0$.*

(e) *There exist* $\kappa \in (0,1)$ *and* $C > 0$ *such that*

$$||\Delta_n^k||_\theta \le C\kappa^k$$

for all $n \ge 0$ *sufficiently large and all* $k \ge 0$*. In particular,*

$$||\Delta_n^k g||_\infty \le ||\Delta_n^k g||_\theta \le C\kappa^k ||g||_\theta$$

for all $g \in \mathcal{B}_\theta$*.*

(f) $\lim_{n\to\infty} \lambda_n = 1$.

(g) *Enlarging the above constant* $C > 0$ *if necessary, we have*

$$||Q_n||_\theta \le C.$$

In particular,

$$||Q_n g||_\infty \le ||Q_n g||_\theta \le C||g||_\theta$$

for all $g \in \mathcal{B}_\theta$*.*

(h) $\lim_{n\to\infty} |||Q_n - Q_\varphi||| = 0$.

Proof. As there exists exactly one Borel probability σ-invariant measure absolutely continuous with respect to the Gibbs measure μ_φ, the number 1 is a simple eigenvalue of the operator $\mathcal{L} : \mathcal{B}_\theta \to \mathcal{B}_\theta$. Invoking also (5.9) and the fact that $1 > \theta$, the existence of eigenvalues λ_n and items (a) and (f) follow immediately from Corollary 24.10 and (5.9).

The operators $Q_n : \mathcal{B}_\theta \to \mathcal{B}_\theta$ are defined by formula (24.8). The items (b) and (h) then directly follow from Corollary 24.11. The items (c), (d), (e), and (g) follow from corresponding items (1), (2), (4), and (3) of Corollary 24.12. The proof is complete. □

From now on for all $n \ge 0$ sufficiently large as following from Proposition 5.2 we denote

$$g_n := Q_n \mathbb{1}. \tag{5.10}$$

Then $g_n \ne 0$ generates the range of the projector operator $Q_n : \mathcal{B}_\theta \to \mathcal{B}_\theta$ and

$$\mathcal{L}_n g_n = \lambda_n g_n. \tag{5.11}$$

The proof of the next proposition is fairly standard. We provide it here for the sake of completeness.

Proposition 5.3. *All eigenvalues* λ_n *produced in Proposition* 5.2 *are real and positive, and all operators* $Q_n : \mathcal{B}_\theta \to \mathcal{B}_\theta$ *preserve* $\mathcal{B}_\theta(\mathbb{R})$ *and* $\mathcal{B}_\theta^+(\mathbb{R})$*, the subsets of* $\mathcal{B}_\theta$ *consisting, respectively, of real–valued functions and positive real–valued functions.*

Proof. Let $\rho_n \in \mathcal{B}_\theta$ be an eigenfunction of the eigenvalue λ_n. Write $\lambda_n = |\lambda_n| e^{i\gamma_n}$, with $\gamma_n \in [0, 2\pi)$. It follows from (b), (c), and (d) of Proposition 5.2 that

$$|\lambda_n|^k e^{-ik\gamma_n} \mathcal{L}_n^k \mathbb{1} = Q_n \mathbb{1} + \lambda_n^{-k} \Delta_n^k \mathbb{1}. \tag{5.12}$$

By (1) of Corollary 1 in [25] we have that $Q_n \mathbb{1} \ne 0$ for all $n \ge 0$ large enough (so after disregarding finitely many terms, we can assume this for all $n \ge 0$) and $|\lambda_n| > (1+\kappa)/2$. Since also $\mathcal{L}_n^k \mathbb{1}$ is a real–valued function, it therefore follows from (5.12) and (e) that the arguments of $Q_n \mathbb{1}(\omega)$ are the same (mod 2π) whenever $Q_n \mathbb{1} \ne 0$. This in turn implies that the set of accumulation points of the sequence $(k\gamma_n)_{k=0}^{\infty}$ is a singleton (mod 2π). This yields $\gamma_n = 0$ (mod 2π). Thus $\lambda_n \in \mathbb{R}$, and, as λ_n is close to 1 (by Proposition 5.2), it is positive. Knowing this and assuming $g \geq 0$, the equality

$$Q_n g = \lambda_n^{-k} \mathcal{L}_n^k g - \lambda_n^{-k} \Delta_n^k(g),$$

along with (e) of Proposition 5.2, non-negativity of $\mathcal{L}_n^k g$, and inequality $|\lambda_n| > (1+\kappa)/2$, yield $Q_n g \geq 0$. Finally, for $g \in \mathcal{B}_\theta(\mathbb{R})$, write canonically $g = g_+ - g_-$ with $g_+, g_- \in \mathcal{B}_\theta^+(\mathbb{R})$ and apply the invariance of $\mathcal{B}_\theta^+(\mathbb{R})$ under the action of $\mathcal{L}_n$. The proof is complete. □

As an immediate consequence of this proposition and Proposition 5.2, we get the following.

Corollary 5.4. *The function $g_n : \mathcal{B}_\theta \to \mathcal{B}_\theta$ belongs to $\mathcal{B}_\theta^+(\mathbb{R})$.*

Corollary 5.5. *The projector operator $Q_n : \mathcal{B}_\theta \to \mathcal{B}_\theta$ gives rise to the bounded linear positive functional $\widehat{Q}_n : \mathcal{B}_\theta \to \mathbb{R}$, uniquely determined by the formula*

$$Q_n(g) = \widehat{Q}_n(g) g_n.$$

Remark 5.6. We would like to note that unlike [20], we did not use the dynamics (i.e., the interpretation of $\log \lambda_n$ as some topological pressure) to demonstrate item (f) of Proposition 5.2 and to prove Proposition 5.3. We instead used the full power of the perturbation results from [25]. **(+++ Redo this remark from here)** The dynamical interpretation will eventually emerge, and will be important for us, but not until Section 16. Therein Lemma **??** will provide, at least in part, a dynamical interpretation.

6. An Asymptotic Formula for λ_ns, the Leading Eigenvalues of Perturbed Operators

In this section we keep the setting of the previous sections. Our goal here is to estblish the asymptotic behavior of eigenvalues λ_n as n diverges to $+\infty$. Let

$$U_\infty := \bigcap_{n=0}^{\infty} \overline{U}_n.$$

In addition to (U0), (U1), and (U2), we now also assume that:

(U3) U_∞ is a finite set.

(U4) Either

(U4A)

$$U_\infty \cap \bigcup_{n=1}^{\infty} \sigma^n(U_\infty) = \emptyset$$

or

(U4B) $U_\infty = \{\xi\}$, where ξ is a periodic point of σ of prime period equal to some integer $p \geq 1$, the pre-concatenation by the first p terms of ξ with elements of U_n satisfy

$$\xi|_p U_n \subseteq U_n \tag{6.1}$$

for all $n \geq 0$, and

$$\lim_{n\to\infty} \sup\{|\varphi(\omega) - \varphi(\xi)| : \omega \in U_n\} = 0. \tag{6.2}$$

(U5) There are no integer $l \geq 1$, no sequence $\left(\omega^{(n)}\right)_{n=0}^{\infty}$ of points in E_A^∞, and no increasing sequence $\left(s_n\right)_{n=0}^{\infty}$ of positive integers with the following properties:

(U5A)
$$\omega^{(n)}, \sigma^l(\omega^{(n)}) \in U_{s_n}$$
for all $n \geq 0$,

(U5B)
$$\varliminf_{n\to\infty} d_\theta(\omega^{(n)}, U_\infty) \begin{cases} > 0 & \text{if (U4A) holds,} \\ > \theta^l & \text{if (U4B) holds,} \end{cases}$$

(U5C)
$$\varlimsup_{n\to\infty} \sum_{i=1}^{l} \omega_i^{(n)} < +\infty,$$
for fixed l, where we identify E with the natural numbers to give $\omega_i^{(n)}$ their numerical values.

These conditions may seem somewhat artificial and a little bit weird at the first look. In fact these are tailored for the needs of Part 3 devoted to graph directed Markov systems. Their meaning will be fully transparent when we pass to deal with these systems. At the moment we would like only to make the following short comments:

- If U_∞ happens to be a singleton (a very common case) then condition (U4A) just means that this singleton is not a periodic point of the shift map σ; periodic points are dealt with in (U4B).
- Condition (6.1) holds always if for example U_n is contained in some cylinder $[\xi|_q]$ and contains the cylinder $[\xi|_{q+p}]$.
- Condition (6.2) is just the continuity of φ at the point ξ if the sets U_n form a sequence of neighborhoods of ξ with diameters converging to 0. However, alluding to the context of graph directed Markov systems from Part 3 and furthers, U_n will be then inverse images of some neighborhoods of singletons in the limit set under the natural projection from the symbol space, and these inverse images may consist of more than one point. More to it:
- If
$$\text{(U5}'\text{)} \quad \limsup_{n\to\infty} \sup\{\text{dist}_\theta(\omega, U_\infty) : \omega \in U_n\} = 0,$$
then (U5B) cannot occur and condition (U5) is trivially satisfied. If however the alphabet E is infinite, then even if the sets U_n are inverse images of some small neighborhoods of singletons in the limit set under the natural projection from the symbol space, their diameters, i.e. of the sets U_n, need not converge to 0, nor even

(U5') needs to hold. Note that condition (U2) does not rule out the possibility of such phenomenon to happen. However even if (U5') fails, condition (U5), being a kind of its surrogate, will be satisfied for sets U_n considered in Part 3. This condition will be used several times in our proofs, perhaps most transparently, in the proof of Lemma 6.2.

Having proved all the perturbation results of the previous section, we shall now derive several further relations between measure μ_φ, the operators $\mathcal{L}$ and $\mathcal{L}_n$, and their respective eigenvalues. Unlike the previous sections we formulate these results for unnormalized operators since this is the form (i.e. involving unnormalized operators), these results are most suitable for applications in later sections. We however in the proofs assume frequently anyway, without loss of generality, that the operators are normalized. We start with the following analogue of Proposition 4.1 in [20], which is our main result concerning the asymptotic behavior of eigenvalues λ_n as $n \to +\infty$.

Proposition 6.1. *With the setting of Sections* 3 *and* 4*, assume that* (U0)–(U5) *hold. Then*

$$\lim_{n\to\infty} \frac{\lambda - \lambda_n}{\mu_\varphi(U_n)} = \begin{cases} \lambda & \text{if (U4A) holds,} \\ \lambda(1 - \lambda^{-p} e^{\varphi_p(\xi)}) & \text{if (U4B) holds,} \end{cases}$$

where λ and λ_n are respective eigenvalues of original (i. e., we recall, not normalized) operators $\mathcal{L}$ and $\mathcal{L}_n$.

This proposition will follow from a sequence of several lemmas we shall prove now. We need a bit of preparation. For every integer $n \geq 0$ let ν_n be μ_φ-conditional measure on U_n, i. e.:

$$\nu_n := \frac{\mu_\varphi|_{U_n}}{\mu_\varphi(U_n)}.$$

We denote

$$1\!\!1_n^c := 1\!\!1_{U_n} = 1\!\!1 - 1\!\!1_n^1.$$

We start with the following.

Lemma 6.2. *If* (U0)–(U4A) *and* (U5) *hold, then*

$$\lim_{n\to\infty} \frac{\int Q_n\big(\mathcal{L}1\!\!1_n^c\big)\, d\nu_n}{\lambda - \lambda_n} = \lim_{n\to\infty} \int_{E_A^\infty} Q_n 1\!\!1 \, d\nu_n = 1.$$

Proof. Assume without loss of generality that $\mathcal{L}$ is normalized so that $\lambda = 1$ and $\mathcal{L}1\!\!1 = 1\!\!1$. With an aim to prove the first equality, we note that

$$\begin{aligned}\int Q_n(\mathcal{L}1\!\!1_n^c)\,d\nu_n &= \int Q_n(\mathcal{L}1\!\!1 - \mathcal{L}1\!\!1_n^1)\,d\nu_n = \int Q_n(1\!\!1 - \mathcal{L}_n1\!\!1)\,d\nu_n \\ &= \int Q_n1\!\!1\,d\nu_n - \int Q_n\mathcal{L}_n1\!\!1\,d\nu_n = \int Q_n1\!\!1\,d\nu_n - \int \mathcal{L}_nQ_n1\!\!1\,d\nu_n \\ &= \int Q_n1\!\!1\,d\nu_n - \lambda_n\int Q_n1\!\!1\,d\nu_n \\ &= (1-\lambda_n)\int Q_n1\!\!1\,d\nu_n,\end{aligned}$$

using Proposition 5.2. and the first equality is established. Now, fix an arbitrary integer $k \geq 1$. For every $\omega \in U_n$ let

$$\sigma_0^{-k}(\omega) := \{\tau \in \sigma^{-k}(\omega) : \exists_{0\leq j\leq k-1}\,\sigma^j(\tau) \in U_n\} \tag{6.3}$$

and

$$\sigma_c^{-k}(\omega) := \sigma^{-k}(\omega) \setminus \sigma^{-k}(\omega). \tag{6.4}$$

If $\tau \in \sigma_0^{-k}(\omega)$, then $\sigma^j(\tau) \in U_n$ for some $0 \leq j \leq k-1$. Denote $\sigma^j(\tau)$ by γ. Then

$$\gamma \in U_n \text{ and } \sigma^{k-j}(\gamma) \in U_n;\ 1 \leq k-j \leq k.$$

Fix an arbitrary $M > 0$. We claim that for all $n \geq 0$ sufficiently large, say $n \geq N := N_k(M)$, we have that

$$\sum_{i=1}^{k-j} \gamma_i \geq Mk \tag{6.5}$$

for any $\gamma = \sigma^j(\tau)$ for any $\tau \in \sigma_0^{-k}(\omega)$ Indeed, seeking a contradiction we assume that there exist an increasing sequence $(s_n)_0^\infty$ of positive integers, a sequence $\left(\gamma^{(n)}\right)_0^\infty \subseteq E_A^\infty$, and an integer $l \in [1,k]$ such that

$$\gamma^{(n)}, \sigma^l(\gamma^{(n)}) \in U_{s_n}, \tag{6.6}$$

and

$$\sum_{i=1}^{l} \gamma_i^{(n)} < Mk$$

for all $n \geq 0$. It then follows from conditions (U4A) and (U5) that the contrapositive of (U5B) holds, i.e.:

$$\varliminf_{n\to\infty} d_\theta(\gamma^{(n)}, U_\infty) = 0.$$

Hence, from continuity of the shift map $\sigma : E_A^\infty \to E_A^\infty$ and from the finiteness of the set U_∞ (by (U3)),

$$\varliminf_{n\to\infty} d_\theta\big(\sigma^l(\gamma^{(n)}), \sigma^l(U_\infty)\big) = 0.$$

So, passing to a subsequence, and invoking finiteness of the set $\sigma^l(U_\infty)$, we may assume without loss of generality that the sequence $\left(\sigma^l(\gamma^{(n)})\right)_0^\infty$ has a limit, call it β, and then $\beta \in \sigma^l(U_\infty)$. But, since the sequence $\left(\overline{U}_n\right)_0^\infty$ is descending, it follows from (6.6) that $\beta \in \overline{U}_q$ for every $q \geq 0$. Thus $\beta \in \bigcap_{q=0}^\infty \overline{U}_q = U_\infty$. We have therefore obtained that $U_\infty \cap \sigma^l(U_\infty) \neq \emptyset$ as this set contains β. This contradicts (U4A) and finishes the proof of (6.5). So, letting $n \geq N_k(M)$ and $\omega \in U_n$, we get

$$
\begin{aligned}
\mathcal{L}_n^k 1\!\!1(\omega) &= \mathcal{L}^k(1\!\!1_n^1)(\omega) \\
&= \sum_{\tau\in\sigma_c^{-k}(\omega)} 1\!\!1_n^1(\tau)e^{\varphi_k(\tau)} + \sum_{\tau\in\sigma_0^{-k}(\omega)} 1\!\!1_n^1(\tau)e^{\varphi_k(\tau)} \\
&= \sum_{\tau\in\sigma_c^{-k}(\omega)} e^{\varphi_k(\tau)} = \mathcal{L}^k 1\!\!1(\omega) - \sum_{\tau\in\sigma_0^{-k}(\omega)} e^{\varphi_k(\tau)} \\
&= 1\!\!1(\omega) - \sum_{\tau\in\sigma_0^{-k}(\omega)} e^{\varphi_k(\tau)}.
\end{aligned}
\tag{6.7}
$$

Now, if $\tau \in \sigma_0^{-k}(\omega)$, then $\gamma := \sigma^{j_\tau}(\tau) \in U_n$ with some $0 \leq j_\tau \leq k-1$, and using (6.5), we get

$$
\begin{aligned}
S_0(\omega) :&= \sum_{\tau\in\sigma_0^{-k}(\omega)} e^{\varphi_k(\tau)} \preceq \sum_{\tau\in\sigma_0^{-k}(\omega)} \mu_\varphi([\tau]) = \mu_\varphi\left(\sum_{\tau\in\sigma_0^{-k}(\omega)} [\tau]\right) \\
&\leq \mu_\varphi\left(\bigcup_{j=0}^{k-1} \sigma^{-j}\left(\bigcup_{e\geq M} [e]\right)\right) \\
&\leq \sum_{j=0}^{k-1} \mu_\varphi\left(\sigma^{-j}\left(\bigcup_{e\geq M} [e]\right)\right) = \sum_{j=0}^{k-1} \mu_\varphi\left(\bigcup_{e\geq M} [e]\right) \\
&= k\mu_\varphi\left(\bigcup_{e\geq M} [e]\right).
\end{aligned}
\tag{6.8}
$$

This means that there exists a constant $C > 0$ such that

$$
S_0(\omega) \leq Ck\mu_\varphi\left(\bigcup_{e\geq M} [e]\right).
$$

Denote the number $C\mu_\varphi\left(\bigcup_{e\ge M}[e]\right)$ by η_M. Using (6.8), (6.7), and Proposition 5.2, we get the following.

$$\begin{aligned}\left|1-\int Q_n\mathbb{1}\,\nu_n\right| &= \left|\int \mathbb{1}\,d\nu_n-\int Q_n\mathbb{1}\,d\nu_n\right| = \left|\int\left(\mathcal{L}_n^k\mathbb{1}+S_0\right)d\nu_n-\int Q_n\mathbb{1}\,\nu_n\right| \\ &= \left|\int\left(\mathcal{L}_n^k-\lambda_n^kQ_n\right)\mathbb{1}\,d\nu_n+\int(\lambda_n^k-1)Q_n\mathbb{1}\,d\nu_n+\int S_0\,d\nu_n\right| \\ &\le \int|\Delta_n^k\mathbb{1}|\,d\nu_n+|\lambda_n^k-1|\cdot||Q_n\mathbb{1}||_\infty+\int S_0\,d\nu_n \\ &\le C\kappa^n+C|\lambda_n^k-1|+k\eta_M.\end{aligned}$$

Now, fix $\varepsilon>0$. Take then $n\ge 1$ so large that $C\kappa^n<\varepsilon/3$. Next, take $M\ge 1$ so large that $k\eta_M<\varepsilon/3$. Finally take any $n\ge N_k(M)$ so large that $C|\lambda_n^k-1|<\varepsilon/3$. Then $\left|1-\int Q_n\mathbb{1}\,\nu_n\right|<\varepsilon$, and the proof is complete. □

The proof of the next lemma, corresponding to Lemma 4.3 in [20], goes through unaltered in the case of an infinite alphabet. We include it here for the sake of completeness and for the convenience of the reader.

Lemma 6.3. *If* (U1)-(U4A) *and* (U5) *hold, then*

$$\lim_{n\to\infty}\frac{\int Q_n\left(\mathcal{L}\mathbb{1}_n^c\right)d\nu_n}{\mu_\varphi(U_n)}=\lambda.$$

Proof. We assume without loss of generality that $\lambda=1$. Let $\tau_n:U_n\to U_n$ be the first return time from U_n to U_n under the shift map $\sigma:E_A^\infty\to E_A^\infty$. It is defined as

$$\tau_n(\omega):=\inf\{k\ge 1:\sigma^k(\omega)\in U_n\}.$$

By Poincaré's Recurrence Theorem, $\tau_n(\omega)<+\infty$ for μ_φ–a.e. $\omega\in E_A^\infty$. We deal with the concept of first return time and first return time more thoroughly in Sections 19, 20, and 21. We have

$$\begin{aligned}\int_{U_n}\tau_n\,d\nu_n &= \sum_{i=1}^\infty i\nu_n(\tau_n^{-1}(i))=\sum_{i=1}^\infty i\nu_n\left(\mathbb{1}_{\tau_n^{-1}(i)}\right)=\nu_n(\tau_n^{-1}(1))+\sum_{i=2}^\infty i\nu_n\left(\mathbb{1}_n^{i-1}\circ\sigma\cdot\mathbb{1}_n^c\circ\sigma^i\right) \\ &= \nu_n(\tau_n^{-1}(1))+\sum_{i=2}^\infty\frac{i}{\mu_\varphi(U_n)}\mu_\varphi\left(\mathbb{1}_n^{i-1}\circ\sigma\cdot\mathbb{1}_n^c\circ\sigma^i\right) \\ &= \nu_n(\tau_n^{-1}(1))+\sum_{i=2}^\infty\frac{i}{\mu_\varphi(U_n)}\mu_\varphi\left(\mathcal{L}^i\left(\left(\mathbb{1}_n^{i-1}\circ\sigma\cdot\mathbb{1}_n^c\circ\sigma^i\right)\right)\right).\end{aligned}$$

Now using several times the property $\mathcal{L}^j(f\cdot g\circ\sigma^j)=g\mathcal{L}^j(f)$, a formal calculation leads to

$$\int_{U_n}\tau_n\,d\nu_n=\nu_n(\tau_n^{-1}(1))+\sum_{i=2}^\infty i\nu_n\left(\mathcal{L}_n^{i-1}(\mathcal{L}(\mathbb{1}_n^c))\right).$$

Invoking at this point Proposition 5.2, we further get

$$\begin{aligned}\int_{U_n} \tau_n \, d\nu_n &= \nu_n(\tau_n^{-1}(1)) + \sum_{i=2}^{\infty} i\nu_n\big(\lambda_n^{i-1} Q_n \mathcal{L}(1\!\!1_n^c) + \Delta_n^{i-1}\mathcal{L}(1\!\!1_n^c)\big) \\ &= \nu_n(\tau_n^{-1}(1)) + \nu_n\big(Q_n\mathcal{L}(1\!\!1_n^c)\big)\sum_{i=2}^{\infty} i\lambda_n^{i-1} + \sum_{i=2}^{\infty} i\nu_n\big(\Delta_n^{i-1}\mathcal{L}(1\!\!1_n^c)\big) \\ &= \nu_n(\tau_n^{-1}(1)) + \nu_n\big(Q_n\mathcal{L}(1\!\!1_n^c)\big)\left(\frac{1}{(1-\lambda_n)^2} - 1\right) + \sum_{i=2}^{\infty} i\nu_n\big(\Delta_n^{i-1}(\mathcal{L}1\!\!1 - \mathcal{L}1\!\!1_n)\big) \\ &= \nu_n(\tau_n^{-1}(1)) + \nu_n\big(Q_n\mathcal{L}(1\!\!1_n^c)\big)\left(\frac{1}{(1-\lambda_n)^2} - 1\right) + \sum_{i=2}^{\infty} i\nu_n\big(\Delta_n^{i-1}(\mathcal{L}1\!\!1 - \mathcal{L}_n 1\!\!1)\big) \\ &= \nu_n(\tau_n^{-1}(1)) + \nu_n\big(Q_n\mathcal{L}(1\!\!1_n^c)\big)\left(\frac{1}{(1-\lambda_n)^2} - 1\right) + \\ &\qquad + \sum_{i=2}^{\infty} i\nu_n\big(\Delta_n^{i-1}(\mathcal{L}1\!\!1)\big) - \sum_{i=2}^{\infty} i\nu_n\big(\Delta_n^{i} 1\!\!1\big).\end{aligned}$$

Since, By Kac's Theorem, $\int_{U_n} \tau_n \, d\nu_n = 1/\mu_\varphi(U_n)$, multiplying both sides of this formula by $\nu_n\big(Q_n\mathcal{L}(1\!\!1_n^c)\big)$, we thus get

$$\begin{aligned}\frac{\nu_n\big(Q_n\mathcal{L}(1\!\!1_n^c)\big)}{\mu_\varphi(U_n)} = \left(\frac{\nu_n\big(Q_n\mathcal{L}(1\!\!1_n^c)\big)}{1-\lambda_n}\right)^2 + \nu_n\big(Q_n\mathcal{L}(1\!\!1_n^c)\big)\Big(\nu_n(\tau_n^{-1}(1)) - \nu_n\big(Q_n\mathcal{L}(1\!\!1_n^c)\big) + \\ + \sum_{i=2}^{\infty} i\nu_n\big(\Delta_n^{i-1}(\mathcal{L}1\!\!1)\big) - \sum_{i=2}^{\infty} i\nu_n\big(\Delta_n 1\!\!1\big)\Big).\end{aligned}$$

Since, by Lemma 6.2,

$$\lim_{n\to\infty} \frac{\nu_n\big(Q_n\mathcal{L}(1\!\!1_n^c)\big)}{1-\lambda_n} = 1,$$

we have that $\lim_{n\to\infty} \nu_n\big(Q_n\mathcal{L}(1\!\!1_n^c)\big) = 0$, and since, applying Proposition 5.2 again, we deduce that the four terms in the big parentheses above are bounded, we get that

$$\lim_{n\to\infty} \frac{\nu_n\big(Q_n\mathcal{L}(1\!\!1_n^c)\big)}{\mu_\varphi(U_n)} = 1.$$

The proof is complete. □

We shall prove the following.

Lemma 6.4. *If* (U1)-(U3), (U4B) *and* (U5) *hold, then*

$$\lim_{n\to\infty} \frac{\int Q_n\big(\mathcal{L}1\!\!1_n^c\big)\, d\nu_n}{\lambda - \lambda_n} = \lim_{n\to\infty} \int_{E_A^\infty} Q_n 1\!\!1 \, d\nu_n = 1 - \lambda^{-p} e^{\varphi_p(\xi)}.$$

Proof. Assume again without loss of generality that $\mathcal{L}$ is normalized so that $\lambda = 1$ and $\mathcal{L}1\!\!1 = 1\!\!1$. The first equality is general and has been established at the beginning of the proof of Lemma 6.2. We will thus concentrate on the second one. So, fix $\omega \in U_n$ and k, an integral multiple of p, say $k = qp$ with $q \geq 0$. Define the sets $\sigma_0^{-k}(\omega)$ and $\sigma_c^{-k}(\omega)$ exactly as in the proof of Lemma 6.2, i.e. by formulae (6.3) and (6.4). We further repeat the proof of Lemma 6.2 verbatim until formula (6.5), which now takes on the form:

$$\text{Either both} \quad k - j \geq p \ \text{ and } \ \gamma|_{k-j} = \xi|_{k-j} \ \text{ or else } \ \sum_{i=1}^{k-j} \gamma_i \geq Mk.$$

Indeed, this is an immediate consequence of (U4B) and (U5). In other words

$$\sigma_0^{-k}(\omega) = \sigma_1^{-k}(\omega) \cup \sigma_2^{-k}(\omega),$$

where

$$\sigma_1^{-k}(\omega) := \Big\{\tau \in \sigma_0^{-k}(\omega) : \exists\, (0 \leq j \leq q-1)\ \sigma^{pj}(\tau) \in U_n \text{ and } \sigma^{pj}(\tau)|_{p(q-j)} = (\xi|_p)^{q-j}\Big\}$$

and

$$\begin{aligned}\sigma_2^{-k}(\omega) &= \sigma_0^{-k}(\omega) \setminus \sigma_1^{-k}(\omega) \\ &\subseteq \Big\{\tau \in \sigma_0^{-k}(\omega) : \exists\, (0 \leq j \leq k-1)\ \sigma^{j}(\tau) \in U_n \text{ and } \sum_{i=j+1}^{k} \tau_i \geq Mk\Big\}.\end{aligned}$$

Now, we shall prove that

$$\sigma_1^{-k}(\omega) = Z := \Big\{\tau \in \sigma_0^{-k}(\omega) : \sigma^{k-p}(\tau) \in [\xi|_p]\Big\}. \tag{6.9}$$

Indeed, denote the set on the right-hand side of this equality by Z. If $\tau \in Z$, then $\sigma^{p(q-1)}(\tau)|_p = \xi|_p$ and

$$\sigma^{p(q-1)}(\tau) = \big(\sigma^{p(q-1)}(\tau)\big)|_p \sigma^{pq}(\tau) = \xi|_p \omega \in \xi|_p U_n \subseteq U_n,$$

where the last inclusion is due to (U4B). Thus, taking $j = q-1$, we see that $\tau \in \sigma_1^{-k}(\omega)$. So, the inclusion

$$Z \subseteq \sigma_1^{-k}(\omega) \tag{6.10}$$

has been established. In order to prove the opposite inclusion, let $\tau \in \sigma_1^{-k}(\omega)$. Then there exists $j \in \{0, 1, \ldots, q-1\}$ such that $\sigma^{pj}(\tau) \in U_n$ and $\sigma^{pj}(\tau)|_{p(q-j)} = (\xi|_p)^{q-j}$. Then

$$\sigma^{k-p}(\tau)|_p = \big(\sigma^{p(q-j-1)} \circ \sigma^{pj}(\tau)\big)|_p = \sigma^{pj}(\tau)|_{p(q-j-1)+1}^{p(q-j)+1} = \xi|_p,$$

and so, $\tau \in Z$. This establishes the inclusion $\sigma_1^{-k}(\omega) \subseteq Z$, and, together with (6.10) completes the proof of (6.9).

Therefore, keeping $\omega \in U_n$ and using (6.9) and (6.7)we can write

$$
\begin{aligned}
\mathcal{L}_n^k \mathbb{1}(\omega) &= \mathcal{L}^k(\mathbb{1}_n^1)(\omega) \\
&= \mathbb{1}(\omega) - \sum_{\tau \in \sigma_1^{-k}(\omega)} e^{\varphi_k(\tau)} - \sum_{\tau \in \sigma_2^{-k}(\omega)} e^{\varphi_k(\tau)} \\
&= \mathbb{1}(\omega) - \sum_{\tau \in \sigma^{-k}(\omega)} \mathbb{1}_{[\xi|_p]} \circ \sigma^{p(q-1)}(\tau) e^{\varphi_k(\tau)} - \sum_{\tau \in \sigma_2^{-k}(\omega)} e^{\varphi_k(\tau)} \\
&= \mathbb{1}(\omega) - \mathcal{L}^{pq}\big(\mathbb{1}_{[\xi|_p]} \circ \sigma^{p(q-1)}\big)(\omega) - \sum_{\tau \in \sigma_2^{-k}(\omega)} e^{\varphi_k(\tau)} \\
&= \mathbb{1}(\omega) - \mathcal{L}^{p}\big(\mathbb{1}_{[\xi|_p]}\big)(\omega) - \sum_{\tau \in \sigma_2^{-k}(\omega)} e^{\varphi_k(\tau)}.
\end{aligned} \tag{6.11}
$$

Putting

$$S_2(\omega) := \sum_{\tau \in \sigma_2^{-k}(\omega)} e^{\varphi_k(\tau)}$$

and keeping η_M the same as in the proof of Lemma 6.2, the same estimates as in (6.8), give us

$$S_2(\omega) \le k\eta_M.$$

Hence, using also (6.11), we get

$$
\begin{aligned}
\left|1 - e^{\varphi_p(\xi)} - \int \mathcal{L}_n^k \mathbb{1}\, d\nu_n\right| &= \left|\int \mathcal{L}^p\big(\mathbb{1}_{[\xi|_p]}\big)\, d\nu_n - e^{\varphi_p}(\xi) + \int S_2\, d\nu_n\right| = \\
&= \left|\int \big(e^{\varphi_p}(\xi|_p\omega) - e^{\varphi_p}(\xi)\big) d\nu_n + \int S_2\, d\nu_n\right| \\
&\le \int \big|e^{\varphi_p}(\xi|_p\omega) - e^{\varphi_p}(\xi)\big| d\nu_n + \int S_2\, d\nu_n \\
&\le \varepsilon_n + k\eta_M,
\end{aligned}
$$

with some $\varepsilon_n \to 0$ resulting from the last item of (U4B). Hence, keeping k fixed and letting M and then n to infinity, we obtain

$$\lim_{n\to\infty} \int \mathcal{L}_n^k \mathbb{1}\, d\nu_n = 1 - e^{\varphi_p}(\xi) \tag{6.12}$$

for every $k = qp \ge 1$. Using Proposition 5.2, we get

$$
\begin{aligned}
\left|\int \mathcal{L}_n^k \mathbb{1}\, d\nu_n - \int Q_n \mathbb{1}\, d\nu_n\right| &= \left|\int \big(\mathcal{L}_n^k - \lambda_n^k Q_n\big) \mathbb{1}\, d\nu_n + \int \big(\lambda_n^k - 1\big) Q_n \mathbb{1}\, d\nu_n\right| \\
&\le \left\|\big(\mathcal{L}_n^k - \lambda_n^k Q_n\big)\mathbb{1}\right\|_\infty + \left|\lambda_n^k - 1\right| \cdot \left\|Q_n \mathbb{1}\right\|_\infty \\
&\le \left\|\Delta_n^k\right\|_\infty + C\left|\lambda_n^k - 1\right| \\
&\le C\kappa^k + C\left|\lambda_n^k - 1\right|.
\end{aligned}
$$

So, fixing $\varepsilon > 0$, we first take and fix $k \geq 1$ large enough so that $C\kappa^k < \varepsilon/2$, and then using Proposition 5.2, we take $n \geq 1$ large enough so that $C\big|\lambda_n^k - 1\big| < \varepsilon/2$. Combining this with (6.12), we finally get the desired equality

$$\lim_{n\to\infty} \int Q_n \mathbb{1} \, d\nu_n = 1 - e^{\varphi^{(p)}(\xi)}.$$

The proof is complete. □

Applying Lemma 6.4 and proceeding along the lines of the proof of Lemma 6.3 (or Lemma 4.3 in [20]), we get the following analogue of Lemma 4.5 from [20].

Lemma 6.5. *If* (U1)-(U3), (U4B) *and* (U5) *hold, then*

$$\lim_{n\to\infty} \frac{\int Q_n\big(\mathcal{L}\mathbb{1}_n^c\big)\, d\nu_n}{\mu_\varphi(U_n)}. = \lambda\big(1 - \lambda^{-p} e^{\varphi_p(\xi)}\big)^2$$

Having proved Lemmas 6.2, 6.3, 6.4, and 6.5, Proposition 6.1 follows.

Part 2. Symbol Escape Rates and the Survivor Set $K(U_n)$

7. The Existence and Values of Symbol Escape Rates $R_{\mu_\varphi}(U_n)$ and Their Asymptotics as $n \to \infty$

We first recall the basic escape rates definitions. Let G be an arbitrary subset of E_A^∞. We set

$$\underline{R}_{\mu_\varphi}(G) := -\varlimsup_{k\to+\infty} \frac{1}{k} \log \mu_\varphi\Big(\big\{\omega \in E_A^\infty : \sigma^i(\omega) \notin G \text{ for all } i = 0, 1, 2, \cdots, k-1\big\}\Big) \tag{7.1}$$

and

$$\overline{R}_{\mu_\varphi}(G) := -\varliminf_{k\to+\infty} \frac{1}{k} \log \mu_\varphi\Big(\big\{\omega \in E_A^\infty : \sigma^i(\omega) \notin G \text{ for all } i = 0, 1, 2, \cdots, k-1\big\}\Big). \tag{7.2}$$

We call $\underline{R}_{\mu_\varphi}(G)$ and $\overline{R}_{\mu_\varphi}(G)$ respectively the lower and the upper escape rate of G. Of course

$$\underline{R}_{\mu_\varphi}(G) \leq \overline{R}_{\mu_\varphi}(G),$$

and if these two numbers happen to be equal, we denote their common value by

$$R_{\mu_\varphi}(G)$$

and call it the escape rate of G. We provide here for the sake of completeness and convenience of the reader the short elegant proof, entirely taken from [20], of the following.

Theorem 7.1. *If* (U0)-(U5) *hold, then for all integers $n \geq 0$ large enough the escape rates $R_{\mu_\varphi}(U_n)$ exist, and moreover*

$$R_{\mu_\varphi}(U_n) = \log\lambda - \log\lambda_n.$$

Proof. Assume without loss of generality that the Perron-Frobenius operator $\mathcal{L} : \mathcal{B}_\theta \to \mathcal{B}_\theta$ is fully normalized so that $\lambda = 1$ and $\mathcal{L}\mathbb{1} = \mathbb{1}$. By virtue of Proposition 5.2 (b), (c), and (d), we have for every $n \geq 0$ large enough and for all $k \geq 1$ that

$$\begin{aligned}
&\mu_\varphi\Big(\big\{\omega \in E_A^\infty : \sigma^i(\omega) \notin U_n \text{ for all } i = 0, 1, 2, \cdots, k-1\big\}\Big) = \\
&\qquad = \mu_\varphi\left(\bigcap_{j=0}^{k-1} \sigma^{-j}(U_n^c)\right) = \int_{E_A^\infty} \mathbb{1}_n^k \, d\mu_\varphi = \int_{E_A^\infty} \mathcal{L}^k\big(\mathbb{1}_n^k\big) \, d\mu_\varphi \\
&\qquad = \int_{E_A^\infty} \mathcal{L}_n^k(\mathbb{1}) \, d\mu_\varphi = \int_{E_A^\infty} \big(\lambda_n^k Q_n \mathbb{1} + \Delta_n^k \mathbb{1}\big) \, d\mu_\varphi \\
&\qquad = \lambda_n^k \int_{E_A^\infty} Q_n \mathbb{1} \, d\mu_\varphi + \int_{E_A^\infty} \Delta_n^k \mathbb{1} \, d\mu_\varphi.
\end{aligned} \tag{7.3}$$

So, employing Proposition 5.2 (b) and Proposition 5.3, the latter to make sure that $\lambda_n \in (0, +\infty)$ and $\int_{E_A^\infty} Q_n \mathbb{1} \, d\mu_\varphi \in (0, +\infty)$, we conclude from (7.3) with the help of Proposition 5.2 (e) and (g), that the limit

$$\lim_{k \to +\infty} \frac{1}{k} \log \mu_\varphi\Big(\big\{\omega \in E_A^\infty : \sigma^i(\omega) \notin U_n \text{ for all } i = 1, \cdots, k\big\}\Big)$$

exists and is equal to $\log \lambda_n$. The proof is complete. □

Now we are in position to prove the following main result of this section.

Proposition 7.2. *With the setting of Sections* 3 *and* 4*, assume that* (U0)–(U5) *hold. Then*

$$\lim_{n \to \infty} \frac{R_{\mu_\varphi}(U_n)}{\mu_\varphi(U_n)} = \begin{cases} 1 & \text{if (U4A) holds,} \\ 1 - \exp\big(\varphi_p(\xi) - p\mathrm{P}(\varphi)\big) & \text{if (U4B) holds.} \end{cases}$$

Proof. By Theorem 7.1 we have

$$R_{\mu_\varphi}(U_n) = \frac{\log \lambda - \log \lambda_n}{\mu_\varphi(U_n)} = -\frac{\log \lambda_n - \log \lambda}{\lambda_n - \lambda} \cdot \frac{\lambda_n - \lambda}{\mu_\varphi(U_n)}.$$

Therefore, invoking Proposition 6.1, we get that

$$\begin{aligned}
\lim_{n \to \infty} \frac{R_{\mu_\varphi}(U_n)}{\mu_\varphi(U_n)} &= \lim_{n \to \infty} \frac{\log \lambda_n - \log \lambda}{\lambda_n - \lambda} \cdot \lim_{n \to \infty} \frac{\lambda - \lambda_n}{\mu_\varphi(U_n)} \\
&= \frac{1}{\lambda} \begin{cases} \lambda & \text{if (U4A) holds,} \\ \lambda(1 - \lambda^{-p} \exp\big(\varphi_p(\xi)\big) & \text{if (U4B) holds.} \end{cases} \\
&= \begin{cases} 1 & \text{if (U4A) holds,} \\ 1 - \exp\big(\varphi_p(\xi) - p\mathrm{P}(\varphi)\big) & \text{if (U4B) holds.} \end{cases}
\end{aligned}$$

The proof is complete. □

8. Conditionally Invariant Measures on U_n^c

Following [9] we call a Borel probability measure ν on U_n^c conditionally invariant if there exists $\alpha \in (0, +\infty)$ such that

$$\nu(U_n^c \cap \sigma^{-1}(A)) = \alpha\nu(A) \tag{8.1}$$

for every Borel set $A \subset U_n^c$. In slightly different terms, a Borel probability measure ν on E_A^∞ is conditionally invariant with respect to U_n^c if $\nu(U_n^c) = 1$ and

$$\nu \circ \sigma^{-1} = \alpha\nu. \tag{8.2}$$

We will frequently treat conditionally invariant measures in this way, i.e., as Borel probability measures on E_A^∞ with support on U_n^c. Precisely, the phrase "a Borel probability measure ν on U_n^c" will mean a Borel probability measure ν on E_A^∞ with $\nu(U_n) = 0$.

From (8.2) and the fact that $\nu(U_n^c) = 1$ we get

$$\alpha = \alpha\nu(U_n^c)) = \nu(\sigma^{-1}(U_n^c)). \tag{8.3}$$

For the sake of completeness we shall prove the following two facts relating conditionally invariant measures with the action of truncated Perron-Frobenius operator operators $\mathcal{L}_n$

Lemma 8.1. *A Borel probability measure ν on U_n^c absolutely continuous with respect to the equilibrium state $\mu = \mu_\varphi$, with Radon–Nikodym derivative h, is conditionally invariant if and only if*

$$\mathcal{L}_n(h)|_{U_n^c} = \alpha h|_{U_n^c}$$

for some $\alpha \in (0, 1]$.

Proof. The measure $h\mu$ is conditionally invariant if and only if for every Borel set $A \subset U_n^c$ we have that

$$\begin{aligned}\alpha \int_\Lambda h d\mu = \alpha\nu(A) &= \nu(\sigma^{-1}A) \\ &= \nu(1\!\!1_n 1\!\!1_A \circ \sigma) = \int_{E_A^\infty} h 1\!\!1_n 1\!\!1_A \circ \sigma \, d\mu \\ &= \int_{E_A^\infty} \mathcal{L}(h 1\!\!1_n 1\!\!1_A \circ \sigma) \, d\mu = \int_A \mathcal{L}(1\!\!1_n h) \, d\mu = \int_A \mathcal{L}_n(h) d\mu.\end{aligned}$$

But this holds if and only if

$$\mathcal{L}_n(h) = \alpha h$$

μ–a.e., U_n^c. This completes the proof. □

Corollary 8.2. *If ν is a conditionally invariant measure on U_n^c absolutely continuous with respect to the equilibrium state $\mu = \mu_\varphi$ with Radon-Nikodym derivative $h = d\nu/dm$ and escaping factor α, then*

$$\mathcal{L}_n^k(h)|_{U_n^c} = \alpha^k h|_{U_n^c}$$

for every integer $k \geq 0$.

Proof. We proceed by induction. For $k=0$ the statement is trivially true. Next, suppose that it holds for some integer $k\ge 0$. Then $\mathcal{L}_n^k h|_{U_n^c}$ is a scalar multiple of $h|_{U_n^c}$. But then by linearity of the operator $\mathcal{L}_n$ and Lemma 8.1 we get that $\mathcal{L}_n(\mathcal{L}_n^k h)|_{U_n^c}=\alpha\mathcal{L}_n h|_{U_n^c}$. The inductive hypothesis then gives that

$$\mathcal{L}_n^{k+1}h|_{U_n^c}=\alpha^{k+1}h|_{U_n^c}$$

and the proof is complete. □

We shall prove the following theorem about conditionally invariant measures.

Theorem 8.3. *If $n\ge 0$ is big enough as required in Proposition 5.2 then*

$$\widehat{\mu}_n:=(\mu(g_n1\!\!1_n))^{-1}g_n|_{U_n^c}\mu_\varphi|_{U_n^c}$$

is a unique conditionally invariant measure on U_n^c absolutely continuous with respect to $\mu_\varphi|_{U_n^c}$ whose Radon-Nikodyn derivative $d\widehat{\mu}_n/d\mu_\varphi$ belongs to $\mathcal{B}_\theta$. In addition, the coefficient α of (8.1) and (8.2) is equal to $\lambda_n(=\widehat{\mu}_n(\sigma^{-1}(U_n^c)))$ and for every Borel set $B\subset U_n^c$ we have that

$$\lim_{k\to+\infty}\frac{\mu_\varphi(\sigma^{-k}(B)\cap U_n^c)}{\mu_\varphi(\sigma^{-k}(U_n^c)\cap U_n^c)}=\widehat{\mu}_n(B).$$

Proof. Because of Corollary 5.4 $\widehat{\mu}_n$ is a Borel (positive) probability measure on U_n^c. Denote

$$\beta:=(\mu_\varphi(g_n1\!\!1_n))^{-1}.$$

Using (5.10) we get that for every Borel set $B\subset U_n^c$ that

$$\begin{aligned}\widehat{\mu}_n(\sigma^{-1}(B))&=\beta\mu_\varphi(1\!\!1_n g_n 1\!\!1_B\circ\sigma)=\beta\mu_\varphi(\mathcal{L}(1\!\!1_n g_n 1\!\!1_B\circ\sigma))\\&=\beta\mu_\varphi(1\!\!1_B\mathcal{L}(1\!\!1_n g_n))=\beta\mu_\varphi(1\!\!1_B\mathcal{L}_n(g_n))\\&=\beta\mu_\varphi(1\!\!1_B\lambda_n g_n)=\lambda_n\beta\mu_\varphi(1\!\!1_B g_n)\\&=\lambda_n\widehat{\mu}_n(B).\end{aligned}$$

Thus, also

$$\lambda_n=\lambda_n\widehat{\mu}_n(U_n^c)=\mu_n(\sigma^{-1}(U_n^c))$$

Now we shall show the uniqueness of $\widehat{\mu}_n$. So suppose that $h\mu$ is a conditionally invariant measure with $h:E_A^\infty\to[0,+\infty)$ belonging to $\mathcal{B}_\theta$ and identically equal to zero on U_n. Then by Proposition 5.2, Corollary 5.5, and Corollary 8.2, we have that for every integer $k\ge 0$ that

$$\alpha^k h|_{U_n^c}=(\lambda_n^k Q_n(h)+\Delta_n^k h)|_{U_n^c}=(\lambda_n^k\widehat{Q}_n(h)g_n+\Delta_n^k h)|_{U_n^c}. \tag{8.4}$$

Assuming that $n\ge 0$ is large enough that it follows from Proposition 5.2 (e) and (f) that $\alpha=\lambda_n$ and $\lim_{k\to+\infty}\|\lambda_n^{-k}\Delta_n^k h\|_\infty=0$. Therefore, after dividing both sides of (8.4) by λ_n^k and letting $k\to+\infty$, we conclude that $h|_{U_n^c}=\widehat{Q}_n(h)g_n|_{U_n^c}$, The proof of the first assertion of our theorem is complete.

The second assertion, $\alpha=\lambda_n=\widehat{\mu}_n(\sigma^{-1}(U_n^c))$ is now an immediate consequence of (8.3).

The third and final assertion of the theorem follows from the following calculation.

$$
\begin{aligned}
\lim_{k\to+\infty}\frac{\mu_\varphi\Big(\sigma^{-k}(B)\cap\bigcap_{j=0}^{k-1}\sigma^{-j}(U_n^c)\Big)}{\mu_\varphi\Big(\sigma^{-k}(U_n)\cap\bigcap_{j=0}^{k-1}\sigma^{-j}(U_n^c)\Big)} &= \lim_{k\to+\infty}\frac{\mu_\varphi(1\!\!1_B\circ\sigma^k 1\!\!1_U^k)}{\mu_\varphi(1\!\!1_{U_n^c}\circ\sigma^k 1\!\!1_n^k)} = \\
&= \lim_{k\to+\infty}\frac{\mu_\varphi(\mathcal{L}_\varphi(1\!\!1_B\circ\sigma^k 1\!\!1_n^k))}{\mu_\varphi(\mathcal{L}_\varphi(1\!\!1_{U_n^c}\circ\sigma^k 1\!\!1_n^k)} = \lim_{k\to+\infty}\frac{\mu_\varphi(1\!\!1_B\mathcal{L}_\varphi^k(1\!\!1_n^k))}{\mu_\varphi(1\!\!1_{U_n^c}\mathcal{L}_\varphi^k(1\!\!1_n^k)} \\
&= \lim_{k\to+\infty}\frac{\mu_\varphi(1\!\!1_B\mathcal{L}_n(1\!\!1)}{\mu_\varphi(1\!\!1_{U_n^c}\mathcal{L}_n(1\!\!1)} \\
&= \frac{\mu_\varphi(1\!\!1_B g_n)}{\mu_\varphi(1\!\!1_{U_n^c} g_n} \\
&= \widehat{\mu}_n(B).
\end{aligned}
$$

This complete the proof. ☐

We cannot really do much better with the uniqueness part of this theorem; the hypothesis that the Radon-Nikodyn derivative $d\widehat{\mu}_n/d\mu_\varphi$ belongs to $\mathcal{B}_\theta$ is important. Indeed, it follows from Theorem 3.1 in [14], that for every $\alpha\in(0,1)$ there are uncountably (a continuum) of many conditionally invariant measures absolutely continuous with respect to μ_φ. Moreover, if $\alpha\in(0,1)$ is sufficiently small, then the Radon-Nikodym derivatives of all these measures with respect to μ_φ are bounded.

9. Symbol Escape Rates of U_ns, I: The Variational Principle on the Survivor Sets $K(U_n)$

We assume again, what is very natural in the context of this section, that the Perron–Frobenius $\mathcal{L}_\varphi$ is fully normalized; in particular its leading eignevalue (simultaneously the spectral radius) $\lambda=1$. Theorem 7.1 then asserts that

$$
R_{\mu_\varphi}(U_n) = -\log\lambda_n. \tag{9.1}
$$

for all integers $n\ge 0$ large enough.

Since the survivor sets $K(U_n)$, $n\ge 1$, are closed and forward invariant with respect to the shift map $\sigma: E_A^\infty\to E_A^\infty$ we can consider the dynamical system

$$
\sigma|_{K(U_n)}: K(U_n)\to K(U_n).
$$

Let $\mathcal{M}(\sigma)$ denote the space of all Borel probability σ-invariant measures on E_A^∞ endowed with the the weak convergence topology. Let

$$
\mathcal{M}_n(\sigma) := \big\{\nu\in\mathcal{M}(\sigma) : \nu(K(U_n))=1\big\},
$$

$$
\mathcal{M}_n^+(\sigma) := \big\{\nu\in\mathcal{M}(\sigma) : \int\varphi\,d\nu > -\infty\big\},
$$

and let $\mathcal{M}_n^e(\sigma)$, $\mathcal{M}_n^e(\sigma)$ and $\mathcal{M}_n^{+,e}(\sigma)$ denote the respective subspaces of ergodic measures. Our goal in the upcoming two sections is to prove the following two results.

Theorem 9.1. *(Variational Principle) If $(U_n)_{n=0}^\infty$ is a sequence of open subsets of E_A^∞ satisfying conditions (U0-U5), then*

$$\sup\Big\{\mathrm{h}_\nu(\sigma)+\int_{K(U_n)}\varphi\,d\nu : \nu\in\mathcal{M}_n^+(\sigma)\Big\}=\sup\Big\{\mathrm{h}_\nu(\sigma)+\int_{K(U_n)}\varphi\,d\nu : \nu\in\mathcal{M}_n^{+e}(\sigma)\Big\}=\log\lambda_n$$

for all $n\geq 1$ large enough.

