Conformal graph directed Markov systems on Carnot groups

Vasilis Chousionis

Jeremy T. Tyson

Mariusz Urbański

Department of Mathematics, University of Connecticut, 196 Auditorium Road U-3009, Storrs, CT 06269-3009
E-mail address: vasileios.chousionis@uconn.edu

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801
E-mail address: tyson@illinois.edu

Department of Mathematics, University of North Texas, General Academics Building 435, 1155 Union Circle 311430, Denton, TX 76203-5017
E-mail address: urbanski@unt.edu

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Abstract. We develop a comprehensive theory of conformal graph directed Markov systems in the non-Riemannian setting of Carnot groups equipped with a sub-Riemannian metric. In particular, we develop the thermodynamic formalism and show that, under natural hypotheses, the limit set of a Carnot conformal GDMS has Hausdorff dimension given by Bowen's parameter. We illustrate our results for a variety of examples of both linear and nonlinear iterated function systems and graph directed Markov systems in such sub-Riemannian spaces. These include the Heisenberg continued fractions introduced by Lukyanenko and Vandehey as well as Kleinian and Schottky groups associated to the non-real classical rank one hyperbolic spaces.

Contents

Introduction

In this monograph we lay the foundation for a theory of conformal dynamical systems in nilpotent stratified Lie groups (Carnot groups) equipped with a sub-Riemannian metric. In particular, we develop a thermodynamic formalism for conformal graph directed Markov systems which permits us to identify the Hausdorff dimension of the limit set as the zero of a suitable pressure function. We also consider the structure of conformal measures and investigate when the Hausdorff and packing measures of the limit set are either positive or finite. Finally, we extend to the sub-Riemannian context a result, first proved by Schief and later extended by Peres–Rams–Simon–Solomyak, on the equivalence between the open set condition, strong open set condition, and positivity of the Hausdorff measure of the limit set. The well established theory of Euclidean conformal graph directed Markov systems (see [**46**]) serves as a model for our investigations.

We formulate the preceding theory in the general setting of a Carnot group equipped with a left-invariant homogeneous metric. However, our results are of particular interest in the special situation of Iwasawa groups. Such groups, a particular class of Carnot groups of step at most two, arise as boundaries at infinity of the classical rank one symmetric spaces, or alternatively as nilpotent components in the Iwasawa decomposition of real rank one simple Lie groups.

Provided the dimension of the ambient Iwasawa group is at least three, a version of Liouville's rigidity theorem holds: every locally defined conformal self-map is the restriction of a Möbius map, acting on the (conformally equivalent) one-point compactification equipped with a suitable spherical metric. As in the Euclidean case, the space of Möbius maps is finite-dimensional, and may be identified with a group of matrices acting by isometries on the corresponding hyperbolic space. The action of this group is sufficiently rich, for instance, it is two-point transitive with nontrivial stabilizer subgroups.

A recent rigidity theorem of Cowling and Ottazzi (see Theorem 2.13) asserts that every conformal mapping defined on domains in a Carnot group which is not of Iwasawa type is necessarily an affine similarity mapping. Moreover, the theory which we develop applies to families of contractive similarities of any homogeneous metric on any Carnot group, which need not be conformal, see also Remark 4.8. Our results therefore encompass two settings, graph directed Markov systems (GDMS) consisting of either contractive similarities or contractive conformal mappings, in any Carnot group. We stress that a number of our results are new in the setting of similarity GDMS in general Carnot groups, especially in the case of countably infinite systems. Finite self-similar iterated function systems in Carnot groups have previously been studied in detail by the second author and his collaborators [**4**], [**7**], [**9**]. The results of the present monograph apply equally well either to finite or countably infinite self-similar iterated function systems in arbitrary Carnot groups.

See Chapter 5 for further information. Moreover, all of the results of this monograph apply to non-affine conformal GDMS in Iwasawa groups; such a setting is entirely new.

We briefly indicate the structure of this monograph. Chapter 1 briefly reviews the algebraic, metric and geometric structure of general Carnot groups and the primary morphisms of interest: the contact mappings. In Chapter 2 we describe in detail the class of Iwasawa Carnot groups which plays a particularly important role throughout this monograph. We also recall the definition of conformal mappings in Carnot groups, and describe both the classification of conformal maps on Iwasawa groups as well as the Cowling–Ottazzi rigidity result mentioned above. In Chapter 3 we develop a series of metric and geometric properties of conformal mappings. Throughout the remainder of this monograph, we will primarily only make use of such properties in our consideration of Carnot conformal GDMS.

Chapter 4 introduces abstract graph directed Markov systems, as well as conformal graph directed Markov systems in Iwasawa groups. A brief interlude (Chapter 5) describes several examples of Iwasawa conformal GDMS. In the self contained Chapter 6 we provide a detailed treatment of thermodynamic formalism in the context of countable alphabet dynamics. In Chapters 7 and 8 we employ the thermodynamic formalism from Chapter 6 and the distortion estimates obtained in Chapter 3 to establish formulas for the Hausdorff and packing dimensions of invariant sets of Iwasawa conformal GDMS. Moreover we investigate the structure of conformal measures on such sets. We revisit the examples of Chapter 5 in Chapter 9, where we apply the results of earlier chapters to compute or estimate the dimensions of several invariant sets. In Chapter 10 we investigate finer properties of limit sets. We study the finiteness or positivity of Hausdorff or packing measure on the invariant set. We also extend a dynamical formula for the Hausdorff dimension of invariant measures, known as the *Volume Lemma*, in the context of Carnot conformal GDMS. In Chapter 11 we establish the equivalence between the open set condition, the strong open set condition, and the positivity of the Hausdorff measure of the limit set of finite Carnot conformal GDMS, extending a well known theorem of Peres, Rams, Simon and Solomyak [**51**].

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CHAPTER 1

Carnot groups

In this chapter we introduce Carnot groups and their sub-Riemannian geometry. We discuss the differential geometric, metric, and measure-theoretic structure of such groups. We also describe the class of contact mappings of a Carnot group, that is, mappings which preserve the inherent stratified structure. Typical examples of contact mappings include left translations, dilations, and automorphisms, which together generate the family of orientation-preserving similarities. We recall one of the most basic examples of a Carnot group: the (complex) Heisenberg group. In the following chapter we introduce the so-called Iwasawa groups, which comprise the Heisenberg groups over the complex numbers, over the quaternions and over the Cayley numbers. Each Iwasawa group is endowed with a natural conformal inversion mapping which ensures that the full class of conformal self-maps is larger than the group of similarities.

We conclude this chapter by reviewing the Dimension Comparison Theorem which relates the spectra of Hausdorff measures in a Carnot group defined with respect to either a sub-Riemannian metric or a Euclidean metric. The Dimension Comparison Theorem will be used later to deduce Euclidean dimension estimates for limit sets of conformal graph directed Markov systems in Carnot groups.

1.1. Carnot groups

A *Carnot group* is a connected and simply connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_\iota \tag{1.1}$$

into vector subspaces satisfying the commutation rules

$$[\mathfrak{v}_1, \mathfrak{v}_i] = \mathfrak{v}_{i+1} \tag{1.2}$$

for $1 \le i < \iota$ and $[\mathfrak{v}_1, \mathfrak{v}_\iota] = (0)$. In particular, the full Lie algebra is generated via iterated Lie brackets of elements of the lowest layer $\mathfrak{v}_1$ of the stratification. This layer is known as the *horizontal layer* of the Lie algebra, and elements of $\mathfrak{v}_1$ are known as *horizontal tangent vectors*. The integer ι is known as the *step* of the group $\mathbb{G}$.

We identify the elements of $\mathfrak{g}$ as either tangent vectors to $\mathbb{G}$ at the neutral element o, or as left invariant vector fields on $\mathbb{G}$. This identification is obtained as follows: any left invariant vector field X on $\mathbb{G}$ defines an element $X(o)$ in $T_o\mathbb{G}$, while any vector $v \in T_o\mathbb{G}$ induces a left invariant vector field X by the rule $X(q) := (\ell_q)_*(v)$ for all $q \in \mathbb{G}$, where $\ell_q : \mathbb{G} \to \mathbb{G}$, $\ell_q(p) = q * p$, denotes left translation by q. In this way, the horizontal layer $\mathfrak{v}_1$ corresponds to a distribution $H\mathbb{G}$ in the tangent bundle $T\mathbb{G}$ given by the rule $H_p\mathbb{G} = \{X(p) : X \in \mathfrak{v}_1\}$. The distribution $H\mathbb{G}$ is known as the *horizontal distribution*. The bracket generating condition (1.2) implies that $H\mathbb{G}$ is completely nonintegrable. According to a fundamental theorem of Chow and Rashevsky, see e.g. [**47**], the complete nonintegrability of $H\mathbb{G}$ ensures that any two points of $\mathbb{G}$ can be joined by a horizontal curve, i.e., a piecewise smooth curve γ such that $\gamma'(s) \in H_{\gamma(s)}\mathbb{G}$ whenever $\gamma'(s)$ is defined.

Since $\mathbb{G}$ is connected, simply connected and nilpotent, the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a global diffeomorphism and $\mathbb{G}$ is naturally identified with a Euclidean space $\mathfrak{g} = \mathbb{R}^N$. Nilpotence implies that the group law on $\mathbb{G}$ is given by polynomial operations in the coordinates of the corresponding Euclidean space $\mathbb{R}^N$. The precise formula for the group law can be derived from the *Baker–Campbell–Hausdorff–Dynkin formula*

$$\exp(U) * \exp(V) = \exp(U + V + \tfrac{1}{2}[U,V] + \tfrac{1}{12}([U,[U,V]] + [V,[V,U]]) + \cdots), \tag{1.3}$$

valid for $U, V \in \mathfrak{g}$. (See, for instance, [**12**, Theorem 2.2.13].) Note that nilpotence ensures that the series occurring on the right hand side of (1.3) is a finite sum. For instance, in the case when $\mathbb{G}$ has step two, (1.3) reads as

$$\exp(U) * \exp(V) = \exp(U + V + \tfrac{1}{2}[U,V]). \tag{1.4}$$

Introducing *exponential coordinates of the first kind* $p = (U_1, U_2)$, where $p = \exp(U_1 + U_2)$ with $U_1 \in \mathfrak{v}_1$ and $U_2 \in \mathfrak{v}_2$, we deduce from (1.4) that

$$p * q = (U_1 + V_1, U_2 + V_2 + \tfrac{1}{2}[U_1, V_1]) \tag{1.5}$$

if $p = (U_1, U_2)$ and $q = (V_1, V_2)$.

For $r > 0$ we define the *dilation* with *scale factor* r, δ_r, to be the automorphism of $\mathfrak{g}$ which is given on the subspace $\mathfrak{v}_i$ by the rule $\delta_r(X) = r^i X$, $X \in \mathfrak{v}_i$. Conjugation with the exponential map transfers this map δ_r to an automorphism of the group $\mathbb{G}$ which we continue to call a *dilation* and continue to denote by δ_r, thus

$$\delta_r(p) = \exp(\delta_r(\exp^{-1}(p))). \tag{1.6}$$

1.1.1. Homogeneous metrics. Each Carnot group can be equipped with a geodesic metric, the *Carnot–Carathéodory metric* d_{cc}. To define this metric, we fix an inner product $\langle \cdot, \cdot \rangle$ on the horizontal layer $\mathfrak{v}_1$ of the Lie algebra, which we promote to a left invariant family of inner products via left invariance of the elements of $\mathfrak{g}$. Relative to this inner product, we fix an orthonormal basis of vector fields $X_1, \ldots, X_m \in \mathfrak{v}_1$, and we define the *horizontal norm* of a vector $v \in H_p\mathbb{G}$ to be

$$|v|_{p,\mathbb{G}} = \left(\sum_{j=1}^m \langle v, X_j(p) \rangle^2 \right)^{1/2}.$$

The *horizontal length* $\ell_{cc}(\gamma)$ of a horizontal curve $\gamma : [a,b] \to \mathbb{G}$ is computed by integrating the horizontal norm of the tangent vector field along γ:

$$\ell_{cc}(\gamma) = \int_a^b |\gamma'(s)|_{\gamma(s),\mathbb{G}}\, ds.$$

Taking the infimum of $\ell_{cc}(\gamma)$ over all horizontal curves γ joining p to q defines the *Carnot–Carathéodory distance* $d_{cc}(p,q)$. It is well known that d_{cc} is a geodesic metric on $\mathbb{G}$. We record that a *CC-geodesic* connecting two points $p, q \in \mathbb{G}$ is a length minimizing horizontal curve $\gamma : [0,T] \to \mathbb{G}$, such that $\gamma(0) = p$, $\gamma(T) = q$.

Explicit formulas for the Carnot–Carathéodory metric are difficult to come by and are known only in very specific cases. For many purposes it suffices to consider any bi-Lipschitz equivalent metric on $\mathbb{G}$. A large class of such metrics is given by the homogeneous metrics. A metric d on $\mathbb{G}$ is said to be *homogeneous* if $d : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ is continuous with respect to the Euclidean topology, is left invariant and is 1-homogeneous with respect to the dilations $(\delta_r)_{r>0}$. The 1-homogeneity of d means that

$$d(\delta_r(p), \delta_r(q)) = r\, d(p,q)$$

for all $p, q \in \mathbb{G}$ and all $r > 0$. We note in particular that the Carnot–Carathéodory metric d_{cc} is a homogeneous metric. The Korányi (gauge) metric on an Iwasawa group (defined in subsection 2.1.4) provides another example of a homogeneous metric. Any two homogeneous metrics d_1 and d_2 on a given Carnot group $\mathbb{G}$ are equivalent in the sense that there exists a constant $L > 0$ so that

$$L^{-1} d_1(p,q) \le d_2(p,q) \le L d_1(p,q) \tag{1.7}$$

for all $p, q \in \mathbb{G}$; this is an easy consequence of the assumptions.

1.1.2. Haar measure and Hausdorff dimension. By a theorem of Mitchell (see also [**47**]), the Hausdorff dimension of $\mathbb{G}$ in any homogeneous metric is equal to the *homogeneous dimension*

$$Q = \sum_{i=1}^{\iota} i \dim \mathfrak{v}_i. \tag{1.8}$$

In all nonabelian examples (i.e., when the step ι is strictly greater than one), we have $Q > N$ where N denotes the topological dimension of $\mathbb{G}$. It follows that the Carnot–Carathéodory metric d_{cc} is never bi-Lipschitz equivalent to any Riemannian metric on the underlying Euclidean space $\mathbb{R}^N$.

We denote the Haar measure of a set E in a Carnot group $\mathbb{G}$ by $|E|$. The Haar measure in $\mathbb{G}$ is proportional to the Lebesgue measure in the underlying Euclidean space $\mathbb{R}^N$. Moreover, there exists a constant c_0 so that for any $p \in \mathbb{G}$ and $r > 0$ we have

$$|B(p,r)| = c_0 r^Q, \tag{1.9}$$

where Q is as in (1.8). One way to see this is to note that $B(o,1)$ is mapped onto $B(p,r)$ by the composition of left translation by p and the dilation δ_r with scale factor $r > 0$. Observe that the Jacobian determinant of the (smooth) map δ_r is everywhere equal to r^Q. It follows that (1.9) holds with $c_0 = |B(o,1)|$.

1.1.3. The Heisenberg group. We briefly record a simple example of a nonabelian Carnot group: the first (complex) Heisenberg group. This group is the lowest-dimensional example in a class of groups, the Iwasawa groups, which we will describe in detail in the following chapter.

The underlying space for the first Heisenberg group **Heis** is $\mathbb{R}^3$, which we also view as $\mathbb{C} \times \mathbb{R}$. We endow $\mathbb{C} \times \mathbb{R}$ with the group law

$$(z;t) * (z';t') = (z + z'; t + t' + 2\,\mathrm{Im}(z\overline{z'})),$$

where we denote elements of **Heis** by either $(z;t) \in \mathbb{C} \times \mathbb{R}$ or $(x,y;t) \in \mathbb{R}^3$. The identity element in **Heis** is the origin in $\mathbb{R}^3$, and the group inverse of $p \in$ **Heis** coincides with its Euclidean additive inverse $-p$.

The vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t} \quad \text{and} \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t},$$

together with

$$T = \frac{\partial}{\partial t},$$

form a left invariant basis for the tangent bundle of **Heis**. The horizontal bundle H**Heis** is the non-integrable subbundle (*horizontal bundle*) which is spanned, at each point $p \in$ **Heis**, by the values of X and Y at p. Consequently, **Heis** has the structure of a Carnot group of step two, with two-dimensional horizontal space $\mathfrak{v}_1$ and one-dimensional vertical space (center) $\mathfrak{v}_2$. For each $r > 0$ the dilation δ_r of **Heis** takes the form

$$\delta_r(x, y; t) = (rx, ry; r^2 t). \tag{1.10}$$

1.2. Contact mappings

Let Ω and Ω' be domains in a Carnot group $\mathbb{G}$. A diffeomorphism $f : \Omega \to \Omega'$ is said to be *contact* if its differential preserves the horizontal bundle (and hence preserves each stratum in the decomposition (1.1)). More precisely, f is contact if f_{*p} maps $H_p\mathbb{G}$ bijectively to $H_{f(p)}\mathbb{G}$ for each $p \in \Omega$. Since f is a diffeomorphism, its action on vector fields is well defined, and the preceding statement implies that f_* maps $\mathfrak{v}_1$ to itself (recall that $H_p\mathbb{G} = \{X(p) : X \in \mathfrak{v}_1\}$. Moreover, f_* preserves the Lie bracket and hence maps $\mathfrak{v}_j$ to itself for each $j = 1, \ldots, \iota$. Hence when f is contact we obtain

$$f_{*p} : H_p^j\mathbb{G} \to H_{f(p)}^j\mathbb{G} \qquad \forall\, p \in \Omega,\ j = 1, \ldots, \iota,$$

where $H_p^j\mathbb{G} = \{X(p) : X \in \mathfrak{v}_j\}$.

