THERMODYNAMIC FORMALISM AND INTEGRAL MEANS SPECTRUM OF LOGARITHMIC TRACTS FOR TRANSCENDENTAL ENTIRE FUNCTIONS

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Abstract. We provide the full theory of thermodynamic formalism for a very general collection of entire functions in class $\mathcal{B}$. This class overlaps with the collection of all entire functions for which thermodynamic formalism has been so far established and contains many new functions.

The key point is that we introduce an integral means spectrum for logarithmic tracts which takes care of the fractal behavior of the boundary of the tract near infinity. It turns out that this spectrum behaves well as soon as the tracts have some sufficiently nice geometry which, for example, is the case for quasicircle, John or Hölder tracts. In this case we get a good control of the corresponding transfer operators, leading to full thermodynamic formalism along with its applications such as exponential decay of correlations, central limit theorem and a Bowen's formula for the Hausdorff dimension of radial Julia sets.

Our approach applies in particular to every hyperbolic function from any Eremenko-Lyubich analytic family of Speiser class $\mathcal{S}$ provided this family contains at least one function with Hölder tracts. The latter is, for example, the case if the family contains a Poincaré linearizer.

1. Introduction

The dynamics of a holomorphic function heavily depends on the behavior of the singular set. The singular set $S(f)$ of an entire function $f : \mathbb{C} \to \mathbb{C}$ is the closure of the set of critical values and finite asymptotic values of f. Eremenko-Lyubich [15] introduced and studied class $\mathcal{B}$ consisting of all entire functions with bounded singular sets. It has as a subclass Speiser class $\mathcal{S}$ consisting of entire functions with finite singular sets. In this paper we develop the full theory of thermodynamic formalism for a large collection of entire functions in Eremenko-Lyubich class $\mathcal{B}$.

When developing the thermodynamic formalism for transcendental functions, one encounters immediately two major difficulties: one has to deal with the essential singularity at infinity and to check whether the transfer operator, which is given by an infinite series, is well defined, i.e. converges, and has sufficiently good properties.

The first work on thermodynamic formalism for transcendental functions is due to Barański [1] who considered the tangent family. Other particular, mainly periodic, functions have been treated in the sequel, see for example [19], [42] and [43]. The first and, up

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to now, the only unified approach appeared in [23] and in [24]. Theses papers deal with a large class of functions that satisfy a condition on the derivative called balanced growth condition. The key point there was to employ Nevanlinna Theory and to make a judicious choice of Riemannian metric. Here we keep this choice of metric but then we proceed totally differently avoiding any use of Nevanlinna Theory. By introducing integral means spectrum for logarithmic tracts we built the theory of thermodynamic formalism for many other entire functions from class $\mathcal{B}$.

The main object of this paper is to show that the transfer operator behaves well depending on the geometry of the logarithmic tracts over infinity. Consider $f \in \mathcal{B}$ and suppose that the bounded set $S(f)$ is contained in the unit disk. Then, the components Ω_j of $f^{-1}(\{|z| > 1\})$ are the tracts, in fact logarithmic tracts of f over infinity. We assume that there are only finitely many of them: see the definition of class $\mathcal{D}$ in the next section.

Let us consider here in this introduction the case where $f^{-1}(\{|z| > 1\})$ consists only in one tract Ω. Then, $f_{|\Omega}$ has the particular form

$$f = e^{\tau},$$

with τ a conformal map from Ω onto the half plane $\mathcal{H} = \{\Re z > 0\}$ such that

$$\varphi := \tau^{-1} : \mathcal{H} \to \Omega \tag{1.1}$$

is a proper map [15]. Although $\partial\Omega$ is an analytic curve, near infinity it often resembles more and more a fractal curve. Typically, going to infinity on $\partial\Omega$ is like considering Green lines that are closer and closer to the boundary of possibly fractal domains. To make this precise, consider the rectangles

$$Q_T := \big\{\xi \in \mathbb{C} : 0 < \Re\xi < 4T \ \text{ and } \ -4T < \Im\xi < 4T\big\} \tag{1.2}$$

and then the domains

$$\Omega_T := \varphi(Q_T)\,, \quad T \geq 1\,.$$

The domains Ω_T form natural exhaustions of Ω and the fractality near infinity of $\partial\Omega$ can be observed by considering Ω_T rescaled by the factors $1/|\varphi(T)|$ as $T \to \infty$.

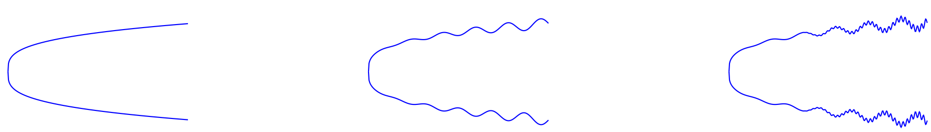

FIGURE 1. Example of $\frac{\Omega_T}{|\varphi(T)|}$ for $T = 1$, $T = 5$ and $T = 20$.

The corresponding rescaled map is

$$\varphi_T := \frac{1}{|\varphi(T)|}\,\varphi \circ T$$

and one can consider the integral means

$$\beta_{\varphi_T}(r,t) := \frac{\log \int_{1\le|y|\le 2} |\varphi_T'(r+iy)|^t dy}{\log 1/r} .$$

Starting from this formula we naturally assign to the tract Ω an integral means spectrum $t \mapsto \beta_\infty(t)$ which measures the fractal behavior of the tract at infinity. As in the classical setting, the important function will be the convex one:

$$b_\infty(t) := \beta_\infty(t) - t + 1 \ \ , \ t \ge 0 .$$

This function has always a smallest zero $\Theta_f > 0$ and, in the good cases, b_∞ has only a unique zero. In this latter case, we will say that the function f has *negative spectrum*.

We provide in addition a very general and easily verifiable condition that implies negative spectrum namely the *Hölder tract* property. It essentially means that the domains Ω_T are uniformly Hölder, see Definition 5.3 for the precise definition. For example, if the tract itself Ω is a quasidisk then it is a Hölder tract.

Proposition 1.1. *Let $f \in \mathcal{B}$ be an entire function having finitely many tracts. If the tracts are Hölder then f has negative spectrum.*

Disjoint type is a particular form of hyperbolicity. We work under this assumption but then, using standard bounded distortion arguments, for functions in class $\mathcal{S}$ our results carry over to a much more general class of hyperbolic functions (see Section 10.1). Class $\mathcal{D}$ essentially consists in disjoint type functions of class $\mathcal{B}$ having finitely many tracts (see Definition 2.1) with some additional properties.

Theorem 1.2. *Let $f \in \mathcal{D}$ be a function having negative spectrum and let $\Theta_f \in]0,2]$ be the smallest zero of b_∞. Then, the following holds:*

- *For every $t > \Theta_f$, the whole thermodynamic formalism, along with its all usual consequences holds: the Perron-Frobenius-Ruelle Theorem, the Spectral Gap property along with its applications: Exponential Mixing, Exponential Decay of Correlations and Central Limit Theorem (see Section 8).*
- *For every $t < \Theta_f$, the series defining the transfer operator $\mathcal{L}_t$ (see (4.3)) is divergent.*

Therefore, thermodynamic formalism is crystal clear for functions in class $\mathcal{D}$ with negative spectrum. The proof is based on Theorem 4.1 which is valid for all functions in class $\mathcal{B}$ without any further assumptions.

This also leads to geometric applications provided that the topological pressure, as defined in Section 9, has a zero $h > \Theta_f$. The following result completes the picture on various Bowen's Formulas (see [23] but especially [4] which contains a very general version of it).

Theorem 1.3 (Bowen's Formula)**.** *Let $f \in \mathcal{D}$ have negative spectrum and be such that the topological pressure* $\mathrm{P}(t)$ *has a zero $h > \Theta_f$. Then, the hyperbolic dimension* $\mathrm{HypDim}(f)$ *of f is equal to the unique zero $h > \Theta_f$ of the topological pressure.*

In conclusion, we get a complete, natural and quite elementary approach for the thermodynamic formalism for entire functions having negative spectrum. It covers many functions that satisfy the balanced growth condition, i.e. many functions of [23, 24]. They have negative spectrum and, even more, they are *elementary* in the sense that the integral means spectrum is as simple as possible: namely $\beta_\infty \equiv 0$ and $\Theta_f = 1$.

There is a very general result of approximating a model function by entire functions due to Bishop [9, 10]. His work is motivated by former results of Rempe-Gillen [38]. We show in Proposition 6.3 that the Hölder tract property is preserved when passing from the model to the approximating entire function. In fact, as Lemma 6.1 demonstrates, the Hölder tract property is a quasiconformal invariant. This has a second important application: for entire functions of class $\mathcal{S}$ the Hölder tract property is in fact a property of an analytic family of functions and not only of a single function. More precisely, if $g \in \mathcal{S}$ then Eremenko-Lyubich [15] naturally associated to g an analytic family of entire functions $\mathcal{M}_g \subset \mathcal{S}$. Proposition 10.1 states that every function of $\mathcal{M}_g$ has Hölder tracts if a function, for example g, has. A concrete application of all of this is the following.

Theorem 1.4. *Let $g \in \mathcal{S}$ be any function having finitely many tracts over infinity and assume that they are Hölder. Then every function $f \in \mathcal{M}_g$ has negative spectrum and the thermodynamic formalism holds for every hyperbolic map from $\mathcal{M}_g$.*

We also study a particular family of entire functions called Poincaré functions studied previously in [12, 28, 14] among others. If $p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a polynomial set and if $z_0 \in \mathcal{J}_p$ is a repelling fixed point of p then there exists an entire function $f : \mathbb{C} \to \mathbb{C}$ such that

$$f \circ |p'(z_0)| = p \circ f.$$

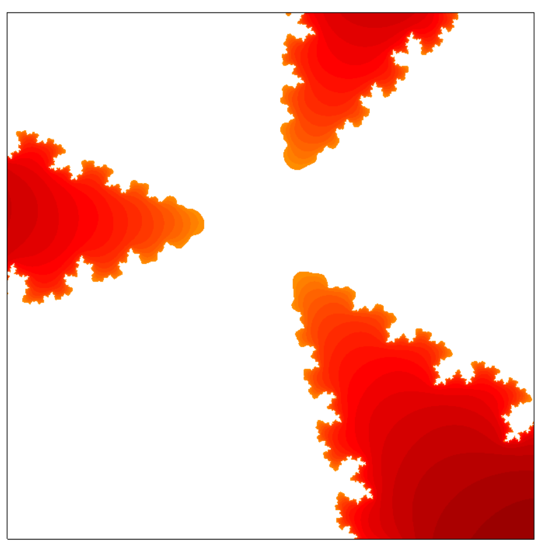

FIGURE 2. Three fractal and Hölder tracts from a linearizer of Douady's Rabbit.

For all entire functions f that obey such a particular linearizing functional equation such that the involved polynomial p has a connected Julia set we show, by a direct calculation in Theorem 7.9, that the transfer operator behaves well. But not all of them have negative spectrum. Based on work of Graczyk, Przytycki, Rivera-Letelier and Smirnov [17, 33], we show that Poincaré functions have Hölder tracts if and only if the corresponding linearizing polynomial is topological Collet-Eckmann (TCE). In addition, such a linearizer is in class $\mathcal{S}$ if and only if the polynomial is post-critically finite (thus TCE). Therefore, such functions can be taken as generating function of the analytic family in Theorem 1.4. They are particularly intriguing since it follows from Zdunik's Theorem 7.8 in [45] that the tracts of Poincaré functions are fractals except for the case of polynomials of the form $z \mapsto z^d$, $d \geq 2$, or Tchebychev ones.

Corollary 1.5. *Let $g \in \mathcal{S}$ be a Poincaré function of a polynomial having connected Julia set. Then every function $f \in \mathcal{M}_g$ has negative spectrum and the thermodynamic formalism holds for every hyperbolic map from $\mathcal{M}_g$.*

Here is an other concrete application of the present approach to this particular family.

Theorem 1.6 (Real analyticity of hyperbolic dimension)**.** *Let $p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a hyperbolic polynomial with connected Julia set, let $z_0 \in \mathcal{J}_p$ be a repelling fixed point of p and let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f \circ |p'(z_0)| = p \circ f$. Then, for all $\kappa \in \mathbb{C}$ with sufficiently small moduli, the function*

$$\kappa \longmapsto \mathrm{HypDim}(f_\kappa) \quad , \quad f_\kappa := f \circ \kappa \, ,$$

is real analytic and $\mathrm{HypDim}(f_\kappa) > \mathrm{Hdim}(\mathcal{J}_p)$.

Additional Remark. After having sent out the first version of this paper, Dezotti and Rempe-Gillen informed us that they are actually finishing the preprint [13] and supplied us with its preliminary version. Concerning thermodynamic formalism, they establish its version for hyberbolic Poincaré functions of TCE polynomials. In particular, they show that our Proposition 9.3 holds for TCE polynomials.

2. The Setting

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and let $S(f)$ be the closure of the set of critical values and finite asymptotic values of f. The different type of singularities of an entire function, in fact Iversen's classification, are all very well explained in [7]. We consider functions of the Eremenko–Lyubich class $\mathcal{B}$ which consists of entire functions for which $S(f)$ is a bounded set. It contains the subclass of Speiser functions $\mathcal{S}$, i.e. the ones for which the set $S(f)$ is finite.

The dynamical setting is the following. An arbitrary entire function $f : \mathbb{C} \to \mathbb{C}$ is called *hyperbolic* if $f \in \mathcal{B}$ and if there is a compact set K such that

$$f(K) \subset \mathrm{Int}(K)$$

and $f : f^{-1}(\mathbb{C}\setminus K) \to \mathbb{C}\setminus K$ is a covering map. According to Theorem 1.3 in [40], an entire function f is hyperbolic if and only if the postsingular set

$$\mathrm{P}(f) := \overline{\bigcup_{n\geq 0} f^n(S(f))}$$

is a compact subset of the Fatou set of f. In particular, we have then

$$\mathrm{dist}\,(S(f), \mathcal{J}_f) \geq \mathrm{dist}\,(\mathrm{P}(f), \mathcal{J}_f) > 0\,. \tag{2.1}$$

Here and in the sequel, $\mathcal{J}_f$ stands for the Julia set of f defined in the usual way (see for example the survey [6]).

Concerning the radial Julia set, there are several definitions in the literature (see [24, 36]). It is explained in Remark 4.1 of [24] that these definitions lead to different sets whose difference is dynamically insignificant. In particular they have same Hausdorff dimension. Since we deal only with hyperbolic entire functions, the following definition fits best to our context:

$$\mathcal{J}_r(f) = \{z \in \mathcal{J}(f) \;:\; \liminf_{n\to\infty} f^n(z) < \infty\}\,.$$

The *hyperbolic dimension* of f is the Hausdorff dimension of this set:

$$\mathrm{HypDim}(f) = Hdim(\mathcal{J}_r(f))\,.$$

Of crucial importance for us is the concept of *disjoint type*. It first implicitly appeared in [2] and has been explicitly studied in several papers including [41, 37, 39]. In these papers it meant that the compact set K in the definition of a hyperbolic function can be taken to be connected. In this case, the Fatou set of f is connected. We will use its normalized form described below.

For every $r > 0$ let $\mathbb{D}_r := \mathbb{D}(0, r)$ be the open disk centered at the origin with radius r and $\mathbb{D}_r^* = \mathbb{C}\setminus\overline{\mathbb{D}}_r$ for the complement of its closure. We denote $A(r, R) := \mathbb{D}_R\setminus\overline{\mathbb{D}}_r$ the annulus centered at 0 with the inner radius r and the outer radius R. We further write $\mathbb{D} := \mathbb{D}_1$ for the unit disk in $\mathbb{C}$ and $\mathbb{D}^* := \mathbb{D}_1^*$ for the complement of its closure.

If

$$S(f) \subset \mathbb{D}$$

then $f^{-1}(\mathbb{D}^*)$ consists of mutually disjoint unbounded Jordan domains Ω with real analytic boundaries such that $f : \Omega \to \mathbb{D}^*$ is a covering map (see [15]). In terms of the classification of singularities, this means that f has only logarithmic singularities over infinity. These connected components of $f^{-1}(\mathbb{D}^*)$ are called tracts and the restriction of f to any of these tracts Ω has the special form

$$f_{|\Omega} = exp \circ \tau \quad \text{where} \quad \varphi = \tau^{-1} : \mathcal{H} := \{z \in \mathbb{C} : \Re(z) > 0\} \to \Omega \tag{2.2}$$

is a conformal proper map. We will always assume that f has only finitely many tracts:

$$f^{-1}(\mathbb{D}^*) = \bigcup_{j=1}^{N} \Omega_j\,. \tag{2.3}$$

Notice that this is always the case if the function f has finite order. Indeed, if f has finite order then the Denjoy-Carleman-Ahlfors Theorem (see [29, p. 313]) states that f can have only finitely many direct singularities and so, in particular, only finitely many logarithmic singularities over infinity.

If $f \in \mathcal{B}$ is such that

$$S(f) \subset \mathbb{D} \quad \text{and} \quad \bigcup_{j=1}^{N} \overline{\Omega}_j \cap \overline{\mathbb{D}} = \emptyset \quad \Big(\text{equivalently}: f^{-1}(\overline{\mathbb{D}^*}) = \bigcup_{j=1}^{N} \overline{\Omega}_j \subset \mathbb{D}^*\Big), \tag{2.4}$$

then we will call f a function of *disjoint type*. This is consistent with the disjoint type models in Bishop's paper [9]. The function f is then indeed of disjoint type in the sense of [2, 41, 37] described above as one can take for K the set $\overline{\mathbb{D}}$. Throughout this paper we will always understand the concept of disjoint type in it more restrictive form of (2.4).