This is the first result. The other one is to show that there exists a unique measure in $\mathcal{M}_n^+(\sigma)$ (and that it belongs to $\mathcal{M}_n^{+e}(\sigma)$) maximizing the above suprema, and to define it explicitly. We will call this measure the surviving equilibrium state for U_n. In this section we will show that

$$\sup\Big\{\mathrm{h}_\nu(\sigma)+\int_{K(U_n)}\varphi\,d\nu : \nu\in\mathcal{M}_n^+(\sigma)\Big\}\leq\log\lambda_n. \tag{9.2}$$

All other assertions we mentioned above will be proved in the next two sections. Let $\nu\in\mathcal{M}_n^+(\sigma)$. For every $k\geq 1$ let

$$\mathcal{F}_n(k):=\big\{\omega\in E_A^n:\ [\omega]\cap K(U_n)\neq\emptyset\big\}$$

and let

$$Z_n(k):=\sum_{\omega\in\mathcal{F}_n(k)}\exp(\sup(\varphi_k|_{[\omega]})).$$

Denote by α the partition of E_A^∞ into cylinders of length one, i.e,

$$\alpha=\{[e]\}_{e\in E}.$$

Then for every $k\geq 1$,

$$\alpha^k:=\alpha\vee\sigma^{-1}(\alpha)\vee\sigma^{-2}(\alpha)\vee\cdots\vee\sigma^{-(k-1)}(\alpha)=\{[\omega]\}_{\omega\in E_A^k}.$$

Denote by h the function $(0,+\infty)\ni x\mapsto -x\log x\in\mathbb{R}$. Since this function is concave, the following calculation, standard in thermodynamic formalism, gives us that for every $\nu\in\mathcal{M}_n^+(\sigma)$ and every integer $k\geq 1$:

(9.3)

$$\begin{aligned}
\mathrm{H}_\nu(\alpha^k)+\int\varphi_k d\nu &\leq \sum_{\omega\in\mathcal{F}_n(k)}\nu([\omega])\big(\sup(\varphi_k|_{[\omega]})-\log\nu([\omega])\big)\\
&= Z_n(k)\sum_{\omega\in\mathcal{F}_n(k)}Z_n(k)^{-1}\exp\big(\sup(\varphi_k|_{[\omega]})\big)\,h\big(\nu([\omega]\exp(-\sup(\varphi_k|_{[\omega]})))\big)\\
&\leq Z_n(k)h\left(\sum_{\omega\in\mathcal{F}_n(k)}Z_n(k)^{-1}\exp\big(\sup(\varphi_k|_{[\omega]})\big)\,\nu([\omega]\exp\big(-\sup(\varphi_k|_{[\omega]})\big)\right)\\
&= Z_n(k)h(Z_n^{-1}(k))=\log Z_n(k)
\end{aligned}$$

Now our goal is to estimate $Z_n(k)$, Since the potential $\varphi : E_A^\infty \to \mathbb{R}$ is summable, there exists $l \geq 1$ such that

$$\sum_{e \geq l+1}^{\infty} \exp(\sup(\varphi|_{[e]})) < \frac{1}{2}.$$

We know from Proposition 5.2 that for every $n \geq 1$ large enough

$$\lambda_n > \frac{1}{2}.$$

We have the estimate

$$\begin{aligned} Z_n(k) &\leq \sum_{j=0}^{k-1} \sum_{s \leq l} \sum_{\substack{\omega \in \mathcal{F}_n(k) \\ \omega_j = s}} \exp\left(\sup(\varphi_j|_{[\omega]})\right) \sum_{\tau (\mathbb{N}_l^c)_A^{k-j}} \exp\left(\sup(\varphi_{k-j}|_{[\tau]})\right) \\ &\leq \sum_{j=0}^{k-1} \sum_{s \leq l} \mathcal{L}_n^j \mathbb{1}(\gamma_s) \left(\frac{1}{2}\right)^{k-j} \end{aligned}$$

where $\gamma_s \in K(U_n)$ is fixed such that $A_{s(\gamma_s)_0} = 1$. Applying Proposition 5.2 we further estimate

$$\begin{aligned} Z_n(k) &\leq \sum_{j=0}^{k} \sum_{s \leq l} \left(\lambda_n^j Q_n \mathbb{1}(\gamma_s) + \Delta_n^k \mathbb{1}(\gamma_0) \left(\frac{1}{2}\right)^{k-j} \right) \\ &\leq \Gamma l \sum_{j=0}^{k-1} \lambda_n^j \left(\frac{1}{2}\right)^{k-j} = \frac{\lambda_n^k - \left(\frac{1}{2}\right)^k}{\lambda_n - \frac{1}{2}} \leq 4\lambda_n^k \end{aligned}$$

if $n \geq 1$ is large enough and Γ is a constant. Therefore,

$$\limsup_{k \to +\infty} \frac{1}{n} \log Z_n(k) \leq \log \lambda_n.$$

Inserting this into (9.3) we get

$$\mathrm{h}_\nu(\sigma) + \int \varphi d\nu = \lim_{k \to +\infty} \frac{1}{k} \left(\mathrm{H}_\nu(\alpha^k) + \int \varphi_k d\nu \right) \leq \log \lambda_n.$$

This establishes formula (9.2).

10. Symbol Escape Rates of U_ns, II: The Variational Principle and Equilibrium States on the Survivor Sets $K(U_n)$; Their Existence and Stochastic Properties

Because of Proposition 5.2 and Proposition 5.3, for every $n \geq 1$ large enough the formula

$$\mathcal{B}_\theta \ni g \mapsto Q_n(g_n g) = \mu_n(g) g_n \tag{10.1}$$

where, we recall, $g_n = Q_n \mathbb{1}$, defines a linear continuous positive functional $\mu_n : \mathcal{B}_\theta \to \mathbb{R}$.

Speaking a little vaguely, the ultimate goal of this section is to prove that this functional gives rise to a shift invariant Borel probability measure on $K(U_n)$ which maximizes the supremum in (9.2) with the value equal to $\log \lambda_n$. Our first step in achieving this goal is to prove the following.

Lemma 10.1. *The linear functional $\mathcal{B}_\theta \ni g \mapsto \mu_n(g) \in \mathbb{R}$ restricted to $\mathrm{H}_\theta^b(A)$ extends (uniquely) to a positive linear functional from $C_b(E_A^\infty)$ to $\mathbb{R}$ and thus it represents a Borel finite measure on E_A^∞. We use the same notation of μ_n for this extension.*

The most natural way to prove this lemma would be to apply the Daniell-Stone Representation theorem, but we do not see any reasonable way to show that if a monotone decreasing sequence of positive bounded Hölder continuous functions converges pointwise to zero, then the sequence of respective values of the functional μ_n also converges to zero. We therefore take a somewhat different way. We first approximate each set U_n from above by some suitable sets $U_n(q)$, $q \geq 1$, apply the corresponding analogue of Proposition 5.2 for perturbations of the operator $\mathcal{L}_n$ and define the appropriate measures on the, what will turn out to be compact, shift invariant sets $K(U_n(q))$ by means of the Stone Representation Theorem of positive linear operators "on compact spaces"; the application of this theorem does not require to show that the continuity (pointwise convergence) hypothesis of the Daniell-Stone Representation Theorem is satisfied.

Then we will show that these shift-invariant measures on $K(U_n(q))$ converge weakly as q goes to infinity. The resulting weak limit is necessarily shift–invariant and supported on $K(U_n)$. We will show that this is the required extension of μ_n.

So, given an integer $l \geq 1$ we denote

$$\mathbb{N}_l := \{1, 2, \ldots, l\}.$$

Given also $n \geq 0$ we set

$$U_n^{(l)} := U_n \cup \bigcup_{j=0}^{n-1} \sigma^{-j}(\mathbb{N}_l^c).$$

Given $q \geq n$ let $l_q \geq 1$ be the least integer such that

$$\mu_\varphi(\mathbb{N}_{l_q}^c) \leq \rho^q/n.$$

Then

$$\mu_\varphi\left(\bigcup_{j=0}^{n-1} \sigma^{-j}(\mathbb{N}_l^c)\right) \leq \rho^q. \tag{10.2}$$

Set

$$U_n(q) := U_n^{l_q}.$$

Of course each open set $U_n(q)$ is a disjoint union of cylinders of length q so that condition (U1) is satisfied for the sequence $(U_n(q))_{q=n}^\infty$. $\mathcal{L} := \mathcal{L}_\varphi$ is now the fully normalized transfer operator associated to φ. As in Section 4 we define the operators

$$\mathcal{L}_{n,q}(g) := \mathcal{L}\big(\mathbb{1}_{U_n^c(q)}g\big).$$

The space $\mathcal{B}_\theta$ and the norm $||\cdot||_\theta$ remain unchanged. We however naturally adjust the seminorm $|\cdot|_*$ to depend on our sequence $(U_n(q))_{q=n}^\infty$. We set for $g\in\mathcal{B}$:

$$|g|_n^* := \sup_{i\ge 0}\sup_{m\ge 1}\left\{\theta^{-m}\int_{\sigma^{-i}\left(\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_{l_m}^c)\right)}|g|\,d\mu_\varphi\right\}$$

and

$$||g||_n^* := ||g||_1 + |g|_n^*.$$

We intend to apply Keller and Liverani (see [25]) perturbation results. Because of (10.2), Lemma 4.1 goes through for the norm $||\cdot||_n^*$. We put

$$\mathbb{1}_{n,q}^k := \prod_{j=0}^{k-1}\mathbb{1}_{\sigma^{-j}(U_n^c(q))} = \prod_{j=0}^{k-1}\mathbb{1}_{U_n^c(q)}\circ\sigma^j,$$

$$\mathbb{1}_{n,q}^{k,*} := \prod_{j=0}^{k-1}\mathbb{1}_{\sigma^{-j}\left(\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_l^c)\right)} = \prod_{j=0}^{k-1}\mathbb{1}_{\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_l^c)}\circ\sigma^j,$$

and note that

$$\mathbb{1}_{n,q}^k = \mathbb{1}_n^k\cdot\mathbb{1}_{n,q}^{k,*}.$$

The proof of Lemma 4.2 goes the same way for the operators $\mathcal{L}_{n,q}^k$ with only formal change of $\mathbb{1}_n^k$ by $\mathbb{1}_{n,q}^k$ and U_m by $U_m(q)$. It gives:

Lemma 10.2. *For every $k\ge 1$ and for every $q\ge n$, we have that*

$$||\mathcal{L}_{n,q}^k||_n^* \le 1.$$

Lemmas 4.3, 4.4, and Corollary 4.5 used only the (U1) property of the sequence $(U_n)_{n=0}^\infty$, and therefore these apply to the sets $U_n(q)$, $q\ge n$, and the operators $\mathcal{L}_{n,q}^k$ (to be clear, the role of n is in these three results is now played by the pair (n,q)). Fix $a,b>1$ such that $\frac{1}{a}+\frac{1}{b}=1$ and

$$\rho^{1/a} < \theta.$$

We shall prove the following analogue of Lemma 5.1.

Lemma 10.3. *For every $n\ge 0$ we have*

$$|||\mathcal{L}_n - \mathcal{L}_{n,q}||| \le 2(\rho^{1/b})^q.$$

Proof. Fix an arbitrary $g\in\mathcal{B}_\theta$ with $||g||_\theta\le 1$. Using Lemma 3.1 and (10.2), we get

$$\begin{aligned}
||(\mathcal{L}_n - \mathcal{L}_{n,q})g||_1 &= ||\mathcal{L}(\mathbb{1}_{U_n^c\setminus U_n^c(q)}g)||_1 = ||\mathbb{1}_{U_n^c\setminus U_n^c(q)}g||_1 \le \mu_\varphi(U_n^c\setminus U_n^c(q))||g||_\infty \\
&= \mu_\varphi(U_n^c\cap U_n(q))||g||_\infty = \mu_\varphi\left(U_n^c\cap\left(U_n\cup\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_l^c)\right)\right)||g||_\infty \\
&= \mu_\varphi\left(U_n^c\cap\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_l^c)\right)||g||_\infty \le \mu_\varphi\left(\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_l^c)\right)||g||_\theta \\
&\le \rho^q||g||_\theta \le \rho^q \le (\rho^{1/b})^q.
\end{aligned} \tag{10.3}$$

Also, using Cauchy-Schwarz Inequality, we get

$$\theta^{-m}\int_{\sigma^{-i}\left(\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_{l_m}^c)\right)}\left|(\mathcal{L}_n-\mathcal{L}_{n,q})g\right|d\mu_\varphi =$$

$$\begin{aligned}
&= \theta^{-m}\int_{E_A^\infty}1\!\!1_{\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_{l_m}^c)}\circ\sigma^i\left|\mathcal{L}\left(1\!\!1_{U_n^c\setminus U_n^c(q)}g\right)\right|d\mu_\varphi \\
&\le \theta^{-m}||g||_\infty\int_{E_A^\infty}1\!\!1_{\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_{l_m}^c)}\circ\sigma^i\mathcal{L}\left(1\!\!1_{U_n^c\setminus U_n^c(q)}\right)d\mu_\varphi \\
&= \theta^{-m}||g||_\infty\int_{E_A^\infty}\mathcal{L}\left(1\!\!1_{\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_{l_m}^c)}\circ\sigma^{i+1}1\!\!1_{U_n^c\setminus U_n^c(q)}\right)d\mu_\varphi \\
&= \theta^{-m}||g||_\infty\int_{E_A^\infty}1\!\!1_{\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_{l_m}^c)}\circ\sigma^{i+1}1\!\!1_{U_n^c\setminus U_n^c(q)}\,d\mu_\varphi \\
&\le \theta^{-m}||g||_\infty\int_{E_A^\infty}1\!\!1_{\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_{l_m}^c)}\circ\sigma^{i+1}1\!\!1_{\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_{l_q}^c)}\,d\mu_\varphi \\
&= \theta^{-m}||g||_\infty\int_{E_A^\infty}1\!\!1_{\sigma^{-(i+1)}\left(\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_{l_m}^c)\right)}1\!\!1_{\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_{l_q}^c)}\,d\mu_\varphi \\
&\le \theta^{-m}||g||_\theta\mu_\varphi^{1/a}\left(\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_{l_m}^c)\right)\mu_\varphi^{1/b}\left(\bigcup_{j=0}^{n-1}\sigma^{-j}(\mathbb{N}_{l_q}^c)\right) \\
&\le ||g||_\theta(\rho^{1/a}/\theta)^m\rho^{q/b}\le\rho^{q/b}||g||_\theta\le\rho^{q/b}.
\end{aligned}$$

Therefore, $\left|(\mathcal{L}_n-\mathcal{L}_{n,q})g\right|_n^*\le\rho^{q/b}$, and together with (10.3), this completes the proof of our lemma. ☐

Having all of this, particularly the last lemma, and taking into account the considerations between the end of the proof of Lemma 5.1 and Proposition 5.2, we get the following analogue of the latter for the operator $\mathcal{L}$ replaced by $\mathcal{L}_n$, and the operators $\mathcal{L}_n$ replaced by $\mathcal{L}_{n,q}$.

Lemma 10.4. *For all integers $n\ge 0$ large enough and for all $q\ge n$ large enough there exist two bounded linear operators $Q_{n,q},\Delta_{n,q}:\mathcal{B}_\theta\to\mathcal{B}_\theta$ and complex numbers $\lambda_{n,q}\ne 0$ with the following properties:*

(a) *$\lambda_{n,q}$ is a simple eigenvalue of the operator $\mathcal{L}_{n,q}:\mathcal{B}_\theta\to\mathcal{B}_\theta$.*

(b) *$Q_{n,q}:\mathcal{B}_\theta\to\mathcal{B}_\theta$ is a projector ($Q_{n,2}^2=Q_{n,q}$) onto the 1–dimensional eigenspace of $\lambda_{n,q}$.*

(c) *$\mathcal{L}_{n,q}=\lambda_{n,q}Q_{n,q}+\Delta_{n,q}$.*

(d) *$Q_{n,q}\circ\Delta_{n,q}=\Delta_{n,q}\circ Q_{n,q}=0$.*

(e) *There exist $\kappa_n \in (0,1)$ and $C>0$ such that for every integer $k \geq 0$ we have that*

$$||\Delta_{n,n}^k||_\theta \leq C(\kappa_n\lambda_n)^k.$$

In particular,

$$||\Delta_{n,q}^k g||_\infty \leq ||\Delta_{n,q}^k g||_\theta \leq C(\kappa_n\lambda_n)^k||g||_\theta$$

for all $g \in \mathcal{B}_\theta$.

(f) $\lim_{q\to\infty} \lambda_{n,q} = \lambda_n$.

(g) *Enlarging the above constant $C>0$ if necessary, we have*

$$||Q_{n,q}||_\theta \leq C.$$

In particular,

$$||Q_{n,q}g||_\infty \leq ||Q_{n,q}g||_\theta \leq C||g||_\theta$$

for all $g \in \mathcal{B}_\theta$.

(h) $\lim_{q\to\infty} |||Q_{n,q} - Q_n||| = 0$.

The following lemma can be proved in exactly the same way as was Proposition 5.3.

Lemma 10.5. *All eigenvalues $\lambda_{n,q}$ produced in Lemma* 10.4 *are real and positive, and all operators $Q_{n,q} : \mathcal{B}_\theta \to \mathcal{B}_\theta$ preserve $\mathcal{B}_\theta(\mathbb{R})$ and $\mathcal{B}_\theta^+(\mathbb{R})$, the subspaces of $\mathcal{B}_\theta$ consisting, respectively, of real–valued functions and positive real–valued functions.*

Remark 10.6. How large n needs to be in Lemmas 10.4 and 10.5 is determined by the requirement that the assertions of Proposition 5.2 hold for such n.

Now, let us consider the dynamical systems $\sigma : K(U_n(q)) \to K(U_n(q))$, where, we recall,

$$K(U_n(q)) = \bigcap_{k=0}^{\infty} \sigma^{-k}(U_n^c(q)),$$

and we denote

$$K_n(q) := \pi\big(K(U_n(q))\big).$$

Note that all sets $K(U_n(q))$ are compact. A straightforward elementary calculation shows that if $f, g \in \mathcal{B}_\theta$, then

$$||fg||_\theta \leq 3||f||_\theta||g||_\theta. \tag{10.4}$$

Hence in particular $fg \in \mathcal{B}_\theta$. This allows us to define a linear functional $\mu_{n,q} : \mathcal{B}_\theta \to \mathbb{R}$ by the requirement that

$$Q_{n,q}(gg_{n,q}) = \mu_{n,q}(g)g_{n,q}.$$

Since, by Lemma 10.5, $Q_{n,q}$ is a positive ($Q_{n,q}(\mathcal{B}_\theta^+(\mathbb{R})) \subseteq \mathcal{B}_\theta^+(\mathbb{R})$) operator and $Q_{n,q} \neq 0$ all $q \geq n$ large enough, it follows that $\mu_{n,q}$ is a positive ($\mu_{n,q}(\mathcal{B}_\theta^+(\mathbb{R})) \subseteq [0,+\infty)$) functional and

$$\mu_{n,q}(1\!\!1) = 1. \tag{10.5}$$

Positivity of $\mu_{n,q}$ immediately implies its monotonicity in the sense that if $f, g \in \mathcal{B}_\theta$ and $f(x) \le g(x)$ μ_φ-a.e. in E_A^∞, then

$$\mu_{n,q}(f) \le \mu_{n,q}(g). \tag{10.6}$$

Now, let $C_b^u(E_A^\infty)$ be the vector subspace of $C_b(E_A^\infty)$ consisting of all functions that are uniformly continuous with respect to the metric d_θ. Let us define a function $\hat{\mu}_{n,q} : C_b^u(E_A^\infty) \to [0, +\infty)$ by the following formula:

$$\hat{\mu}_{n,q}(g) := \sup\big\{\mu_{n,q}(f) : f \le g \ \text{ and } \ f \in \mathrm{H}_\theta^b(A)\big\}. \tag{10.7}$$

Of course by (10.6) we get that

$$\mu|_{\mathrm{H}_\theta^b(A)} = \mu_{n,q}|_{\mathrm{H}_\theta^b(A)}. \tag{10.8}$$

Given $g \in C_b^u(E_A^\infty)$ and $k \ge 1$ define two functions

$$\underline{g}_k(\omega) := \inf\{g(\tau) : \tau \in [\omega|k]\} \ \text{ and } \ \overline{g}_k(\omega) := \sup\{g(\tau) : \tau \in [\omega|k]\}.$$

Of course

$$\underline{g}_k \le g \le \overline{g}_k$$

and

$$\lim_{k\to\infty} ||g - \underline{g}_k||_\infty = \lim_{k\to\infty} ||g - \overline{g}_k||_\infty = 0. \tag{10.9}$$

We shall prove that for every $g \in C_b^u(E_A^\infty)$ we have that

$$\hat{\mu}_{n,q}(g) = \overline{\mu}_{n,q}(g) := \inf\big\{\mu_{n,q}(f) : f \ge g \ \text{ and } \ f \in \mathrm{H}_\theta^b(A)\big\}. \tag{10.10}$$

Then for every $k \ge 1$ we have that

$$\begin{aligned}\hat{\mu}_{n,q}(g) &\le \mu_{n,q}(\overline{g}_k) = \mu_{n,q}\big(\underline{g}_k + (\overline{g}_k - \underline{g}_k)\big) = \mu_{n,q}(\underline{g}_k) + \mu_{n,q}(\overline{g}_k - \underline{g}_k) \\ &\le \hat{\mu}_{n,q}(g) + \mu(||\overline{g}_k - \underline{g}_k||_\infty) \\ &= \hat{\mu}_{n,q}(g) + ||\overline{g}_k - \underline{g}_k||_\infty,\end{aligned}$$

and invoking (10.9), we obtain $\hat{\mu}_{n,q}(g) \le \overline{\mu}_{n,q}(g) \le \hat{\mu}_{n,q}(g)$, completing the proof of (10.10). We now prove the next axiliary fact.

Lemma 10.7. *The function $\hat{\mu}_{n,q} : C_b^u(E_A^\infty) \to \mathbb{R}$ is a positive linear functional such that $\hat{\mu}_{n,q}(1\!\!1) = 1$ and $\hat{\mu}_{n,q}|_{\mathrm{H}_\theta^b(A)} = \mu_{n,q}|_{\mathrm{H}_\theta^b(A)}$.*

Proof. Positivity is immediate from formula (10.7). It is also immediate from this formula that

$$\hat{\mu}_{n,q}(\alpha g) = \alpha\hat{\mu}_{n,q}(g) \tag{10.11}$$

for every $\alpha \ge 0$. Employing also (10.10), we get that

$$\begin{aligned}\hat\mu_{n,q}(-g) &= \inf\big\{\mu_{n,q}(f) : f \ge -g \ \text{ and } \ f \in \mathrm{H}_\theta^b(A)\big\}\\ &= \inf\big\{-\mu_{n,q}(-f) : -f \le g \ \text{ and } \ f \in \mathrm{H}_\theta^b(A)\big\}\\ &= \inf\big\{-\mu_{n,q}(f) : f \le g \ \text{ and } \ f \in \mathrm{H}_\theta^b(A)\big\}\\ &= -\sup\big\{\mu_{n,q}(f) : f \le g \ \text{ and } \ f \in \mathrm{H}_\theta^b(A)\big\}\\ &= -\hat\mu_{n,q}(g).\end{aligned}$$

Along with (10.12) this implies that

$$\hat\mu_{n,q}(\alpha g) = \alpha\hat\mu_{n,q}(g) \tag{10.12}$$

for every $g \in C_b^u(E_A^\infty)$ and all $\alpha \in \mathbb{R}$. Now fix two functions $f, g \in C_b^u(E_A^\infty)$. Because of (10.10) and (10.7) there exist four sequences $(f_k^-)_1^\infty$, $(f_k^+)_1^\infty$, $(g_k^-)_1^\infty$, and $(g_k^+)_1^\infty$ of elements of $\mathrm{H}_\theta^b(A)$ such that

$$f_k^- \le f \le f_k^+, \quad g_k^- \le g \le g_k^+,$$

and

$$\lim_{k\to\infty}\mu_{n,q}(f_k^-) = \lim_{k\to\infty}\mu_{n,q}(f_k^+) = \hat\mu_{n,q}(f) \quad \text{and} \quad \lim_{k\to\infty}\mu_{n,q}(g_k^-) = \lim_{k\to\infty}\mu_{n,q}(g_k^+) = \hat\mu_{n,q}(g).$$

Therefore, applying again (10.10) and (10.7), we obtain

$$\hat\mu_{n,q}(f+g) \ge \varlimsup_{k\to\infty}\mu_{n,q}(f_k^- + g_k^-) = \lim_{k\to\infty}\mu_{n,q}(f_k^-) + \lim_{k\to\infty}\mu_{n,q}(g_k^-) = \hat\mu_{n,q}(f) + \hat\mu_{n,q}(g)$$

and

$$\hat\mu_{n,q}(f+g) \le \varliminf_{k\to\infty}\mu_{n,q}(f_k^+ + g_k^+) = \lim_{k\to\infty}\mu_{n,q}(f_k^+) + \lim_{k\to\infty}\mu_{n,q}(g_k^+) = \hat\mu_{n,q}(f) + \hat\mu_{n,q}(g).$$

Hence,

$$\hat\mu_{n,q}(f+g) = \hat\mu_{n,q}(f) + \hat\mu_{n,q}(g),$$

and along with (10.12) this finishes the proof of Lemma 10.7 (the last two assertions of this lemma are immediate consequences of (10.5) and (10.8). □

Now we shall prove the following auxiliary fact.

Lemma 10.8. *If $g \in C_b^u(E_A^\infty)$ and $g|_{\tilde K_{n,q}} = 0$, then $\hat\mu_{n,q}(g) = 0$.*

Proof. Let

$$\mathcal{F}_{n,q} = \big\{\omega \in E_A^n : [\omega] \subseteq U_{n,q}^c\big\},$$

and note that $\mathcal{F}_{n,q}$ is a finite set. For every $k \ge 1$ let

$$U_{n,q}^{ck} := \bigcap_{j=0}^{k-1}\sigma^{-j}(U_n^c).$$

We shall prove the following.

Claim 1: There exists $p \ge 1$ such that if $\omega \in E_A^{kn}$ and $[\omega] \subseteq U_{n,q}^{ck}$, then $[\omega|_{kn-pn}] \cap K(U_n(q)) \ne \emptyset$.

Proof. Let

$$p := \#\mathcal{F}_{n,q} + 1 < +\infty. \tag{10.13}$$

Seeking a contradiction suppose that $k > p$ and

$$[\omega|_{(k-p)n}] \cap K(U_n(q)) = \emptyset \tag{10.14}$$

for some $\omega \in E_A^{kn}$ with $[\omega] \subseteq U_{n,q}^{ck}$. Because $|\omega|_{(k-p)n+1}^{kn}| = pn$ and because $\omega|_{(k-p)n+1}^{kn}$ is a concatenation of non-overlapping blocks from $\mathcal{F}_{n,q}$, it follows from (10.13) that there are two non-overlapping subblocks of $\omega|_{(k-p)n+1}^{kn}$ forming the same element of $\mathcal{F}_{n,q}$. Let $\omega|_{(l-1)n+1}^{ln}$, $k-p \le l-1 \le k-1$ be the latter of these two blocks, and let the former, denote it by τ, have the last coordinate j ($j \le (l-1)n$). But then the infinite word

$$\omega|_{ln}(\omega|_{j+1}^{ln})^\infty = \omega|_{j-n}(\tau\omega|_{j+1}^{(l-1)n+1})^\infty$$

is an element of E_A^∞ and each of its subblocks of length n is a subblock of length n of ω. So, $\omega|_{ln}(\omega|_{j+1}^{ln})^\infty \in K(U_n(q))$. Thus, $[\omega|_{ln}] \cap K(U_n(q)) \neq \emptyset$. As $l \ge k-p$, this contradicts (10.14) and finishes the proof of Claim 1. □

Now passing to the direct proof of our lemma, fix $\varepsilon > 0$ arbitrary. Since $g|_{K(U_n(q))} = 0$ and $g \in C_b^u(E_A^\infty)$, there exists $l \ge 1$ sufficiently large that

$$|g|\big|_{[\omega]} \le \varepsilon/2 \tag{10.15}$$

if $|\omega| \ge l$ ($\omega \in E_A^*$) and $[\omega] \cap K(U_n(q)) \neq \emptyset$. Take any $k \ge l+p$ so large that $||\overline{g}_{kn} - g||_\infty \le \varepsilon/2$. Employing Claim 1, (10.15), Lemma 10.7, and (10.8), we get

$$\begin{aligned}
\hat{\mu}_{n,q}(g)g_{n,q} &\le \hat{\mu}_{n,q}(\overline{g}_{kn})g_{n,q} = \mu_{n,q}(\overline{g}_{kn})g_{n,q} = Q_{n,q}(\overline{g}_{kn}g_{n,q}) = \lambda_{n,q}^{-kn}\mathcal{L}_{n,q}^{kn}Q_{n,q}(\overline{g}_{kn}g_{n,q}) \\
&= \lambda_{n,q}^{-kn}Q_{n,q}\mathcal{L}_{n,q}^{kn}(\overline{g}_{kn}g_{n,q}) \\
&= \lambda_{n,q}^{-kn}Q_{n,q}\Big(\tau \longmapsto \sum_{[\omega]\subseteq U_{n,q}^{ck}:A_{\omega_{kn}\tau_0}=1} \overline{g}_{kn}(\omega\tau)g_{n,q}(\omega\tau)e^{\varphi_{kn}(\omega\tau)}\Big) \\
&\le \lambda_{n,q}^{-kn}Q_{n,q}\Big(\tau \longmapsto \varepsilon \sum_{A_{\omega_{kn}\tau_0}=1} 1\!\!1_n^{kn}(\omega\tau)g_{n,q}(\omega\tau)e^{\varphi_{kn}(\omega\tau)}\Big) \\
&= \varepsilon\lambda_{n,q}^{-kn}Q_{n,q}\mathcal{L}_{n,q}^{kn}(g_{n,q}) = \varepsilon Q_{n,q}(g_{n,q}) \le \varepsilon||g_{n,q}||_\infty Q_{n,q}(1\!\!1) \\
&= \varepsilon||g_{n,q}||_\infty g_{n,q} \le \varepsilon||g_{n,q}||_\theta g_{n,q}.
\end{aligned}$$

Hence,

$$\hat{\mu}_{n,q}(g) \le ||g_{n,q}||_\theta\varepsilon.$$

Likewise, -$\hat{\mu}_{n,q}(g) = \hat{\mu}_{n,q}(-g) \le ||g_{n,q}||_\theta\varepsilon$, and in consequence.

$$|\hat{\mu}_{n,q}(g)| \le ||g_{n,q}||_\theta\varepsilon.$$

Letting $\varepsilon \searrow 0$ we thus get that $\hat{\mu}_{n,q}(g) = 0$ finishing the proof of Lemma 10.8 □

Since every function $g \in C\big(K(U_n(q))\big)$ is uniformly continuous, it extends to some uniformly continuous function $\tilde{g} : E_A^\infty \to \mathbb{R}$. The value

$$\widetilde{\mu}_{n,q}(g) := \hat{\mu}_{n,q}(\tilde{g})$$

is then, by virtue of, Lemma 10.8, independent of the choice of extension $\tilde{g} \in C_b^u(E_A^\infty)$ of g. By Lemma 10.7, we get the following.

Lemma 10.9. *The function $\widetilde{\mu}_{n,q} : C(K(U_n(q))) \to \mathbb{R}$ (also denoted in the sequel just by $\mu_{n,q}$) is a positive linear functional such that $\mu(\mathbb{1}) = 1$. Thus by the Riesz Representation Theorem $\tilde{\mu}$ represents a Borel probability measure on $K(U_n(q))$.*

We shall prove the following.

Lemma 10.10. *The measure $\widetilde{\mu}_{n,q}$ (as indicated above also denoted in the sequel just by $\mu_{n,q}$) on $K(U_n(q))$ is σ-invariant.*

Proof. Let $g \in C\big(K(U_n(q))\big)$. Let $\tilde{g} \in C_b^u(E_A^\infty)$ be an extension of g. Then $\tilde{g} \circ \sigma \in C_b^u(E_A^\infty)$ and it extends $g \circ \sigma$. Fix $\varepsilon > 0$ and take $\tilde{g}_+$ and $\tilde{g}_-$ both in $\mathrm{H}_\theta^b(A)$, such that $\tilde{g}_- \le \tilde{g} \le \tilde{g}_+$ and

$$\mu_{n,q}(\tilde{g}_+) - \varepsilon \le \hat{\mu}_{n,q}(\tilde{g}) \le \mu_{n,q}(\tilde{g}_-) + \varepsilon.$$

Of course then we also have $\tilde{g}_+ \circ \sigma, \tilde{g}_- \circ \sigma \in \mathrm{H}_\theta^b(A)$ and $\tilde{g}_- \circ \sigma \le \tilde{g} \circ \sigma \le \tilde{g}_+ \circ \sigma$. We thus get

$$\begin{aligned}
\tilde{\mu}_{n,q}(g \circ \sigma) g_{n,q} &= \hat{\mu}_{n,q}(\tilde{g} \circ \sigma) g_{n,q} \le \mu_{n,q}(\tilde{g}_+ \circ \sigma) g_{n,q} = Q_{n,q}(g_{n,q} \tilde{g}_+ \circ \sigma) \\
&= \lambda_{n,q}^{-kn} \mathcal{L}_{n,q}^{kn} Q_{n,q}(g_{n,q} \tilde{g}_+ \circ \sigma) \\
&= \lambda_{n,q}^{-kn} Q_{n,q} \mathcal{L}_{n,q}^{kn}(g_{n,q} \tilde{g}_+ \circ \sigma) \\
&= \lambda_{n,q}^{-kn} Q_{n,q}\big(\tilde{g}_+ \mathcal{L}_{n,q}^{kn}(g_{n,q})\big) \\
&= Q_{n,q}(\tilde{g}_+ g_{n,q}) \\
&= \mu_{n,q}(\tilde{g}_+) g_{n,q} \\
&\le (\hat{\mu}_{n,q}(\tilde{g}) + \varepsilon) g_{n,q} = (\widetilde{\mu}_{n,q}(\tilde{g}) + \varepsilon) g_{n,q}.
\end{aligned}$$

Hence, $\tilde{\mu}_{n,q}(g \circ \sigma) \le \widetilde{\mu}_{n,q}(\tilde{g}) + \varepsilon$. By letting $\varepsilon \searrow 0$ this yields $\tilde{\mu}_{n,q}(g \circ \sigma) \le \widetilde{\mu}_{n,q}(\tilde{g})$. Likewise, working with $\tilde{g}_-$ instead of $\tilde{g}_+$, we get $\tilde{\mu}_{n,q}(g \circ \sigma) \ge \hat{\mu}_{n,q}(\tilde{g})$. Thus $\tilde{\mu}_{n,q}(g \circ \sigma) = \widetilde{\mu}_{n,q}(\tilde{g})$ and the proof is complete. ☐

As we have already indicated, our goal now is to prove that the sequence $(\mu_{n,q})_{q=1}^\infty$ converges weakly. For this we bring in the concept of Wasserstein metric. We denote it by d_W and recall that in the setting of the symbol space E_A^∞ it is defined by the following formula:

$$d_W(\nu_1, \nu_2) = \sup\big\{|\nu_2(g) - \nu_1(g)| : g \in \mathrm{H}_\theta^b(A) \text{ and } \mathrm{H}_\theta(g) \le 1\big\},$$

where ν_1 and ν_2 are Borel probability measures on E_A^∞.

The Wasserstein metric d_W induces the weak convergence topology on E_A^∞. We shall prove the following.

Lemma 10.11. *The sequence $(\mu_{n,q})_{q=1}^\infty$ is fundamental (Cauchy) with respect to the Wasserstein metric d_W.*

Proof. Fix $p, q \geq 1$ large enough so that Lemma 10.4 holds. Fix $g \in \mathrm{H}_\theta^b(A)$ with $\|g\|_\theta \leq 1$. Using item (g) of this lemma, Proposition 5.2 and formula (5.10) we get for μ_φ–a.e. $\omega \in E_A^\infty$ (even on the zero sets of $g_{n,q}$ and $g_{n,p}$ since in the formula just above the first equality below, both $g_{n,q}$ and $g_{n,p}$ cancel out) that

$$
\begin{aligned}
|\mu_{n,q}(g) - \mu_{n,p}(g)| &= \left| \frac{Q_{n,q}(gg_{n,q})(\omega)}{g_{n,q(\omega)}} - \frac{Q_{n,p}(gg_{n,p})(\omega)}{g_{n,p(\omega)}} \right| = \\
&= \left| \frac{Q_{n,q}(gg_{n,q})(\omega)g_{n,p}(\omega) - Q_{n,p}(gg_{n,p})(\omega)g_{n,q}(\omega)}{g_{n,q}(\omega)g_{n,p}(\omega)} \right| \\
&= \left| \frac{g_{n,p}(\omega)\big(Q_{n,q}(gg_{n,q})(\omega) - Q_{n,p}(gg_{n,p})(\omega)\big) + Q_{n,p}(gg_{n,p})(\omega)\big(g_{n,p}(\omega) - g_{n,q}(\omega)\big)}{g_{n,q}(\omega)g_{n,p}(\omega)} \right| \\
&\leq \frac{|(Q_{n,q}(gg_{n,q})(\omega) - Q_{n,p}(gg_{n,p})(\omega))|}{g_{n,q}(\omega)} + \frac{|Q_{n,p}(gg_{n,p})(\omega)||g_{n,p}(\omega) - g_{n,q}(\omega)|}{g_{n,q}(\omega)g_{n,p}(\omega)} \\
&\leq \frac{\big|(Q_{n,q}\big(gg_{n,q} - gg_{n,p}\big)(\omega) + (Q_{n,q} - Q_{n,p})(gg_{n,p})(\omega)\big|}{g_{n,q}(\omega)} + \\
&\qquad\qquad + \frac{Q_{n,p}(\|g\|_\infty g_{n,p})(\omega)|}{g_{n,q}(\omega)g_{n,p}(\omega)}|g_{n,p}(\omega) - g_{n,q}(\omega)| \\
&\leq \frac{\|g\|_\infty Q_{n,q}(|g_{n,q} - g_{n,p}|)(\omega)}{g_{n,p}(\omega)} + \frac{|(Q_{n,q} - Q_{n,p})(gg_{n,p})(\omega)|}{g_{n,q}(\omega)} + \\
&\qquad\qquad + \frac{\|g\|_\infty g_{n,p}(\omega)}{g_{n,q}(\omega)g_{n,p}(\omega)}|g_{n,q}(\omega) - g_{n,p}(\omega)| \\
&\leq \frac{Q_{n,q}(|g_{n,q} - g_{n,p}|)(\omega)}{g_{n,q}(\omega)} + \frac{|(Q_{n,q} - Q_{n,p})(gg_{n,p})(\omega)|}{g_{n,q}(\omega)} + \\
&\qquad\qquad + \frac{|g_{n,p}(\omega) - g_{n,q}(\omega)|}{g_{n,q}(\omega)} \\
&\leq g_{n,q}^{-1}(\omega)\big(Q_{n,q}(|g_{n,q} - g_{n,p}|)(\omega)) + |(Q_{n,q} - Q_{n,p})(gg_{n,p})(\omega)| + |(g_{n,p} - g_{n,q})(\omega)|\big).
\end{aligned}
$$

Multiplying both sides of this inequality by $g_{n,q}(\omega)$ we thus get that

(10.16)

$$
\begin{aligned}
&|\mu_{n,q}(g) - \mu_{n,p}(g)|g_{n,q}(\omega) \leq \\
&\qquad \leq Q_{n,q}(|g_{n,q} - g_{n,p}|)(\omega) + |(Q_{n,q} - Q_{n,p})(gg_{n,p})(\omega)| + |(g_{n,p} - g_{n,q})(\omega)| \\
&\qquad \leq Q_{n,q}(|g_{n,q} - g_n|)(\omega) + Q_{n,q}(g_{n,p} - g_n)(\omega)| + |(Q_{n,q} - Q_n)(gg_{n,p})(\omega)| + \\
&\qquad\qquad + |(Q_{n,p} - Q_n)(gg_{n,p})(\omega)| + |g_{n,q} - g_n|(\omega) + |g_{n,p} - g_n|(\omega).
\end{aligned}
$$

Now we integrate both sides of this inequality with respect to the measure μ_φ. We treat each each summand seperately. We start with the last terms. Fix $\varepsilon \in (0, \|g_n\|_{L^1(\mu_\varphi)}/2)$.

Then by Lemma 10.4 (h), we get for all $q \geq 1$ large enough, say $q \geq l_1$, that

$$\begin{aligned} \int |g_{n,q} - g_n| \, d\mu_\varphi = \int |(Q_{n,q} - Q_n)1\!\!1| \, d\mu_\varphi &\leq \|(Q_{n,q} - Q_n)1\!\!1\|^* \\ &\leq |||(Q_{n,q} - Q_n)|||.\|1\!\!1\|_\theta \\ &= |||(Q_{n,q} - Q_n)||| \\ &< \varepsilon. \end{aligned} \tag{10.17}$$

So, also

$$\int |g_{n,p} - g_n| d\mu_\varphi < \varepsilon \tag{10.18}$$

if $p \geq l_1$. Next, by Lemma 10.4 (g), we get

$$\|g_{n,p}\|_\theta = \|Q_{n,p}1\!\!1\|_\theta \leq \|Q_{n,p}\|_\theta \|1\!\!1\|_\theta \leq C \tag{10.19}$$

if $p \geq 1$ is large enough, say $p \geq l_2 \geq l_1$. Hence, by (10.4)we get that

$$\|gg_{n,p}\| \leq 3C.$$

We therefore get for all $q, p \geq l_2$ that

$$\begin{aligned} \int |(Q_{n,q} - Q_n)(gg_{n,p})| \, d\mu_\varphi &\leq \|(Q_{n,q} - Q_n)(gg_{n,p})\|_* \\ &\leq |||(Q_{n,q} - Q_n)||| \cdot \|gg_{n,p}\|_\theta \\ &\leq 3C\varepsilon \end{aligned}$$

and in the same way

$$\int |(Q_{n,p} - Q_n)(gg_{n,p})| \, d\mu_\varphi \leq 3C\epsilon \tag{10.20}$$

Next using (10.18) and Lemma 10.4 (g) we get

$$\begin{aligned} \int Q_{n,q}(|g_{n,p} - g_n|) \, d\mu_\varphi &\leq \|Q_{n,q}\|_{L^1(\mu_\varphi)} \int |g_{n,p} - g| \, d\mu_\varphi \\ &\leq \|Q_{n,q}\|_\theta \int |g_{n,p} - g| \, d\mu_\varphi \\ &\leq C\varepsilon \end{aligned}$$

and in the same way, with (10.18) replaced by (10.17), we get that

$$\int Q_{n,q}(|g_{n,q} - g_n|) \, d\mu_\varphi \leq C\varepsilon. \tag{10.21}$$

As the (almost) last step we get from (10.17) and the choice of $\varepsilon > 0$, that

$$\begin{aligned} \int g_{n,q} d\mu_\varphi &= \int g_n d\mu_\varphi + \int (g_{n,q} - g_n) d\mu_\varphi \\ &> \|g_n\|_{L^1(\mu_\varphi)} - \epsilon \geq \frac{1}{2}\|g\|_{L^1(\mu_\varphi)}. \end{aligned}$$

Therefore, we conclude from the above inequalities, that after integrating both sides of (10.16) we get that

$$|\mu_{n,q}(g)-\mu_{n,p}(g)|\le 2\|g_n\|^{-1}_{L^1(\mu_q)}(8C+2)\varepsilon$$

for all $p,q\ge l_2$. Hence

$$d_W(\mu_{n,q},\mu_{n,p})<\|g_n\|^{-1}_{L^1(\mu_q)}(8C+2)\varepsilon$$

for all $p,q\ge l_2$. The proof is complete. □

Having this lemma, we can easily prove the following.

Proposition 10.12. *For every $n\ge 1$ large enough the sequence $(\mu_{n,q})_{q=1}^{\infty}$ converges weakly. Denoting its limit by μ_n, we have the following:*

(a) $\mu_n(K(U_n))=1$;
(b) $\mu_n\circ\sigma^{-1}=\mu_n$, *i.e., the measure μ_n is shift invariant; and*
(c) $Q_n(g_ng)=\mu_n(g)g_n$ *for every* $g\in \mathrm{H}_\theta^b(A)$.

Proof. Since the Wasserstein metric is complete if the underlying metric space is complete (more precisely, the underlying topological space is completely metrizable) and the space E_A^∞ is completely metrizable, the convergence of the sequence $(\mu_{n,q})_{q=1}^{\infty}$ follows from Lemma 10.11. The item 9a) then follows from the fact that $K(U_n)$ is a closed subset of E_A^∞ and $K(U_n(q))\subset K(U_n)$ for every $q\ge 1$. Item (b) follows from Lemma 10.10 and the fact that μ_n is the weak limit of the measures $\mu_{n,q}$, $q\ge 1$.

In order to prove item (c) first note that by the definition of the measures $\mu_{n,q}$, $q\ge 1$, and by the the very first assertion of the present proposition, we have that for every $g\in \mathrm{H}_\theta^b(A)$ that $\mu_n(g)=\lim_{q\to+\infty}\mu_{n,q}(g)$ and then, after multiplying both sides of this equality by $g_{n,q}$, we get

$$0=\lim_{q\to\infty}(\mu_n(g)g_{n,q}-\mu_{n,q}(g)g_{n,q})=\lim_{q\to\infty}(\mu_n(g)g_{n,q}-Q_{n,q}(gg_{n,q})) \tag{10.22}$$

μ_φ–a.e. But since all the functions involved are uniformly bounded, the formula (10.27) also holds if the limit is understood to be in the space $L^1(\mu_\varphi)$ and because of (10.20) along with (10.21), we have that

$$\lim_{q\to+\infty}Q_{n,q}(gg_{n,q})=Q_n(gg_{n,q})$$

in $L^1(\mu_\varphi)$. In conclusion:

$$Q_n(g_ng)=\mu_n(g)g_n$$

in $L^1(\mu_\varphi)$. The proof of item (c) is thus complete. We are done. □

Now we shall prove the following.

Proposition 10.13. . *If $\varphi\in \mathrm{H}_\theta(A)$ is a summable potential and $\int(-\varphi)d\mu_\varphi<+\infty$, then*

$$\int(-\varphi)d\mu_n<+\infty$$

for all $n\ge 0$ large enough.

Proof. Since the potential $\varphi : E_A^\infty \to \mathbb{R}$ is summable, there exists an integer $l \geq 1$ such that for all $e \geq l$,

$$\varphi|_{[e]} \leq 0.$$

Therefore, by Proposition 5.2 we have for every $e \geq l$ and every $\omega \in E_A^\infty$ that

$$\mathcal{L}(g_n\big(-\varphi)1\!\!1_{[e]}\big)(\omega) \geq \mathcal{L}_n\big(g_n(-\varphi)1\!\!1_{[e]}\big)(\omega) = \lambda_n Q_n\big(g_n(-\varphi)1\!\!1_{[e]}\big)(\omega) + \Delta_n\big(g_n(-\varphi)1\!\!1_{[e]}\big)(\omega).$$

So, by Proposition 5.3,

$$\begin{aligned} Q_n\mathcal{L}\big(g_n(-\varphi)1\!\!1_{[e]}\big)(\omega) &\geq \lambda_n Q_n^2\big(g_n(-\varphi)1\!\!1_{[e]}\big)(\omega) = \lambda_n Q_n\big(g_n(-\varphi)1\!\!1_{[e]}\big)(\omega) \\ &\geq \lambda_n \inf(-\varphi)|_{[e]} Q_n(g_n 1\!\!1_{[e]}), \end{aligned}$$

where writing the last inequality sign of this formula we also used the inequality $(-\varphi)1\!\!1_{[e]} \geq \inf(-\varphi)|_{[e]}1\!\!1_{[e]}$. By Proposition5.2 there exists $p \geq 1$ such that for all $n \geq p$ we have that $\lambda_n \geq 1/2$. Therefore,

$$\inf(-\varphi)|_{[e]} Q_n(g_n 1\!\!1_{[e]})(\omega) \leq 2 Q_n \mathcal{L}(g_n(-\varphi)1\!\!1_{[e]})(\omega).$$

Hence

$$\sum_{e\geq l} \inf(-\varphi)|_{[e]} Q_n(g_n 1\!\!1_{[e]})(\omega) \leq 2 \sum_{e\geq l} Q_n \mathcal{L}(g_n(-\varphi)1\!\!1_{[e]})(\omega).$$

Equivalently,

$$\sum_{e\geq l} \inf(-\varphi)|_{[e]} \mu_n([e]) g_n(\omega) \leq 2 \sum_{e\geq l} Q_n \mathcal{L}(g_n(-\varphi)1\!\!1_{[e]})(\omega) \tag{10.23}$$

Furthermore,

$$2\sum_{e\geq l} \mathcal{L}(g_n(-\varphi)1\!\!1_{[e]})(\omega) \leq \sum_{e\geq l} \sup(-\varphi)|_{[e]} \sup(g_n|_{[e]}) e^{-\inf(\varphi|_{[e]})} \preceq \|g_n\|_\infty \sum_{e\geq l} \sup(-\varphi)|_{[e]} \mu_\varphi([e]).$$

So, by Proposition 5.3 again and by Proposition5.2 (g), we get that

$$\begin{aligned} 2\sum_{e\geq l} Q_n\mathcal{L}(g_n(-\varphi)1\!\!1_{[e]})(\omega) &\preceq \|g_n\|_\infty \sum_{e\geq l} \sup(-\varphi)|_{[e]} \mu_\varphi([e]) Q_n 1\!\!1(\omega) \\ &\leq C\|g_n\|_\infty \sum_{e\geq l} \sup(-\varphi)|_{[e]} \mu_\varphi([e]) \\ &\leq C^2 \sum_{e\geq l} \sup(-\varphi)|_{[e]} \mu_\varphi([e]). \end{aligned}$$

Inserting this to (10.23), gives

$$\sum_{e\geq l} \inf(-\varphi)|_{[e]} \mu_n([e]) g_n(\omega) \preceq \sum_{e\geq l} \sup(-\varphi)|_{[e]} \mu_\varphi([e]). \tag{10.24}$$

Now, recalling that

$$C' := \sup_{e\in E} \big\{ \sup(-\varphi) - \inf(-\varphi) \big\} = \sup_{e\in E} \big\{ \sup(\varphi) - \inf(\varphi) \big\} < +\infty \tag{10.25}$$

and that $\|g_n\|_\infty \le C$ for all $n \ge 1$ large enough, because of (10.24), we get that

$$\begin{aligned}\sum_{e\ge l}\inf(-\varphi)|_{[e]}\mu_n([e])g_n(\omega) &\preceq \sum_{e\ge l}\inf(-\varphi)|_{[e]}\mu_\varphi([e]) + \sum_{e\ge l}\mu_\varphi([e]) \\ &\le \int_{[l,+\infty)}(-\varphi)d\mu_\varphi.\end{aligned} \tag{10.26}$$

Now fix $\omega \in E_A^\infty$ such that $g_n(\omega) > 0$. We then and obtain from (10.25) and (10.26) that

$$\begin{aligned}\int_{[l,+\infty)}(-\varphi)d\mu_n &\le \sum_{e\ge l}\sup(-\varphi)|_{[e]}\mu_n(1\!\!1_{[e]}) \le \sum_{e\ge l}(C' + \inf(-\varphi)|_{[e]})\mu_n(1\!\!1_{[e]}) \\ &\le C' + \sum_{e\ge l}\inf(-\varphi)|_{[e]})\mu_n([e]) \\ &\preceq g_n^{-1}(\omega)\int_{[l,+\infty)}(-\varphi)d\mu + C'.\end{aligned}$$

Therefore,

$$\int_{E_A^\infty}(-\varphi)d\mu_n = \int_{[1,l-1]}(-\varphi)d\mu_n + \int_{[l,+\infty]}(-\varphi)d\mu_n < +\infty,$$

and the proof is finished. ∎

Proposition 10.14. *Assume* $\int(-\varphi)d\mu_\varphi < +\infty$. *If* $n \ge 1$ *is large enough and* μ_n *is the measure produced in Proposition* 10.12, *then* μ_n *is the surviving equilibrium state for* U_n, *i.e.,* $\int(-\varphi)d\mu_n < +\infty$ *and*

$$\mathrm{h}_{\mu_n}(\sigma) + \int_{K(U_n)}\varphi\, d\mu_n = \log\lambda_n \tag{10.27}$$

Proof. That $\int(-\varphi)d\mu_n < +\infty$ was proved in Proposition 10.13. Because of (9.2) we are now only left to show that

$$\mathrm{h}_{\mu_n}(\sigma) + \int_{K(U_n)}\varphi\, d\mu_n \ge \log\lambda_n. \tag{10.28}$$

Indeed, if $\tau \in E_A^k$, $k \ge 1$, then

$$\begin{aligned}\mu_n([\tau])g_n &= Q_n(g_n 1\!\!1_{[\tau]}) = Q_nQ_n(g_n 1\!\!1_{[\tau]}) \\ &= \lambda_n^{-k}(\mathcal{L}_n^k\big(Q_n(g_n 1\!\!1_{[\tau]})) - \Delta_n^k(Q_n(g_n 1\!\!1_{[\tau]}))\big) \\ &= \lambda_n^{-k}\mathcal{L}_n^k\big(Q_n(g_n 1\!\!1_{[\tau]})\big) \\ &= \lambda_n^{-k}Q_n\mathcal{L}_n^k(g_n 1\!\!1_{[\tau]}).\end{aligned}$$

Now,

$$\mathcal{L}_n^k(g_n 1\!\!1_{[\tau]}) \le \|g_n\|_\infty\mathcal{L}_n^k(1\!\!1_{[\tau]}) \le \|g_n\|_\infty \exp\big(-\inf(\varphi_k|_{[\tau]})\big).$$

Therefore, invoking Proposition 5.3 we get that

$$Q_n\mathcal{L}_n^k(g_n 1\!\!1_{[\tau]}) \le \|g_n\|_\infty \exp\big(-\inf(\varphi_k|_{[\tau]})\big)Q_n(1\!\!1) = \|g_n\|_\infty \exp\big(-\inf(\varphi_k|_{[\tau]})\big)g_n.$$

Hence,

$$\mu_n([\tau]) \le \|g_n\|_\infty \lambda_n^{-k} \exp\big(-\inf(\varphi_k|_{[\tau]})\big). \tag{10.29}$$

So, using also the Bounded Distortion Property for φ_k we get

$$\begin{aligned}
-\log\mu([\tau]) &\ge k\log\lambda_n + \inf\big(\varphi_k|_{[\tau]}\big) - \log\|g_n\|_\infty \\
&\ge k\log\lambda_n + \sup\big(\varphi_k|_{[\tau]}\big) - \log\|g_n\|_\infty - C
\end{aligned}$$

with some constant $C > 0$. Therefore

$$\begin{aligned}
\mathrm{H}_{\mu_n}(\alpha^k) &= \sum_{\tau\in E_A^k} -\mu_n([\tau])\log\mu_n([\tau]) \\
&\ge k\log\lambda_n + \sum_{\tau\in E_A^k} \mu_n([\tau])\sup(\varphi_k|_{[\tau]}) - \log\|g_n\|_\infty - C \\
&\ge k\log\lambda_n + \int \varphi_k d\mu_n - \log\|g_n\|_\infty - C \\
&= k\log\lambda_n + k\int \varphi d\mu_n - \log\|g_n\|_\infty - C.
\end{aligned}$$

Hence

$$\mathrm{H}_{\mu_n}(\sigma) = \mathrm{H}_{\mu_n}(\sigma,\alpha) = \lim_{k\to+\infty}\frac{1}{k}\mathrm{H}_{\mu_n}(\alpha^k) \ge \log\lambda_n + \int\varphi d\mu_n,$$

and the proof of (10.28) is complete. Simultaneously, the proof of Proposition 10.14 is complete. □

We would like to end this section with showing how stochastically sound are the measures μ_n. Indeed, it directly follows from Proposition 5.2 and Proposition 10.12 that conditions (2.1), (2.2), and (I) of Gouëzel, from [21] are all satisfied for the measure–preserving dynamical systems $(\sigma : K(U_n) \to K(U_n), \mu_n)$, and therefore Theorem 2.1 from [21] (comp. also [1] for a more dynamical setting) applies to give the following.