Examples of contact mappings of Carnot groups include left translations, dilations and homogeneous automorphisms. For instance, denoting by ℓ_q the left translation of $\mathbb{G}$ by a point q, $\ell_q(p) = q * p$, we note that $(\ell_q)_{*p}(Y_p) = Y_{q*p}$ for all left invariant vector fields Y and any $p \in \mathbb{G}$. In particular, $(\ell_q)_*$ acts as the identity on each stratum $\mathfrak{v}_j$ of $\mathfrak{g}$. Similarly, the dilation δ_r acts on the level of the Lie algebra as follows:

$$(\delta_r)_{*p}(Y_1 + \cdots + Y_\iota) = r(Y_1)_{\delta_r(p)} + \cdots + r^\iota(Y_\iota)_{\delta_r(p)}, \qquad \text{where } Y_j \in \mathfrak{v}_j,$$

and hence $(\delta_r)_*$ acts on $\mathfrak{v}_j$ as multiplication by r^j. An automorphism of $\mathbb{G}$ is a bijective map L which preserves the Lie group law: $L(p * q) = L(p) * L(q)$. We say that L is homogeneous if it commutes with dilations: $L(\delta_r(p)) = \delta_r(L(p))$ for all $p \in \mathbb{G}$ and $r > 0$. The differential of a homogeneous automorphism is an automorphism of Lie algebras which is again homogeneous and strata-preserving.

EXAMPLE 1.1. In the Heisenberg group **Heis**, the maps

$$R_\theta : \mathbf{Heis} \to \mathbf{Heis}, \qquad R_\theta(z, t) = (e^{\mathbf{i}\theta} z, t), \qquad \theta \in \mathbb{R}$$

are homogeneous automorphisms. The action of the differential of R_θ on the Lie algebra has the matrix representation

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

when expressed in the basis X, Y, T.

1.3. Dimension comparison in Carnot groups

Each Carnot group is equipped with two distinct metric structures: the sub-Riemannian geometry defined by the Carnot–Carathéodory metric and the underlying Euclidean metric geometry. These two metrics are topologically—even bi-Hölder—equivalent, but are not bi-Lipschitz equivalent (except in the trivial case when the group is abelian). It is natural to ask for precise estimates describing the relationship between the Hausdorff dimensions defined by these two metrics. This *Dimension Comparison Problem* was first considered by Balogh, Rickly and Serra-Cassano in the Heisenberg group [5] (see also [7] for a continuation of this line of research) and later by Balogh, Warhurst and the second author in general Carnot groups [8], [9].

We first state the Dimension Comparison Theorem in its most general form (on arbitrary Carnot groups), and then specialize to the case of step two groups. Denote by $\mathbb{G}$ a Carnot group of arbitrary step $\iota \geq 1$, with stratified Lie algebra $\mathfrak{g} = \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_\iota$. Let m_i be the dimension of the vector subspace $\mathfrak{v}_i$ in $\mathfrak{g} \simeq \mathbb{R}^N$. Recall that the topological dimension N and the homogeneous dimension Q of $\mathbb{G}$ satisfy

$$N = \sum_{j=1}^{\iota} m_j$$

and

$$Q = \sum_{j=1}^{\iota} j\, m_j.$$

For convenience, we also set $m_0 = 0$ and $m_{\iota+1} = 0$. We define the *upper* and *lower dimension comparison functions* $\beta_+ = \beta_+^{\mathbb{G}}$ and $\beta_- = \beta_-^{\mathbb{G}}$ as follows:

$$\beta_- : [0, N] \to [0, Q], \qquad \beta_-(\alpha) = \sum_{j=0}^{\ell_-} j\, m_j + (1 + \ell_-) \left(\alpha - \sum_{j=0}^{\ell_-} m_j \right)$$

where $\ell_- = \ell_-(\alpha)$ is the unique integer in $\{0, \ldots, \iota - 1\}$ such that

$$\sum_{j=0}^{\ell_-} m_j < \alpha \leq \sum_{j=0}^{1+\ell_-} m_j,$$

and

$$\beta_+ : [0, N] \to [0, Q], \qquad \beta_+(\alpha) = \sum_{j=\ell_+}^{\iota+1} j\, m_j + (-1 + \ell_+) \left(\alpha - \sum_{j=\ell_+}^{\iota+1} m_j \right)$$

where $\ell_+ = \ell_+(\alpha)$ is the unique integer in $\{2, \ldots, \iota + 1\}$ such that

$$\sum_{j=\ell_+}^{\iota+1} m_j < \alpha \leq \sum_{j=-1+\ell_+}^{\iota+1} m_j.$$

The integers ℓ_+ and ℓ_- can be interpreted as weighted versions of the usual greatest integer function $\lfloor x \rfloor$, $x \in \mathbb{R}$. The integer $\ell_-(\alpha)$ is the largest number of layers of the Lie algebra, starting from the lowest layer $\mathfrak{v}_1$, for which the cumulative dimension $\sum_{j=0}^{\ell} m_j$ is less than α. The integer $\ell_+(\alpha)$ has a similar interpretation, starting from the highest layer $\mathfrak{v}_\iota$. The dimension comparison functions $\beta_\pm(\alpha)$ each involve two terms, one of which gives the sum of the weighted dimensions of the relevant subspaces $\mathfrak{v}_j$ determined by the value of $\ell_\pm(\alpha)$, and the other of which gives the fractional value of the weighted dimension of the 'boundary' subspace $\mathfrak{v}_{1+\ell_-}$ or $\mathfrak{v}_{-1+\ell_+}$. Note that the formulas for the dimension comparison functions $\beta_\pm$ involve only the dimensions of the subspaces $\mathfrak{v}_j$, and do not depend in any way on the precise algebraic relationships (e.g., commutation relations) involving vector fields determining bases for these subspaces.

In the special case of step two groups (which will be of primary interest), these formulas simplify as follows. Let $\mathbb{G}$ be a step two Carnot group with topological dimension $N = m_1 + m_2$ and homogeneous dimension $Q = m_1 + 2m_2$, where $\mathfrak{g} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$ and $m_j = \dim \mathfrak{v}_j$. The dimension comparison functions $\beta_\pm$ satisfy

$$\beta_-(\alpha) = \begin{cases} \alpha, & \text{if } 0 \le \alpha \le m_1, \\ 2\alpha - m_1, & \text{if } m_1 \le \alpha \le N, \end{cases} \tag{1.11}$$

and

$$\beta_+(\alpha) = \begin{cases} 2\alpha, & \text{if } 0 \le \alpha \le m_2, \\ \alpha + m_2, & \text{if } m_2 \le \alpha \le N. \end{cases} \tag{1.12}$$

Note that β_- and β_+ are continuous and strictly increasing piecewise linear functions from $[0,N]$ to $[0,Q]$ satisfying the symmetry relation $\beta_+(N-\alpha) = Q - \beta_-(\alpha)$, $0 \le \alpha \le N$. (These facts are true for the dimension comparison functions of Carnot groups of arbitrary step.)

For instance, in the first Heisenberg group **Heis**, where $\iota = 2$, $m_1 = 2$ and $m_2 = 1$, we have

$$\beta_-(\alpha) = \max\{\alpha, 2\alpha - 2\}$$

and

$$\beta_+(\alpha) = \min\{2\alpha, \alpha + 1\}$$

for $0 \le \alpha \le 3$.

We are now ready to state the Dimension Comparison Theorem. See [**5**] for a proof in the Heisenberg group and [**9**] for a proof in arbitrary Carnot groups. In the statement of the theorem, we have denoted by $\dim_{\mathcal{H},cc}$, resp. $\dim_{\mathcal{H},E}$, the Hausdorff dimension with respect to the metric d_{cc}, resp. d_E on a Carnot group $\mathbb{G}$.

Theorem 1.2 (Dimension Comparison in Carnot groups). *Let $\mathbb{G}$ be a Carnot group of topological dimension N and homogeneous dimension Q. Let d_{cc}, resp. d_E, denote the Carnot–Carathéodory, resp. Euclidean, metrics on $\mathbb{G}$. For any set $S \subset \mathbb{G}$,*

$$\beta_-(\dim_{\mathcal{H},E} S) \le \dim_{\mathcal{H},cc} S \le \beta_+(\dim_{\mathcal{H},E} S). \tag{1.13}$$

The estimates in (1.13) are sharp in the following sense: for each ordered pair $(\alpha,\beta) \in [0,N] \times [0,Q]$ such that $\beta_-(\alpha) \le \beta \le \beta_+(\alpha)$, there exists a compact subset $S_{\alpha,\beta} \subset \mathbb{G}$ such that $\dim_{\mathcal{H},cc} S_{\alpha,\beta} = \beta$ and $\dim_{\mathcal{H},E} S_{\alpha,\beta} = \alpha$. Invariant sets of self-similar iterated function systems provide a large class of examples of sets S which often realize the lower bound $\dim_{\mathcal{H},cc} S = \beta_-(\dim_{\mathcal{H},E} S)$. For more details and more precise statements, see [**9**].

CHAPTER 2

Carnot groups of Iwasawa type and conformal mappings

In this chapter we consider Carnot groups of Iwasawa type equipped with a sub-Riemannian metric. Such groups occur as the nilpotent components in the Iwasawa decomposition of real rank one simple Lie groups. The one-point compactifications of these groups, equipped with suitable sub-Riemannian metrics, arise as boundaries at infinity of the classical rank one symmetric spaces. The study of conformal (and more generally, quasiconformal) mappings in Iwasawa groups dates back to the foundational work of Mostow [**48**], [**49**] on rigidity of hyperbolic manifolds, with significant later contributions by Korányi and Reimann [**36**], [**37**] and Pansu [**50**]. In this chapter we recall the definitions and basic analytic and geometric properties of conformal mappings of Carnot groups, especially Iwasawa groups.

2.1. Iwasawa groups of complex, quaternionic or octonionic type

In this section we describe a collection of examples of Carnot groups, the so-called Iwasawa groups of complex, quaternionic and octonionic type. These are precisely the Carnot groups which admit a sufficiently rich family of conformal self-mappings (see Theorem 2.13 for further explanation).

We give a unified description which covers both the complex and quaternionic cases. In subsection 2.1.1 we recall the definition of these objects, and in subsection 2.1.2 we indicate how to view them as sub-Riemannian Carnot groups. In subsection 2.1.4 we describe an explicit homogeneous metric on these groups (the so-called gauge metric) which is closely tied to conformal geometry.

The octonionic case is more subtle. While the definitions and basic formulas, e.g., for the gauge metric, are ostensibly identical to those in the complex and quaternionic cases, the nonassociativity of the octonions leads to many complexities in the derivation of those formulas. We make brief remarks about the octonionic case in subsection 2.1.3, where we also provide copious references to the literature for those wishing to learn more about this fascinating construction.

2.1.1. The group law. Let $\mathbb{K}$ denote either the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$. We denote by $k = \dim_{\mathbb{R}} \mathbb{K} \in \{2, 4\}$ the dimension of $\mathbb{K}$ as an $\mathbb{R}$-vector space. The *Heisenberg group* $\mathbf{Heis}^n_{\mathbb{K}}$ is modeled by the space $\mathbb{K}^n \times \mathrm{Im}(\mathbb{K})$ equipped with the non-abelian group law

$$(z;\tau) * (z';\tau') = (z + z'; \tau + \tau' + 2\,\mathrm{Im} \sum_{\nu=1}^{n} \overline{z'_\nu} z_\nu).$$

Here $z = (z_1, \ldots, z_n)$ and $z' = (z'_1, \ldots, z'_n)$ lie in $\mathbb{K}^n$ and $\tau, \tau' \in \mathrm{Im}(\mathbb{K}) := \{\mathrm{Im}(z) : z \in \mathbb{K}\}$. We recall that $\overline{z}$ denotes the conjugate of an element $z \in \mathbb{K}$, i.e., $\overline{z} = x - \mathbf{i}y$ if $z = x + \mathbf{i}y \in \mathbb{C}$ and $\overline{z} = a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d$ if $z = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d \in \mathbb{H}$, while

$\mathrm{Im}(z)$ denotes the imaginary part[1] of $z \in \mathbb{K}$, i.e., $\mathrm{Im}(z) = \mathbf{i}y$ if $z = x + \mathbf{i}y \in \mathbb{C}$ and $\mathrm{Im}(z) = \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ if $z = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d \in \mathbb{H}$.

For the benefit of the reader we recall that the space of quaternions is the four-dimensional real vector space $\mathbb{H}$ spanned by indeterminates $\mathbf{e}_j$, $j = 0, 1, 2, 3$, and equipped with an $\mathbb{R}$-linear noncommutative multiplication rule $\cdot$. We adopt the standard convention, denoting $\mathbf{e}_0 = 1$ for the multiplicative identity, $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$ and $\mathbf{e}_3 = \mathbf{k}$, and summarize the multiplication rule in the following diagram:

$\cdot$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_1$	-1	$\mathbf{e}_3$	$-\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_3$	-1	$\mathbf{e}_1$
$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	-1

We call $\mathbf{Heis}^n_{\mathbb{K}}$ either the *complex Heisenberg group* or the *quaternionic Heisenberg group*. We often abbreviate $\mathbf{Heis}^n = \mathbf{Heis}^n_{\mathbb{C}}$ and $\mathbf{Heis} = \mathbf{Heis}^1$. Note that $\mathbf{Heis}^n_{\mathbb{K}}$ is identified with the Euclidean space $\mathbb{R}^N$ where $N = kn + k - 1$, $k = \dim_{\mathbb{R}} \mathbb{K}$. We briefly pause to explicitly record the group law in real coordinates. In the complex Heisenberg group $\mathbb{C}^n \times \mathrm{Im}(\mathbb{C}) \leftrightarrow \mathbb{R}^{2n+1}$ we obtain

$$(x, y; t) * (x', y'; t') = (x + x', y + y'; t + t' + 2(x' \cdot y - x \cdot y')),$$

where $x, x', y, y' \in \mathbb{R}^n$ and $t, t' \in \mathbb{R}$. In the quaternionic Heisenberg group $\mathbb{H}^n \times \mathrm{Im}(\mathbb{H}) \leftrightarrow \mathbb{R}^{4n+3}$ we obtain

$$(x, y, z, w; t, u, v) * (x', y', z', w'; t', u', v') = (x'', y'', z'', w''; t'', u'', v'')$$

with $x'' = x + x'$, $y'' = y + y'$, $z'' = z + z'$, $w'' = w + w'$,

$$t'' = t + t' + 2(x' \cdot y - x \cdot y' + w' \cdot z - w \cdot z'),$$
$$u'' = u + u' + 2(x' \cdot z - x \cdot z' + y' \cdot w - y \cdot w'),$$

and

$$v'' = v + v' + 2(x' \cdot w - x \cdot w' + z' \cdot y - z \cdot y').$$

In this case, $x, x', y, y', z, z', w, w' \in \mathbb{R}^n$ and $t, t', u, u', v, v' \in \mathbb{R}$.

The identity element in $\mathbb{G}$ is the origin of the corresponding Euclidean space $\mathbb{R}^{kn+k-1}$; we denote this element by o. The inverse of $p \in \mathbb{G}$ coincides with its Euclidean additive inverse $-p$, however, we denote this element by p^{-1} to emphasize the nonabelian character of the group $\mathbb{G}$.

2.1.2. Left invariant vector fields and the horizontal distribution. In this section we indicate how to view the Heisenberg group $\mathbf{Heis}^n_{\mathbb{K}}$ as a Carnot group. We first consider the complex Heisenberg groups. The vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, n,$$

together with

$$T = \frac{\partial}{\partial t},$$

form a left invariant basis for the tangent bundle of $\mathbf{Heis}^n$. For convenience we also write $X_{n+j} = Y_j$ for $j = 1, \ldots, n$. The sub-Riemannian geometry of the

[1] We define the imaginary part of z to be $\mathbf{i}y$ rather than just y in order to be consistent with the quaternionic convention.

Heisenberg group is defined by the non-integrable subbundle (*horizontal bundle*) $H\mathbf{Heis}^n$, where

$$H_p\mathbf{Heis}^n = \mathrm{span}\{X_1(p), \dots, X_n(p), Y_1(p), \dots, Y_n(p)\} = \mathrm{span}\{X_1(p), \dots, X_{2n}(p)\}.$$

The non-integrability of this distribution is clear, since $[X_j, Y_j] = -4T$ for every $j = 1, \dots, n$. The Carnot–Carathéodory metric on $\mathbf{Heis}^n$ is defined by fixing on $H\mathbf{Heis}^n$ a frame such that the vector fields $X_1, Y_1, \dots, X_n, Y_n$ are orthonormal.

The Heisenberg group $\mathbf{Heis}^n$ is naturally equipped with the structure of a contact manifold. Define a 1-form α on $\mathbf{Heis}^n$ as follows:

$$\alpha = dt + 2\sum_{j=1}^{n}(x_j\,dy_j - y_j\,dx_j) \tag{2.1}$$

The 1-form α is a contact form, which means that $\alpha \wedge (d\alpha)^n$ is a nonzero multiple of the volume form. Observe also that $d\alpha = 4\sum_{j=1}^{n} dx_j \wedge dx_{n+j}$ is a multiple of the standard symplectic form in $\mathbb{R}^{2n}$, and that the horizontal space $H_p\mathbf{Heis}^n$ coincides with the kernel at p of α.

For each $r > 0$ the dilation δ_r of $\mathbf{Heis}^n$ takes the form

$$\delta_r(x, y; t) = (rx, ry; r^2t). \tag{2.2}$$

A similar story can be told in the quaternionic Heisenberg group. The vector fields

$$\begin{aligned}
X_j &= \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t} + 2z_j\frac{\partial}{\partial u} + 2w_j\frac{\partial}{\partial v},\\
Y_j = X_{n+j} &= \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t} - 2w_j\frac{\partial}{\partial u} + 2z_j\frac{\partial}{\partial v},\\
Z_j = X_{2n+j} &= \frac{\partial}{\partial z_j} + 2w_j\frac{\partial}{\partial t} - 2x_j\frac{\partial}{\partial u} - 2y_j\frac{\partial}{\partial v},
\end{aligned}$$

and

$$W_j = X_{3n+j} = \frac{\partial}{\partial w_j} - 2z_j\frac{\partial}{\partial t} + 2y_j\frac{\partial}{\partial u} - 2x_j\frac{\partial}{\partial v} \qquad j = 1, \dots, n,$$

form a left invariant basis for the horizontal distribution $H\mathbf{Heis}^n_{\mathbb{H}}$. This distribution is not integrable, since the Lie span of the vector fields X_j, Y_j, Z_j, W_j contains the vertical subspace spanned by $\partial/\partial t$, $\partial/\partial u$ and $\partial/\partial v$. Again, the Carnot–Carathéodory metric on $\mathbf{Heis}^n_{\mathbb{H}}$ is defined by introducing a frame on $H\mathbf{Heis}^n_{\mathbb{H}}$ for which the vector fields X_j, Y_j, Z_j, W_j, $j = 1, \dots, n$, are orthonormal.