It is well known that for every $f \in \mathcal{B}$ and every $\lambda \in \mathbb{C}^*$ the function λf is of disjoint type provided $|\lambda|$ is small enough.

In our present paper we focus on the following class $\mathcal{D}$ of entire functions.

Definition 2.1. *An entire function $f : \mathbb{C} \to \mathbb{C}$ belongs to class $\mathcal{D}$ if the following holds:*

(1) *f has only finitely many tracts, i.e.* (2.3) *holds.*
(2) *f is of disjoint type in the sense of* (2.4)*; in particular f belongs to class $\mathcal{B}$.*
(3) *The corresponding function φ of* (1.1) *satisfies the following very general geometric condition: there exists a constant $M \in (0, +\infty)$ such that for every $T \geq 1$ large enough,*

$$|\varphi(\xi)| \leq M|\varphi(\xi')| \quad \textit{for all} \quad \xi, \xi' \in Q_T \setminus Q_{T/8}. \tag{2.5}$$

Frequently, only the dynamics of the restriction of f to the union of the tracts will be relevant. We recall from (2.2) that such a restriction is given on each component Ω_j by a proper conformal map $\varphi_j : \mathcal{H} \to \Omega_j$.

Definition 2.2. *A model (τ, Ω) is a finite union $\Omega = \bigcup_{j=1}^{N} \Omega_j$ of simply connected unbounded domains Ω_j along with conformal maps $\tau_{|\Omega_j} = \tau_j : \Omega_j \to \mathcal{H}$ such that $\varphi_j = \tau_j^{-1}$ extends continuously to infinity:*

$$\textit{If } \xi_n \in \mathcal{H} \textit{ with } \lim_{n\to\infty} |\xi_n| = \infty \quad \textit{then} \quad \lim_{n\to\infty} |\varphi_j(\xi_n)| = \infty\,.$$

Associated to (τ, Ω) is the model function $f = e^{\tau}$ and we say that $f \in \mathcal{D}$ if f is a disjoint type model in the sense that $\overline{\Omega} \cap \overline{\mathbb{D}} = \emptyset$.

The Julia set of a model f is defined by

$$\mathcal{J}_f := \{z \in \mathbb{D}^*\,;\;\; f^n(z) \in \mathbb{D}^* \;\text{ for every }\; n \geq 1\}\,.$$

By Proposition 2.2 in [39], this definition coincides with the usual definition of the Julia set in case of a disjoint type entire function.

Given these definitions, we will write in this paper $f \in \mathcal{D}$ for either an entire or a model function f having the properties of class $\mathcal{D}$. Model functions can be approximated by entire functions of class $\mathcal{B}$. Rempe–Gillen [38, Theorem 1.7] has a very precise result on uniform approximation. He has proved good estimates for the difference between the model and the approximating entire functions which certainly could be exploited in our context. A weaker notion of approximation of a model function f by an entire function g is when there exists a quasiconformal map φ of the plane such that $f = g \circ \varphi$. Bishop in [9, 10] has established the existence of such quasiconformal approximations in full generality. In his results Ω can be an arbitrary disjoint union of tracts and he can approximate by functions in class $\mathcal{B}$ and even in class $\mathcal{S}$. We will come back to this in the Section 5.2 dealing with Hölder tracts.

If f is of disjoint type, either entire or model, then

$$\mathcal{J}_f \subset \bigcup_{j=1}^{N} \Omega_j$$

and the Julia set $\mathcal{J}_f$ is entirely determined by the dynamics of f in the tracts. So, for disjoint type functions we can work indifferently either with a model or a global entire function. It follows from (2.3) and (2.4) that for such functions

$$f^{-1}\Big(\bigcup_{j=1}^{N} \overline{\Omega}_j\Big) \subset \bigcup_{j=1}^{N} \Omega_j, \tag{2.6}$$

and that there exists $\gamma \in (0,1)$ such that

$$\mathcal{J}_f \subset \Omega = \bigcup_{j=1}^{N} \Omega_j \subset \mathbb{D}^*_{e^\gamma}. \tag{2.7}$$

As said, throughout the whole paper we restrict our attention to the functions in class $\mathcal{D}$, so, in particular, to those of disjoint type. One can extend all our considerations and results to the case of hyperbolic entire functions belonging to Speiser class $\mathcal{S}$, i.e. replacing (2.4) by mere hyperbolicity in Definition 2.1 and assuming class $\mathcal{S}$. This is, quite easily, done in Section 10.1 by using Koebe's Distortion Theorem only (of course plus all what we did for disjoint type functions).

Here and in the sequel we use the classical notation such as

$$A \asymp B.$$

It means, as usually, that the ratio A/B is bounded below and above by strictly positive and finite constants that do not depend on the parameters involved. The corresponding inequalities up to a multiplicative constant are denoted by

$$A \preceq B \quad \text{and} \quad A \succeq B.$$

With this notation we have the following. We recall that the rectangle Q_2 has been defined in (1.2).

Theorem 2.3 (Bounded distortion)**.** *If $\varphi : Q_2 \to \mathbb{C}$ is a univalent holomorphic map then, for every $0 < r < 1$ and every $-2 \le y \le 2$, we have that*

$$|\varphi'(1)|(1-r) \preceq |\varphi'((1 \pm r) + iy)| \preceq |\varphi'(1)| \frac{1}{(1-r)^3} \; . \tag{2.8}$$

If φ is a univalent holomorphic map defined on the entire half-plane $\mathcal{H}$ then, for every $x > 1$,

$$|\varphi'(1)| \frac{1}{x^3} \preceq |\varphi'(x)| \preceq |\varphi'(1)| x \; . \tag{2.9}$$

In here, the multiplicative constants involved are absolute.

Proof. This is simply a fairly straightforward application of Koebe's Distortion Theorem. Let $g : Q_2 \to \mathbb{D}$ be conformal, i.e univalent and holomorphic surjection. It has a holomorphic extension to a neighborhood of Q_1 in $\mathbb{C}$ and thus $g_{|Q_1} : Q_1 \to g(Q_1)$ is a bi–Lipschitz map. It suffices thus to apply Theorem 1.3 in [30] to $\varphi \circ g^{-1}$ in order to deduce (2.8). The inequalities in (2.9) also follow since for every univalent map φ on $\mathcal{H}$ the map $z \mapsto \varphi(xz)$ is a univalent map on Q_2 and one can apply (2.8). □

Let $\varphi : \mathcal{H} \to \Omega$ be a conformal homeomorphism. Then (2.9) implies for every $T \ge 1$ that

$$|\varphi(T) - \varphi(1)| \le \int_1^T |\varphi'(x)| dx \le |\varphi'(1)| \, T^2 \, . \tag{2.10}$$

3. Fractal behavior of $\partial\Omega$ at infinity

We first analyze what happens for one single tract. So, we consider a model (τ, Ω) with Ω a simply connected domain. In Figure 1 we illustrated the possible fractal behavior of a tract Ω near infinity by considering rescalings of the exhaustion domains Ω_T of Ω. Associated to these rescaled domains are the rescaled conformal maps

$$\varphi_T := \frac{1}{|\varphi(T)|} \, \varphi \circ T : \mathcal{H} \to \mathbb{C}. \tag{3.1}$$

We will frequently treat the maps φ_T as restricted to the set Q_2 and will use the same symbol φ_T for this restriction. In symbols, we will consider the maps

$$\varphi_T = \frac{1}{|\varphi(T)|} \, \varphi \circ T : Q_2 \longrightarrow \frac{1}{|\varphi(T)|} \Omega_T \quad , \quad T \ge \gamma \tag{3.2}$$

where, as always, γ comes from (2.7). In particular

$$|\varphi_T(1)| = 1. \tag{3.3}$$

We denote by $\mathcal{F}_\Omega$ the family of all the functions φ_T, $T \ge \gamma$. Since asymptotic properties of this family will be crucial, we now make some elementary observations. Let us recall here that we always work under the standard assumption (2.5).

Lemma 3.1. *Suppose* (2.5) *holds. Then,* $\mathcal{F}_\Omega$ *is a normal family, in the sense of Montel, on* $Q_1 \setminus Q_{1/8}$ *and furthermore*

$$\frac{1}{T^4} \preceq |\varphi_T'(1)| \preceq 1 \quad , \quad T \geq \gamma, \tag{3.4}$$

Proof. It follows from (2.5) and (3.3) that for every $T \geq \gamma$ it holds

$$\varphi_T(Q_1 \setminus Q_{1/8}) \subset \mathbb{D}(0, M). \tag{3.5}$$

Normality of $\mathcal{F}_\Omega$ follows thus directly from Montel's Theorem. The left hand side of (3.4) is a straightforward consequence of the left hand side of item (2.9) of the distortion Theorem 2.3 along with (2.10), both applied to the map $\varphi : \mathcal{H} \to \mathbb{C}$. Indeed, using them, we get

$$|\varphi_T'(1)| = \frac{|\phi'(T)|}{|\varphi(T)}T \succeq |\varphi'(1)| \frac{T}{T^3|\varphi(T)|} = |\varphi'(1)| \frac{1}{T^2|\varphi(T)|} \asymp \frac{1}{T^2|\varphi(T)|} \succeq \frac{1}{T^4},$$

with (2.10) invoked for the last inequality sign. Since $Q_1 \setminus Q_{1/8} \supset \mathbb{D}(1, 1/2)$, it follows from (3.5) that $\varphi_T(\mathbb{D}(1, 1/2)) \subset \mathbb{D}(0, M)$. But by Koebe's $\frac{1}{4}$–Distortion Theorem, $\varphi_T(\mathbb{D}(1, 1/2)) \supset \mathbb{D}(\varphi_T(1), |\varphi_T'(1)|/8)$. Therefore, $|\varphi_T'(1)| \leq 8M$, formula (3.4) is proved. □

Information of the boundary of the image domain can be obtained by considering integral means spectrum (see [21] and [30] for the classical case which concerns conformal mappings defined on the unit disk). In order to do so, let $h : Q_2 \to U$ be a conformal map onto a bounded domain U and define

$$\beta_h(r, t) := \frac{\log \int_I |h'(r + iy)|^t dy}{\log 1/r} \quad , \quad r \in (0, 1) \ \text{ and } \ t \in \mathbb{R}\,. \tag{3.6}$$

The integral is taken over $I = [-2, -1] \cup [1, 2]$ since this corresponds to the part of the boundary of U that is important for our purposes.

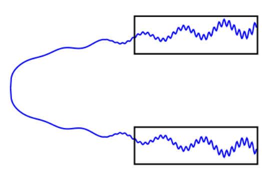

FIGURE 3. The part of the boundary in the boxes can resemble more and more a fractal as $T \to \infty$.

A well known application of the distortion Theorem 2.3 shows that there exists $K_1 > 0$ such that, for all $t \geq 0$,

$$-t + \frac{t \log(|h'(1)|) - K_1}{\log 1/r} \leq \beta_h(r, t) \leq 3t + \frac{t \log\big(|h'(1)|\big) + K_1}{\log 1/r} \tag{3.7}$$

and a corresponding inequality holds for all $t < 0$. Replacing now h by the conformal maps of the family $\mathcal{F}_\Omega$, one first has the following observation.

Lemma 3.2. *Let $t \in \mathbb{R}$, $R \geq \gamma$ and set $T = R/r$, $r > 0$. Then*

$$\limsup_{r\to 0} \beta_{\varphi_{R/r}}(r,t) = \limsup_{T\to+\infty} \beta_{\varphi_T}(1/T,t)$$

is finite and does not dependent on $R \geq \gamma$.

Proof. It suffices to treat the case $t > 0$ since $t < 0$ can be treated the same way and for $t = 0$ there is nothing to show. So, let $t > 0$ and let $R \geq \gamma$. Finiteness of $\limsup_{r\to 0} \beta_{\varphi_{R/r}}(r,t)$ directly results from (3.4) and (3.7). In order to study the dependence on R of this expression, observe that

$$\beta_{\varphi_{R/r}}(r,t) = \frac{\log \int_I |\varphi'_{R/r}(r+iy)|^t dy}{\log 1/r} = \frac{\log \int_I \left(\frac{R/r}{|\varphi(R/r)|}\right)^t |\varphi'(R+i\frac{R}{r}y)|^t dy}{\log 1/r}\,.$$

Since $0 < \gamma < 1$, it suffices to compare this with the corresponding expression with $R' = 1$. If $r' = r/R$ then

$$\beta_{\varphi_{1/r'}}(r',t) = \frac{\log \int_I |\varphi'_{1/r'}(r'+iy)|^t dy}{\log 1/r'} = \frac{\log \int_I \left(\frac{R/r}{|\varphi(R/r)|}\right)^t |\varphi'(1+i\frac{R}{r}y)|^t dy}{\log 1/r'}\,.$$

By Theorem 2.3 there exists $K \geq 1$ such that

$$\frac{1}{KR^3}\left|\varphi'\left(1+i\tilde{y}\right)\right| \leq \left|\varphi'\left(R+i\tilde{y}\right)\right| \leq \left|\varphi'\left(1+i\tilde{y}\right)\right| KR$$

for every $\tilde{y} \in \mathbb{R}$ and every $R \geq 1$. For $\gamma \leq R < 1$ a corresponding estimation holds, we omit this detail and consider $R \geq 1$. Then

$$\frac{-t\log(KR^3)}{\log 1/r} + \beta_{\varphi_{1/r'}}(r',t)\frac{\log 1/r'}{\log 1/r} \leq \beta_{\varphi_{R/r}}(r,t) \leq \frac{t\log(KR)}{\log 1/r} + \beta_{\varphi_{1/r'}}(r',t)\frac{\log 1/r'}{\log 1/r}$$

from which follows that

$$\limsup_{r\to 0} \beta_{\varphi_{R/r}}(r,t) = \limsup_{r'=\frac{r}{R}\to 0} \beta_{\varphi_{1/r'}}(r',t)\,.$$

□

We are now ready to introduce the function

$$\beta_\infty(t) = \limsup_{r\to 0} \beta_{\varphi_{1/r}}(r,t) = \limsup_{T\to+\infty} \beta_{\varphi_T}(1/T,t) \quad , \quad t \in \mathbb{R}\,. \tag{3.8}$$

Lemma 3.2 justifies that $t \mapsto \beta_\infty(t)$ is a well defined finite function on $\mathbb{R}$ and that

$$\beta_\infty(t) = \sup_{R\geq\gamma}\ \limsup_{r\to 0} \beta_{\varphi_{R/r}}(r,t) \quad , \quad t \in \mathbb{R}\,. \tag{3.9}$$

Up to now we considered a single tract. In the general case we deal with a function $f \in \mathcal{D}$ and so $f^{-1}(\mathbb{D}^*)$ is a disjoint union of finitely many tracts Ω_j, $j = 1, ..., N$. Denoting $\beta_{\infty,j}(t)$ the function of (3.8) defined in the tract Ω_j, we can associate to f the function

$$\beta_\infty = \beta_{\infty,f} := \max_{j=1,\dots,N} \beta_{\infty,j}\,.$$

Now we continue dealing with one fixed tract and we skip the index j.

Proposition 3.3. *The function* $\beta_\infty : \mathbb{R} \to \mathbb{R}$ *is convex with*

$$\beta_\infty(0) = 0 \quad \textit{and} \quad \beta_\infty(2) \leq 1.$$

Proof. All involved β–functions are convex by a classical application of Hölder's inequality (see for example p.176 in [30]). It is trivially obvious that $\beta_\infty(0) = 0$ while $\beta_\infty(2) \leq 1$ results from the well known area estimate. Indeed, let $0 < r < 1$. For every integer $0 \leq k < 1/r$ set $y_k^+ := 1 + kr$,

$$U_k^+ := \left\{ z \in \mathbb{C} : r < \Re z < 2r \ \text{ and } \ y_k^+ < \Im z < y_{k+1}^+ \right\} \quad \text{and} \quad U_k^- := \left\{ \overline{z}\ ,\, z \in U_k^+ \right\}.$$

Then,

$$\int_I |\varphi_T'(r+iy)|^2 dy \asymp \sum_{\substack{0 \leq k < [1/r] \\ \varepsilon \in \{+,-\}}} \frac{1}{r} area(\varphi_T(U_k^\varepsilon)) \leq \frac{1}{r} area(\varphi_T(Q_1 \setminus Q_{1/8})) \preceq \frac{1}{r}$$

since $\operatorname{diam}\big(\varphi_T(Q_1 \setminus Q_{1/8})\big) \preceq 1$. Applying logarithms, dividing by $\log(1/r)$ and letting $r \to 0$ gives that $\beta_\infty(2) \leq 1$. □

A function related β_∞, that will be crucial in the sequel, is the following:

$$b_\infty(t) := \beta_\infty(t) - t + 1 \quad , \quad t \in \mathbb{R}\,. \tag{3.10}$$

As an immediate consequence of Proposition 3.3 we get the following.

Proposition 3.4. *The function* $b_\infty : \mathbb{R} \to \mathbb{R}$ *is also convex, thus continuous, with*

$$b_\infty(0) = 1 \ \text{ and } \ b_\infty(2) \leq 0.$$

Consequently, the function b_∞ has at least one zero in $]0, 2]$ and we can introduce a number $\Theta_f \in (0, 2]$ by

$$\Theta_f := \inf\{t > 0 \ :\ b_\infty(t) = 0\} = \inf\{t > 0 \ :\ b_\infty(t) \leq 0\}\,. \tag{3.11}$$

Again, in the case of a function f with finitely many tracts Ω_j we thus have finitely many numbers $\Theta_{f,j}$ and then we set

$$\Theta_f := \max_j \Theta_{f,j}.$$

We will consider below various situations and examples illustrating the behavior of b_∞ and of Θ_f. Notice also that the paper [22] also is based on β_∞ along with the zero (called also Θ) of b_∞.