Theorem 10.15. *Suppose that $(U_n)_{n=0}^\infty$ is a sequence of open subsets of E_A^∞ satisfying conditions (U0)–(U5). Let $d \ge 1$ be an integer. Fix an integer $n \ge 0$ so large as required in Proposition 5.2. Let $g : K(U_n) \to \mathbb{R}^d$ be a bounded Hölder continuous function. Then there exists a matrix $\Sigma^2 : \{1, 2, \dots, d\}^2 \to \mathbb{R}^d$ such that the process*

$$\Big(g\circ\sigma^k - \int_{K(U_n)} g\, d\mu_n\Big)_{k=1}^\infty$$

satisfies an almost sure invariance principle with the limiting covariance Σ^2. In particular, the sequence

$$\Big(\sum_{j=0}^{k-1} g\circ\sigma^j - k\int_{K(U_n)} g\, d\mu_n\Big)_{k=1}^\infty$$

converges in distribution to the Gaussian (normal) distribution $\mathcal{N}(0,\sigma^2)$. In addition, if $d=1$ then the Law of Iterated Logarithm holds in the form that for μ_n–a.e. $\omega\in K(U_n)$, we have that

$$\limsup_{k\to+\infty}\frac{\sum_{j=0}^{k-1} g\circ\sigma^j(\omega)-k\int_{K(U_n)} g\,d\mu_n}{\sqrt{k\log\log k}}=\sqrt{2\pi}\sigma,$$

where $\sigma^2:=\Sigma^2$ is a non-negative number. It is positive if an only if the function $g: K(U_n)\to\mathbb{R}$ is not cohomologous to a constant in $L^2(\mu_n)$.

As the last stochastic law following from Proposition 5.2 and Proposition 10.12 in a standard way, we record the following exponential decay of correlations.

Theorem 10.16. *Suppose that $(U_n)_{n=0}^\infty$ is a sequence of open subsets of E_A^∞ satisfying conditions (U0)–(U5). Fix an integer $n\ge 0$ so large as required in Proposition 5.2. Then there exist $\kappa\in(0,1)$ and $C\in(0,+\infty)$ such that if $g: K(U_n)\to\mathbb{R}$ is a bounded Hölder continuous function and $h\in L^1(\mu_n)$, then*

$$\Big|\int_{K(U_n)}(g\circ\sigma^k\cdot h)\,d\mu_n-\int_{K(U_n)} g\,d\mu_n\int_{K(U_n)} h\,d\mu_n\Big|\le C\kappa^n\|g\|_\theta\int_{K(U_n)}|f|\,d\mu_n$$

for every integer $k\ge 0$.

11. Symbol Escape Rates of U_ns, III: The Variational Principle and Equilibrium States on the Survivor Sets $K(U_n)$; Uniqueness

The ultimate goal of the last two sections and the current one is to prove the following.

Theorem 11.1. *Assume that $\int(-\varphi)\,d\mu_\varphi<+\infty$. If $(U_n)_{n=0}^\infty$ is a sequence of open subsets of E_A^∞ satisfying conditions (U0)–(U5) then*

$$\sup\Big\{\mathrm{h}_\nu(\sigma)+\int_{K(U_n)}\varphi\,d\nu:\nu\in\mathcal{M}_n^+(\sigma)\Big\}=\sup\Big\{\mathrm{h}_\nu(\sigma)+\int_{K(U_n)}\varphi\,d\nu:\nu\in\mathcal{M}_n^{+e}(\sigma)\Big\}=\log\lambda_n$$

for all $n\ge 1$ large enough.

Moreover, μ_n is a unique (surviving) equilibrium state on the survivor set $K(U_n)$, i.e., the unique (ergodic) σ-invariant Borel probability measure on $K(U_n)$ for which

$$\mathrm{h}_{\mu_n}(\sigma)+\int_{K(U_n)}\varphi\,d\mu_n=\log\lambda_n.$$

We first shall prove the following.

Proposition 11.2. *If $n\ge 1$ is large enough, then the shift invariant measure μ_n on $K(U_n)$ is ergodic.*

Proof. We first shall prove a weak version of ergodicity. Precisely,

Claim 1^0. If $g,h\in \mathrm{H}_\theta^b(A)$ then

$$\lim_{k\to+\infty}\mu_n(g\circ\sigma^k h)=\mu_n(g)\mu_n(h)$$

In particular,

$$\lim_{k\to+\infty}\mu_n\left(g\frac{1}{k}S_kh\right)=\mu_n(g)\mu_n(h).$$

Proof. We have

$$\begin{aligned}\mu_n(g\circ\sigma^kh)g_n &= Q_n(g\circ\sigma^khg_n)=Q_nQ_n(g\circ\sigma^khg_n)\\ &= Q_n\big(\lambda_n^{-k}\mathcal{L}_n^k(g\circ\sigma^khg_n)-\lambda_n^{-k}\Delta_n^k(g\circ\sigma^khg_n)\big)\\ &= \lambda_n^{-k}Q_n\mathcal{L}_n^k(g\circ\sigma^khg_n)\\ &= \lambda_n^{-k}Q_n(g\mathcal{L}_n^k(hg_n))\\ &= \lambda_n^{-k}Q_n\big(g(\lambda_n^kQ_n(hg_n)+\Delta_n^k(hg_n))\big) \qquad (11.1)\\ &= Q_n\big(gQ_n(hg_n)\big)+\lambda_n^{-k}Q_n\big(g\Delta_n^k(hg_n)\big)\\ &= Q_n(g\mu_n(h)g_n)+\lambda_n^{-k}Q_n\big(g\Delta_n^k(hg_n)\big)\\ &= \mu_n(h)Q_n(gg_n)+\lambda_n^{-k}Q_n\big(g\Delta_n^k(hg_n))\big)\\ &= \mu_n(h)\mu_n(g)g_n+\lambda_n^{-k}Q_n\big(g\Delta_n^k(hg_n))\big)\end{aligned}$$

Now, because of Proposition 5.2 (g) and (e) and (10.4), we have that

$$\begin{aligned}\|\lambda_n^{-k}Q_n\big(g\Delta_n^k(hg_n)\big)\|_\infty &\le \|\lambda_n^{-k}Q_n\big(g\Delta_n^k(hg_n)\big)\|_\theta\le\lambda_n^{-k}C\|g\Delta_n^k(hg_n)\|_\theta\\ &\le 3C\|g\|_\theta\lambda_n^{-k}\|\Delta_n^k(hg_n))\|_\theta\\ &\le 3C\|g\|_\theta\|hg_n\|_\theta\kappa^k\lambda_n^{-k}.\end{aligned}$$

Therefore, knowing that $\kappa<1$ and invoking Proposition 5.2 (f) we conclude that for all $n\ge 1$ large enough $\lim_{n\to+\infty}\|\lambda_n^{-k}Q_n\big(g\Delta_n^k(hg_n)\big)\|_\infty=0$. Inserting this into (11.1) we get that

$$\lim_{k\to+\infty}\mu_n(g\circ\sigma^kh)=\mu_n(g)\mu_n(h)$$

and the proof of Claim 1^0 is now complete. □

Now, we shall prove the following.

Claim 2^0. The vector space $\mathrm{H}_\theta^b(A)$ is dense in $L^\infty_{\mu_n}(E_A^\infty)$ with respect to the $L^1(\mu_n)$-norm on $L^\infty_{\mu_n}(E_A^\infty)$.

Proof. Let $h:E_A^\infty\to\mathbb{R}$ be a Borel bounded function. Since E_A^∞ is a completely metrizable topological space, for every $\varepsilon>0$ there exists a compact set $K_\varepsilon\subset E_A^\infty$ such that

$$\mu_n(E_A^\infty\backslash K_\varepsilon)\le\frac{\varepsilon}{6M},$$

where $M:=\max\{1,\|h\|_\infty\}$ and $h|_{K_\varepsilon}$ is continuous. Since K_ε is compact, $h|_{K_\varepsilon}$ is thus uniformly continuous. Let $q\ge 1$ be so large that

$$\sup\big(h|_{K_\varepsilon\cap[\omega]}\big)-\inf(h|_{K_\varepsilon\cap[\omega]}\big)<\frac{\varepsilon}{6M}$$

for every $\omega \in E_A^q$. Define the function $\widehat{h}_\epsilon : E_A^\infty \to [0, +\infty)$ as follows:

$$\widehat{h}_\epsilon(\omega) = \begin{cases} \sup\left(h|_{K_\varepsilon \cap [\omega|q]}\right) & \text{if } K_\varepsilon \cap [\omega|q] \neq \emptyset \\ 0 & \text{if } K_\varepsilon \cap [\omega|q] = \emptyset \end{cases}$$

Then $\widehat{h}_\varepsilon \in \mathrm{H}_\theta(E_A^\infty)$, $\|\widehat{h}_\varepsilon\|_\infty \leq 2M$, $|\widehat{h}_\varepsilon - h| \leq \frac{\varepsilon}{6M}$ on K_ϵ and

$$\begin{aligned} \|\widehat{h}_\epsilon - h\|_{L^1(\mu_n)} &= \int_{E_A^\infty} |\widehat{h}_\varepsilon - h| d\mu_n + \int_{K_\varepsilon} |\widehat{h}_\varepsilon - h| d\mu_n = \int_{E_A^\infty \setminus K_\varepsilon} |\widehat{h}_\varepsilon - h| d\mu_n \\ &\leq \frac{\varepsilon}{2M} \mu_n(K_\varepsilon) + 3M\mu_n(E_A^\infty \setminus K_\varepsilon) \\ &\leq \frac{\varepsilon}{2} + 3M\frac{\varepsilon}{6M} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The proof of Claim 2^0 is thus complete. □

Claim 3^0**.** If $g \in \mathrm{H}_\theta^b(A)$ then for all $n \geq 1$ large enough,

$$\lim_{k \to +\infty} \frac{1}{k} \sum_{j=0}^{k-1} g \circ \sigma^j(\omega) = \mu_n(g)$$

for μ_n–a.e. $\omega \in E_A^\infty$ (or $K(U_n)$ equivalently).

Proof. By Birkhoff's Ergodic Theorem there exists $\widetilde{g} \in L_{\mu_n}^\infty(E_A^\infty)$ such that $\widetilde{g} \circ \sigma = \widetilde{g}$ and

$$\lim_{k \to +\infty} \frac{1}{k} \sum_{j=0}^{k-1} g \circ \sigma^j(\omega) = \widetilde{g}(\omega) \tag{11.2}$$

for μ_n–a.e. $\omega \in E_A^\infty$. By subtracting $\mu_n(g)$ from g we may assume without loss of generality that

$$\mu_n(g) = 0 \tag{11.3}$$

Assume for a contraction that $\widetilde{g} \neq 0$ in $L_{\mu_n}^\infty(E_A^\infty)$. Then there exist $\varepsilon > 0$ and a compact set $L_\varepsilon \subset E_A^\infty$ such that

$$\widetilde{g}(\omega) \geq \varepsilon \ \text{ and } \ \mu_n(L_\epsilon) \geq \varepsilon$$

for every $\omega \in L_\varepsilon$. By Claim 2^0 there exists $h \in \mathrm{H}_\theta^b(A)$ such that

$$\int_{E_A^\infty} |h - \mathbb{1}_{L_\varepsilon}| d\mu_n < \frac{1}{2}\varepsilon^2.$$

Then

$$\int h\widetilde{g} d\,\mu_n = \int \mathbb{1}_{L_\varepsilon} \widetilde{g}\, d\mu_n + \int (h - \mathbb{1}_{L_\varepsilon}) \widetilde{g}\, d\mu_n \geq \varepsilon^2 - \frac{1}{2}\varepsilon^2 = \frac{1}{2}\varepsilon^2 > 0.$$

Inserting this into (11.2) we thus get that

$$\lim_{k\to+\infty} \mu_n\left(\frac{1}{k}\sum_{j=0}^{k-1} g\circ\sigma^j h\right) = \mu_n(h\widetilde{g}) = \frac{\varepsilon^2}{2} > 0.$$

On the other hand, it follows from Claim 1 and (11.3) that

$$\lim_{k\to+\infty} \mu_n\left(\frac{1}{k}\sum_{j=0}^{k-1} g\circ\sigma^j h\right) = \mu_n(g)\mu_n(h) = 0.\cdot\mu_n(h) = 0.$$

This contradiction finishes the proof of Claim 3^0. □

Claim 4^0**.** If $g \in L^\infty_{\mu_n}(E_A^\infty)$ then

$$\lim_{k\to+\infty} \mu_n\left(\left|\frac{1}{k}\sum_{j=0}^{k-1} g\circ\sigma^j - \mu_n(g)\right|\right) = 0$$

for μ_n–a.e. $\omega \in E_A^\infty$ (or equivalently on $K(U_n)$).

Proof. We can assume without loss of generality that $\mu_n(g) = 0$. Fix $\varepsilon > 0$. By Claim 2^0 there exists $g_\varepsilon \in \mathrm{H}^b_\theta(A)$ such that

$$\int_{E_A^\infty} |g_\varepsilon - g|d\mu_n < \varepsilon/2.$$

Then

$$|\mu_n(g_\varepsilon)| < \varepsilon/2.$$

It therefore follows from Claim 3^0 and Lebesgue Dominated Convergence Theorem that

$$\begin{aligned}
\overline{\lim_{k\to+\infty}}\, \mu_n\left(\left|\frac{1}{k}\sum_{j=0}^{k-1} g\circ\sigma^j\right|\right) &= \overline{\lim_{k\to+\infty}}\, \mu_n\left(\left|\frac{1}{k}\sum_{j=0}^{k-1} g_\varepsilon\circ\sigma^j + \frac{1}{k}\sum_{j=0}^{k-1}(g-g_\varepsilon)\circ\sigma^j\right|\right) \\
&\le \overline{\lim_{k\to+\infty}}\, \mu_n\left(\left|\frac{1}{k}\sum_{j=0}^{k-1} g_\varepsilon\circ\sigma^j\right|\right) + \overline{\lim_{k\to+\infty}}\, \mu_n\left(\left|\frac{1}{k}\sum_{j=0}^{k-1}(g-g_\varepsilon)\circ\sigma^j\right|\right) \\
&= \mu_n(|\mu_n(g_\varepsilon)|) + \overline{\lim_{k\to+\infty}}\, \frac{1}{k}\sum_{j=0}^{k-1}\mu_n(|g-g_n|) \\
&\le \mu_n(|g_\varepsilon|) + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Hence

$$\lim_{k\to+\infty} \mu_n\left(\left|\frac{1}{k}\sum_{j=0}^{k-1} g\circ\sigma^j\right|\right) = 0.$$

This completes the proof of Claim 4^0. □

With Claim 4^0 having been proved, the proof of Proposition 11.2 is complete. □

We know from (9.2) and from Proposition 10.14 that for all $n \geq 1$ large enough μ_n is a surviving equilibrium state on $K(U_n)$. Thus, in order to conclude the proof of Theorem 11.1, we are only left to establish the following.

Proposition 11.3. *For all $n \geq 1$ large enough ν_n is a unique surviving equilibrium state on $K(U_n)$.*

Proof. In order to prove this proposition we follow the reasoning taken from the proof of Theorem 1 in [16]. So, suppose that $\nu \neq \mu_n$ is a surviving conditional equilibrium state for the potential $\varphi : E_A^\infty \to \mathbb{R}$ on $K(U_n)$. Applying the ergodic decomposition theorem, we may assume that ν is ergodic. Then, using (10.29) and the Bounded Distortion Property for φ_l, we get, with an appropriate constant $C \in (0, +\infty 0$ for every integer $l \geq 1$ that:

$$\begin{aligned}
0 &= l\Big(\mathrm{h}_\nu(\sigma) + \int_{K(U_n)} (\varphi - \log\lambda_n)\, d\nu\Big) \leq \mathrm{H}_\nu(\alpha^l) + \int_{K(U_n)} (\varphi_l - \log\lambda_n l)\, d\nu \\
&= -\sum_{|\omega|=l} \nu([\omega]) \left(\log\nu([\omega]) - \frac{1}{\nu([\omega])}\int_{[\omega]} (\varphi_l - \log\lambda_n l)\, d\nu\right) \\
&\leq -\sum_{|\omega|=l} \nu([\omega]) \big(\log\nu([\omega]) - (\varphi_l(\tau_\omega) - \log\lambda_n l)\big) \text{ for a suitable } \tau_\omega \in [\omega] \cap K(U_n) \\
&= -\sum_{|\omega|=l} \nu([\omega]) \Big(\log\big(\nu([\omega]) \exp\big(\log\lambda_n l - \varphi_l(\tau_\omega)\big)\big)\Big) \\
&\leq -\sum_{|\omega|=l} \nu([\omega]) \Big(\log\big(\nu([\omega])(\mu_n([\omega])C)^{-1}\big)\Big) \\
&= \log C - \sum_{|\omega|=l} \nu([\omega]) \log\left(\frac{\nu([\omega])}{\mu_n([\omega])}\right).
\end{aligned}$$

Therefore, in order to conclude the proof, it suffices to show that

$$\lim_{l\to\infty}\left(-\sum_{|\omega|=l} \nu([\omega]) \log\left(\frac{\nu([\omega])}{\mu_n([\omega])}\right)\right) = -\infty.$$

Since both measures ν and μ_n are ergodic, the former by assumption, the latter by Proposition 11.2, and $\nu \neq \mu_n$, the measures ν and μ_n are therefore mutually singular. In particular,

$$\lim_{l\to\infty} \nu\left(\left\{\omega \in K(U_n) : \frac{\nu([\omega|_l])}{\mu_n([\omega|_l])} \leq S\right\}\right) = 0$$

for every $S > 0$. For every $j \in \mathbb{Z}$ and every $l \geq 1$, set

$$F_{l,j} = \left\{\omega \in K(U_n) : e^{-j} \leq \frac{\nu([\omega|_l])}{\mu_n([\omega|_l])} < e^{-j+1}\right\}.$$

Then

$$\nu(F_{l,j}) = \int_{F_{l,j}} \frac{\nu([\omega|_l])}{\mu_n([\omega|_l])} d\mu_n(\omega) \le e^{-j+1}\mu_n(F_{l,j}) \le e^{-j+1}.$$

Notice

$$-\sum_{|\omega|=l} \nu([\omega]) \log\left(\frac{\nu([\omega|_l])}{\mu_n([\omega|_l])}\right) = -\int \log\left(\frac{\nu([\omega|_l])}{\mu_n([\omega|_l])}\right) d\nu(\omega) \le \sum_{j\in\mathbb{Z}} j\widetilde{\nu}(F_{l,j}).$$

Now, for each $k = -1, -2, -3, \ldots$ we have

$$\begin{aligned}-\sum_{|\omega|=l} \nu([\omega]) \log\left(\frac{\nu([\omega|_l])}{\mu_n([\omega|_l])}\right) &\le k\sum_{j\le k} \widetilde{\nu}(F_{l,j}) + \sum_{j\ge 1} je^{-j+1} \\ &= k\nu\left(\left\{\omega \in K(U_n) : \frac{\nu([\omega|_l])}{\mu_n([\omega|_l])} \ge e^{-k}\right\}\right) + \sum_{j\ge 1} je^{-j+1}.\end{aligned}$$

Thus, we have for each negative integer k,

$$\limsup_{n\to\infty}\left(-\sum_{|\omega|=n} \nu([\omega]) \log\left(\frac{\nu([\omega])}{\mu_n([\omega])}\right)\right) \le k + \sum_{j\ge 1} je^{-j+1}.$$

So, letting k go to $-\infty$ completes the proof. □

Combining formula (9.2), Proposition 10.14, and Proposition 11.3, we conclude that the proof of Theorem 11.1 is complete.

Part 3. **Escape Rates for Conformal GDMSs and IFSs**

Our approach to proving results on escape rates for conformal graph directed Markov systems and conformal iterated function systems is based on the symbolic dynamics, more precisely, the symbolic thermodynamic formalism, developed in the preceding sections.

12. Preliminaries on Conformal GDMSs

A Graph Directed Markov System (abbr. GDMS) consists of a directed multigraph and an associated incidence matrix, (V, E, i, t, A). As earlier A is the incidence matrix, i. e.

$$A : E \times E \to \{0, 1\}$$

The multigraph consists of a finite set V of vertices and a countable (either finite or infinite) set of directed edges E and two functions $i, t : E \to V$ together with a set of nonempty compact metric spaces $\{X_v\}_{v\in V}$, a number s, $0 < s < 1$, and for every $e \in E$, a 1-to-1 contraction $\varphi_e : X_{t(e)} \to X_{i(e)}$ with a Lipschitz constant $\le s$. For brevity, the set

$$S = \{\varphi_e : X_{t(e)} \to X_{i(e)}\}_{e\in E}$$

is called a Graph Directed Markov System (abbr. GDMS). The main object of interest in this book will be the limit set of the system S and objects associated to this set. We now describe the limit set. For each $\omega \in E_A^*$, say $\omega \in E_A^n$, we consider the map coded by ω:

$$\varphi_\omega = \varphi_{\omega_1} \circ \ldots \circ \varphi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)}.$$

For $\omega \in E_A^\infty$, the sets $\{\varphi_{\omega|_n}(X_{t(\omega_n)})\}_{n\geq 1}$ form a descending sequence of non-empty compact sets and therefore $\bigcap_{n\geq 1} \varphi_{\omega|_n}(X_{t(\omega_n)}) \neq \emptyset$. Since for every $n \geq 1$, $\mathrm{diam}\big(\varphi_{\omega|_n}(X_{t(\omega_n)})\big) \leq s^n \mathrm{diam}\big(X_{t(\omega_n)}\big) \leq s^n \max\{\mathrm{diam}(X_v) : v \in V\}$, we conclude that the intersection

$$\bigcap_{n\geq 1} \varphi_{\omega|_n}\big(X_{t(\omega_n)}\big)$$

is a singleton and we denote its only element by $\pi(\omega)$. In this way we have defined the map

$$\pi : E_A^\infty \longrightarrow X := \bigoplus_{v\in V} X_v$$

from E_A^∞ to $\bigoplus_{v\in V} X_v$, the disjoint union of the compact sets X_v. The set

$$J = J_S = \pi(E_A^\infty)$$

will be called the limit set of the GDMS S.

In order to pass to geometry, we call a GDMS conformal (CGDMS) if the following conditions are satisfied.

(a) For every vertex $v \in V$, X_v is a compact connected subset of a euclidean space $\mathbb{R}^d$ (the dimension d common for all $v \in V$) and $X_v = \overline{\mathrm{Int}(X_v)}$.

(b) For every vertex $v \in V$ there exists an open connected set $W_v \supseteq X_v$ (where $X = \cup_{v\in V} X_v$) such that for every $e \in I$ with $t(e) = v$, the map φ_e extends to a C^1 conformal diffeomorphism of W_v into $W_{i(e)}$.

(c) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\left| \frac{|\varphi_e'(y)|}{|\varphi_e'(x)|} - 1 \right| \leq L||y - x||^\alpha.$$

for every $e \in E$ and every pair of points $x, y \in X_{t(e)}$, where $|\varphi_\omega'(x)|$ means the norm of the derivative.

(d) (Open Set Condition) For all $a, b \in E$, $a \neq b$,

$$\varphi_a(\mathrm{Int}(X_{t(a)})) \cap \varphi_b(\mathrm{Int}(X_{t(b)})) = \emptyset,$$

(e) (Geometric Condition) At least one of the following two conditions hold:

(e1) (Strong Open Set Condition)

$$J_{\mathcal{S}} \cap \mathrm{Int}(X) \neq \emptyset.$$

or

(e2) There exist two numbers $l > 0$ and $\kappa \in (0,1]$ such that for every $x \in \partial X \subseteq \mathbb{R}^d$ and every $r \in (0,l]$,

$$\mathrm{Leb}_d\big(B(x,r) \cap X\big) \geq \kappa \mathrm{Leb}_d\big(B(x,r)\big).$$

Remark 12.1. If $d \geq 2$ and a family $S = \{\varphi_e\}_{e\in E}$ satisfies the conditions (a) and (c), then, due to Koebe's Distortion Theorem in dimension $d = 2$ and the Louisville Representation Theorem (stating that if $d \geq 3$ then each conformal map is necessarily a Möbius transformation) it also satisfies condition (d) with $\alpha = 1$.

Remark 12.2. In the papers [30] and [31] there appeared also the following condition called the Cone Condition:

There exist two numbers $\gamma, l > 0$ such that for every $x \in \partial X \subseteq \mathbb{R}^d$ there exists an open cone $Con(x,\gamma,l) \subseteq \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude l.

This condition was however exclusively needed in [30] and [31] (and essentially all related papers) to have (e2). We will comment more on the Geometric Condition (e) in Remark 12.12 at the end of this section.

We will frequently need to use the concept of incomparable words. We call two words $\omega, \tau \in E^*$ incomparable if none of them is an extension of the other; equivalently

$$[\omega] \cap [\tau] = \emptyset.$$

What concerns geometric applications, we will be dealing throughout the manuscript with projections of equilibrium states μ_φ from the symbol space E_A^∞ to the limit set $J_{\mathcal{S}}$ via the projection map $\pi_{\mathcal{S}} : E_A^\infty \to J_{\mathcal{S}}$. We begin to deal with such projections now. The following theorem was proved in [31], although its formulation there involved the Cone condition rather than (e2).

Theorem 12.3. *If (e2) holds and μ is a Borel shift-invariant ergodic probability measure on E_A^∞, then*

$$\mu \circ \pi^{-1}\big(\varphi_\omega(X_{t(\omega)}) \cap \varphi_\tau\big(X_{t(\tau)}\big)\big) = 0 \tag{12.1}$$

for all incomparable words $\omega, \tau \in E^*$.

This theorem is of particular importance if measure μ is a Gibbs state of a Hölder continuous function. The following slight strengthening of Theorem 12.3 however immediately follows from the Strong Open Set Condition.

Theorem 12.4. *If the Strong Open Set Condition (e1) holds and μ is a Borel shift-invariant ergodic probability measure on E_A^∞ with full topological support, then*

$$\mu \circ \pi^{-1}\big(\varphi_\omega(X_{t(\omega)}) \cap \varphi_\tau\big(X_{t(\tau)}\big)\big) = 0 \tag{12.2}$$

for all incomparable words $\omega, \tau \in E^*$.

Proof. Indeed, the Strong Open Set Condition ensures that for such measures μ

$$\mu(\mathrm{Int}(X)) > 0$$

and, since we clearly have,

$$\sigma^{-1}(\pi^{-1}(\mathrm{Int}(X)) \subseteq \mathrm{Int}(X),$$

we thus conclude from ergodicity of μ that $\mu(\mathrm{Int}(X)) = 1$. The assertion of Theorem 12.4 thus follows. ☐

Note now that all Gibbs states are of full topological support, so for them either condition (e1) or (e2) is fine.

Moving more toward geometry let $\zeta : E_A^\infty \to \mathbb{R}$ be defined by the formula

$$\zeta(\omega) := \log |\varphi'_{\omega_0}(\pi(\sigma(\omega))|. \tag{12.3}$$

For every $s \geq 0$ we denote:

$$\mathrm{P}(s) := \mathrm{P}(\sigma, s\zeta) \in (-\infty, +\infty].$$

We call $\mathrm{P}(s)$ the topological pressure of s. We recall from [30] and [31] the following other definitions:

$$\theta_{\mathcal{S}} := \inf \Gamma_{\mathcal{S}}, \quad \text{where} \quad \Gamma_{\mathcal{S}} := \left\{ s \geq 0 : \sum_{e \in E} ||\varphi'_e||_\infty^s < +\infty \right\}.$$

The proofs of the following two statements can be found in [31].

Proposition 12.5. *If $\mathcal{S}$ is an irreducible conformal* GDMS*, then for every $s \geq 0$ we have that*

$$\Gamma_{\mathcal{S}} = \{s \geq 0 : \mathrm{P}(s) < +\infty\}.$$

In particular,

$$\theta_{\mathcal{S}} := \inf \{s \geq 0 : \mathrm{P}(s) < +\infty\}.$$

Theorem 12.6. *If $\mathcal{S}$ is a finitely irreducible conformal* GDMS*, then the function $\Gamma_{\mathcal{S}} \ni s \longmapsto \mathrm{P}(s)\imath\mathbb{R}$ is*

- *strictly decreasing,*
- *real-analytic,*
- *convex, and*
- $\lim_{s \to +\infty} \mathrm{P}(s) = -\infty$.

We also introduce the following important characteristic of the system $\mathcal{S}$.

$$b_{\mathcal{S}} := \inf\{s \geq 0 : \mathrm{P}(s) \leq 0\} \geq \theta_{\mathcal{S}}.$$

We call $b_{\mathcal{S}}$ the Bowen's parameter of the system $\mathcal{S}$. The following theorem, providing a geometrical interpretation of this parameter has been proved in [31].

Theorem 12.7. *If $\mathcal{S}$ is an finitely irreducible conformal* GDMS, *then*

$$\mathrm{HD}(J_{\mathcal{S}}) = b_{\mathcal{S}} \geq \theta_{\mathcal{S}}{}^1.$$

Following [30] and [31] we call the system $\mathcal{S}$ regular if there exists $s \in (0, +\infty)$ such that

$$\mathrm{P}(s) = 0.$$

Then by Theorem12.6, such a zero is unique and if exists it is equal to $b_{\mathcal{S}}$.

We call the system $\mathcal{S}$ strongly regular if there exists $s \in [0, +\infty)$ (in fact in $(\gamma_{\mathcal{S}}, +\infty)$) such that

$$0 < \mathrm{P}(s) < +\infty.$$

By Theorem 12.6 each strongly regular conformal GDMS is regular. We need one concept more:

Let $F = \{f^{(e)} : X_{t(e)} \to \mathbb{R} : e \in E\}$ be a family of real-valued functions. For every $n \geq 1$ and $\beta > 0$ let

$$V_n(F) = \sup_{\omega \in E^n} \sup_{x,y \in X_{t(\omega)}} \{|f^{(\omega_0)}(\varphi_{\sigma(\omega)}(x)) - f^{(\omega_0)}(\varphi_{\sigma(\omega)}(y))|\} e^{\beta(n-1)},$$

We have made the conventions that the empty word $\emptyset$ is the only word of length 0 and $\varphi_\emptyset = \mathrm{Id}_X$. Thus, $V_1(F) < \infty$ simply means the diameters of the sets $f^{(e)}(X)$ are uniformly bounded. The collection F is called a Hölder family of functions (of order β) if

$$V_\beta(F) = \sup_{n \geq 1}\{V_n(F)\} < \infty. \tag{12.4}$$

We call the Hölder family F, summable (of order β) if (12.4) is satisfied and

$$\sum_{e \in E} \exp\left(\sup\left(f|_{[e]}\right)\right) < +\infty. \tag{12.5}$$

In order to get the link with the previous sections on thermodynamic formalism on symbol spaces, we introduce now a potential function or amalgamated function, $f : E_A^\infty \to \mathbb{R}$, induced by the family of functions F as follows.

$$f(\omega) = f^{(\omega_0)}(\pi(\sigma(\omega))).$$

Our convention will be to use lower case letters for the potential function corresponding to a given Hölder system of functions. The following lemma is a straightforward, see [31] for a proof.

Lemma 12.8. *If F is a Hölder family (of order β) then the amalgamated function f is Hölder continuous (of order β). If F is summable, then so is f.*

Let us record the following obvious observation.

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[1] As the relevant proof in [31] shows, the Geometric Condition (e) is not needed at all for this theorem; comp. Remark 12.12

Observation 12.9. For every $t \geq 0$, $t\zeta : E_A^\infty \to \mathbb{R}$ is the amalgamated function of the following family of functions:

$$t\Xi := \{X_{t(e)} \ni x \mapsto t \log |\varphi_e'(x)| \in \mathbb{R}\}_{e \in E}.$$

The following proposition is easy to prove; see [31, Proposition 3.1.4] for complete details.

Proposition 12.10. *For every real $t \geq 0$ the function $t\zeta : E_A^\infty \to \mathbb{R}$ is Hölder continuous and $t\Xi$ is a Hölder continuous family of functions.*

Observation 12.11. For every $t \geq 0$ we have that $t \in \Gamma_{\mathcal{S}}$ if and only if the Hölder continuous potential $t\zeta$ is summable and this happens if and only if the Hölder continuous family of functions $t\Xi$ is summable.

For every $t \in \Gamma_{\mathcal{S}}$ we denote by μ_t the unique equilibrium state of the potential $t\zeta : E_A^\infty \to \mathbb{R}$ and by m_t the probability eigenmeasure of the dual operator $\mathcal{L}_t^* := \mathcal{L}_{t\zeta}*$. Of particular geometric importance for us will be the measures $\mu_{b_{\mathcal{S}}}$ and $m_{b_{\mathcal{S}}}$ for regular systems $\mathcal{S}$. Lots of our geometric and dynamical considerations throughout the rest of the manuscript will concern equilibrium states μ_t and their projections from the symbol space E_A^∞ to the limit set $J_{\mathcal{S}}$ via the projection map $\pi_{\mathcal{S}} : E_A^\infty \to J_{\mathcal{S}}$.

Remark 12.12. We want to emphasize the following.

(1) As the relevant proof in [31] shows the Geometric Condition (e) is not needed at all for Theorem 12.7.

(2) The primary power of the Geometric Condition (e) is that it yields Theorems 12.3 and 12.4 that in turn have significant dynamical and geometric consequences; for example that the measures $\mu \circ \pi_{\mathcal{S}}^{-1}$ are dimensional exact and

$$\mathrm{HD}\big(\mu \circ \pi_{\mathcal{S}}^{-1}\big) = \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu},$$

where $\mathrm{h}_\mu(\sigma)$ is the Kolmogorov–Sinaj metric entropy of the dynamical system $\sigma : E_A^\infty \to E_A^\infty$ with respect to the measure μ and

$$\chi_\mu := -\int_{E_A^\infty} \log \big|\varphi_{\omega_0}'\big(\pi_{\mathcal{S}}(\sigma(\omega))\big)\big| \, d\mu(\omega)$$

is the corresponding Lyapunov exponent.

(3) Another result, important for us, for which the Geometric Condition (e) is needed (via Theorems 12.3 and 12.4), is that if $f : E_A^\infty \to \mathbb{R}$ is the amalgamated function of a Hölder family of F functions, then

$$m_f \circ \pi_{\mathcal{S}}^{-1}\big(\varphi_\omega(H)\big) = \int_H \exp\big(S_\omega(F) - \mathrm{P}(f)|\omega|\big) \, dm_f \circ \pi_{\mathcal{S}}^{-1},$$

for every $\omega \in E_A^*$ and every Borel set $H \subseteq X_{t(\omega)}$, where

$$S_\omega(F) = \sum_{j=1}^n f^{(\omega_j)} \circ \varphi_{\sigma^j \omega}.$$

In particular, we have then for every $t \in \Gamma_{\mathcal{S}}$ that

$$m_t \circ \pi_{\mathcal{S}}^{-1}(\varphi_\omega(H)) = e^{-\mathrm{P}(t)} \int_A |\varphi_\omega'|^t dm_t \circ \pi_{\mathcal{S}}^{-1}.$$

(4) The Strong Open Set Condition (e1) is natural and easy to occur; it is equivalent to the condition that

$$\varphi_\omega(X_{t(\omega)}) \subseteq \mathrm{Int}(X_{i(\omega)})$$

for some $\omega \in E_A^*$. And this one holds for example if

$$\varphi_e(X_{t(e)}) \subseteq \mathrm{Int}(X_{i(e)})$$

for some $e \in E$.

(5) Condition (e2) is also easy to have. For example it holds if the boundaries ∂X_v, $v \in V$, are piecewise smooth or the sets X_v are convex.

(6) No Geometric Condition (e) is needed at all if the alphabet E is finite.

We would like however to complete this comment by saying that in the case of finite alphabet E the Open set Condition alone suffices, and the item (b2) is not needed at all. It is not needed in the case of infinite alphabet either as long as we are only interested in the Hausdorff dimension of the limit set, i. e. as long as we only want prove Bowen's Formula.

13. More Technicalities on Conformal GDMSs

We keep the setting and notation from the previous section.

- We call a point $z \in X$ pseudo-periodic for $\mathcal{S}$ if there exists $\omega \in E_A^*$ such that $z \in X_{t(\omega_0)}$ and $\varphi_\omega(z) = z$.
- We call a point $z \in \mathcal{S}$ periodic for $\mathcal{S}$ if $z = \pi(\omega)$ for some periodic element $\omega \in E_A^\infty$.
- Of course every periodic point is pseudo-periodic. Also obviously, for maximal graph directed Markov systems, in particular for conformal iterated function systems, periodic points and pseudo-periodic points coincide.
- We call a periodic point $z \in J_{\mathcal{S}}$ uniquely periodic if $\pi^{-1}(z)$ is a singleton and there is exactly one $\xi \in E_A^*$ such that the infinite concatenation $\xi^\infty \in E_A^\infty$, $\varphi_\xi(z) = z$, and if $\varphi_\alpha(z) = z$ for some $\alpha \in E_A^*$, then $\alpha = \xi^q$, the concatenation of q copies of ξ for some integer $q \geq 1$.

We shall prove the following.

Lemma 13.1. *If $z \in J_{\mathcal{S}}$ is not pseudo-periodic for $\mathcal{S}$, then*

$$\pi^{-1}(z) \cap \bigcup_{n=1} \sigma^n(\pi^{-1}(z)) = \emptyset.$$

Proof. Assume for a contradiction that there exists $\omega \in \pi^{-1}(z)$ such that $\sigma^n(\omega) \in \pi^{-1}(z)$ for some $n \geq 1$. We then have

$$\varphi_{\omega|_n}(z) = \varphi_{\omega|_n}\big(\pi(\sigma^n(\omega))\big) = \pi\big(\omega|_n\sigma^n(\omega)\big) = \pi(\omega) = z.$$

So, z is pseudo-periodic, and this contradiction finishes the proof. □

In fact, we will need more:

Lemma 13.2. *Assume that $z \in J_{\mathcal{S}}$ is not pseudo-periodic for the system $\mathcal{S}$. If $k \geq 1$ is an integer, $(l_n)_{n=1}^\infty$ is a sequence of integers in $\{1, 2, \dots, k\}$, and $\big(\tau^{(n)}\big)_{n=1}^\infty$ is a sequence of points in E_A^∞ such that*

$$\lim_{n\to\infty} \pi\big(\tau^{(n)}\big) = \lim_{n\to\infty} \pi\big(\sigma^{l_n}(\tau^{(n)})\big) = z,$$

then

$$\lim_{n\to\infty} \sum_{i=0}^{l_n} \tau_i^{(n)} = +\infty.$$

Proof. Seeking a contradiction suppose that

$$\underline{\lim}_{n\to\infty} \sum_{i=0}^{l_n} \tau_i^{(n)} < +\infty.$$

Passing to a subsequence, we may assume without loss of generality

$$\overline{\lim}_{n\to\infty} \sum_{i=0}^{l_n} \tau_i^{(n)} < +\infty.$$

There then exists $M \in (0, +\infty)$ such that

$$\sum_{i=0}^{l_n} \tau_i^{(n)} \leq M$$

for all $n \geq 1$. Hence,

$$\tau_i^{(n)} \leq M$$

for all $n \geq 1$ and all $i = 0, 1, 2, \dots, l_n$. So, passing to yet another subsequence, we may further assume that the sequence $(l_n)_{n=1}^\infty$ is constant, say $l_n = l$ for all $n \geq 1$, and that for every $i = 0, 1, 2, \dots, l_n$ the sequence $\big(\tau^{(n)}\big)_{n=1}^\infty$ is constant, say $\tau_i^{(n)} = \tau_i \leq M$ for all $n \geq 1$. Let

$$\tau := \tau_1\tau_2 \dots \tau_l.$$

it then follows from our hypothesis that

$$\begin{aligned}\varphi_\tau(z) &= \varphi_\tau\big(\lim_{n\to\infty} \pi\big(\sigma^l(\tau^{(n)})\big)\big) = \lim_{n\to\infty} \varphi\big(\tau\big(\pi\big(\sigma^l(\tau^{(n)})\big)\big)\big) \\ &= \lim_{n\to\infty} \pi\big(\sigma^l(\tau^{(n)})\big) \\ &= \lim_{n\to\infty} \pi\big(\tau^{(n)}\big) = z.\end{aligned}$$

Thus z is a pseudo-periodic point for the graph directed Markov system $\mathcal{S}$, and this contradiction finishes the proof. □

A statement corresponding to Lemma 13.2 in the case of periodic points is the following.

Lemma 13.3. *Assume that $z \in J_{\mathcal{S}}$ is uniquely periodic for the system $\mathcal{S}$ (i.e., $\pi^{-1}(z)$ is a singleton and that there exists a unique point $\xi \in E_A^*$ such that $\xi^\infty \in E_A^\infty$, $\varphi_\xi(z) = z$, and if $\varphi_\alpha(z) = z$ for some $\alpha \in E_A^*$, then $\alpha = \xi^q$ for some integer $q \geq 1$). Then if $k \geq 1$ is an integer, $(l_n)_{n=1}^\infty$ is a sequence of integers in $\{1, 2, \dots, k\}$, and $\left(\tau^{(n)}\right)_{n=1}^\infty$ is a sequence of points in E_A^∞ such that*

(a)
$$\lim_{n\to\infty} \pi\big(\tau^{(n)}\big) = \lim_{n\to\infty} \pi\big(\sigma^{l_n}(\tau^{(n)})\big) = z,$$
and

(b)
$$\overline{\lim_{n\to\infty}} \sum_{i=0}^{l_n} \tau_i^{(n)} < +\infty,$$
then l_n is an integral multiple of $|\xi|$, say $l_n = q_n|\xi|$, and
$$\tau^{(n)}|_{l_n} = \xi^{q_n}$$
for all $n \geq 1$ large enough.

Proof. It follows from item (b) that there exists $M \geq 1$ such that $\tau_i^{(n)} \leq M$ for all $n \geq 1$ and all $1 \leq i \leq l_n$. Assuming the contrapositive statement to our claim and passing to a subsequence, we may assume without loss of generality that the sequence $(l_n)_{n=1}^\infty$ is constant, say $l_n = l$ for all $n \geq 1$, and we may further assume that for every $1 \leq i \leq l_n$ the sequence $\left(\tau^{(n)}\right)_{n=1}^\infty$ is constant, say $\tau_i^{(n)} = \tau_i \in \{1, 2, \dots, M\}$ for all $n \geq 1$ and

$$\tau_j^{(n)} \neq \xi_j \tag{13.1}$$
for all $n \geq 1$ and some $1 \leq j \leq l$. Let
$$\tau := \tau_1\tau_2\dots\tau_l.$$
We now conclude, in exactly the same way as in the proof of Lemma 13.2 that $\varphi_\tau(z) = z$. Therefore, since z is uniquely pseudo-periodic, we get $\tau = \xi^q$ with some $q \geq 1$. In particular $q|\xi| = l$, and so, using (13.1), we deduce that $\tau \neq \xi^q$. This contradiction finishes the proof. □

14. Weakly Boundary Thin (WBT) Measures and Conformal GDMSs

In this section we first introduce the concept of Weakly Bounded Thin (WBT) measures. Roughly speaking, this notion relates the measure of an annulus to the measure of the ball it encloses. We prove some basic properties of (WBT) and provide some sufficient conditions for (WBT) to hold for a large class of measures on the limit set of a CGDMS. We were able to establish these properties, mainly due to the progress achieved in [36], Let μ be a Borel probability measure on a separable metric space (X,d). For all $\beta>0$, $x\in X$ and $r>0$, let

$$A_\mu^\beta(x,r):=A\big(x;r-\mu(B(x,r))^\beta,r+\mu(B(x,r))^\beta\big),$$

where, in general,

$$A(z;r,R):=B(z,R)\setminus B(z,r)$$

is the annulus centered at z with the inner radius r and the outer radius R; in order not to overlook a possible cases of negative numbers $r-\mu(B(x,r))^\beta$, we naturally declare $B(z,r)$ to be the empty set if $r\le 0$. We say that the measure μ is weakly boundary thin (WBT) with exponent β at the point x if

$$\lim_{r\to 0}\frac{\mu\big(A_\mu^\beta(x,r)\big)}{\mu(B(x,r))}=0.$$

We simply say that measure μ is weakly boundary thin (WBT) if it is is weakly boundary thin with some exponent $\beta>0$. Given $\alpha>0$, we further define:

$$A_\mu^{\beta,\alpha}(x,r):=A\big(x;r-\alpha\mu(B(x,r))^\beta,r+\alpha\mu(B(x,r))^\beta\big).$$

The following proposition is obvious.

Proposition 14.1. *If μ is a Borel probability measure on a separable metric space X, then for every point $x\in\operatorname{supp}(\mu)$, the following are equivalent.*

(a) μ *is* (WBT) *at* x.

(b) *There exists $\beta>0$ such that the measure μ is* (WBT) *at x with exponent $\gamma>0$ either if and only if $\gamma\in(\beta,+\infty)$ or if and only if $\gamma\in[\beta,+\infty)$. Denote this β by $\beta_\mu(x)$.*

(c) *There exist $\alpha,\beta>0$ such that*

$$\lim_{r\to 0}\frac{\mu\big(A_\mu^{\beta,\alpha}(x,r)\big)}{\mu(B(x,r))}=0.$$

(d) *For every $\alpha>0$ and every $\beta\in(\beta_\mu(x),+\infty)$,*

$$\lim_{r\to 0}\frac{\mu\big(A_\mu^{\beta,\alpha}(x,r)\big)}{\mu(B(x,r))}=0.$$

We say that a measure is weakly boundary thin (WBT) if it is (WBT) at every point of its topological support. We also say that a measure is weakly boundary thin almost everywhere (WBTAE) if it is (WBT) at almost every point. Of course (WBT) implies (WBTAE).

Now we aim to provide sufficient conditions for a Borel probability measure to be (WBT) and (WBTAE). Let μ be an arbitrary Borel probability measure on a separable metric space.

Let $\alpha > 0$. We say that μ is α-upper Ahlfors (α-up) at a point $x \in X$ if there exists a constant $C > 0$ (which may depend on x) such that

$$\mu(B(x,r)) \le Cr^\alpha$$

for all radii $r > 0$. Equivalently, for all radii $r > 0$ sufficiently small. Following [42], the measure μ is said to have the Thin Annuli Property (TAP) at a point $x \in X$ if there exists $\kappa > 0$ (which may depend on x) such that

$$\lim_{r\to 0} \frac{\mu(A(x;r,r+r^\kappa))}{\mu(B(x,r))} = 0$$

We shall easily show the following.

Proposition 14.2. *Let (X,d) be a separable metric space, let μ be a Borel probability measure on X and let $\alpha > 0$. If μ is α-upper Ahlfors with the Thin Annuli Property* (TAP) *at some $x \in X$, then μ is* (WBT) *at x.*

Proof. Taking $\beta > 0$ so large that $C^\beta r^{\beta\alpha} \le r^\kappa$ for $r > 0$ small enough. Then for such radii $r > 0$ we have that $A_\mu^\beta(x,r) \subset A(x;r,r+r^\kappa)$ and thus

$$0 \le \limsup_{r\to 0} \frac{\mu(A_\mu^\beta(x,r))}{\mu(B(x,r))} \le \limsup_{r\to 0} \frac{\mu(A(x;r,r+r^\kappa))}{\mu(B(x,r))} = 0$$

The proof is then complete. □

We recall from the book [42] that

$$\mathrm{HD}_*(\mu) = \inf\{\mathrm{HD}(Y) : Y \subset X \text{ is Borel and } \mu(Y) > 0\}.$$

We call $\mathrm{HD}_*(\mu)$ the lower Hausdorff dimension of μ. The Hausdorff Dimension of μ is commonly defined to be

$$\mathrm{HD}(\mu) = \inf\{\mathrm{HD}(Y) : Y \subset X \text{ is Borel and } \mu(Y) = 1\}$$

The reader should be aware that in [42] the above infimum is denoted $\mathrm{HD}^*(\mu)$ and is called the upper Hausdorff Dimension of μ. We however, will use the more commonly accepted tradition rather than the point of view taken in [42]. Referring to the well-known fact (see [42] for instance) that if $\mu(B(x,t)) \ge Cr^\gamma$ for the points x belonging to some Borel set $F \subset X$ then $\mathrm{HD}(F) \le \gamma$, we immediately obtain the following.

Lemma 14.3. *If $\mathrm{HD}_*(\mu) > 0$ then μ is α-upper Ahlfors for every $\alpha \in (0, \mathrm{HD}_*(\mu))$ and μ–a.e. $x \in X$ with some constant $C \in (0,+\infty)$ and every $r > 0$ small enough.*

Definition 14.4. We say that a set $J \subseteq \mathbb{R}^d$, $d \ge 1$, is geometrically irreducible if it is not contained in any countable union of conformal images of hyperplanes or spheres of dimension $\le d-1$.

Observation 14.5. Every set $J \subseteq \mathbb{R}^d$, $d \ge 1$, with $\mathrm{HD}(J) > d-1$ is geometrically irreducible.

Observation 14.6. If a set $J \subseteq \mathbb{C}$ is not contained in any countable union of real analytic curves, then J is geometrically irreducible.

Now we can apply the results obtained above, in the context of CGMS. We shall prove the following.