The horizontal space $H_p\mathbf{Heis}^n_{\mathbb{H}}$ coincides with the kernel at p of the $\mathrm{Im}\,\mathbb{H}$-valued quaternionic contact form

$$\begin{aligned}
\alpha = \Bigg(& dt + 2\sum_{j=1}^{n}(x_j\,dy_j - y_j\,dx_j + z_j\,dw_j - w_j\,dz_j),\\
& du + 2\sum_{j=1}^{n}(x_j\,dz_j - z_j\,dx_j - y_j\,dw_j + w_j\,dy_j),\\
& dv + 2\sum_{j=1}^{n}(x_j\,dw_j - w_j\,dx_j + y_j\,dz_j - z_j\,dy_j)\Bigg).
\end{aligned} \tag{2.3}$$

We will not use the quaternionic contact structure of $\mathbf{Heis}^n_{\mathbb{H}}$ in what follows.

As for the complex Heisenberg groups, the dilation δ_r of $\mathbf{Heis}^n_{\mathbb{H}}$ takes the form

$$\delta_r(x, y, z, w; t, u, v) = (rx, ry, rz, rw; r^2 t, r^2 u, r^2 v). \tag{2.4}$$

2.1.3. The first octonionic Heisenberg group. The octonionic Heisenberg group $\mathbf{Heis}^1_{\mathbb{O}}$ is defined in a manner analogous to its complex or quaternionic cousins, however, the nonassociativity of the octonions introduces significant complications into the derivation of basic metric features of this space and its connection to octonionic hyperbolic geometry. We give here a brief description and refer the reader to [1], [2], [3] and [41] for more detailed information and background regarding the octonionic hyperbolic plane $H^2_{\mathbb{O}}$ and the first[2] octonionic Heisenberg group $\mathbf{Heis}^1_{\mathbb{O}}$.

Recall that the octonions (also known as the *Cayley numbers*) can be defined as the eight-dimensional real vector space $\mathbb{O}$ spanned by indeterminates $\mathbf{e}_j$, $j = 0, \ldots, 7$, and equipped with a certain $\mathbb{R}$-linear nonassociative multiplication rule $\cdot$. By convention we take the first indeterminate $\mathbf{e}_0$ to be the identity element for this multiplication rule, i.e., $\mathbf{e}_0 \cdot \mathbf{e}_j = \mathbf{e}_j \cdot \mathbf{e}_0 = \mathbf{e}_j$ for all $j = 0, \ldots, 7$. We write $\mathbf{e}_0 = 1$ and summarize the remaining relations in the following multiplication table:

$\cdot$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	-1	$\mathbf{e}_4$	$\mathbf{e}_7$	$-\mathbf{e}_2$	$\mathbf{e}_6$	$-\mathbf{e}_5$	$-\mathbf{e}_3$
$\mathbf{e}_2$	$-\mathbf{e}_4$	-1	$\mathbf{e}_5$	$\mathbf{e}_1$	$-\mathbf{e}_3$	$\mathbf{e}_7$	$-\mathbf{e}_6$
$\mathbf{e}_3$	$-\mathbf{e}_7$	$-\mathbf{e}_5$	-1	$\mathbf{e}_6$	$\mathbf{e}_2$	$-\mathbf{e}_4$	$\mathbf{e}_1$
$\mathbf{e}_4$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_6$	-1	$\mathbf{e}_7$	$\mathbf{e}_3$	$\mathbf{e}_5$
$\mathbf{e}_5$	$-\mathbf{e}_6$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_7$	-1	$\mathbf{e}_1$	$\mathbf{e}_4$
$\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_7$	$\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_1$	-1	$\mathbf{e}_2$
$\mathbf{e}_7$	$\mathbf{e}_3$	$\mathbf{e}_6$	$-\mathbf{e}_1$	$-\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_2$	-1

We note that this multiplication table can be conveniently recalled via its connection to the Fano plane as indicated in Figure 2.1.3. Observe that each of the basis elements $\mathbf{e}_j$, $1 \le j \le 7$, satisfy $\mathbf{e}_j^2 = -1$ and anticommute ($\mathbf{e}_j \cdot \mathbf{e}_k = -\mathbf{e}_k \cdot \mathbf{e}_j$, $1 \le j, k \le 7$, $j \ne k$).

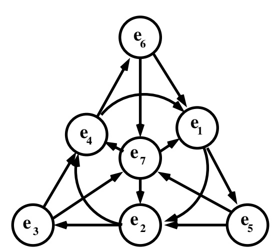

FIGURE 1. The Fano plane

[2]Note that while the octonionic Heisenberg group $\mathbf{Heis}^n_{\mathbb{O}}$ can be defined for any n, it is only the first octonionic Heisenberg group $\mathbf{Heis}^1_{\mathbb{O}}$ which arises as an Iwasawa group. This is due to complexities stemming from the nonassociativity of the octonions, which preclude the definition of octonionic hyperbolic spaces of dimension strictly greater than two.

Expressing a typical element $z \in \mathbb{O}$ in the form $z = z_0 + \sum_{j=1}^{7} z_j \mathbf{e}_j$ with $z_0, \ldots, z_7 \in \mathbb{R}$ we recall that the real part of z is $\mathrm{Re}(z) = z_0$ and the imaginary part is $\mathrm{Im}(z) = \sum_{j=1}^{7} z_j \mathbf{e}_j$. The conjugate of $z = \mathrm{Re}(z) + \mathrm{Im}(z)$ is $\overline{z} = \mathrm{Re}(z) - \mathrm{Im}(z)$ and the modulus is $|z| = \sqrt{\overline{z}z} = \sqrt{z\overline{z}}$. Note that

$$\mathrm{Re}(xy) = \mathrm{Re}(yx)$$

for all $x, y \in \mathbb{O}$.

Since the octonions are not associative, one must be careful when performing computations. It is useful to observe that certain associativity-type formulas hold. For instance, any product of octonions involving only two octonions is associative; this is a result of Artin. For instance, if $x \in \mathbb{O}$ and μ is an imaginary unit quaternion (so that $\overline{\mu} = -\mu$), then the triple product

$$\mu x \overline{\mu} \tag{2.5}$$

is unambiguously defined. Furthermore, the *Moufang identities*

$$\mathrm{Re}((xy)z) = \mathrm{Re}(x(yz)), \tag{2.6}$$

$$(\mu x \overline{\mu})(\mu y) = \mu(xy) \tag{2.7}$$

and

$$(x\mu)(\overline{\mu} y \mu) = (xy)\mu \tag{2.8}$$

hold true for all $x, y, z \in \mathbb{O}$ and all imaginary unit quaternions μ. In view of (2.6) we may unambiguously denote by $\mathrm{Re}(xyz)$ the quantity specified in that formula, despite the nonassociativity of $\mathbb{O}$.

The first octonionic Heisenberg group is modeled as $\mathbf{Heis}^1_{\mathbb{O}} = \mathbb{O} \times \mathrm{Im}\,\mathbb{O}$ (in real coordinates, as $\mathbb{R}^{15}$). Here $\mathrm{Im}\,\mathbb{O}$ denotes the space of imaginary octonions. The group law, by analogy with the complex and quaternionic cases, is

$$(z; \tau) * (z'; \tau') = (z + z'; \tau + \tau' + 2\,\mathrm{Im}(\overline{z'}z)).$$

The group $\mathbf{Heis}^1_{\mathbb{O}}$ is equipped with an 8-dimensional horizontal distribution $H\mathbf{Heis}^1_{\mathbb{O}}$ defined by a left-invariant basis of horizontal vector fields $X_1, \ldots, X_8$; we omit the explicit formulas for these vector fields in real coordinates. By analogy with (1.10), the dilation automorphisms δ_r of $\mathbf{Heis}^1_{\mathbb{O}}$ are given by

$$\delta_r(z; \tau) = (rz; r^2\tau), \qquad (z; \tau) \in \mathbb{O}.$$

2.1.4. The gauge metric of an Iwasawa group. The conformal geometry of an Iwasawa group $\mathbb{G}$ is naturally described in terms of a more computationally friendly homogeneous metric called the *gauge metric* (also known as the *Cygan* or *Korányi metric*)

$$d_H(p, q) = ||p^{-1} \cdot q||_H,$$

where

$$||(z; \tau)||_H = (|z|^4 + |\tau|^2)^{1/4} \qquad \text{for } (z; \tau) \in \mathbf{Heis}^n_{\mathbb{K}}.$$

The fact that d_H is a homogeneous metric is well known. See [**20**, p. 18] for a proof in the case $\mathbb{G} = \mathbf{Heis}^n$. In the case $\mathbb{G} = \mathbf{Heis}^1_{\mathbb{O}}$ a proof can be found in [**41**, Section 3.3]. However, since the presentation of $\mathbf{Heis}^1_{\mathbb{O}}$ in [**41**] is slightly different from ours, we give here a direct proof which is valid for any Iwasawa group.

PROPOSITION 2.1. *The expression d_H defines a metric on any Iwasawa group $\mathbb{G} = \mathbf{Heis}^n_{\mathbb{K}}$.*

PROOF. It suffices to show that $||(z;\tau)*(z';\tau')||_H \le ||(z;\tau)||_H + ||(z';\tau')||_H$ for all $(z;\tau)$ and $(z';\tau')$ in $\mathbb{G}$. We take advantage of the identity

$$||(z;\tau)||_H^4 = \left|-|z|^2+\tau\right|^2.$$

We compute

$$\begin{aligned}||(z;\tau)*(z';\tau')||_H^4 &= ||(z+z';\tau+\tau'+\overline{z'}z-\overline{z}z')||_H^4\\ &= \left|-|z+z'|^2+\tau+\tau'+\overline{z'}z-\overline{z}z'\right|^2\\ &= \left|(-|z|^2+\tau)-2\overline{z}z'+(-|z'|^2+\tau')\right|^2\\ &\le \left|||(z;\tau)||_H^2+2|z|\,|z'|+||(z';\tau')||_H^2\right|^2\\ &\le (||(z;\tau)||_H+||(z';\tau')||_H)^4.\end{aligned}$$

□

Throughout this monograph, we will denote by d a general (unspecified) homogeneous metric in a Carnot group. We reserve the notations d_{cc} and d_H to denote the particular examples of the Carnot–Carathéodory metric (defined in any Carnot group) and the gauge metric (defined in any Iwasawa group). We will denote by $B(p,r)$, resp. $\overline{B}(p,r)$ the open, resp. closed, ball with center p and radius r in the metric space $(\mathbb{G},d)$, with similar notations for the metrics d_{cc} and d_H. Diameters of and distances between sets will be denoted diam and dist, and suitably adorned with subscripts when necessary.

REMARKS 2.2. Let $\mathbb{G}$ be an Iwasawa group. It follows from (1.7) that

$$B_{cc}(p_0,r/L)\subset B_H(p_0,r)\subset B_{cc}(p_0,Lr) \tag{2.9}$$

for all $p_0\in\mathbb{G}$ and $r>0$. Here L denotes a comparison constant as in (1.7) relating the Carnot–Carathéodory and gauge metrics.

We also record the following elementary fact:

$$\operatorname{diam}_H \overline{B}_H(p,r) = 2r \qquad \text{for all } p\in\mathbb{G} \text{ and } r>0, \tag{2.10}$$

Equation (2.10) follows from the observation that through every point $p\in\mathbb{G}$ there exist horizontal embedded lines $p*\exp(\mathbb{R}\,V)$, $V\in H_o\mathbb{G}$, $|V|_{o,\mathbb{G}}=1$. The restriction of d_H to such a line coincides with the Euclidean metric, whence $p*\exp(\pm rV)\in \overline{B}_H(p,r)$ are two points whose distance is equal to $2r$.

Among the many remarkable features of the gauge metric is the following *Ptolemaic inequality*:

$$d_H(p_1,p_2)d_H(p_3,p_4)\le d_H(p_1,p_3)d_H(p_2,p_4)+d_H(p_2,p_3)d_H(p_1,p_4) \tag{2.11}$$

for all p_1,p_2,p_3,p_4 in $\mathbb{G}$. We refer the reader, for instance, to the paper [**52**] for a proof of the Ptolemaic inequality for the gauge metric in Iwasawa groups. See also [**24**] and [**15**] for more information about Ptolemaic geometry and its relation to Iwasawa groups.

2.1.5. Extended Iwasawa groups and cross ratios. Let $\mathbb{G}$ be an Iwasawa group. We denote by $\overline{\mathbb{G}} = \mathbb{G} \cup \{\infty\}$ the one-point compactification of $\mathbb{G}$. Note that $\overline{\mathbb{G}}$ is homeomorphic to the sphere $\mathbb{S}^N$. The gauge metric d_H induces a spherical metric $\overline{d}_H$ on $\overline{\mathbb{G}}$, given by the formula

$$\overline{d}_H(p,q) = \begin{cases} \frac{2d_H(p,q)}{\sqrt{1+d_H(p,o)^2}\sqrt{1+d_H(q,o)^2}} & \text{if } p \neq \infty \text{ and } q \neq \infty, \\ \frac{2}{\sqrt{1+d_H(p,o)^2}} & \text{if } q = \infty. \end{cases}$$

The quantity $\overline{d}_H$ is a natural analog of the usual spherical metric on the Euclidean sphere $\mathbb{S}^n = \overline{\mathbb{R}^n}$. The fact that $\overline{d}_H$ is a metric on $\overline{\mathbb{G}}$ follows from the Ptolemiac inequality (2.11).

The extended Iwasawa group $M = \overline{\mathbb{G}}$ can be equipped with the structure of a sub-Riemannian manifold locally modeled on $\mathbb{G}$. In the complex case, this structure arises from the usual CR structure on the sphere $\mathbb{S}^{2n+1}$, while in the quaternionic case, it arises from the quaternionic CR structure on the sphere $\mathbb{S}^{4n+3}$. The Carnot–Carathéodory metric d_{cc} on $\overline{\mathbb{G}}$ coming from this sub-Riemannian structure is the length metric generated by, and is bi-Lipschitz equivalent, to the spherical metric $\overline{d}_H$ defined above.

The *cross ratio* of four points $p_1, p_2, p_3, p_4 \in \overline{\mathbb{G}}$, at most two of which are equal, is

$$[p_1 : p_2 : p_3 : p_4] := \frac{d_H(p_1,p_3)d_H(p_2,p_4)}{d_H(p_1,p_4)d_H(p_2,p_3)}. \tag{2.12}$$

This quantity is well-defined as an element of $[0,\infty]$, where we make the standard conventions $\frac{a}{0} = +\infty$ for $a > 0$ and $\frac{a}{+\infty} = 0$ for $a \geq 0$. If any one of the points p_j is equal to the point at infinity, we modify the definition of $[p_1 : p_2 : p_3 : p_4]$ by deleting the terms in the numerator and denominator containing that term. The value of the cross ratio in (2.12) is unchanged if the spherical metric $\overline{d}_H$ is used in place of d_H.

2.1.6. Summary. To summarize, the Iwasawa groups $\mathbb{G} = \mathbf{Heis}^n_{\mathbb{K}}$ consist of the following:

- the *(complex) Heisenberg group* $\mathbf{Heis}^n = \mathbf{Heis}^n_{\mathbb{C}}$ for some $n \geq 1$,
- the *quaternionic Heisenberg group* $\mathbf{Heis}^n_{\mathbb{H}}$ for some $n \geq 1$,
- the *first octonionic Heisenberg group* $\mathbf{Heis}^1_{\mathbb{O}}$.

Each of these is a Carnot group of step two, with an even-dimensional horizontal layer and a second layer of dimension 1 (resp. 3 or 7). The group $\mathbb{G}$ is equipped with a sub-Riemannian structure, coming either from the Carnot–Carathéodory metric d_{cc} or from the bi-Lipschitz equivalent Korányi (gauge) metric d_H. Moreover, the gauge metric d_H induces a natural spherical metric $\overline{d}_H$ on the one-point compactification $\overline{\mathbb{G}}$. Later we will see that conformal mappings of $\mathbb{G}$ interact naturally with the gauge metric d_H, extend to $\overline{\mathbb{G}}$ as Möbius transformations, and preserve cross ratios.

Henceforth we denote by $m = kn$ the rank of the horizontal bundle $H\mathbb{G}$, by $N = kn + (k-1)$ the topological dimension of $\mathbb{G}$ (and also of $\overline{\mathbb{G}}$), and by $Q = kn + 2(k-1)$ the homogeneous dimension of $\mathbb{G}$.

2.2. Conformal mappings on Carnot groups

Conformal mappings of Carnot groups admit several mutually equivalent descriptions. We adopt an abstract metric definition for conformality. All conformal mappings are smooth and contact, and a Liouville-type conformal rigidity theorem holds: every conformal map between domains in an Iwasawa group $\mathbb{G}$ is the restriction of a globally defined Möbius map acting on the compactified space $\overline{\mathbb{G}}$. We give an explicit description of all such mappings and a list of basic geometric and analytic properties which will be used in subsequent chapters.

2.2.1. Definition and basic properties. We adopt as our definition for conformal maps the standard *metric definition* for 1-quasiconformal maps.