In order to perform full thermodynamic formalism we need the following crucial property.

Definition 3.5. *A function* $f \in \mathcal{D}$ *has negative spectrum if*

$$b_\infty(t) < 0 \ \ \textit{for all} \ \ t > \Theta_f\,.$$

As we will see in Section 5, this property does hold if the tracts have some nice geometry.

4. Transfer operator

In the sequel f will be either an entire function in $\mathcal{D}$ or a model map in $\mathcal{D}$ and we will work with the Riemannian metric

$$|dz|/|z|. \tag{4.1}$$

This metric is conformally equivalent with the standard Euclidean one, it has singularity at 0 but is tailor crafted for our analysis of Perron–Frobenius operators. With respect to this metric these operators are at least well defined (a big advantage over the ordinary ones for which the defining series is usually divergent) but we will in the sequel prove much more about them. The derivative of a holomorphic function h calculated with respect to the metric of (4.1) at a point z in the domain of h is denoted by $|h'(z)|_1$ and is given by the formula

$$|h'(z)|_1 = |h'(z)|\frac{|z|}{|h(z)|}. \tag{4.2}$$

So, given a real number $t \geq 0$, we define the transfer operator $\mathcal{L}_t$ by the usual formula:

$$\mathcal{L}_t g(w) := \sum_{f(z)=w} |f'(z)|_1^{-t} g(z) \quad \text{for every} \quad w \in \overline{\Omega}\,. \tag{4.3}$$

where g is any function in $\mathcal{C}_b(\overline{\Omega})$, the vector space of all continuous bounded functions defined on Ω. The norm on this space, making it a Banach space, will be the usual sup-norm $\|\cdot\|_\infty$. Note that if $w \in \overline{\Omega}$, then $f^{-1}(w) \subset \Omega$, whence $f'(z)$ is well defined for all $z \in f^{-1}(w)$ and, in consequence, all terms of the above series are also well defined.

Theorem 4.1. *Let f be a model or an entire function of class $\mathcal{B}$ such that $S(f) \subset \mathbb{D}$. Assume that there exists $s > 0$ and $w_0 \in \mathbb{D}^*$ such that*

$$\mathcal{L}_t \mathbb{1}(w_0) < \infty \quad \textit{for every} \quad t > s\,.$$

Let $\tilde{\gamma} \in (0,1)$. Then

$$\sup_{|w|>e^{\tilde{\gamma}}} \mathcal{L}_t \mathbb{1}(w) < \infty \quad \textit{for every} \quad t > s\,.$$

In addition, for all $t > s$ and $p > 1$ such that $\frac{1}{p} < \frac{t}{s} - 1$, there exists a constant $C_{p,t}$ such that

$$\mathcal{L}_t \mathbb{1}(w) \leq \frac{C_{p,t}}{(\log|w|)^{1/p}} \quad \textit{for all} \quad w \in \mathbb{D}^*_{e^{\tilde{\gamma}}}\,. \tag{4.4}$$

In this key result, no dynamical hypothesis nor finiteness of the number of tracts is assumed. If we restrict to functions where Ω is backward invariant then it tells us that the transfer operators $\mathcal{L}_t$ are bounded.

Corollary 4.2. *Let $f \in \mathcal{D}$. If there exists $s > 0$ and $w_0 \in \Omega$ such that $\mathcal{L}_t \mathbb{1}(w_0) < \infty$ for all $t > s$, then all $\mathcal{L}_t$, $t > s$, are bounded operators of $\mathcal{C}_b(\overline{\Omega})$, satisfying in addition* (4.4).

We will explain in Section 8 that the conclusion of this result combined with our previous work [24] lead to full thermodynamic formalism along with all its usual consequences.

Proof. Although f may have infinitely many tracts, it suffices to consider the case of a single tract Ω since the estimates we obtain generalize directly. If $w \in \overline{\mathbb{D}}^*_{e^{\tilde\gamma}}$ then

$$\mathcal{L}_t 1\!\!1(w) = \sum_{f(z)=w} |f'(z)|_1^{-t}. \tag{4.5}$$

The function f restricted to Ω is of the form $f = e^\tau$. Thus

$$|f'(z)|_1 = \frac{|f'(z)|}{|f(z)|}|z| = |\tau'(z)||z|.$$

Since $\varphi = \tau^{-1}$, we have that

$$|f'(z)|_1 = \left|\frac{\varphi(\xi)}{\varphi'(\xi)}\right|,$$

where $\xi = \tau(z)$. In the series of (4.5) z runs through the preimages of w under f, thus ξ runs through the set $\exp^{-1}(w)$. Let

$$R := \log|w| = \Re(\xi)$$

for every $\xi \in \exp^{-1}(w)$. We have $R \geq \tilde\gamma > 0$. We have

$$\mathcal{L}_t 1\!\!1(w) = \sum_{\xi\in\exp^{-1}(w)} \left|\frac{\varphi'(\xi)}{\varphi(\xi)}\right|^t = \sum_{\xi\in\exp^{-1}(w)} |(\log\varphi)'(\xi)|^t \tag{4.6}$$

with an arbitrary choice of a holomorphic branch of the logarithm of φ. Koebe's Distortion Theorem applied to the conformal map $\log\varphi : \mathcal{H} \to \log\Omega$ gives

$$\mathcal{L}_t 1\!\!1(w) \asymp \int_{\mathbb{R}} |(\log\varphi)'(R+iy)|^t\, dy \tag{4.7}$$

On the other hand, $\log\varphi$ is an inverse branch of the logarithmic coordinates of the function f as defined in Section 2 of [15]. Hence, Lemma 1 of [15] applies and yields

$$|(\log\varphi)'(\xi)| \leq \frac{4\pi}{\Re\,\xi} \quad , \quad \xi \in \mathcal{H}.$$

In particular, the holomorphic function $u : \mathcal{H} \to \mathbb{C}$, defined by

$$u(z) := (\log\varphi)'(\frac{\tilde\gamma}{2} + z),$$

is bounded and the function $z \mapsto |u(z)|^t$ is subharmonic, continuous on $\overline{\mathcal{H}}$ and bounded. We can therefore compare it with its harmonic majorant as it is done in [27, Corollary 10.15]:

$$|u(z)|^t \leq \frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{(y-s)^2+x^2}|u(is)|^t ds \quad , \quad z = x+iy \in \mathcal{H}. \tag{4.8}$$

Integrating this inequality and using Fubini's Theorem gives

$$\int_{\mathbb{R}} |u(x+iy)|^t\, dy \leq \int_{\mathbb{R}} \frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{(y-s)^2+x^2} dy\, |u(is)|^t ds = \int_{\mathbb{R}} |u(is)|^t ds < +\infty \tag{4.9}$$

where the last inequality holds since, by Koebe's Distortion Theorem and (4.7), the integral on the right hand side is comparable to $\mathcal{L}_t 1\!\!1(w_0)$, where $w_0 \in \Omega$ is the point of the assumptions in Theorem 4.1. Therefore, (4.7) along with (4.9) imply that

$$\mathcal{L}_t 1\!\!1(w) \preceq \mathcal{L}_t 1\!\!1(w_0) \quad \text{for every} \quad w \in \mathbb{D}^*_{e^{\tilde{\gamma}}}.$$

We have thus proved that $\mathcal{L}_t$ is uniformly bounded on $\overline{\Omega}$.

It remains to show the additional property (4.4). Write $t = \tau + \delta$ where $\tau > s$ and $0 < \delta < t - s$. With $x = \log|w| - \tilde{\gamma}/2$, formulas (4.7) and (4.9) imply that

$$\mathcal{L}_t 1\!\!1(w) \preceq \frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{x}{(y-s)^2+x^2}\,|u(is)|^{\delta} ds\,|u(x+iy)|^{\tau} dy\,.$$

For every $p, q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $p\delta > \Theta_f$, Hölder's inequality yields

$$\int_{\mathbb{R}} \frac{x}{(y-s)^2+x^2}\,|u(is)|^{\delta} ds \le \left(\int_{\mathbb{R}}\left(\frac{x}{(y-s)^2+x^2}\right)^q ds\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}}|u(is)|^{\delta p} ds\right)^{\frac{1}{p}} = O(x^{-\frac{1}{p}})\,.$$

Combinig the last two displayed formulas, we get

$$\mathcal{L}_t 1\!\!1(w) \le \frac{C_1}{x^{1/p}}\int_{\mathbb{R}} |u(x+iy)|^{\tau} dy \le \frac{C_2}{x^{1/p}} \quad \text{for every} \quad w \in \mathbb{D}^*_{e^{\tilde{\gamma}}}$$

for appropriate constants C_1, C_2 depending on δ and p. The proof is now complete since $x = \log|w| - \gamma/2 \asymp \log|w|$ for every $w \in \mathbb{D}^*_{e^{\tilde{\gamma}}}$. □

Given Theorem 4.1, the essential question is to decide whether, for a given function $f \in \mathcal{D}$ and parameter t, the transfer operator, evaluated at $1\!\!1$, is finite at or not at some point. This is where our new geometric tools come into play. Aiming to prove Theorem 1.2 we first reformulate $\mathcal{L}_t$ in terms of the β–functions.

Proposition 4.3. *If $f \in \mathcal{D}$ and $t \ge 0$, then*

$$\mathcal{L}_t 1\!\!1(w) \asymp (\log|w|)^{1-t}\left\{\int_{-1}^{1}\left|\varphi'_{\log|w|}(1+iy)\right|^t dy + \sum_{n\ge1} 2^{n\left(1-t+\beta_{\varphi_{2^n\log|w|}}(2^{-n},t)\right)}\right\}$$

for every $w \in \Omega$ with the above series being possibly divergent.

The issue of convergence of the above mentioned series will be the next step.

Proof of Proposition 4.3. From the proof of Theorem 4.1 we already have the reformulation of the transfer operator that we need in (4.6). Hence

$$\mathcal{L}_t 1\!\!1(w) = \sum_{\xi\in\exp^{-1}(w)}\left|\frac{\varphi'(\xi)}{\varphi(\xi)}\right|^t = \sum_{\substack{\xi\in\exp^{-1}(w)|\\ |\Im\xi|<R}}\left|\frac{\varphi'(\xi)}{\varphi(\xi)}\right|^t + \sum_{n\ge0}\sum_{\substack{\xi\in\exp^{-1}(w)\\ 2^nR\le|\Im\xi|<2^{n+1}R}}\left|\frac{\varphi'(\xi)}{\varphi(\xi)}\right|^t.$$

Applying to each of these sums (2.5) respectively with $T = R$ and $T = 2^{n+1}$, $n \ge 0$, we get

$$\mathcal{L}_t 1\!\!1(w) \asymp \frac{1}{|\varphi(R)|^t}\sum_{\substack{\xi\in\exp^{-1}(w)|\\ |\Im\xi|<R}}|\varphi'(\xi)|^t + \sum_{n\ge0}\frac{1}{|\varphi(2^{n+1}R)|^t}\sum_{\substack{\xi\in\exp^{-1}(w)|\\ 2^nR\le|\Im\xi|<2^{n+1}R}}|\varphi'(\xi)|^t\,. \tag{4.10}$$

Since two consecutive elements of $\exp^{-1}(w)$ are at distance 2π and since $\Re\xi = R \geq \gamma > 0$, Koebe's Distortion Theorem yields

$$\mathcal{L}_t\mathbb{1}(w) \asymp \frac{1}{|\varphi(R)|^t}\int_{-R}^{R} |\varphi'(R+iy)|^t\, dy + \sum_{n\geq 0} \frac{1}{|\varphi(2^{n+1}R)|^t}\int_{I_{n,R}} |\varphi'(R+iy)|^t\, dy$$

where

$$I_{n,R} = [-2^{n+1}R, -2^nR] \cup [2^nR, 2^{n+1}R].$$

Remember that $I = [-2,-1] \cup [1,2]$ and that we have introduced the rescaled functions φ_R in (3.2). A change of variables gives now

$$\mathcal{L}_t\mathbb{1}(w) \asymp\ R^{1-t}\int_{-1}^{1} |\varphi_R'(1+iy)|^t\, dy + \sum_{n\geq 0} \left(2^{n+1}R\right)^{1-t}\int_I \left|\varphi'_{2^{n+1}R}\left(\frac{1}{2^{n+1}}+iy\right)\right|^t dy\,.$$

With invoking the definition (3.6) this completes the proof of Proposition 4.3. □

Passing to functions having negative spectrum, we can now fully describe the behavior of their transfer operators. As it will be explained in Section 8, this then allows us to prove Theorem 1.2, and its usual consequences, following [24]. We recall that for functions with negative spectrum Θ_f is the unique zero of b_∞.

Theorem 4.4. *If $f \in \mathcal{D}$ is a function with negative spectrum, then:*

- *For every $t > \Theta_f$, $\|\mathcal{L}_t\mathbb{1}\|_\infty < +\infty$ and (4.4) holds.*
- *For every $t < \Theta_f$, the series defining $\mathcal{L}_t\mathbb{1}$ is divergent at every point.*

Proof. Let $w_0 \in \Omega$ be any point and set $R = \log|w_0| > \gamma$. Since f has negative spectrum, $b_\infty(t) = -2a_t < 0$ for every $t > \Theta_f$. It thus follows right from the definition of β_∞ in (3.8) that there exist $n_{R,t} > 0$ such that

$$\beta_{\varphi_{2^nR}}(2^{-n},\, t) - t + 1 \leq -a_t < 0 \quad \text{for all} \quad n \geq n_{R,t}\,.$$

Applying Proposition 4.3, we get that

$$\mathcal{L}_t\mathbb{1}(w_0) < \infty$$

for all $t > \Theta_f$. We therefore have checked the hypotheses of Theorem 4.1. It implies that Theorem 4.4 holds for all $t > \Theta_f$.

Let now $t < \Theta_f$, and $w \in \Omega$ be any point. Set again

$$R := \log|w|.$$

Then $b_\infty(t) = 4a_t > 0$ and thus, by Lemma 3.2,

$$\limsup_{r\to 0} \beta_{\varphi_{R/r}}(r,t) - t + 1 = 4a_t > 0\,.$$

Fix a sequence $r_j \searrow 0$ such that $\beta_{\varphi_{R/r_j}}(r_j,t) - t + 1 \geq 2a_t$ for all $j \geq 1$. Then associate to every $j \geq 0$ an integer n_j such that $2^{-n_j-1} < r_j \leq 2^{-n_j}$. Writing down the definition of

$\beta_{\varphi_T}(r,t)$ and employing (2.5) along with bounded distortion, one gets

$$\lim_{j\to\infty} \frac{\beta_{\varphi_{R/r_j}}(r_j,t)}{\beta_{\varphi_{R2^{n_j}}}(2^{-n_j},t)} = 1\,.$$

Thus, $\beta_{\varphi_{R2^{n_j}}}(2^{-n_j},t)-t+1 \geq a_t$ for all sufficiently large j. This implies that the coefficients in the series in Proposition 4.3 do not converge to zero, whence $\mathcal{L}_t 1\!\!1(w) = \infty$. □

5. Functions with negative spectrum and Hölder tracts.

Theorem 1.2 decisively shows that the transfer operators $\mathcal{L}_t$ of negative spectrum entire functions behave sufficiently well so that a fairly complete account of the corresponding thermodynamic formalism can be derived. In the current section we want to get some idea of which functions in class $\mathcal{D}$ may have negative spectrum. We will start with considering some classical examples such as exponential functions and we will see that the class of balanced functions in [23, 24] behaves like these classical examples (Proposition 5.2); it has the simplest possible b_∞ spectrum, namely $b_\infty(t) = 1-t$. Functions with such spectrum will be called elementary.

Then we will show that a function has negative spectrum as soon as its tracts have some nice geometry. For us it will be Hölder domains. Particular examples of such tracts are quasidisks or tracts having the John or Hölder property used in [22]. We finally show that functions of infinite order can also have negative spectrum, and thus the thermodynamic formalism applies to them too.

5.1. Classical functions and balanced growth.

The most classical transcendental family is certainly λe^z or, more generally, λe^{z^d}, $\lambda\in\mathbb{C}\setminus\{0\}$, $d\geq 1$. By a straightforward calculation, all these functions have a trivial integral means spectrum $\beta_\infty\equiv 0$ and thus

$$b_\infty(t) = 1-t \quad , \quad t\geq 0\,. \tag{5.1}$$

In particular, they have negative spectrum with $\Theta_f = 1$ and the tracts of λe^{z^d} are not fractal at all. This is also clear when we consider the rescalings. For any tract of such a function, the part of its boundary depicted in the boxes in Figure 3 converges to a straight line segment as $T\to\infty$.

The thermodynamic formalism has been for first time developed for some transcendental meromorphic functions by Krzysztof Barański in [1]. He did it for for the tangent family. Then this theory has been established for several other families of meromorphic functions. One should mention a quite large and general class of meromorphic functions considered in [19] where, however, as in [1], there where no singular values of f^{-1} in the Julia set considered as a compact subset of $\hat{\mathbb{C}}$, and [42, 43], where for the first time the thermodynamic formalism was built for a transcendental meromorphic function having such singularity in the Julia set, precisely it was ∞ as the asymptotic value of hyperbolic exponential functions. The most general, actually the only general, framework comprising all the classes mentioned above and much more, for which a full fledged thermodynamic formalism has been developed, is up to now the one of [23, 24]. Indeed, these two works cover many classes of entire and meromorphic functions, that include such classical functions as exponential family, the ones of the sine and cosine-root family, elliptic functions, and all the functions having polynomial Schwarzian derivative. It is based on a condition for the derivative which, for entire functions takes on the following form.