Theorem 14.7. *Let $\mathcal{S} = \{\varphi_e\}_{e \in E}$ be a finitely primitive* CGDS *with a phase space $X \subset \mathbb{R}^d$. Let $\psi : E_A^{\mathbb{N}} \to \mathbb{R}$ be a Hölder continuous strongly summable potential, the latter meaning that*

$$\sum_{e \in E} \exp\left(\inf(\varphi|_{[e]})\right) \|\varphi_e'\|^{-\beta} < +\infty \tag{14.1}$$

for some $\beta > 0$. As usual, let μ_ψ denote its unique equilibrium state. If the limit set of $J_{\mathcal{S}}$ is geometrically irreducible, then

(a) $\mathrm{HD}_*(\mu_\psi \circ \pi_{\mathcal{S}}^{-1}) = \mathrm{HD}(\mu_\psi \circ \pi_{\mathcal{S}}^{-1}) > 0$*;*
(b) *The measure $\mu_\psi \circ \pi_{\mathcal{S}}^{-1}$ satisfies the Thin Annuli Property* (TAP) *at $\mu_\psi \circ \pi_{\mathcal{S}}^{-1}$ a.e. point of $\mathcal{S}$*
(c) *$\mu_\psi \circ \pi_{\mathcal{S}}^{-1}$ is* (WBT) *at μ_ψ a.e. point of $J_{\mathcal{S}}$.*

Proof. The proof of Theorem 4.4.2 in [30] [31] gives in fact that the measure μ_ψ is dimensionally exact, i.e., that

$$\lim_{r \to 0} \frac{\log \mu_\psi \circ \pi_{\mathcal{S}}^{-1}(B(x,r))}{\log r}$$

exists for $\mu_\psi \circ \pi_{\mathcal{S}}^{-1}$ for a.e. $x \in J_{\mathcal{S}}$ and is equal to $\mathrm{h}_{\mu_\psi}(\sigma)/\chi_{\mu_\psi} > 0$. A complete proof with all the details can be found in the last section of [7]. Therefore, property (a) is established. Property (b) follows now immediately from Theorem 30 in [36]. Condition (c) is now an immediate consequence of (a),(b), Lemma 14.3 and Proposition 14.2. ☐

Remark 14.8. Condition 14.1 is satisfied for instance for all potentials of the form

$$E_A^{\mathbb{N}} \ni \omega \mapsto t\zeta(\omega) = t \log\left|(\varphi_{\mathcal{S}}'(\pi_{\omega_0}(\sigma(\omega)))\right| \in \mathbb{R},$$

where $t \in \Gamma_{\mathcal{S}}$.

Now we shall deal with the case of a finite alphabet. We shall show that in the case of a finite alphabet (under a mild geometric condition in dimension $d \geq 2$) the equilibrium states of all Hölder continuous potentials satisfy (WBT) at every point of the limit set. Thus our approach is complete in the case of the finite alphabet and present paper entirely covers the case of conformal IFSs (even GDMSs) with finite alphabet. We shall prove the following.

Theorem 14.9. *Let E be a finite set and let $\mathcal{S} = \{\varphi_e\}_{e \in E}$ be a primitive conformal* GDMS *acting in the space $\mathbb{R}$. If $\psi : E_A^\infty \to \mathbb{R}$ is an arbitrary Hölder continuous (with the phase space sets $X_v \subset W_v \subset \mathbb{R}$, $v \in V$) and μ_φ is the corresponding equilibrium state on E_A^∞ then the projection measure $\mu_\psi \circ \pi_{\mathcal{S}}^{-1}$ is* (WBT) *at every point of $J_{\mathcal{S}}$.*

Proof. Put

$$u := K^{-1} \min\left\{\|\varphi_e'\| : e \in E\right\}$$

so that

$$|\varphi_e'(x)| \geq u$$

for all $e \in E$ and all $x \in X_{t(e)}$. For ease of notation we denote

$$\widehat{\mu}_\psi := \mu_\psi \circ \pi_{\mathcal{S}}^{-1}.$$

Fix $\beta > 0$. Consider the family

$$\mathcal{F}_\psi^\beta(z,r) := \{\omega \in E_A^k : A_{\mu_\psi}^\beta(z,r) \cap \varphi_\omega(J_{\omega_{|\omega|-1}}) \neq \emptyset \text{ and } \|\varphi_\omega'\| \geq \mu_\psi(B(z,r)^\beta)\}$$

Now consider $\widehat{\mathcal{F}}_\psi^\beta(z,r)$, the family of all words in $\mathcal{F}_\psi^\beta(z,r)$ that have no extensions to elements in $\mathcal{F}_\psi^\beta(z,r)$, where we don't consider a finite word to be an extension of itself. Note that then:

(a) $\widehat{\mathcal{F}}_\psi^\beta(z,r)$ consists of mutually incomparable words;

(b) $\bigcup_{\omega\in\widetilde{\mathcal{F}}_\psi^\beta(z,r)}[w] \supset \pi_{\mathcal{S}}^{-1}(A_{\mu_\psi}^\beta(z,r))$; and

(c) $\forall\omega \in \widehat{\mathcal{F}}_\psi^\beta(z,r)$, $\mu_\psi(B(z,r))^\beta \leq \|\varphi_\omega'\| \leq u^{-1}\mu_\psi(B(z,r))^\beta$

Therefore the family

$$\{\varphi_\omega(\mathrm{Int}(X_{t(\omega)})) : \omega \in \widehat{\mathcal{F}}_\psi^\beta(z,r)\}$$

consists of mutually disjoint open sets each of which contains a ball of radius $K^{-1}R\mu_\psi(B(z,r))^\beta$ where R is as in the proof of Lemma 14.13. Since also

$$\bigcup_{\omega\in\widehat{\mathcal{F}}_\psi^\beta(z,r)} \varphi_\omega(X_{t(\omega)}) \subset A(z, r-(1+DM^{-1})\mu_\psi(B(z,r))^\beta, r-(1+DM^{-1})\mu_\psi(B(z,r))^\beta)$$

we obtain that

(14.2)

$$\begin{aligned}\#\widehat{\mathcal{F}}_\psi^\beta(z,r) &\leq \frac{\mathrm{Leb}_1(A(z,r-(1+DM^{-1})\mu_\psi(B(z,r)^\beta), r-(1+Du^{-1})\mu_\psi(B(z,r))^\beta)}{2K^{-1}R\mu_\psi(B(z,r))^\beta} \\ &\approx \frac{\mu_\psi^\beta(B(z,r))}{\mu_\psi^\beta(B(z,r))} = 1.\end{aligned}$$

So we have shown that the number of elements of $\widetilde{\mathcal{F}}_\psi^\beta(z,r)$ is uniformly bounded above, and in order to estimate $\widehat{\mu}_\psi(A_{\mu_\psi}^\beta(z,r))$. i.e. in order to complete the proof we now only need a sufficiently good upper bound on $\mu_\psi([\omega])$ for all $\omega \in \widehat{\mathcal{F}}_\psi^\beta(z,r)$. We will do so now. It is well known (see [30], [31]) that there are two constants $\eta \in (0,+\infty)$ and $C \in (0,+\infty)$ such that

$$\mu_\psi([\tau]) \leq C\exp(-\eta(|\tau|+1)) \tag{14.3}$$

for all $\tau \in E_A^*$. Fix $\omega \in \widehat{\mathcal{F}}_\psi^\beta(z,r)$. Denote $k := |\omega|$. Invoking (c) we get that $u^k \le \|\varphi_\omega'\| \le u^{-1}\mu_\psi^\beta(B(z,r))$, whence

$$k+1 \ge \frac{\beta \log \mu_\psi(B(z,r))}{\log u}.$$

Inserting this into (14.3) we get that

$$\mu_\psi([\omega]) \le C \exp\left(-\beta\eta \frac{\log \mu_\psi(B(z,r))}{\log u}\right) = C\mu^{\gamma\beta}(B(z,r))$$

where $\gamma = \frac{\eta}{\log(1/u)} \in (0,+\infty)$. Having this and invoking (b) and (14.2) we obtain that

$$\frac{\widehat{\mu}_\psi(A^\beta_{\mu_\psi}(z,r))}{\mu_\psi(B(z,r))} \le \frac{\widehat{\mu}_\psi(B(z,r))^{\gamma\beta}}{\widehat{\mu}_\psi(B(z,r))} \le \mu_\psi(B(z,r))^{\gamma\beta-1}$$

and the proof is complete by noting that $\lim_{r\to 0} \mu_\psi(B(z,r))^{\gamma\beta-1} = 0$ provide that we take $\gamma > 1/\beta$. □

Now we pass to the case of $d \ge 2$. We get the same full result as in the case of $d = 1$ but with a small additional assumption that the conformal system $\mathcal{S}$ is geometrically irreducible.

Theorem 14.10. *Let E be a finite set and let $\mathcal{S} = \{\varphi_e\}_{e\in E}$ be a primitive geometrically irreducible conformal* GDMS *with the phase space sets $X_v \subset W_v \subset \mathbb{R}^d$. If $\psi : E_A^{\mathbb{N}} \to \mathbb{R}$ is an arbitrary Hölder continuous potential and μ_φ is the corresponding equilibrium state then the projection measure $\mu_\psi \circ \pi_{\mathcal{S}}^{-1}$ is* (WBT) *at every point of $J_{\mathcal{S}}$*

Proof. The meaning of $\widehat{\mu}_\psi$ is exactly the same as in the proof of the previous theorem. The proof of the current theorem is entirely based on the following.

Claim 1: There are a constant $\alpha > 0$ and $C \in (0,+\infty)$ such that

$$\widehat{\mu}_\psi\big(A(z; R-r, R+r)\big) \le Cr^\alpha$$

for all $z \in \mathbb{R}^d$ and all radii $r, R > 0$.

This claim is actually a sub-statement of formula (2.19) from [54] in a more specific setting. In particular:

(a) [54] deals with finite IFSs rather than finite alphabet CGDMS;

(b) [54] deals with Hölder continuous families of functions and their corresponding equilibrium states rather than Hölder continuous potentials on the symbol space $E_A^{\mathbb{N}}$ and their projections; and

(c) with the restrictions of (a) and (b) Claim 1 is a sub-statement of formula (2.19) from [54] only in the case of $d \ge 3$.

However, a close inspection of arguments leading to (2.19) of [54] indicates that the difference of (a) is inessential for these arguments, and for (b) that the only property of equilibrium states of Hölder continuous families of functions was that of being projections of Hölder continuous potentials from the symbol space E_A^∞. Concerning (c) it only remains to consider the case $d = 2$. We then redefine the family $\mathcal{F}_0$ from section 2, page 225 of [54] to conclude that also all the intersections of the form $X \cap L$, where $L \subset \mathbb{C}$ where is a round circle (of arbitrary center and radius). The argument in [54] leading to (2.19) goes through with obvious minor modifications. Claim 1 is then established.

Using this claim, we obtain

$$\frac{\widehat{\mu}_\psi(A^\beta_{\mu_\psi}(z,r))}{\widehat{\mu}_\psi(B(x,r))} \le \frac{C\widehat{\mu}_\psi^{\alpha\beta}(B(z,r))}{\widehat{\mu}_\psi(B(x,r))} = C\widehat{\mu}_\psi^{\alpha\beta-1}(B(z,r))$$

and the proof is complete and by noting that $\lim_{r\to 0}\mu_\psi^{\alpha\beta-1}(B(z,r))$ for every $\beta > 1/\alpha$. ☐

Fixing a $\kappa > 0$ let

$$N_\kappa(x,r) := \left[-\frac{1}{\kappa}\log\mu(B(x,r)),\right] \in \mathbb{N}\cup\{+\infty\}$$

where $[t]$, $t \in \mathbb{R}$, denotes the integer part of t. Let us make right away an immediately evident, but extremely important, observation.

Observation 14.11. If μ is a Borel probability measure on X, then for every $r > 0$, we have that

$$e^{-\kappa N_\kappa(x,r)} \le \mu(B(x,r)) \le e^\kappa e^{-\kappa N_\kappa(x,r)}.$$

Now, let in addition $\mathcal{S} = \{\varphi_e\}_{e\in E}$ be a finitely primitive CGDMS with a phase space $X \subseteq \mathbb{R}^d$. For every $x \in X$ and $r > 0$ and an integer $n \ge 0$, let

$$A_n^*(x,r) := \bigcup\big\{\varphi_\omega(J_\mathcal{S}) : \omega \in E_A^n,\ \ \varphi_\omega(J)\cap B(x,r) \ne \emptyset \ \text{ and } \ \varphi_\omega(J_\mathcal{S})\cap B^c(x,r) \ne \emptyset\big\}.$$

We say that the measure μ is dynamically boundary thin (DBT) at the point $x \in \overline{J}_\mathcal{S}$ if for some $\kappa > 0$

$$\lim_{r\to 0}\frac{\mu\big(A^*_{N_\kappa(x,r)}(x,r)\big)}{\mu(B(x,r))} = 0. \tag{14.4}$$

We say that the measure μ is Dynamically Boundary Thin (DBT) almost everywhere if the set of points where it fails to be (DBT) is measure zero, and that the measure μ is Dynamically Boundary Thin (DBT) if it is (DBT) at every point of its topological support. We shall prove the following.

Proposition 14.12. *If a Borel probability measure μ on $\overline{J}_\mathcal{S}$ is* (WBT) *at some point $x \in \overline{J}_\mathcal{S}$, then it is* (DBT) *at x.*

Proof. Let $\beta > 0$ be as in the definition of (WBT) of μ at x. Since $\mathcal{S}$ is a conformal GDMS, there exist constants $\eta > 0$ and $D \geq 1$ such that

$$\mathrm{diam}(\varphi_\omega(X)) \leq D\eta^{-\eta|\omega|}$$

for all $\omega \in E_A^*$. Therefore, if $\kappa > 0$ is sufficiently small, then for every $x \in \overline{J}_{\mathcal{S}}$ and every $r > 0$ we have

$$\begin{aligned} A^*_{N_\kappa(x,r)}(x,r) &\subseteq A\big(x; r - De^{-\kappa N_\kappa(x,r)}, r + De^{-\kappa N_\kappa(x,r)}\big) \\ &\subseteq A\big(x; r - D(\mu(B(x,r))^{\eta/\kappa}, r + D(\mu(B(x,r))^{\eta/\kappa}\big) \\ &= A_\mu^{\eta/\kappa, De}(x,r). \end{aligned}$$

For every $r > 0$, sufficiently small, we then have

$$\frac{\mu\big(A^*_{N_\kappa(x,r)}(x,r)\big)}{\mu(B(x,r))} \leq \frac{\mu\big(A_\mu^{\eta/\kappa, De}(x,r)\big)}{\mu(B(x,r))}.$$

Now, if $\kappa > 0$ is sufficiently small, then $\eta/\kappa > \beta$ and, in consequence,

$$0 \leq \lim_{r\to 0} \frac{\mu\big(A^*_{N_\kappa(x,r)}(x,r)\big)}{\mu(B(x,r))} \leq \lim_{r\to 0} \frac{\mu\big(A_\mu^{\eta/\kappa, De}(x,r)\big)}{\mu(B(x,r))} = 0.$$

This means that μ is (DBT) at x and the proof is complete. $\square$

Now we shall provide some sufficient conditions, different than (WBT), for (DBT) to hold at every point of $J_{\mathcal{S}}$. We will do it by developing the reasoning of Lemma 5.2 in [5]. We will not really make use of these conditions in the current manuscript but these are very close to the subject matter of the current section and will not occupy too much space. These may be needed in some future. We shall first prove the following.

Lemma 14.13. *Let $\mathcal{S} = \{\varphi_e\}_{e\in E}$ be a finitely primitive* CGDMS *satisfying* SOSC*. Assume that a number $t > \max\{\theta_{\mathcal{S}}, d-1\}$ satisfies*

$$t > d - 1 + \frac{\mathrm{P}(t)}{\log s} \tag{14.5}$$

Recallthat μ_t is the unique equilibrium state of the potential $E_A^\infty \ni \omega \mapsto t\log|\varphi_{\omega_1}(\pi(\sigma\omega))|$. Then there exists constants $\alpha > 0$ and $C \in [0, +\infty]$ such that

$$\mu_t \circ \pi_{\mathcal{S}}^{-1}(A_k^*(z,r)) \leq Ce^{-\alpha k}$$

for all $z \in \overline{J_{\mathcal{S}}}$, all radii $r > 0$ and all integers $n \geq 0$.

Proof. For all $a \in E$, let $r > 0$. Set

$$J_a := \bigcup_{b\in E:A_{ab}=1} \pi_{\mathcal{S}}([b]).$$

$r \in (0,1]$. For $k \geq 0$ consider the set

$$E_A^k(z,r) := \Big\{\omega \in E_A^k : \varphi_\omega(J_{w_{|n|-1}}) \cap B(z,r) \neq \emptyset \ \text{ and } \ \varphi_\omega(J_{w_{|w|-1}}) \cap B(z,r)^c \neq \emptyset\Big\}.$$

Furthermore, for every $k \geq 0$ let

$$E_A^k(z,r;n) := \{\omega \in E_A^k(z,r) : s^{n+1} < \|\varphi_w'\| \leq s^n\}$$

Then the family

$$\mathcal{F}_k(z,r;n) := \big\{\varphi_\omega(\mathrm{Int}(X)) : \omega \in E_A^k(z,r;n)\big\}$$

consists of mutually disjoint open sets contained in

$$A(z; r - Ds^n, r + Ds^n)$$

each of which contains a ball of radius $K^{-1}Rs^{n+1}$ where $R > 0$ is the radius of an open ball entirely contained in $\mathrm{Int}X_v$ for all $v \in V$. So, then

$$\#\mathcal{F}_k(z,r;n) \leq \frac{\mathrm{Leb}_d(A(z;r-Ds^n,r+Ds^n))}{\mathrm{Leb}_d(0,k^{-1}Rs^{n+1})} \leq C_1\frac{r^{d-1}s^n}{s^{nd}} = C_1 r^{d-1}s^{(1-d)n} \leq C_1 s^{(1-d)n}$$

with the same universal constant $C_1 \in (0,+\infty)$. Since $E_A^k(z,r,n) = \emptyset$ for every $n < k$, then knowing that $t > \max\{\theta_{\mathcal{S}}, d-1\}$ gives that

$$\begin{aligned}
\mu_t \circ \pi_{\mathcal{S}}^{-1}(A_k^*(z,v)) &= \sum_{n=k}^{\infty} \mu_t\left(\bigcup_{\omega \in E_A^k(z,r;k)} \varphi_\omega(\mathcal{J}_{\omega_{|\omega|-1}})\right) \\
&\leq \sum_{n=k}^{\infty} \#\mathcal{F}_k(z,r;)\sup\{\mu_t(\varphi_\omega(X)) : \omega \in E_A(z,r;n)\} \\
&\leq C_1 \sum_{n=k}^{\infty} s^{(1-d)n}e^{-P(t)k}s^{tn} \\
&= C_1 e^{-\mathrm{P}(t)k}\sum_{n=k}^{\infty} s^{(t+1-d)n} \\
&= C_1(1-s^{t+1-d})^{-1}e^{-P(t)k}s^{(t+d-1)k} \\
&= C_1(1-s^{t+d-1})^{-1}\exp(((t+1-d)\log s - \mathrm{P}(t))k)
\end{aligned}$$

But $(t+1-d)\log s - \mathrm{P}(t) < 0$ by virtue of (14.5) and the proof is complete. ☐

As an immediate consequence of this lemma we get the following.

Theorem 14.14. *Let* $\mathcal{S} = \{\varphi_e\}_{e \in E}$ *be a finitely primitive* CGDMS*. If a number* $t > \max\{\theta_{\mathcal{S}}, d-1\}$ *satisfies*

$$t > d - 1 + \frac{\mathrm{P}(t)}{\log s} \tag{14.6}$$

then $\mu_t \circ \pi_{\mathcal{S}}^{-1}$*, the projection of the corresponding equilibrium state* μ_t *on* E_A^∞*, is* DBT *at every point of* $\overline{J_{\mathcal{S}}}$

Proof. Because of the Lemma 14.13 for all $z \in \overline{J_{\mathcal{S}}}$ and all radii $r > 0$, we have that

$$\begin{aligned}\mu_t \circ \pi_{\mathcal{S}}^{-1}\left(A^*_{N_\kappa(z,r)}(z,r)\right) &\le C\exp(-\alpha N_\kappa(z,k)) \\ &\le C\exp\left(-\alpha\left(\frac{1}{\kappa}\log\mu_t\circ\pi_{\mathcal{S}}^{-1}(B(z,r)) - 1\right)\right) \\ &= Ce^\alpha(\mu_t\circ\pi_{\mathcal{S}}(B(z,r)))^{\alpha/\kappa} \\ &= Ce^\alpha\left(\mu_t\circ\pi_{\mathcal{S}}^{-1}(B(z,r))\right)^{\frac{\alpha}{\kappa}-1}\mu_t\circ\pi_{\mathcal{S}}^{-1}(B(z,r))\end{aligned}$$

Equivalently,

$$\frac{\mu_t\circ\pi_{\mathcal{S}}^{-1}(A^*_{N_k(z,r)}(z,r))}{\mu_t\circ\pi_{\mathcal{S}}^{-1}(B(z,r))} \le Ce^\alpha\left(\mu_t\circ\pi_{\mathcal{S}}^{-1}(B(z,r))\right)^{\frac{\alpha}{\kappa}-1}$$

and the proof is complete since the right hand-side of this inequality converges to 0 as $r \to 0$ for every $\kappa \in (0,\alpha)$.

□

As an immediate consequence of this theorem we get the following.

Corollary 14.15. *Let $\mathcal{S}$ be a finitely primitive strongly regular* CGDMS*. Then there exists $\eta > 0$ such that if $t \in (\max\{\theta_{\mathcal{S}}, d-1\}, \mathrm{HD}(J_{\mathcal{S}})+\eta)$, then $\mu_t\circ\pi_{\mathcal{S}}^{-1}$, the projection of the corresponding equilibrium state μ_t on E_A^∞, is* DBT *at every point of $\overline{J_{\mathcal{S}}}$.*

Proof. We only need to check that if $t \in (\max\{\theta_{\mathcal{S}}, d-1\}, \mathrm{HD}(J_{\mathcal{S}})+\eta)$ for some $\eta > 0$ sufficiently small then (14.6) holds. Indeed, since $\mathrm{P}(b_{\mathcal{S}}) = 0$ (by strong regularity of $\mathcal{S}$) this is an immediate consequence of continuity of the function $(\theta_{\mathcal{S}}, +\infty) \ni t \mapsto \mathrm{P}(t) \in \mathbb{R}$. □

15. Escape Rates for Conformal GDMSs; Measures

In this section we continue the analysis from the previous section and we prove our first main results concerning escape rates; the one for conformal GDMSs and equilibrium/Gibbs measures. We first work for a while in full generality. Indeed, let μ be an arbitrary Borel probability measure on a metric space (X,d). Fix $\kappa > 0$. Fix $z \in X$. Let

$$\Gamma := \Gamma_\kappa(z) := \{N_\kappa(z,r) : 0 < r \le 2\mathrm{diam}(X)\}.$$

Represent Γ as a strictly increasing sequence $(l_n)_{n=0}^\infty$ of non-negative integers. Let us record the following.

Observation 15.1. If $z \in \mathrm{supp}(\mu)$, then $\Gamma_\kappa(z) \subseteq \mathbb{N}$. Moreover, the set Γ is infinite if and only z is not an atom of μ.

We shall prove the following.

Lemma 15.2. *If μ is a Borel probability measure on X which is* (WBT) *at some point $z \in X$, then the set $\Gamma_\kappa(z)$ has bounded gaps, precisely meaning that*

$$\Delta l(z) := \sup_{n\ge0}\{l_{n+1} - l_n\} < +\infty$$

Proof. Fix $n \geq 0$. There then exists $r_n > 0$ such that

$$N_\kappa(z,r) \leq l_n + 1$$

for all $r > r_n$, and

$$N_\kappa(z,r) \geq l_{n+1}$$

for all $r < r_n$. Therefore, by Observation 14.11

$$\mu(B(z,r_n)) \leq e^\kappa \exp\big(-\kappa l_{n+1}\big)$$

and

$$\mu(B(z,r_n)) \geq \exp\big(-\kappa(l_n+1)\big).$$

Hence,

$$\frac{\mu\big(B\big(z,r_n+\mu^\beta\big(B(z,r_n)\big)\big)\big)}{\mu(B(z,r_n))} \geq e^{-\kappa}\exp\big(\kappa(l_{n+1}-(l_n+1))\big).$$

for all $\beta > 0$, in particular for $\beta > \beta_\mu(z)$. But since the measure μ is (WBT) at z, we therefore have that

$$\overline{\lim_{n\to\infty}} \exp\big(\kappa(l_{n+1}-(l_n+1))\big) \leq e^\kappa \lim_{n\to\infty} \frac{\mu\big(B\big(z,r_n+\mu^\beta\big(B(z,r_n)\big)\big)\big)}{\mu(B(z,r_n))} \leq e^\kappa.$$

Thus

$$\overline{\lim_{n\to\infty}} (l_{n+1}-(l_n+1)) < +\infty$$

and the proof is complete. $\square$

For every $n \geq 0$ let

$$\mathcal{R}_n := \{r \in (0, 2\mathrm{diam}(X)) : N_\kappa(z,r) = l_n\},$$

and, given in addition $0 \leq m \leq n$, let

$$\mathcal{R}(m,n) := \bigcup_{k=m}^{n} \mathcal{R}_k. \tag{15.1}$$

Now we make an additional substantial assumption that

$$\mathcal{S} = \{\varphi_e\}_{e\in E},$$

a conformal GDMS is given, and

$$\mu\big(J_\mathcal{S}\big) = 1. \tag{15.2}$$

For any $z \in J_\mathcal{S}$ and $r \in (0, 2\mathrm{diam}(X))$, define

$$W^-(z,r) := B_{J_\mathcal{S}}(z,r) \setminus A^*_{N_\kappa(z,r)}(z,r) \ \text{ and } \ W^+(z,r) := B_{J_\mathcal{S}}(z,r) \cup A^*_{N_\kappa(z,r)}(z,r). \tag{15.3}$$

Let us record the following two immediate consequences of this definition.

Observation 15.3. For every $z \in J_\mathcal{S}$ and $r \in (0, 2\mathrm{diam}(X))$, we have

$$W^-(z,r) \subseteq B_{J_\mathcal{S}}(z,r) \subseteq W^+(z,r).$$

Observation 15.4. For every $z \in J_\mathcal{S}$ and $r \in (0, 2\mathrm{diam}(X))$ both sets $W^-(z,r)$ and $W^+(z,r)$ can be represented as unions of cylinders of length $N_\kappa(z,r)$.

In the sequel we will frequently, without explicit mentioning it, use the formula

$$\mu\big(B_{J_{\mathcal{S}}}(z,r)\big) = \mu(B(z,r)),$$

which follows immediately from (15.2).

Fix $\kappa > 0$ so small that (14.4) holds and so that $\eta/\kappa > \beta_\mu(z)$. We shall prove the following.

Lemma 15.5. *For all $k \geq 0$ large enough, if $n - k \geq 2$, then*

$$W^+(z,s) \subseteq W^-(z,r)$$

for all $r \in \mathcal{R}_k$ and all $s \in \mathcal{R}_n$.

Proof. The assertion of our lemma is equivalent to the statement that

$$W^+(z,s) \cap A^*_{l_k}(z,r) = \emptyset.$$

Assume for a contradiction that there are sequences $\big(n_j\big)_{j=0}^\infty$ and $\big(k_j\big)_{j=0}^\infty$ of positive integers such that $\lim_{j\to\infty} k_j = +\infty$ and $n_j - k_j \geq 2$ for all $j \geq 0$, and also there are radii $r_j \in \mathcal{R}_{k_j}$ and $s_j \in \mathcal{R}_{n_j}$ such that

$$W^+(z,s_j) \cap A^*_{l_{k_j}}(z,r_j) \neq \emptyset,$$

for all $j \geq 0$. Since we know that for each $\omega \in E^*_A$,

$$\operatorname{diam}\big(\varphi_\omega(J)\big) \leq De^{-\eta|\omega|},$$

using Observation 14.11, we therefore conclude that

$$\begin{aligned} s_j + D\mu^{\eta/\kappa}(B(z,r_j)) &\geq s_j + D\exp\big(-\eta N_\kappa(z,r_j)\big) \geq s_j + D\varepsilon^{-\eta l_{k_j}} \\ &\geq s_j + D\varepsilon^{-\eta l_{n_j}} \geq r_j - D\varepsilon^{-\eta l_{k_j}} \\ &= r_j - D\exp\big(-\eta N_\kappa(z,r_j)\big) \\ &\geq r_j - D\mu^{\eta/\kappa}(B(z,r_j)). \end{aligned}$$

Hence, $s_j \geq r_j - 2D\mu^{\eta/\kappa}(B(z,r_j))$, and therefore,

$$\frac{\mu(B(z,s_j))}{\mu(B(z,r_j))} \geq \frac{\mu(B(z,r_j)) - \mu\big(A_\mu^{\eta/\kappa,2D}(z,r_j)\big)}{\mu(B(z,r_j))} = 1 - \frac{\mu\big(A_\mu^{\eta/\kappa,2D}(z,r_j)\big)}{\mu(B(z,r_j))}. \tag{15.4}$$

On the other hand,it follows from Observation 14.11 that

$$\mu(B(z,s_j)) \leq e^\kappa e^{-\kappa l_{n_j}} \quad \text{and} \quad \mu(B(z,r_j)) \geq e^{-\kappa l_{k_j}}.$$

This yields

$$\frac{\mu(B(z,s_j))}{\mu(B(z,r_j))} \leq e^\kappa \exp\big(-\kappa(l_{n_j} - l_{k_j})\big) \leq e^\kappa \exp\big(-\kappa(n_j - k_j)\big) \leq e^\kappa e^{-2\kappa} = e^{-\kappa}.$$

Along with (15.4) this implies that

$$\frac{\mu\big(A_\mu^{\eta/\kappa,2D}(z,r_j)\big)}{\mu(B(z,r_j))} \geq 1 - e^{-\kappa}. \tag{15.5}$$

However, since $\lim_{j\to\infty} r_j = 0$, since the measure μ is (WBT), and since $\kappa > 0$ was taken so small that $\eta/\kappa > \beta_\mu(z)$, we conclude that (15.5) may hold for finitely many integers $j \ge 0$ only, and the proof of Lemma 15.5 is complete. $\square$

As an immediate consequence of this lemma and Observation 15.3, we get the following.

Lemma 15.6. *For all integers $k \ge 0$ large enough, if $n - k \ge 2$, then*

$$W^-(z,s) \subseteq W^-(z,r) \quad \textit{and} \quad W^+(z,s) \subseteq W^+(z,r)$$

for all $r \in \mathcal{R}_k$ and all $s \in \mathcal{R}_n$.

Now we shall prove the following.

Proposition 15.7. *Let $\mathcal{S}$ be a conformal* GDMS*. Let μ be a Borel probability measure supported on $\overline{J}_{\mathcal{S}}$. Suppose that μ is* (WBT) *at some point $z \in J_{\mathcal{S}}$ which is not an atom of μ. Let $\mathcal{R}$ be an arbitrary countable set of positive reals containing* 0 *in its closure. Then there exists $(n_j = n_j(\mathcal{R}))_{j=0}^\infty$, a strictly increasing sequence of non-negative integers, with the following properties.*

(a) $n_{j+1} - n_j \le 4$,

(b) $n_{j+1} - n_j \ge 2$,

(c) *The set $\mathcal{R} \cap \mathcal{R}_{n_j} \neq \emptyset$ for infinitely many js.*

Proof. We construct the sequence $(n_j)_{j=0}^\infty$ inductively. Assume without loss of generality that $r_0 = 2\mathrm{diam}\big(\overline{J}_{\mathcal{S}}\big)$ and set $n_0 := 0$. For the inductive step suppose that $n_j \ge 0$ with some $j \ge 0$ has been constructed. Look at the set $\mathcal{R}(n_j+2, n_j+4)$; see (15.1) for its definition. If

$$\{l_k : n_j + 2 \le k \le n_j + 4\} \cap \{N_\kappa(z,r) : r \in \mathcal{R}\} \neq \emptyset,$$

take n_{j+1} to be an arbitrary number from $\{n_j+2, n_j+3, n_j+4\}$ such that

$$l_{n_{j+1}} \in \{N_\kappa(z,r) : r \in \mathcal{R}\}.$$

If, on the other hand,

$$\{l_k : n_j + 2 \le k \le n_j + 4\} \cap \{N_\kappa(z,r) : r \in \mathcal{R}\} = \emptyset,$$

set

$$n_{j+1} = n_j + 2.$$

Properties (a) and (b) are immediate from our construction. In order to prove (c) suppose on the contrary that

$$\mathcal{R} \cap \bigcup_{j=p}^\infty \mathcal{R}_{n_j} = \emptyset$$

with some $p \ge 0$. This yields $n_{j+1} = n_j + 2$ for all $j \ge p$, i.e. $n_j = n_p + 2(j-p)$ and

$$\bigcup_{j=p}^\infty \{l_k : n_p + 2(j+1-p) \le k \le n_p + 2(j+2-p)\} \cap \{N_\kappa(z,r) : r \in \mathcal{R}\} = \emptyset$$

But
$$\bigcup_{j=p}^{\infty}\{l_k : n_p+2(j+1-p)\le k\le n_p+2(j+2-p)\}=\{l_k : k\ge n_p+2\}.$$
Thus, $N_\kappa(z,r)\le n_p+1$ for all $r\in\mathcal{R}$. By Observation 14.11 this gives that $\mu(B(z,r))\ge \exp\big(-\kappa(n_p+1)\big)$ for all $r\in\mathcal{R}$, contrary to the facts that $0\in\overline{\mathcal{R}}$ and that z is not an atom of μ. We are done. ☐

Keep $\mathcal{R}$ arbitrary, with properties as in Proposition 15.7, until the proof of Theorem 15.10, where it will be determined. Let $n_j=n_j(\mathcal{R})$, $j\ge 1$, be the integers produced in Proposition 15.7. For every $j\ge 0$ fix arbitrarily $r_j\in\mathcal{R}_{n_j}$ requiring in addition that $r_j\in\mathcal{R}$ if $\mathcal{R}\cap\mathcal{R}_{n_j}\ne\emptyset$. Set

$$U^-_{l_{n_j}}(z):=\pi^{-1}\big(W^-(z,r_j)\big)\ \text{ and }\ U^+_{l_{n_j}}(z):=\pi^{-1}\big(W^+(z,r_j)\big). \tag{15.6}$$

These sets are well defined as the function $l:\mathbb{N}\to\mathbb{N}$ is 1-to-1 and, by (b), the function $j\mapsto n_j$ is also 1-to-1. Furthermore, for every $j\ge 0$ and every $l_{n_j}\le k<l_{n_{j+1}}$, define

$$U^\pm_k(z):=U^\pm_{l_{n_j}}(z). \tag{15.7}$$

In this way we have well-defined two sequences of open neighborhoods of $\pi^{-1}(z)$. We shall prove the following.

Proposition 15.8. *With hypotheses exactly as in Proposition* 15.7*, both* $\big(U^\pm_k(z)\big)_{k=0}^\infty$ *are descending sequences of open subsets of* E_A^∞ *satisfying conditions* (U0)–(U2).

Proof. (U0) is immediate from the very definition. If $k\ge 0$ and then $j=j_k\ge 0$ is uniquely chosen so that $l_{n_j}\le k<l_{n_{j+1}}$, then $U^\pm_k(z):=U^\pm_{l_{n_j}}(z)$, and both sets are disjoint unions of cylinders of length n_j by Observation 15.4 and since $r_j\in\mathcal{R}_{n_j}$, so also of length k as $k\ge l_{n_j}$. Thus (U1) holds. That both sequences $\big(U^\pm_k(z)\big)_{k=0}^\infty$ are descending follows immediately from Lemmas 15.6, property (b) of Proposition 15.7, and formulas (15.7) and (15.6). Applying formulas (15.7) and (15.6) along with Proposition 15.7 (b), Lemma 15.5, Observation 15.3, Observation 14.11, Lemma 15.2, and Proposition 15.7 (a), we get

$$\begin{aligned}
\mu\big(U^\pm_k(z)\big) &\le \mu\big(U^+_k(z)\big)=\mu\big(U_{l_{n_j}}(z)\big)=\mu\big(\pi^{-1}\big(W^+(z,r_j)\big)\big)\le\mu\big(\pi^{-1}\big(W^+(z,r_{j-1})\big)\big)\\
&=\mu\big(\pi^{-1}\big(W^-(z,r_{j-1})\big)\big)\le\mu\big(\pi^{-1}\big(B(z,r_{j-1})\big)\big)\le e^\kappa\exp\big(-\kappa N_\kappa(z,r_{j-1})\big)\\
&=e^\kappa e^{-l_{n_{j-1}}}=e^\kappa e^{-l_{n_{j+1}}}\exp\big(\kappa(l_{n_{j+1}}-l_{n_{j-1}})\big)\\
&\le e^\kappa e^{-l_{n_{j+1}}}\exp\big(\kappa\Delta l(z)\big(n_{j+1}-n_{j-1}\big)\big)\le e^\kappa e^{8\kappa\Delta l(z)}\exp\big(-\kappa l_{n_{j+1}}\big)\\
&\le\exp\big(\kappa((1+8\Delta l(z))\big)e^{-\kappa k},
\end{aligned}\tag{15.8}$$

and thus condition (U2) is satisfied with any $\rho\in(e^{-\kappa},1)$ sufficiently close to 1. The proof is complete. ☐

Proposition 15.9. *With hypotheses exactly as in Proposition* 15.7*, both* $\big(U^\pm_k(z)\big)_{k=0}^\infty$ *satisfy condition* (U3)*. If in addition either z is not pseudo-periodic for $\mathcal{S}$ or it is uniquely periodic* *and* $z \in \mathrm{Int} X$*, then* (U5) *holds. In the former case also* (U4) *holds while in the latter case it holds if in addition* μ *is an equilibrium state of the amalgamated function of a summable Hölder continuous system of functions.*

Proof. With the same arguments as in (15.8) we get that

$$\pi^{-1}(z) \subseteq \bigcap_{k=0}^{\infty} \overline{U_n^-(z)} \subseteq \bigcap_{k=0}^{\infty} \overline{U_n^+(z)} \subseteq \bigcap_{j=1}^{\infty} \pi^{-1}\big(\overline{B\big(z, r_{j-1}\big)}\big) \subseteq \pi^{-1}(z). \tag{15.9}$$

So (U3) holds as $\pi^{-1}(z)$ is a finite set. Assume now that z is not pseudo-periodic. Then condition (U4A) holds because of Lemma 13.1 and (15.9), while (U5) directly follows from Lemma 13.2 and the inclusion $U^{\pm}_{l_{n_j}}(z) \subseteq \pi^{-1}\big(B(z, r_{j-1})\big)$.

Assume in turn that $z \in \mathrm{Int}X$ is uniquely periodic point of $\mathcal{S}$ with prime period p. Then U_∞ consists of a periodic point, call it ξ, of period p because of (15.9). So, $\xi = \tau^\infty$ for a unique point $\tau \in E_A^\infty$. Condition (U5) directly follows from Lemma 13.3. Now we shall show that the sequence $\big(U_i^+(z)\big)_{i=0}^\infty$ satisfies the property (U4B). Indeed, without loss of generality we may assume that $i = l_k$, where $k = n_j$, $j \geq 0$. Take an arbitrary $\omega \in U^+_{l_k}(z)$. This means that $\omega|_{l_k} \in E_A^{l_k}$ and $\varphi_{\omega|_{l_k}}(J) \cap B(z, r_j) \neq \emptyset$. Then

$$\varphi_{\tau\omega|_{l_k}}(J) \cap B(z, r_j) \supseteq \varphi_{\tau\omega|_{l_k}}(J) \cap \varphi_\tau(B(z, r_j)) = \varphi_\tau\big(\varphi_{\omega|_{l_k}}(J) \cap B(z, r_j)\big) \neq \emptyset.$$

Hence, $\varphi_{\omega|_{l_k}}(J) \cap B(z, r_j) \neq \emptyset$, meaning that $\tau\omega \in U^+_{l_k}(z)$. So, the inclusion $\tau U^+_{l_k}(z) \subseteq U^+_{l_k}(z)$ has been proved and (6.1) of (U4B) holds for the sequence $\big(U_i^+(z)\big)_{i=0}^\infty$.

In order to establish (6.1) of (U4B) for the sequence $\big(U_i^-(z)\big)_{i=0}^\infty$, recall that $\eta > 0$ is so small that $||\varphi'_\omega|| \leq e^{-\eta|\omega|}$ for all $\omega \in E_A^*$. Take now $\kappa > 0$ as small as previously required and furthermore so small that $\beta\eta\kappa^{-1} > 2$. On the other hand, for every $k := n_j$, $j \geq 1$, we have

$$\varphi_\tau(W^-(z, r_j)) \subseteq \varphi_\tau(B(z, r_j)) \subseteq B\big(z, |\varphi'_\tau(z)| r_j\big)\big) \subseteq B\big(z, e^{-\eta|\tau|} r_j\big). \tag{15.10}$$

On the other hand, by (15.3) and the definition of l_{n_j}, we have that

$$\begin{aligned} W^-(z, r_j) &\supseteq B\big(z, r_j - De^{-\eta l_j}\big) \supseteq B\big(z, r_j - D\mu^{\eta/\kappa}(B(z, r_j))\big) \\ &\supseteq B\big(z, r_j - DC^{\eta/\kappa} r_j^{\beta\eta/\kappa}\big) \\ &\supseteq B\big(z, e^{-\eta|\tau|} r_j\big) \end{aligned}$$

provided that $\kappa > 0$ is taken sufficiently small (independently of j). Along with (15.10) this gives,

$$\varphi_\tau\big(W^-(z, r_j)\big) \subseteq W^-(z, r_j)).$$

Hence,

$$\begin{aligned} \pi\big(\tau U^-_{l_k}(z)\big) &= \pi\big(\tau\pi^{-1}(W^-(z, r_j))\big) = \varphi_\tau\big(\pi\big(\pi^{-1}(W^-(z, r_j))\big)\big) \\ &= \varphi_\tau\big(W^-(z, r_j)\big). \end{aligned}$$

Thus

$$\tau U^-_{l_k}(z) \subseteq \pi^{-1}(W^-(z, r_j)) = U^-_{l_k}(z).$$

Thus, the part (6.1) of (U4B) is established. In order to prove (6.2) of (U4B), let $k \geq 0$ and $j_k \geq 0$ be as in the proof of Proposition 15.8. The proof of this proposition gives that

$$U^\pm_k(z) \subseteq \pi^{-1}\big(W^-(z, r_{j_k-1})\big).$$

Since we now assume that $\varphi(\omega) = f^{\omega_0}(\pi(\sigma(\omega)))$, $\omega \in E_A^\infty$, where $\big(f^e\big)_{e\in E}$ is a Hölder continuous summable system of functions, condition (6.2) of (U4B) follows from continuity of the function $f^{\tau_0} : X_{t(\tau_0)} \to \mathbb{R}$ and the fact that $\lim_{k\to\infty} j_k = +\infty$. The proof of our proposition is complete. □

Now, we are in position to prove the following main result of this section, which is also one of the main results of the entire paper. Recall that the lower and upper escape rates $\underline{R}_\mu$ and $\overline{R}_\mu$ have been defined by formulas (7.1) and (7.2).

Theorem 15.10. *Let $\mathcal{S} = \{\varphi_e\}_{e\in E}$ be a finitely primitive Conformal Graph Directed Markov System. Let $\varphi : E_A^\infty \to \mathbb{R}$ be a Hölder continuous summable potential. As usual, denote its equilibrium/Gibbs state by μ_φ. Assume that the measure $\mu_\varphi \circ \pi_\mathcal{S}^{-1}$ is* (WBT) *at a point $z \in J_\mathcal{S}$. If z is either*

(a) *not pseudo-periodic,*

or

(b) *uniquely periodic, it belongs to* Int$X$ *(and $z = \pi(\xi^\infty)$ for a (unique) irreducible word $\xi \in E_A^*$), and φ is the amalgamated function of a summable Hölder continuous system of functions,*

then, with $\underline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon)) := \underline{R}_{\mu_\varphi}\big(\pi_\mathcal{S}^{-1}(B(z,\varepsilon))\big)$ and $\overline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon)) := \overline{R}_{\mu_\varphi}\big(\pi_\mathcal{S}^{-1}(B(z,\varepsilon))\big)$, we have that

$$\lim_{\varepsilon\to 0} \frac{\underline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon))}{\mu_\varphi \circ \pi_\mathcal{S}^{-1}(B(z,\varepsilon))} = \lim_{\varepsilon\to 0} \frac{\overline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon))}{\mu_\varphi \circ \pi_\mathcal{S}^{-1}(B(z,\varepsilon))} = \tag{15.11}$$

$$= d_\varphi(z) := \begin{cases} 1 & \text{if (a) holds} \\ 1 - \exp\big(\varphi_p(\xi) - p\mathrm{P}(\varphi)\big) & \text{if (b) holds,} \end{cases}$$

where in (b), *$\{\xi\} = \pi_\mathcal{S}^{-1}(z)$ and $p \geq 1$ is the prime period of ξ under the shift map.*

Proof. Assume for a contradiction that (15.11) does not hold. This means that there exists a strictly decreasing sequence $s_n(z) \to 0$ of positive reals such that at least one of the sequences

$$\left(\frac{\underline{R}_{\mathcal{S},\varphi}(B(z,s_n(z)))}{\mu_\varphi \circ \pi_\mathcal{S}^{-1}(B(z,s_n(z)))}\right)_{n=0}^\infty \quad \text{or} \quad \left(\frac{\overline{R}_{\mathcal{S},\varphi}(B(z,s_n(z)))}{\mu_\varphi \circ \pi_\mathcal{S}^{-1}(B(z,s_n(z)))}\right)_{n=0}^\infty$$

does not have $d_\varphi(z)$ as its accumulation point. Let

$$\mathcal{R} := \{s_n(z) : n \geq 0\}.$$

Let $\big(U_n^\pm(z)\big)_{n=0}^\infty$ be the corresponding sequence of open subsets of E_A^∞ produced in formula (15.7). Then, because of both Proposition 15.8 and Proposition 15.9, Proposition 7.2 applies to give

$$\lim_{n\to\infty}\frac{R_{\mu_\varphi}(U_n^\pm(z))}{\mu_\varphi(U_n^\pm(z))}=d_\varphi(z). \tag{15.12}$$

Let $(n_j)_{j=0}^\infty$ be the sequence produced in Proposition 15.7 with the help of $\mathcal{R}$ defined above. By this proposition there exists an increasing sequence $(j_k)_{k=0}^\infty$ such that $\mathcal{R}\cap\mathcal{R}_{n_{j_k}}\neq\emptyset$ for all $k\ge 1$. For every $k\ge 1$ pick one element $r_k\in\mathcal{R}\cap\mathcal{R}_{n_{j_k}}$. Set $q_k:=l_{n_{j_k}}$. By Observation 15.3 and formula (15.6), we then have

$$\begin{aligned}\frac{R_{\mu_\varphi}(U_{q_k}^-(z))}{\mu_\varphi(U_{q_k}^-(z))}\cdot\frac{\mu_\varphi(U_{q_k}^-(z))}{\mu_\varphi(B(z,r_k))} &\le \frac{\underline{R}_{\mathcal{S},\varphi}(B(z,r_k))}{\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}(B(z,r_k))}\le\frac{\overline{R}_{\mathcal{S},\varphi}(B(z,r_k))}{\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}(B(z,r_k))}\le\\ &\le\frac{R_{\mu_\varphi}(U_{q_k}^+(z))}{\mu_\varphi(U_{q_k}^+(z))}\cdot\frac{\mu_\varphi(U_{q_k}^+(z))}{\mu_\varphi(B(z,r_k))}.\end{aligned}\tag{15.13}$$

But, since $\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}$ is (WBT) at z, it is (DBT) at z by Proposition 14.12, and it therefore follows from (14.4) along with formulas (15.3) and (15.6) that

$$\lim_{k\to\infty}\frac{\mu_\varphi(U_{q_k}^-(z))}{\mu_\varphi(B(z,r_k))}=1=\lim_{k\to\infty}\frac{\mu_\varphi(U_{q_k}^+(z))}{\mu_\varphi(B(z,r_k))}.$$

Inserting this to (15.12) and (15.13), yields:

$$\lim_{k\to\infty}\frac{\underline{R}_{\mathcal{S},\varphi}(B(z,r_k))}{\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}(B(z,r_k))}=\lim_{k\to\infty}\frac{\overline{R}_{\mathcal{S},\varphi}(B(z,r_k))}{\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}(B(z,r_k))}=d_\varphi(z).$$

Since $r_k\in\mathcal{R}$ for all $k\ge 1$, this implies that $d_\varphi(z)$ is an accumulation point of both sequences $\big(\underline{R}_{\mathcal{S},\varphi}(B(z,r_k))/\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}(B(z,r_k))\big)_{n=1}^\infty$, $\big(\overline{R}_{\mathcal{S},\varphi}(B(z,r_k))/\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}(B(z,r_k))\big)_{n=1}^\infty$, and this contradiction finishes the proof of Theorem 15.10. □

Now, as an immediate consequence of Theorem 15.10 and Theorem 14.7, we get the following.

Theorem 15.11. *Assume that $\mathcal{S}$ is a finitely primitive conformal GDMS whose limit set $J_{\mathcal{S}}$ is geometrically irreducible. Let $\varphi:E_A^\infty\to\mathbb{R}$ be a Hölder continuous strongly summable potential. As usual, denote its equilibrium/Gibbs state by μ_φ. Then*

$$\lim_{\varepsilon\to0}\frac{\underline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon))}{\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))}=\lim_{\varepsilon\to0}\frac{\overline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon))}{\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))}=1 \tag{15.14}$$

for $\mu_\varphi\circ\pi_{\mathcal{S}}^{-1}$–a.e. point z of $\mathcal{J}_{\mathcal{S}}$.

In the realm of finite alphabets E, by virtue of Theorem 15.10 and both Theorem 14.9 and Theorem 14.10, we get the following stronger result.

Theorem 15.12. *Let $\mathcal{S} = \{\varphi_e\}_{e\in E}$ be a primitive Conformal Graph Directed Markov System with a finite alphabet E acting in the space $\mathbb{R}^d$, $d \geq 1$. Assume that either $d = 1$ or that the system $\mathcal{S}$ is geometrically irreducible. Let $\varphi : E_A^\infty \to \mathbb{R}$ be a Hölder continuous potential. As usual, denote its equilibrium/Gibbs state by μ_φ. Let $z \in J_{\mathcal{S}}$ be arbitrary. If either z is*

(a) *not pseudo-periodic,*
or
(b) *uniquely periodic, it belongs to* Int$X$ *(and $z = \pi(\xi^\infty)$ for a (unique) irreducible word $\xi \in E_A^*$), and φ is the amalgamated function of a summable Hölder continuous system of functions,*

then

$$\lim_{\varepsilon\to 0} \frac{\underline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))} = \lim_{\varepsilon\to 0} \frac{\overline{R}_{\mathcal{S},\varphi}(B(z,\varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z,\varepsilon))} = \tag{15.15}$$
$$= d_\varphi(z) := \begin{cases} 1 & \text{if (a) holds} \\ 1 - \exp\big(\varphi_p(\xi) - p\mathrm{P}(\varphi)\big) & \text{if (b) holds,} \end{cases}$$

where in (b), *$\{\xi\} = \pi_{\mathcal{S}}^{-1}(z)$ and $p \geq 1$ is the prime period of ξ under the shift map.*

16. The derivatives $\lambda_n'(t)$ and $\lambda_n''(t)$ of Leading Eigenvalues

In this section we have $\mathcal{S} = \{\varphi_e\}_{e\in E}$, a finitely primitive strongly regular conformal graph directed Markov system. We keep a parameter $t > \theta_{\mathcal{S}}$ and consider the Hölder continuous summable potential $\varphi_t : E_A^\infty \to \mathbb{R}$ given by the formula

$$\varphi_t(\omega) := t \log|\varphi'_{\omega_0}(\pi(\sigma(\omega)))|.$$

We further assume that a sequence $(U_n)_{n=0}^\infty$ of open subsets of E_A^∞ is given satisfying the conditions (U0)-(U5). The eigenvalues λ and λ_n along with other objects associated to the potential φ_t are now indicated with the letter/number t.

Our goal in this section is to calculate the asymptotic behavior of derivatives $\lambda_n'(t)$ and $\lambda_n''(t)$ of leading eigenvalues of unnormalized operators $\mathcal{L}_{t,n}$ when the integer $n \geq 0$ diverges to infinity and the parameter t approaches $b_{\mathcal{S}}$. Because of dealing with derivatives and because of wanting/needing our results about them to be of full strength and generality, we do impose in this section no normalizations on the operators $\mathcal{L}$ and $\mathcal{L}_n$. Note for example that the normalization $\lambda(t) = 1$ for all $t > \theta_{\mathcal{S}}$ "artificially" yields the derivatives of all orders of $\lambda(t)$ equal to 0. Also, the equation $\lambda(t) = 1$ (with no normalizations), known as Bowen's equation, has a rich geometric meaning; its (unique) solution is the Hausdorff dimension of the limit set $J_{\mathcal{S}}$, the meaning entirely lost after normalization.