DEFINITION 2.3. Let $f : \Omega \to \Omega'$ be a homeomorphism between domains in a Carnot group $\mathbb{G}$. The map f is said to be *conformal* if

$$\lim_{r\to 0} \frac{\sup\{d_{cc}(f(p), f(q)) \,:\, d_{cc}(p,q) = r\}}{\inf\{d_{cc}(f(p), f(q')) \,:\, d_{cc}(p,q') = r\}} = 1 \tag{2.13}$$

for all $p \in \Omega$.

REMARKS 2.4. (1) For topological reasons, the value on the left hand side of (2.13) is unchanged if the quotient is replaced by

$$\frac{\sup\{d_{cc}(f(p), f(q)) \,:\, d_{cc}(p,q) \le r\}}{\inf\{d_{cc}(f(p), f(q')) \,:\, d_{cc}(p,q') \ge r\}}. \tag{2.14}$$

The formulation of conformality using (2.14) is more suitable in the setting of general metric spaces, where the sphere $\{y : d(x,y) = r\}$ may be empty for certain choices of x and $r > 0$. See [**33**] for a detailed discussion of quasiconformality in metric spaces.

(2) It is clear that every Carnot–Carathéodory similarity map (i.e., every map f which distorts all distances by a fixed scale factor $r > 0$) is conformal. Examples of such maps include left translations, dilations, and certain automorphisms. In the following section we will give a complete classification of all conformal maps in Iwasawa groups (henceforth termed *Iwasawa conformal maps*).

(3) Definition 2.3 is formulated in terms of the Carnot–Carathéodory metric. In Iwasawa groups, the same class of maps is obtained if the Carnot–Carathéodory metric is replaced throughout by the gauge metric d_H. In other words, f is conformal if and only if

$$\lim_{r\to 0} \frac{\sup\{d_H(f(p), f(q)) \,:\, d_H(p,q) = r\}}{\inf\{d_H(f(p), f(q')) \,:\, d_H(p,q') = r\}} = 1 \tag{2.15}$$

for all $p \in \Omega$. See the introduction of [**37**] for a discussion of this matter in the complex Heisenberg groups; the rationale given there extends to the remaining Iwasawa groups without any complication.

(4) No a priori regularity is assumed in Definition 2.3. However, every homeomorphism f satisfying (2.13) at all points p in Ω is necessarily C^∞.[3] This fact was proved by Capogna [**17**], [**18**] in the case of the complex Heisenberg group, and by Capogna and Cowling [**19**] in arbitrary Carnot groups.

(5) Pansu [**50**] proved that if two domains $\Omega \subset \mathbb{G}$ and $\Omega' \subset \mathbb{G}'$ are equivalent by a conformal (or more generally, a quasiconformal) map, then $\mathbb{G} = \mathbb{G}'$.

[3]Regularity is understood with respect to the underlying Euclidean structure.

Moreover, every quasiconformal map between domains Ω, Ω' in either the quaternionic Heisenberg group $\mathbf{Heis}^n_{\mathbb{H}}$ or the first octonionic Heisenberg group $\mathbf{Heis}^1_{\mathbb{O}}$ is 1-quasiconformal and hence conformal in the sense of Definition 2.3. In the complex Heisenberg groups $\mathbf{Heis}^n_{\mathbb{C}}$ there exist plenty of nonconformal quasiconformal maps. An extensive theory of quasiconformal mappings in the complex Heisenberg group can be found in the papers [36] and [37] of Korányi and Reimann.

(6) According to a classical theorem of Liouville , conformal maps of a Euclidean space $\mathbb{R}^n$ of dimension at least three are the restrictions of Möbius maps acting on $\mathbb{S}^n = \overline{\mathbb{R}^n}$. In Liouville's original formulation, the maps in question were assumed a priori to be of classs C^4. Gehring [31] relaxed the smoothness assumption by showing that the same conclusion holds for 1-quasiconformal maps (assumed initially only to lie in the local Sobolev space $W^{1,n}_{loc}$). In particular, 1-quasiconformal maps of Euclidean domains are C^∞. Analogous rigidity results hold for Iwasawa conformal maps, see Theorem 2.10.

For a proof of the following result, see [37] for the complex case and [25] for the remaining cases.

Proposition 2.5. *Every Iwasawa conformal map is a contact mapping.*

In the following subsections, we give a number of examples of conformal mappings of Iwasawa groups. Later, we will see that the full group of orientation-preserving conformal mappings is generated by these mappings.

2.2.2. Classification of Iwasawa conformal maps I: similarities.

Example 2.6. Left translations and dilations are conformal maps in any Carnot group.

Example 2.7 (Rotations). (1) The rotations of the (complex) Heisenberg group $\mathbf{Heis}^n$ are the automorphisms $R_A : \mathbf{Heis}^n \to \mathbf{Heis}^n$, $A \in U(n)$, given by

$$R_A(z;t) = (Az;t).$$

These were already discussed in $\mathbf{Heis}^1$ in Example 1.1. The complex Heisenberg group $\mathbf{Heis}^n$ is equipped with another mapping $\rho : \mathbf{Heis}^n \to \mathbf{Heis}^n$ given by $\rho(z,t) = (\overline{z}, -t)$. The map ρ is also an automorphism. The rotations of $\mathbf{Heis}^n$ are the mappings generated by R_A, $A \in U(n)$, and ρ.

(2) Rotations of the quaternionic Heisenberg group have a more complicated structure. The class of all rotations of $\mathbf{Heis}^n_{\mathbb{H}}$ include both the "horizontal rotations" $R_A : \mathbf{Heis}^n_{\mathbb{H}} \to \mathbf{Heis}^n_{\mathbb{H}}$, $A \in \mathrm{Sp}(n)$, given by

$$R_A(z;\tau) = (Az;\tau)$$

as well as the "vertical rotations" $R^V_B : \mathbf{Heis}^n_{\mathbb{H}} \to \mathbf{Heis}^n_{\mathbb{H}}$, $B \in \mathrm{Sp}(1)$, given by

$$R^V_B(z;\tau) = (Bz\overline{B}; B\tau\overline{B}).$$

Here we identified $\mathrm{Sp}(1)$ with the group of unit quaternions. The full group of rotations is generated by these two types.

(3) For each imaginary unit octonion μ, the map $R_\mu : \mathbf{Heis}^1_{\mathbb{O}} \to \mathbf{Heis}^1_{\mathbb{O}}$ given by

$$R_\mu(z;\tau) = (z\overline{\mu}, \mu\tau\overline{\mu})$$

defines a rotation. Note that the expression $\mu\tau\overline{\mu}$ is well-defined; see (2.5). The collection of maps R_μ generates the full rotation group of $\mathbf{Heis}^1_{\mathbb{O}}$. However, the function $\mu \mapsto R_\mu$ is not a homomorphism, and the full rotation group is larger than the space $\mathbb{S}^6$ of imaginary unit quaternions. In fact, the first octonionic rotation group is isomorphic to Spin(7). For more information, see [41, Section 3.2].

Rotations, left translations and dilations generate the full *similarity group* $\mathrm{Sim}(\mathbb{G})$ of each Iwasawa group $\mathbb{G}$.

2.2.3. Classification of Iwasawa conformal maps II: inversion. Each Iwasawa group is equipped with a conformal inversion map which serves as an analog of the Euclidean inversion map $x \mapsto x/|x|^2$.

EXAMPLE 2.8 (Inversion). Let $\mathbb{G}$ be an Iwasawa group $\mathbb{G}$. The *inversion* $\mathcal{J}$: $\mathbb{G} \setminus (o) \to \mathbb{G} \setminus (o)$ is defined as follows. On the complex Heisenberg group $\mathbf{Heis}^n$,

$$\mathcal{J}(z;t) := \left(\frac{z}{|z|^2 - \mathbf{i}t}; \frac{-t}{||(z,t)||^4} \right),$$

while on $\mathbf{Heis}^n_{\mathbb{H}}$ or $\mathbf{Heis}^1_{\mathbb{O}}$

$$\mathcal{J}(z,\tau) := \left(z(|z|^2 - \tau)^{-1}; -||(z,\tau)||^{-4}\,\tau \right).$$

Observe that $\mathcal{J}$ is an involution ($\mathcal{J}^2$ is the identity). It is an easy exercise (see also Remark 2.9 below) to check that

$$d_H(\mathcal{J}(p), o) = \frac{1}{d_H(p,o)} \qquad \text{for all } p \in \mathbb{G} \setminus \{o\}. \tag{2.16}$$

More generally,

$$d_H(\mathcal{J}(p), \mathcal{J}(q)) = \frac{d_H(p,q)}{d_H(p,o) d_H(q,o)} \qquad \text{for all } p, q \in \mathbb{G} \setminus \{o\}. \tag{2.17}$$

See Proposition 3.2 for a more general statement.

REMARK 2.9. In the case of the first octonionic Heisenberg group, equations (2.16) and (2.17) are derived in Proposition 3.5 of [41]. However, as already mentioned, the presentation of $\mathbf{Heis}^1_{\mathbb{O}}$ is slightly different in that reference. Namely, points of $\mathbf{Heis}^1_{\mathbb{O}}$ are considered as pairs $(x,y) \in \mathbb{O}^2$ such that $x + x' + |y|^2 = 0$, with group law $(x,y) * (x',y') = (x + x' - \overline{y}y', y + y')$. In that presentation, the inversion map is given by $(x,y) \mapsto (|x|^{-2}\overline{x}, -|x|^{-2}y\overline{x})$. We leave it as an exercise to the reader to verify that the choice $x = -|z|^2 + \tau$, $y = \sqrt{2}\,z$ identifies our model of $\mathbf{Heis}^1_{\mathbb{O}}$ with that of [41], and that the expressions for the conformal inversion coincide when this identification is taken into account.

Conformality can also be defined for maps between domains in an extended Iwasawa group $\overline{\mathbb{G}}$, equipped with either the spherical metric $\overline{d}$ or the Carnot–Carathéodory metric d_{cc}. As was the case in Definition 2.3, the class of conformal maps is the same. Each similarity of $\mathbb{G}$ extends to a map of $\overline{\mathbb{G}}$ preserving the point at infinity; the extended map is still conformal. In particular, the inversion $\mathcal{J}$ extends to a conformal self-map of $\overline{\mathbb{G}}$ interchanging the neutral element o and ∞.

2.2.4. Classification of Iwasawa conformal maps III: Liouville's theorem. Liouville's rigidity theorem in Iwasawa groups reads as follows.

THEOREM 2.10. *Let $f : \Omega \to \Omega'$ be a conformal map between domains in an Iwasawa group $\mathbb{G}$. Then $f = F|_\Omega$ where F is a conformal self-map of $\overline{\mathbb{G}}$. Moreover, the group of conformal maps of $\overline{\mathbb{G}}$ is generated by the similarities of $\mathbb{G}$ and the conformal inversion $\mathcal{J}$.*

In complex Heisenberg groups, Theorem 2.10 is due to Korányi and Reimann [**36**], [**37**] for smooth conformal maps and to Capogna [**17**], [**18**] for general 1-quasiconformal maps. In other Iwasawa groups, it follows from the regularity theorem of Capogna–Cowling [**19**] together with recent work of Cowling–Ottazzi [**25**].

In general, we will denote by $\mathrm{Conf}(\overline{\mathbb{G}})$ the group of conformal self-maps of an extended Iwasawa group $\overline{\mathbb{G}}$.

The decomposition of conformal maps in the following proposition corresponds to the Bruhat decomposition of the corresponding matrix group acting on the associated rank one symmetric space (cf. [**36**, p. 312]).

PROPOSITION 2.11. *If $f : \Omega \to \Omega'$ is a conformal map between domains in an Iwasawa group $\mathbb{G}$, then $f = F|_\Omega$, where*

$$F = \ell_b \circ \delta_r \circ R \circ \mathcal{J}^\epsilon \circ \ell_{a^{-1}}, \tag{2.18}$$

where $a \in \mathbb{G} \setminus \Omega$, $b \in \mathbb{G}$, $r > 0$, $\epsilon \in \{0, 1\}$, and R is a rotation of $\mathbb{G}$.

The scaling factor r is uniquely determined by the map f; we denote it by r_f. For the benefit of the reader we provide a short proof.

PROOF. Let $f : \Omega \to \Omega'$ be a conformal map between domains in an Iwasawa group $\mathbb{G}$. By Theorem 2.10, $f = F|_\Omega$ for some F which is conformal on $\overline{\mathbb{G}}$. If $F(\infty) = \infty$ then F, hence also f, is a similarity. In this case, (2.18) holds with $\epsilon = 0$, $a = o$, $b = F(o)$, r the scaling factor of F, and for a suitable choice of a rotation R. If $F(\infty) \neq \infty$, let $a := F^{-1}(\infty) \in \mathbb{G}$. Then $\tilde{F} := F \circ \ell_a \circ \mathcal{J} \in \mathrm{Conf}(\overline{\mathbb{G}})$ and $\tilde{F}(\infty) = \infty$. It follows that $\tilde{F}$ is a similarity of $\mathbb{G}$, whence, as in the first case, $\tilde{F} = \ell_b \circ \delta_r \circ R$ for suitable $b \in \mathbb{G}$, $r > 0$ and a rotation R. The proof is complete. □

REMARK 2.12. It follows from (2.17) and the explicit description of $\mathrm{Conf}(\overline{\mathbb{G}})$ that each conformal map f preserves cross ratios of arbitrary quadruples of points, i.e., is a *Möbius transformation*. That is,

$$[f(p_1) : f(p_2) : f(p_3) : f(p_4)] = [p_1 : p_2 : p_3 : p_4] \tag{2.19}$$

for all quadruples p_1, p_2, p_3, p_4 of points in $\overline{\mathbb{G}}$ for which the expressions are defined. For the definition of the cross ratio, see (2.12).

The extended Iwasawa group $\overline{\mathbb{G}}$ may be identified with the boundary at infinity of a rank one symmetric space. The classification of such spaces is well-known: every such space is either real hyperbolic space $H^{n+1}_{\mathbb{R}}$, complex hyperbolic space $H^{n+1}_{\mathbb{C}}$, quaternionic hyperbolic space $H^{n+1}_{\mathbb{H}}$, or the octonionic hyperbolic plane $H^2_{\mathbb{O}}$.[4]

Elements of $\mathrm{Conf}(\overline{\mathbb{G}})$ act as isometries on the corresponding hyperbolic space, conversely, each isometry of one of the preceding hyperbolic spaces extends to a

[4]We omit real hyperbolic space $H^{n+1}_{\mathbb{R}}$ from the ensuing discussion as its boundary at infinity is the standard round sphere $\mathbb{S}^n$ equipped with its Euclidean metric and we are interested in nonabelian Iwasawa groups, i.e., Iwasawa groups of step two.

conformal self-map of the corresponding extended Iwasawa group. Each of these isometry groups is finite dimensional, obtained via the action of a suitable matrix group. More precisely,

$$\mathrm{Conf}(\overline{\mathbf{Heis}^n}) \simeq \mathrm{Isom}(\mathcal{H}_{\mathbb{C}}^{n+1}) \simeq \mathrm{PSU}(n,1),$$
$$\mathrm{Conf}(\overline{\mathbf{Heis}_{\mathbb{H}}^n}) \simeq \mathrm{Isom}(\mathcal{H}_{\mathbb{H}}^{n+1}) \simeq \mathrm{PSp}(n,1),$$

and

$$\mathrm{Conf}(\overline{\mathbf{Heis}_{\mathbb{O}}^1}) \simeq \mathrm{Isom}(\mathcal{H}_{\mathbb{O}}^2) \simeq F_4^{(-20)}.$$

Here $\mathrm{PSU}(n,1)$, resp. $\mathrm{PSp}(n,1)$, denotes the component of the identity in the non-compact Lie group of $(n+1)\times(n+1)$ complex, resp. quaternionic, matrices preserving an indefinite Hermitian form, resp. quaternionic Hermitian form, of signature $(n,1)$. For the identification of the exceptional Lie group $F_4^{(-20)}$ as the isometry group of the octonionic hyperbolic plane, see for instance [**2**].

2.2.5. Classification of conformal mappings on Carnot groups. The following theorem of Cowling and Ottazzi [**25**] indicates that Iwasawa groups are the only Carnot groups which admit non-affine conformal maps. In view of this theorem, the theory of conformal dynamical systems which we develop in this paper provides for a nontrivial generalization of self-similar dynamical systems only in the Iwasawa group case. See the introduction for further information and discussion.

THEOREM 2.13 (Cowling–Ottazzi). *Let $f:\Omega\to\Omega'$ be a conformal map between domains in a Carnot group $\mathbb{G}$. Then either f is the restriction of an affine similarity of $\mathbb{G}$ or $\mathbb{G}$ is an Iwasawa group.*

By an affine similarity of a Carnot group $\mathbb{G}$ equipped with the metric d_{cc}, we mean the composition of a left translation, a dilation, and an isometric automorphism. Note that not all automorphisms of Carnot groups (even of the Heisenberg group) are necessarily isometries.

CHAPTER 3

Metric and geometric properties of conformal maps

In subsequent chapters, we will only need to use a number of basic metric and geometric facts about Carnot conformal mappings. We collect these facts in the present chapter. Most of the results which we need derive from certain fundamental identities (3.1), (3.2), (3.3) which hold for arbitrary conformal mappings. We introduce the pointwise stretch factor $||Df(p)||$ of a conformal map f at a point p. This quantity coincides with the norm of the horizontal differential of f at p, and also with the local Lipschitz constant. Lemma 3.6 states a relative continuity estimate for the function $p \mapsto ||Df(p)||$ which plays a key role in later chapters. In section 3.2 we use Lemma 3.6 to derive estimates for Iwasawa conformal mappings in the spirit of the classical Koebe distortion theorem.

We reiterate that all of the results of the present chapter are valid in arbitrary Carnot groups, however, the main content comes in the Iwasawa group case. In particular, if the group in question is not of Iwasawa type, then all conformal mappings are similarities, and many of the estimates stated here become trivial. To simplify the exposition, we do not dwell on this point.

REMARK 3.1. Throughout this chapter, $\mathbb{G}$ denotes a general Carnot group and f denotes a conformal map of $\mathbb{G}$. If $\mathbb{G}$ is an Iwasawa group, then d will denote the gauge metric d_H. If $\mathbb{G}$ is not an Iwasawa group, then d will denote the Carnot–Carathéodory metric d_{cc}.