Definition 5.1. *An entire function $f:\mathbb{C}\to\mathbb{C}$ is said to be of balanced growth if it has finite order, denoted in the sequel by $\rho=\rho(f)$, and if*

$$|f'(z)| \asymp |f(z)|\,|z|^{\rho-1} \quad , \quad z\in\mathcal{J}_f\,. \tag{5.2}$$

The examples in [23, 24] that satisfy this condition have non fractal tracts precisely as the classical exponential functions λe^{z^d}. This is a general fact for balanced functions. They are elementary in the sense that their integral means spectrum β_∞ is most trivial possible. In the next result we have slightly stronger assumptions than simply balanced growth but all the examples in [23, 24] satisfy them.

Proposition 5.2. *If $f\in\mathcal{D}$ satisfies the balanced condition* (5.2) *in Ω, then f is elementary in the sense that*

$$b_\infty(t)=\beta_\infty(t)-t+1=1-t \quad , \quad t\geq 0\,.$$

In particular, f has negative spectrum with $\Theta_f=1$.

Proof. It suffices to consider an entire function or a model function $f\in\mathcal{D}$ with Ω being one single tract. Then

$$f_{|\Omega}=e^\tau,$$

where

$$\tau=\varphi^{-1}:\Omega\to\mathcal{H}$$

is a conformal map. Shrinking Ω if necessary, we may assume that τ is a continuous map defined on $\overline{\Omega}$. We also may assume that $0\notin\Omega$ and that a holomorphic branch of $z\mapsto z^\rho$ can be well defined on Ω, where ρ comes from (5.2). This allows us to introduce a map

$$h:=\varphi^\rho:\mathcal{H}\to\Omega^\rho:=\{z^\rho\ :\ z\in\Omega\}\,.$$

By assumption, f satisfies (5.2) and $f'=f\,\tau'$. Therefore,

$$|\varphi'(\xi)|=\frac{1}{|\tau'(\varphi(\xi))|}\asymp|\varphi(\xi)|^{1-\rho},$$

which implies

$$|h'(\xi)|=\rho|\varphi(\xi)|^{\rho-1}|\varphi'(\xi)|\asymp 1 \quad , \quad \xi\in\mathcal{H}\,. \tag{5.3}$$

As immediate consequence we get that $|h(T)-h(0)|\preceq T$ which implies

$$|\varphi(T)|\preceq T^{1/\rho} \quad \text{for} \quad T\geq T_0 \tag{5.4}$$

where $T_0\geq 1$ is such that $|h(T)|\geq 2|h(0)|$ for $T\geq T_0$; it is finite due to (5.3) and Koebe's $\frac{1}{4}$–Distortion Theorem.

Let $T \geq T_0$ and consider φ_T the rescaled map from (3.2). We have to estimate

$$|\varphi_T'(\xi)| = \frac{T}{|\varphi(T)|}|\varphi'(T\xi)| \asymp \frac{T}{|\varphi(T)|}|\varphi(T\xi)|^{1-\rho} \quad , \quad \xi \in Q_1 \setminus Q_{1/8}\,.$$

Assumption (2.5) implies $|\varphi(T\xi)| \asymp |\varphi(T)|$ which then gives

$$|\varphi_T'(\xi)| \asymp \frac{T}{|\varphi(T)|^\rho} \succeq 1 \quad , \quad \xi \in Q_1 \setminus Q_{1/8}\,,$$

by (5.4). On the other hand, $\frac{T}{|\varphi(T)|^\rho}$ is independent of ξ. Thus $|\varphi_T'(\xi)| \asymp |\varphi_T'(1)|$ and we know from Lemma 3.1 that $|\varphi_T'(1)| \preceq 1$. Combining all of this gives $|\varphi_T'| \asymp 1$ on $Q_1 \setminus Q_{1/8}$ for $T \geq T_0$. This readily implies that $\beta_\infty \equiv 0$. □

5.2. **Hölder tracts.** Let f be a model as defined in Definition 2.2 or an entire function of class $\mathcal{B}$. Assume that Ω is a single tract of f and that $\varphi = \tau^{-1} : \mathcal{H} \to \Omega$ the associated conformal map. The rectangles Q_T have been introduced in (1.2) and $\Omega_T = \varphi(Q_T)$, $T \geq 1$. A conformal map $g : Q_1 \to U$ is called (H,α)–Hölder if

$$|g(z_1) - g(z_2)| \leq H|g'(1)||z_1 - z_2|^\alpha \quad \text{for all} \quad z_1, z_2 \in Q_1\,. \tag{5.5}$$

The factor $|g'(1)|$ has been introduced in this definition in order to make this Hölder condition scale invariant in the range of g.

Definition 5.3. *The tract Ω is called Hölder, more precisely (H,α)–Hölder, if* (2.5) *holds and if there exists $T_0 \geq 1$ such that*

$$\varphi \circ T : Q_2 \to \Omega_T$$

satisfies (5.5) *for every $T \geq T_0$. We say that f has Hölder tracts if for some $R \geq 1$ the components of $f^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}}_R)$ are Hölder tracts.*

The main point of this definition is that the tract Ω is exhausted by a family of uniformly Hölder domains Ω_T. We shall prove the following.

Lemma 5.4. *If f is a model or an entire function of class $\mathcal{B}$ and if Ω is a single Hölder tract of f, then*

$$|\varphi(T)| \asymp \mathrm{diam}(\Omega_T) \asymp |(\varphi \circ T)'(1)| \quad , \quad T \geq 1\,, \tag{5.6}$$

Proof. Inequality $\mathrm{diam}(\Omega_T) \preceq |(\varphi \circ T)'(1)|$ is immediate from (5.5). Inequality $\mathrm{diam}(\Omega_T) \succeq |(\varphi \circ T)'(1)|$ is immediate from Koebe's $\frac{1}{4}$–Distortion Theorem. Hence, also

$$\begin{aligned}|\varphi(T)| &\leq |\varphi(T) - \varphi(1)| + |\varphi(1)| \leq H|(\varphi \circ T)'(1)|\left|1 - \frac{1}{T}\right|^\alpha + |\varphi(1)| \\ &\preceq |\varphi(1)| + \mathrm{diam}(\Omega_T) \\ &\preceq \mathrm{diam}(\Omega_T),\end{aligned}$$

where the last inequality was written assuming that $T \geq T_0$ is large enough, say $T \geq T_1 \geq T_0 \geq 1$. We are thus left to show that

$$\mathrm{diam}(\Omega_T) \leq |\varphi(T)|. \tag{5.7}$$

Indeed, proving this inequality, it follows from (5.5) and the, already proven, right–hand side of (5.6), that

$$\operatorname{diam}\big(\Omega_{8^{-q}T}\big) \le \frac{1}{4}\operatorname{diam}(\Omega_T)$$

with some integer $q \ge 1$ and $T \ge T_1$ large enough, say $T \ge T_2 \ge T_1$. Therefore, there exist two points $z_1, z_2 \in \Omega_T \setminus \Omega_{8^{-q}T}$ such that

$$|\varphi(Tz_2) - \varphi(Tz_1)| \ge \frac{1}{4}\operatorname{diam}(\Omega_T).$$

So, applying (2.5) we get that

$$\begin{aligned}\operatorname{diam}(\Omega_T) &\le 4|\varphi(Tz_2) - \varphi(Tz_1)| \le 4|\varphi(Tz_2)| + |\varphi(Tz_1)| \\ &\le 4(M^q|\varphi(T)| + M^q|\varphi(T)|) \\ &= 8M^q|\varphi(T)|,\end{aligned}$$

whence formula (5.6) constituting Lemma 5.4 is established. □

Remark 5.5. *Invoking Lemma 5.4 we conclude that for a Hölder tract Ω, the functions $\varphi \circ T$ are uniformly Hölder if and only if the rescaled functions φ_T satisfy uniformly* (5.5) *without the factor $|g'(1)|$.*

We also note that if the components of $f^{-1}(\mathbb{D}^*_{R_0})$ are Hölder for some $R_0 \ge 1$ then the components of $f^{-1}(\mathbb{D}^*_R)$ are Hölder for all $R \ge R_0$. A very important feature of Hölder tracts is expressed by the following.

Proposition 5.6. *All models or functions $f \in \mathcal{B}$ with finitely many Hölder tracts have negative spectrum and*

$$1 \le \Theta_f \le \operatorname{HypDim}(f) \le 2\ . \tag{5.8}$$

In addition, if the corresponding Hölder exponent $\alpha \in (0,1]$ is larger than $1/2$, then $\Theta_f < 2$.

Proof. A classical argument (see [30] or the proof of Proposition 3.3 in [22]) applies word by word showing that

$$\beta_\infty(t+s) \le (1-\alpha)s + \beta_\infty(t), \tag{5.9}$$

where, we recall, α is a Hölder exponent of the tract Ω. Therefore,

$$b_\infty(t+s) \le b_\infty(t) - \alpha s. \tag{5.10}$$

Thus $b_\infty(t) < 0$ for all $t > \Theta_f$ which shows that f has negative spectrum.

As explained before, the paper [22] also employs the same b_∞–function along with the zero Θ_f. It is shown in [22] that that the inequalities of (5.8) hold for Hölder tracts.

The second to the last assertion of Proposition 3.4 is that $b_\infty(0) = 1$. So, with $t = 0$ and $s = 2$, it follows from (5.10) that $b_\infty(2) \le 1 - 2\alpha < 0$ whenever $\alpha > 1/2$, and thus that $\Theta_f < 2$. The proof is complete. □

All the elementary functions enjoy the property that $\Theta_f = 1$ but in general Hölder tracts are fractal in the sense that

$$\Theta_f > 1\,.$$

Models with fractal tracts have been considered in [22]. A particular family of entire functions having fractal tracts is studied in detail in the forthcoming Section 7.

5.3. **Functions of infinite order.** Let us finally consider one other family of examples having totally different behavior than the preceding ones. They have tracts that are not Hölder, they are of infinite order and also the family of rescalings $\mathcal{F}_\Omega$ has only constant limit functions. Nevertheless, we will see that they have negative spectrum and thus they are first examples of infinite order for which the thermodynamic formalism is developed.

Consider functions $f \in \mathcal{D}$, no matter whether entire or model, having the following properties:

- f has negative spectrum.
- f has a Hölder tract $\Omega_f \subset \{\Re z \geq 3\} \subset \mathcal{H}$.

We will associate to such a function a model function F defined on the domain $\Omega_F = \log(\Omega_f)$, log meaning any, or even finitely many, arbitrary branches of the logarithm. The definition of F is this.

$$F := f \circ \exp : \Omega_F \to \mathbb{C}\,.$$

To such a function Theorem 1.2 applies since we have the following.

Proposition 5.7. *The infinite order function $F = f \circ \exp : \Omega_F \to \mathbb{C}$ belongs to $\mathcal{D}$ and has negative spectrum with $\Theta_F \leq \Theta_f$.*

Proof. The disjoint type property follows since $S(F) = S(f) \subset \mathbb{D}$ and $F^{-1}(\mathbb{D}^*) = \Omega_f \subset \{\Re z \geq 3\} \subset \mathbb{D}^*$. Let $\varphi = \tau^{-1} : \mathcal{H} \to \Omega_f$ be conformal such that $f = e^\tau$ on Ω_f. Then

$$F = \exp \circ (\tau \circ \exp).$$

It suffices to consider the case where Ω_F is a single tract so that $\tau \circ \exp : \Omega_F \to \mathcal{H}$ is a conformal map with inverse $\Phi = \log \circ \varphi$. Since $f \in \mathcal{D}$ it satisfies (2.5) and since $\Omega_f \subset \{\Re z \geq 3\}$ we have

$$\frac{|\Phi(\xi_1)|}{|\Phi(\xi_2)|} = \frac{\log|\varphi(\xi_1)|}{\log|\varphi(\xi_2)|} \leq 1 + \frac{\log M}{\log|\varphi(\xi_2)|} \leq 1 + \frac{\log M}{\log 3} \quad \text{for all} \quad \xi_1, \xi_2 \in Q_T \setminus Q_{T/8}\,.$$

Thus Φ satisfies (2.5), completing the argument that F is in $\mathcal{D}$.

It remains to estimate $\beta_{\infty,F}$. For $T \geq 1$, $\Phi_T = \frac{1}{|\Phi(T)|} \log \circ \varphi \circ T$ hence

$$\Phi_T'(\xi) = \frac{T}{|\Phi(T)|} \frac{\varphi'(T\xi)}{\varphi(T\xi)} = \frac{1}{|\log\varphi(T)|} \frac{|\varphi(T)|}{\varphi(T\xi)}\, \varphi_T'(\xi)$$

thus

$$|\Phi_T'(\xi)| \asymp \frac{1}{\log|\varphi(T)|} |\varphi_T'(\xi)| \quad \text{for every} \quad \xi \in Q_1 \setminus Q_{1/8}\,.$$

The factor $|\varphi(T)|$ can be estimated as follows. Still since $\Omega_f \subset \{\Re z \geq 3\}$ we have $|\varphi(T)| \geq 3$. On the other hand we have from (2.10)

$$|\varphi(T)| \preceq |\varphi(T) - \varphi(1)| \leq T^2 |\varphi'(1)| \, .$$

It follows that there exists a constant $C \geq 0$ such that

$$\frac{e^{-C}}{\log T} \int_I |\varphi_T'(r+iy)|^t dy \leq \int_I |\Phi_T'(r+iy)|^t dy \leq e^C \int_I |\varphi_T'(r+iy)|^t dy$$

for every $r \in (0,1)$ and $T \geq 1$. This shows that

$$\frac{-C - \log\log \gamma/r}{\log 1/r} + \beta_{\varphi_{\gamma/r}}(r,t) \leq \beta_{\infty,F}(r,t) \leq \beta_{\infty,f}(r,t) \tag{5.11}$$

which immediately implies that $t_{*,F} \leq t_{*,f}$ and that $\beta_{\infty,F}(r,t) < 0$ for $t > t_{*,f}$ since f has negative spectrum. □

6. Quasiconformal invariance of Hölder tracts

Quasiconformal maps have good Hölder continuity properties and thus preserve Hölder tracts. Let us make this precise.

Lemma 6.1. *Let $g, f \in \mathcal{B}$ have finitely many tracts and let $R \geq 1$ be such that all the connected components of $g^{-1}(D_R^*)$ are Hölder. If $\Phi : \mathbb{C} \to \mathbb{C}$ is a quasiconformal homeomorphism such that*

$$f^{-1}(D_R^*) = \Phi\left(g^{-1}(D_R^*)\right) \quad \text{and} \quad f \circ \Phi = g \quad \text{on} \quad g^{-1}(D_R^*), \, .$$

then all the connected components of $f^{-1}(D_R^*)$ are Hölder.

Proof. It suffices to consider the case where the functions have just one tract $\Omega_g = g^{-1}(D_R^*)$ and $\Omega_f = \Phi(\Omega_g)$. We may also assume without loss of generality that $R = 1$. Then there are conformal maps $\varphi_g : \mathcal{H} \to \Omega_g$ and $\varphi_f : \mathcal{H} \to \Omega_f$ such that, with appropriate holomorphic branches of logarithms, the holomorphic maps $\log g : \Omega_g \to \mathcal{H}$ and $\log f : \Omega_f \to \mathcal{H}$ are the respective inverses of φ_g and φ_f, and, in addition,

$$\varphi_f = \Phi \circ \varphi_g \, .$$

By our hypotheses, φ_g satisfies the conditions of Definition 5.3 and we have to show that φ_f does it too. The condition (2.5) is satisfied by φ_g and $|\varphi_g(T)| \to \infty$ as $T \to \infty$. Thus

$$|\varphi_g(\xi_1) - \varphi_g(0)| \asymp |\varphi_g(\xi_1)| \leq M |\varphi_g(\xi_2)| \asymp |\varphi_g(\xi_2) - \varphi_g(0)| \tag{6.1}$$

for all $\xi_1, \xi_2 \in Q_T \setminus Q_{T/8}$, $T \geq 8$. Since the map Φ is quasiconformal, it is quasisymmetric. This means that there exists a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that $|a - b| \leq t|a - c|$ yields

$$|\Phi(a) - \Phi(b)| \leq \eta(t) |\Phi(a) - \Phi(c)|.$$

Along with (6.1), this gives that

$$\begin{aligned}|\varphi_f(\xi_1)| &= |\Phi(\varphi_g(\xi_1))| \asymp |\Phi(\varphi_g(\xi_1)) - \Phi(\varphi_g(0))| \\ &\le \eta(M')|\Phi(\varphi_g(\xi_2)) - \Phi(\varphi_g(0))| \\ &\asymp |\Phi(\varphi_g(\xi_2))| \\ &= |\varphi_f(\xi_2)|\end{aligned}$$

for all $\xi_1, \xi_2 \in Q_T \setminus Q_{T/8}$, $T \ge 8$, where M' is a constant witnessing the comparability of the very left and the very right sides of (6.1). In other words, φ_f satisfies (2.5).

We know that for some $T_0 \ge 1$ the family of rescalings $\varphi_{g,T}$, $T \ge T_0$, of φ_g is uniformly Hölder and it remains to show that

$$\varphi_{f,T} = \frac{1}{|\Phi(\varphi_g(T))|}\, \Phi \circ |\varphi_g(T)| \circ \varphi_{g,T} \quad , \quad T \ge T_0\,,$$

has the same property. All the mappings

$$\hat{g}_T := \frac{1}{|\Phi(\varphi_g(T))|}\, \Phi \circ |\varphi_g(T)|$$

are K–quasiconformal, where K is the quasiconformal constant of Φ and they are normalized by $\hat{g}_T(\infty) = \infty$. We shall prove the following.