The task of calculate the asymptotic behavior of derivatives $\lambda_n'(t)$ and $\lambda_n''(t)$ is truly tedious and technically involved. This is partially due to unboundedness of the function φ_t in the supremum norm and partially due to lack of uniform topological mixing on the sets $K_z(\varepsilon)$ introduced below.

The main theorems of this section form the crucial ingredients in the escape rates considerations of the next section, i.e. Section 17.

We start with the following.

Theorem 16.1. *For every* $0 \le n \le +\infty$*, the function* $(\theta_{\mathcal{S}}, +\infty) \ni t \mapsto \lambda_n(t) \in (0, +\infty)$ *is real analytic and*

$$\lambda'(t) = \lim_{n\to\infty} \lambda_n'(t). \tag{16.1}$$

Proof. By extending the transfer operators $\mathcal{L}_{t,n} : \mathcal{B}_\theta \to \mathcal{B}_\theta$ in the natural way to complex operators for all $t \in \mathbb{C}$ with $\mathrm{Re}(t) < \theta_{\mathcal{S}}$, and applying Kato-Rellich Perturbation Theorem (see [55]), along with Proposition 5.2, we see that for every $0 \le n \le +\infty$ there exists V_n, an open neighborhood of $(\theta_{\mathcal{S}}, +\infty)$, such that each function $(\theta_{\mathcal{S}}, +\infty) \ni t \mapsto \lambda_n(t) \in (0, +\infty)$ extends (and we keep the same symbol λ_n for this extension) to a holomorphic function from V_n to $\mathbb{C}$, and also each function $(\theta_{\mathcal{S}}, +\infty) \ni t \mapsto Q_n^{(t)} 1\!\!1 \in \mathcal{B}_\theta$ extends to a holomorphic function from V_n to $\mathbb{C}$ belonging to $\mathcal{B}_\theta$. Denote these latter extensions by

$$g_n : V_n \to \mathbb{C}, \; n \ge 0.$$

It is also a part of Kato-Rellich Theorem that

$$\mathcal{L}_{t,n} g_n(t) = \lambda_n(t) g_n(t) \tag{16.2}$$

for all $0 \le n \le +\infty$ and all $t \in V_n$. In particular, all the functions $(\theta_{\mathcal{S}}, +\infty) \ni t \mapsto \lambda_n(t) \in (0, +\infty)$, $0 \le n \le +\infty$, are real analytic. In order to prove (16.1), we shall derive first a "thermodynamical" formula for $\lambda_n'(t)$. Differentiating both sides of (16.2), we obtain

$$\mathcal{L}_{t,n}' g_n(t) + \mathcal{L}_{t,n} g_n'(t) = \lambda_n'(t) g_n(t) + \lambda_n(t) g_n'(t), \tag{16.3}$$

where

$$\mathcal{L}_{t,n}'(u)(\omega) := \sum_{\varepsilon \in E : A_{e\omega_0} = 1} 1\!\!1_{U_n^c}(e\omega) u(e\omega) \log |\varphi_e'(\pi(u))| \cdot |\varphi_e'(\pi(u))|^t, \tag{16.4}$$

and all four terms involved in (16.3) belong to $\mathcal{B}_\theta$. Applying the operator $Q_n^{(t)}$ to both sides of this equation, we get

$$Q_n^{(t)}\big(\mathcal{L}_{t,n}' g_n(t)\big) + Q_n^{(t)} \mathcal{L}_{t,n}\big(g_n'(t)\big) = \lambda_n'(t) Q_n^{(t)}\big(g_n(t)\big) + \lambda_n(t) Q_n^{(t)}\big(g_n'(t)\big).$$

Since

$$Q_n^{(t)}\big(g_n(t)\big) = g_n(t)$$

and

$$Q_n^{(t)} \mathcal{L}_{t,n}\big(g_n'(t)\big) = \mathcal{L}_{t,n} Q_n^{(t)}\big(g_n'(t)\big) = \lambda_n(t) Q_n^{(t)}\big(g_n'(t)\big),$$

we thus get

$$\lambda_n'(t) g_n(t) = Q_n^{(t)}\big(\mathcal{L}_{t,n}' g_n(t)\big). \tag{16.5}$$

Since in addition $Q_n^{(t)}$ is a projector onto the 1-dimensional space $\mathbb{C} g_n(t)$, this operator gives rise, similarly as in Section 10, to a unique bounded positive linear functional

$$\nu_{t,n} : \mathcal{B}_\theta \to \mathbb{C}$$

determined by the property that

$$Q_n^{(t)}(u) = \nu_{t,n}(u) g_n(t)$$

for every $u \in \mathcal{B}_\theta$. So, we can write (16.5) in the form

$$\lambda_n'(t) = \nu_{t,n}\big(\mathcal{L}_{t,n}' g_n(t)\big). \tag{16.6}$$

Now, writing

$$\ell(\omega) := \log |\varphi_{\omega_0}'(\pi(\sigma(\omega))|,$$

formula (16.4) readily gives

$$\mathcal{L}_{t,n}'(u) = \mathcal{L}_{t,n}(u\ell),$$

so that (16.5) takes on the form

$$\lambda_n'(t) = \nu_{t,n}\big(\mathcal{L}_{t,n}(\ell g_n(t))\big). \tag{16.7}$$

Keeping $t \in (\theta_{\mathcal{S}}, +\infty)$ and for the rest of the proof set

$$\psi_n := \mathcal{L}_{t,n}(\ell g_n(t)) \quad \text{and} \quad \psi := \mathcal{L}_{t,n}(\ell g(t)),$$

but remember that all ψ_n and ψ depend on t too. Now, we have

$$\begin{aligned} Q_n^{(t)}(\psi_n) - Q^{(t)}(\psi) &= \nu_{t,n}(\psi_n) g_n(t) - \nu_t(\psi) g(t) \\ &= \big(\nu_{t,n}(\psi_n) - \nu_t(\psi)\big) g(t) + (g_n(t) - g(t))\nu_{t,n}(\psi_n), \end{aligned}$$

and keep in mind that all objects considered here are understood in their unnormalized meaning. Hence,

$$\big(\nu_{t,n}(\psi_n) - \nu_t(\psi)\big) g(t) = Q_n^{(t)}(\psi_n) - Q^{(t)}(\psi) + (g(t) - g_n(t))\nu_{t,n}(\psi_n).$$

Therefore, recalling that the function $g(t)$ is everywhere positive and that $\nu_t(g(t)) = 1$, we get the following.

$$\begin{aligned} \Big|\nu_{t,n}(\psi_n) - \nu_t(\psi)\Big| &= \int \Big| Q_n^{(t)}(\psi_n) - Q^{(t)}(\psi) + \nu_{t,n}(\psi_n)(g(t) - g_n(t))\Big| d\,\nu_t \\ &\le \int \Big| Q_n^{(t)}(\psi_n) - Q^{(t)}(\psi)\Big| d\,\nu_t + \nu_{t,n}(-\psi_n) \int |g_n(t) - g(t)| d\,\nu_t. \end{aligned}$$

But, because of Proposition 5.2 (h),

$$\begin{aligned} \int |g_n(t) - g(t)| d\,\nu_t \le ||g_n(t) - g(t)||_* &= \Big\| Q_n^{(t)}(1\!\!1) - Q^{(t)}(1\!\!1)\Big\|_* \\ &\le \Big|\Big|\Big| Q_n^{(t)}(1\!\!1) - Q^{(t)}\Big|\Big|\Big| \|1\!\!1\|_\theta \\ &= \Big|\Big|\Big| Q_n^{(t)}(1\!\!1) - Q^{(t)}\Big|\Big|\Big| \longrightarrow 0 \end{aligned} \tag{16.8}$$

as $n \to 0$. Hence, in view (16.7), in order to conclude the theorem, it suffices to show that

$$\lim_{n\to\infty} \int \Big| Q_n^{(t)}(\psi_n) - Q^{(t)}(\psi)\Big| d\,\nu_t = 0 \tag{16.9}$$

and

$$M := \sup_{n\ge1}\{\nu_{t,n}(-\psi_n)\} < +\infty. \tag{16.10}$$

We first deal with the latter. We note that it follows from Proposition 5.2 (g) that

$$||g_n(t)||_\infty \le ||g_n(t)||_\theta = ||Q_n^{(t)}(1\!\!1)||_\theta \le C(t) < +\infty. \tag{16.11}$$

with some constant $C(t) \in (0,+\infty)$ and all integers $n \ge 0$. Now, since $t > \theta_{\mathcal{S}}$, it directly follows from the inequality

$$\big|\log|\varphi_e'|\big| \le ||\varphi_e'||^{-\varepsilon} \tag{16.12}$$

for every $\varepsilon > 0$ and all $e \in \mathbb{N}$ large enough that

$$\begin{aligned} ||\psi_n||_\infty &= ||\mathcal{L}_{t,n}\big(\ell g_n(t)\big)||_\infty \le ||g_n(t)||_\infty ||\mathcal{L}_{t,n}\ell||_\infty \\ &\le C(t)||\mathcal{L}_{t,n}\ell||_\infty \\ &\le C(t)||\mathcal{L}_t\ell||_\infty < +\infty \end{aligned} \tag{16.13}$$

for all $n \ge 1$ (including infinity) large enough. Now note that the same argument (only easier) as the one leading to Lemma 10.1 shows that all $\nu_{t,n}$ are in fact positive measure on E_A^∞, and, by (16.11), are uniformly bounded above by $C(t)$. So, (16.10) immediately follows from (16.13).

Now we shall prove that (16.9) holds. Write, as usually, $||h||_1 := ||h||_{L^1(\nu_t)}$ for all $h \in L^1(\nu_t)$. With the use of (16.18) we then estimate

$$\begin{aligned} ||Q_n^{(t)}(\psi_n) - Q^{(t)}(\psi)||_1 &= ||(Q_n^{(t)} - Q^{(t)})\psi_n + Q^{(t)}(\psi_n - \psi)||_1 \\ &\le ||(Q_n^{(t)} - Q^{(t)})\psi_n||_1 + ||Q^{(t)}(\psi_n - \psi)||_1 \\ &\le ||(Q_n^{(t)} - Q^{(t)})\psi_n||_* + ||Q^{(t)}||_1||\psi_n - \psi||_1 \\ &\le |||(Q_n^{(t)} - Q^{(t)})||| \cdot ||\psi_n||_\theta + ||Q^{(t)}||_1||\psi_n - \psi||_1 \\ &\le M_t|||(Q_n^{(t)} - Q^{(t)})||| + ||Q^{(t)}||_1||\psi_n - \psi||_1. \end{aligned}$$

Hence, applying Proposition 5.2 (h), we see that in order to prove that (16.9) holds, and by having done this, to conclude the proof of Theorem 16.1, it suffices to show that

$$\lim_{n\to\infty} ||\psi_n - \psi||_1 = 0. \tag{16.14}$$

It is well-known, and follows easily from (16.12) that $\ell \in L^p(\nu_t)$ for all real $p > 0$. Using Cauchy-Schwarz inequality we then estimate:

$$\begin{aligned}
||\psi_n - \psi||_1 &= \big\|\mathcal{L}_t(\ell\gamma_n) - \mathcal{L}_t(\ell\gamma)\big\|_1 = \big\|\mathcal{L}_t(\ell\gamma_n - \ell\gamma)\big\|_1 = ||\ell(\gamma_n - \gamma)||_1 \\
&\le ||\ell||_2||\gamma_n - \gamma||_2 = ||\ell||_2||g_n(t)\mathbb{1} - g(t)||_2 \\
&= ||\ell||_2||\mathbb{1}_n(\gamma_n(t) - \gamma(t)) + \gamma(t)(\mathbb{1}_n - \mathbb{1})||_2 \\
&\le ||\ell||_2\big(||\mathbb{1}_n(\gamma_n(t) - \gamma(t))||_2 + ||\gamma(t)\mathbb{1}_{U_n}||_2\big) \\
&\le ||\ell||_2\big(||\gamma_n(t) - \gamma(t)||_2 + ||\mathbb{1}_{U_n}||_2\big) \\
&\le ||\ell||_2\big(||\gamma_n(t) - \gamma(t)||_2 + \sqrt{\nu_t(U_n)}\big) \\
&\le ||\ell||_2\big(||\gamma_n(t) - \gamma(t)||_4 + \sqrt{\nu_t(U_n)}\big)
\end{aligned}$$

But $\lim_{n\to\infty} \nu_t(U_n) = 0$ and $\lim_{n\to\infty} ||g_n(t) - g(t)||_4 = 0$ because of (16.8) and (16.11). Hence, the formula (16.14) holds and the proof of Theorem 16.1 is complete. □

Now our goal is to show that the derivatives $\lambda_n''(t)$ are uniformly bounded above in appropriate domains of t and n. In order to do this we will need several auxiliary results. Our strategy is to apply the results of [25] for the family of operators

$$\big(\mathcal{L}_{t,n} : t \in (s-\delta, s+\delta), n \ge 0\big),$$

where $s > \theta_{\mathcal{S}}$ and $\delta > 0$ is small enough. It is evident from the form of our potentials $\varphi_t(\omega) = t \log|\varphi'_{\omega_0}(\pi(\sigma(\omega)))|$ that the distortion constants M_φ of Lemma 2.2 and Lemma 2.3 can be taken of common value for all $t \in (0, 2s - \theta_{\mathcal{S}}]$. Denote this common constant by M_s. An inspection of the proof of Lemma 3.2 leads to the following.

Lemma 16.2. *For every $\delta \in (0, s - \theta_{\mathcal{S}})$ there exists a constant $C_\delta \in (0, +\infty)$ such that for every $t \in [s-\delta, s+\delta]$, every integer $k \ge 0$, and every $g \in \mathcal{B}_\theta$, we have*

$$|\mathcal{L}_t^k g|_\theta \le C_\delta(\theta\lambda(t))^k(t)|g|_\theta + \lambda^k(t)||g||_1.$$

Since the function $(\theta_{\mathcal{S}}, +\infty) \ni t \mapsto \lambda(t)$ is strictly decreasing, denoting $\lambda(s-\delta)$ by M, as an immediate consequence of Lemma 16.2 we get the following.

Lemma 16.3. *For every $\delta \in (0, s - \theta_{\mathcal{S}})$, every $t \in [s-\delta, s+\delta]$, every integer $k \ge 0$, and every $g \in \mathcal{B}_\theta$, we have*

$$|\mathcal{L}_t^k g|_\theta \le C_\delta(\theta M)^k|g|_\theta + M^k||g||_1.$$

Lemma 4.2 directly translates into the following.

Lemma 16.4. *For every $\delta \in (0, s - \theta_{\mathcal{S}})$, every $t \in [s-\delta, s+\delta]$, every integer $k \ge 0$, and every $n \ge 0$, we have*

$$||\mathcal{L}_{t,n}^k||_* \le \lambda^k(t) \le M^k.$$

The proof of Corollary 4.5 provides exact estimates of constants, and gives the following.

Lemma 16.5. *For every $\delta \in (0, s - \theta_{\mathcal{S}})$, every $t \in [s-\delta, s+\delta]$, every integer $k \geq 0$, every integer $n \geq 0$, and every $g \in \mathcal{B}_\theta$, we have*

$$||\mathcal{L}_{t,n}^k g||_\theta \leq (C_\delta + 1)(\theta M)^k ||g||_\theta + (C_\delta + 1)(1 + \theta(1-\theta)^{-1}) M^k ||g||_*,$$

with some constant $C\delta \in (0, +\infty)$.

From now on throughout the entire section we assume that condition (U2) holds in the following uniform version:

(U2*) There exists $\rho \in (0,1)$ such that for some $\delta > 0$ and for all integers $n \geq 0$ we have

$$\sup\left\{\mu_t(U_n) : t \in [s-\delta, s+\delta]\right\} \leq \rho^n.$$

We now have the following.

Lemma 16.6. *For every $\delta \in (0, s - \theta_{\mathcal{S}})$, every $t \in [s-\delta, s+\delta]$ and every integer $n \geq 0$, we have*

$$|||\mathcal{L} - \mathcal{L}_n||| \leq 2\lambda(t)(\rho^{1/q})^n \leq 2M\rho^{n/q}.$$

Now, Lemmas 16.4, 16.2, and 16.6, along with formula (5.8), and compactness (in fact finite dimensionality) of the operators $\pi_k : \mathcal{B}_\theta \to \mathcal{B}_\theta$ imply that Theorem 1 in [25] along with all corollaries therein, applies to the family of operators

$$\big(\mathcal{L}_{t,n} : t \in (s-\delta, s+\delta), n \geq 0\big),$$

(i. e. $\mathcal{L}_s$ corresponds to P_0 and $\mathcal{L}_{t,n}$ correspond to operators P_ε) with

$$(t,n) \to s \quad \Leftrightarrow \quad t \to s \;\text{ and }\; n \to +\infty$$

to give the following extension of Proposition 5.2.

Proposition 16.7. *Fix $s > \theta_{\mathcal{S}}$. Then there exist $\delta \in (0, s - \theta_{\mathcal{S}})$ sufficiently small and an integer $n_s \geq 0$ sufficiently large such that for all $(t,n) \in (s-\delta, s+\delta) \times \{n_s, n_s+1, \ldots,\}$ there exist bounded operators $Q_n^{(t)}, \Delta_n^{(t)} : \mathcal{B}_\theta \to \mathcal{B}_\theta$ and complex numbers $\lambda_n(t) \neq 0$ with the following properties:*

(a) *$\lambda_n(t)$ is a simple eigenvalue of the operator $\mathcal{L}_{t,n} : \mathcal{B}_\theta \to \mathcal{B}_\theta$.*

(b) *$Q_t^{(n)} : \mathcal{B}_\theta \to \mathcal{B}_\theta$ is a projector ($Q_t^{(n)2} = Q_t^{(n)}$) onto the 1–dimensional eigenspace of $\lambda_n(t)$.*

(c) *$\mathcal{L}_{t,n} = \lambda_n(t) Q_t^{(n)} + \Delta_{t,n}$.*

(d) *$Q_t^{(n)} \circ \Delta_{t,n} = \Delta_{t,n} \circ Q_t^{(n)} = 0$.*

(e) *There exist* $\kappa \in (0,1)$ *and* $C > 0$ *(independent of* $(t,n) \in (s-\delta, s+\delta) \times \{n_s, n_s + 1, \ldots, \}$*) such that*

$$||\Delta_{t,n}^k||_\theta \le C(\kappa\lambda(t))^k$$

for all $k \ge 0$*. In particular,*

$$||\Delta_{t,n}^k g||_\infty \le ||\Delta_{t,n}^k g||_\theta \le C(\kappa\lambda(t))^k ||g||_\theta$$

for all $g \in \mathcal{B}_\theta$*.*

(f) $\lim_{(t,n)\to s} \lambda_n(t) = \lambda(s)$.

(g) *Enlarging the above constant* $C > 0$ *if necessary, we have*

$$||Q_t^{(n)}||_\theta \le C.$$

In particular,

$$||Q_t^{(n)} g||_\infty \le ||Q_t^{(n)} g||_\theta \le C||g||_\theta$$

for all $g \in \mathcal{B}_\theta$*.*

(h) $\lim_{(t,n)\to s} |||Q_t^{(n)} - Q_s||| = 0$.

We now pass to deal with the second derivatives $\lambda_n''(t)$. We start with close scrutiny of $\mathcal{L}_{t,n}\big(\ell\gamma\big)$, $\gamma \in \mathcal{B}_\theta$. We estimate for every $\gamma \in \mathcal{B}_\theta$:

$$
\begin{aligned}
||\mathcal{L}_{t,n}\big(\ell\gamma\big)||_1 &\le ||\mathcal{L}_{t,n}\big(\ell\gamma\big)||_\infty \le ||\mathcal{L}_{t,n}\big(\ell \cdot ||\gamma||_\infty\big)||_\infty \\
&= ||\gamma||_\infty ||\mathcal{L}_{t,n}\big(\ell\big)||_\infty \\
&\le ||\mathcal{L}_{t,n}\ell||_\infty ||\gamma||_\theta \\
&\le ||\mathcal{L}_t\ell||_\infty ||\gamma||_\theta.
\end{aligned}
\tag{16.15}
$$

We have already got a uniform upper bound on $|\psi_n|_\infty$. Let us now also estimate $|\psi_n|_\theta$. Write

$$\gamma_n := g_n(t)1\!\!1_{U_n^c}. \tag{16.16}$$

Then

$$\psi_n = \mathcal{L}_{t,n}(\ell g_n(t)) = \mathcal{L}_t(\ell\gamma_n). \tag{16.17}$$

We will in fact proceed more generally than merely estimating $|\psi_n|_\theta$. We shall prove, and we will need, the following.

Lemma 16.8. *There exists a constant* $C_t > 0$ *such that for every* $\gamma \in \mathcal{B}_\theta$*, we have that*

$$||\mathcal{L}_t(\ell\gamma)||_\theta \le C_t ||\gamma||_\theta.$$

Proof. By virtue of 16.15 it suffices to estimate $|\mathcal{L}_t(\ell\gamma)|_\theta$. Fix an integer $m \ge 0$, $\omega \in E_A^\infty$, and $\alpha, \beta \in [\omega|_m]$. Let $e \in E$ be such that $A_{e\alpha_0} = A_{e\beta_0} = 1$. Then

$$\begin{aligned}
&\Big|\ell(e\beta)\gamma(e\beta)|\varphi_e'(\pi(\beta))|^t - \ell(e\alpha)\gamma(e\alpha)|\varphi_e'(\pi(\alpha))|^t\Big| \\
&\quad = \Big|\ell(e\beta)\big(\gamma(e\beta)|\varphi_e'(\pi(\beta))|^t - \gamma(e\alpha)|\varphi_e'(\pi(\alpha))|^t\big) + \gamma(e\alpha)|\varphi_e'(\pi(\alpha))|^t\big(\ell(e\beta) - \ell(e\alpha)\big)\Big| \\
&\quad \le \Big|\ell(e\beta)\big(\gamma(e\beta)\big(|\varphi_e'(\pi(\beta))|^t - |\varphi_e'(\pi(\alpha))|^t\big) + |\varphi_e'(\pi(\alpha))|^t(\gamma(e\beta) - \gamma(e\alpha))\big)\Big| \\
&\qquad\qquad + \mathrm{osc}_{m+1}(\ell)(e\omega)\gamma(e\alpha)|\varphi_e'(\pi(\alpha))|^t \\
&\quad \le A\theta^{2m}|\ell(e\beta)\gamma(e\beta)| \cdot |\varphi_e'(\pi(\beta))|^t + |\varphi_e'(\pi(\alpha))|^t \mathrm{osc}_{m+1}(\gamma)(e\omega) + \\
&\qquad\qquad + A\theta^{2m}\gamma(e\alpha)|\varphi_e'(\pi(\alpha))|^t
\end{aligned}$$

with some constant $A \in (0, +\infty)$ and some constant $\theta \in (0,1)$ sufficiently close to 1. Hence, using also (16.15) and Lemma 3.1, we get

$$\begin{aligned}
&\big|\mathcal{L}_t(\ell\gamma)(\beta) - \mathcal{L}_t(\ell\gamma)(\alpha)\big| \\
&\qquad \le A\theta^{2m}\big(|\mathcal{L}_t(|\ell|\gamma)(\beta) + \mathcal{L}_t(\gamma)(\alpha)\big) + K^t\mathcal{L}_t\big(\mathrm{osc}_{m+1}(\gamma)\big)(\omega) \\
&\qquad \le A\theta^{2m}\big(||\mathcal{L}_t(\ell)||_\infty||\gamma||_\theta + ||\mathcal{L}_t(\gamma)||_\infty\big) + K^t\mathcal{L}_t\big(\mathrm{osc}_{m+1}(\gamma)\big)(\omega) \\
&\qquad \le A\theta^{2m}\big(||\mathcal{L}_t(\ell)||_\infty||\gamma||_\theta + ||\mathcal{L}_t(\gamma)||_\theta\big) + K^t\mathcal{L}_t\big(\mathrm{osc}_{m+1}(\gamma)\big)(\omega) \\
&\qquad \le A\theta^{2m}\big(||\mathcal{L}_t(\ell)||_\infty||\gamma||_\theta + ||\mathcal{L}_t||_\theta||\gamma||_\theta\big) + K^t\mathcal{L}_t\big(\mathrm{osc}_{m+1}(\gamma)\big)(\omega) \\
&\qquad \le A\theta^{2m}\big(||\mathcal{L}_t||_\theta + ||\mathcal{L}_t(\ell)||_\infty\big)||\gamma||_\theta + K^t\mathcal{L}_t\big(\mathrm{osc}_{m+1}(\gamma)\big)(\omega)
\end{aligned}$$

Therefore,

$$\mathrm{osc}\big(\mathcal{L}_t(\ell\gamma)\big)(\omega) \le A\theta^{2m}\big(||\mathcal{L}_t||_\theta + ||\mathcal{L}_t(\ell)||_\infty\big)||\gamma||_\theta + K^t\mathcal{L}_t\big(\mathrm{osc}_{m+1}(\gamma)\big)(\omega).$$

Thus, after integrating against measure ν_t, we get

$$\begin{aligned}
||\mathrm{osc}_m\big(\mathcal{L}_t(\ell\gamma)\big)||_{L^1(\nu_t)} &\le A\theta^{2m}\big(||\mathcal{L}_t||_\theta + ||\mathcal{L}_t(\ell)||_\infty\big)||\gamma||_\theta + K^t\int \mathcal{L}_t\big(\mathrm{osc}_{m+1}(\gamma)\big)\,d\nu_t \\
&= A\theta^{2m}\big(||\mathcal{L}_t||_\theta + ||\mathcal{L}_t(\ell)||_\infty\big)||\gamma||_\theta + K^t\int \mathrm{osc}_{m+1}(\gamma)\,d\nu_t \\
&\le A\theta^{2m}\big(||\mathcal{L}_t||_\theta + ||\mathcal{L}_t(\ell)||_\infty\big)||\gamma||_\theta + K^t\theta^{-(m+1)}|\gamma|_\theta.
\end{aligned}$$

Therefore,

$$\theta^{-2m}||\mathrm{osc}_m\big(\mathcal{L}_t(\ell\gamma)\big)||_{L^1(\nu_t)} \le \big(A(||\mathcal{L}_t||_\theta + ||\mathcal{L}_t(\ell)||_\infty) + K^t\theta^{-1}\big)||\gamma||_\theta.$$

Combining this with (16.15) we finally get

$$||\mathcal{L}_t(\ell\gamma)||_\theta \le \big(||\mathcal{L}_t\ell||_\infty + A(||\mathcal{L}_t||_\theta + ||\mathcal{L}_t(\ell)||_\infty) + K^t\theta^{-1}\big)||\gamma||_\theta.$$

So, the proof is complete. ∎

As a fairly straightforward consequence of this lemma we get the following.

Corollary 16.9. *There exists a constant $C_t' > 0$ such that for every $\gamma \in \mathcal{B}_\theta$ and all $n \ge 1$, we have that*

$$||\mathcal{L}_{t,n}(\ell\gamma)||_\theta \le C_t'||\gamma||_\theta.$$

Proof. By virtue of Lemma 4.4 (with $k = 1$) and Lemma 4.1 we get

$$|\mathbb{1}_n\gamma|_\theta \le |\gamma|_\theta + (1-\theta)^{-1}||\gamma||_* \le (1 + 2(1-\theta)^{-1})||\gamma||_\theta.$$

Of course,

$$||\mathbb{1}_n\gamma||_{L^1(\nu_t)} \le ||\gamma||_{L^1(\nu_t)} \le ||\gamma||_\theta.$$

Hence,

$$||\mathbb{1}_n\gamma||_\theta \le 2(1 + (1-\theta)^{-1})||\gamma||_\theta.$$

As

$$\mathcal{L}_{t,n}(\ell\gamma) = \mathcal{L}_t(\ell\gamma\mathbb{1}_n) = \mathcal{L}_t(\ell(\mathbb{1}_n\gamma)),$$

applying Lemma 16.8, we thus get

$$||\mathcal{L}_{t,n}(\ell\gamma)||_\theta = ||\mathcal{L}_t(\ell(\mathbb{1}_n\gamma))||_\theta \le 2C_t(1 + (1-\theta)^{-1})||\gamma||_\theta.$$

The proof is complete. □

It immediately follows from this corollary, along with (16.17) and (16.11), that

$$||\psi_n||_\theta \le M_t \tag{16.18}$$

with some constant $M_t \in (0, \infty)$ and all integers $n \ge 0$.

Now we are ready to prove the following.

Lemma 16.10. *For every $s > \theta_{\mathcal{S}}$ there exists $\eta \in (0,1)$ such that*

$$\Gamma := \sup_{n \ge n_s} \sup\{\lambda_n''(t) : t \in (s-\eta, s+\eta)\} < +\infty.$$

Proof. Throughout the whole proof we always assume that $t \in (s-\delta, s+\delta)$ and $n \ge n_s$, where $\delta > 0$ is the one produced in Proposition 16.7. Fix an integer $N \ge 1$ and differentiate the eigenvalue equation

$$\mathcal{L}_{t,n}^N g_n(t) = \lambda_n^N(t) g_n(t)$$

with respect the variable t two times. This gives in turn

$$(\mathcal{L}_{t,n}^N)'(g_n(t)) + \mathcal{L}_{t,n}^N(g_n'(t)) = N\lambda_n'(t)\lambda_n^{N-1}(t)g_n(t) + \lambda_n^N(t)g_n'(t)$$

and

$$\begin{aligned}(\mathcal{L}_{t,n}^N)''(g_n(t)) + (\mathcal{L}_{t,n}^N)'(g_n'(t)) + (\mathcal{L}_{t,n}^N)'(g_n'(t)) + \mathcal{L}_{t,n}^N(g_n''(t)) = \\ = N(N-1)\lambda_n^{N-2}(t)(\lambda_n'(t))^2 g_n(t) + N\lambda_n^{N-1}(t)\lambda_n'(t)g_n'(t) + \\ + N\lambda_n^{N-1}(t)\lambda_n''(t)g_n(t) + N\lambda_n^{N-1}(t)\lambda_n'(t)g_n'(t) + \lambda_n^N(t)g_n''(t).\end{aligned}$$

Equivalently:

$$\begin{aligned}(\mathcal{L}_{t,n}^N)''(g_n(t)) + 2(\mathcal{L}_{t,n}^N)'(g_n'(t)) + \mathcal{L}_{t,n}^N(g_n''(t)) =\\ = N(N-1)\lambda_n^{N-2}(t)(\lambda_n'(t))^2 g_n(t) + 2N\lambda_n^{N-1}(t)\lambda_n'(t)g_n'(t) +\\ + N\lambda_n^{N-1}(t)\lambda_n''(t)g_n(t) + \lambda_n^N(t)g_n''(t).\end{aligned}$$

Noting that

$$Q_t^{(n)}\mathcal{L}_{t,n}^N(g_n''(t)) = \mathcal{L}_{t,n}^N Q_t^{(n)}(g_n''(t)) = \lambda_n^N(t)Q_t^{(n)}(g_n''(t))$$

and applying to both sides of this equality the linear operator $Q_t^{(n)}$, we get

$$\begin{aligned}Q_t^{(n)}(\mathcal{L}_{t,n}^N)''(g_n(t)) + 2Q_t^{(n)}(\mathcal{L}_{t,n}^N)'(g_n'(t)) =\\ = N(N-1)\lambda_n^{N-2}(t)(\lambda_n'(t))^2 g_n(t) + 2N\lambda_n^{N-1}(t)\lambda_n'(t)Q_t^{(n)}(g_n'(t)) + N\lambda_n^{N-1}(t)\lambda_n''(t)g_n(t).\end{aligned}$$

Now since $(\mathcal{L}_{t,n}^N)'(g_n(t)) = \mathcal{L}_{t,n}^N(g_n(t)S_N\ell)$, and so $(\mathcal{L}_{t,n}^N)''(g_n(t)) = \mathcal{L}_{t,n}^N(g_n(t)(S_N\ell)^2)$, and since also $(\mathcal{L}_{t,n}^N)'(g_n'(t)) = \mathcal{L}_{t,n}^N(g_n'(t)S_N\ell)$, we thus get

(16.19)

$$\begin{aligned}Q_t^{(n)}\mathcal{L}_{t,n}^N(g_n(t)(S_N\ell)^2) + 2Q_t^{(n)}\mathcal{L}_{t,n}^N(g_n'(t)S_N\ell) =\\ = N(N-1)\lambda_n^{N-2}(t)(\lambda_n'(t))^2 g_n(t) + 2N\lambda_n^{N-1}(t)\lambda_n'(t)Q_t^{(n)}(g_n'(t)) + N\lambda_n^{N-1}(t)\lambda_n''(t)g_n(t).\end{aligned}$$

We first deal with the term $Q_t^{(n)}\mathcal{L}_{t,n}^N(g_n'(t)S_N\ell)$. We have

$$\begin{aligned}Q_t^{(n)}\mathcal{L}_{t,n}^N(g_n'(t)S_N\ell) &= Q_t^{(n)}\mathcal{L}_{t,n}^N\left(g_n'(t)\sum_{j=0}^{N-1}\ell\circ\sigma^j\right) = \sum_{j=0}^{N-1}Q_t^{(n)}\mathcal{L}_{t,n}^N\big(g_n'(t)\ell\circ\sigma^j\big)\\ &= \sum_{j=0}^{N-1}Q_t^{(n)}\mathcal{L}_{t,n}^{N-j}\big(\ell\mathcal{L}_{t,n}^j)'(g_n'(t))\big)\\ &= \sum_{j=0}^{N-1}\mathcal{L}_{t,n}^{N-j-1}Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\mathcal{L}_{t,n}^j)'(g_n'(t))\big)\\ &= \sum_{j=0}^{N-1}\lambda_n(t)^{N-j-1}Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\mathcal{L}_{t,n}^j)'(g_n'(t))\big).\end{aligned} \tag{16.20}$$

Now, by virtue of Proposition 16.7, particularly by its parts (c) and (e), and by Corollary 16.9, we get

$$\begin{aligned}
\Big\|\mathcal{L}_{t,n}\big(\ell\mathcal{L}^j_{t,n}(g'_n(t)\big) - \lambda_n(t)^j\mathcal{L}_{t,n}\big(\ell Q_t^{(n)}(g'_n(t)\big)\Big\|_\infty &\leq \\
&\leq \Big\|\mathcal{L}_{t,n}\big(\ell\mathcal{L}^j_{t,n}(g'_n(t)\big) - \lambda_n(t)^j\mathcal{L}_{t,n}\big(\ell Q_t^{(n)}(g'_n(t)\big)\Big\|_\theta \\
&= \Big\|\mathcal{L}_{t,n}\big(\ell\big(\mathcal{L}^j_{t,n}(g'_n(t)\big) - \lambda_n(t)^j Q_t^{(n)}(g'_n(t)\big)\big)\big)\Big\|_\theta \\
&= \big\|\mathcal{L}_{t,n}\big(\ell\Delta_n^j(g'_n(t))\big)\big\|_\theta \\
&\leq C'||\Delta_n^j(g'_n(t))||_\theta \\
&\leq C'C(\kappa\lambda(t))^j||g'_n(t)||_\theta.
\end{aligned}$$

Therefore, by item (g) of Proposition 16.7 we get

$$\Big\|Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\mathcal{L}^j_{t,n}(g'_n(t))\big) - \lambda_n^j(t)Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell Q_t^{(n)}(g'_n(t))\big)\Big\|_\infty \leq C'C^2(\kappa\lambda(t))^j||g'_n(t)||_\theta. \tag{16.21}$$

On the other hand, because of (16.7), we get

$$\begin{aligned}
\lambda'_n(t)Q_t^{(n)}(g'_n(t)) = \nu_{t,n}(g'_n(t))\lambda'_n(t)g_n(t) &= \nu_{t,n}(g'_n(t))Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell g_n(t)\big) \\
&= Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\nu_{t,n}(g'_n(t))g_n(t)\big) \\
&= Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell Q_t^{(n)}(g'_n(t))\big)
\end{aligned}$$

Therefore, using (16.20) and (16.21), we get

$$\begin{aligned}
\Big\|\lambda_n^{1-N}(t)N^{-1}\Big(Q_t^{(n)}\mathcal{L}_{t,n}\big(g'_n(t)S_N(\ell)\big) - N\lambda_n^{N-1}(t)\lambda'_n(t)Q_t^{(n)}(g'_n(t))\Big)\Big\|_\infty &= \\
&= \Big\|\frac{1}{N}\sum_{j=0}^{N-1}\lambda_n(t)^{-j}Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\mathcal{L}^j_{t,n}(g'_n(t))\big) - Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell Q_t^{(n)}(g'_n(t))\big)\Big\|_\infty \\
&= \Big\|\frac{1}{N}\sum_{j=0}^{N-1}\Big[\lambda_n(t)^{-j}Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\mathcal{L}^j_{t,n}(g'_n(t))\big) - Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell Q_t^{(n)}(g'_n(t))\big)\Big]\Big\|_\infty \\
&\leq \frac{1}{N}\sum_{j=0}^{N-1}\Big\|\lambda_n(t)^{-j}Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\mathcal{L}^j_{t,n}(g'_n(t))\big) - Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell Q_t^{(n)}(g'_n(t))\big)\Big\|_\infty \\
&\leq \frac{1}{N}\sum_{j=0}^{N-1}C'C^2||g'_n(t)||_\theta\kappa^j.
\end{aligned}$$

Therefore, for all pairs (t,n) sufficiently close to s, we have

$$\lim_{N\to\infty}\Big\|2N^{-1}\lambda_n^{1-N}(t)\Big(Q_t^{(n)}\mathcal{L}^N_{t,n}\big(g'_n(t)S_N(\ell)\big) - N\lambda_n^{N-1}(t)\lambda'_n(t)Q_t^{(n)}(g'_n(t))\Big)\Big\|_\infty = 0$$

where the convergence, in the supremum norm $||\cdot||_\infty$ is uniform with respect to all t sufficiently close to s. Inserting this to (16.19), we thus get

(16.22)

$$\lambda_n''(t)g_n(t) = \lim_{n\to\infty}\Big[N^{-1}\lambda_n^{1-N}(t)Q_t^{(n)}\mathcal{L}_{t,n}^N\big(g_n(t)(S_N(\ell))^2\big)\Big] - \lambda_n^{-1}(t)(N-1)g_n(t)(\lambda_n'(t))^2g_n(t),$$

with the same meaning of convergence as above.

Let us first deal with the term $Q_t^{(n)}\mathcal{L}_{t,n}^N(g_n(t)(S_N\ell)^2)$. We have

(16.23)

$$\begin{aligned}
&Q_t^{(n)}\mathcal{L}_{t,n}^N(g_n(t)(S_N\ell)^2) = \\
&\quad = 2\sum_{i=0}^{N-1}\sum_{j=i+1}^{N-1} Q_t^{(n)}\mathcal{L}_{t,n}^N\big(g_n(t)\ell\circ\sigma^i\cdot\ell\circ\sigma^j\big) + \sum_{j=0}^{N-1} Q_t^{(n)}\mathcal{L}_{t,n}^N\big(g_n(t)\ell^2\circ\sigma^j\big) \\
&\quad = \sum_{j=0}^{N-1} Q_t^{(n)}\mathcal{L}_{t,n}^N\big(g_n(t)\ell^2\circ\sigma^j\big) + 2\sum_{i=0}^{N-1}\sum_{j=i+1}^{N-1} Q_t^{(n)}\mathcal{L}_{t,n}^N\big(g_n(t)(\ell\cdot\ell\circ\sigma^{j-i})\circ\sigma^i\big) \\
&\quad = \sum_{j=0}^{N-1} Q_t^{(n)}\mathcal{L}_{t,n}^{N-j}\big(\ell^2\mathcal{L}_{t,n}^j(g_n(t))\big) + 2\sum_{i=0}^{N-1}\sum_{j=i+1}^{N-1} Q_t^{(n)}\mathcal{L}_{t,n}^{N-i}\big(\ell\cdot\ell\circ\sigma^{j-i}\mathcal{L}_{t,n}^i(g_n(t))\big) \\
&\quad = \lambda_n^j(t)\sum_{j=0}^{N-1} Q_t^{(n)}\mathcal{L}_{t,n}^{N-j}\big(g_n(t)\ell^2\big) + 2\lambda_n^i(t)\sum_{i=0}^{N-1}\sum_{j=i+1}^{N-1} Q_t^{(n)}\mathcal{L}_{t,n}^{N-i}\big(g_n(t)\ell\cdot\ell\circ\sigma^{j-i})\big) \\
&\quad = \lambda_n^j(t)\sum_{j=0}^{N-1} Q_t^{(n)}\mathcal{L}_{t,n}^{N-j}\big(g_n(t)\ell^2\big) + 2\lambda_n^i(t)\sum_{i=0}^{N-1}\sum_{j=i+1}^{N-1} Q_t^{(n)}\mathcal{L}_{t,n}^{N-j}\big(\ell\mathcal{L}_{t,n}^{j-i}(\ell g_n(t))\big) \\
&\quad = \lambda_n^{N-1}(t)\sum_{j=0}^{N-1} Q_t^{(n)}\mathcal{L}_{t,n}\big(g_n(t)\ell^2\big) + 2\sum_{i=0}^{N-1}\sum_{j=i+1}^{N-1} \lambda_n^{N+i-(j+1)}(t)Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\mathcal{L}_{t,n}^{j-i}(\ell g_n(t))\big) \\
&\quad = \lambda_n^{N-1}(t)Q_t^{(n)}\mathcal{L}_{t,n}\big(g_n(t)\ell^2\big) + 2\sum_{k=1}^{N-1} \lambda_n^{N-k-1}(t)(N-k)Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\mathcal{L}_{t,n}^k(\ell g_n(t))\big).
\end{aligned}$$

Now, using Proposition 16.7 (c) (and (d)) and denoting $\psi_n = \mathcal{L}_{t,n}\ell g_n(t))$, we get

$$\begin{aligned}
Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\mathcal{L}_{t,n}^k(\ell g_n(t))\big) &= Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\mathcal{L}_{t,n}^{k-1}\big(\mathcal{L}_{t,n}\ell g_n(t))\big)\big) = Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\mathcal{L}_{t,n}^{k-1}(\psi_n)\big)\\
&= Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell(\lambda_n^{k-1}(t)Q_t^{(n)}(\psi_n) + \Delta_n^{k-1}(\psi_n))\big)\\
&= \lambda_n^{k-1}(t)Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell Q_t^{(n)}(\psi_n)\big) + Q_t^{(n)}\mathcal{L}_{t,n}\big(\Delta_n^{k-1}(\psi_n)\big)\\
&= \lambda_n^{k-1}(t)Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\nu_{t,n}(\psi_n)g_n(t)\big) + Q_t^{(n)}\mathcal{L}_{t,n}\big(\Delta_n^{k-1}(\psi_n)\big)\\
&= \lambda_n^{k-1}(t)\nu_{t,n}(\psi_n)Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell g_n(t)\big) + Q_t^{(n)}\mathcal{L}_{t,n}\big(\Delta_n^{k-1}(\psi_n)\big)\\
&= \lambda_n^{k-1}(t)\nu_{t,n}(\psi_n)Q_t^{(n)}(\psi_n) + Q_t^{(n)}\mathcal{L}_{t,n}\big(\Delta_n^{k-1}(\psi_n)\big)\\
&= \lambda_n^{k-1}(t)\big(\nu_{t,n}(\psi_n)\big)^2 g_n(t) + Q_t^{(n)}\mathcal{L}_{t,n}\big(\Delta_n^{k-1}(\psi_n)\big)\\
&= \lambda_n^{k-1}(t)\big(\lambda_n'(t)\big)^2 g_n(t) + Q_t^{(n)}\mathcal{L}_{t,n}\big(\Delta_n^{k-1}(\psi_n)\big).
\end{aligned}$$

Therefore, using Proposition 16.7 again, we get

$$\begin{aligned}
&2\sum_{k=1}^{N-1}\lambda_n^{N-k-1}(t)(N-k)Q_t^{(n)}\mathcal{L}_{t,n}\big(\ell\mathcal{L}_{t,n}^k(\ell g_n(t))\big) =\\
&\quad= \lambda_n^{N-2}(t)N(N-1)\big(\lambda_n'(t)\big)^2 g_n(t) + 2\sum_{k=1}^{N-1}\lambda_n^{N-k-1}(t)Q_t^{(n)}\mathcal{L}_{t,n}\big(\Delta_n^{k-1}(\psi_n)\big)\\
&\quad= \lambda_n^{N-2}(t)N(N-1)\big(\lambda_n'(t)\big)^2 g_n(t) + 2\sum_{k=1}^{N-1}\lambda_n^{N-k-1}(t)\mathcal{L}_{t,n}Q_t^{(n)}\big(\Delta_n^{k-1}(\psi_n)\big)\\
&\quad= \lambda_n^{N-2}(t)N(N-1)\big(\lambda_n'(t)\big)^2 g_n(t) + 2\lambda_n^{N-2}(t)\mathcal{L}_{t,n}Q_t^{(n)}(\psi_n)\\
&\quad= \lambda_n^{N-2}(t)N(N-1)\big(\lambda_n'(t)\big)^2 g_n(t) + 2\lambda_n^{N-2}(t)\mathcal{L}_{t,n}\big(\nu_{t,n}(\psi_n)g_n(t)\big)\\
&\quad= \lambda_n^{N-2}(t)N(N-1)\big(\lambda_n'(t)\big)^2 g_n(t) + 2\lambda_n^{N-2}(t)\nu_{t,n}(\psi_n)\mathcal{L}_{t,n}\big(g_n(t)\big)\\
&\quad= \lambda_n^{N-2}(t)N(N-1)\big(\lambda_n'(t)\big)^2 g_n(t) + 2\lambda_n^{N-1}(t)\lambda_n'(t)g_n(t).
\end{aligned}$$

In consequence, denoting by $T_N(t,n)$ the function whose limit (as $n\to\infty$) is calculated in (16.22), and utilizing (16.23), we get

$$T_N(t,n) = Q_t^{(n)}\mathcal{L}_{t,n}\big(g_n(t)\ell^2\big) + \frac{2}{N}\lambda_n'(t)g_n(t)$$

It thus follows from (16.22) that

$$\lambda_n''(t)g_n(t) = Q_t^{(n)}\mathcal{L}_{t,n}\big(g_n(t)\ell^2\big). \tag{16.24}$$

Since, by Proposition 5.3, all the operators $Q_t^{(n)} : \mathcal{B}_\theta \to \mathcal{B}_\theta$ are positive, and because of this also non-decreasing, and $g_n(t) = Q_t^{(n)}1\!\!1$ is non-negative, the formula (16.24) yields the following.

$$\begin{aligned}\lambda_n''(t)g_n(t) &\le ||g_n(t)||_\infty Q_t^{(n)}\big(\mathcal{L}_{t,n}(\ell^2)\big) \le ||g_n(t)||_\infty ||\mathcal{L}_{t,n}(\ell^2)||_\infty Q_t^{(n)}1\!\!1 \\ &\le ||g_n(t)||_\infty ||\mathcal{L}_t(\ell^2)||_\infty g_n(t) \\ &\le 2||g_n(t)||_\infty ||\mathcal{L}_s(\ell^2)||_\infty g_n(t),\end{aligned}$$

where the last inequality was written for t sufficiently close to s. Canceling out $g_n(t)$ and noticing that by Proposition 16.7 (g), $||g_n(t)||_\infty = ||Q_t^{(n)}1\!\!1||_\infty \le C$, we now finally obtain that

$$\lambda_n''(t) \le 2C||\mathcal{L}_s(\ell^2)||_\infty,$$

and the proof is complete. □

Now we shall prove the following.

Lemma 16.11. *We have*

(a) *For every $n \ge 1$ the function $(\theta_{\mathcal{S}}, +\infty) \ni t \mapsto \lambda_n(t)$ is decreasing.*

(b) *For every $s \in (\theta_{\mathcal{S}}, +\infty)$ and for every $n \ge 1$ large enough there exists $\delta > 0$ such that the function $\lambda_n|_{(s-\delta,s+\delta)}$ is strictly decreasing, in fact $\lambda_n' \le \frac{1}{4}\lambda'(s)$ on $(s-\delta, s+\delta)$.*

(c) *For every $t \in (\theta_{\mathcal{S}}, +\infty)$ and for every $n \ge 1$, $\lambda_n(t) \le \lambda(t)$.*

(d) *For every $n \ge 1$, $\lim_{t\to+\infty} \lambda_n(t) = 0$.*

(e) *For every $n \ge 1$ large enough there exists a unique $b_n > 0$ such that $\lambda_n(b_n) = 1$.*

Proof. For part (a), Proposition 16.7 implies that

$$\lambda_n(t) = \lim_{k\to\infty} \big|\big|\mathcal{L}_{t,n}^k 1\!\!1\big|\big|_\infty^{1/k}, \tag{16.25}$$

and since for each $n \ge 1$ the function $t \mapsto \big|\big|\mathcal{L}_{t,n}^k 1\!\!1\big|\big|_\infty$ is decreasing, item (a) follows immediately. For part (b) note that $\lambda'(s) < 0$. Hence, by Theorem 16.1, $\lambda_n'(s) < \frac{1}{2}\lambda'(s) < 0$ for all $n \ge 1$ large enough, say $n \ge N_1$. Take now $\delta \in (0, \eta)$ so small that $\Gamma\delta \le -\frac{1}{4}\lambda'(s)$, where $\Gamma \ge 0$ is the constant coming from Lemma 16.10. By the Mean Value Theorem $\lambda_n'(t) = \lambda_n'(s) + \lambda_n''(u)(t-s)$ for every $t \in (s-\delta, s+\delta)$ and some $u \in (s-\delta, s+\delta)$ depending on t. Hence, applying Lemma 16.10, we get for all $n \ge N_1$ and all $t \in (s-\delta, s+\delta)$ that

$$\lambda_n'(t) < \frac{1}{2}\lambda'(s) + \Gamma\delta < 0.$$

Thus item (b) is proved. Similarly as in item (a), item (c) immediately follows from (16.25) and inequality $\mathcal{L}_{t,n}^k 1\!\!1 \le \mathcal{L}_t^k 1\!\!1$. Item (d) is an immediate consequence of item (c) and the well-known fact (see [31]) that $\lim_{t\to+\infty} \lambda(t) = 0$. Proving (e), it is well-known (see again [31]) that there exists a unique $b \in (\theta_{\mathcal{S}}, +\infty)$ such that

$$\lambda(b) = 1.$$

Let $\delta > 0$ be the value produced in item (b) for $s = b$. We know that

$$\lambda\big(b - \frac{1}{2}\delta\big) > 0 \ \text{ and } \ \lambda\big(b + \frac{1}{2}\delta\big) < 0.$$

It the follows from Proposition 16.7 (f) that

$$\lambda_n\big(b - \frac{1}{2}\delta\big) \geq \frac{1}{2}\lambda\big(b - \frac{1}{2}\delta\big) > 0 \quad \text{and} \quad \lambda_n\big(b + \frac{1}{2}\delta\big) \leq \frac{1}{2}\lambda\big(b + \frac{1}{2}\delta\big) < 0$$

for all $n \geq 1$ large enough, say $n \geq N_2$. Because of the choice of $\delta > 0$ and because of item (b), we may also have $N_2 \geq 1$ so large that the function $\lambda_n\big|_{[b-\frac{1}{2}\delta, b+\frac{1}{2}\delta]}$ is strictly decreasing for every $n \geq N_2$. Therefore, for every $n \geq N_2$ the function $\lambda_n\big|_{[b-\frac{1}{2}\delta, b+\frac{1}{2}\delta]}$ has a unique zero. Along with item (a) this finishes the proof of item (e). The proof of Lemma 16.11 is complete. □

Remark 16.12. With the help of Proposition 16.7 we could have strengthened Theorem 16.1 to show uniform convergence with respect to t ranging over compact subsets of $(\theta_{\mathcal{S}}, +\infty)$. However, we really do not need this in the current paper.

By analogy to the unperturbed case, we call the numbers b_n produced in this lemma Bowen's parameters. Now we can prove the following.