3.1. Norm of the horizontal differential and local Lipschitz constant

The following result can be derived from Proposition 2.1 of [41]. It can also be easily deduced from Proposition 2.11. Note that if $\mathbb{G}$ is not of Iwasawa type, then only (3.1) arises.

PROPOSITION 3.2. *Let $f \in \mathrm{Conf}(\overline{\mathbb{G}})$ and let $r_f > 0$ be the scaling factor from Proposition 2.11. If $f(\infty) = \infty$ then*

$$d(f(p), f(q)) = r_f\, d(p,q) \qquad \text{for all } p, q \in \mathbb{G}. \tag{3.1}$$

If $f(\infty) \neq \infty$ then

$$d(f(p), f(q)) = \frac{r_f\, d(p,q)}{d(p, f^{-1}(\infty))\, d(q, f^{-1}(\infty))} \quad \text{for all } p, q \in \mathbb{G} \setminus \{f^{-1}(\infty)\}, \tag{3.2}$$

and

$$d(f(p), f(\infty)) = \frac{r_f}{d(p, f^{-1}(\infty))} \qquad \text{for all } p \in \mathbb{G} \setminus \{f^{-1}(\infty)\}. \tag{3.3}$$

As a corollary, we observe that all elements of $\mathrm{Conf}(\overline{\mathbb{G}})$ act conformally in a metric fashion, as follows.

Corollary 3.3. *Let $f \in \mathrm{Conf}(\overline{\mathbb{G}})$ and let $p \in \mathbb{G}$, $p \neq f^{-1}(\infty)$. Then*

$$\lim_{q\to p} \frac{d(f(p), f(q))}{d(p,q)} \tag{3.4}$$

exists and is positive.

Note that existence of the limit in (3.4) is a more restrictive condition than the 1-quasiconformality at p as in (2.13) or (2.15). Indeed, if the limit in (3.4) exists, then

$$\lim_{r\to 0} \frac{\sup\{d(f(p), f(q)) \,:\, d(p,q) = r\}}{\inf\{d(f(p), f(q')) \,:\, d(p,q') = r\}} = \frac{\lim_{r\to 0} \sup\{d(f(p), f(q))/r \,:\, d(p,q) = r\}}{\lim_{r\to 0} \inf\{d(f(p), f(q'))/r \,:\, d(p,q') = r\}}$$

which is equal to one.

The quantity in (3.4) is the local stretching factor of f at the point p. In Proposition 3.5 we will see that it agrees with the operator norm of $Df(p)$, the restriction of the differential f_* to the horizontal tangent space at p. Note that the horizontal differential $Df(p)$ maps $H_p\mathbb{G}$ to $H_{f(p)}\mathbb{G}$. Recalling Subsection 2.1.6 we interpret $Df(p)$ as an automorphism of the vector space $\mathbb{R}^{kn}$ and denote its operator norm by

$$||Df(p)||.$$

Corollary 3.3 follows from the identities (3.1) and (3.2) upon dividing by $d(p,q)$ and letting $q \to p$. In the case when $f(\infty) = \infty$ we obtain

$$||Df(p)|| = r_f \qquad \text{for all } p \in \mathbb{G},$$

while in the case when $f(\infty) \neq \infty$ we obtain

$$||Df(p)|| = \frac{r_f}{d(p, f^{-1}(\infty))^2} \qquad \text{for all } p \in \mathbb{G} \setminus \{f^{-1}(\infty)\}. \tag{3.5}$$

It also follows from Corollary 3.3 that the quantity $||Df(\cdot)||$ satisfies the Leibniz rule

$$||D(f\circ g)(p)|| = ||Df(g(p))||\,||Dg(p)|| \tag{3.6}$$

whenever $p \notin \{g^{-1}(\infty), (f\circ g)^{-1}(\infty)\}$.

Next, we provide an analytic formulation of conformality in Carnot groups as well as the promised relationship between the operator norm $||Df(p)||$ and the limit in (3.4).

Theorem 3.4. *Let $f : \Omega \to \Omega'$ be a Carnot conformal map. Then*

$$||Df(p)||^{kn} = \det Df(p) \tag{3.7}$$

and

$$||Df(p)||^{Q} = \det f_*(p) \tag{3.8}$$

for all $p \in \Omega$.

Proposition 3.5. *Let $f : \Omega \to \Omega'$ be a Carnot conformal map. Then*

$$||Df(p)|| = \lim_{q\to p} \frac{d(f(p), f(q))}{d(p,q)}. \tag{3.9}$$

We clarify that in the previous proposition d denotes the gauge metric when $\mathbb{G}$ is an Iwasawa group and $d = d_{cc}$ when $\mathbb{G}$ is not an Iwasawa group. Recalling (3.1) we will also denote $r_f = \|Df(p)\|$ when f is a metric similarity in $(\mathbb{G}, d)$ and d is any homogeneous metric.

PROOFS OF THEOREM 3.4 AND PROPOSITION 3.5. First assume that f is a similarity mapping. (Recall that this is automatically the case if $\mathbb{G}$ is not an Iwasawa group). Then f is a composition of left translations, dilations and automorphisms. In this case, (3.7), (3.8) and (3.9) are trivial.

Suppose then that $\mathbb{G}$ is an Iwasawa group and that f is not a similarity mapping. Since the Jacobian determinants $\det Df$ and $\det f_*$ as well as the stretch factor $||Df(p)||$ are multiplicative under composition, it suffices to verify (3.7), (3.8) and (3.9) for the inversion mapping. This is an elementary computation which we leave to the reader. □

The following lemma states an inequality of Harnack type for the norm of the horizontal differential of a conformal mapping.

LEMMA 3.6. *Let S be a compact subset of a domain $\Omega \subset \mathbb{G}$. Then there exists a constant $K_1 = K_1(\delta)$ depending only on $\delta = \operatorname{diam}(S)/\operatorname{dist}(S, \partial\Omega)$ so that*

$$\left| \frac{||Df(p)||}{||Df(q)||} - 1 \right| \le K_1 \frac{d(p,q)}{\operatorname{diam}(S)} \tag{3.10}$$

whenever $p, q \in S$ and $f : \Omega \to \mathbb{G}$ is conformal. In particular,

$$||Df(p)|| \le K\, ||Df(q)|| \tag{3.11}$$

for all $p, q \in S$, where $K = K(\delta)$ also depends only on δ.

PROOF. Let f be conformal with $a = f^{-1}(\infty) \notin \Omega$ and let $p, q \in S \subset \Omega$. Then

$$\frac{d(q,a)}{d(p,a)} \le \frac{d(q,p) + d(p,a)}{d(p,a)} \le 1 + \frac{d(p,q)}{\operatorname{dist}(S,\partial\Omega)}$$

and so

$$\left(\frac{d(q,a)}{d(p,a)}\right)^2 \le 1 + \left(\frac{2}{\operatorname{dist}(S,\partial\Omega)} + \frac{\operatorname{diam}(S)}{\operatorname{dist}(S,\partial\Omega)^2}\right) d(p,q).$$

Reversing the roles of p and q yields

$$\left| \left(\frac{d(q,a)}{d(p,a)}\right)^2 - 1 \right| \le K_1(\delta) \frac{d(p,q)}{\operatorname{diam}(S)}$$

with $K_1(\delta) = 2\delta + \delta^2$. If $f(\infty) \neq \infty$ then an application of (3.5) completes the proof of (3.10). If $f(\infty) = \infty$ then (3.10) follows by Proposition 3.2 (in particular (3.1)) and Proposition 3.5. Equation (3.11) follows from (3.10) with $K(\delta) = K_1(\delta) + 1$. □

In view of (3.5), the maximal stretching factor $||Df||$ of a conformal map f is continuous whenever it is defined. For $f : \Omega \to \mathbb{G}$ conformal and S a compact subset of Ω, we denote by

$$||Df||_S := \max\{||Df(p)|| \,:\, p \in S\}. \tag{3.12}$$

From (3.11) we immediately deduce

COROLLARY 3.7. *Let S be a compact subset of a domain Ω. Then*

$$||Df||_S \le K\, ||Df(p)||$$

for all $p \in S$, where $K = K(\delta)$ denotes the constant from Lemma 3.6.

For an arc length parameterized γ defined on $[a,b]$ and taking values in a metric space, we denote by $\int_\gamma g\,ds := \int_a^b g(\gamma(t))\,dt$ the line integral of a real-valued Borel function g along γ.

The following lemma is a standard fact. For technical reasons we temporarily work with the Carnot–Carathéodory metric d_{cc}.

LEMMA 3.8 ($||Df||$ is an upper gradient for f). *Let $f:\Omega\to\Omega'$ be a conformal map between domains in a Carnot group $\mathbb{G}$. Let $\gamma:[a,b]\to\Omega$ be a horizontal curve with $\gamma(a)=p$ and $\gamma(b)=q$, parameterized with respect to arc length in (Ω,d_{cc}). Then*

$$d_{cc}(f(p),f(q)) \le \int_\gamma ||Df(\gamma(s))||\,ds.$$

PROOF. The curve $f\circ\gamma$ is a horizontal curve joining $f(p)$ to $f(q)$. Hence

$$d_{cc}(f(p),f(q)) \le \ell_{cc}(f\circ\gamma) \le \int_\gamma \langle|Df(\gamma(s))\gamma'(s)|\,ds \le \int_\gamma ||Df(\gamma(s))||\,ds.$$

□

3.2. Koebe distortion theorems for Carnot conformal mappings

The estimates of the previous section imply certain theorems in the spirit of the Koebe distortion theorem for conformal mappings of Carnot groups. We first state such theorems for the Carnot–Carathéodory metric, then derive corresponding results for general homogeneous metrics.

PROPOSITION 3.9. *Let $p_0\in\Omega$ and let $r>0$ so that $S:=\overline{B}_{cc}(p_0,2r)\subset\Omega$. Then*

$$f(B_{cc}(p_0,r)) \subset B_{cc}(f(p_0),||Df||_S\,r) \tag{3.13}$$

and

$$\operatorname{diam} f(B_{cc}(p_0,r)) \le ||Df||_S \operatorname{diam}(B_{cc}(p_0,r)). \tag{3.14}$$

In particular, if $\overline{B}_{cc}(p_0,3r)\subset\Omega$, then

$$f(B_{cc}(p_0,r)) \subset B_{cc}(f(p_0),K\,||Df(p_0)||\,r) \tag{3.15}$$

and

$$\operatorname{diam}_{cc} f(B_{cc}(p_0,r)) \le K\,||Df(p_0)|| \operatorname{diam}_{cc}(B_{cc}(p_0,r)) \tag{3.16}$$

for some absolute constant K.

PROOF. Given two points $p,q\in B_{cc}(p_0,r)$ there exists a Carnot–Carathéodory geodesic γ connecting p to q and contained in $B_{cc}(p_0,2r)$. By Lemma 3.8,

$$d_{cc}(f(p),f(q)) \le ||Df||_S\,d_{cc}(p,q).$$

Choosing $q=p_0$ leads to (3.13). Taking the supremum over all $p,q\in B(p_0,r)$ leads to (3.14). Conclusions (3.15) and (3.16) follow via Corollary 3.7, noting that the constant K in that corollary is uniformly bounded for points p_0 such that $\overline{B}_{cc}(p_0,3r)\subset\Omega$ when $S=\overline{B}_{cc}(p_0,2r)$. □

The following distortion estimate is related to the so-called *egg yolk principle* for quasiconformal mappings, see for instance [**32**, p. 93]. We will state the result first for the Carnot–Carathéodory metric, and derive a similar statement for the gauge metric on Iwasawa groups as a corollary. In what follows, L will denote a comparison constant between the Carnot–Carathéodory and gauge metrics in case $\mathbb{G}$ is an Iwasawa group (cf. (1.7)). If $\mathbb{G}$ is not an Iwasawa group, we may take $L = 1$.

PROPOSITION 3.10 (Koebe distortion theorem for conformal maps). *Let* $f : \Omega \to \Omega'$ *be a conformal map between domains in an Iwasawa group* $\mathbb{G}$*, let* $p_0 \in \Omega$ *and let* $r > 0$ *be such that* $\overline{B}_{cc}(p_0, 3r) \subset \Omega$*. Then*

$$B_{cc}(f(p_0), c^{-1}\,||Df(p_0)||r) \subset f(B_{cc}(p_0, r)) \subset B_{cc}(f(p_0), c\,||Df(p_0)||\,r) \tag{3.17}$$

for some constant c *depending only on the constant* K *from Proposition 3.9 and the constant* L *defined above.*

In the proof of Proposition 3.10 we will make use of the following lemma, which provides a quasisymmetry-type estimate for conformal maps in compact subsets of their domain.

LEMMA 3.11. *Let* $f : \Omega \to \Omega'$ *be a conformal map between domains in a Carnot group* $\mathbb{G}$*, let* $p_0 \in \Omega$ *and let* $r > 0$ *be such that* $\overline{B}_{cc}(p_0, 3r) \subset \Omega$*. For* $p_1, p_2 \in \overline{B}_{cc}(p_0, r) \subset \Omega$ *define*

$$\tau := \frac{d_{cc}(f(p_0), f(p_1))}{d_{cc}(f(p_0), f(p_2))}.$$

Then

$$d_{cc}(p_0, p_2) \geq \frac{1}{2\tau L^4} d_{cc}(p_0, p_1). \tag{3.18}$$

PROOF. If f is a similarity the result is obvious. Assume that $\mathbb{G}$ is an Iwasawa group and that f is not a similarity. We appeal to the preservation of cross-ratios by conformal maps. By assumption $q = f^{-1}(\infty) \neq \infty$. Since $f(\Omega) \subset \mathbb{G}$, we must have $q \notin \Omega$. Let d_H denote the gauge metric in $\mathbb{G}$. Appealing to the preservation of the cross ratio, (2.12), by conformal maps (see Remark 2.12), we deduce that

$$\frac{d_H(f(p_0), f(p_1))}{d_H(f(p_0), f(p_2))} = \frac{d_H(p_0, p_1)\, d_H(p_2, q)}{d_H(p_0, p_2)\, d_H(p_1, q)}$$

and so, by (1.7),

$$\tau = \frac{d_{cc}(f(p_0), f(p_1))}{d_{cc}(f(p_0), f(p_2))} \geq \frac{1}{L^4} \frac{d_{cc}(p_0, p_1)\, d_{cc}(p_2, q)}{d_{cc}(p_0, p_2)\, d_{cc}(p_1, q)}.$$

Observe that $d_{cc}(p_2, q) \geq 2r$ since $d_{cc}(p_0, q) \geq 3r$. Therefore,

$$\begin{aligned} \tau L^4 &\geq \frac{d_{cc}(p_2, q)}{d_{cc}(p_1, p_2) + d_{cc}(p_2, q)} \frac{d_{cc}(p_0, p_1)}{d_{cc}(p_0, p_2)} \\ &\geq \frac{d_{cc}(p_2, q)}{2r + d_{cc}(p_2, q)} \frac{d_{cc}(p_0, p_1)}{d_{cc}(p_0, p_2)} \geq \frac{d_{cc}(p_0, p_1)}{2\, d_{cc}(p_0, p_2)}. \end{aligned}$$

□

PROOF OF PROPOSITION 3.10. Since the right hand inclusion in (3.17) has already been proved in Proposition 3.9, it suffices to prove the left hand inclusion. We accomplish this by applying the previous argument to the map $g = f^{-1}$.

Let f, p_0 and $r > 0$ be as in the statement of the proposition. From (3.6) we conclude that $||Dg(f(p_0))|| = ||Df(p_0)||^{-1}$. Let $R > 0$ be the maximal radius such that

$$B_{cc}(f(p_0), 3R) \subset f(B_{cc}(p_0, r)). \tag{3.19}$$

For topological reasons, we conclude that

$$\partial f^{-1}(B_{cc}(f(p_0), 3R)) \cap \partial B_{cc}(p_0, r) \neq \emptyset. \tag{3.20}$$

Moreover, $\overline{B}_{cc}(f(p_0), 3R) \subset f(\overline{B}_{cc}(p_0, r)) \subset \Omega'$. Applying the conclusions of Proposition 3.9 for $g = f^{-1}$ gives

$$f^{-1}(B_{cc}(f(p_0), R)) \subset B_{cc}(p_0, K\,||Dg(f(p_0))||\,R) = B_{cc}(p_0, K||Df(p_0)||^{-1}R)$$

and so

$$B_{cc}(f(p_0), R) \subset f(B_{cc}(p_0, K||Df(p_0)||^{-1}R)). \tag{3.21}$$

In order to use this conclusion in combination with the choice of R we employ Lemma 3.11. Let $s > 0$ be the minimal radius such that

$$B_{cc}(f(p_0), R) \subset f(B_{cc}(p_0, s)).$$

Again for topological reasons we conclude that

$$\partial B_{cc}(p_0, s) \cap \partial f^{-1}(B_{cc}(f(p_0), R)) \neq \emptyset. \tag{3.22}$$

By (3.20) and (3.22) it is possible to select p_1 and p_2 so that

$$d_{cc}(p_0, p_1) = r, \quad d_{cc}(p_0, p_2) = s, \quad d_{cc}(f(p_0), f(p_1)) = 3R$$

and

$$d_{cc}(f(p_0), f(p_2)) = R.$$

Applying Lemma 3.11 with $\tau = 3$ yields $s \geq \frac{1}{6L^4} r$. By the choice of s and (3.21), we obtain $s \leq K||Df(p_0)||^{-1}R$, and so

$$R \geq \frac{1}{6KL^4}||Df(p_0)||\,r.$$

Therefore, using also (3.19)

$$B_{cc}(f(p_0), \frac{1}{6KL^4}||Df(p_0)||\,r) \subset B_{cc}(f(p_0), R) \subset f(B_{cc}(p_0, r)).$$

This completes the proof. □

Finally, we convert the statement of the Koebe distortion theorem for conformal maps back to the gauge metric on Iwasawa groups, for convenience in later work.