Claim 1^0: There exists a constant $\kappa \in (0, 1]$ such that

$$|\hat{g}_T(0)| \le \kappa \le 2\kappa \le |\hat{g}_T(1)| \le 1/\kappa \quad \text{ for all } T \ge T_0.$$

Proof. We have

$$|\Phi(|\varphi_g(T)|)| \asymp |\Phi(|\varphi_g(T)|) - \Phi(0)|| \quad \text{and} \quad |\Phi(\varphi_g(T))| \asymp |\Phi(\varphi_g(T)) - \Phi(0)||,$$

and obviously $|\varphi_g(T) - 0| = \big||\varphi_g(T)| - 0\big|$. Therefore, invoking again quasisymmetricity of Φ, witnessed by the homeomorphism $\eta : [0, \infty) \to [0, \infty)$, we consecutively get

$$\frac{1}{\eta(1)} \le \frac{|\Phi(|\varphi_g(T)|) - \Phi(0)||}{|\Phi(\varphi_g(T)) - \Phi(0)||} \le \eta(1)$$

and

$$\frac{|\Phi(|\varphi_g(T)|)}{|\Phi(\varphi_g(T))|} \asymp 1.$$

This means that $|\hat{g}_T(1)| \asymp 1$. The proof of Claim 1^0 is complete. □

In conclusion,

$$\mathcal{G} := \{g_T : T \ge T_0\}$$

is a uniformly quasiconformal and normalized family. By Remark 5.5 there exists $R > 1$ such that

$$\varphi_{g,T}(Q_1) \subset \overline{\mathbb{D}}_R$$

for all $T \ge T_0$. These two fact imply (see Theorem 4.3 in [20]) that the family $\mathcal{G}$ restricted to $\overline{\mathbb{D}}_R$ is uniformly Hölder. Therefore, $\varphi_{f,T}$ is uniformly Hölder as a composition of two Hölder functions whose Hölder exponents and constants do not depend on $T \ge T_0$. □

We provide two important applications of the quasiconformal invariance of Hölder tracts. The first one we present right now and it concerns quasiconformal approximation. The second application will be in Section 10 on analytic families of functions in Speiser class $\mathcal{S}$.

6.1. **Quasiconformal approximation.** As already mentioned, Bishop [9, 10] considered quasiconformal approximations of most general models where Ω can be an arbitrary union of simply connected unbounded domains. Keeping our definition of a model, the following result is a simplified version of Theorem 1.1 in [9].

Theorem 6.2 ([9]). *Let (τ, Ω) be a tract model and $f = e^\tau$ the corresponding function. Fix $R > 0$. Then there exist an entire function $F \in \mathcal{B}$ and a quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$ with ψ conformal out of $\{z \in \mathbb{C} : R < \Re\tau(z) < 2R\}$ such that*

$$e^\tau = F \circ \psi$$

on $\Omega(2R) = \{z \in \mathbb{C} : \Re\tau(z) > 2R\}$. Moreover, the components of $\{|F| > e^R\}$ are in a 1-to-1 correspondence with the components of Ω via ψ.

If the initial model is of disjoint type one can adjust $R > 0$ such that

$$\mathcal{J}_f \subset \Omega(3R) = \{z \in \mathbb{C} : \Re\tau(z) > 3R\}\,. \tag{6.2}$$

Then we can assume that F also is of disjoint type since otherwise it suffices to compose ψ with an affine map. So, we can consider for the map F the set of tracts $\Omega_F = \psi(\Omega(2R))$ and suppose that $\mathcal{J}_F \subset \psi(\Omega(3R))$.

Proposition 6.3. *Suppose $f \in \mathcal{D}$ is a model having only Hölder tracts and suppose that F is a disjoint type entire function given by Bishop's Theorem 6.2 with $R > 0$ small enough such that* (6.2) *holds. Then, $F \in \mathcal{D}$, the tracts Ω_F of F are Hölder and, consequently, F has negative spectrum.*

Proof. Follows directly from Lemma 6.1. □

7. Poincaré functions

In this section now consider a, quite particular, family of entire functions. They are obtained by linearization of a polynomial at a repelling fixed point and are often called linearizers or Poincaré functions (see [12, 28, 14]). Let $p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a polynomial having a repelling fixed point z_0 with multiplier λ. By the Koenigs-Poincaré linearization theorem there exists an entire function $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(0) = z_0 \quad \text{and} \quad f \circ \lambda = p \circ f \tag{7.1}$$

with

$$f'(0) \neq 0.$$

Remark 7.1. *One could consider here a much more general family. Instead of linearizing the dynamics at a repelling fixed point one can consider limits of rescalings at conical limit points. For example, if p is a hyperbolic polynomial with connected Julia set then there exist entire linearizers at any point of the Julia set $\mathcal{J}_p$ (see* [5, Theorem 2.10]*).*

If f has the property (7.1), then every other solution of (7.1) is of the form $f_\kappa = f \circ \kappa$ for some $\kappa \in \mathbb{C}^*$.

Lemma 7.2. *If p is a polynomial with connected Julia set and if κ is sufficiently small in modulus, then, up to normalization, f_κ satisfies the first two conditions of Definition 2.1 of $\mathcal{D}$ and*

$$\mathcal{J}_f \subset \mathbb{D}^* \subset A_p(\infty),$$

where $A_p(\infty)$ the attracting basin of infinity of the polynomial p.

Proof. Linearizers of polynomials are entire functions of finite order [44] and the Denjoy–Carleman–Ahlfors Theorem asserts that finite order functions have only finitely many tracts.

Because of Proposition 3.2 in [28] the set of singular values of f_κ is equal to the post–singular set of the polynomial p. By assumption p has connected Julia set and thus its post–singular set is bounded. Therefore f_κ belongs to class $\mathcal{B}$.

Let $R > 0$ be so large that the Julia set of the polynomial $\mathcal{J}_p \subset \mathbb{D}_R$. Consider

$$\Omega := f^{-1}(\mathbb{D}_R^*\}) . \tag{7.2}$$

Since $f(0) = z_0 \in \mathcal{J}_p$, we conclude that $0 \notin \overline{\Omega}$. Therefore, if $|\kappa|$ is sufficiently small then

$$\Omega_\kappa := f_\kappa^{-1}(\mathbb{D}_R^*) = \kappa^{-1}\Omega \subset \mathbb{D}_R^* .$$

This proves the well known fact that f_κ is of disjoint type and that $\mathcal{J}_f \subset \mathbb{D}_R^* \subset A_p(\infty)$ provided $|\kappa|$ is small enough. Normalizing the whole picture allows us to take $R = 1$, and thus we showed that f_κ satisfies the first two conditions of Definition 2.1 of $\mathcal{D}$ for small values of $|\kappa|$. □

Concerning the attracting basin of infinity, it has nice geometry as long as the polynomial has some expansion. Carleson, Jones and Yoccoz [11] have shown that $A_p(\infty)$ is a John domain if and only if p is semi-hyperbolic. Graczyk and Smirnov [17] considered Collet-Eckmann rational functions. Their result states that attracting and super-attracting components of the Fatou set are Hölder if and only if the function is Collet-Eckmann. There is a useful concepts capturing essential features of Collet-Eckmann maps, the one of Topological Collet-Eckmann rational functions. There are various characterizations of such functions. Several of them have been provided in the paper [33] by Przytycki, Rivera-Letelier and Smirnov. A partial version of their results is this.

Theorem 7.3 ([17], [33]). *Let $p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a polynomial and $A_p(\infty)$ its attracting basin of infinity. Then, the following conditions are equivalent:*

(1) *The polynomial p is TCE,*

(2) *$A_p(\infty)$ is a Hölder domain,*

(3)
$$\chi_{\inf}(p) := \inf\left\{\chi_\mu(p) := \int \log|p'|\, d\mu\right\} > 0,$$
where the infimum is taken over all Borel probability p–invariant measures on $\mathcal{J}_p$.

For more about characterizations of TCE maps see [17, 33].

Proposition 7.4. *Let $p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a* TCE *polynomial with connected Julia set. Let z_0 be a repelling fixed point of p and let f be a corresponding linearizer that satisfies all the assertions of Lemma 7.2.*

Then $f \in \mathcal{D}$ and all the connected components of Ω, defined by (7.2), are Hölder tracts. Consequently, by Proposition 5.6, $f : \mathbb{C} \to \mathbb{C}$ has negative spectrum.

Proof. As always, we assume $R = 1$ and then $\mathcal{J}_p \subset \mathbb{D}$ or, equivalently, $\overline{\mathbb{D}}^* \subset A_p(\infty)$. Conjugating p by a affine map if necessary, we also may assume without loss of generality that $z_0 \neq 0$ and that $0 \notin A_p(\infty)$.

Since f is TCE, Theorem 7.3 applies and yields that there exists a Hölder continuous conformal homeomorphism $h : \mathbb{D}^* \to A_p(\infty)$ such that

$$h(1) = z_0 \quad \text{and} \quad p \circ h(z) = h(z^d) \tag{7.3}$$

for all $z \in \mathbb{D}^*$, where $d \geq 2$ is the degree of the polynomial p. The map h can be lifted, via the exponential map, to a conformal homeomorphism

$$H : \mathcal{H} \to \hat{\mathcal{H}} := \exp^{-1}(A_p(\infty))$$

that commutes with translation by $2\pi i$, i.e. such that

$$H(z + 2\pi i) = H(z) + 2\pi i \tag{7.4}$$

for all $z \in \mathcal{H}$, and

$$\varphi \circ H(0) = 0 \quad \text{and} \quad \exp \circ H = h \circ \exp \quad \mathit{on} \quad \mathcal{H}\,. \tag{7.5}$$

We shall prove the following.

Claim 7.5. *The inverse conformal map $H^{-1} : \hat{\mathcal{H}} \to \mathcal{H}$ is bi–Lipschitz on the half–space $\{\Re z > s\}$, whenever $s \in \mathbb{R}$ is such that $\{z \in \mathbb{C} : \Re z \geq s\} \subset \hat{\mathcal{H}}$. Denote a common Lipschitz constant of both $H^{-1} : \{z \in \mathbb{C} : \Re z \geq s\} \to \mathcal{H}$ and and its inverse H by L_s.*

Proof. Indeed, with $\tilde{h}(z) := 1/h(1/z)$, $z \in \mathbb{D}$,

$$H'(z) = \frac{e^{-z}}{\tilde{h}(e^{-z})}\tilde{h}'(e^{-z}) \longrightarrow 1 \quad \text{when} \quad \Re z \to +\infty\,.$$

This and (7.4) imply the announced bi–Lipschitz property. □

Since $\overline{\mathbb{D}}^* \subset A_p(\infty)$, there exists $t < 0$ such that

$$\overline{\mathcal{H}} \subset \{\Re z > t\} \subset \hat{\mathcal{H}}\,. \tag{7.6}$$

Consequently, H^{-1} is uniformly bi–Lipschitz on all the rectangles Q_T, $T \geq 1$.

Let now Ω be a tract of f, i.e. a connected component of $f^{-1}(\mathbb{D}^*)$, and let $\hat{\Omega}$ be the component of $f^{-1}(A_p(\infty))$ containing Ω. Recalling that $0 \notin A_p(\infty)$ we see that the function f restricted to $\hat{\Omega}$ is again of the form

$$f = e^{\varphi^{-1}},$$

where $\varphi : \hat{\mathcal{H}} \to \hat{\Omega}$ is a conformal homeomorphism and $\Omega = \varphi(\mathcal{H})$.

Recall that λ is the multiplier of p at the repelling fixed point z_0. Since f conjugates multiplication by λ and the polynomial p, since h satisfies (7.3) and the right hand sided part of (7.5), and since the exponential map lifts $z \mapsto z^d$ to multiplication by d, we get for all $z \in \mathcal{H}$ that

$$f\big(\varphi \circ H(dz)\big) = h\big(e^{dz}\big) = p\big(h(e^z)\big) = p\big(f(\varphi \circ H)(z)\big) = f\big(\lambda(\varphi \circ H)(z)\big)\,.$$

Along with the left hand sided part of (7.5) this implies that

$$\big(\varphi \circ H\big)(dz) = \lambda\,\big(\varphi \circ H\big)(z) \quad , \quad z \in \mathcal{H}\,. \tag{7.7}$$

Indeed, both $\varphi \circ H \circ d$ and $\lambda \circ \varphi \circ H$ are conformal mappings from $\mathcal{H}$ onto $\hat{\Omega}$ and (7.7) holds near the origin. In conclusion, we have the commutative diagram of Figure 4.

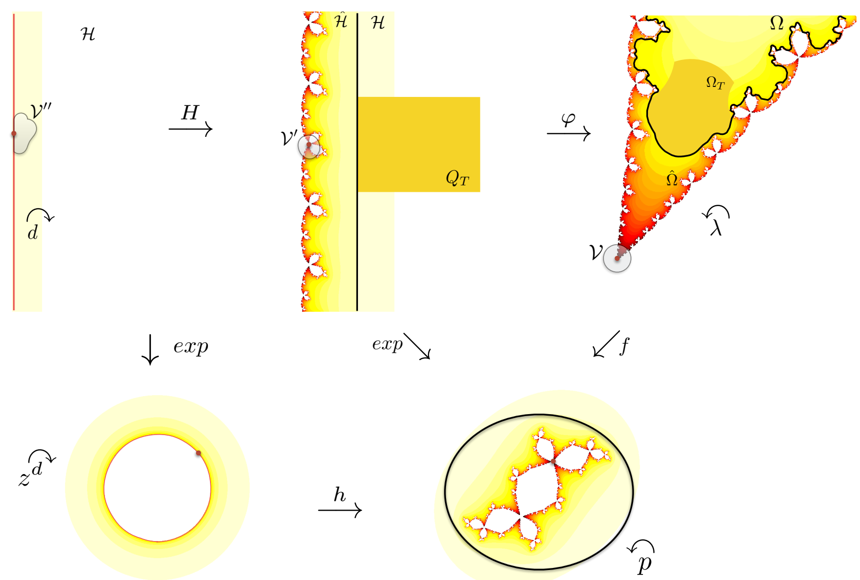

FIGURE 4. Linearizing Douady's Rabbit

If $\exp_*^{-1}$ is the holomorphic inverse branch of the exponential map, defined near z_0, such that $\exp_*^{-1}(z_0) = \tau(0)$, then $\exp_*^{-1} \circ f$ extends $\tau = \varphi^{-1}$ to some bounded open neighborhood, call it V, of the origin. Since $f'(0) \neq 0$, the neighborhood V can be chosen such that $\tau : V \to V' = \tau(V)$ is bi–Lipschitz. Recall that $\tau(0) \in \partial\hat{\mathcal{H}}$ and denote

$$V'' = H^{-1}(V' \cap \hat{\mathcal{H}}) = H^{-1}(V').$$

By the left hand sided part of (7.5) V'' is a bounded open set whose boundary contains 0.

We are to verify Definition 5.3 for f. In order to do this, let $T \geq 1$ and consider the rescaled map

$$\varphi_T = \frac{1}{|\varphi(T)|}\varphi \circ T : Q_1 \to \mathbb{C}.$$

Let $N = N(T)$ be the minimal integer such that

$$\lambda^{-N}\Omega_T = \lambda^{-N}\varphi(Q_T) \subset V\,.$$

Claim 7.6. *$T \asymp d^N$ uniformly in $T \geq 1$.*

Proof. First note that there exist two constants $0 < c < C$ such that

$$\mathbb{D}(0,c) \cap \mathcal{H} \subset V'' \quad \text{and} \quad V'' \subset \mathbb{D}(0,C).$$

By the choice of N and the commutative diagram above, specifically by (7.7), we get

$$\begin{aligned} d^{-N} \circ H^{-1}(Q_T) &= d^{-N} \circ H^{-1}(\varphi^{-1}(\Omega_T)) = (\varphi \circ H \cdot d^N)^{-1}(\Omega_T) = (\lambda^n \varphi \circ H)^{-1}(\Omega_T) \\ &= H^{-1}(\varphi^{-1}(\lambda^{-N}\Omega_T)) = H^{-1}(\varphi^{-1}(\lambda^{-N}\varphi(Q_T))) \\ &\subset H^{-1}(\varphi^{-1}(V)) = H^{-1}(V') \\ &= V'', \end{aligned} \tag{7.8}$$

and there exists $z \in d^{-(N-1)} \circ H^{-1}(Q_T) \setminus V''$. Therefore,

$$|d^{-N} \circ H^{-1}(T)| \leq C. \tag{7.9}$$

Invoking Claim 7.5 and (7.6) we thus get

$$d^{-N}T \preceq |d^{-N}H^{-1}(T) - d^{-N}H^{-1}(0)| \leq |d^{-N}H^{-1}(T)| + |d^{-N}H^{-1}(0)| \preceq C.$$

On the other hand, since $z \notin V''$ and since $z = d^{-(N-1)} \circ H^{-1}(\xi)$ for some $\xi \in Q_T$, invoking Claim 7.5 and (7.6) again, we get that

$$c \leq |z| = d^{-(N-1)}|H^{-1}(\xi)| \preceq d^{-N}|\xi| \preceq d^{-N}T\,.$$

The proof of the Claim 7.6 is complete. □

Denote

$$G_T := d^{-N} \circ (H^{-1}) \circ T : \mathcal{H} \to \mathcal{H}$$

Claim 7.5 and Claim 7.6 yield the following.