Proposition 16.13. *With the settings of the current section (in particular with the stronger condition* (U2*) *replacing* (U2)*), we have*

$$\lim_{n\to\infty} \frac{b - b_n}{\mu_b(U_n)} = \begin{cases} 1/\chi_{\mu_b} & \text{if (U4A) holds} \\ \big(1 - |\varphi'_\xi(\pi(\xi^\infty))|\big)/\chi_{\mu_b} & \text{if (U4B) holds} \end{cases}$$

Proof. Since the functions $(\theta_{\mathcal{S}}, +\infty) \ni t \mapsto \lambda_n(t)$, $n \geq 1$, are all real-analytic by the Kato-Rellich Perturbation Theorem, making use of Lemma 16.10, we can apply Taylor's Theorem to get

$$1 = \lambda_n(b_n) = \lambda_n(b) + \lambda'(b) + \mathcal{O}((b - b_n)^2).$$

Equivalently,

$$\frac{1 - \lambda_n(b)}{b - b_n} = -\lambda'(b) + \mathcal{O}(b - b_n).$$

Denoting by $d(\xi)$ the right-hand side of the formula appearing in Proposition 6.1, and using this proposition along with the fact that $\lambda(b) = 1$, we thus get

$$\lim_{n\to\infty} \frac{\mu_b(U_n)}{b - b_n} = -\lambda'(b) d^{-1}(\xi).$$

Equivalently,

$$\lim_{n\to\infty} \frac{b - b_n}{\mu_b(U_n)} = -\frac{1}{\lambda'(b)} d(\xi). \tag{16.26}$$

But expanding (16.7) with $n = \infty$, we get $\lambda'(b) = -\lambda(b)\chi_{\mu_b} = -\chi_{\mu_b}$, and inserting this into (16.26) completes the proof. □

Now we shall link Bowen's parameters b_n to geometry. We shall prove the following.

Theorem 16.14. *Let $\mathcal{S}=\{\varphi_e\}_{e\in E}$ be a finitely primitive strongly regular conformal graph directed Markov system. Let $(U_n)_{n=0}^\infty$ be a nested sequence of open subsets of E_A^∞ satisfying conditions* (U0), (U1), *and* (U2*) *with $s=b_{\mathcal{S}}$. Recall that*

$$K(U_n)=\bigcap_{k=0}^\infty \sigma^{-k}(U_n^c)=\big\{\omega\in E_A^\infty:\forall_{(k\ge 0)}\ \sigma^k(\omega)\notin U_n\big\}$$

for all $n\ge 0$ and denote

$$K_n:=\pi_{\mathcal{S}}(K(U_n)).$$

Then

$$\mathrm{HD}(K_n)=b_n$$

for all $n\ge 0$ large enough.

Proof. Put

$$h_n:=\mathrm{HD}(K_n).$$

We first shall prove that

$$h_n\le b_n$$

for all $n\ge 0$ large enough. Assume that $\delta>0$ is chosen so small that the conclusion of Lemma 16.11 (b)holds. Take then an arbitrary $t>b_n$. Fix any $q\ge 1$. Define

$$K_q(U_n):=\big\{\omega\in K(U_n):\omega_n=q \text{ for infinitely many } n\big\}$$

and

$$K_n(q):=\pi(K_q(U_n)).$$

Our first goal is to show that

$$\mathrm{HD}(K_n(q))\le b_n \tag{16.27}$$

for all $n\ge 0$ large enough. Indeed, for every $k\ge 1$ let

$$\tilde E_k(q):=\big\{\omega|_k:\omega\in K_q(U_n)\ \text{ and }\ \omega_{k+1}=q\big\}.$$

Fix an arbitrary $\alpha\in E_A^\infty$ such that $q\alpha\in E_A^\infty$. Then, using (BDP), Proposition 16.7 (c), (e), and (g), along with Lemma 3.1, we get

$$\begin{aligned}
\sum_{\tau\in\tilde E_k(q)}\mathrm{diam}^t\big(\varphi_\tau(X_{t(\tau)})\big)&=\sum_{\tau\in\tilde E_k(q)}\mathrm{diam}^t\big(\varphi_\tau(X_{t(q)})\big)\asymp\sum_{\tau\in\tilde E_k(q)}||\varphi_\tau'||^t\\
&\asymp\sum_{\tau\in\tilde E_k(q)}|\varphi_\tau'(\pi(q\alpha))|^t\le\mathcal{L}_t^k(1\!\!1_n^k)(q\alpha)=\mathcal{L}_{t,n}^k(1\!\!1)(q\alpha)\\
&=\lambda_n^k(t)Q_t^{(n)}(1\!\!1)(q\alpha)+S^k(1\!\!1)(q\alpha)\\
&\le\lambda_n^k(t)||Q_t^{(n)}(1\!\!1)||_\infty+||S^k(1\!\!1)||_\infty\\
&\le C\lambda_n^k(t)+C(\kappa\lambda(t))^k\\
&=C(\lambda_n^k(t)+(\kappa\lambda(t))^k).
\end{aligned}$$

Therefore, for every $k \geq 0$, using the facts that $\lambda_n(t) < 1$ (Lemma 16.11(b)), that $\kappa < 1$, and that $\kappa\lambda(t) < 1$ if $n \geq 1$ is sufficiently large so that the perturbed Bowen's parameter b_n is sufficiently close to the (unperturbed) Bowen's parameter b, we get

$$\begin{aligned}(16.28)\qquad \sum_{k=l}^{\infty}\sum_{\tau\in\tilde{E}_k(q)}\text{diam}^t\big(\varphi_\tau(X_{t(\tau)})\big) &\leq C\sum_{k=l}^{\infty}(\lambda_n^k(t)+(\kappa\lambda(t))^k)\\ &\leq C\big(1-\lambda_n(t))^{-1}\lambda_n^l(t)+(1-\kappa\lambda(t))^{-1}(\kappa\lambda(t))^l\big).\end{aligned}$$

Since $\bigcup_{k=l}^{\infty}\bigcup_{\tau\in\tilde{E}_k(q)}\varphi_\tau(X_{t(\tau)})$ is a cover of $K_n(q)$ whose diameters converge (exponentially fast) to zero as $l\to\infty$, formula (16.28) yields $\text{H}_t(K_n(q)) = 0$.Therefore, $\text{HD}(K_n(q)) \leq t$. As $t > b_n$ was arbitrary, this gives formula (16.27). Let

$$\begin{aligned}K_\infty(U_n) :&= \big\{\omega\in K(U_n) : \text{at least one } q\in\mathbb{N} \text{ appears in } \omega \text{ infinitely many times}\big\}\\ &=\bigcup_{q=1}^{\infty}K_q(U_n)\end{aligned}$$

and let

$$K_n(\infty) := \pi\big(K_\infty(U_n)\big) = \bigcup_{q=1}^{\infty}K_n(q).$$

Formula (16.27) and σ-stability of Hausdorff dimension then imply that

$$(16.29)\qquad \text{HD}(K_n(\infty)) \leq b_n.$$

Now, for every integer $l \geq 1$ let

$$K_l^*(U_n) := \big\{\omega\in E_A^\infty : \text{the letters } 1,2,\ldots,l \text{ appear in } \omega \text{ only finitely many times}\big\}$$

and

$$K_l^0(U_n) := \big\{\omega\in E_A^\infty : \text{the letters } 1,2,\ldots,l \text{ do not appear in } \omega \text{ at all}\big\}.$$

Furthermore,

$$K_n^*(l) := \pi\big(K_l^*(U_n)\big) \quad\text{and}\quad K_n^0(l) := \pi\big(K_l^0(U_n)\big).$$

But

$$K_n^*(l) \subseteq \bigcup_{\omega\in E_A^*}\varphi_\omega(K_n^0(l)),$$

and therefore

$$\text{HD}(K_n^*(l)) = \text{HD}(K_n^0(l)).$$

But $K_n \setminus K_n(\infty) \subseteq \bigcap_{l=1}^{\infty}K_n^*(l)$. Hence, applying Theorem 4.3.6 in [31], we get

$$\text{HD}(K_n\setminus K_n(\infty)) \leq \inf_{l\geq 1}\{\text{HD}(K_n^*(l))\} = \inf_{l\geq 1}\{\text{HD}(K_n^0(l))\} = \theta_{\mathcal{S}} < b_{\mathcal{S}}(=b).$$

Since $\lim_{n\to\infty}b_n = b$, this implies that for all $n \geq 1$ large enough $\text{HD}(K_n\setminus K_n(\infty)) < b_n$. Along with (16.29) this yields

$$(16.30)\qquad \text{HD}(K_n) \leq b_n.$$

Passing to proving the opposite inequality, let $\mu_{b_n,n}$ be the shift–invariant ergodic measure supported on $K(U_n)$ produced in for the potential

$$E_A^\infty \ni \omega \longmapsto -b_n \log\left|\varphi'_{\omega_0}\big(\pi_{\mathcal{S}}(\sigma(\omega))\big)\right| \in \mathbb{R}.$$

Then $\mu_{b_n,n} \circ \pi_{\mathcal{S}}^{-1}$ is a Borel probability measure on K_n and by the definition of b_n, by Theorem 11.1, and by Theorem 4.4.2 in [31], we get that

$$\text{(16.31)}\quad \mathrm{HD}(K_n) \ge \mathrm{HD}\big(\mu_{b_n,n} \circ \pi_{\mathcal{S}}^{-1}\big) = \frac{\mathrm{h}_{\mu_{b_n,n}}(\sigma)}{\chi_{\mu_{b_n,n}}} = \frac{\log\lambda_n(b_n) + b_n\chi_{\mu_{b_n,n}}}{\chi_{\mu_{b_n,n}}} = \frac{b_n\chi_{\mu_{b_n,n}}}{\chi_{\mu_{b_n,n}}} = b_n.$$

Along with (16.30), this completes the proof of Theorem 16.14. □

As a direct consequence of this theorem and Proposition 16.13, we get the following.

Proposition 16.15. *With the hypotheses of Theorem* 16.14 *we have that*

$$\text{(16.32)}\qquad \lim_{n\to\infty} \frac{\mathrm{HD}(J_{\mathcal{S}}) - \mathrm{HD}(K_n)}{\mu_b(U_n)} = \begin{cases} 1/\chi_{\mu_b} & \text{if (U4A) holds} \\ \big(1 - |\varphi'_\xi(\pi(\xi^\infty))|\big)/\chi_{\mu_b} & \text{if (U4B) holds}. \end{cases}$$

17. Escape Rates for Conformal GDMSs; Hausdorff Dimension

This mini-section is the main fruit of the labor in the previous section. It pertains to the rate of decay of Hausdorff dimension of the set of avoiding/survivor points. It contains, in particular, Theorem 17.1, the second main result of this manuscript. Given $z \in J_{\mathcal{S}}$ and $r > 0$ let

$$K_z(r) := \pi_{\mathcal{S}}(\tilde{K}_z(r)),$$

where

$$\tilde{K}_z(r) := \big\{\omega \in E_A^\infty : \forall_{n\ge 0}\, \sigma^n(\omega) \notin \pi^{-1}(B(z,r)\big\} = \bigcap_{n=0}^\infty \sigma^{-n}\big(\pi^{-1}(B^c(z,r))\big).$$

More generally, given a set $G \subseteq \mathbb{R}^d$, we denote

$$\tilde{K}(G) := \big\{\omega \in E_A^\infty : \forall_{n\ge 0}\, \sigma^n(\omega) \notin \pi^{-1}(G)\big\} = \bigcap_{n=0}^\infty \sigma^{-n}\big(\pi^{-1}(G^c)\big)$$

and

$$K(G) := \pi_{\mathcal{S}}(\tilde{K}(G)).$$

We say that a parameter $t > \theta_{\mathcal{S}}$ is powering at a point $z \in J_{\mathcal{S}}$ if there exist $\alpha > 0$, $C > 0$, and $\delta > 0$ such that

$$\text{(17.1)}\qquad \mu_s \circ \pi^{-1}\big(B(z,r)\big) \le C\big(\mu_t \circ \pi^{-1}\big(B(z,r)\big)\big)^\alpha$$

for every $s \in (t-\delta, t+\delta)$ and for all radii $r > 0$ small enough. The constant α is called the powering exponent of t and z. The following is one of the main results of our paper.

Theorem 17.1. *Let $\mathcal{S}$ be a finitely primitive strongly regular conformal GDMS. Assume that both $\mathcal{S}$ is (*WBT*) and parameter $b_{\mathcal{S}}$ is powering at some point $z \in J_{\mathcal{S}}$ which is either*

(a) *not pseudo-periodic or else*

(b) *uniquely periodic and belongs to* Int$X$ *(and* $z = \pi(\xi^\infty)$ *for a (unique) irreducible word* $\xi \in E_A^*$*).*

Then

$$\lim_{r\to 0}\frac{\mathrm{HD}(J_{\mathcal S})-\mathrm{HD}(K_z(r))}{\mu_b\big(\pi^{-1}(B(z,r))\big)}=\begin{cases}1/\chi_{\mu_b} & \text{if (a) holds}\\ \big(1-|\varphi'_\xi(z)|\big)/\chi_{\mu_b} & \text{if (b) holds}\end{cases}. \tag{17.2}$$

Proof. Denote the right-hand side of (16.32) by $\xi(z)$. Put

$$h := \mathrm{HD}(J_{\mathcal S}) = b_{\mathcal S} \quad \text{and} \quad h_r := \mathrm{HD}(K_z(r)).$$

Seeking contradiction assume that (17.2) fails to hold at some point $z \in J_{\mathcal S}$. This means that there exists a strictly decreasing sequence $(s_n(z))_{n=0}^\infty$ of positive reals such that the sequence

$$\left(\frac{h-h_{s_n(z)}}{\mu_b\big(\pi^{-1}(B(\pi(z),s_n(z)))\big)}\right)_{n=0}^\infty$$

does not have $\xi(z)$ as its accumulation point. Let

$$\mathcal R := \{s_n(z) : n \ge 0\}.$$

Let $(U_n^\pm(z))_{n=0}^\infty$ be the corresponding sequence of open subsets of E_A^∞ produced in formula (15.7). We shall prove the following.

Claim 1^0: Both sequences $(U_n^\pm(z))_{n=0}^\infty$ satisfy the (U2*) condition for the parameter h.

Proof. Let $\alpha > 0$ be a powering exponent of $h = b_{\mathcal S}$ at z and let $\delta > 0$ come from this powering property. Let $s \in (h-\delta, h+\delta)$. Applying then formula (15.8) to the measure μ_h, we get, with notation used in this formula, that

$$\mu_s\big(U_k^\pm(z)\big) \le \mu_s\circ\pi^{-1}(B(z,r_{j-1})) \le C\big(\mu_h\circ\pi^{-1}(B(z,r_{j-1}))\big)^\alpha \le C\exp^\alpha\big(\kappa(1+8\Delta l(z))e^{-\alpha\kappa k}.$$

The claim is proved. $\square$

By this claim and because of Propositions 15.8 and 15.9, Proposition 16.15 applies to give

$$\lim_{n\to\infty}\frac{h-h_n^\pm}{\mu_b(U_n^\pm(z))} = \xi(z), \tag{17.3}$$

where $h_n^\pm := \mathrm{HD}(K(U_n^\pm(z))$. Let $(n_j)_{j=0}^\infty$ be the sequence produced in Proposition 15.7 with the help of $\mathcal R$. By virtue of this proposition there exists an increasing sequence $(j_k)_{k=1}^\infty$ such that $\mathcal R \cap \mathcal R_{n_{j_k}} \ne \emptyset$ for all $k \ge 1$. For every $k \ge 1$ pick one element $r_k \in \mathcal R \cap \mathcal R_{n_{j_k}}$. Set $q_k := l_{n_{j_k}}$. By Observation 15.3 and formula (15.6), we have

$$\begin{aligned}\frac{h-h_{q_k}^-}{\mu_b(U_{q_k}^-(z))}\cdot\frac{\mu_b(U_{q_k}^-(z))}{\mu_b\big(\pi^{-1}(B(\pi(z),r_k))\big)} &\le \frac{h-h_{r_k}}{\mu_b\big(\pi^{-1}(B(\pi(z),r_k))\big)} \le\\ &\le \frac{h-h_{q_k}^+}{\mu_b(U_{q_k}^+(z))}\cdot\frac{\mu_b(U_{q_k}^+(z))}{\mu_b\big(\pi^{-1}(B(\pi(z),r_k))\big)}\end{aligned} \tag{17.4}$$

But since $\mu_b \circ \pi^{-1}$ is WBT, it is DBT by Proposition 14.12, and it therefore follows from (14.4) along with formulas (15.3) and (15.6) that

$$\lim_{k\to\infty} \frac{\mu_b(U^-_{q_k}(z))}{\mu_b\big(\pi^{-1}(B(\pi(z),r_k))\big)} = 1 = \lim_{k\to\infty} \frac{\mu_b(U^+_{q_k}(z))}{\mu_b\big(\pi^{-1}(B(\pi(z),r_k))\big)}.$$

Inserting this and (17.3) to (17.4) yields

$$\lim_{k\to\infty} \frac{h-h_{r_k}}{\mu_b\big(\pi^{-1}(B(\pi(z),r_k))\big)} = \xi(z).$$

Since $r_k \in \mathcal{R}$ for all $k \geq 1$, this implies that $\xi(z)$ is an accumulation point of the sequence‘

$$\left(\frac{h-h_{s_n(z)}}{\mu_b\big(\pi^{-1}(B(\pi(z),s_n(z)))\big)}\right)_{n=0}^{\infty},$$

and this contradiction finishes the proof of Theorem 17.1. □

We have discussed at length the (WBT) condition in Section 14, particularly in Theorem 14.7; we now would like also to note that since any two measures μ_t, $t > \theta_{\mathcal{S}}$, are either equal or mutually singular, the standard covering argument gives the following simple but remarkable result.

Proposition 17.2. *If $\mathcal{S}$ is a finitely primitive regular conformal* GDMS*, then every parameter $t > \theta_{\mathcal{S}}$ is powering with exponent 1 at $\mu_t \circ \pi^{-1}$–a.e. point of $J_{\mathcal{S}}$.*

Now, as an immediate consequence of Theorem 17.1, Theorem 14.7, and Proposition 17.2, we get the following result, also one of our main.

Corollary 17.3. *If $\mathcal{S}$ be a finitely primitive strongly regular conformal* GDMS *whose limit set $J_{\mathcal{S}}$ is geometrically irreducible, then*

$$\lim_{r\to 0} \frac{\mathrm{HD}(J_{\mathcal{S}}) - \mathrm{HD}(K_z(r))}{\mu_b\big(\pi^{-1}(B(z,r))\big)} = \frac{1}{\chi_{\mu_b}} \tag{17.5}$$

at $\mu_{b_{\mathcal{S}}} \circ \pi^{-1}$–a.e. point z of $J_{\mathcal{S}}$.

In the case of finite alphabet E, we can say much more for the parameter $b_{\mathcal{S}}$ than established in Proposition 17.2. Namely, we shall prove the following.

Proposition 17.4. *If $\mathcal{S}$ is a finite alphabet primitive conformal* GDMS*, then $\mathcal{S}$ is powering at the parameter $b_{\mathcal{S}}$ at each point $\xi \in J_{\mathcal{S}}$.*

Proof. The proof of Theorem 7.20 in [7] (see also Theorem 7.17 therein for the main geometric ingredient of this proof) produces for every radius $r \in \big(0, \frac{1}{2}\min\{\mathrm{diam}(X_v) : v \in V\}$ a family $Z(r) \subseteq E_A^*$ consisting of mutually incomparable words with the following properties.

(1) $C_1^{-1} r \leq \|\varphi_\omega'\|_\infty, \mathrm{diam}\big(\varphi_\omega(X_{t(\omega)})\big) \leq C_1 r$ for all $\omega \in Z(r)$

(2) $\varphi_\omega(X_{t(\omega)}) \cap B(\xi, r) \neq \emptyset$ for all $\omega \in Z(r)$

(3) $\pi_{\mathcal{S}}^{-1}(B(\xi,r)) \subseteq \bigcup_{\omega\in Z(r)}[\omega]$,

(4) $\#Z(r) \le C_2$,

where C_1 and C_2 are some finite positive constants independent of ξ and r. Abbreviate

$$b := b_{\mathcal{S}}.$$

It easily follows from [31] that there exist a constant $\delta \in (0, b_{\mathcal{S}}/4)$ and a constant $Q \in (1,+\infty)$ such that

$$Q^{-1} \le \frac{\mu_s([\tau])}{e^{-\mathrm{P}(s)|\tau|}\|\varphi_\tau'\|_\infty^s} \le Q$$

for every $s \in (b-\delta, b+\delta)$ and for all $\tau \in E_A^*$. We therefore get for every $s \in (b-\delta, b+\delta)$ and all $\omega \in Z(r)$ that

$$\mu_s([\omega]) \le Qe^{-\mathrm{P}(s)|\omega|}\|\varphi_\omega'\|_\infty^s \le QC_1^s e^{-\mathrm{P}(s)|\omega|} r^s \tag{17.6}$$

and

$$\mu_b([\omega]) \ge QC_1^{-b} r^b. \tag{17.7}$$

It is also known from [7] that, with perhaps larger $Q \ge 1$:

$$\mu_b \circ \pi_{\mathcal{S}}^{-1}\big(B(\xi,r)\big) \ge Q^{-1} r^b \tag{17.8}$$

This formula follows for example from (17.7) applied to a sufficiently small fixed fraction of r. If $b/2 \le s \le b$, then $\mathrm{P}(s) \ge 0$, and we get

$$\begin{aligned} \mu_s([\omega]) &\le QC_1^s r^s \le QC_1^b r^s = QC_1^b\big(r^b\big)^{s/b} \\ &\le QC_1^b Q^{\frac{s}{b}} \mu_b^{\frac{s}{b}} \circ \pi_{\mathcal{S}}^{-1}\big(B(\xi,r)\big) \\ &\le Q^2 C_1^b \mu_b^{\frac{s}{b}} \circ \pi_{\mathcal{S}}^{-1}\big(B(\xi,r)\big) \\ &\le Q^2 C_1^b \mu_b^{\frac{1}{2}} \circ \pi_{\mathcal{S}}^{-1}\big(B(\xi,r)\big). \end{aligned} \tag{17.9}$$

Now we assume that $s \ge b$. We set

$$\kappa := \max\{\|\varphi_e'\|_\infty : e \in E\} < 1,$$

and we recall that

$$\chi_b := \chi_{\mu_b} = -\int_{E_A^\infty} \log|\varphi_{\omega_0}'(\pi_{\mathcal{S}}(\sigma(\omega)))|\,d\mu_b(\omega) > 0.$$

By taking $\delta \in (0, b/4)$ small enough, we will have

$$\frac{s-\frac{b}{2}}{s-b} \ge \frac{2\chi_b}{log(1/\kappa)} \quad \text{and} \quad \mathrm{P}(s) \ge -2\chi_b(s-b)$$

for all $s \in (b, b+\delta)$. Hence

$$\left(s-\frac{b}{2}\right)\log\kappa \le -2\chi_b(s-b) \le \mathrm{P}(s).$$

Equivalently $\kappa^{\left(s-\frac{b}{2}\right)} \le e^{\mathrm{P}(s)}$. Thus

$$\kappa^{\left(s-\frac{b}{2}\right)|\omega|} \le e^{\mathrm{P}(s)|\omega|}.$$

As $\|\varphi_\omega'\|_\infty \le \kappa^{\omega|}$ and $s \ge b$, we therefore get

$$\begin{aligned}\mu_s([\omega]) &\le Qe^{-\mathrm{P}(s)|\omega|}\|\varphi_\omega'\|_\infty^s \le Q\|\varphi_\omega'\|_\infty^{\frac{b}{2}} \le QC_1^{\frac{b}{2}}r^{\frac{b}{2}} \\ &\le Q^2C_1^{\frac{b}{2}}Q^{\frac{b}{2}}\mu_b^{\frac{1}{2}} \circ \pi_{\mathcal{S}}^{-1}\big(B(\xi,r)\big) \\ &= Q^{3/2}C_1^{\frac{b}{2}}\mu_b^{\frac{1}{2}} \circ \pi_{\mathcal{S}}^{-1}\big(B(\xi,r)\big) \\ &\le Q^2C_1^b\mu_b^{\frac{1}{2}} \circ \pi_{\mathcal{S}}^{-1}\big(B(\xi,r)\big).\end{aligned}$$

Combining this along with (17.9) we get that

$$\mu_s([\omega]) \le Q^2C_1^b\mu_b^{\frac{1}{2}} \circ \pi_{\mathcal{S}}^{-1}\big(B(\xi,r)\big).$$

for all $s \in (b-\delta, b+\delta)$ and all $\omega \in Z(r)$. Thus, looking also up at (4) and (3), this yields

$$\mu_s \circ \pi_{\mathcal{S}}^{-1}\big(B(\xi,r)\big) \le C_2Q^2C_1^b\mu_b^{\frac{1}{2}} \circ \pi_{\mathcal{S}}^{-1}\big(B(\xi,r)\big)$$

for all $s \in (b-\delta, b+\delta)$ and all radii $r \in \big(0, \frac{1}{2}\min\{\mathrm{diam}(X_v) : v \in V\}\big)$. The proof of Proposition 17.4 is complete. □

As an immediate consequence of Theorem 17.1, Theorem 14.9, and Proposition 17.4, we get the following considerably stronger/fuller result.

Theorem 17.5. *Let $\mathcal{S} = \{\varphi_e\}_{e\in E}$ be a primitive Conformal Graph Directed Markov System with a finite alphabet E acting in the space $\mathbb{R}^d$, $d \ge 1$. Assume that either $d = 1$ or that the system $\mathcal{S}$ is geometrically irreducible. Let $z \in J_{\mathcal{S}}$ be arbitrary. If either z is*

(a) *not pseudo-periodic or else*
(b) *uniquely periodic and belongs to* $\mathrm{Int}X$ *(and $z = \pi(\xi^\infty)$ for a (unique) irreducible word $\xi \in E_A^*$).*

Then

$$\lim_{r\to 0}\frac{\mathrm{HD}(J_{\mathcal{S}}) - \mathrm{HD}(K_z(r))}{\mu_b\big(\pi^{-1}(B(z,r))\big)} = \begin{cases} 1/\chi_{\mu_b} & \text{if (a) holds} \\ \big(1-|\varphi_\xi'(z)|\big)/\chi_{\mu_b} & \text{if (b) holds}. \end{cases} \tag{17.10}$$

18. Escape Rates for Conformal Parabolic GDMSs

In this section, following [32] and [31], we first shall provide the appropriate setting and basic properties of conformal parabolic iterated function systems, and more generally of parabolic graph directed Markov systems. We then prove for them the appropriate theorems on escaping rates.

As in Section 12 there are given a directed multigraph (V, E, i, t) (E countable, V finite), an incidence matrix $A : E \times E \to \{0, 1\}$, and two functions $i, t : E \to V$ such that $A_{ab} = 1$ implies $t(b) = i(a)$. Also, we have nonempty compact metric spaces $\{X_v\}_{v\in V}$. Suppose further that we have a collection of conformal maps $\varphi_e : X_{t(e)} \to X_{i(e)}$, $e \in E$, satisfying the following conditions:

(1) (Open Set Condition) $\varphi_i(\mathrm{Int}(X)) \cap \varphi_j(\mathrm{Int}(X)) = \emptyset$ for all $i \neq j$.

(2) $|\varphi_i'(x)| < 1$ everywhere except for finitely many pairs (i, x_i), $i \in E$, for which x_i is the unique fixed point of φ_i and $|\varphi_i'(x_i)| = 1$. Such pairs and indices i will be called parabolic and the set of parabolic indices will be denoted by Ω. All other indices will be called hyperbolic. We assume that $A_{ii} = 1$ for all $i \in \Omega$.

(3) $\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in E^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then φ_ω extends conformally to an open connected set $W_{t(\omega_n)} \subseteq \mathbb{R}^d$ and maps $W_{t(\omega_n)}$ into $W_{i(\omega_n)}$.

(4) If i is a parabolic index, then $\bigcap_{n \geq 0} \varphi_{i^n}(X) = \{x_i\}$ and the diameters of the sets $\varphi_{i^n}(X)$ converge to 0.

(5) (Bounded Distortion Property) $\exists K \geq 1$ $\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ $\forall x, y \in V$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then

$$\frac{|\varphi_\omega'(y)|}{|\varphi_\omega'(x)|} \leq K.$$

(6) $\exists s < 1$ $\forall n \geq 1$ $\forall \omega \in E_A^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then $||\varphi_\omega'|| \leq s$.

(7) (Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subseteq \mathbb{R}^d$ there exists an open cone $\mathrm{Con}(x, \alpha, l) \subseteq \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure α, and altitude l.

(8) There exists a constant $L \geq 1$ such that

$$\big||\varphi_i'(y)| - |\varphi_i'(x)|\big| \leq L||\varphi_i'|||y - x|$$

for every $i \in I$ and every pair of points $x, y \in V$.

We call such a system of maps

$$\mathcal{S} = \{\varphi_i : i \in E\}$$

a subparabolic iterated function system. Let us note that conditions (1),(3),(5)-(7) are modeled on similar conditions which were used to examine hyperbolic conformal systems. If $\Omega \neq \emptyset$, we call the system $\{\varphi_i : i \in E\}$ parabolic. As declared in (2) the elements of the set $E \setminus \Omega$ are called hyperbolic. We extend this name to all the words appearing in (5) and (6). It follows from (3) that for every hyperbolic word ω,

$$\varphi_\omega(W_{t(\omega)}) \subseteq W_{t(\omega)}.$$

Note that our conditions ensure that $\varphi_i'(x) \neq 0$ for all $i \in E$ and all $x \in X_{t(i)}$. It was proved (though only for IFSs but the case of GDMSs can be treated completely similarly) in [32] (comp. [31]) that

$$\lim_{n \to \infty} \sup_{|\omega| = n} \{\mathrm{diam}(\varphi_\omega(X_{t(\omega)}))\} = 0. \tag{18.1}$$

As its immediate consequence, we record the following.

Corollary 18.1. *The map* $\pi : E_A^\infty \to X := \bigoplus_{v\in V} X_v$, $\{\pi(\omega)\} := \bigcap_{n\ge 0} \varphi_{\omega|_n}(X)$, *is well defined, i.e. this intersection is always a singleton, and the map* π *is uniformly continuous.*

As for hyperbolic (attracting) systems the limit set $J = J_{\mathcal{S}}$ of the system $\mathcal{S} = \{\varphi_e\}_{e\in e}$ is defined to be

$$J_{\mathcal{S}} := \pi(E_A^\infty)$$

and it enjoys the following self-reproducing property:

$$J = \bigcup_{e\in E} \varphi_e(J).$$

We now, following still [54] and [31], want to associate to the parabolic system $\mathcal{S}$ a canonical hyperbolic system $\mathcal{S}^*$. The set of edges is this.

$$E_* := \left\{ i^n j : n \ge 1,\ i \in \Omega,\ i \ne j \in E,\ A_{ij} = 1 \right\} \cup (E \setminus \Omega) \subseteq E_A^*.$$

We set

$$V_* = t(E_*) \cup i(E_*)$$

and keep the functions t and i on E_* as the restrictions of t and i from E_A^*. The incidence matrix $A_* : E_* \times E_* \to \{0,1\}$ is defined in the natural (the only reasonable) way by declaring that $A_{ab}^* = 1$ if and only if $ab \in E_A^*$. Finally

$$\mathcal{S}^* := \{\varphi_e : X_{t(e)} \to X_{t(e)} :\ e \in E_*\}.$$

It immediately follows from our assumptions (see [54] and [31] for details) that the following is true.

Theorem 18.2. *The system* $\mathcal{S}^*$ *is a hyperbolic conformal GDMS and the limit sets* $J_{\mathcal{S}}$ *and* $J_{\mathcal{S}^*}$ *differ only by a countable set.*

We have the following quantitative result, whose complete proof can be found in [48].

Proposition 18.3. *Let* $\mathcal{S}$ *be a conformal parabolic GDMS. Then there exists a constant* $C \in (0, +\infty)$ *and for every* $i \in \Omega$ *there exists some constant* $\beta_i \in (0, +\infty)$ *such that for all* $n \ge 1$ *and for all* $z \in X_i := \bigcup_{j\in I\setminus\{i\}} \varphi_j(X)$,

$$C^{-1} n^{-\frac{\beta_i+1}{\beta_i}} \le |\varphi'_{i^n}(z)| \le C n^{-\frac{\beta_i+1}{\beta_i}}.$$

In fact we know more: if $d = 2$ *then all constants* β_i *are integers* ≥ 1 *and if* $d \ge 3$ *then all constants* β_i *are equal to* 1.

Let

$$\beta = \beta_{\mathcal{S}} := \min\{\beta_i :\in\in \Omega\}$$

Passing to equilibrium/Gibbs states and their escape rates, we now describe the class of potentials we want to deal with. This class is somewhat narrow as we restrict ourselves to geometric potentials only. There is no obvious natural larger class of potentials for which our methods would work and trying to identified such classes would be of dubious value and unclear benefits. We thus only consider potentials of the form

$$E_A^\infty \ni \omega \mapsto \zeta_t(\omega) := t\log\left|\varphi'_{\omega_0}(\pi_{\mathcal{S}}(\sigma(\omega)))\right| \in \mathbb{R}, \quad t \geq 0.$$

We then define the potential $\zeta_t^* : E_{*A^*}^\infty \to \mathbb{R}$ as

$$\zeta_t^*(i^n j\omega) = \sum_{k=0}^{n} \zeta_t(\sigma^k(i^n j\omega)), \quad i \in \Omega,\ n \geq 0,\ j \neq i \ \text{ and } \ i^n j\omega \in E_{*A^*}^\infty.$$

We shall prove the following.

Proposition 18.4. *If $\mathcal{S}$ is a conformal parabolic GDMS, then the potential ζ_t^* is Hölder continuous for each $t \geq 0$ it is summable if and only if*

$$t > \frac{\beta}{\beta+1}$$

Proof. Hölder continuity of potentials ζ_t^*, $t \geq 0$, follows from the fact that the system $\mathcal{S}^*$ is hyperbolic, particularly from its distortion property, while the summability statement immediately follows from Proposition 18.3. □

So, for every $t > \frac{\beta}{\beta+1}$ we can define μ_t^* to be the unique equilibrium/Gibbs state for the potential ζ_t^* with respect to the shift map $\sigma_* : E_{*A^*}^\infty \to E_{*A^*}^\infty$. We will not use this information in the current paper but we would like to note that μ_t^* gives rise to a Borel σ-finite, unique up to multiplicative constant, σ-invariant measure μ_t on E_A^∞, absolutely continuous, in fact equivalent, with respect to μ_t^*; see [31] for details in the case of $t = b_{\mathcal{S}} = b_{\mathcal{S}^*}$, the Bowen's parameter of the systems $\mathcal{S}$ and $\mathcal{S}^*$ alike. The case of all other $t > \frac{\beta}{\beta+1}$ can be treated similarly. It follows from [31] that the measure μ_t is finite if and only if either

(a) $t \in \left(\frac{\beta}{\beta+1}, b_{\mathcal{S}}\right)$ or

(b) $t = b_{\mathcal{S}}$ and $b_{\mathcal{S}} > \frac{2\beta}{\beta+1}$.

Now having all of this, as an immediate consequence of theorems Theorem 15.10 and Theorem 15.11 we get the following two results.

Theorem 18.5. *Let $\mathcal{S} = \{\varphi_e\}_{e\in E}$ be a parabolic Conformal Graph Directed Markov System. Fix $t > \frac{\beta}{\beta+1}$ and assume that the measure $\mu_t^* \circ \pi_{\mathcal{S}^*}^{-1}$ is* (WBT) *at a point $z \in J_{\mathcal{S}^*}$. If z is either*

(a) *not pseudo-periodic with respect to the system $\mathcal{S}^*$,*
or

(b) *uniquely periodic with respect to $\mathcal{S}^*$, it belongs to* Int$X$ *(and $z = \pi_{\mathcal{S}^*}(\xi^\infty)$ for a (unique) irreducible word $\xi \in E_{*A^*}^*$),* *then, with* $\underline{R}_{\mathcal{S}^*,\mu_t^*}(B(z,\varepsilon) := \underline{R}_{\mu_t^*}\big(\pi_{\mathcal{S}^*}^{-1}(B(z,\varepsilon))\big)$ *and* $\overline{R}_{\mathcal{S}^*,\mu_t^*}(B(z,\varepsilon) := \overline{R}_{\mu_t^*}\big(\pi_{\mathcal{S}^*}^{-1}(B(z,\varepsilon))\big)$, *we have*

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\underline{R}_{\mathcal{S}^*,\mu_t^*}(B(z,\varepsilon))}{\mu_t^*\circ\pi_{\mathcal{S}^*}^{-1}(B(z,\varepsilon))} &= \lim_{\varepsilon\to 0}\frac{\overline{R}_{\mathcal{S}^*,\mu_t^*}(B(z,\varepsilon))}{\mu_t^*\circ\pi_{\mathcal{S}^*}^{-1}(B(z,\varepsilon))} \\
&= d_\varphi(z) := \begin{cases} 1 & \text{if (a) holds} \\ 1-\big|\varphi_\xi'(z)\big|e^{-p\mathrm{P}_{\mathcal{S}^*}(t)} & \text{if (b) holds,} \end{cases}
\end{aligned} \tag{18.2}
$$

where in (b), $\{\xi\} = \pi_{\mathcal{S}^*}^{-1}(z)$ *and* $p \geq 1$ *is the prime period of* ξ *under the shift map* $\sigma_* : E_{*A^*}^\infty \to E_{*A^*}^\infty$.

Theorem 18.6. *Let* $\mathcal{S} = \{\varphi_e\}_{e\in E}$ *be a parabolic Conformal Graph Directed Markov System whose limit set* $J_{\mathcal{S}}$ *is geometrically irreducible. If* $t > \frac{\beta}{\beta+1}$ *then*

$$
\lim_{\varepsilon\to 0}\frac{\underline{R}_{\mathcal{S}^*,\mu_t^*}(B(z,\varepsilon))}{\mu_t^*\circ\pi_{\mathcal{S}^*}^{-1}(B(z,\varepsilon))} = \lim_{\varepsilon\to 0}\frac{\overline{R}_{\mathcal{S}^*,\mu_t^*}(B(z,\varepsilon))}{\mu_t^*\circ\pi_{\mathcal{S}^*}^{-1}(B(z,\varepsilon))} = 1 \tag{18.3}
$$

for $\mu_t\circ\pi^{-1}$*–a.e. point of* $J_{\mathcal{S}}$.

Sticking to notation of Section 17, given $z \in E_{*A^*}^\infty$ and $r > 0$ let

$$
K_z^*(r) := \pi_{\mathcal{S}^*}(\tilde{K}_z^*(r)),
$$

where

$$
\tilde{K}_z^*(r) := \big\{\omega \in E_{*A^*}^\infty : \forall_{n\geq 0}\, \sigma_*^n(\omega) \notin \pi_{\mathcal{S}^*}^{-1}(B(\pi_{\mathcal{S}^*}(z), r)\big\} = \bigcap_{n=0}^\infty \sigma_*^{-n}\big(\pi_{\mathcal{S}^*}^{-1}(B(\pi_{\mathcal{S}^*}(z), r))\big).
$$

As immediate consequences respectively of Theorem 17.1 and Corollary 17.3, we get the following two results.

Theorem 18.7. *Let* $\mathcal{S} = \{\varphi_e\}_{e\in E}$ *be a parabolic Conformal Graph Directed Markov System. Assume that both* $\mathcal{S}^*$ *is* (WBT) *and parameter* $b_{\mathcal{S}}$ *is powering at some point* $z \in J_{\mathcal{S}^*}$*. If* z *is either*

(a) *not pseudo-periodic with respect to the system* $\mathcal{S}^*$,
   or

(b) *uniquely periodic with respect to* $\mathcal{S}^*$*, it belongs to* $\mathrm{Int}X$ *(and* $z = \pi_{\mathcal{S}^*}(\xi^\infty)$ *for a (unique) irreducible word* $\xi \in E_{*A^*}^*$*),*

then,

$$
\lim_{r\to 0}\frac{\mathrm{HD}(J_{\mathcal{S}}) - \mathrm{HD}(K_z^*(r))}{\mu_b^*\big(\pi_{\mathcal{S}^*}^{-1}(B(z,r))\big)} = \begin{cases} 1/\chi_{\mu_b^*} & \textit{if (a) holds} \\ \big(1-|\varphi_\xi'(z)|\big)/\chi_{\mu_b^*} & \textit{if (b) holds}. \end{cases} \tag{18.4}
$$

Theorem 18.8. *Let* $\mathcal{S} = \{\varphi_e\}_{e\in E}$ *be a parabolic Conformal Graph Directed Markov System whose limit set* $J_{\mathcal{S}}$ *is geometrically irreducible. Then*

$$
\lim_{r\to 0}\frac{\mathrm{HD}(J_{\mathcal{S}}) - \mathrm{HD}(K_z^*(r))}{\mu_b^*\big(\pi_{\mathcal{S}^*}^{-1}(B(z,r))\big)} = \frac{1}{\chi_{\mu_b^*}} \tag{18.5}
$$

for $\mu_{b^*_{\mathcal{S}}} \circ \pi^{-1}$*–a.e. point* z *of* $J_{\mathcal{S}}$.

Part 4. Applications: Escape Rates for Multimodal Interval Maps and One–Dimensional Complex Dynamics

Our goal in this part of the manuscript is to get the existence of escape rates in the sense of (1.3) and (1.4) for a large class of topologically exact piecewise smooth multimodal maps of the interval $[0,1]$, many rational functions of the Riemann sphere $\widehat{\mathbb{C}}$ with degree ≥ 2, and a vast class of transcendental meromorphic functions from $\mathbb{C}$ to $\widehat{\mathbb{C}}$. In order to do this we employ two primary tools. The first one is formed by the escape rates results for the class of all countable alphabet conformal graph directed Markov systems obtained in Sections 15 and 17. The other one is the method based on the first return (induced) map developed in Section 19, Section 20, and Section 21 of this part. This method closely relates the escape rates of the original map and the induced one. It turns out that for the above mentioned class of systems one can find a set of positive measure which gives rise to the first returned map which is isomorphic to a countable alphabet conformal IFS or full shift map; the task highly non-trivial and technically involved in general. In conclusion, the existence of escape rates in the sense of (1.3) and (1.4) follows.

19. First Return Maps

Let (X,ρ) be a metric space and let $F\subseteq X$ be a Borel set. Let $T:X\to X$ be a Borel map. Define

$$F_\infty := F\cap\bigcap_{k=0}^\infty\bigcup_{n=k}^\infty T^{-k}(F),$$

i. e. F_∞ is the set of all those points in F that return to F infinitely often under the iteration of the map T. Then for every $x\in F_\infty$ the number

$$\tau_F(x) := \min\{n\geq 1 : T^n(x)\in F\} = \min\{n\geq 1 : T^n(x)\in F_\infty\}$$

is well-defined, i. e. it is finite. The number $\tau_F(x)$ is called the first return of x to F under the map F. Having the function $\tau_F : F_\infty\to\mathbb{N}_1$ defined one defines the first return map $T_F : F_\infty\to F_\infty$ by the formula

$$T_F(x) : T^{\tau_F(x)}(x)\in F_\infty\subseteq F. \tag{19.1}$$

Let B be a Borel subsets of F. As in the two previous sections let

$$K(B)=K_T(B) := \bigcap_{n=0}^\infty T^{-n}(X\setminus B) \ \text{ and } \ K_F(B) := \bigcap_{n=0}^\infty T_F^{-n}(F_\infty\setminus B).$$

In particular, this definition applies for the set $B=F$. A straightforward observation is that $K_F(B)=F_\infty\cap K(B)$, so that we have the following.

$$K_F(B)=F_\infty\cap K(B)\subseteq F\cap K(B). \tag{19.2}$$

We shall prove the following.

Theorem 19.1. *If the map $T : X \to X$ is locally bi-Lipschitz and $B \subseteq F$ are Borel subsets of X, then*

$$\mathrm{HD}(K(B)) = \max\{\mathrm{HD}(K_F(B)), \mathrm{HD}(K(F))\}.$$

Proof. Since $K(F) \subseteq K(B)$, we have that $\mathrm{HD}(K(F)) \le \mathrm{HD}(K(B))$, and by (19.2) we have $\mathrm{HD}(K_F(B)) \le \mathrm{HD}(K(B))$. We are thus let to show only that

$$\mathrm{HD}(K(B)) \le \max\{\mathrm{HD}(K_F(B)), \mathrm{HD}(K(F))\}.$$

Indeed, fix $x \in K(B)$. Let

$$N_x := \{n \ge 0 : T^n(x) \in F\}.$$

Consider two cases:

Case 1^0: The set N_x is finite. Denote then by n_x its largest element. Then $T^{n_x+1}(x) \in K(F)$. Hence

$$x \in \bigcup_{n=0}^{\infty} T^{-n}(K(F));$$

note that this relation holds even if $N_x = \emptyset$.

Case 1^0: The set N_x is infinite. Then there exists $m_x \ge 0$ such that $T^{m_x}(x) \in F_\infty$. Hence,

$$x \in \bigcup_{n=0}^{\infty} T^{-n}(F_\infty).$$

In conclusion

$$K(B) \subseteq \bigcup_{n=0}^{\infty} T^{-n}(K(F)) \cup \bigcup_{n=0}^{\infty} T^{-n}(F_\infty).$$

But then, using (19.2), we get

$$\begin{aligned}
K(B) &\subseteq \left(\bigcup_{n=0}^{\infty} T^{-n}(K(F))\right) \cup \left(K(B) \cap \bigcup_{n=0}^{\infty} T^{-n}(F_\infty)\right) \\
&\subseteq \bigcup_{n=0}^{\infty} T^{-n}(K(B) \cap K(F)) \bigcup \bigcup_{n=0}^{\infty} T^{-n}(K(B) \cap F_\infty) \\
&= \bigcup_{n=0}^{\infty} T^{-n}(K(F)) \bigcup \bigcup_{n=0}^{\infty} T^{-n}(K_F(B))
\end{aligned}$$

Therefore, using σ-stability of Hausdorff dimension and local bi-Lipschitzness of T, we get

$$\begin{aligned}
\mathrm{HD}(K(B)) &\le \sup_{n\ge 0} \big\{\max\{\mathrm{HD}(T^{-n}(K(F))), \mathrm{HD}(T^{-n}(K_F(B)))\}\big\} \\
&\le \sup_{n\ge 0} \big\{\max\{\mathrm{HD}(T^{-n}(K(F))), \mathrm{HD}(T^{-n}(K_F(B)))\}\big\} \\
&= \max\{\mathrm{HD}(K(F)), \mathrm{HD}(K_F(B))\}
\end{aligned}$$

The proof of Theorem 19.1 is complete. □

As an immediate consequence of this theorem we get the following.

Corollary 19.2. *If the map $T : X \to X$ is locally bi-Lipschitz, $B \subseteq F$ are Borel subsets of X, and $\mathrm{HD}(K(F)) < \mathrm{HD}(K(B))$, then*

$$\mathrm{HD}(K(B)) = \mathrm{HD}(K_F(B)).$$

20. First Return Maps and Escaping Rates, I

As in Section 19 (X, ρ) is a metric space, $F \subseteq X$ be a Borel set and $T : X \to X$ is a Borel map. The symbols F_∞, τ_F, and T_F have the same meaning as in Section 19. Now in addition we also assume that the system $T : X \to X$ preserves a Borel probability measure μ on X. It is well-known that then the first return map $T_F : F_\infty \to F_\infty$ preserves the conditional measure μ_F on F (or F_∞ alike). This measure is given by the formula

$$\mu_F(A) = \frac{\mu(A)}{\mu(F)}$$

for every Borel set $A \subseteq F$. The famous Kac's Formula tells us that

$$\int_F \tau_F \, d\mu_F = \frac{1}{\mu(F)}.$$

For every $n \geq 1$ denote

$$\tau_F^{(n)} := \sum_{j=0}^{n-1} \tau_F \circ T_F^j,$$

so that

$$T_F^n(x) = T^{\tau_F^{(n)}(x)}(x).$$

If B, as in Section 19, is a Borel subset of F, then for every $n \geq 1$ we denote

$$B_n^c := \bigcap_{j=0}^{n-1} T^{-j}(X \setminus B), \quad B_n^c(F) := F_\infty \cap B_n^c, \quad \text{and} \quad B_n^c(T_F) := \bigcap_{j=0}^{n-1} T_F^{-j}(X \setminus B).$$

For every $\eta \in (0,1)$ and every integer $k \geq 1$ denote

$$F_{k-1}(\eta) := \left\{ x \in F_\infty : \left(\frac{1}{\mu(F)} - \eta \right) k \leq \tau_F^{(k)}(x) \leq \left(\frac{1}{\mu(F)} + \eta \right) k \right\}.$$

Let us record the following straightforward observation.

$$F_{n-1}(\eta) \cap B^c_{(\frac{1}{\mu(F)}+\eta)n} \subseteq F_{n-1}(\eta) \cap B_n^c(T_F) \subseteq F_{n-1}(\eta) \cap B^c_{(\frac{1}{\mu(F)}-\eta)n}. \tag{20.1}$$

This simple relation will be however our starting point for relating the escape rates of B with respect to the map T and the first return map $T_F : F_\infty \to F_\infty$.

Definition 20.1. We say that the pair (T, F) satisfies the large deviation property (LDP) if for all $\eta \in (0, 1)$ there exist two constants $\hat{\eta} > 0$ and $C_\eta \in [1, +\infty)$ such that

$$\mu(F_n^c(\eta)) \le C_\eta e^{-\hat{\eta} n}$$

for all integers $n \ge 1$.

In what follows we will need one (standard) concept more. We define for every $x \in X$ the number

$$E_F(x) := \min\big\{n \in \{0, 1, 2, \dots, \infty\} : T^n(x) \in F\big\}.$$

This number is called the first entrance time to F under the map T and it is closely related to τ_F,

$$\tau_F(x) = E_F(T(x)) + 1$$

if $x \in F$, but of course it is different.

Definition 20.2. We say that the pair (T, F) has exponential tail decay (ETD) if

$$\mu\big(E_F^{-1}([n, +\infty])\big) \le Ce^{-\alpha n}$$

for all integers $n \ge 0$ and some constants $C, \alpha \in (0, +\infty)$.

Let B be a Borel subset of F. Following the previous sections denote respectively by $R_{T,\mu}(B)$ and $R_{T_F,\mu}(B)$ the respective escape rates of B by the maps $T : X \to X$ and $T_F : F_\infty \to F_\infty$, i. e.

$$\underline{R}_{T,\mu}(B) := -\varlimsup_{n\to\infty} \frac{1}{n} \log \mu(B_n^c) \quad \le \quad \overline{R}_{T,\mu}(B) := -\varliminf_{n\to\infty} \frac{1}{n} \log \mu(B_n^c),$$

and

$$\underline{R}_{T_F,\mu}(B) := -\varlimsup_{n\to\infty} \frac{1}{n} \log \mu_F(B_n^c(T_F)) = -\varlimsup_{n\to\infty} \frac{1}{n} \log \mu(B_n^c(T_F)),$$

$$\overline{R}_{T_F,\mu}(B) := -\varliminf_{n\to\infty} \frac{1}{n} \log \mu_F(B_n^c(T_F)) = -\varliminf_{n\to\infty} \frac{1}{n} \log \mu(B_n^c(T_F)),$$

with obvious inequality

$$\underline{R}_{T_F,\mu}(B) \le \overline{R}_{T_F,\mu}(B).$$

We shall prove the following.

Theorem 20.3. *Assume that a pair (T, F) satisfies the large deviation property* (LDP) *and has exponential tail decay* (ETD). *Let $(B_k)_{k=0}^\infty$ be a sequence of Borel subsets of F such that*

(a) $\lim_{k\to\infty} \mu(B_k) = 0$,

(b) *The limits*

$$\lim_{k\to\infty} \frac{\underline{R}_{T_F,\mu}(B_k)}{\mu_F(B_k)} \quad \text{and} \quad \lim_{k\to\infty} \frac{\overline{R}_{T_F,\mu}(B_k)}{\mu_F(B_k)}$$

exist, are equal, and belong to $(0, +\infty)$; denote their common value by $R_F(\mu)$.