COROLLARY 3.12 (Koebe distortion for conformal maps, gauge metric version). *Let $f : \Omega \to \Omega'$ be a conformal map between domains in an Iwasawa group $\mathbb{G}$, let $p_0 \in \Omega$ and let $r > 0$ be such that $\overline{B}_H(p_0, 3Lr) \subset \Omega$. Then*

$$B_H(f(p_0), C^{-1}\,||Df(p_0)||r) \subset f(B_H(p_0, r)) \subset B_H(f(p_0), C\,||Df(p_0)||\,r) \tag{3.23}$$

for some constant C depending only on the constants K and L as before.

The corollary follows immediately from Proposition 3.10 and the inclusions in (2.9).

Remark 3.13. All of the distortion estimates discussed in this chapter continue to hold in an arbitrary Carnot group $\mathbb{G}$ equipped with an arbitrary homogeneous metric d, provided that the mapping $f : \Omega \to \Omega'$ is assumed to be a metric similarity. In this case, (3.1) of Proposition 3.2 is just the definition of a metric similarity, and the remainder of the chapter follows. In particular $r_f = \|Df(p)\| = \|Df\|_\infty$, for $p \in \Omega$. We will return to this point in connection with Definition 4.7 and Remark 4.8, see also subsection 5.1.

CHAPTER 4

Conformal graph directed Markov systems

In this chapter we introduce conformal graph directed Markov systems (GDMS) in Carnot groups. In Section 4.1 we define graph directed Markov systems in metric spaces and we also discuss one important subclass of such systems, namely maximal systems, which will also occur in several subsequent chapters. In Section 4.2 we define conformal GDMS in general Carnot groups and we employ results from Chapter 3 to obtain fundamental distortion properties of conformal GDMS in our setting.

4.1. Graph Directed Markov Systems

A *graph directed Markov system* (GDMS)

$$\mathcal{S} = \big\{V, E, A, t, i, \{X_v\}_{v\in V}, \{\phi_e\}_{e\in E}\big\}$$

consists of

- a directed multigraph (E, V) with a countable set of edges E, frequently referred to also as alphabet, and a finite set of vertices V,
- an incidence matrix $A : E \times E \to \{0, 1\}$,
- two functions $i, t : E \to V$ such that $t(a) = i(b)$ whenever $A_{ab} = 1$,
- a family of non-empty compact metric spaces $\{X_v\}_{v\in V}$,
- a number s, $0 < s < 1$, and
- a family of injective contractions

  $$\{\phi_e : X_{t(e)} \to X_{i(e)}\}_{e\in E}$$

  such that every ϕ_e, $e \in E$, has Lipschitz constant no larger than s.

We will always assume that the alphabet E contains at least two elements, because otherwise the system is trivial. For the sake of brevity we will frequently use the notation $\mathcal{S} = \{\phi_e\}_{e\in E}$ for a GDMS. We will also assume that for every $v \in V$ there exist $e, e' \in E$ such that $t(e) = v$ and $i(e') = v$. A GDMS is called *finite* if E is a finite set. In the particular case when V is a singleton and for every $e_1, e_2 \in E$, $A_{e_1e_2} = 1$ if and only if $t(e_1) = i(e_2)$, the GDMS is called an *iterated function system* (IFS).

We set

$$E^* = \bigcup_{n=0}^{\infty} E^n,$$

and for every $\omega \in E^*$, we denote by $|\omega|$ the unique integer $n \geq 0$ such that $\omega \in E^n$. We call $|\omega|$ the length of ω. We make the convention that $E^0 = \{\emptyset\}$. If $\omega \in E^{\mathbb{N}}$ and $n \geq 1$, we put

$$\omega|_n = \omega_1 \dots \omega_n \in E^n.$$

If $\tau \in E^*$ and $\omega \in E^* \cup E^{\mathbb{N}}$, we define

$$\tau\omega = (\tau_1, \dots, \tau_{|\tau|}, \omega_1, \dots).$$

Given $\omega, \tau \in E^{\mathbb{N}}$, we define $\omega \wedge \tau \in E^{\mathbb{N}} \cup E^*$ to be the longest initial block common to both ω and τ.

Given the $\{0,1\}$ valued matrix $A : E \times E \to \{0,1\}$ we set

$$E_A^{\mathbb{N}} := \{\omega \in E^{\mathbb{N}} : \ A_{\omega_i \omega_{i+1}} = 1 \text{ for all } i \in \mathbb{N}\}.$$

Elements of $E_A^{\mathbb{N}}$ are called *A-admissible*. We also set

$$E_A^n := \{w \in E^{\mathbb{N}} : \ A_{\omega_i \omega_{i+1}} = 1 \text{ for all } 1 \le i \le n-1\}, \qquad n \in \mathbb{N},$$

and

$$E_A^* := \bigcup_{n=0}^{\infty} E_A^n.$$

The elements of these sets are also called A-admissible. For every $\omega \in E_A^*$, we put

$$[\omega] := \{\tau \in E_A^{\mathbb{N}} : \ \tau_{|_{|\omega|}} = \omega\}.$$

Definition 4.1. Let E be a countable alphabet. The $\{0,1\}$ valued matrix $A : E \times E \to \{0,1\}$ is said to be *irreducible* if there exists $\Phi \subset E_A^*$ such that for all $i, j \in E$ there exists $\omega \in \Phi$ for which $i\omega j \in E_A^*$.

Of course if A is irreducible then E_A^* witnesses irreducibility. However, one is naturally interested in sets Φ which witness irreducibility and are as small as possible. Most notably, we say that the matrix A is *finitely irreducible* if there exist finite sets Φ witnessing irreducibility. Finite irreducible systems abound and are natural. The simplest example of a finitely irreducible matrix is the one whose all entries are equal to 1. In that case Φ can be taken to be the empty set. Furthermore, if A is co-finite, i.e. if the set of its zero entries is finite, then it is finitely irreducible; one can take Φ to be any sufficiently "large" element of E. In Chapters 5 and 9 we study in detail several natural examples of Carnot GDMS with finitely irreducible alphabets.

Note that if the alphabet E is finite, then irreducibility and finite irreducibility of A coincide. In dynamical terms irreducibility means that the symbolic dynamical system $\sigma : E_A^{\mathbb{N}} \to E_A^{\mathbb{N}}$ (see Chapter 6 for more details) is *topologically mixing*, i.e. for any two non-empty open sets $U, V \subset E_A^{\mathbb{N}}$ there exists an integer $n \ge 0$ such that

$$\sigma^n(U) \cap V \neq \emptyset.$$

Finite irreducibility is more subtle, it means that there exists a finite set of cylinders (the ones corresponding to the elements of Φ) such that if n is the length of the shortest cylinder contained in U, then $\sigma^n(U)$ contains one of these finitely many cylinders.

We use finite irreducibility to develop a fully fledged thermodynamic formalism in Chapter 6. Using machinery from that chapter (in particular the ergodic theory of Gibbs and equilibrium states) we will obtain various properties of invariant sets of conformal GDMS in Chapters 7, 8 and 10, including Bowen's formula for their Hausdorff dimension. Without irreducibility the corresponding ergodic theory is significantly weaker, less elegant, and more cumbersome hence it will be left untreated in the present monograph.

A GDMS $\mathcal{S}$ is said to be *finitely irreducible* if its associated incidence matrix A is finitely irreducible. Notice that if $\mathcal{S}$ is a finite irreducible GDMS then it is finitely irreducible.

For $\omega \in E_A^*$ we consider the map coded by ω:

$$\phi_\omega = \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)} \qquad \text{if } \omega \in E_A^n. \tag{4.1}$$

For the sake of convenience we will write $t(\omega) = t(\omega_n)$ and $i(\omega) = i(\omega_1)$ for ω as in (4.1).

For $\omega \in E_A^{\mathbb{N}}$, the sets $\{\phi_{\omega|_n}(X_{t(\omega_n)})\}_{n=1}^{\infty}$ form a descending sequence of non-empty compact sets and therefore have nonempty intersection. Since

$$\operatorname{diam}(\phi_{\omega|_n}(X_{t(\omega_n)})) \le s^n \operatorname{diam}(X_{t(\omega_n)}) \le s^n \max\{\operatorname{diam}(X_v) : v \in V\}$$

for every $n \in \mathbb{N}$, we conclude that the intersection

$$\bigcap_{n\in\mathbb{N}} \phi_{\omega|_n}(X_{t(\omega_n)})$$

is a singleton and we denote its only element by $\pi(\omega)$. In this way we define the coding map

$$\pi : E_A^{\mathbb{N}} \to \bigoplus_{v\in V} X_v, \tag{4.2}$$

the latter being a disjoint union of the sets X_v, $v \in V$. The set

$$J = J_{\mathcal{S}} := \pi(E_A^{\mathbb{N}})$$

will be called the *limit set* (or *attractor*) of the GDMS $\mathcal{S}$.

For each $\alpha > 0$, we define a metric d_α on $E^{\mathbb{N}}$ by setting

$$d_\alpha(\omega, \tau) = \mathrm{e}^{-\alpha|\omega\wedge\tau|}. \tag{4.3}$$

It is easy to see that all these metrics induce the same topology. We record the following standard fact concerning the coding map.

Proposition 4.2. *The coding map $\pi : E_A^{\mathbb{N}} \to \bigoplus_{v\in V} X_v$ is Hölder continuous, when $E_A^{\mathbb{N}}$ is equipped with any of the metrics d_α as in (4.3) and $\bigoplus_{v\in V} X_v$ is equipped with the direct sum metric δ.*

To see why Proposition 4.2 is true, observe that if $\omega, \tau \in E_A^{\mathbb{N}}$ with $\rho = \omega \wedge \tau \in E^n$, then

$$d(\pi(\omega), \pi(\tau)) \le \operatorname{diam} \phi_\rho(X_{t(\rho_n)}) \le s^n \max\{\operatorname{diam} X_v : v \in V\}.$$

This also shows that π is Lipschitz continuous from $(E_A^{\mathbb{N}}, d_\alpha)$ to $\bigoplus_{v\in V} X_v$, when $\alpha = \log(1/s)$ because

$$d_\alpha(\omega, \tau) = e^{-\alpha n} = e^{-\log(1/s)^n} = s^n.$$

Definition 4.3. Given a GDMS $\mathcal{S}$ with an incidence matrix A, we define the matrix $\hat{A} : E \times E \to \{0, 1\}$ by

$$\hat{A}_{ab} = \begin{cases} 1 & \text{if } t(a) = i(b) \\ 0 & \text{if } t(a) \ne i(b). \end{cases}$$

The GDMS $\hat{\mathcal{S}}$ is then defined by means of the incidence matrix $\hat{A}$.

Of course,

$$E_A^n \subset E_{\hat{A}}^n, \quad E_A^* \subset E_{\hat{A}}^*, \quad E_A^{\mathbb{N}} \subset E_{\hat{A}}^{\mathbb{N}},$$

and

$$J_{\mathcal{S}} \subset J_{\hat{\mathcal{S}}}.$$

DEFINITION 4.4. A GDMS $\mathcal{S}$ with an incidence matrix A is called *maximal* if $\hat{\mathcal{S}} = \mathcal{S}$. This equivalently means that $A = \hat{A}$ or further equivalently that $A_{ab} = 1$ if and only if $t(a) = i(b)$.

The following proposition asserts that for the study of finite GDMS it is actually enough to restrict our attention to maximal systems.

PROPOSITION 4.5. *If $\mathcal{S}$ is a finite GDMS, then there exists a maximal, finite GDMS $\hat{\mathcal{S}}$ such that $J_{\mathcal{S}} = J_{\hat{\mathcal{S}}}$. Moreover if $\mathcal{S}$ is irreducible then $\hat{\mathcal{S}}$ is also irreducible.*

PROOF. Let $\mathcal{S} = \{V, E, A, t, i, \{X_v\}_{v\in V}, \{\phi_e\}_{e\in E}\}$ be a GDMS. Let

$$\hat{\mathcal{S}} = \{\hat{V}, \hat{E}, \hat{A}, \hat{t}, \hat{i}, \{\hat{X}_v\}_{v\in\hat{V}}, \{\hat{\phi}_e\}_{e\in\hat{E}}\}$$

where

(1) $\hat{V} = E$,
(2) $\hat{E} = \{(a, b) \in E^2 : A_{ab} = 1\}$,
(3) if $\mathtt{e} \in \hat{E}$, $\mathtt{e} = (\mathtt{e}^1, \mathtt{e}^2) \in E^2$,

$$\hat{t}(\mathtt{e}) := \mathtt{e}^2 \text{ and } \hat{i}(\mathtt{e}) := \mathtt{e}^1,$$

(4) the matrix $\hat{A} : \hat{E} \times \hat{E} \to \{0, 1\}$ is defined by

$$\hat{A}_{\mathtt{e},\mathtt{f}} = \begin{cases} 1 & \text{if } \hat{t}(\mathtt{e}) = \hat{i}(\mathtt{f}) \\ 0 & \text{if } \hat{t}(\mathtt{e}) \neq \hat{i}(\mathtt{f}), \end{cases}$$

(5) $\hat{X}_e = \phi_e(X_{t(e)})$ for $e \in \hat{V} = E$,
(6) if $\mathtt{e} = (\mathtt{e}^1, \mathtt{e}^2) \in \hat{E}$, then

$$\hat{\phi}_{\mathtt{e}} = \phi_{\mathtt{e}^1} : \phi_{\mathtt{e}^2}(X_{t(\mathtt{e}^2)}) \to \phi_{\mathtt{e}^1}(X_{t(\mathtt{e}^1)}).$$

Observe that if $\mathtt{ef} \in \hat{E}_A^*$ then $\hat{t}(\mathtt{e}) = \hat{i}(\mathtt{f})$, that is $\mathtt{e}^2 = \mathtt{f}^1$.

The system $\hat{\mathcal{S}}$ is maximal by definition. We will now show that $J_{\mathcal{S}} = J_{\hat{\mathcal{S}}}$. Trivially $J_{\hat{\mathcal{S}}} \subset J_{\mathcal{S}}$. For any $\omega \in E_A^{\mathbb{N}}$ define $\hat{\omega} \in \hat{E}_{\hat{A}}^{\mathbb{N}}$ by $\hat{\omega} = (\hat{\omega}_n)_{n\in\mathbb{N}}$ where $\hat{\omega}_n = (\omega_n, \omega_{n+1}) \in E_A^2$. Notice that

$$\bigcap_{n\in\mathbb{N}} \phi_{\omega|_n}(X_{t(\omega_n)}) = \bigcap_{n\in\mathbb{N}} \hat{\phi}_{\hat{\omega}|_n}(\hat{X}_{\hat{t}(\hat{\omega}_n)}),$$

hence $J_{\mathcal{S}} \subset J_{\hat{\mathcal{S}}}$.

Now suppose that $\mathcal{S}$ is irreducible. Recall that since $\mathcal{S}$ is finite, it is finitely irreducible. Let $\Phi \subset E_A^*$ be a finite set witnessing irreducibility for $\mathcal{S}$. Let $\mathtt{e}, \mathtt{f} \in \hat{E}$. Then there exists some $\tau = (\tau_1, \ldots, \tau_{|\tau|}) \in \Phi$ such that $\mathtt{e}^2 \tau \mathtt{f}^1 \in E_A^*$. Hence

$$\mathtt{e}\tau^0\tau^1 \ldots \tau^{|\tau|}\mathtt{f} \in \hat{E}_{\hat{A}}^*,$$

where $\tau^0 = (\mathtt{e}^2, \tau_1)$, $\tau^m = (\tau_m, \tau_{m+1})$ for $m = 1, \ldots, |\tau| - 1$ and $\tau^{|\tau|} = (\tau_{|\tau|}, \mathtt{f}^1)$. Therefore $\hat{\mathcal{S}}$ is finitely irreducible and the proof is complete. □

We end this section with the following obvious observation.

Remark 4.6. A GDMS is an IFS if and only if it is maximal and the set of vertices is a singleton.

4.2. Carnot conformal graph directed Markov systems

We now introduce the primary objects of study in this monograph.

Definition 4.7. A graph directed Markov system is called *Carnot conformal* if the following conditions are satisfied.

(i) For every vertex $v \in V$, X_v is a compact connected subset of a fixed Carnot group $(\mathbb{G}, d)$ and $X_v = \overline{\mathrm{Int}(X_v)}$.

(ii) (*Open set condition* or *OSC*). For all $a, b \in E$, $a \neq b$,
$$\phi_a(\mathrm{Int}(X_{t(a)})) \cap \phi_b(\mathrm{Int}(X_{t(b)})) = \emptyset.$$

(iii) For every vertex $v \in V$ there exists an open connected set $W_v \supset X_v$ such that for every $e \in E$ with $t(e) = v$, the map ϕ_e extends to a conformal diffeomorphism of W_v into $W_{i(e)}$.

A graph directed Markov system is called *weakly Carnot conformal* if only conditions (i) and (iii) from Definition 4.7 are required to be satisfied; (ii) may hold or not.

Remark 4.8. As previously indicated in Remark 3.13, Definition 4.7 applies to conformal mappings of Iwasawa groups with the gauge metric, or to affine similarities of Carnot groups with the Carnot–Carathéodory metric. Nevertheless, the subsequent theory also applies to graph directed Markov systems comprised of contractive metric similarities of Carnot groups equipped with any homogeneous metrics. It would be natural to term such systems *Carnot similarity GDMS*. Abusing terminology, we choose to use the term *Carnot conformal GDMS* to refer to all of these cases. Hence our theory applies when

- $(\mathbb{G}, d)$ is an Iwasawa group, $d = d_H$ and the maps ϕ_e are conformal.
- $(\mathbb{G}, d)$ is a Carnot group not of Iwasawa type, $d = d_{cc}$ and the maps ϕ_e are conformal.
- $(\mathbb{G}, d)$ is a Carnot group, d is **any** homogeneous metric and the maps ϕ_e are metric similarities.