Claim 7.7. *The conformal maps $G_T : \overline{\mathcal{H}} \to G_T(\overline{\mathcal{H}})$ are uniformly bi–Lipschitz with respect to $T \geq 1$. Denote by L the corresponding Lipschitz constant.*

Hence,

$$G_T(B(1,1)) \supset B\big(G_T(1), L^{-1}\big) \subset H(V'') \subset \mathcal{H},$$

the (last) inclusion following from (7.8) and the definition of V''. Thus,

$$\Re(G_T(1)) \geq L^{-1}.$$

Therefore, invoking also (7.9), we get that

$$\Re(z_T) \asymp 1 \quad \text{where} \quad z_T = G_T(1) = d^{-N} \circ H^{-1} \circ T(1) \in V''\,. \tag{7.10}$$

Now consider the map

$$g_T := H \circ G_T|_{Q_T} : Q_1 \to Q_{1,T} := g_T(Q_1) \subset V' \cap \hat{\mathcal{H}}. \tag{7.11}$$

The map g_T is the composition of the bi–Lipschitz map G_T and $H_{|V''}$. By assumption the polynomial p is TCE and thus the conformal map $h : \mathbb{D}^* \to A_p(\infty)$ is Hölder. Therefore $H_{|V''}$ is also a Hölder map and, consequently, g_T is uniformly Hölder. Its Hölder exponent is α and denote by Γ its Hölder constant. Since the modulus of derivative of G_T is uniformly bounded above and uniformly separated from zero, looking up at (7.10), which implies that $|H'(z_T)| \asymp 1$, we then obtain that

$$|g_T'(1)| \asymp 1, \quad T \geq 1. \tag{7.12}$$

We thus showed that there exists (H, α) such that g_T satisfies (5.5) for all $T \geq 1$.

Making use of (7.7), we get the following:

$$\varphi \circ T = \lambda^N \circ \varphi_{|Q_{1,T}} \circ g_T \quad \text{on} \quad Q_1 \,. \tag{7.13}$$

Since, by (7.11), $Q_{1,T} \subset V'$ and since the map $\varphi : V' \to V$ is bi–Lipschitz as the inverse of τ, we get that

$$|\varphi'| \asymp 1 \tag{7.14}$$

on $Q_{1,T}$. From this and (7.12) it follows that $|(\varphi \circ T)'(1)| \asymp |\lambda|^N$. Then, making use of (7.13), we get for all points $z_1, z_2 \in Q_1$ that

$$|\varphi \circ T(z_1) - \varphi \circ T(z_2)| \asymp |\lambda|^N |g_T(z_1) - g_T(z_2)| \preceq |(\varphi \circ T)'(1)| \, \Gamma \, |z_1 - z_2|^\alpha \,.$$

It follows from Claim 7.7 that

$$\mathrm{dist}(G_T(0), G_T(Q_1 \setminus Q_{1/8})) \succeq 1.$$

uniformly with respect to $T \geq 1$. Since $\lim_{T\to\infty} G_T(0) = \lim_{T\to\infty} d^{-N(T)} \circ H^{-1}(0) = 0$, applying the map H, and invoking Claim 7.5 (bi–Lipschitzness of H^{-1} on $\overline{\mathcal{H}}$), we get that

$$\mathrm{dist}(H(0), g_T(Q_1 \setminus Q_{1/8})) \succeq 1.$$

uniformly with respect to $T \geq 1$. Furthermore, because of (7.14), and the first assertion of (7.5), we obtain

$$\mathrm{dist}(0, \varphi \circ g_T(Q_1 \setminus Q_{1/8})) \succeq 1. \tag{7.15}$$

Now, uniform Hölder continuity of g_T and (7.14) imply that

$$|\varphi \circ g_T(z_1) - \varphi \circ g_T(z_2)| \preceq 1$$

for all $z_1, z_2 \in Q_1 \setminus Q_{1/8}$. Therefore, using (7.15) and (7.13), we get that

$$\left|\frac{\varphi \circ T(z_1)}{\varphi \circ T(z_2)}\right| = \left|\frac{\varphi \circ g_T(z_1)}{\varphi \circ g_T(z_2)}\right| \preceq 1 + \frac{1}{|\varphi \circ g_T(z_1)|} \preceq 1.$$

This means that (2.5) has been established, so $f \in \mathcal{D}$, and the proof is complete. □

This proof opens the door to a much finer result. The advantage we have now is that we get a good expression for φ_T from (7.13). It allows us to relate the β_∞ function of the rescalings to the classical integral means spectrum

$$\beta_h(t) = \limsup_{r\to 1^+} \frac{\log \int_{|z|=1} |h'(rz)|^t |dz|}{-\log(r-1)}$$

of a Riemann map $h : \{|z| > 1\} \to A_p(\infty)$. For this function there is a formula holding for all polynomials with connected Julia sets (see [8], see also [35] for the expanding case):

$$\beta_h(t) - t + 1 = \frac{\mathrm{P}(t)}{\log d} \tag{7.16}$$

where $d := \deg(p)$ and $\mathrm{P}(t)$ is the topological pressure of the potential $-t\log|p'|$ with respect to the polynomial p. In fact $\mathrm{P}(t)$ is the tree pressure in the general non-expanding case, see [31, 34]. Since this formula does not directly hold here, we provide its suitable variant along with all the details in the Appendix A.

Recall that a polynomial is called exceptional if it is either of the form $z \mapsto z^d$, $d \geq 2$, or it is a Tchebychev polynomial. Such polynomials are special in the sense that Zdunik [45] has shown that they are the only polynomials for which the harmonic measure, viewed from infinity, is not singular with respect to the natural Hausdorff measure of the Julia set.

Theorem 7.8. *Let $p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a polynomial with connected Julia set, let z_0 be a repelling fixed point of p and let $f : \mathbb{C} \to \mathbb{C}$ be a corresponding linearizer that satisfies the conclusion of Lemma 7.2. Then,*

(1) $f \in \mathcal{D}$.
(2) $\Theta_f = \mathrm{HypDim}(p)$, *the hyperbolic dimension of p.*
(3) *The following are equivalent:*
  (a) *p is Topological Collett–Eckmann polynomial (equivalent to $A_p(\infty)$ being a Hölder domain).*
  (b) *All the connected components of $f^{-1}(\mathbb{D}^*)$ are Hölder tracts.*
  (c) *f has negative spectrum.*
(4) *The tracts of f are fractal in the sense that*

$$\Theta_f > 1$$

if and only if p is not an exceptional polynomial.

Proof. The setting is the same as in the proof of Proposition 7.4. So we will use the notations of that proof and the parts of it that do not rely on the TCE hypothesis. For example, the uniform bi–Lipschitz property of H^{-1}, Claim 7.6, and the last part of the proof yielding (2.5) do not rely at all on the TCE hypothesis. The last remark directly implies that for every polynomial p having connected Julia set the tracts of a Poincaré function f linearizing p at a repelling fixed point satisfy (2.5).

Out of Formula (7.13) we get the expression

$$\varphi_T = \frac{\lambda^N}{|\varphi(T)|}\,\varphi_{|Q_{1,T}} \circ g_T = \frac{\lambda^N}{|\varphi(T)|}\,\varphi_{|Q_{1,T}} \circ H \circ G_T = \frac{\lambda^N}{|\varphi(T)|}\,\varphi_{|Q_{1,T}} \circ H \circ \left(d^{-N} \circ H^{-1} \circ T\right) .$$

Concerning the integer N we adjust it as follows. Since we are after the behavior of β_∞, we are only interested in the values of φ_T on $Q_1 \setminus Q_{1/8}$. Therefore, we define now $N = N(T)$ the minimal integer such that

$$\lambda^{-N}(\Omega_T \setminus \Omega_{T/8}) \subset V.$$

From (2.5) follows then that $|\lambda|^N \asymp |\varphi(T)|$. Notice also that this modification of N does not affect Claim 7.6. It suffices indeed to replace in its proof the set Q_T by $Q_T \setminus Q_{T/8}$. This also explains that this modification does not really affect the integer N.

Now, coming back to the above expression of φ_T, even without the TCE assumption we have that the map $G_T = d^{-N} \circ H^{-1} \circ T$ is uniformly bi–Lipschitz on Q_1. Along with (7.14) this yields

$$|\varphi_T'| \asymp |H' \circ G_T| \quad \text{on} \quad Q_1 \setminus Q_{1/8}\,. \tag{7.17}$$

Remember that in the definition of β_{φ_T} one integrates over the set

$$I_r = \left\{r + iy :\ 1 < |y| < 2\right\} \quad , \quad 0 < r < 1\,.$$

Fix $r \in (0,1)$ and focus on the set

$$I_r^+ = \{z \in I_r :\ \Im(z) > 0\}$$

and, as in the definition of $\beta_\infty(r,t)$, consider in what follows $T \geq \gamma/r$. We claim that then there exists $L \geq 1$ such that

$$G_T(I_r^+) \subset \left\{z \in \mathbb{C} : 0 < \Re z < Lr\right\} \quad \text{for every} \quad T \geq \gamma/r\,. \tag{7.18}$$

Indeed, since $H^{-1}(z + 2\pi i) = H^{-1}(z) + 2\pi i$, we have that $\Re(H^{-1}(z + 2\pi i)) = \Re(H^{-1}(z))$, and therefore there exists a constant $K \geq \max\{1, L_0\}$ such that $\Re(H^{-1}(iy)) \leq K$ for all $y \in \mathbb{R}$. By Claim 7.5 we have

$$|H^{-1}(Tr + iy) - H^{-1}(iy)| \leq KTr$$

for all $y \in \mathbb{R}$. Thus,

$$\Re H^{-1}(Tr + iy) \leq \left|\Re H^{-1}(Tr + iy) - \Re H^{-1}(iy)\right| + \left|\Re H^{-1}(iy)\right| \leq K(Tr + 1).$$

Since $T \asymp d^N$ (Claim 7.6), this shows that

$$\begin{aligned} 0 < \Re G_T(r + iy) &= \Re\big(d^{-N} H^{-1}(Tr + iTy)\big) = d^{-N}\Re\big(H^{-1}(Tr + iTy)\big) \\ &\asymp T^{-1}\Re\big(H^{-1}(Tr + iTy)\big) \\ &\leq K(r + T^{-1}), \end{aligned}$$

and (7.18) is established. Let now $\sigma \subset (\{\Re z = Lr\})$ be a sufficiently long compact line segment so that $\gamma := G_T^{-1}(\sigma)$ is a cross-cut of $\{\xi \in \mathbb{C} : 1 \leq \Im\xi \leq 2\}$. For every $k \in \{0, ..., [1/r]\}$ set

$$a_k(r) := r + i(1 + kr)$$

and let $b_k(r) \in \gamma$ with

$$\Im(b_k(r)) = 1 + kr.$$

We can choose the points b_k such that $\Im c_{k+1} > \Im c_k$, where $c_k := G_T(b_k)$. It follows from (7.18) and (7.7) that the Hausdorff distance between I_r^+ and γ is bounded above by a multiple of r and, in addition, these two sets are disjoint and (recalling that G_T is orientation preserving since conformal) $\Re(w) > r$ for all $w \in \gamma$. Hence, there exists $\kappa > 1$ and for every $r \in (0, 1)$ and every $k \in \{0, ..., [1/r]\}$ there exists a rectangle $\Delta_k(r)$ whose ratio of the longer to the lower edge is uniformly bounded above such that $a_k(r), b_k(r) \in \Delta_k(r)$ and $\kappa\Delta_k(r) \subset \mathcal{H}$. Therefore we can apply Koebe's Distortion Theorem to the map $\varphi_T|_{\kappa\Delta_k(r)}$ to conclude that

$$|\varphi_T'(a_k(r))| \asymp |\varphi_T'(b_k(r))|$$

with a comparability constant independent of r and k. Then,

$$\int_{I_r^+} |\varphi_T'(\xi)|^t |d\xi| \asymp \sum_{k=0}^{[1/r]} |\varphi_T'(a_k)|^t r \asymp \sum_{k=0}^{[1/r]} |\varphi_T'(b_k)|^t r \asymp \sum_{k=0}^{[1/r]} |H'(c_k)|^t r$$

where the last comparability sign follows directly from (7.17). On the other hand,

$$\int_\sigma |H'(z)|^t |dz| \asymp \sum_{k=0}^{[1/r]} |H'(c_k)|^t |c_{k+1} - c_k| \asymp \sum_{k=0}^{[1/r]} |H'(c_k)|^t \, r$$

since $|c_{k+1} - c_k| \asymp |b_{k+1} - b_k| \asymp r$. This shows that

$$\int_{I_r^+} |\varphi_T'(\xi)|^t |d\xi| \asymp \int_\sigma |H'(z)|^t |dz|.$$

Having this, an elementary calculation (Chain Rule) based on formula (7.5) and on the fact that $z_0 \neq 0$, yields

$$\int_{I_r^+} |\varphi_T'(\xi)|^t |d\xi| \asymp \int_{C_r} |h'(z)|^t |dz| \quad \text{where} \quad C_r = \exp(\sigma)\,. \tag{7.19}$$

The conclusion comes now from Formula 7.16, in fact from Proposition A.1. In order to be able to apply it notice that there exists $c > 0$ such that $\operatorname{diam}(C_r) \geq c$ for every $1 < r < 2$ since $C_r = \exp \circ G_T(\gamma)$, since all the maps $\exp \circ G_T$, $t \geq 1$, are uniformly bi–Lipschitz and since $\operatorname{diam}(\gamma) \geq 1$. Therefore,

$$\beta_\infty(t) = \beta_h(t) = t - 1 + \frac{\mathrm{P}(t)}{\log d}\,.$$

The behavior of the pressure function is perfectly understood thanks to [33] and [32]. Proposition 2.1 in the latter paper asserts that $\mathrm{P}(t)$ is affine with slope $-\chi_{\inf}(p)$ (see Theorem 7.3) for $t \geq t_+$ where the freezing point $t_+ \geq \mathrm{HypDim}(p)$ with strict inequality if and only if p is TCE. The hyperbolic dimension $\mathrm{HypDim}(p)$ is the first zero of the pressure function P (see [31]).

In conclusion, if p is TCE then $\chi_{\inf}(p) > 0$ (Theorem 7.3) and thus b_∞ is strictly decreasing on $[0, \infty[$. Then f has negative spectrum. If p is not TCE then $\chi_{inf}(p) = 0$ (Theorem 7.3) and the freezing point $t_+ = \mathrm{HypDim}(p)$. In this case Θ_f, the first zero of b_∞, equals $t_+ = \mathrm{HypDim}(p)$ and $b \equiv 0$ on $[t_+, \infty[$. The function f does not have negative spectrum. This shows item (1).

The equality $\Theta_f = \mathrm{HypDim}(p)$ has been shown by Przytycki in [31, Appendix 2] and Item (2) follows from this equality and from Zdunik's work [45]. □

7.1. **Poincaré functions without negative spectrum.** For functions with negative spectrum, the convergence of the series defining the transfer operator converges exponentially fast. In general, i.e. without assuming negative spectrum, this series can still converge. We illustrate this here by considering arbitrary Poincaré functions, i.e. also those without negative spectrum. They are very particular entire functions because of the functional equation (7.1). This equation allows us to do direct calculations even without using integral means. Epstein and Rempe-Gillen [14] have exploited (7.1) in order to relate $\mathcal{L}_t 1\!\!1(w)$ to the Poincaré series of the linearized polynomial. Their argument gives finiteness of $\mathcal{L}_t 1\!\!1$ at any point. We prove the following.

Theorem 7.9. *Let $f \in \mathcal{D}$ be a linearizer of a polynomial $p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with connected Julia set such that $\mathcal{J}_f \subset A_p(\infty)$. Then, there exists a neighborhood $\mathcal{V}$ of the Julia set $\mathcal{J}_f$ such that for every $t > \Theta_f = \mathrm{HypDim}(p)$ the Perron–Frobenius operator $\mathcal{L}_t$ is well defined and bounded on $\mathcal{C}_b(\mathcal{V})$. Moreover,*

$$\mathcal{L}_t 1\!\!1(w) \asymp (\log |w|)^{1-t} \quad \text{for every} \quad w \in \mathcal{V},$$

with comparability constant depending on the whole initial data.