Then the limits

$$\lim_{k\to\infty}\frac{\underline{R}_{T,\mu}(B_k)}{\mu(B_k)} \quad \textit{and} \quad \lim_{k\to\infty}\frac{\overline{R}_{T,\mu}(B_k)}{\mu(B_k)}$$

also exist, and, denoting their common value by $R_T(\mu)$, we have that

$$R_T(\mu)=R_F(\mu).$$

Proof. Fix $\eta,\varepsilon\in(0,1)$. Fix two integers $k,n\geq 1$. Denote the sets $(B_k)^c_{(\frac{1}{\mu(F)}+\eta)n}$ and $(B_k)^c_{(\frac{1}{\mu(F)}-\eta)n}$ respectively by $B^-_{k,n}(\eta)$ and $B^+_{k,n}(\eta)$. Because of (20.1), we have

$$\mu\big(F_{n-1}(\eta)\cap(B_k)^c_n(T_F)\big)\leq\mu\big(B^+_{k,n}(\eta)\big). \tag{20.2}$$

Fix $M_1\geq 1$ so large that

$$(1-\varepsilon)R_F(\mu)\leq\frac{\underline{R}_{T_F,\mu}(B_k)}{\mu_F(B_k)}\leq\frac{\overline{R}_{T_F,\mu}(B_k)}{\mu_F(B_k)}\leq(1+\varepsilon)R_F(\mu) \tag{20.3}$$

and

$$4R_F(\mu)\mu_F(B_k)\leq\min\{\varepsilon,\hat\eta/2\} \tag{20.4}$$

for all $k\geq M_1$. Fix such a k. Fix then $N_k\geq 1$ so large that

$$\exp\big(-(1+\varepsilon)\overline{R}_{T_F,\mu}(B_k)n\big)\leq\mu\big((B_k)^c_n(T_F)\big)\leq\exp\big(-(1-\varepsilon)\underline{R}_{T_F,\mu}(B_k)n\big)$$

for all $n\geq N_k^{(1)}$. Along with (20.3) this gives

$$\exp\big(-(1+\varepsilon)^2R_F(\mu)\mu_F(B_k)n\big)\leq\mu\big((B_k)^c_n(T_F)\big)\leq\exp\big(-(1-\varepsilon)^2R_F(\mu)\mu_F(B_k)n\big). \tag{20.5}$$

Therefore, using also Definition 20.1, we get for all $k\geq M_1$ and all $n\geq N_k^{(1)}$ that

$$\begin{aligned}\frac{\mu\big(F_{n-1}(\eta)\cap(B_k)^c_n(T_F)\big)}{\mu\big((B_k)^c_n(T_F)\big)}&=\frac{\mu\big((B_k)^c_n(T_F)\big)-\mu\big(F^c_{n-1}(\eta)\cap(B_k)^c_n(T_F)\big)}{\mu\big((B_k)^c_n(T_F)\big)}\\&\geq\frac{\mu\big((B_k)^c_n(T_F)\big)-\mu\big(F^c_{n-1}(\eta)\big)}{\mu\big((B_k)^c_n(T_F)\big)}=1-\frac{\mu\big(F^c_{n-1}(\eta)\big)}{\mu\big((B_k)^c_n(T_F)\big)}\\&\geq 1-\frac{C_\eta e^{-\hat\eta(n-1)}}{\exp\big(-4R_F(\mu)\mu_F(B_k)n\big)}\\&=1-C_\eta e^{\hat\eta}\exp\big((4R_F(\mu)\mu_F(B_k)-\hat\eta)n\big)\\&\geq 1-C_\eta e^{\hat\eta}\exp\big(-\frac{1}{2}\hat\eta n\big)\geq 1/2,\end{aligned}$$

where the last inequality holds for all $n\geq N_k^{(1)}$ large enough, say $n\geq N_k^{(2)}\geq N_k^{(1)}$. Along with (20.2) this gives

$$\mu\big(B^+_{k,n}(\eta)\big)\geq\frac{1}{2}\mu\big((B_k)^c_n(T_F)\big).$$

Hence

$$-\varliminf_{n\to\infty}\frac{1}{n}\log\mu\big(B^+_{k,n}(\eta)\big)\leq\overline{R}_{T_F,\mu}(B_k).$$

Since

$$(B_k)_n^c \supseteq B^+_{k,\left[\frac{n}{\frac{1}{\mu(F)}-\eta}\right]+1},$$

we get

$$\varliminf_{n\to\infty} \frac{1}{n}\log\mu\big(B^+_{k,n}(\eta)\big) \le \left(\frac{1}{\mu(F)}-\eta\right)\varliminf_{n\to\infty}\frac{1}{n}\log\mu\big((B_k)_n^c\big).$$

Therefore,

$$\overline{R}_{T_F,\mu}(B_k) \ge \left(\frac{1}{\mu(F)}-\eta\right)\overline{R}_{T,\mu}(B_k).$$

Dividing both sides of this inequality by $\mu(B_k)$ and passing to the limit with $k\to\infty$, this entails

$$R_F(\mu) \ge (1-\eta\mu(F))\varlimsup_{k\to\infty}\frac{\overline{R}_{T,\mu}(B_k)}{\mu(B_k)}.$$

By letting in turn $\eta\searrow 0$, this yields

$$R_F(\mu) \ge \varlimsup_{k\to\infty}\frac{\overline{R}_{T,\mu}(B_k)}{\mu(B_k)}. \tag{20.6}$$

Passing to the proof of the opposite inequality, denote $\left(\frac{1}{\mu(F)}+\eta\right)n$ by n^+ and $\left(\frac{1}{\mu(F)}+\eta\right)^{-1}n$ by n^-. We have

$$\begin{aligned} B^-_{k,n}(\eta) &= \bigcup_{j=0}^{n^+}\big(B^-_{k,n}(\eta)\cap E_F^{-1}(j)\big)\cup E_F^{-1}((n^+,+\infty]) \\ &= E_F^{-1}((n^+,+\infty])\cup\bigcup_{j=0}^{n^+}E_F^{-1}(j)\cap T^{-j}\big((B_k)^c_{n^+-j}\big). \end{aligned} \tag{20.7}$$

Now,

(20.8)

$$\begin{aligned} E_F^{-1}(j)\cap T^{-j}\big((B_k)^c_{n^+-j}\big) &= E_F^{-1}(j)\cap T^{-j}(F)\cap T^{-j}\big((B_k)^c_{n^+-j}\big) \\ &= E_F^{-1}(j)\cap T^{-j}\big(F\cap(B_k)^c_{n^+-j}\big) \\ &= E_F^{-1}(j)\cap\Big(T^{-j}\big(F_{(n^+-j)^--1}(\eta)\cap(B_k)^c_{n^+-j}\big)\cup T^{-j}\big(F^c_{(n^+-j)^--1}(\eta)\cap(B_k)^c_{n^+-j}\big)\Big). \end{aligned}$$

By (20.1) we have

$$\begin{aligned} F^c_{(n^+-j)^--1}(\eta)\cap(B_k)^c_{n^+-j} &= F^c_{(n^+-j)^--1}(\eta)\cap(B_k)^c_{((n^+-j)^-)^+} \\ &\subseteq F^c_{(n^+-j)^--1}(\eta)\cap(B_k)^c_{(n^+-j)^-}(T_F) \\ &\subseteq (B_k)^c_{(n^+-j)^-}(T_F). \end{aligned} \tag{20.9}$$

Now take $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. By applying Hölder inequality, T-invariance of the measure μ, (20.5), and making use of Definition 20.2, we get for all $0 \le j \le N_k^{(1)}$, that

$$\mu\Big(E_F^{-1}(j) \cap T^{-j}\big(F_{(n^+-j)^- -1}(\eta) \cap (B_k)^c_{n^+-j}\big)\Big) \le \mu\Big(E_F^{-1}(j) \cap T^{-j}\big((B_k)^c_{(n^+-j)^-}(T_F)\big)\Big) =$$

$$\begin{aligned}
&= \int_X \mathbb{1}_{E_F^{-1}(j)} \mathbb{1}_{T^{-j}\left((B_k)^c_{(n^+-j)^-}(T_F)\right)}\, d\mu \\
&\le \left(\int_X \mathbb{1}_{E_F^{-1}(j)}\, d\mu\right)^{1/p} \left(\int_X \mathbb{1}_{T^{-j}\left((B_k)^c_{(n^+-j)^-}(T_F)\right)}\, d\mu\right)^{1/q} \\
&= \mu^{1/p}(E_F^{-1}(j))\mu^{1/q}\big(T^{-j}\big((B_k)^c_{(n^+-j)^-}(T_F)\big)\big) \\
&= \mu^{1/p}(E_F^{-1}(j))\mu^{1/q}\big((B_k)^c_{(n^+-j)^-}(T_F)\big) \\
&\le C^{1/p} e^{-\frac{\alpha}{p}j} \exp\left(-\frac{(1-\varepsilon)^2}{q} R_F(\mu)\mu_F(B_k)(n^+-j)^-\right) \\
&= C^{1/p} e^{-\frac{\alpha}{p}j} \exp\left(-\frac{(1-\varepsilon)^2}{q} R_F(\mu)\mu_F(B_k)\left(\frac{1}{\mu(F)}+\eta\right)^{-1}\left(\left(\frac{1}{\mu(F)}+\eta\right)n - j\right)\right) \\
&= C^{1/p} \exp\left(-\frac{(1-\varepsilon)^2}{q} R_F(\mu)\mu_F(B_k) n\right) \cdot \\
&\qquad\qquad \cdot \exp\left(-\left(\frac{\alpha}{p} - \frac{(1-\varepsilon)^2}{q}\left(\frac{1}{\mu(F)}+\eta\right)^{-1} R_F(\mu)\mu_F(B_k)\right) j\right)
\end{aligned}$$

Together with the left-hand side of (20.5) this gives that

$$\begin{aligned}
&\frac{\mu\Big(E_F^{-1}(j) \cap T^{-j}\big(F_{(n^+-j)^- -1}(\eta) \cap (B_k)^c_{n^+-j}\big)\Big)}{\mu\big((B_k)^c_n(T_F)\big)} \le \\
&\qquad \le C^{1/p} \exp\left(R_F(\mu)\mu_F(B_k)\big((1+e)^2 - \frac{1}{q}(1-\varepsilon)^2\big)n\right) \cdot \\
&\qquad\qquad \cdot \exp\left(-\left(\frac{\alpha}{p} - \frac{(1-\varepsilon)^2}{q}\left(\frac{1}{\mu(F)}+\eta\right)^{-1} R_F(\mu)\mu_F(B_k)\right) j\right).
\end{aligned}$$

Taking now $q > 1$ sufficiently close to 1 and looking at (20.4) we will have for every $\varepsilon > 0$ small enough that

$$(1+e)^2 - \frac{1}{q}(1-\varepsilon)^2 \le 1 - \frac{1}{q} + 6\varepsilon \le 7\varepsilon \quad \text{and} \quad \frac{\alpha}{p} - \frac{(1-\varepsilon)^2}{q}\left(\frac{1}{\mu(F)}+\eta\right)^{-1} R_F(\mu)\mu_F(B_k) > \frac{\alpha}{2p}.$$

Therefore, for all $k \ge M_1$ and for all $0 \le j \le n^+ - N_k^{(1)}$, we have that

$$\frac{\mu\Big(E_F^{-1}(j) \cap T^{-j}\big(F_{(n^+-j)^- -1}(\eta) \cap (B_k)^c_{n^+-j}\big)\Big)}{\mu\big((B_k)^c_n(T_F)\big)} \le C^{1/p} e^{7\varepsilon n} e^{-\frac{\alpha}{2p}j}. \tag{20.10}$$

For $n^+ - N_k^{(1)} < j \le n^+$, using (20.4), the left-hand side of (20.5), and looking up at Definition 20.2, we have the easier estimate:

$$
\begin{aligned}
\frac{\mu\Big(E_F^{-1}(j)\cap T^{-j}\big(F_{(n^+-j)^--1}(\eta)\cap (B_k)^c_{n^+-j}\big)\Big)}{\mu\big((B_k)^c_n(T_F)\big)} &\le \frac{\mu\big(E_F^{-1}(j)\big)}{\mu\big((B_k)^c_n(T_F)\big)} \le \\
&\le C^{-\alpha j}\exp\big((1+\varepsilon)^2R_F(\mu)\mu_F(B_k)n\big)\\
&\le C\exp\big(-\alpha(n^+-N_k^{(1)})\big)\exp\big((1+\varepsilon)^2R_F(\mu)\mu_F(B_k)n\big)\\
&= Ce^{\alpha N_k^{(1)}}\exp\left(\left((1+\varepsilon)^2R_F(\mu)\mu_F(B_k)-\alpha\left(\frac{1}{\mu(F)}+\eta\right)\right)n\right)\\
&\le Ce^{\alpha N_k^{(1)}}.
\end{aligned}
\tag{20.11}
$$

Now we can estimate the second part of (20.8). We note that

$$
E_F^{-1}(j)\cap T^{-j}\big(F_{(n^+-j)^--1}(\eta)\cap (B_k)^c_{n^+-j}\big) \subseteq E_F^{-1}(j)\cap T^{-j}\big(F^c_{(n^+-j)^--1}(\eta)\big),
$$

and use again Hölder inequality, T-invariance of measure μ, and Definitions 20.2 and 20.1, to estimate:

$$
\begin{aligned}
&\mu\Big(E_F^{-1}(j)\cap T^{-j}\big(F^c_{(n^+-j)^--1}(\eta)\cap (B_k)^c_{n^+-j}\big)\Big) \le\\
&\qquad\le \mu\Big(E_F^{-1}(j)\cap T^{-j}\big(F^c_{(n^+-j)^--1}(\eta)\big)\Big) = \int_X \mathbb{1}_{E_F^{-1}(j)}\mathbb{1}_{T^{-j}\left(F^c_{(n^+-j)^--1}(\eta)\right)}\,d\mu\\
&\qquad\le \mu^{1/p}(E_F^{-1}(j))\cdot\nu^{1/q}\big(T^{-j}\big(F^c_{(n^+-j)^--1}(\eta)\big)\big)\\
&\qquad= \mu^{1/p}(E_F^{-1}(j))\cdot\nu^{1/q}\big(F^c_{(n^+-j)^--1}(\eta)\big)\\
&\qquad\le C^{1/p}e^{-\frac{\alpha}{p}j}C_\eta^{1/q}e^{-\frac{\hat\eta}{q}\left((n^+-j)^--1\right)}\\
&\qquad= C^{1/p}C_\eta^{1/q}e^{\frac{\hat\eta}{q}}e^{-\frac{\alpha}{p}j}\exp\left(-\left(\frac{\alpha}{p}-\frac{\hat\eta}{q}\left(\frac{1}{\mu(F)}+\eta\right)^{-1}\right)j\right).
\end{aligned}
$$

Combining this with the left-hand side of (20.5) this gives that

$$
\begin{aligned}
&\frac{\mu\Big(E_F^{-1}(j)\cap T^{-j}\big(F^c_{(n^+-j)^--1}(\eta)\cap (B_k)^c_{n^+-j}\big)\Big)}{\mu\big((B_k)^c_n(T_F)\big)}\\
&\qquad\le C^{1/p}C_\eta^{1/q}e^{\frac{\hat\eta}{q}}\exp\left(-\left(\frac{\hat\eta}{q}-(1+\varepsilon)^2R_F(\mu)\mu_F(B_k)\right)n\right)\cdot\\
&\qquad\qquad\cdot\exp\left(-\left(\frac{\alpha}{p}-\frac{\hat\eta}{q}\left(\frac{1}{\mu(F)}+\eta\right)^{-1}\right)j\right).
\end{aligned}
\tag{20.12}
$$

Now, first take $q > 1$ so large that

$$
\frac{\alpha}{p}-\frac{\hat\eta}{q}\left(\frac{1}{\mu(F)}+\eta\right)^{-1} > \frac{\alpha}{2}.
$$

Then take $k \geq M_1$, say $k \geq M_{1,q} \geq M_1$ so large that

$$\frac{\hat{\eta}}{q} - (1+\varepsilon)^2 R_F(\mu)\mu_F(B_k) \geq \frac{\hat{\eta}}{2q}.$$

Inserting these two inequalities into (20.12), yields

$$\frac{\mu\Big(E_F^{-1}(j) \cap T^{-j}\big(F^c_{(n^+-j)^- -1}(\eta) \cap (B_k)^c_{n^+-j}\big)\Big)}{\mu\big((B_k)^c_n(T_F)\big)} \leq C^{1/p} C_\eta^{1/q} e^{\frac{\hat{\eta}}{q}} e^{-\frac{\hat{\eta}}{2q}n} e^{-\frac{\alpha}{2}j}. \tag{20.13}$$

Finally, by Definition 20.2, the left-hand side of (20.5), and (20.4),

$$\begin{aligned}\frac{m\big(E_F^{-1}((n^+, +\infty])\big)}{\mu\big((B_k)^c_n(T_F)\big)} &\leq Ce^{-\alpha n^+} \exp\big((1+\varepsilon)^2 R_F(\mu)\mu_F(B_k)n\big) \\ &= C\exp\left(-\left(\alpha\left(\frac{1}{\mu(F)}+\eta\right) - (1+\varepsilon)^2 R_F(\mu)\mu_F(B_k)n\right)\right) \\ &\leq C\end{aligned}$$

for every $\varepsilon > 0$ small enough and $n \geq M_1$. Combining this inequality, (20.10), (20.11), (20.9), (20.8), and (20.7), we get for every $k \geq 1$ large enough, every $e > 0$, and every $n \geq N_k^{(1)}$, that

$$\frac{\mu\big(B^-_{k,n}(\eta)\big)}{\mu\big((B_k)^c_n(T_F)\big)} \leq C\big(1 + e^{\alpha N_k^{(1)}}\big) + C' \sum_{j=0}^{n^+} e^{-\frac{\alpha}{2}j} + C'' e^{7\varepsilon n} \sum_{j=0}^{n^+-N_k^{(1)}} e^{-\frac{\alpha}{2p}j} \leq C''' e^{7\varepsilon n}$$

with some constants $C, C', C'', C''' \in (0, +\infty)$ and $p > 1$ independent of $k \geq 1$ large enough, $n \geq N_k^{(1)}$, and $\varepsilon \in (0,1)$ small enough. Hence,

$$-\varlimsup_{n\to\infty} \frac{1}{n} \log \mu\big(B^-_{k,n}(\eta)\big) \geq \underline{R}_{T_F,\mu}(B_k) - 7\varepsilon$$

for every $\varepsilon > 0$ and every $k \geq 1$ large enough. Therefore,

$$-\varlimsup_{n\to\infty} \frac{1}{n} \log \mu\big(B^-_{k,n}(\eta)\big) \geq \underline{R}_{T_F,\mu}(B_k)$$

for every $k \geq 1$ large enough. Since

$$(B_k)^c_n \subseteq B^-_{k,\left[\frac{n}{\frac{1}{\mu(F)}+\eta}\right]},$$

we get

$$\varlimsup_{n\to\infty} \frac{1}{n} \log \mu\big((B_k)^c_n\big) \leq \frac{1}{\frac{1}{\mu(F)}+\eta} \varlimsup_{n\to\infty} \frac{1}{n} \log \mu\big(B^-_{k,n}(\eta)\big).$$

Therefore,

$$\underline{R}_{T_F,\mu}(B_k) \leq \left(\frac{1}{\mu(F)}+\eta\right) \underline{R}_{T,\mu}(B_k)$$

for every $k \geq 1$ large enough. Dividing both sides of this inequality by $\mu(B_k)$ and passing to the limit with $k \to \infty$, this gives

$$R_F(\mu) \leq (1 + \eta\mu(F)) \varliminf_{k\to\infty} \frac{R_{T,\mu}(B_k)}{\mu(B_k)}.$$

By letting in turn $\eta \searrow 0$, this yields

$$R_F(\mu) \leq \varliminf_{k\to\infty} \frac{R_{T,\mu}(B_k)}{\mu(B_k)}.$$

Together with (20.6) this finishes the proof of Theorem 20.3. □

21. First Return Maps and Escaping Rates, II

In this section we keep the settings of Section 20; more specifically that described between its beginning until formula (20.1). In particular, we do not assume appriori that (LDP) holds. In fact our goal in this section is provide natural sufficient conditions for (LDP) to hold. Let $\varphi : X \to \mathbb{R}$ be a Borel measurable function. We define the function $\varphi_F : F \to \mathbb{R}$ by the formula

$$\varphi_F(x) := \sum_{j=0}^{\tau_F(x)-1} \varphi \circ T^j(x). \tag{21.1}$$

It is well-known

$$\int_X \varphi \, d\mu = \mu(F) \int_F \varphi_F \, d\mu_F. \tag{21.2}$$

In particular,

$$1\!\!1_F = \tau_F,$$

and, inserting this to (21.2), we obtain the familiar, discussed in the previous section, Kac's Formula

$$\int_F \tau_F \, d\mu_F = \frac{1}{\mu(F)}.$$

Definition 21.1. We say that a pentacle (X, T, F, μ, φ), or just T, is of symbol return type (SRT) if the following conditions are satisfied:

(a) $F = E_A^\infty$ for some countable alphabet E and some finitely irreducible incidence matrix A.

(b) $T_F = \sigma : E_A^\infty \to E_A^\infty$.

(c) $\varphi_F : F \to \mathbb{R}$ is a Hölder continuous summable potential.

(d) $\mathrm{P}(\varphi_F) = 0$.

(e) $\mu = \mu_{\varphi_F}$ is the Gibbs/equilibrium state for the potential $\varphi_F : F \to \mathbb{R}$

(f) There are two constants $C, \alpha > 0$ such that
$$\mu\big(\tau_F^{-1}(n)\big) \le Ce^{-\alpha n}$$
for all integers $n \ge 1$.

Since
$$\tau_F^{-1}(n) \subseteq T^{-1}(E_F^{-1}(n-1))$$
and since the measure μ is T-invariant, we immediately obtain the following.

Observation 21.2. If a pentacle (X, T, F, μ, φ) satisfies all conditions (a)–(e) of Definition 21.1 and it also has exponential tail decay (ETD), then (X, T, F, μ, φ) also satisfies condition (f) of Definition 21.1; thus in conclusion, the pentacle (X, T, F, μ, φ) is then of symbol return type (SRT).

Given $\theta \in \mathbb{R}$ we consider the potential
$$\varphi_\theta := \varphi_F + \theta\tau_F : F \to \mathbb{R}.$$
We shall prove several lemmas. We start with the following.

Lemma 21.3. *If T is an* (SRT) *system, then the potential $\varphi_\theta : F \to \mathbb{R}$ is summable for every $\theta < \alpha$.*

Proof. Since T is SRT, we have that
$$\begin{aligned}
\sum_{e\in E} \exp\big(\sup\big(\varphi_\theta|_{[e]}\big)\big) &= \sum_{e\in E} \exp\big(\sup\big((\varphi_F + \theta\tau_F)|_{[e]}\big)\big) \\
&= \sum_{e\in E} \exp\big(\sup\big(\varphi_F|_{[e]}\big)\big) \exp\big(\theta\tau_F(e)\big) \asymp \sum_{e\in E} \mu([e]) \exp\big(\theta\tau_F(e)\big) \\
&= \sum_{n=1}^{\infty} \sum_{\tau_F(e)=n} \mu([e])e^{\theta n} = \sum_{n=1}^{\infty} e^{\theta n} \sum_{\tau_F(e)=n} \mu([e]) \\
&= \sum_{n=1}^{\infty} e^{\theta n}\mu\big(\tau_F^{-1}(n)\big) \le C\sum_{n=1}^{\infty} \exp\big((\theta-\alpha)n\big) < +\infty,
\end{aligned}$$
whenever $\theta < \alpha$. The proof is complete. □

Lemma 21.4. *If T is an* (SRT) *system, then the function $(-\infty, \alpha) \ni \theta \mapsto \mathrm{P}(\varphi_\theta) \in \mathbb{R}$ is real-analytic.*

Proof. In the terminology of Corollary 2.6.10 in [31], condition (c) of Definition 21.1 says that $\varphi_F \in \mathcal{K}_\beta$, where $\beta > 0$ is the Hölder exponent of φ_F. Of course $\tau_F \in \mathcal{K}_\beta$ since τ_F is constant on cylinders of length one. Lemma 21.3 says that $\varphi_\theta \in \mathcal{K}_\beta$ for all $\theta < \alpha$; in fact the proof of this lemma shows that $\varphi_\theta \in \mathcal{K}_\beta$ for all $\theta \in \mathbb{C}$ with $\mathrm{Re}(\theta) < \alpha$. This now means that all hypotheses of Corollary 2.6.10 from [31] are satisfied. The upshot of this corollary is that the function
$$\{\theta \in \mathbb{C} : \mathrm{Re}(\theta) < \alpha\} \ni \theta \mapsto \mathcal{L}_{\varphi_\theta} \in L(\mathcal{K}_\beta)$$

is holomorphic, where $\mathcal{L}_{\varphi_\theta}$ is the Perron-Frobenius operator associated to the potential φ_θ and the shift map $\sigma = T_F$. The proof is now concluded by applying Kato-Rellich perturbation Theorem and the fact that $\exp\big(\mathrm{P}(\varphi_\theta)\big)$ is a simple isolated eigenvalue of $\mathcal{L}_{\varphi_\theta}$ for all real $\theta < \alpha$ (it is not really relevant here but in fact $\exp\big(\mathrm{P}(\varphi_\theta)\big)$ is equal to the spectral radius of the operator $\mathcal{L}_{\varphi_\theta} \in L(\mathcal{K}_\beta)$), see the paragraph of [31] located between Remark 2.6.11 and Theorem 2.6.12 for more details. □

Because of Lemma 21.3, for every $\theta < \alpha$ there exists a unique Gibbs/equilibrium state μ_θ for the potential $\varphi_\theta : F \to \mathbb{R}$. Having the previous two lemmas, Proposition 2.6.13 in [31] applies to give the following.

Lemma 21.5. *If T is an* (SRT) *system, then*

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{P}(\varphi_\theta) = \int_F \tau_F \, d\mu_\theta$$

for every $\theta < \alpha$.

Now having all the three previous lemmas along with Definition 21.1, employing the standard (by now) tools of [31], exactly the same proof as in [17] yields the following.

Theorem 21.6. *If T is an* (SRT) *system, then for every $\theta < \alpha$ we have that*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu\Big(\Big\{x \in F : \mathrm{sgn}(\theta)\tau_F^{n)}(x) \geq \mathrm{sgn}(\theta) n \int_F \tau_F \, d\mu_\theta\Big\}\Big) = -\theta \int_F \tau_F \, d\mu_\theta + \mathrm{P}(\varphi_\theta).$$

In order to make use of this theorem we shall prove the following.

Lemma 21.7. *If T is an* (SRT) *system and the first return map function $\tau_F : F_\infty \to \mathbb{N}$ is unbounded, then for every non-zero $\theta < \alpha$ we have that*

$$\mathrm{P}(\varphi_\theta) - \theta \int_F \tau_F \, d\mu_\theta < 0.$$

Proof. Since μ_θ is an equilibrium state for φ_θ, we have that

$$\begin{aligned} \mathrm{P}(\varphi_\theta) - \theta \int_F \tau_F \, d\mu_\theta &= \mathrm{h}_{\mu_\theta}(\sigma) + \int_F \varphi_F \, d\mu_\theta + \theta \int_F \tau_F \, d\mu_\theta - \theta \int_F \tau_F \, d\mu_\theta \\ &= \mathrm{h}_{\mu_\theta}(\sigma) + \int_F \varphi_F \, d\mu_\theta \\ &\leq \mathrm{P}(\varphi_F) = 0. \end{aligned}$$

Hence, in order to complete the proof we only need to show that the inequality sign above is strict. In order to do this suppose for a contradiction that $\mathrm{h}_{\mu_\theta}(\sigma) + \int_F \varphi_F \, d\mu_\theta = \mathrm{P}(\varphi_F)$. But then the fact that μ is the only equilibrium state for φ_F, implies that $\mu_\theta = \mu$. But because of Theorem 2.2.7 in [31] this in turn implies that the function $\varphi_\theta - \varphi_F$ is cohomologous to a constant in the class of Hölder continuous functions defined on $E_A^\infty = F$. But $\varphi_\theta - \varphi_F = \theta\tau_F$ is, by our hypotheses, unbounded unless $\theta = 0$. This finishes the proof. □

Now we can prove the main result of this section:

Lemma 21.8. *If T is an* (SRT) *system and the first return map function $\tau_F : F_\infty \to \mathbb{N}$ is unbounded, then the pair (T_F, F) satisfies the large deviation property* (LDP).

Proof. Fix $\eta \in (0,1)$. It follows from Lemma 21.4 and Lemma 21.5 that the function

$$(-\infty, \alpha) \ni \theta \mapsto \int_F \tau_F \, d\mu_\theta \in [1, +\infty)$$

is continuous. Therefore, there exists $\delta \in (0, \alpha)$ such that

$$\int_F \tau_F \, d\mu - \eta \leq \int_F \tau_F \, d\mu_\delta, \int_F \tau_F \, d\mu_{-\delta} \leq \int_F \tau_F \, d\mu + \eta.$$

Equivalently:

$$\mu(F)^{-1} - \eta \leq \int_F \tau_F \, d\mu_\delta, \int_F \tau_F \, d\mu_{-\delta} \leq \mu(F)^{-1} + \eta.$$

Hence for every $k \geq 1$:

$$F^c_{k-1}(\eta) \subseteq \big\{x \in F : \tau_F^{k)}(x) \geq k \int_F \tau_F \, d\mu_\delta\big\} \cup \big\{x \in F : \tau_F^{k)}(x) \leq k \int_F \tau_F \, d\mu_{-\delta}\big\}.$$

So, denoting

$$\hat{\eta} := \frac{1}{2} \min\Big\{\delta \int_F \tau_F \, d\mu_\delta - \mathrm{P}(\varphi_d), -\delta \int_F \tau_F \, d\mu_{-\delta} - \mathrm{P}(\varphi_{-\delta})\Big\},$$

which is positive by Lemma 21.7, we conclude from Theorem 21.6, that

$$\mu\big(F^c_{k-1}(\eta)\big) \leq C_\eta e^{-\hat{\eta} k}$$

for all $k \geq 1$. The proof is complete. □

22. Escape Rates for Interval Maps

In this and the next sections we will reap the benefits of our work in the previous sections, most notably of that on escape rates of conformal countable alphabet IFSs and of that on the first return map techniques including large deviations. This section is devoted to the study of the multimodal smooth maps of an interval.

We start with the definition of the class of dynamical systems and potentials we consider.

Definition 22.1. Let $I = [0,1]$ be the closed interval. Let $T : I \to I$ be a C^3 differentiable map with the following properties:

(a) T has only a finitely many maximal closed intervals of monotonicity; or equivalently $\mathrm{Crit}(T) = \{x \in I : T'(x) = 0\}$, the set of all critical points of T is finite.

(b) The dynamical system $T : I \to I$ is topologically exact, meaning that for every non-empty subset U of I there exists an integer $n \geq 0$ such that $T^n(U) = I$.

(c) All critical points are non-flat.

(d) $T : I \to I$ is a *topological Collet-Eckmann map*, meaning that

$$\inf\{(|(T^n)'(x)|)^{1/n} : T^n(x) = x \text{ for } n \geq 1\} > 1$$

where the infimum is taken over all integers $n \geq 1$ and all fixed points of T^n.

We then call $T : I \to I$ a *topologically exact topological Collet-Eckmann map* (teTCE). If (c) and (d) are relaxed and only (a) and (b) are assumed then T is called a topologically exact multimodal map.

We set

$$\mathrm{PC}(T) := \bigcup_{n=1}^{\infty} T^n(\mathrm{Crit}(T))$$

and call this the *postcritical* set of T. We say that the map $T : I \to I$ is *tame* if

$$\overline{\mathrm{PC}(T)} \neq I.$$

The following theorem is due to many authors and a detailed and readable discussion on this topic can be found, for example, in [40]

Theorem 22.2 (Exponential Shrinking Property)**.** *If $T : I \to I$ satisfies conditions* (a)–(c) *of Definition* 22.1*, then T is a* (te)TCE*, i.e. condition* (d) *holds if and only if there exist $\delta > 0$, $\gamma > 0$ and $C > 0$ such that if $z \in I$ and $n \geq 0$ then*

$$\mathrm{diam}(W) \leq Ce^{-\gamma n}$$

for each connected component W of $T^{-n}(B(z, 2\delta))$.

The hard part of this theorem is its "if" part. The converse is easy. There are more conditions equivalent to teTCE, but we need only the above Exponential Shrinking Property (ESP) and we do not bring them up here. We now however articulate two standard sufficient conditions for (ESP) to hold. It is implied by the Collet-Eckmann condition which requires that there exist $\lambda > 1$ and $C > 0$ such that for every integer $n \geq 0$ we have that

$$|(f^n)'(f(c))| \geq C\lambda^n.$$

If also suffices to assume that the map T is semi-hyperbolic, i.e., that no critical points c in the Julia belongs to its own omega limit set $\omega(c)$ for (ESP) to hold. This so for example, if T is a classical unimodal map of the form $I \ni x \mapsto \lambda x(1-x)$, with $0 < \lambda \leq 4$ and the critical point $1/2$ is not in its own omega limit set, i.e., $1/2 \notin \omega(1/2)$.

We call a potential $\psi : I \to \mathbb{R}$ *acceptable* if it is Lipschitz continuous and

$$\sup(\psi) - \inf(\psi) < h_{\mathrm{top}}(T).$$

We would also like to mention that for the purposes of this section it would suffice that $\psi : I \to \mathbb{R}$ is Hölder continuous (with any exponent) and of bounded variation. We denote by BV_I the vector space of all functions in $L^1(\lambda)$, where λ denotes Lebesgue measure on I, that have a version of bounded variation. This vector space becomes a Banach space when endowed with the norm

$$\|g\|_{BV} := \|g\|_{\mathrm{Leb}_1} + v_I(g)$$

where $v_I(g)$ denotes the variation of g on I. For every $g \in \mathrm{BV}_I$ define the *Perron-Frobenius operator* associated to ψ by

$$\mathcal{L}_\psi(g)(x) = \sum_{y \in T^{-1}(x)} g(y) e^{\psi(y)}.$$

It is well known and easy to check that $\mathcal{L}_\psi(\mathrm{BV}_I) \subset \mathrm{BV}_I$ and $\mathcal{L}_\psi : \mathrm{BV}_I \to \mathrm{BV}_I$ is a bounded linear operator.

The following theorem collects together some fundamental results of [22] and [23]

Theorem 22.3. *If $T : I \to I$ is a topologically exact multimodal map and $\psi : I \to \mathbb{R}$ is an acceptable potential then*

(a) *there exists a Borel probability eigenmeasure m_ψ for the dual operator $\mathcal{L}_\psi^*$ whose corresponding eigenvalue is equal to $e^{\mathrm{P}(\psi)}$. It then follows that* $\mathrm{supp}(m_\psi) = I$.
(b) *there exists a unique Borel T-invariant probability measure μ_ψ on I absolutely continuous with respect to m_ψ. Furthermore, μ_ψ is equivalent to m_ψ;*
(c) *$h_{\mu_\psi}(T) + \int_I \psi d\mu_\psi = P(\psi)$, meaning that μ_ψ is an (ergodic) equilibrium state for $\psi : I \to \mathbb{R}$ with respect to the dynamical system $T : I \to I$.*
(d) *The Perron-Frobenius $\mathcal{L}_\psi : \mathrm{BV}_I \to \mathrm{BV}_I$ is quasi-compact.*
(e) $r(\mathcal{L}_\psi) = e^{P(\psi)}$.
(f) $\mathrm{sp}(\mathcal{L}_\psi) \cap \partial B(0, e^{P(\psi)}) = \{e^{P(\varphi)}\}$
(g) *The number $e^{P(\psi)}$ is a simple isolated eigenvalue (this follows from (f), (e) and (f)) of $\mathcal{L}_\psi : \mathrm{BV}_I \to \mathrm{BV}_I$ with eigenfunction $\rho_\psi := \frac{d\mu_\psi}{dm_\psi}$ which is Lipschitz continuous and* log*-bounded.*

We shall use the commonly accepted convention, used throughout this article, that for every $r \in (0,1]$ and every bounded interval $\Delta \subset \mathbb{R}$ we denote by $r\Delta$ the (smaller) interval of length $r|\Delta|$ centered at the same point as Δ. We now consider the following version of the bounded distortion property taken from [40] whose proof has a long history and is well documented therein.

Theorem 22.4. *Let $T : I \to I$ be a teTCE. Then for every $r \in (0,1)$ there exists $K(r) \in (0, +\infty)$ such that if $\Delta \subset I$ is an interval, $n \geq 0$ is an integer, the map $T^n|_\Delta$ is* 1*-to-*1*, and $x, y \in \Delta$ are such that $T^n(x), T^n(y) \in rT^n(\Delta)$, then*

$$\left| \frac{(T^n)'(y)}{(T^n)'(x)} - 1 \right| \leq K(r) |(T^n)(y) - (T^n)(x)|.$$

We next recall the following definition.

Definition 22.5. An interval $V \subset I$ is called a *nice set* for a multimodal map $T : I \to I$ if

$$\mathrm{int}(V) \cap \bigcup_{n=0}^{\infty} T^n(\partial V) = \emptyset$$

The proof of the following theorem is both standard and straightforward, and has been presented in various similar settings. We provide the proof below because of the critical importance for us of the theorem it proves and the brevity of the proof, for the sake of completeness, and for the convenience of the reader.

Theorem 22.6. *If $T : I \to I$ is topologically exact multimodal map then for every point $\xi \in (0,1)$ and every $R > 0$ there exists a nice set $V \subset I$ such that $\xi \in V \subset B(\xi, R)$.*

Proof. Since the map $T : I \to I$ is topologically exact it has a dense set of periodic points. Fix one periodic point ω, say of prime period $p \geq 1$, such that $\xi \notin \cup_{k=0}^{\infty} T^{-k}(\{T^j(\omega) : 0 \leq j \leq p-1\})$. Again because of topological exactness of T,

$$\xi \in \overline{(0,\xi) \cap \bigcup_{k=0}^{\infty} T^{-k}(\{T^j(\omega) : 0 \leq j \leq p-1\})}$$

and

$$\xi \in \overline{(\xi,1) \cap \bigcup_{k=0}^{\infty} T^{-k}(\{T^j(\omega) : 0 \leq j \leq p-1\})}.$$

For every $n \geq 1$, sufficiently large denote by $\xi_n^- \in I$ the point closest to ξ in

$$\overline{(0,\xi) \cap \bigcup_{k=0}^{n} T^{-k}(\{T^j(\omega) : 0 \leq j \leq p-1\})}$$

and by $\xi_n^+ \in I$ the point closest to ξ in

$$\overline{(\xi,1) \cap \bigcup_{k=0}^{n} T^{-k}(\{T^{-j}(\omega) : 0 \leq j \leq p-1\})}.$$

We then denote

$$V_n := (\xi_n^-, \xi_n^+).$$

Then obviously $\xi \in V_n$, $T^k(\xi_n^\pm) \notin (\xi_n^-, \xi_n^+)$ for all $k = 0, 1, \cdots, n-1$, and $T^k(\xi_n^\pm) \in \{T^j(w) : 0 \leq j \leq n-1\}$ for all $k \geq n$. Since $\lim_{n\to+\infty} \xi_n^\pm = \xi$ it then follows that $T^k(\xi_n^\pm) \notin V_n$ for all $k \geq n$. In conclusion, V_n are the required nice sets for all integers $n \geq 1$. Since in addition $\lim_{n\to+\infty} \operatorname{diam}(V_n) = 0$ the proof is complete. □

Given a set $F \subseteq I$ and an integer $n \geq 0$, we denote by $\mathcal{C}_F(n)$ the collection of all connected components of $T^{-n}(F)$. From their definitions, nice sets enjoy the following property.

Theorem 22.7. *If V is a nice set for a multimodal map $T : I \to I$, then for every integer $n \geq 0$ and every $U \in \mathcal{C}_V(n)$ either*

$$U \cap V = \emptyset \quad \text{or} \quad U \subset V.$$

From now on throughout this section we assume that $T : I \to I$ is a tame teTCE map. Fix a point $\xi \in I \backslash \overline{PC(T)}$. By virtue of Theorem 22.4 there is a nice set V such that

$$\xi \in V \quad \text{and} \quad 2V \cap \overline{\mathrm{PC}(T)} = \emptyset.$$

The nice set V canonically gives rise to a countable alphabet conformal iterated function system in the sense considered in the previous sections of the present paper. Namely, put

$$\mathcal{C}_V^* := \bigcup_{n=1}^{\infty} \mathcal{C}_V(n).$$

For every $U \in \mathcal{C}_V^*$ let $\tau_V(U) \geq 1$ the unique integer $n \geq 1$ such that $U \in \mathcal{C}_V(n)$. Put further

$$\varphi_U := f_U^{-\tau_V(U)} : V \to U$$

and keep in mind that

$$\varphi_U(V) = U.$$

Denote by E_V the subset of all elements U of $\mathcal{C}_V^*$ such that

(a) $\varphi_U(V) \subset V$,
(b) $f^k(U) \cap V = \emptyset$ for all $k = 1, 2, \ldots, \tau_V(U) - 1$.

The collection

$$\mathcal{S}_V := \{\varphi_U : V \to V\}$$

of all such inverse branches forms obviously an iterated function system in the sense considered in the previous sections of the present paper. In other words the elements of $\mathcal{S}_V$ are formed by all inverse branches of the first return map $f_V : V \to V$. In particular, $\tau_V(U)$ is the first return time of all points in $U = \varphi_U(V)$ to V. We define the function $N_V : E_V^\infty \to \mathbb{N}_1$ by setting

$$N_V(\omega) := \tau_V(\omega_1).$$

Let

$$\pi_V : E_V^\infty \to \mathbb{R}$$

be the canonical projection induced by the iterated function system $\mathcal{S}_V$. Let

$$J_V := \pi_V\big(E_V^\infty\big)$$

be the limit set of the system $\mathcal{S}_V$. Clearly

$$J_V \subseteq I.$$

It is immediate from our definitions that

$$\tau_V(\pi(\omega)) = N_V(\omega)$$

for all $\omega \in E_V^{\mathbb{N}}$.

We shall now prove the following.

Proposition 22.8. *Let $T : I \to I$ be a tame* teTCE *map. Let $\psi : I \to \mathbb{R}$ be an acceptable potential. If V is a nice set for T, then*

(a) $\widetilde{\psi}_V := \psi_V \circ \pi_V - P(\psi)N_V : E^{\mathbb{N}} \to \mathbb{R}$ *is a summable Hölder continuous potential;*
(b) $P(\sigma, \widetilde{\psi}_V) = 0$ *for the pressure for the shift map* $\sigma : E_V^{\mathbb{N}} \to E_V^{\mathbb{N}}$*;*
(c) $\mu_{\varphi,V} = \mu_{\widetilde{\psi}_V} \circ \pi_V^{-1}$, *where $\mu_{\widetilde{\psi}_V}$ is the equilibrium state for $\widetilde{\psi}_V$ and the shift map* $\sigma : E_V^{\mathcal{N}} \to E_V^{\mathcal{N}}$*;*

(d) *In addition, ψ_V is the amalgamated function of a summable Hölder continuous system of functions.*

Proof. Hölder continuity of $\widetilde{\psi}_V$ follows directly from Theorem 22.2 (the Exponential Shrinking Property) and the fact that the function N_V is constant on cylinders of length 1. Hölder continuity of $\widetilde{\psi}_V$ follows directly from Theorem 22.2 (the Exponential Shrinking Property) and the fact that the function N_V is constant on cylinders of length 1. We define a Hölder continuous system of functions $G = \{g^{(l)} : V \to \mathbb{R}\}_{e\in E}$ by putting

$$g^{(e)} := \big(\psi_V - \mathrm{P}(\varphi)\tau_V\big) \circ \varphi_e, \ e \in E.$$

Theorem 22.3 then implies the system G is summable, $\mathrm{P}(G) = 0$, and $m_{\psi,V}$ is the unique G-conformal measure for the IFS $\mathcal{S}_V$. According to [31], $g : E_V^{\mathbb{N}} \to \mathbb{R}$, the amalgamated function of G is defined by the formula

$$\begin{aligned} g(\omega) = g^{(\omega_1)}(\pi_V(\sigma(\omega))) &= \psi_V \circ \varphi_{\omega_1}(\pi_V(\sigma(\omega))) - \mathrm{P}(\psi)\tau_V \circ \varphi_{\omega_1}(\pi_V(\sigma(\omega))) \\ &= \psi_V \circ \pi_V(\omega) - \mathrm{P}(\psi)N_V(\omega) \\ &= \tilde{\psi}_V(\omega). \end{aligned}$$

By Proposition 3.1.4 in [31] we thus have that

$$\mathrm{P}\big(\sigma, \tilde{\psi}_V\big) = \mathrm{P}(G) = 0.$$

Now, since $\pi_V \circ \sigma = T_V \circ \pi_V$, i.e. since the dynamical system $T_V : J_V \to J_V$ is a factor of the shift map $\sigma : E_V^{\mathbb{N}} \to E_V^{\mathbb{N}}$ via the map $\pi_V : E_V^{\mathbb{N}} \to J_V$, we see that $\mu_{\tilde{\psi}_V} \circ \pi_V^{-1}$ is a Borel f_V-invariant probability measure on J_V equivalent to $m_{\tilde{\psi}_V} \circ \pi_V^{-1} = m_g \circ \pi^{-1} = m_G = m_{\psi,V}$. Since $m_{\psi,V}$ is equivalent to $\mu_{\psi,V}$, we thus conclude that the measures $m_{\tilde{\psi}_V} \circ \pi_V^{-1}$ and $\mu_{\psi,V}$ are equivalent. Since both these measures are T_V-invariant and $\mu_{\psi,V}$ is ergodic, they must be equal. The proof is thus complete. □

Since $\pi_V : E_V^{\mathbb{N}} \to J_V = V_\infty$, where, we recall the latter is the set of points returning infinitely often to V, is a measurable isomorphism sending the σ-invariant measure $\mu_{\tilde{\psi}_V}$ to the f_V-invariant probability measure $\mu_{\psi,V}$, by identifying the sets $E_V^{\mathbb{N}}$ and $V_\infty (= J_V)$, we can prove the following.

Lemma 22.9. *With all the hypotheses of Proposition* 22.8*, the pentacle $(I, T, V, \tilde{\psi}_V, \mu_{\tilde{\psi}_V})$ is an* SRT *system having exponential tail decay* (ETD)*, where we recall that V_∞ is identified with $E_V^{\mathbb{N}}$, $\tilde{\psi}_V$ is identified with $\psi_V - \mathrm{P}(\psi)\tau_V$, and $\mu_{\tilde{\psi}_V}$ is identified with $\mu_{\psi,V}$.*

Proof. By virtue of Proposition 22.8 and Observation 21.2 we only need to prove that the pentacle $(I, T, V, \tilde{\psi}_V, \mu_{\tilde{\psi}_V})$ has exponential tail decay (ETD). We can assume without loss of generality that $\psi : I \to \mathbb{R}$ is normalized so that

$$\mathrm{P}(\psi) = 0 \quad \text{and} \quad m_\psi = \mu_\psi.$$

Now define

$$\mathcal{C}_V^0(n) := \big\{U \in \mathcal{C}_V(n) : \forall_{(0\le k\le n-1)}\, T^k(U) \cap V = \emptyset\big\}$$

and

$$\mathcal{C}_V^*(n) := \{U \in \mathcal{C}_V(n) : U \subseteq V\} = \{U \in \mathcal{C}_V(n) : U \cap V \neq \emptyset\}.$$

Since the map $T : I \to I$ is topologically exact, there exists an integer $q \geq 1$ such that

$$T^q(V) \supseteq I.$$

Therefore for every $e \in \mathcal{C}_V(n)$ there exists (at least one) $\hat{e} \in \mathcal{C}_V^*(n+q)$ such that

$$T^q \circ \varphi_{\hat{e}} = \varphi_e.$$

By conformality of the measure μ_ψ, for every $e \in \mathcal{C}_V(n)$, we have

$$\mu_\psi\big(\varphi_{\hat{e}}(V)\big) \geq \exp(-q||\psi||_\infty)\mu_\psi(\varphi_e(V)).$$

So, since

$$\bigcup_{a \in \mathcal{C}_V^0(n+q)} \varphi_a(V) \subseteq \bigcup_{\substack{b \in \mathcal{C}_V(n+q) \\ T^q \circ \varphi_b \in \mathcal{C}_V^0(n)}} \varphi_b(V) \setminus \bigcup_{e \in \mathcal{C}_V^0(n)} \varphi_{\hat{e}}(V),$$

we therefore get

$$\begin{aligned}
\mu_\psi\left(\bigcup_{a \in \mathcal{C}_V^0(n+q)} \varphi_a(V)\right) &\leq \mu_\psi\left(\bigcup_{\substack{b \in \mathcal{C}_V(n+q) \\ T^q \circ \varphi_b \in \mathcal{C}_V^0(n)}} \varphi_b(V) \setminus \bigcup_{e \in \mathcal{C}_V^0(n)} \varphi_{\hat{e}}(V)\right) \\
&= \mu_\psi\left(\bigcup_{\substack{b \in \mathcal{C}_V(n+q) \\ f^q \circ \varphi_b \in \mathcal{C}_V^0(n)}} \varphi_b(V)\right) - \mu\left(\bigcup_{e \in \mathcal{C}_V^0(n)} \varphi_{\hat{e}}(V)\right) \\
&= \mu_\psi\left(T^{-q}\left(\bigcup_{c \in \mathcal{C}_V^0(n)} \varphi_c(V)\right)\right) - \sum_{e \in \mathcal{C}_V^0(n)} \mu_\psi\big(\varphi_{\hat{e}}(V)\big) \\
&= \mu_\psi\left(\bigcup_{c \in \mathcal{C}_V^0(n)} \varphi_c(V)\right) - \sum_{e \in \mathcal{C}_V^0(n)} \mu_\psi\big(\varphi_{\hat{e}}(V)\big) \\
&\leq \mu_\psi\left(\bigcup_{c \in \mathcal{C}_V^0(n)} \varphi_c(V)\right) - \exp(-q||\psi||_\infty) \sum_{e \in \mathcal{C}_V^0(n)} \mu_\psi\big(\varphi_e(V)\big) \\
&= \gamma\mu_\psi\left(\bigcup_{c \in \mathcal{C}_V^0(n)} \varphi_c(V)\right),
\end{aligned}$$

where $\gamma := 1 - \exp(-q||\psi||_\infty) \in [0,1)$. An immediate induction then yields

$$\mu_\psi\left(\bigcup_{e\in\mathcal{C}_V^0(n)} \varphi_e(V)\right) \le \gamma^{-1}\gamma^{n/q}$$

for all $n \ge 0$. But, as

$$E_V^{-1}([n,+\infty]) = E_V^{-1}(\{+\infty\}) \cup \bigcup_{k=n}^{\infty} \bigcup_{e\in\mathcal{C}_V^0(k)} \varphi_e(V)$$

and since $\mu_\psi\big(E_V^{-1}(\{+\infty\})\big) = 0$ by ergodicity of μ_ψ and of $\mu_\psi(V) > 0$, we therefore get that

$$\mu_\psi\big(E_V^{-1}([n,+\infty])\big) \le \big(\gamma(1-\gamma^{1/q})\big)^{-1}\gamma^{n/q} \tag{22.1}$$

for all $n \ge 0$. This just means that the pentacle $(I, T, V, \tilde\psi_V, \mu_{\tilde\psi_V})$ has exponential tail decay (ETD), and the proof is complete. □

Denote by $I_R(T)$ the set of all recurrent points of T in I. Formally

$$I_R(T) := \{z \in I : \underline{\lim}_{n\to\infty} |T^n(z) - z| = 0\}.$$

Of course $I_R(T) \subseteq J_T$ and $\mu_\psi(I \setminus I_R(T)) = 0$ because of Poincaré's Recurrence Theorem. The set $I_R(T)$ is significant for us since

$$I_R(T) \cap V \subseteq J_V.$$

Now we can harvest the fruits of the work we have done. As a direct consequence of Theorem 15.10, Theorem 15.11, Proposition 22.8, Lemma 22.9, Lemma 21.8, and Theorem 20.3, we get the following two results.

Theorem 22.10. *Let $T : I \to I$ be a tame* teTCE *map. Let $\psi : I \to \mathbb{R}$ be an acceptable potential. Let $z \in I_R(T)\backslash\overline{\mathrm{PC}(T)}$.*

Assume that the equilibrium state μ_ψ is (WBT) *at z. Then*

$$\lim_{\varepsilon\to0}\frac{\underline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} = \lim_{\varepsilon\to0}\frac{\overline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} =$$

$$= \begin{cases} & \text{if } z \text{ is not any periodic point of } T,\\ 1-\exp\big(S_p\psi(z) - p\mathrm{P}(f,\psi)\big) & \text{if } z \text{ is a periodic point of } T.\end{cases}$$

Theorem 22.11. *Let $T : I \to I$ be a tame* teTCE *map. Let $\psi : I \to \mathbb{R}$ be an acceptable potential. Then*

$$\lim_{\varepsilon\to0}\frac{\underline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} = \lim_{\varepsilon\to0}\frac{\overline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} = 1$$

for μ_ψ–a.e. point $z \in I$.