For each $v \in V$, we select a compact set S_v such that $X_v \subset \mathrm{Int}(S_v) \subset S_v \subset W_v$. Moreover the sets $S_v, v \in V$, are chosen to be pairwise disjoint. The assumption that the compact sets $(X_v)_{v \in V}$ are pairwise disjoint is not essential. One could modify the definition of a GDMS in order to avoid this, see Remark 4.20 for more details. We also set

$$X := \bigcup_{v \in V} X_v \text{ and } S := \bigcup_{v \in V} S_v. \tag{4.4}$$

Since $\max\{\mathrm{diam}(X_v) : v \in V\}$ is finite and $\min\{\mathrm{dist}(X_v, \mathbb{G} \setminus \mathrm{Int}(S_v)) : v \in V\}$ is positive, the following is an immediate consequence of Lemma 3.6.

Lemma 4.9 (Bounded Distortion Property). *Let $S = \{\phi_e\}_{e \in E}$ be a weakly Carnot conformal GDMS on $(\mathbb{G}, d)$. There exists a constant K so that*
$$\left| \frac{||D\phi_\omega(p)||}{||D\phi_\omega(q)||} - 1 \right| \leq K d(p, q)$$

and

$$K^{-1} \leq \frac{||D\phi_\omega(p)||}{||D\phi_\omega(q)||} \leq K$$

for every $\omega \in E_A^*$ and every pair of points $p, q \in S_{t(\omega)}$.

Recalling (3.12), for $\omega \in E_A^*$ we denote

$$||D\phi_\omega||_\infty := ||D\phi_\omega||_{S_{t(\omega)}}.$$

From Lemma 4.9 and (3.6) we easily see that

$$K^{-1}||D\phi_{\omega|_{(n-1)}}||_\infty\,||D\phi_{\omega_n}||_\infty \leq ||D\phi_\omega||_\infty \leq ||D\phi_{\omega|_{(n-1)}}||_\infty\,||D\phi_{\omega_n}||_\infty \tag{4.5}$$

whenever $\omega \in E_A^n$. More generally if $\omega \in E_A^*$ and $\omega = \tau\upsilon$ for some $\tau, \upsilon \in E_A^*$,

$$K^{-1}||D\phi_\tau||_\infty\,||D\phi_\upsilon||_\infty \leq ||D\phi_\omega||_\infty \leq ||D\phi_\tau||_\infty\,||D\phi_\upsilon||_\infty. \tag{4.6}$$

We record the following consequence of Corollary 3.12 and (3.11).

Corollary 4.10. *Let $S = \{\phi_e\}_{e\in E}$ be a weakly Carnot conformal GDMS on a Carnot group $(\mathbb{G}, d)$. For all finite words $\omega \in E_A^*$, all $p \in X_{t(\omega)}$ and all $0 < r < \operatorname{dist}(X_{t(\omega)}, \partial S_{t(\omega)})/3\,L$,*

$$B(\phi_\omega(p), (K\,C)^{-1}\|D\phi_\omega\|_\infty r) \subset \phi_\omega(B(p,r)) \subset B(\phi_\omega(p), C\|D\phi_\omega\|_\infty r). \tag{4.7}$$

We shall now prove the following Lipschitz estimate.

Lemma 4.11. *Let $S = \{\phi_e\}_{e\in E}$ be a weakly Carnot conformal GDMS on $(\mathbb{G}, d)$. There exists a constant $\Lambda \geq 1$ such that*

$$d\big(\phi_\omega(p), \phi_\omega(q)\big) \leq \Lambda\,||D\phi_\omega||_\infty d(p,q) \tag{4.8}$$

for all finite words $\omega \in E_A^*$ and all $p, q \in X_{t(\omega)}$. In particular,

$$\operatorname{diam}\big(\phi_\omega(X_{t(\omega)})\big) \leq \Lambda M \|D\phi_\omega\|_\infty, \tag{4.9}$$

where $M := \operatorname{diam} X$.

Remark 4.12. Note that if the set $X_{t(\omega)}$ were geodesically convex, the previous result would follow immediately from Lemma 3.8. However, there are very few nontrivial geodesically convex sets in nonabelian Carnot groups. Convexity in sub-Riemannian Carnot groups remains a topic of intense focus. We refer the interested reader to the foundational papers by Danielli–Garofalo–Nhieu [**26**] and Lu–Manfredi–Stroffolini [**39**], which introduced the nowadays established notion of *horizontal convexity (H-convexity)* in Carnot groups.

Proof of Lemma 4.11. Fix

$$\eta_{\mathcal{S}} := \min\{\operatorname{dist}(X_v, \partial S_v) : v \in V\}/3L > 0 \tag{4.10}$$

where, as before, L denotes a quasiconvexity constant for $(\mathbb{G}, d)$. Fix also some $\omega \in E_A^*$. If $d(p,q) < \eta_{\mathcal{S}}$ then there exists some $\varepsilon \in (0,1)$ such that $(1+\varepsilon)d(p,q) < \eta_{\mathcal{S}}$ and Corollary 4.10 implies that

$$\phi_\omega(B(p,(1+\varepsilon)d(p,q))) \subset B(\phi_\omega(p), C\|D\phi_\omega\|_\infty(1+\varepsilon)d(p,q)).$$

Thus

$$d(\phi_\omega(p), \phi_\omega(q)) \leq 2C\|D\phi_\omega\|_\infty d(p,q) \tag{4.11}$$

and (4.8) follows in this case. Hence we can assume $d(p,q) \geq \eta_{\mathcal{S}}$. Since each X_v is compact and connected and the vertex set V is finite, there exists an integer

$N \geq 1$ so that for each $v \in V$, the space X_v can be covered by finitely many balls $\mathcal{B}_v := \{B(p_{v,1}, \eta_{\mathcal{S}}/2), \dots, B(p_{v,N}, \eta_{\mathcal{S}}/2)\}$ with centers $p_{v,1}, \dots, p_{v,N}$ in X_v and with the property that any two points of X_v lie in a connected union of balls chosen from $\mathcal{B}_v$.

Therefore, for every vertex $v \in V$ and all points $p, q \in X_v$ there are $k \leq N$ points $p = z_0, z_1, \dots, z_k = q$ in S_v such that for all $i = 0, \dots, k-1$ the consecutive points z_i, z_{i+1} belong to some ball $B(p_{v,n_i}, \eta_{\mathcal{S}}/2) \in \mathcal{B}_v$. Now by (2.9) $z_i, z_{i+1} \in B_{cc}(p_{v,n_i}, L\eta_{\mathcal{S}}/2)$. Hence if $\gamma_{z_i,z_{i+1}}$ is the geodesic horizontal curve joining the points z_i and z_{i+1}, we deduce, for example by the *segment property* [**12**, Corollary 5.15.6], that $\gamma_{z_i,z_{i+1}} \in B_{cc}(p_{v,n_i}, 3L\eta_{\mathcal{S}}/2)$. Then again by (2.9) and the choice of $\eta_{\mathcal{S}}$ we deduce that $\gamma_{z_i,z_{i+1}} \subset \mathrm{Int}(S_v)$. Thus for $v = t(\omega)$ an application of Lemma 3.8 gives that for all $i = 0, \dots, k-1$

$$d_{cc}(\phi_\omega(z_i), \phi_\omega(z_{i+1})) \leq \|D\phi\|_\infty d_{cc}(z_i, z_{i+1}). \tag{4.12}$$

Moreover note that $d(z_i, z_{i+1}) \leq \eta_{\mathcal{S}} \leq d(p,q)$ for all $i = 1, 2, \dots, k$. Using (4.12) and (1.7) we get

$$\begin{aligned} d(\phi_\omega(p), \phi_\omega(q)) &\leq \sum_{i=0}^{k-1} d(\phi_\omega(z_i), \phi_\omega(z_{i+1})) \\ &\leq \sum_{i=0}^{k-1} L||D\phi_\omega||_\infty d(z_i, z_{i+1}) \\ &\leq Lk||D\phi_\omega||_\infty d(p,q) \leq LN||D\phi_\omega||_\infty d(p,q). \end{aligned}$$

Recalling also (4.11) the proof is complete upon setting $\Lambda := \max\{2C, LN\}$. ☐

Let $R_{\mathcal{S}} > 0$ be the radius of the largest open ball that can be inscribed in any of the sets X_v, $v \in V$. Let $p_v \in \mathrm{Int}(X_v)$ be the centers of balls of this radius inscribed in the sets X_v. As an immediate consequence of Corollary 4.10 and (2.10) we get the following conclusions.

LEMMA 4.13. *Let $S = \{\phi_e\}_{e \in E}$ be a weakly Carnot conformal GDMS on $(\mathbb{G}, d)$. Let $\tilde{R}_{\mathcal{S}} = \min\{R_{\mathcal{S}}, \eta_{\mathcal{S}}\}$. For all finite words $\omega \in E_A^*$ we have*

$$\phi_\omega(\mathrm{Int}(X_{t(\omega)})) \supset B(\phi_\omega(p_{t(\omega)}), (K\,C)^{-1}\|D\phi_\omega\|_\infty \tilde{R}_{\mathcal{S}}), \tag{4.13}$$

and hence

$$\mathrm{diam}(\phi_\omega(X_{t(\omega)})) \geq 2(K\,C)^{-1}\|D\phi_\omega\|_\infty \tilde{R}_{\mathcal{S}}. \tag{4.14}$$

LEMMA 4.14. *Let $S = \{\phi_e\}_{e \in E}$ be a weakly Carnot conformal GDMS. Then for every $\omega \in E_A^*$ and every pair of points $p, q \in X_{t(\omega)}$,*

$$d(\phi_\omega(p), \phi_\omega(q)) \geq (L^2 K)^{-1} \kappa_0 \|D\phi_\omega\|_\infty d(p,q),$$

where

$$\kappa_0 = \min\left\{\left\{\frac{\mathrm{dist}(X_v, \partial S_v)}{\mathrm{diam}(X_v)}\right\}_{v \in V}, 1\right\}.$$

PROOF. Let $\omega \in E_A^*$ and $p, q \in X_{t(\omega)}$. We will consider two cases. We first assume that the the arc of the Carnot–Carathéodory geodesic curve joining $\phi_\omega(p)$

and $\phi_\omega(q)$ is contained in $\phi_\omega(S_{t(\omega)})$. Then by Lemma 3.8 and (3.6) there exists some $\xi \in S_{t(\omega)}$ such that,

$$\begin{aligned} d_{cc}(p,q) &= d_{cc}(\phi_\omega^{-1}(\phi_\omega(p)), \phi_\omega^{-1}(\phi_\omega(q))) \\ &\leq \|D\phi_\omega^{-1}(\phi_\omega(\xi))\|\, d_{cc}(\phi_\omega(p), \phi_\omega(q)) \\ &= \|D\phi_\omega(\xi)\|^{-1} d_{cc}(\phi_\omega(p), \phi_\omega(q)). \end{aligned}$$

Hence by Lemma 4.9,

$$\begin{aligned} Ld(\phi_\omega(p), \phi_\omega(q)) &\geq d_{cc}(\phi_\omega(p), \phi_\omega(q)) \\ &\geq \|D\phi_\omega(\xi)\|\, d_{cc}(\phi_\omega(p), \phi_\omega(q)) \\ &\geq K^{-1}\|D\phi_\omega\|_\infty d(p,q). \end{aligned} \tag{4.15}$$

Therefore if the arc of the geodesic connecting $\phi_\omega(p)$ and $\phi_\omega(q)$ lies inside $\phi_\omega(S_{t(\omega)})$,

$$d(\phi_\omega(p), \phi_\omega(q)) \geq (KL)^{-1}\|D\phi_\omega\|_\infty d(p,q). \tag{4.16}$$

If the arc of the geodesic $\gamma : [0,T] \to \mathbb{G}$ connecting $\phi_\omega(p)$ and $\phi_\omega(q)$ is not contained in $\phi_\omega(S_{t(\omega)})$, let

$$t_0 := \min\{t \in (0,T) : \gamma(t) \in \partial\phi_\omega(S_{t(\omega)})\}.$$

Hence if $z = \gamma(t_0) \in \partial\phi_\omega(S_{t(\omega)})$ there exists some $\zeta \in \partial S_{t(\omega)}$ such that $z = \phi_\omega(\zeta)$. Using (4.16) and the segment property [**12**, Corollary 5.15.6] of CC-geodesics, we have

$$\begin{aligned} Ld(\phi_\omega(p), \phi_\omega(q)) &\geq d_{cc}(\phi_\omega(p), \phi_\omega(q)) \geq d_{cc}(\phi_\omega(p), \phi_\omega(\zeta)) \\ &\geq (LK)^{-1}\|D\phi_\omega\|_\infty d_{cc}(p,\zeta) \\ &\geq (LK)^{-1}\|D\phi_\omega\|_\infty d(p,\zeta) \\ &\geq (LK)^{-1}\|D\phi_\omega\|_\infty \operatorname{dist}(X_{t(\omega)}, \partial S_{t(\omega)}) \\ &\geq (LK)^{-1}\|D\phi_\omega\|_\infty d(p,q)\frac{\operatorname{dist}(X_{t(\omega)}, \partial S_{t(\omega)})}{\operatorname{diam}(X_{t(\omega)})}. \end{aligned}$$

Thus,

$$d(\phi_\omega(p), \phi_\omega(q)) \geq (L^2K)^{-1}\kappa_0\|D\phi_\omega\|_\infty d(p,q),$$

and the proof follows. □

PROPOSITION 4.15. *Let $S = \{\phi_e\}_{e\in E}$ be a weakly Carnot conformal GDMS such that $\sharp(J_{\mathcal{S}} \cap X_v) > 1$ for all $v \in V$. Then for every $\omega \in E_A^*$,*

$$\operatorname{diam}(\phi_\omega(J_{\mathcal{S}} \cap X_{t(\omega)})) \geq (2L^2K)^{-1}\kappa_0\mu_0\|D\phi_\omega\|_\infty, \tag{4.17}$$

where κ_0 is as in Lemma 4.14 and $\mu_0 = \min\{\operatorname{diam}(J_{\mathcal{S}} \cap X_v)\}$.

PROOF. Notice that for every $v \in V$ there exist points $p_v, q_v \in J_{\mathcal{S}} \cap X_v$ such that $d(p_v, q_v) \geq \mu_0/2$. Hence if $\omega \in E_A^*$, Lemma 4.14 implies that

$$\begin{aligned} \operatorname{diam}(\phi_\omega(J_{\mathcal{S}} \cap X_{t(\omega)})) &\geq d(\phi_\omega(p_{t(\omega)}), \phi_\omega(q_{t(\omega)})) \\ &\geq (L^2K)^{-1}\kappa_0\|D\phi_\omega\|_\infty d(p_{t(\omega)}, q_{t(\omega)}) \\ &\geq (2L^2K)^{-1}\kappa_0\mu_0\|D\phi_\omega\|_\infty, \end{aligned}$$

and the proof is complete. □

Lemmas 4.9 and 4.11 imply that the function $p \mapsto \log\|D\phi_\omega(p)\|$ is locally Lipschitz. This fact is proved in the following lemma.

LEMMA 4.16. *If $S = \{\phi_e\}_{e \in E}$ is a weakly Carnot conformal GDMS, then*

$$\big|\log \|D\phi_\omega(p)\| - \log \|D\phi_\omega(q)\|\big| \leq \frac{\Lambda K}{1-s} d(p,q)$$

for all $\omega \in E_A^*$ and all $p,q \in X_{t(\omega)}$. Here K denotes the constant from Lemma 4.9 while Λ denotes the constant from Lemma 4.11.

PROOF. For every $\omega \in E_A^*$, say $\omega \in E_A^n$, and every $z \in X_{t(\omega)}$ put

$$z_k = \phi_{\omega_{n-k+1}} \circ \phi_{\omega_{n-k+2}} \circ \cdots \circ \phi_{\omega_n}(z)$$

Put also $z_0 = z$. In view of Lemma 4.9 and Lemma 4.11, for any points $p, q \in X_{t(\omega)}$, we get

$$\begin{aligned}
\big|\log \|D\phi_\omega(p)\| - \log \|D\phi_\omega(q)\|\big| &= \left| \sum_{j=1}^{n} \big(\log \|D\phi_{\omega_j}(p_{n-j})\| - \log \|D\phi_{\omega_j}(q_{n-j})\|\big) \right| \\
&\leq \sum_{j=1}^{n} \big| \log \|D\phi_{\omega_j}(p_{n-j})\| - \log \|D\phi_{\omega_j}(q_{n-j})\| \big| \\
&\leq \sum_{j=1}^{n} \frac{\big|\|D\phi_{\omega_j}(p_{n-j})\| - \|D\phi_{\omega_j}(q_{n-j})\|\big|}{\min\{\|D\phi_{\omega_j}(p_{n-j})\|, \|D\phi_{\omega_j}(q_{n-j})\|\}} \\
&\leq \sum_{j=1}^{n} K \frac{\big|\|D\phi_{\omega_j}(p_{n-j})\| - \|D\phi_{\omega_j}(q_{n-j})\|\big|}{||D\phi_{\omega_j}||_\infty} \\
&\leq \sum_{j=1}^{n} \Lambda K d(p_{n-j}, q_{n-j}) \\
&\leq \Lambda K \sum_{j=1}^{n} s^{n-j} d(p,q) \leq \frac{\Lambda K}{1-s} d(p,q).
\end{aligned}$$

The proof is complete. □

In several instances we are going to need slightly stronger versions of Lemmas 4.11 and 4.16. We gather them in the following remark.