Proof. Let $h : \mathbb{D}^* \to A_p(\infty)$ be the Riemann map such that

$$h(z^d) = p \circ h(z) \ \text{ in } \ z \in \mathbb{D}^* . \tag{7.20}$$

Since $\mathcal{J}_f \subset A(\infty)$, there exists $R_0 > 1$ such that $\mathcal{J}_f \subset h(\mathbb{D}^*_{R_0})$. Denote

$$G_0 := h\big(A(R_0^{1/d}, R_0)\big) .$$

This is a fundamental annulus for the action of p in $A_p(\infty)$. We also need such an annulus for the action of p near the repelling fixed point z_0:

$$V_0 := f\big(A(r, |\lambda| r)\big),$$

where $r > 0$ is so small that f is univalent on $\mathbb{D}_{2|\lambda|r}$ and

$$f\big(\mathbb{D}_{2|\lambda|r}\big) \subset \mathbb{C} \setminus h\big(\mathbb{D}^*_{R_0^{1/d}}\big) . \tag{7.21}$$

Notice that $V_0 \cap \mathcal{J}_p \neq \emptyset$ since otherwise z_0 would be an isolated point of $\mathcal{J}_p$. Therefore there exists $M \geq 1$ such that

$$p^M(V_0) \supset h\big(A(1, R_0)\big) . \tag{7.22}$$

Let in the following w be an arbitrary point of

$$\mathcal{V} := h\big(\mathbb{D}^*_{R_0^{1/d}}\big) \supset \mathcal{J}_f.$$

Then there exists a unique integer $N_w \geq 0$ such that $w \in p^{N_w}(G_0)$. It then follows from (7.20), iterated N_w times, that

$$d^{N_w} \asymp \log|w|\,. \tag{7.23}$$

In order to estimate $\mathcal{L}_t1\!\!1(w)$ we have to estimate $|f'(z)|_1$ for all $z \in f^{-1}(w)$. If z is such a pre-image, i.e. if $z \in f^{-1}(w)$, then there exists a unique integer $n \geq 1$ such that

$$\lambda^{-n}z \in A(r, |\lambda|r).$$

Then $w = f(z) = p^n \circ f \circ \lambda^{-n}(z)$. By (7.21), $n > N_w$. Setting $N := n - N_w$, $\eta := f(\lambda^{-n}z) \in V_0$ and $\xi := p^N(\eta) \in G_0$, we get

$$f'(z) = \left(p^{N_w}\right)'(\xi)\left(p^N\right)'(\eta) f'(\lambda^{-n}z)\lambda^{-n}\,. \tag{7.24}$$

Since f is univalent on $\mathbb{D}_{2|\lambda|r}$ and since $\lambda^{-n}z \in A(r, |\lambda|r) \subset \mathbb{D}_{|\lambda|r}$, we have $|f'(\lambda^{-n}z)| \asymp 1$. The factor $\left(p^{N_w}\right)'(\xi)$ can be estimated as follows. Since $|h'| \asymp 1$ on $\overline{\mathbb{D}^*_{R_0^{1/d}}}$, we have $|h^{-1}(w)| \asymp |w|$, $|(h^{-1})'(\xi)| \asymp 1$ and $|h'\big(a^{d^{N_w}}\big)| \asymp 1$ where $a = h^{-1}(\xi)$. Therefore,

$$\left|\left(p^{N_w}\right)'(\xi)\right| = \left|h'\big(a^{d^{N_w}}\big)\right| d^{N_w}|a|^{d^{N_w}-1}\left|(h^{-1})'(\xi)\right| \asymp \frac{d^{N_w}}{|a|}\left|h^{-1}(w)\right| \asymp d^{N_w}|w|\,.$$

Inserting this into (7.24) leads to

$$|f'(z)|_1 = \left|\frac{f'(z)}{w}z\right| \asymp d^{N_w}\left|\left(p^N\right)'(\eta)\right||\lambda^{-n}z| \asymp d^{N_w}\left|\left(p^N\right)'(\eta)\right|.$$

Finally this gives

$$\mathcal{L}_t1\!\!1(w) \asymp \sum_{\substack{\xi\in G_0\\ p^{N_w}(\xi)=w}} \sum_{N\geq 1} \sum_{\eta\in p^{-N}(\xi)\cap V_0} \left(d^{N_w}\left|\left(p^N\right)'(\eta)\right|\right)^{-t}\,. \tag{7.25}$$

Let

$$\mathcal{P}_t(\xi, V_0) := \sum_{N\geq 1} \sum_{\eta\in p^{-N}(\xi)\cap V_0} \left|\left(p^N\right)'(\eta)\right|^{-t}$$

and let

$$\mathcal{P}_t(\xi) := \sum_{N\geq 1} \sum_{\eta\in p^{-N}(\xi)} \left|\left(p^N\right)'(\eta)\right|^{-t}$$

be the corresponding full Poincaré series of the polynomial p evaluated at the point ξ.

Claim 7.10. *There exists a constant $c_t > 0$ such that*

$$c_t\mathcal{P}_t(\xi) \leq \mathcal{P}_t(\xi, V_0) \leq \mathcal{P}_t(\xi)$$

for all $\xi \in G_0$.

Proof. Recall that the integer M has been introduced in (7.22). We have

$$\mathcal{P}_t(\xi, V_0) \geq \sum_{N>M} \sum_{\eta \in p^{-N}(\xi)\cap V_0} \left| \left(p^N\right)'(\eta)\right|^{-t}.$$

Note that for every integer $k > 0$ we have that $p^{-k}(G_0) \subset A(1, R_0)$ and thus, using (7.22), we conclude that for every $z \in p^{-k}(\xi)$, $\xi \in G_0$, there exists at least one point $\eta \in p^{-M}(z) \cap V_0$. Therefore,

$$\mathcal{P}_t(\xi, V_0) \geq \inf_{\eta \in V_0} \left|\left(p^M\right)'(\eta)\right|^{-t} \sum_{k>0} \sum_{z \in p^{-k}(\xi)\cap V_0} \left|\left(p^k\right)'(z)\right|^{-t} = c_t \mathcal{P}_t(\xi),$$

where $c_t := \inf_{\eta \in V_0} \left|\left(p^M\right)'(\eta)\right|^{-t}$. The other inequality in Claim 7.10 trivially holds and so its proof is complete. □

Fix arbitrarily $\xi_0 \in G_0$. Koebe's Distortion Theorem implies that $\mathcal{P}_t(\xi) \asymp \mathcal{P}_t(\xi_0)$ and thus, by Claim 7.10,

$$\mathcal{P}_t(\xi, V_0) \asymp \mathcal{P}_t(\xi_0) \tag{7.26}$$

for every $\xi \in G_0$. On the other hand, w has exactly d^{N_w} preimages $z \in p^{-N_w}(w)$ and they are all in G_0. We can therefore deduce from (7.25) that

$$\mathcal{L}_t 1\!\!1(w) \asymp d^{-tN_w} \sum_{\substack{\xi \in G_0 \\ p^{N_w}(\xi)=w}} \mathcal{P}_t(\xi_0, V_0) \asymp d^{N_w(1-t)} \mathcal{P}_t(\xi_0) \asymp d^{N_w(1-t)}.$$

The conclusion follows now directly by applying (7.23) □

8. The Classics of Thermodynamic Formalism: Conformal Measures and Beyond

Let $f \in \mathcal{D}$ be a function with negative spectrum. Then the whole thermodynamic formalism can be established for f, word by word, exactly as it was done in [23, 24] except for [24, Lemma 5.13] which is the key point in the construction of conformal measures. Since we provide below, in Proposition 8.7, a proof of this missing point, we finally show that all the relevant results comprising Thermodynamical Formalism, the ones established in [24] and stated below, hold. Combined with Theorem 4.4 this shows Theorem 1.2.

In Section 7.1 we considered entire functions that do not have negative spectrum. As Theorem 7.9 shows, they perfectly satisfy the assumptions of Proposition 8.7. Consequently, all the results of the present section are also valid for these functions.

- The Perron-Frobenius-Ruelle Theorem [24, Theorem 5.15].

Theorem 8.1. *If $f \in \mathcal{D}$ is a function with negative spectrum and $t > \Theta_f$, then the following are true.*

(1) *The topological pressure* $\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_t^n \mathbb{1}(w)$ *exists and is independent of* $w \in \mathcal{J}_f$.
(2) *The function* $(\Theta_f, +\infty) \ni t \longmapsto \mathrm{P}(t) \in \mathbb{R}$ *is convex, thus continuous, in fact real–analytic, strictly decreasing, and* $\lim_{t\to+\infty} \mathrm{P}(t) = -\infty$.
(3) *There exists a unique* $\lambda|f'|_1^t$*–conformal measure* m_t *and necessarily* $\lambda = e^{\mathrm{P}(t)}$. *Also, there exists a unique Gibbs state* μ_t, *i.e.* μ_t *is* f*-invariant and equivalent to* m_t.
(4) *Both measures* m_t *and* μ_t *are ergodic and supported on the radial (or conical) Julia set* $\mathcal{J}_r(f)$.
(4) *The density* $\rho_t := d\mu_t/dm_t$ *is an everywhere positive continuous and bounded function on the Julia set* $\mathcal{J}_f$.

- The Spectral Gap [24, Theorem 6.5]

Theorem 8.2. *If* $f \in \mathcal{D}$ *is a function with negative spectrum and* $t > \Theta_f$, *then the following are true.*

(a) *The number* 1 *is a simple isolated eigenvalue of the operator* $\hat{\mathcal{L}}_t := e^{-\mathrm{P}(t)}\mathcal{L}_t : \mathrm{H}_\beta \to \mathrm{H}_\beta$ *(*$\beta \in (0,1]$ *is arbitrary and* H_β *is the Banach space of real–valued bounded Hölder continuous defined on* $\mathcal{J}_f$*) and all other eigenvalues are contained in a disk of radius strictly smaller than* 1.
(b) *There exists a bounded linear operator* $S : \mathrm{H}_\beta \to \mathrm{H}_\beta$ *such that*

$$\hat{\mathcal{L}}_t = Q_1 + S,$$

where $Q_1 : \mathrm{H}_\beta \to \mathbb{C}\rho$ *is a projector on the eigenspace* $\mathbb{C}\rho$, *given by the formula*

$$Q_1(g) = \left(\int g \, dm_\phi\right) \rho_t,$$

$Q_1 \circ S = S \circ Q_1 = 0$ *and*

$$||S^n||_\beta \le C\xi^n$$

for some constant $C > 0$, *some constant* $\xi \in (0,1)$ *and all* $n \ge 1$.

- [24, Corollary 6.6]

Corollary 8.3. *With the setting and notation of Theorem 8.2 we have, for every* $n \ge 1$, *that* $\hat{\mathcal{L}}^n = Q_1 + S^n$ *and that* $\hat{\mathcal{L}}^n(g)$ *converges to* $\left(\int g \, dm_\phi\right) \rho$ *exponentially fast when* $n \to \infty$. *Precisely,*

$$\left\| \hat{\mathcal{L}}^n(g) - \left(\int g \, dm_\phi\right) \rho \right\|_\beta = \|S^n(g)\|_\beta \le C\xi^n \|g\|_\beta \quad , \quad g \in H_\beta.$$

- Exponential Decay of Correlations [24, Theorem 6.16]

Theorem 8.4. *With the setting and notation of Theorem 8.2 there exists a large class of functions ψ_1 such that for all $\psi_2 \in L^1(m_t)$ and all integers $n \geq 1$, we have that*

$$\left| \int (\psi_1 \circ f^n \cdot \psi_2) \, d\mu_t - \int \psi_1 \, d\mu_t \int \psi_2 \, d\mu_t \right| \leq O(\xi^n),$$

where $\xi \in (0,1)$ comes from Theorem 8.2(b), while the big "O" constant depends on both ψ_1 and ψ_2.

- Central Limit Theorem [24, Theorem 6.17]

Theorem 8.5. *With the setting and notation of Theorem 8.2 there exists a large class class of functions ψ such that the sequence of random variables*

$$\frac{\sum_{j=0}^{n-1} \psi \circ f^j - n \int \psi \, d\mu_t}{\sqrt{n}}$$

converges in distribution with respect to the measure μ_t to the Gauss (normal) distribution $\mathcal{N}(0, \sigma^2)$ with some $\sigma > 0$. Precisely, for every $t \in \mathbb{R}$,

$$\lim_{n \to \infty} \mu_t \left(\left\{ z \in \mathcal{J}_f : \frac{\sum_{j=0}^{n-1} \psi \circ f^j(z) - n \int \psi \, d\mu_t}{\sqrt{n}} \leq t \right\} \right) =$$

$$= \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{u^2}{2\sigma^2} \right) du.$$

- Variational Principle [24, Theorem 6.25]

Theorem 8.6. *If $f \in \mathcal{D}$ is a function with negative spectrum and $t > \Theta_f$, then the f-invariant measure μ_t is the only equilibrium state of the potential $-t \log |f'|_1$, that is*

$$\mathrm{P}(t) = \sup \left\{ \mathrm{h}_\mu(f) - t \int_{\mathcal{J}_f} \log |f'|_1 \, d\mu \right\},$$

where the supremum is taken over all Borel probability f-invariant ergodic measures μ with $\int_{\mathcal{J}_f} \log |f'|_1 \, d\mu > -\infty$, and

$$\mathrm{P}(t) = \mathrm{h}_{\mu_t}(f) - t \int_{\mathcal{J}_f} \log |f'|_1 \, d\mu_t.$$

We will obtain conformal measures following the approach in [24, Section 5.3]. For dynamical systems with compact Julia set theses measures can be produced either as fixed points of dual Perron-Frobenius operators or as weak limits of some atomic measures using the fact that the space of probability measures on the Julia set is weakly compact. In the present setting the Julia set $\mathcal{J}_f$ is an unbounded subset of $\mathbb{C}$ and so the key point is to establish the tightness (in $\mathbb{C}$) of an appropriate sequence of measures. This can be done by following [24, Section 5.3] since we we have the following analogue of [24, Lemma 5.13]:

Proposition 8.7. *Let $f \in \mathcal{D}$ and suppose that there exists $t > 0$ for which $\mathcal{L}_t$ is a bounded operator of $\mathcal{C}_b(\overline{\Omega})$ with*

$$\lim_{w\to\infty\,,\ w\in\Omega} \mathcal{L}_t 1\!\!1(w) = 0\,.$$

Then

$$\lim_{S\to\infty} \|\mathcal{L}_t 1\!\!1_{\mathbb{D}_S^*}\|_\infty = 0\,.$$

Proof. Let $\varepsilon > 0$ and let $R_\varepsilon > e^\gamma$ be so large that $\mathcal{L}_t 1\!\!1(w) < \varepsilon$ for all $w \in \Omega$ with $|w| > R_\varepsilon$. Then clearly $\mathcal{L}_t 1\!\!1_{\mathbb{D}_S^*}(w) \le \varepsilon$ for every $S > 0$ and every $w \in \Omega$ with $|w| > R_\varepsilon$.

We are left to consider points

$$w \in \overline{\Omega} \ \text{ with } |w| \le R_\varepsilon.$$

The set $\overline{\Omega} \cap \overline{\mathbb{D}}_{e^{R_\varepsilon}}$ is compact and thus admits a finite covering by δ–disks with centers in $\overline{\Omega}$. Let $w_1, ..., w_N$ be these centers. Bounded distortion implies that there exists $K < \infty$ such that, for every $S > 0$,

$$\mathcal{L}_t 1\!\!1_{\mathbb{D}_S^*}(w) \le K \mathcal{L}_t 1\!\!1_{\mathbb{D}_S^*}(w_j) \quad \text{for every} \quad w \in \mathbb{D}(w_j, \delta)\ ,\ j = 1, ..., N\,.$$

By its very definition, $\mathcal{L}_t 1\!\!1_{\mathbb{D}_S^c}(w)$ takes into account only the preimages $z \in f^{-1}(w)$ for which $|z| \ge S$. Since $\mathcal{L}_t 1\!\!1(w)$ is convergent, for every $w \in \Omega$, there exists $S > 0$ such that $\mathcal{L}_t 1\!\!1_{\mathbb{D}_S^*}(w_j) < \varepsilon/K$ for every $j = 1, ..., N$. This shows Proposition 8.7. □

9. Thermodynamics: Bowen's Formula

Let $f \in \mathcal{D}$ have negative spectrum. The pressure function introduced in the previous section along with its properties established in Theorem 8.1 (2) allows us to provide a closed formula for the Hausdorff dimension of the radial Julia set of f. This quantity is called the *hyperbolic dimension* of f, is denoted by $\mathrm{HypDim}(f)$ and is also known (see [36]) to be the supremum of all Hausdorff dimensions of the hyperbolic subsets of $\mathcal{J}_f$. Here is a reformulation of Theorem 1.3.

Theorem 9.1 (Bowen's Formula)**.** *Let $f \in \mathcal{D}$ have negative spectrum. Then, the function $(\Theta_f, +\infty) \ni t \mapsto \mathrm{P}(t)$ has a (unique) zero $h > \Theta_f$ if and only if $\mathrm{HypDim}(f) > \Theta_f$. In this case we have*

$$\mathrm{HypDim}(f) = h\,.$$

Proof. Since $f \in \mathcal{D}$ is a function with negative spectrum the thermodynamic formalism of Section 8 applies. In particular, for every $t > \Theta_f$ there exists an $e^{\mathrm{P}(t)}|f'|_1$–conformal measure. If for some $h > \Theta_f$ we have $\mathrm{P}(h) = 0$, then the corresponding conformal measure is frequently called geometric conformal measure, i.e. $e^{\mathrm{P}(h)} = 1$. The proof of Theorem 1.2 in [23] then applies yielding $\mathrm{HypDim}(f) = h (> \Theta_f)$.

Conversely, if $\mathrm{HypDim}(f) > \Theta_f$, then (see [36]) there exists a hyperbolic (compact) set $X \subset \mathcal{J}_f$ such that $\mathrm{HD}(X) > \Theta_f$. Then, see [35] for ex., $\mathrm{P}(f|_X, -\mathrm{HD}(X)\log|f|'_X|) = 0$. Therefore, $\mathrm{P}(\mathrm{HD}(X)) \ge \mathrm{P}(f|_X, -\mathrm{HD}(X)\log|f|'_X|) = 0$. In conjunction with Theorem 8.1 (2) this implies that there exists $h \ge \mathrm{HD}(X)$ such that $\mathrm{P}(h) = 0$. As $\mathrm{HD}(X) > \Theta_f$, the proof is complete. □

The issue yielded by Theorem 9.1 is to be able to tell whether $\mathrm{HypDim}(f) > \Theta_f$. We know that there is a quite general class of functions for which this holds. Indeed, Barański, Karpińska and Zdunik showed in [3] that $\mathrm{HypDim}(f) > 1$ for every function $f \in \mathcal{D}$. Along with Theorem 9.1 this implies the following.