Definition 22.12. A multimodal map $T : I \to I$ is called *subexpanding* if

$$\mathrm{Crit}(T) \cap \overline{\mathrm{PC}(T)} = \emptyset.$$

It is not hard to see (good references for a proof can be found in [40]) that the following it true.

Proposition 22.13. *Any topologically exact multimodal subexpanding map of the interval I is a tame teTCE map.*

Let us quote another well-known result which can be found, for example, in the book of de Melo and van Strien [35].

Theorem 22.14. *If $T : I \to I$ is a topologically exact multimodal subexpanding map, then there exists a unique Borel probability T-invariant measure μ absolutely continuous with respect to Lebesgue measure λ. In fact,*

(a) *μ is equivalent to λ and (therefore)*
(b) *has full topological support.*
(c) *The Radon–Nikodym derivative $\frac{d\mu}{d\lambda}$ is uniformly bounded above and separated from zero on the complement of every fixed neighborhood of $\overline{\mathrm{PC}(T)}$.*
(d) *μ is ergodic, even K-mixing,*
(e) *μ has Rokhlin's natural extension metrically isomorphic to some two sided Bernoulli shift and*
(f) *μ charges with full measure both topologically transitive and radial points of T.*

As an immediate consequence of this theorem, particularly of its item (c), we get the following.

Corollary 22.15. *If $T : I \to I$ is a topologically exact multimodal subexpanding map, then the T-invariant measure μ absolutely continuous with respect to Lebesgue measure λ is (WBT) at every point of $I \setminus \overline{\mathrm{PC}(T)}$.*

Passing to Hausdorff dimension, by a small obvious modification (see [40] for details) of the proof of Theorem 22.6 for all $c \in \mathrm{Crit}(T) \cup \{\xi\}$ there are arbitrarily small open intervals V_c, $c \in V_c$, such that $V_c \cap \overline{PC}(T) = \emptyset$ and the collection T_*^{-n}, $n \geq 1$, of all continuous (equivalently smooth inverse branches of T^n) defined on V_c, $c \in \mathrm{Crit}(T) \cup \{\xi\}$, and such that for some $c' \in \mathrm{Crit}(T) \cup \{\xi\}$,

$$T_*^{-n}(V_c) = V_{c'}$$

and

$$\bigcup_{k=1}^{n-1} T^k(T_*^{-n}(V_c)) \cap \bigcup \big\{V_z : z \in \mathrm{Crit}(T) \cup \{\xi\}\big\} = \emptyset$$

forms a finitely primitive conformal GDS, which we will call $\mathcal{S}_T$, whose limit set contains $\mathrm{Trans}(T)$. Another characterization of $\mathcal{S}_T$ is that its elements are composed of continuous inverse branches of the first return map of f from

$$V := \bigcup \big\{V_z : z \in \mathrm{Crit}(T) \cup \{\xi\}\big\}$$

to V. It has been proved in [40] that $\mathrm{HD}(K(V)) < 1$.

So, since by Theorem 17.1, $\lim_{r\to 0} \mathrm{HD}\big(K(B(\xi,r))\big) = \mathrm{HD}(I) = 1$, we conclude that

$$\mathrm{HD}(K(V)) < \mathrm{HD}\big(K(B(\xi,r))\big)$$

for all $r > 0$ small enough. Therefore, since $b_{\mathcal{S}_T} = 1$ and since $\mu_{h,V} = \mu_{b_{\mathcal{S}_f}}$, applying Theorem 17.1, Corollary 17.3 and Corollary 19.2, we get the following two theorems.

Theorem 22.16. *Let $T : I \to I$ be a topologically exact multimodal subexpanding map. Fix $\xi \in I \backslash \overline{\mathrm{PC}(T)}$. Assume that the parameter 1 is powering at ξ with respect to the conformal* GDS $\mathcal{S}_T$. *Then the following limit exists, is finite, and positive:*

$$\lim_{r\to 0} \frac{1 - \mathrm{HD}(K_\xi(r))}{\mu(B(\xi,r))}.$$

Theorem 22.17. *If $T : I \to I$ is a topologically exact multimodal subexpanding map, then for Lebesgue–a.e. point $\xi \in I \setminus \overline{\mathrm{PC}(T)}$ the following limit exists, is finite and positive:*

$$\lim_{r\to 0} \frac{1 - \mathrm{HD}(K_\xi(r))}{\mu(B(\xi,r))}.$$

23. Escape Rates for Rational Functions of the Riemann Sphere

Now, we will apply the results of sections 14 and 15 to two large classes of conformal dynamical systems in te complex plane: rational functions of the Riemann sphere $\widehat{\mathbb{C}}$ in this section and, in the next section, transcendental meromorphic functions on $\mathbb{C}$. This section considerably overlaps in some of its parts with the previous section on the multimodal interval maps. We provide here its full exposition for the sake of coherent completeness and convenience of the readers not necessarily interested in interval maps.

As said, now we deal with rational functions. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree $d \geq 2$. Let $J(f)$ denote the Julia sets of f and let

$$\mathrm{Crit}(f) := \{c \in \widehat{\mathbb{C}} : f'(c) = 0\}$$

be the set of all critical (branching) points of f. As in the case of interval maps set

$$\mathrm{PC}(f) := \bigcup_{n=1}^{\infty} f^n(\mathrm{Crit}(f))$$

and call it the postcritical set of f. The best understood and the easiest (nowadays) to deal with class of rational functions is formed by expanding (also frequently called hyperbolic) maps. The rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is said to be expanding if the restriction $f|_{J(f)} : J(f) \to J(f)$ satisfies

$$\inf\{|f'(z)| : z \in J(f)\} > 1 \tag{23.1}$$

or, equivalently,

$$|f'(z)| > 1 \tag{23.2}$$

for all $z \in J(f)$. Another, topological, characterization of the expanding property is this.

Fact 23.1. A rational function $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is expanding if and only if

$$J(f)\cap\overline{\mathrm{PC}(f)}=\emptyset.$$

It is immediate from this characterization that all the polynomials $z\mapsto z^d$, $d\geq 2$, are expanding along with their small perturbations $z\mapsto z^d+\varepsilon$; in fact expanding rational functions are commonly believed to form a vast majority amongst all rational functions. This is known at least for polynomials with real coefficients. We however do not restrict ourselves to expanding rational maps only. We start with all rational functions, no restriction whatsoever, and then make some, weaker than hyperbolicity, appropriate assumptions.

Let $\psi:\widehat{\mathbb{C}}\to\mathbb{R}$ be a Hölder continuous function, referred to in the sequel as potential. We say that $\psi:\widehat{\mathbb{C}}\to\mathbb{R}$ has a pressure gap if

$$n\mathrm{P}(\psi)-\sup\big(\psi_n\big)>0 \tag{23.3}$$

for some integer $n\geq 1$, where $\mathrm{P}(\psi)$ denotes the ordinary topological pressure of $\psi|_{J(f)}$ and the Birkhoff's sum ψ_n is also considered as restricted to $J(f)$.

We would like to mention that (23.3) always holds (with all $n\geq 0$ sufficiently large) if the function $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ restricted to its Julia set is expanding (also frequently referred to as hyperbolic).

The probability invariant measure we are interested in comes from the following.

Theorem 23.2 ([18])**.** *If $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is a rational function of degree $d\geq 2$ and if $\psi:\widehat{\mathbb{C}}\to\mathbb{R}$ is a Hölder continuous potential with a pressure gap, then ψ admits a unique equilibrium state μ_ψ, i.e. a unique Borel probability f-invariant measure on $J(f)$ such that*

$$\mathrm{P}(\psi)=\mathrm{h}_{\mu_\psi}(f)+\int_{J(f)}\psi\,d\mu_\psi.$$

In addition,

(a) *the measure μ_ψ is ergodic, in fact K-mixing, and (see* [48]*) enjoys further finer stochastic properties.*
(b) *The Jacobian*

$$J(f)\ni z\longmapsto\frac{d\mu_\psi\circ T}{d\mu_\psi}(z)\in(0,+\infty)$$

is a Hölder continuous function.

In [41] a rational function $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ was called tame (comp. Section 22 devoted to interval maps) if

$$J(f)\setminus\overline{\mathrm{PC}(f)}\neq\emptyset.$$

Likewise, following [44], we adopt the same definition for (transcendental) meromorphic functions $f:\mathbb{C}\to\widehat{\mathbb{C}}$.

Remark 23.3. Tameness is a very mild hypothesis and there are many classes of maps foe which these hold. These include:

(1) Quadratic maps $z\mapsto z^2+c$ for which the Julia set is not contained in the real line;

(2) Rational maps for which the restriction to the Julia set is expansive which includes the case of expanding rational functions; and
(3) Misiurewicz maps, where the critical point is not recurrent.

In this paper the main advantage of dealing with tame functions is that these admit Nice Sets. Let us define and discuss them now.

Analogously as in the case of interval maps, given a set $F \subseteq \widehat{\mathbb{C}}$ and $n \geq 0$, we denote by $\mathcal{C}_F(n)$ the collection of all connected components of $f^{-n}(F)$. J. Rivera-Letelier introduced in [43] the concept of Nice Sets in the realm of the dynamics of rational maps of the Riemann sphere. In [19] N. Dobbs proved their existence for tame meromorphic functions from $\mathbb{C}$ to $\widehat{\mathbb{C}}$. We quote now his theorem.

Theorem 23.4. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a tame meromorphic function. Fix a non-periodic point $z \in J(f) \setminus \overline{\mathrm{PC}(f)}$, $\kappa > 1$, and $K > 1$. Then for all $L > 1$ and for all $r > 0$ sufficiently small there exists an open connected set $V = V(z, r) \subseteq \mathbb{C} \setminus \overline{\mathrm{PC}(f)}$ such that*

(a) *If $U \in \mathcal{C}_V(n)$ and $U \cap V \neq \emptyset$, then $U \subseteq V$.*
(b) *If $U \in \mathcal{C}_V(n)$ and $U \cap v \neq \emptyset$, then, for all $w, w' \in U$,*

$$|(f^n)'(w)| \geq L \quad \textit{and} \quad \frac{|(f^n)'(w)|}{|(f^n)'(w')|} \leq K.$$

(c) $\overline{B(z,r)} \subset U \subset B(z, \kappa r) \subseteq \mathbb{C} \setminus \overline{\mathrm{PC}(f)}$.

Each nice set canonically gives rise to a countable alphabet conformal iterated function system in the sense considered in the previous sections of the present paper. Namely, put

$$\mathcal{C}_V^* = \bigcup_{n=1}^{\infty} \mathcal{C}_V(n).$$

For every $U \in \mathcal{C}_V^*$ let $\tau_V(U) \geq 1$ the unique integer $n \geq 1$ such that $U \in \mathcal{C}_V(n)$. Put further

$$\varphi_U := f_U^{-\tau_V(U)} : V \to U$$

and keep in mind that

$$\varphi_U(V) = U.$$

Denote by E_V the subset of all elements U of $\mathcal{C}_V^*$ such that

(a) $\varphi_U(V) \subseteq V$,

(b) $f^k(U) \cap V = \emptyset \quad \text{for all} \quad k = 1, 2, \ldots, \tau_V(U) - 1.$

The collection

$$\mathcal{S}_V := \{\varphi_U : V \to V\}$$

of all such inverse branches forms obviously a conformal iterated function system in the sense considered in the previous sections of the present paper. In other words the elements of $\mathcal{S}_V$ are formed by all holomorphic inverse branches of the first return map $f_V : V \to V$.

In particular, $\tau_V(U)$ is the first return time of all points in $U = \varphi_U(V)$ to V. We define the function $N_V : E_V^{\mathbb{N}} \to \mathbb{N}_1$ by setting

$$N_V(\omega) := \tau_V(\omega_1).$$

Let

$$\pi_V : E_V^{\mathbb{N}} \to \widehat{\mathbb{C}}$$

be the canonical projection induced by the iterated function system $\mathcal{S}_V$. Let

$$J_V : \pi_V\big(E_V^{\mathbb{N}}\big)$$

be the limit set of the system $\mathcal{S}_V$. Clearly

$$J_V \subseteq J(f).$$

It is immediate from our definitions that

$$\tau_V(\pi(\omega)) = N_V(\omega)$$

for all $\omega \in E_V^{\mathbb{N}}$.

Now, having in addition a Hölder continuous potential $\psi : \widehat{\mathbb{C}} \to \mathbb{R}$ with pressure gap, we already know from the previous sections that $\mu_{\psi,V}$, the conditional measure of μ_ψ on V is f_V-invariant and ergodic.

Definition 23.5. We say that the rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ has the Exponential Shrinking Property (ESP) if there exist $\delta > 0$, $\gamma > 0$, and $C > 0$ such that if $z \in J(f)$ and $n \geq 0$, then

$$\operatorname{diam}(W) \leq Ce^{-\gamma n} \tag{23.4}$$

for each $W \in \mathcal{C}_{B(z,2\delta)}(n)$.

Remark 23.6. This property has been throughly explored in the papers including [39] and the references therein. These papers provide several different characterizations of Exponential Shrinking Property, most notably the one called Topological Collet-Eckmann; one of them being uniform hyperbolicity of periodic points in the Julia set. We do not recall any more of them here as we will only need (ESP).

Exactly as in the case of interval maps, we now provide two standard sufficient conditions for (ESP) to hold. It is implied by the Collet-Eckmann condition which requires that there exist $\lambda > 1$ and $C > 0$ such that for every integer $n \geq 0$ we have that

$$|(f^n)'(f(c))| \geq C\lambda^n.$$

If also suffices for (ESP) to hold to assume that a rational map is semi-hyperbolic, i.e., that no critical point c in the Julia belongs to its own omega limit set $\omega(c)$. This so for example, if T is a classical unimodal map of the form $I \ni x \mapsto \lambda x(1-x)$, with $0 < \lambda \leq 4$ and the critical point $1/2$ is not in its own omega limit set, i.e., $1/2 \notin \omega(1/2)$.

Last observation: all expanding rational functions have the Exponential Shrinking Property (ESP).

We shall prove the following.

Proposition 23.7. *Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a tame rational function satisfying* (ESP)*. Let $\psi:\widehat{\mathbb{C}}\to\mathbb{R}$ be a Hölder continuous potential with pressure gap. If V is a nice set for f, then*

(a)
$$\tilde{\psi}_V := \psi_V\circ\pi_V - \mathrm{P}(\psi)N_V : E_V^{\mathbb{N}}\to\mathbb{R}$$
is a Hölder continuous potential,

(b) $\mathrm{P}\big(\sigma,\tilde{\psi}_V\big)=0$,

(c)
$$\mu_{\psi,V}=\mu_{\tilde{\psi}_V}\circ\pi_V^{-1},$$
where $\mu_{\tilde{\psi}_V}$ is the equilibrium/Gibbs state for the potential $\tilde{\psi}_V$ and the shift map $\sigma:E_V^{\mathbb{N}}\to E_V^{\mathbb{N}}$.

(d) *In addition, $\tilde{\psi}_V$ is the amalgamated function of a summable Hölder continuous system of functions.*

Proof. Hölder continuity of $\tilde{\psi}_V$ follows directly from (ESP) i.e Definition 23.5, and the fact that the function N_V is constant on cylinders of length one. Now, it follows from [18] that there exists a unique $\exp(\mathrm{P}(\psi)-\psi)$-conformal measure on $J(f)$, i.e. a Borel probability measure m_ψ on $J(f)$ such that

$$m_\psi\big(f(A)\big)=e^{\mathrm{P}(\psi)}\int_A e^{-\psi}\,dm_\psi$$

for every Borel set $A\subseteq J(f)$ such that the map $f|_A$ is 1-to-1. In addition m_ψ is equivalent to μ_ψ with logarithmically bounded Hölder continuous Radon-Nikodym derivative. It immediately follows from this formula that for every $e\in E_V$ and every Borel set $A\subseteq V$, we have that

$$m_{\psi,V}\big(\varphi_e(A)\big)=\int_A \exp\big((\psi_V-\mathrm{P}(\psi)\tau_V)\circ\varphi_e\big)\,dm_{\psi,V}, \tag{23.5}$$

where $m_{\psi,V}$ is the conditional measure of m_ψ on V. Now we define a Hölder continuous system of functions $G:=\{g^{(e)}:V\to\mathbb{R}\}_{e\in E}$ by putting

$$g^{(e)}:=(\psi_V-\mathrm{P}(\psi)\tau_V)\circ\varphi_e,\quad e\in E_V.$$

Formula (23.5) thus means that the system G is summable, $\mathrm{P}(G)=0$, and $m_{\psi,V}$ is the unique G-conformal measure for the IFS $\mathcal{S}_V$. According to [31], $g:E_V^{\mathbb{N}}\to\mathbb{R}$, the amalgamated function of G is defined by the formula

$$\begin{aligned}
g(\omega)=g^{(\omega_1)}(\pi_V(\sigma(\omega)))&=\psi_V\circ\varphi_{\omega_1}(\pi_V(\sigma(\omega)))-\mathrm{P}(\psi)\tau_V\circ\varphi_{\omega_1}(\pi_V(\sigma(\omega)))\\
&=\psi_V\circ\pi_V(\omega)-\mathrm{P}(\psi)N_V(\omega)\\
&=\tilde{\psi}_V(\omega).
\end{aligned}$$

By Proposition 3.1.4 in [31] we thus have that

$$\mathrm{P}\big(\sigma, \tilde{\psi}_V\big) = \mathrm{P}(G) = 0.$$

Now, since $\pi_V \circ \sigma = f_V \circ \pi_V$, i.e. since the dynamical system $f_V : J_V \to J_V$ is a factor of the shift map $\sigma : E_V^{\mathbb{N}} \to E_V^{\mathbb{N}}$ via the map $\pi_V : E_V^{\mathbb{N}} \to J_V$, we see that $\mu_{\tilde{\psi}_V} \circ \pi_V^{-1}$ is a Borel f_V-invariant probability measure on J_V equivalent to $m_{\tilde{\psi}_V} \circ \pi_V^{-1} = m_g \circ \pi^{-1} = m_G = m_{\psi,V}$. Since $m_{\psi,V}$ is equivalent to $\mu_{\psi,V}$, we thus conclude that the measures $m_{\tilde{\psi}_V} \circ \pi_V^{-1}$ and $\mu_{\psi,V}$ are equivalent. Since both these measures are f_V-invariant and $\mu_{\psi,V}$ is ergodic, they must be equal. The proof is thus complete. $\square$

Since $\pi_V : E_V^{\mathbb{N}} \to J_V = V_\infty$, where, we recall the latter is the set of points returning infinitely often to V, is a measurable isomorphism sending the σ-invariant measure $\mu_{\tilde{\psi}_V}$ to the f_V-invariant probability measure $\mu_{\psi,V}$, by identifying the sets $E_V^{\mathbb{N}}$ and $V_\infty(= J_V)$, we can prove the following.

Lemma 23.8. *With the hypotheses of Proposition* 23.7*, the pentacle* $(J(f), f, V, \tilde{\psi}_V, \mu_{\tilde{\psi}_V})$ *is an* SRT *system and has exponential tail decay* (ETD)*, where we recall that* V_∞ *is identified with* $E_V^{\mathbb{N}}$*,* $\tilde{\psi}_V$ *is identified with* $\psi_V - \mathrm{P}(\psi)\tau_V$*, and* $\mu_{\tilde{\psi}_V}$ *is identified with* $\mu_{\psi,V}$*.*

Proof. By virtue of Proposition 23.7 and Observation 21.2 we only need to prove that the pentacle $(J(f), f, V, \tilde{\psi}_V, \mu_{\tilde{\psi}_V})$ has exponential tail decay (ETD). We can assume without loss of generality that $\psi : \widehat{\mathbb{C}} \to \mathbb{R}$ is normalized so that

$$\mathrm{P}(\psi) = 0 \quad \text{and} \quad m_\psi = \mu_\psi.$$

Now define

$$\mathcal{C}_V^0(n) := \big\{U \in \mathcal{C}_V(n) : \forall_{(0\le k\le n)}\, f^k(U) \cap V = \emptyset\big\}$$

Since the map $f : J(f) \to J(f)$ is topologically exact, there exists an integer $q \ge 1$ such that

$$f^q(V) \supseteq J(f).$$

Therefore for every $e \in \mathcal{C}_V(n)$ there exists (at least one) $\hat{e} \in \mathcal{C}_V^*(n+q)$ such that

$$f^q \circ \varphi_{\hat{e}} = \varphi_e.$$

By conformality of the measure μ_ψ, for every $e \in \mathcal{C}_V(n)$, we have

$$\mu_\psi\big(\varphi_{\hat{e}}(V)\big) \ge \exp(-q||\psi||_\infty)\mu_\psi(\varphi_e(V)).$$

So, since

$$\bigcup_{a\in\mathcal{C}_V^0(n+q)} \varphi_a(V) \subseteq \bigcup_{\substack{b\in\mathcal{C}_V(n+q)\\ f^q\circ\varphi_b\in\mathcal{C}_V^0(n)}} \varphi_b(V) \setminus \bigcup_{e\in\mathcal{C}_V^0(n)} \varphi_{\hat{e}}(V),$$

we therefore get

$$\begin{aligned}
\mu_\psi\left(\bigcup_{a\in\mathcal{C}_V^0(n+q)}\varphi_a(V)\right) &\le \mu_\psi\left(\bigcup_{\substack{b\in\mathcal{C}_V(n+q)\\ f^q\circ\varphi_b\in\mathcal{C}_V^0(n)}}\varphi_b(V)\setminus\bigcup_{e\in\mathcal{C}_V^0(n)}\varphi_{\hat e}(V)\right)\\
&= \mu_\psi\left(\bigcup_{\substack{b\in\mathcal{C}_V(n+q)\\ f^q\circ\varphi_b\in\mathcal{C}_V^0(n)}}\varphi_b(V)\right) - \mu\left(\bigcup_{e\in\mathcal{C}_V^0(n)}\varphi_{\hat e}(V)\right)\\
&= \mu_\psi\left(f^{-q}\left(\bigcup_{c\in\mathcal{C}_V^0(n)}\varphi_c(V)\right)\right) - \sum_{e\in\mathcal{C}_V^0(n)}\mu_\psi\big(\varphi_{\hat e}(V)\big)\\
&= \mu_\psi\left(\bigcup_{c\in\mathcal{C}_V^0(n)}\varphi_c(V)\right) - \sum_{e\in\mathcal{C}_V^0(n)}\mu_\psi\big(\varphi_{\hat e}(V)\big)\\
&\le \mu_\psi\left(\bigcup_{c\in\mathcal{C}_V^0(n)}\varphi_c(V)\right) - \exp(-q||\psi||_\infty)\sum_{e\in\mathcal{C}_V^0(n)}\mu_\psi\big(\varphi_e(V)\big)\\
&= \gamma\mu_\psi\left(\bigcup_{c\in\mathcal{C}_V^0(n)}\varphi_c(V)\right),
\end{aligned}$$

where $\gamma := 1-\exp(-q||\psi||_\infty)\in[0,1)$. An immediate induction then yields

$$\mu_\psi\left(\bigcup_{e\in\mathcal{C}_V^0(qn)}\varphi_e(V)\right)\le\gamma^n$$

for all $n\ge 0$. An immediate induction then yields

$$\mu_\psi\left(\bigcup_{e\in\mathcal{C}_V^0(n)}\varphi_e(V)\right)\le\gamma^{-1}\gamma^{n/q}$$

for all $n\ge 0$. But, as

$$E_V^{-1}([n,+\infty]) = E_V^{-1}(\{+\infty\})\cup\bigcup_{k=n}^\infty\bigcup_{e\in\mathcal{C}_V^0(k)}\varphi_e(V)$$

and since $\mu_\psi\big(E_V^{-1}(\{+\infty\})\big)=0$ by ergodicity of μ_ψ and of $\mu_\psi(V)>0$, we therefore get that

$$\mu_\psi\big(E_V^{-1}([n,+\infty])\big)\le\big(\gamma(1-\gamma^{1/q})\big)^{-1}\gamma^{n/q} \tag{23.6}$$

for all $n \geq 0$. This just means that the pentacle $(I, f, V, \tilde{\psi}_V, \mu_{\tilde{\psi}_V})$ has exponential tail decay (ETD), and the proof is complete. $\square$

Denote by $J_R(f)$ the set of all recurrent points of f in $J(f)$. Formally

$$J_R(f) := \{z \in J(f) : \varliminf_{n\to\infty} |f^n(z) - z| = 0\}.$$

Of course $J_R(f) \subseteq J_f$ and $\mu_\psi(J(f) \setminus J_R(f)) = 0$ because of Poincaré's Recurrence Theorem. The set $J_R(f)$ is significant for us since

$$J_R(f) \cap V \subseteq J_V.$$

Now we can now apply the conclusions of the work done. As a direct consequence of Theorem 15.10, Proposition 23.7, Lemma 23.8, Lemma 21.8, and Theorem 20.3, we get the following.

Theorem 23.9. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a tame rational function having the exponential shrinking property (ESP). Let $\psi : \widehat{\mathbb{C}} \to \mathbb{R}$ be a Hölder continuous potential with pressure gap. Let $z \in J_R(f) \setminus \overline{\mathrm{PC}(f)}$. Assume that the equilibrium state μ_ψ is* (WBT) *at z. Then*

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{\underline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} &= \lim_{\varepsilon\to 0} \frac{\overline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} \\ &= \begin{cases} 1 & \text{if } z \text{ is not any periodic point of } f, \\ 1 - \exp\big(S_p\psi(z) - p\mathrm{P}(f,\psi)\big) & \text{if } z \text{ is a periodic point of } f. \end{cases}\end{aligned}$$

Remark 23.10. Theorem 23.9holds in fact for a larger set than $J_R(f)$. Indeed, it holds for every point in $V \cap J_{\mathcal{S}_V}$, where V is an arbitrary nice set.

As a fairly immediate consequence of Theorem 23.9 and Theorem 14.7, we get the following.

Corollary 23.11. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a tame rational function having the exponential shrinking property (ESP) whose Julia set $J(f)$ is geometrically irreducible. If $\psi : \widehat{\mathbb{C}} \to \mathbb{R}$ is a Hölder continuous potential with pressure gap, then*

$$\lim_{\varepsilon\to 0} \frac{\underline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} = \lim_{\varepsilon\to 0} \frac{\overline{R}_{\mu_\psi}(B(z,\varepsilon))}{\mu_\psi(B(z,\varepsilon))} = 1$$

for μ_ψ–a.e. $z \in J(f)$.

Indeed in order to prove this corollary it suffices to note that if the Julia set $J(f)$ is geometrically irreducible, then neither is the limit set of the iterated function system constructed in the arguments leading to Theorem 23.9.

Remark 23.12. We would like to note that if the rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is expanding, then it is tame, satisfies (ESP), and each Hölder continuous potential has pressure gap. In particular the two above theorems hold for it.

Now turn to the asymptotics of Hausdorff dimension. We recall the following.

Definition 23.13. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree $d \geq 2$. We say that the map f is sub-expanding if one of the following two equivalent conditions holds:

(a)

$$\overline{\bigcup_{n=0}^{\infty} f^n\big(\mathrm{Crit}(f) \setminus J(f)\big)} \cap J(f) = \emptyset \quad \text{and} \quad \mathrm{Crit}(f) \cap \overline{\bigcup_{n=1}^{\infty} f^n\big(\mathrm{Crit}(f) \cap J(f)\big)} = \emptyset,$$

(b)

$$\mathrm{Crit}(f) \cap \overline{\bigcup_{n=1}^{\infty} f^n\big(\mathrm{Crit}(f) \cap J(f)\big)} = \emptyset \ \text{ and } f \text{ has no rationally indifferent periodic points.}$$

Let

$$h := \mathrm{HD}(J(f)).$$

It was proved in [52] and [53] that there exists a unique h–conformal measure m_h on $J(f)$ for f and a unique f-invariant (ergodic) measure μ_h on $J(f)$ equivalent to m_h. In addition μ_h is supported on the intersection of the transitive and radial points of f. It has been proved in [53] that any subexpanding rational function enjoys ESP. It therefore follows from [39] that there are arbitrarily small open connected sets V_c, $c \in J(f) \cap \mathrm{Crit}(f)$, and V_ξ, respectively containing points c and ξ such that the collection of all holomorphic inverse branches f_*^{-n} of f^n, $n \geq 0$, defined on V_z, $z \in (J(f) \cap \mathrm{Crit}(f)) \cup \{\xi\}$, and such that for some $z' \in (J(f) \cap \mathrm{Crit}(f)) \cup \{\xi\}$,

$$f_*^{-n}(V_z) \subseteq V_{z'}$$

and

$$\bigcup_{k=1}^{n-1} f^k\big(f_*^{-n}(V_z)\big) \ \cap \ \bigcup\big\{V_w : w \in (J(f) \cap \mathrm{Crit}(f)) \cup \{\xi\}\big\} = \emptyset.$$

forms a finitely primitive conformal GDS, call it $\mathcal{S}_f$. Another characterization of $\mathcal{S}_f$ is that its elements are composed of analytic inverse branches of the first return map of f from

$$V := \bigcup\big\{V_w : w \in (J(f) \cap \mathrm{Crit}(f)) \cup \{\xi\}\big\}$$

V. It has been proved in [46] and [47] that the system $\mathcal{S}_f$ is strongly regular. It follows from Lemma 6.2 in [39] that $\mathrm{HD}(K(V)) < h$. So, as by Theorem 17.1, $\lim_{r\to 0} \mathrm{HD}\big(K(B(\xi, r))\big) = h$, we conclude that

$$\mathrm{HD}(K(V)) < \mathrm{HD}\big(K(B(\xi, r))\big)$$

for all $r > 0$ small enough. Therefore, since $h = b_{\mathcal{S}_f}$ and since $\mu_{h,V} = \mu_{b_{\mathcal{S}_f}}$, applying Theorem 17.1, Corollary 17.3, and Corollary 19.2, we get the following two theorems.

Theorem 23.14. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a subexpanding rational function of degree $d \geq 2$. Fix $\xi \in J(f) \setminus \overline{\mathrm{PC}(f)}$. Assume that the measure μ_h is* (WBT) *at ξ and the parameter h is* *powering at* ξ *with respect to the conformal* GDS $\mathcal{S}_f$*. Then the following limit exists, is finite and positive:*

$$\lim_{r\to 0}\frac{\mathrm{HD}(J(f))-\mathrm{HD}(K_\xi(r))}{\mu_h(B(\xi,r))}.$$

Theorem 23.15. *If* $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ *be a subexpanding rational function of degree* $d\geq 2$ *whose Julia set* $J(f)$ *is geometrically irreducible, then for* μ_h*–a.e. point* $\xi\in J(f)\setminus\overline{\mathrm{PC}(f)}$ *the following limit exists, is finite and positive:*

$$\lim_{r\to 0}\frac{\mathrm{HD}(J(f))-\mathrm{HD}(K_\xi(r))}{\mu_h(B(\xi,r))}.$$

Remark 23.16. We would like to note that if the rational function $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is expanding, then it is automatically subexpanding and the two above theorems apply.

24. Escape Rates for Meromorphic Functions on the Complex Plane

We deal in this final section with transcendental meromorphic functions. We also apply here the results on escape rates for conformal GDMS and the techniques of first return maps. Let $f:\mathbb{C}\to\widehat{\mathbb{C}}$ be a meromorphic function. Let $\mathrm{Sing}(f^{-1})$ be the set of all singular points of f^{-1}, i. e. the set of all points $w\in\widehat{\mathbb{C}}$ such that if W is any open connected neighborhood of w, then there exists a connected component U of $f^{-1}(W)$ such that the map $f:U\to W$ is not bijective. Of course if f is a rational function, then $\mathrm{Sing}(f^{-1})=f(\mathrm{Crit}(f))$. As in the case of rational functions, we define

$$\mathrm{PS}(f):=\bigcup_{n=0}^{\infty}f^n(\mathrm{Sing}(f^{-1})).$$

The function f is called *topologically hyperbolic* if

$$\mathrm{dist}_{\mathrm{Euclid}}(J_f,\mathrm{PS}(f))>0,$$

and it is called *expanding* if there exist $c>0$ and $\lambda>1$ such that

$$|(f^n)'(z)|\geq c\lambda^n$$

for all integers $n\geq 1$ and all points $z\in J_f\setminus f^{-n}(\infty)$. Note that every topologically hyperbolic meromorphic function is tame. A meromorphic function that is both topologically hyperbolic and expanding is called *hyperbolic*. The meromorphic function $f:\mathbb{C}\to\widehat{\mathbb{C}}$ is called dynamically *semi-regular* if it is of finite order, commonly denoted by ρ_f, and satisfies the following rapid growth condition for its derivative.

$$|f'(z)|\geq\kappa^{-1}(1+|z|)^{\alpha_1}(1+|f(z)|)^{\alpha_2}\,,\quad z\in J_f, \tag{24.1}$$

with some constant $\kappa>0$ and α_1,α_2 such that $\alpha_2>\max\{-\alpha_1,0\}$. Set $\alpha:=\alpha_1+\alpha_2$.

Remark 24.1. A particularly simple example of such maps are meromorphic functions $f_\lambda(z) = \lambda e^z$ where $\lambda \in (0, 1/e)$ since these maps have an attracting periodic point. A good reference is [33].

Let $h : J_f \to \mathbb{R}$ be a weakly Hölder continuous function in the sense of [34]. The definition, introduced in [34] is somewhat technical and we will not provided it in the current paper. What is important is that each bounded, uniformly locally Hölder function $h : J_f \to \mathbb{R}$ is weakly Hölder. Fix $\tau > \alpha_2$ as required in [34]. For $t \in \mathbb{R}$, let

$$\psi_{t,h} = -t \log |f'|_\tau + h \tag{24.2}$$

where $|f'(z)|_\tau$ is the norm, or, equivalently, the scaling factor, of the derivative of f evaluated at a point $z \in J_f$ with respect to the Riemannian metric

$$|d\tau(z)| = (1 + |z|)^{-\tau} |dz|.$$

Following [34] functions of the form (24.2)(frequently referred to as potentials) are called *loosely tame*. Let $\mathcal{L}_{t,h} : C_b(J_f) \to C_b(J_f)$ be the corresponding *Perron-Frobenius operator* given by the formula

$$\mathcal{L}_{t,h} g(z) := \sum_{w \in f^{-1}(z)} g(w) e^{\psi_{t,h}(w)}.$$

It was shown in [34] that, for every $z \in J_f$ and for the function $1\!\!1 : z \mapsto 1$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_{t,h} 1\!\!1(z)$$

exists and takes on the same common value, which we denote by $\mathrm{P}(t)$ and call *the topological pressure* of the potential ψ_t. The following theorem was proved in [34].

Theorem 24.2. *If $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is a dynamically semi-regular meromorphic function and $h : J_f \to \mathbb{R}$ is a weakly Hölder continuous potential, then for every $t > \rho_f/\alpha$ there exist uniquely determined Borel probability measures $m_{t,h}$ and $\mu_{t,h}$ on J_f with the following properties.*

(a) $\mathcal{L}^*_{t,h} m_{t,h} = m_{t,h}$.

(b) $\mathrm{P}\big(\psi_{t,h}\big) = \sup\big\{\mathrm{h}_\mu(f) + \int \psi_{t,h}\, d\mu : \mu \circ f^{-1} = \mu \ \text{ and } \ \int \psi_{t,h}\, d\mu > -\infty\big\}$.

(c) $\mu_{t,h} \circ f^{-1} = \mu_{t,h}$, $\int \psi_{t,h}\, d\mu_{t,h} > -\infty$, *and* $\mathrm{h}_{\mu_{t,h}}(f) + \int \psi_{t,h}\, d\mu_{t,h} = \mathrm{P}\big(\psi_{t,h}\big)$.

(d) *The measures $\mu_{t,h}$ and $m_{t,h}$ are equivalent and the Radon–Nikodym derivative $\frac{d\mu_{t,h}}{dm_{t,h}}$ has a nowhere-vanishing Hölder continuous version which is bounded above.*

The exact analogue of Theorem 23.4 holds, with the same references, for all hyperbolic meromorphic functions; we will refer to this theorem as Theorem 23.4(M). Also, for the system $\mathcal{S}_V$ and the projection $\pi_V : E_V^{\mathbb{N}} \to J_V$ have the same meaning. As in the case of rational functions denote by $J_R(f)$ the set of all recurrent points of f in $J(f)$. Formally

$$J_R(f) := \{z \in J(f) : \varliminf_{n\to\infty} |f^n(z) - z| = 0\}.$$

Of course $J_R(f) \subseteq J_f$ and $\mu_\psi(J(f) \setminus J_R(f)) = 0$ because of Poincaré's Recurrence Theorem. The set $J_R(f)$ is significant for us since

$$J_R(f) \cap V \subseteq J_V.$$

The Exponential Shrinking Property (ESP) holds since now the function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is expanding. The proof of Proposition 23.7 goes through unchanged except that instead of using [18] we now invoke Theorem 24.2 (a). We also will refer this proposition (23.7) as Proposition 23.7 (M). Lemma 23.8 also carries on to the meromorphic case (we refer to it as Lemma 23.8 (M); the proof of items (a)–(e) Definition 21.1 required by this lemma to hold, follows as in the case of rational functions, from proposition 23.7 (M), while the proof of item (f) of this definition is now a direct consequence of Lemma 4.1 in [45]. Now, in exactly the same way as in the case of rational functions, as a direct consequence of Theorem 15.10, Theorem 15.11, Proposition 23.7 (M), Lemma 23.8 (M), Lemma 21.8, and Theorem 20.3, we get the following two theorems.

Theorem 24.3. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a dynamically semi-regular meromorphic function. Let $t > \rho_f/\alpha$ and let $h : J(f) \to \mathbb{R}$ be a weakly Hölder continuous function. Let $z \in J_R(f)$. Assume that the corresponding equilibrium state $\mu_{t,h}$ is* (WBT) *at z. Then*

$$\lim_{\varepsilon\to 0} \frac{\underline{R}_{\mu_{t,h}}(B(z,\varepsilon))}{\mu_{t,h}(B(z,\varepsilon))} = \lim_{\varepsilon\to 0} \frac{\overline{R}_{\mu_{t,h}}(B(z,\varepsilon))}{\mu_{t,h}(B(z,\varepsilon))} =$$

$$= \begin{cases} 1 & \text{if } z \text{ is not any periodic point of } f, \\ 1 - \exp\big(S_p\psi_{t,h}(z) - p\mathrm{P}(\psi_{t,h})\big) & \text{if } z \text{ is a periodic point of } f. \end{cases}$$

Theorem 24.4. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a dynamically semi-regular meromorphic function whose Julia set is geometrically irreducible. Let $t > \rho_f/a$ and let $h : J(f) \to \mathbb{R}$ be a weakly Hölder continuous function. Then*

$$\lim_{\varepsilon\to 0} \frac{\underline{R}_{\mu_{t,h}}(B(z,\varepsilon))}{\mu_{t,h}(B(z,\varepsilon))} = \lim_{\varepsilon\to 0} \frac{\overline{R}_{\mu_{t,h}}(B(z,\varepsilon))}{\mu_{t,h}(B(z,\varepsilon))} = 1$$

for $\mu_{t,h}$–a.e. $z \in J(f)$.

Remark 24.5. Theorem 24.3 holds in fact for a larger set than $J_R(f)$. Indeed, it holds for every point in $V \cap J_{\mathcal{S}_V}$, where V is an arbitrary nice set.

Turning to the asymptotics of Hausdorff dimension, let $J_r(f)$ be the set of radial (or conical) points in $J(f)$, i. e. the set of all those points in $J(f)$ that do not escape to infinity under the action of the map $f : \mathbb{C} \to \widehat{\mathbb{C}}$. Assume now more, namely that $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is dynamically regular in the sense of [33] and [34]. What at the moment is important for us is that $\mathrm{P}(h_r) = 0$, where

$$h_r := \mathrm{HD}(J_r(f)).$$

We already know that there exists a nice set V containing ξ and the elements of the corresponding conformal IFS $\mathcal{S}_f$ are composed of analytic inverse branches of the the first return map from V to V. Since $\xi \in J_R(f)$, we have that $\xi \in J_V$. Corollary 6.4 in [44] tells us that $\mathrm{HD}(K(V)) < h_r$. So, since by Theorem 17.1, $\lim_{r\to 0}\mathrm{HD}(K(B(\xi,r))) = h_r$, we conclude that

$$\mathrm{HD}(K(V)) < \mathrm{HD}(K(B(\xi,r)))$$

for all $r > 0$ small enough. Therefore, since $h_r = b_{\mathcal{S}_V}$ and since $\mu_{h,V} = \mu_{b_{\mathcal{S}_V}}$, applying Theorem 17.1, Corollary 17.3, and Corollary 19.2, we get the following two theorems.

Theorem 24.6. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a dynamically regular meromorphic function. Fix $\xi \in J_R(f)$. Assume that the measure μ_{h_r} (i.e. $\mu_{h_r,0}$ with the weakly Hölder function h identically equal to 0) is* (WBT) *at ξ and the parameter h_r is powering at ξ with respect to the conformal* IFS *$\mathcal{S}_f$. Then the following limit exists and is finite and positive:*

$$\lim_{r\to 0}\frac{\mathrm{HD}(J_r(f)) - \mathrm{HD}(K_z(r))}{\mu_{h_r}(B(z,r))}.$$

Theorem 24.7. *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a dynamically regular meromorphic function whose Julia set is geometrically irreducible. Then the following limit exists and is finite and positive for μ_{h_r}–a.e. $z \in J(f)$:*

$$\lim_{r\to 0}\frac{\mathrm{HD}(J_r(f)) - \mathrm{HD}(K_z(r))}{\mu_{h_r}(B(z,r))}.$$

Note that the conclusion of Remark 24.5 holds in the case of Theorem 24.6 too.

Appendix: The Keller–Liverani Perturbation Theorem

In this appendix we formulate the Keller–Liverani Perturbation Theorem from [25] in its full generality and, formally speaking, in a slightly more general form than in [25]. We also formulate all its consequences derived in [25] that we need in our manuscript, particularly in Section 5 to prove Proposition 5.2 which is crucial for us. We follow pretty closely the notation, formulations, and enumeration of [25] for the reader to easily compare our text with the original article [25]. We first describe the setting.

Let $(B, \|\cdot\|)$ be a Banach space. The vector space B is also equipped with a second norm $|\cdot| \le \|\cdot\|$ with respect to which B need not be complete. For any bounded linear operator $Q : B \to B$, B understood here with the norm $\|\cdot\|$, let

(KL1) $$|||Q||| := \sup\big\{|Qf| : f \in B, \|f\| \le 1\big\}.$$

Let Λ be a directed set having a largest element which we denote by 0. In [25] $\Lambda = [0, +\infty)$ with the reverse order. For our applications in Section 5 $\Lambda = \mathbb{N} \cup \{+\infty\}$, although actually it suffices to consider $\{n, n+1, \ldots\} \cup \{+\infty\}$ where $n \ge 0$ is large enough, with the natural order. Assume that a family $(\mathrm{P}_\varepsilon)_{\varepsilon\in\Lambda}$ of bounded linear opertors on $(B, \|\cdot\|)$ is given which enjoys the following properties.

(KL2) There are constants $C_1, M > 0$ such that for all $\varepsilon \in \Lambda$

$$|P_\varepsilon^n| \le C_1 M^n$$

for all $n \ge 0$,

(KL3) There are constants $C_2, C_3 > 0$ and $\alpha \in (0, \min\{1, M\})$ such that for all $\varepsilon > 0$,

$$\|P_\varepsilon^n f\| \le C_2 \alpha^n \|f\| + C_3 M^n |f|$$

for all $n \ge 0$ and all $f \in B$,

(KL4) If $z \in \sigma(P_\varepsilon) \cap \overline{B}^c(0, \alpha)$, then z is not in the residual spectrum of P_ε,

(KL5) There exists a net $\tau : \Lambda \to [0, +\infty)$ such that $\tau(0) = 0$, $\tau(\Lambda \setminus \{0\}) \subseteq (0, +\infty)$

$$\lim_{\varepsilon \in \Lambda} \tau(\varepsilon) = 0$$

and

$$|||P_\varepsilon - P_0|| \le \tau(\varepsilon)$$

for all $\varepsilon \in \Lambda$.

These are all hypotheses for the Keller–Liverani Perturbation Theorem. In order to formulate this theorem we need one more piece of notation.

For all $\delta > 0$ and all $r > \alpha$ let

$$V_{\delta,r} := \big\{ z \in \mathbb{C} : |z| \le r \ \text{ or } \ \operatorname{dist}(z, \sigma(P_0) \le \delta \big\}.$$

The actual Keller–Liverani Perturbation Theorem from [25] is about upper bounds on the norms of resolvents $(z - P_\varepsilon)^{-1}$ and continuity at 0 of the latter.

Theorem 24.8 (Keller–Liverani Perturbation Theorem)**.** *Suppose that $(\mathrm{P}_\varepsilon)_{\varepsilon \in \Lambda}$ is a family of bounded linear operators on $(B, \|\cdot\|)$ satisfying conditions (KL2)–(KL5). Fix $\delta > 0$ and $r \in (\alpha, M)$ and let*

$$\eta := \frac{\log(r/\alpha)}{\log(M/\alpha)} > 0.$$

Then there are constants $\varepsilon_0 = \varepsilon_0(\delta, r) > 0$, $a = a(r) > 0$, $b = b(\delta, r) > 0$, $c = c(\delta, r) > 0$, and $d = d(\delta, r) > 0$ such that for every $\varepsilon \ge \varepsilon_0$ and all $z \in \mathbb{C} \setminus V_{\delta,r}$, we have that

$$\|(z - P_\varepsilon)^{-1} f\| \le a\|f\| + b|f| \tag{KL8}$$

and

$$|||(z - P_\varepsilon)^{-1} - (z - P_0)^{-1}||| \le \tau^\eta(\varepsilon)\big(c\|(z - P_0)^{-1}\| + d\|(z - P_0)^{-1}\|^2\big). \tag{KL9}$$

Remark 24.9. This remark is essential for us and corresponds to Remark 3 (and partly Remark 1) in [25]. As Keller and Liverani write in Remark 1 “In nearly all cases the two norms involved have the additional property that

(KL7) the closed unit ball of $(B, \|\cdot\|)$ is $|\cdot|$–compact.”

and this yields condition (KL4) to hold. However in the case of the present paper, with $B = \mathcal{B}_\theta$, $\|\cdot\| = \|\cdot\|_\theta$ and $|\cdot| = \|\cdot\|_*$, (KL7) does fail. The remedy comes from Remark 3 in [25] which we explain now.

Assume there exists a sequence of linear operators $\pi_k : B \to B$, $k \ge 1$, such that

$$\sup_k \{\|\pi_k\|\} < +\infty, \tag{24.3}$$

$$\sup\left\{|f-\pi_k f| : f\in B, \|f\|\le 1\right\} \preceq (\alpha/M)^k \tag{24.4}$$

and

$$P_\varepsilon\pi_k \ \text{ is a compact operator for all } \ k\ge 1. \tag{24.5}$$

Then all the operators $P_\varepsilon : B\to B$ are quasicompact with essential spectral radius $\le\alpha$ and in particular (KL4) holds.

We now list the selected corollaries from Theorem 24.8 derived in [25], the ones needed to have the full proof of Proposition 5.2. The first one is a slightly simplified version of Remark 4 from [25].

Corollary 24.10. *If λ is a simple eigenvalue of P_0 with $|\lambda|>\alpha$ (so isolated), then for every $\varepsilon\in\Lambda$ sufficiently close to 0, there exists a unique simple eigenvalue λ_ε of P_ε such that*

$$\lim_{\varepsilon\to 0}\lambda_\varepsilon=\lambda. \tag{24.6}$$

Let λ be as in this corollary. Take $\eta>0$ so small that

$$\overline{B}(\lambda,\eta)\cap\sigma(P_0)=\{\lambda\}. \tag{24.7}$$

Define for every $\varepsilon\in\Lambda$ sufficiently close to 0:

$$Q_\varepsilon := \frac{1}{2\pi i}\int_{\partial B(\lambda,\eta)}(z-P_\varepsilon)^{-1}\,dz. \tag{24.8}$$

Note that Q_ε does not depend on η as long as (24.7) is satisfied.

As an immediate consequence of the definition of Q_ε and of item 1) of Corollary 1 from [25], we get the following.

Corollary 24.11. *If λ is a simple eigenvalue of P_0 with $|\lambda|>\alpha$ (so isolated), then*

(1) *For every $\varepsilon\in\Lambda$ sufficiently close to 0 the operator $Q_\varepsilon : B\to B$ is a projector (meaning that $Q_\varepsilon^2=Q_\varepsilon$) onto the one–dimensional eigenspace of the eigenvalue λ_ε of P_ε.*

(2)
$$\lim_{\varepsilon\to 0}|||Q_\varepsilon-Q_0|||=0.$$

Now, given $r>\alpha$ define:

$$\Delta_\varepsilon := \frac{1}{2\pi i}\int_{\partial B(0,r)}(z-P_\varepsilon)^{-1}\,dz. \tag{24.9}$$

Before we deal with the next corollary we record the following, technical but crucial, consequence of formula (KL8) of Theorem 24.8.

$$S_{\delta,r} := \sup\left\{\|(z-P_\varepsilon)^{-1}\| : 0\le\varepsilon\le\varepsilon_0(\delta,r), z\in\mathbb{C}\setminus V_{\delta,r}\right\} < +\infty \tag{KL10}$$

for all $\delta>0$ and all $r\in(\alpha,M)$. We shall prove the following.

Corollary 24.12. *Let λ be a simple eigenvalue of P_0 with $|\lambda| > \alpha$ (so isolated). If $\gamma \in (\alpha, \min\{M, |\lambda|\})$ and*

$$\sigma(P_0) \setminus \{\lambda\} \subseteq B(0, \gamma) \tag{24.10}$$

then for every $\varepsilon \in \Lambda$ close enough to 0, we have that

(1)
$$P_\varepsilon = \lambda_\varepsilon Q_e + \Delta_e,$$

(2)
$$Q_\varepsilon \Delta_\varepsilon = \Delta_\varepsilon Q_\varepsilon = 0,$$

(3) *There exists a constant $C \in (0, +\infty)$ such that*
$$\|Q_\varepsilon\| \leq C,$$
and for every $k \geq 0$:

(4)
$$\|\Delta_\varepsilon^k\| \leq C\gamma^k$$

Proof. Items (1) and (2) are immediate consequences of (24.8) and (24.9) and elementary basic properties of Riesz Functional Calculus.

For the convenience of the reader we shall now provide the standard proof of item (4). Since $\gamma \in (\alpha, \min\{M, |\lambda|\})$, it follows from (24.10) there exists $\hat{\gamma} \in (\alpha, \min\{M, |\lambda|, \gamma\})$ such that $\sigma(P_0) \setminus \{\lambda\} \subseteq B(0, \hat{\gamma})$. Therefore there exists $\delta > 0$ so small that $\partial B(0, \hat{\gamma}) \cap B(\sigma(P_0), 2\delta) = \emptyset$. Hence, formula (KL10) applies to give

$$S_{\delta, \hat{\gamma}} < +\infty. \tag{24.11}$$

It follows from (24.9) and the already mentioned basic properties of Riesz Functional Calculus that

$$\Delta_\varepsilon^k := \frac{1}{2\pi i} \int_{\partial B(0, \hat{\gamma})} z^k (z - P_\varepsilon)^{-1} \, dz$$

for every integer $k \geq 0$. Therefore, invoking (24.11), we estimate as follows:

$$\|\Delta_\varepsilon^k\| \leq \frac{1}{2\pi} \int_{\partial B(0, \hat{\gamma})} |z|^k \|(z - P_\varepsilon)^{-1}\| \, |dz| = \frac{\gamma^k}{2\pi} \int_{\partial B(0, \hat{\gamma})} \|(z - P_\varepsilon)^{-1}\| \, |dz| \leq \hat{\gamma} S_{\delta, \hat{\gamma}} \gamma^k,$$

and formula (4) is proved.

Now, we shall prove item (3). It follows from (24.10) that $\overline{B}(\lambda, |\lambda| - \gamma) \cap \sigma(P_0) = \{\lambda\}$. Hence, invoking also (24.8) and (KL10), we get

$$\|Q_\varepsilon\| \leq \frac{1}{2\pi} \int_{\partial B(\lambda, (|\lambda| - \gamma)/2)} \|(z - P_\varepsilon)^{-1}\| \, |dz| \leq (1 - \gamma) S_{(|\lambda| - \gamma)/2, \gamma} < +\infty.$$

The proof of item (3) and, simultaneously, of entire Corollary 24.12 is complete. □

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