Proposition 9.2. *Let $f \in \mathcal{D}$ have negative spectrum with $\Theta_f \leq 1$. Then, the pressure function $(\Theta_f, +\infty) \ni t \mapsto \mathrm{P}(t)$ of f has a unique zero, call it h, and*

$$\mathrm{HypDim}(f) = h\,.$$

The problem when $\Theta_f > 1$ is that then the Θ_f–Hausdorff measure of the boundary of a tract may be zero. If this is quantitatively not the case in the sense that $\varphi_T(I)$ has Θ_f–measure greater than some strictly positive constant and if the tract is Hölder then the hyperbolic dimension can be estimated like in [3]. This has been observed and worked out for a family of examples in Proposition 4.3 of [22]. The model functions of the Proposition 4.3 in [22] are all the property $\mathrm{HypDim}(f) > \Theta_f$.

In general we can use our optimal estimates for the transfer operator in order to show directly that the pressure function has a zero. Here, it is done for a large class of linearizers. Let us mention again that [13] contains a more general version of this result.

Proposition 9.3. *Let $p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a hyperbolic polynomial with connected Julia set. Let $z_0 \in \mathcal{J}_p$ be a repelling fixed point of p and let f be an associated linearizer of disjoint type. Then, the pressure function $(\Theta_f, +\infty) \ni t \mapsto \mathrm{P}(t)$ of f has a zero which we denote by h. In consequence,*

$$\mathrm{HypDim}(f) = h\,.$$

Proof. We are to estimate $\mathcal{L}_t \mathbb{1}$ for $t > \Theta_f$ near Θ_f. It suffices to show that there exists $R > 1$ such that

$$\mathcal{L}_t \mathbb{1}_{\mathbb{D}_R}(w) \geq 2 \quad \text{for all} \quad w \in \mathbb{D}_R \cap \mathcal{J}_f \tag{9.1}$$

since then it follows by induction that $\mathcal{L}_t^n \mathbb{1}(w) \geq 2^n$ for all $w \in \mathbb{D}_R \cap \mathcal{J}_f$; thus that $\mathrm{P}(t) \geq \log 2 > 0$. This, along with Theorem 8.1 (2) entails the existence of a unique zero $h > \Theta_f$.

For the special type of functions f we consider here we have the estimate from Theorem 7.9:

$$\mathcal{L}_t \mathbb{1}(w) \asymp (\log |w|)^{1-t},$$

With the notations of the proof of Theorem 7.9, let

$$A_n := \{z \in \mathbb{C} : |\lambda|^n r < |z| \leq |\lambda|^{n+1} r\} \quad \text{and} \quad R_n = |\lambda|^{n+1} r\,, \qquad n \geq 0\,.$$

Then for every integer $M \geq 1$ we have that

$$\mathcal{L}_t(\mathbb{1}_{\mathbb{D}_{R_M}})(w) = \sum_{n=0}^{M} \sum_{z \in f^{-1}(w) \cap A_n} |f'(z)|_1^{-t}\,.$$

Because of (7.25) and (7.23) we see that the sum over the preimages of w lying in A_n is approximately

$$(\log|w|)^{1-t} \sum_{\eta \in p^{-N}(\xi)\cap V_0} \left| \left(p^N\right)'(\eta) \right|^{-t} \tag{9.2}$$

where $N = n - N_w$. Since the polynomial $p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is hyperbolic, it is well known that the sum of (9.2) is approximately $\lambda_{t,p}^N$ where $\log \lambda_{t,p}$ is the topological pressure of p evaluated at t. Also, $t > \Theta_f = \mathrm{HypDim}(p) = \mathrm{HD}(\mathcal{J}_p)$ because of Theorem 7.8 and again since p is hyperbolic. In particular, $\lambda_{t,p} < 1$ for all $t > \Theta_f$ and $\lim_{t\to\Theta_f} \lambda_{t,p} = \lambda_{\Theta_f,p} = 1$. It follows that

$$\mathcal{L}_t(1\!\!1_{\mathbb{D}_{R_M}})(w) \asymp (\log|w|)^{1-t} \frac{1-\lambda_{t,p}^{M+1}}{1-\lambda_{t,p}} \asymp (\log|w|)^{1-t} M,$$

where the latter comparability holds if we make the particular choice $M = M(t) := \frac{2}{\log \lambda_{t,p}}$. So, if $w \in \mathbb{D}_{R_M} \cap \mathcal{J}_f$ then

$$\mathcal{L}_t(1\!\!1_{\mathbb{D}_{R_M}})(w) \succeq M(t)^{2-t}.$$

Since $\lim_{t\searrow\Theta_f} M(t) = +\infty$, we immediately see that there exists $t > \Theta_f$ such that (9.1) holds with $R = R_{M(t)}$. The proof is complete. □

Proof of Theorem 1.6. This theorem is an immediate consequence of Theorem 7.8, Corollary 8.3, Theorem 9.1 and Theorem 9.11 of [26] (our present theorem is a very special case of Theorem 9.11 as we now consider only deterministic systems). □

10. Functions of Class $\mathcal{S}$

We finally consider functions of the Speiser class $\mathcal{S}$. For this more restrictive class the present theory of thermodynamic formalism can be extended in a straightforward way to hyperbolic functions. This section contains also the promised second application of quasiconformal invariance of Hölder tracts in Proposition 10.1 and provide a prove for Theorem 1.4 of the Introduction.

10.1. **Hyperbolic Functions in Class $\mathcal{S}$.** The object here is to explain how to passe from the disjoint type case to hyperbolic functions. In order to do so, let us consider a function f having negative spectrum and the properties of class $\mathcal{D}$ excepted the disjoint type property. Instead, f is assumed hyperbolic and of class $\mathcal{S}$.

We assume as usual that $S(f) \subset \mathbb{D}$, that $0 \in \mathcal{F}(f)$ and we fix arbitrarily $\gamma > 0$. Then, exactly as in the disjoint type case, the conclusions of Theorem 4.1 hold for every $|w| > e^\gamma$. We are thus to consider only points

$$w \in \overline{\mathbb{D}}_{e^\gamma} \cap \mathcal{J}_f$$

We will compare $\mathcal{L}_t 1\!\!1(w)$ to $\mathcal{L}_t 1\!\!1(\xi)$ where $\xi \in \mathbb{C}\setminus\overline{\mathbb{D}}_{e^\gamma}$ is an arbitrarily fixed point. This goes exactly as in [10, Section 10] (and this is the only point where class $\mathcal{S}$ rather than merely class $\mathcal{B}$ is needed) by employing the bounded distortion argument. Indeed, it suffices to connect w to ξ by a piecewise smooth path σ of Euclidean length uniformly bounded above with respect two points $w \in \overline{\mathbb{D}}_{e^\gamma}$ and such that for some fixed $\delta > 0$, the δ–neighborhood of σ does not intersect $S(f)$. Let us recall from [24, Section 4.2] that there exists good distortion estimates for $|(f^n)'|_1$. In conclusion, all of this shows that Theorem 1.2 holds for f.

10.2. **Analytic families of class $\mathcal{S}$.** Two entire functions f and g are (topologically) equivalent if there exist two homeomorphisms $\Phi, \Psi : \mathbb{C} \to \mathbb{C}$ such that

$$\Psi \circ g = f \circ \Phi \,. \tag{10.1}$$

Given $g \in \mathcal{S}$, Eremenko and Lyubich [15] showed that the set $\mathcal{M}_g$ of all functions $f \in \mathcal{S}$ equivalent to g has a natural structure of a complex analytic manifold. It can be parametrized by the singular values $\{a_1, ..., a_q\} = S(g)$ of g.

Proposition 10.1. *Let $g \in \mathcal{S}$ be a function having finitely many tracts all of which are Hölder. Then all tracts of every function $f \in \mathcal{M}_g$ are Hölder, and thus all functions of $\mathcal{M}_g$ have negative spectrum.*

The proof of this fact relies on a special choice of homeomorphisms in the equivalence relation (10.1). In fact, they can be freely chosen in an isotopy class without changing f, g and, in particular, there is a quasiconformal choice for these homeomorphisms (see again [15]). We first shall prove the following.

Lemma 10.2. *Let $g \in \mathcal{S}$. If $f \in \mathcal{M}_g$, then the homeomorphism Ψ in* (10.1) *can be chosen quasiconformal and such that, for some $R \geq 1$,*

$$\Psi|_{\mathbb{D}_R^*} \equiv \mathrm{Id}|_{\mathbb{D}_R^*} \,.$$

Proof. Assume without loss of generality that $S(g) = \{a_1, ...a_q\} \subset \mathbb{D}$. Let $a_{q+1}, a_{q+2} \in \mathbb{D} \setminus S(g)$ be two arbitrary distinct points and let $\psi : \mathbb{C} \to \mathbb{C}$ be a quasiconformal homeomorphism such that (10.1) holds for the given maps $g \in \mathcal{S}$ and $f \in \mathcal{M}_g$. A standard application of the Ahlfors-Bers-Bojarski Measurable Riemann Mapping Theorem is that ψ^{-1} can be embedded into a holomorphic motion of quasiconformal mappings $\psi_\lambda^{-1} : \mathbb{C} \to \mathbb{C}$, $\lambda \in \mathbb{D}$. In particular, $\psi_0^{-1} = \mathrm{Id}$ and $\psi_{\lambda_0}^{-1} = \psi^{-1}$ for some $\lambda_0 \in \mathbb{D}$. Define R to be any number

$$\geq 2 \max \big\{ \mathrm{diam}(\psi_\lambda(\mathbb{D})) : |\lambda| \leq |\lambda_0| \big\},$$

and consider the holomorphic motion $z \mapsto z_\lambda$ defined on $\psi(\mathbb{D} \cup \mathbb{D}_R^*)$ by

$$z_\lambda = z \text{ if } z \in \psi(\mathbb{D}) \quad \text{and} \quad z_\lambda = \psi_\lambda^{-1}(z) \text{ if } z \in \psi(\mathbb{D}_R^*).$$

By Slodkovski's extension of Mañé-Sad-Sullivan's λ-Lemma, this holomorphic motion has an extension to a holomorphic motion $h_\lambda : \mathbb{C} \to \mathbb{C}$, $|\lambda| \leq |\lambda_0|$. Consider

$$\Psi_\lambda := h_\lambda \circ \psi \quad , \quad \lambda \in \mathbb{D} \,.$$

The map we look for is Ψ_{λ_0} and $t \mapsto \Psi_{t\lambda_0}$, $t \in [0, 1]$ is a homotopy between $\psi = \Psi_0$ and Ψ_{λ_0} that does not move the points a_j, $j = 1, ..., q+2$; in fact does not move any point of $\mathbb{D}$. Following Section 3 in Eremenko–Lyubich's paper [15] and in particular Lemma 2 of that section, we conclude that there exists $\Phi_{\lambda_0} : \mathbb{C} \to \mathbb{C}$ quasiconformal such that (10.1) holds with Ψ and Φ replaced respectively by Ψ_{λ_0} and Φ_{λ_0}. ∎

Proof of Proposition 10.1. Let g, f be as in Proposition 10.1 and let $R > 1$ so large that the assertion of Lemma 10.2 holds and that all the components of $g^{-1}(\overline{\mathbb{D}}_R^c)$ are Hölder. Then, if Ψ is given by Lemma 10.2 and if Φ is the corresponding quasiconformal map such that (10.1) holds, then clearly Φ identifies the components of $g^{-1}(\overline{\mathbb{D}}_R^*)$ with those of $f^{-1}(\overline{\mathbb{D}}_R*)$. Proposition 10.1 follows now from Lemma 6.1. □

10.3. **Proof of Theorem 1.4 and of Corollary 1.5.** If g is as in Theorem 1.4, then Proposition 10.1 yields that every $f \in \mathcal{M}_g$ has Hölder tracts and negative spectrum. Therefore, Theorem 1.2 applies first to any disjoint type map of $\mathcal{M}_g$ and then also to every hyperbolic function of this family $\mathcal{M}_g$ because of the argument of Section 10.1. This proves Theorem 1.4. □

Let now $g \in \mathcal{S}$ be a linearizer of a polynomial p with connected Julia set. The singular set $S(g)$ equals the post-singular set of the polynomial p (see [28]). Since $g \in \mathcal{S}$, p must be post-critically finite hence TCE. On the other hand, we may assume that g is of disjoint type so that it satisfies the conclusion of Lemma 7.2. Otherwise it suffices to replace it by $g \circ \kappa \in \mathcal{M}_g$ with sufficiently small $\kappa \neq 0$. It thus follows from Proposition 7.4 that $g \in \mathcal{D}$ and that g has finitely many Hölder tracts. It suffices now to apply Theorem 1.4 in order to complete the proof of Corollary 1.5. □

Appendix A. Integral means spectrum and pressure

For the sake of completeness we provide here the details related to formula (7.16) in the settings of the proof of Theorem 7.8. Let again $p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a polynomial with connected Julia set and let $h : \mathbb{D}^* \to A_p(\infty)$ be a Riemann map such that

$$h \circ D = p \circ h \quad \text{on} \quad \mathbb{D}^* \quad \text{where} \quad D(z) = z^d \, . \tag{A.1}$$

Suppose we are given a constant $c > 0$ and circular arcs $C_r \subset \{|z| = r\}$ with

$$\operatorname{diam}(C_r) \geq c \quad , \quad r > 1 \, .$$

Define

$$\hat{\beta}_h(t) = \limsup_{r \to 1^+} \frac{\log \int_{C_r} |h'(z)|^t |dz|}{|\log(r-1)|} \, .$$

Consider also the tree pressure

$$\mathrm{P}(t, w) := \limsup_{n \to \infty} \frac{1}{n} \log \sum_{z \in p^{-n}(w)} |(p^n)'(z)|^{-t} \quad , \; t > 0 \;\text{ and }\; w \in A_p(\infty) \, .$$

It has been shown in [31] that this expression does not depend on w. More precisely, Przytycki has shown that $\mathrm{P}(t, w)$ is the same value for every typical $w \in \mathbb{C}$. Since now the polynomial p is assumed to have connected Julia set, every point of $A_p(\infty)$ is typical in the sense of [31]. We can therefore write $\mathrm{P}(t)$ for $\mathrm{P}(t, w)$, for any $w \in A_p(\infty)$.

Proposition A.1. *If $p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a polynomial with connected Julia set and $d \geq 2$ denotes its degree then,*

$$\hat{\beta}_h(t) = \beta_h(t) = \frac{\mathrm{P}(t)}{\log d} + t - 1$$

for every $t \geq 0$.

We adapt the proof given in [35]. There, the second equality is shown in the case of expanding polynomials.

Proof. Given $1 < r < 2$, there exists a unique integer $n \geq 1$ such that $R_r = D^n(r) \in [2, 2^d[$. Obviously $r - 1 \asymp d^{-n}$ and the (arcwise) distance between two consecutive elements of $D^{-n}(R_r)$ is $2\pi r/d^n \asymp d^{-n}$. Therefore, applying Koebe's Distortion Theorem, we get the following.

$$\int_{C_r} |h'(\xi)|^t |d\xi| \asymp d^{-n} \sum_{\xi \in C_r \cap D^{-n}(R_r)} |h'(\xi)|^t.$$

Iterating the functional equation (A.1) and taking derivatives gives

$$h'(R_r)(D^n)'(\xi) = (p^n)'(h(\xi))h'(\xi)$$

for all $\xi \in D^{-n}(R_r)$. Hence, we get for such ξ that

$$|h'(\xi)| \asymp d^n |(p^n)'(h(\xi))|^{-1}.$$

Thus

$$\text{(A.2)} \qquad \begin{aligned} \int_{C_r} |h'(\xi)|^t |d\xi| &\asymp d^{n(t-1)} \sum_{\xi \in C_r \cap D^{-n}(R_r)} |(p^n)'(h(\xi))|^{-t} \\ &= d^{n(t-1)} \sum_{z \in h(C_r) \cap p^{-n}(w_r)} |(p^n)'(z)|^{-t}, \quad w_r = h(R_r). \end{aligned}$$

Since the constant $c > 0$ from the definition of the circular arcs C_r does not depend on $r > 1$, there exists an integer $M \geq 1$ (in fact every integer M large enough is good) such that

$$D^M(C_r) = \{|\xi| = r^{d^M}\}$$

for every $r > 1$. If r is sufficiently close to 1 then $n > M$. Then

$$h(C_r) \cap p^{-n}(w_r) = h(C_r) \cap p^{-M}(p^{-(n-M)}(w_r)).$$

Obviously there exists $R \in (0, +\infty)$ such that

$$p^{-(n-M)}(w_r) \subset \mathbb{D}_R$$

for all $1 < r < 2$. Since

$$K := \sup \left\{ |(p^M)(v)| : v \in \mathbb{D}_R \right\} < \infty,$$

we thus have that

$$\sum_{z\in h(C_r)\cap p^{-n}(w_r)} |(p^n)'(z)|^{-t} \geq K^{-t} \sum_{\xi\in p^{-(n-M)}(w_r)} |(p^{n-M})'(\xi)|^{-t}$$
$$\asymp \sum_{\xi\in p^{-(n-M)}(h(2))} |(p^{n-M})'(\xi)|^{-t}.$$

Therefore

$$\int_{C_r} |h'(\xi)|^t |d\xi| \succeq d^{n(t-1)} \sum_{\xi\in p^{-(n-M)}(h(2))} |(p^{n-M})'(\xi)|^{-t},$$

from which immediately follows that

$$\hat{\beta}_h(t) \geq t-1+\frac{\mathrm{P}(t)}{\log d}.$$

On the other hand, $\hat{\beta}_h(t) \leq \beta_h(t)$ and, arguing exactly as before but with C_r replaced by the full circle $\{|\xi| = r\}$ and skipping the, not needed now, mixing argument based on the existence of the integer M, it follows that

$$\beta_h(t) \geq t-1+\frac{\mathrm{P}(t)}{\log d}.$$

□

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