REMARK 4.17. Let $S = \{\phi_e\}_{e \in E}$ be a weakly Carnot conformal GDMS on $(\mathbb{G}, d)$. Set

$$N_v := B(X_v, \text{dist}(X_v, \partial S_v)/2), v \in V.$$

Arguing exactly as in the proof of Lemma 4.11, one can show that there exists some Λ_0 such that for all $\omega \in E_A^*$ and $p, q \in N_{t(\omega)}$,

$$d(\phi_\omega(p), \phi_\omega(q)) \leq \Lambda_0 \|D\phi_\omega\|_\infty d(p,q). \tag{4.18}$$

Without loss of generality we can assume that $\Lambda_0 \geq \Lambda$. Using (4.18) as in the proof of Lemma 4.16 we also obtain that for all $\omega \in E_A^*$ and all $p, q \in N_{t(\omega)}$

$$\big|\log \|D\phi_\omega(p)\| - \log \|D\phi_\omega(q)\|\big| \leq \frac{\Lambda_0 K}{1-s} d(p,q). \tag{4.19}$$

LEMMA 4.18. *If $S = \{\phi_e\}_{e \in E}$ is a Carnot conformal GDMS, then*

$$\sum_{e \in E} \|D\phi_e\|_\infty^Q < \infty.$$

PROOF. Let $m_0 = \min\{|\mathrm{Int}(X_v)| : v \in V\} > 0$. By the open set condition, Theorem 3.4 and Lemma 4.9

$$|\mathrm{Int}(X)| \geq \sum_{e\in E} |\phi_e(\mathrm{Int}(X_{t(e)}))| = \sum_{e\in E} \int_{\mathrm{Int}(X_{t(e)})} \|D\phi_e(p)\|^Q dp$$
$$\geq \sum_{e\in E} K^{-Q} |\mathrm{Int}(X_{t(e)})| \, \|D\phi_e\|_\infty^Q \geq m_0 K^{-Q} \sum_{e\in E} \|D\phi_e\|_\infty^Q.$$

Therefore

$$\sum_{e\in E} \|D\phi_e\|_\infty^Q \leq K^Q m_0^{-1} |\mathrm{Int}(X)| < \infty.$$

Now by (4.9),

$$\sum_{e\in E} \mathrm{diam}(\phi_e(X_{t(e)}))^Q \leq (\Lambda M)^Q \sum_{e\in E} \|D\phi_e\|_\infty^Q < \infty.$$

□

REMARK 4.19. If $S = \{\phi_e\}_{e\in E}$ is a Carnot conformal GDMS without loss of generality we can identify $E = \mathbb{N}$. In that case Lemma 4.18 implies that $\lim_{n\to\infty} \mathrm{diam}\,\phi_n(X_{t(n)}) = 0$. Several times in the following we will not make this identification explicit, and we will instead write $\lim_{e\in E} \mathrm{diam}\,\phi_e(X_{t(e)}) = 0$.

In this section, as well as in the some subsequent chapters, we are assuming that the sets $X_v, v \in V$, (and as a result the sets S_v as well) are disjoint. Although this assumption simplifies the proofs of some of our results, it is not essential. We will now describe how a GDMS $\mathcal{S}$ can be lifted to a new GDMS $\tilde{\mathcal{S}}$ such that $\mathcal{S}$ and $\mathcal{S}'$ have essentially the same limit sets but the compact sets $\tilde{X}_v$, corresponding to $\tilde{\mathcal{S}}$, are disjoint. For the sake of clarity and in order not to overly complicate the exposition of the material in this monograph, we chose to use GDMS with disjoint corresponding compact sets X_v instead of the more formal route presented in the following remark.

REMARK 4.20. Let $\mathcal{S} = \{V, E, A, t, i, \{X_v\}_{v\in V}, \{\phi_e\}_{e\in E}\}$ be a GDMS such that the sets X_v are compact subsets of a metric space (M, d). Then $\tilde{M} = M \times V$ is a metric space endowed with the product metric $\tilde{d} = d + d_0$ where d_0 denotes the discrete metric on V. Let

$$\tilde{X}_v = X_v \times v, v \in V,$$

and notice that the sets $\tilde{X}_v$ are compact subsets of $\tilde{M}$. For every $e \in E$ we define maps $\tilde{\phi}_e : \tilde{X}_{t(e)} \to \tilde{X}_{i(e)}$ by

$$\tilde{\phi}_e(x, t(e)) = (\phi_e(x), i(e)).$$

Notice that the maps $\tilde{\phi}_e$ are contractions with respect to the metric $\tilde{d}$ and they have the same contraction ratios as the maps ϕ_e. We also define a projection $\tilde{\pi} : E_A^N \to \tilde{M}$ by

$$\tilde{\pi}(\omega) = (\pi(\omega), i(\omega)), \quad \omega \in E_A^{\mathbb{N}}.$$

Then $\tilde{\pi}(E_A^{\mathbb{N}}) = J_{\mathcal{S}} \times V$. We will call

$$\tilde{\mathcal{S}} = \{V, E, A, t, i, \{\tilde{X}_v\}_{v\in V}, \{\tilde{\phi}_e\}_{e\in E}\}$$

the *formal lift* of $\mathcal{S}$.

CHAPTER 5

Examples of GDMS in Carnot groups

This chapter contains a variety of examples of conformal GDMS in Iwasawa groups and similarity GDMS in general Carnot groups. First, we consider self-similar iterated function systems satisfying the open set condition. The main novelty here is that we include also the case of self-similar IFS with infinite generating set. Finite self-similar IFS in general Carnot groups have previously been studied in [7] and [9]. Next, we give a genuinely conformal (i.e., not self-similar) example of a Carnot conformal GDMS whose invariant set is a Cantor set. Following that, we define a class of conformal IFS in Iwasawa groups which are of continued fraction type. Continued fractions in the first Heisenberg group have been studied by Lukyanenko and Vandehey. The examples which we give here correspond to subsystems associated to continued fractions with restricted digits. Finally, we show how conformal GDMS arise in relation to complex hyperbolic Schottky groups. The diversity of these examples justifies our aim of providing a unified framework for the study of conformal dynamical systems in Iwasawa and other Carnot groups.

5.1. Infinite self-similar iterated function systems

Let $\mathbb{G}$ be a Carnot group (not necessarily of Iwasawa type) equipped with any homogeneous metric. Let E be a countable indexing set (either finite or countably infinite) and, for each $e \in E$, let $\phi_e : \mathbb{G} \to \mathbb{G}$ be a contractive metric similarity with contraction ratio $r_e < 1$. For example, ϕ_e could be a contractive homothety (composition of a left translation and a contractive dilation). We always assume that

$$\sup\{r_e : e \in E\} < 1;$$

needless to say, this assumption is automatically satisfied if E is finite. The collection $\{\phi_e : e \in E\}$ is a self-similar iterated function system in $\mathbb{G}$. Assuming the open set condition, it follows (see, e.g., Lemma 4.18) that

$$\sum_{e \in E} r_e^Q < \infty. \tag{5.1}$$

REMARK 5.1. Self-similar IFS with finite index set E were previously studied in [7] and [9], where formulas for the Hausdorff dimension of invariant sets were established. In Chapter 9 we will extend such results to the case of countably infinite index set, as an application of the general dimension theory developed in the following chapters.

5.2. Iwasawa conformal Cantor sets

We now present a very general method for constructing non-trivial, i.e. not consisting of similarities only, conformal iterated function systems in any Iwasawa group $\mathbb{G}$. Let $G \subset \mathbb{G}$ be an open set such that $o \notin \overline{G}$ and $\overline{G}$ is compact. Let

$P := (p_n)_{n=1}^{\infty} \subset G$ be a discrete sequence and let $d_n = \inf_{m \neq n} d(p_n, p_m)$. We will assume that $\lim_{n\to\infty} d_n = 0$ and that $\mathrm{dist}(P, \partial G) > \sup_{n\in\mathbb{N}} d_n$.

We now construct the conformal iterated function system. Note that

$$o \in \ell_{\mathcal{J}(p_n)^{-1}} \circ \mathcal{J}(G)$$

for every $n \in \mathbb{N}$. Let $s \in (0,1)$ and choose real numbers $(r_n)_{n=1}^{\infty}$ such that $r_n < s$ for all $n \in \mathbb{N}$ and

$$r_n \,\mathrm{diam}\, \mathcal{J}(G) = \mathrm{diam}(\delta_{r_n} \circ \ell_{\mathcal{J}(p_n)^{-1}} \circ \mathcal{J}(G)) < d_n/2.$$

We consider the iterated function system

$$\mathcal{S} = \{\phi_n : \overline{G} \to \overline{G}\}_{n\in\mathbb{N}}$$

where

$$\phi_n = \ell_{p_n} \circ \delta_{r_n} \circ \ell_{\mathcal{J}(p_n)^{-1}} \circ \mathcal{J}, \qquad n \in \mathbb{N}.$$

The functions ϕ_n are non-affine conformal maps. Moreover all ϕ_n's are injective contractions with contraction ratios uniformly bounded by $s < 1$. It follows easily that $\mathcal{S}$ satisfies the open set condition since $\phi_n(G) \subset G$ for all $n \in \mathbb{N}$ and $\phi_n(G) \cap \phi_l(G) = \emptyset$ for all $n, l \in \mathbb{N}$, $n \neq l$. By Lemma 4.18,

$$\sum_{n=1}^{\infty} ||D\phi_n||^Q < \infty; \tag{5.2}$$

equation (5.2) can also be easily derived directly:

$$\sum_{n=1}^{\infty} ||D\phi_n||^Q \lesssim \sum_{n=1}^{\infty} r_n^Q \lesssim \sum_{n=1}^{\infty} \left(\frac{d_n}{2}\right)^Q \lesssim |G| < \infty.$$

5.3. Continued fractions in Iwasawa groups

We start this section by introducing a version of integer lattices in Carnot groups of step 2.

5.3.1. Integer lattices in Carnot groups of step two. Let $\mathbb{G} \cong \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ be a Carnot group of step two, equipped with a homogeneous metric d. It follows by [**12**, Section 3.2]—see also the discussion associated to the Baker–Campbell–Hausdorff–Dynkin formula (1.3), especially (1.5)—that the group law $*$ in $\mathbb{G}$ has the following form; if $p, q \in \mathbb{G}$ such that $p = (z,t), z \in \mathbb{R}^{m_1}, t \in \mathbb{R}^{m_2}$, and $q = (w,s), w \in \mathbb{R}^{m_1}, s \in \mathbb{R}^{m_2}$, then

$$p * q = (z + w, t + s + (B^i z \cdot w)_{i=1}^{m_2}) \tag{5.3}$$

where the *structure matrices* B^i's are skew-symmetric $m_1 \times m_1$ matrices with real coefficients and $\cdot$ denotes the usual inner product in $\mathbb{R}^{m_1}$. We remark that if $p = (z,t), q = (w,s) \in \mathbb{G}$ such that $z = 0$ or $w = 0$ then

$$B^i z \cdot w = 0 \text{ for all } i = 1, \dots, m_2. \tag{5.4}$$

Let

$$\mathbb{G}(\mathbb{Z}) = \{p = (z,t) \in \mathbb{G} : z \in \mathbb{Z}^{m_1} \text{ and } t \in \mathbb{Z}^{m_2}\}.$$

In the following we are going to show that if the matrices $(B^i)_{i=1}^{m_2}$ associated with the group law of $\mathbb{G}$ have integer coefficients, then $\mathbb{G}(\mathbb{Z})$ shares several properties with the usual Euclidean integer lattices.

THEOREM 5.2. *Let $\mathbb{G}$ be a Carnot group of step two with group operation $*$ and let d be a homogeneous metric on $\mathbb{G}$. If the structure matrices B^i of (5.3) lie in $\mathbb{Z}^{m_1\times m_1}$ for all $i=1,\dots,m_2$ then*

(i) $\gamma_1 * \gamma_2 \in \mathbb{G}(\mathbb{Z})$ *for all* $\gamma_1,\gamma_2 \in \mathbb{G}(\mathbb{Z})$,

(ii) there exists an absolute positive constant $A_1 = A_1(\mathbb{G}, d)$ such that for all $\gamma \in \mathbb{G}(\mathbb{Z}) \setminus \{o\}$

$$d(\gamma, o) \geq A_1,$$

(iii) there exists an absolute positive constant $A_2 = A_2(\mathbb{G}, d)$ such that for all $p \in \mathbb{G}$ there exists some $\gamma_p \in \mathbb{G}(\mathbb{Z})$ such that

$$d(p, \gamma_p) \leq A_2.$$

PROOF. The proof of (i) is immediate since $B^i \in \mathbb{Z}^{m_1\times m_1}$.

We now move to the proof of (ii). Note that the function $\|\cdot\|_d : \mathbb{G} \to [0,\infty)$ defined by $\|p\|_d = d(p,o)$ for $p \in \mathbb{G}$ is a homogeneous norm; i.e. it is a continuous function with respect to the Euclidean topology of $\mathbb{R}^{m_1+m_2}$, $\|\delta_r(p)\|_d = r\|p\|_d$ for all $r > 0$ and $p \in \mathbb{G}$, $\|p\|_d > 0$ if and only if $p \neq o$, and $\|p^{-1}\|_d = \|p\|_d$ for all $p \in \mathbb{G} \setminus \{o\}$. For $p = (z,t) \in \mathbb{G}$ let

$$\|p\| = (|z|^4 + |t|^2)^{1/4}.$$

It follows easily that $\|\cdot\|$ is a homogeneous quasi-norm in $\mathbb{G}$, [**12**, Section 5.1]. Since all homogeneous quasi-norms are globally equivalent, see e.g. [**12**, Proposition 5.1.4], we conclude that there exists an absolute positive constant A_1 such that

$$A_1\|p\| \leq \|p\|_d \leq A_1^{-1}\|p\|, \tag{5.5}$$

for all $p \in \mathbb{G}$. But $\|\gamma\| \geq 1$ whenever $\gamma \in \mathbb{G}(\mathbb{Z}) \setminus \{o\}$ and hence (ii) follows by (5.5).

We will now prove (iii). Let

$$K_0 = \{p = (w,s) \in \mathbb{G} : w \in [-1/2, 1/2]^{m_1} \text{ and } s \in [-1/2,1/2]^{m_2}\}.$$

We will first show that

$$\mathbb{G} \subset \bigcup_{\gamma \in \mathbb{G}(\mathbb{Z})} \gamma * K_0. \tag{5.6}$$

Let $p = (z,t) \in \mathbb{G}$, then there exists some $\gamma_1 = (\gamma_z, 0) \in \mathbb{G}(\mathbb{Z})$ such that

$$(z,0) \in \gamma_1 * K_0 = \{(\gamma_z + w, t + (B^i\gamma_z \cdot w)_{i=1}^{m_2}) : (w,t) \in K_0\}. \tag{5.7}$$

Now notice that if $\gamma_2 \in \mathbb{G}(\mathbb{Z})$ such that $\gamma_2 = (0, \gamma_t)$, then recalling (5.4)

$$\gamma_2 * \gamma_1 * K_0 = \{(\gamma_z + w, \gamma_t + s + (B^i\gamma_z \cdot w)_{i=1}^{m_2}) : (w,t) \in K_0\}. \tag{5.8}$$

Therefore we can now choose $\gamma_t := ((\gamma_t)_1, \dots, (\gamma_t)_{m_2}) \in \mathbb{Z}^{m_2}$ such that

$$|t_i - B^i\gamma_z \cdot w - (\gamma_t)_i| \leq 1/2$$

for all $i = 1,\dots,m_2$. Therefore there exists some $s \in [-1/2,1/2]^{m_2}$ such that

$$t = \gamma_t + (B^i\gamma_z \cdot w)_{i=1}^{m_2} + s. \tag{5.9}$$

If $\gamma = (\gamma_z, \gamma_t) \in \mathbb{G}(\mathbb{Z})$ then by (5.7), (5.8) and (5.9) we conclude that $p \in \gamma * K_0$ and (5.6) follows. Set $A_2 = \max\{d(q,o) : q \in K_0\}$ and let $p \in \mathbb{G}$. By (5.6) there exists $\gamma_p \in \mathbb{G}(\mathbb{Z})$ and $q \in K_0$ so that $p = \gamma_p * q$. Hence $d(\gamma_p, p) = d(\gamma_p, \gamma_p * q) = d(0,q) \leq A_2$ and (iii) follows. □

REMARK 5.3. Carnot groups of Iwasawa type, equipped with the gauge metric, satisfy the assumption of Theorem 5.2 with $A_1 = 1$.

5.3.2. Continued fractions as conformal iterated function systems. Continued fractions in the first Heisenberg group **Heis** have been considered by Lukyanenko and Vandehey [40], see also subsequent papers of Vandehey developing detailed number-theoretic properties of such continued fraction representations [60], [59]. As in the Euclidean case, see e.g. [44] and [45], continued fractions can be realized as limit sets of conformal iterated function systems. In this section we describe a class of conformal iterated function systems in general Iwasawa groups which generalize those arising in connection with Heisenberg continued fractions.

Let $\mathbb{G}$ be an Iwasawa group and recall that d denotes its Korányi–Cygan metric. Let $\mathbb{G}(\mathbb{Z})$ be the integer lattice of $\mathbb{G}$ and for $\varepsilon \geq 0$ set

$$I_\varepsilon = \mathbb{G}(\mathbb{Z}) \cap B(o, \Delta_\varepsilon)^c$$

for

$$\Delta_\varepsilon = \frac{5}{2} + \varepsilon.$$

By Theorem 5.2(ii) it follows that for $p \in \overline{B}\left(o, \frac{1}{2}\right)$ and $\gamma \in \mathbb{G}(\mathbb{Z}) \setminus \{o\}$,

$$\frac{1}{2} d(\gamma, o) \leq d(\gamma, p) \leq 2d(\gamma, o). \tag{5.10}$$

We now consider the conformal iterated function system

$$\mathcal{S}_\varepsilon = \left\{\phi_\gamma : \overline{B}\,(o, 1/2) \to \overline{B}\,(o, 1/2)\right\}_{\gamma \in I_\varepsilon} \tag{5.11}$$

where

$$\phi_\gamma = \mathcal{J} \circ \ell_\gamma.$$

Recalling the notation of Section 4.2, in particular (4.4), we note that $X = \overline{B}\left(o, \frac{1}{2}\right)$ and we can take $S = \overline{B}\left(o, \frac{2}{3}\right)$. Note that for every $\gamma \in I_\varepsilon$ and every $p \in \overline{B}\left(o, \frac{1}{2}\right)$,

$$d(\phi_\gamma(p), o) = \frac{1}{d(\gamma * p, o)} \leq \frac{1}{d(\gamma, o) - d(o, p)} \leq \frac{1}{2 + \varepsilon} < \frac{1}{2} \tag{5.12}$$

by (2.16) and (5.10), and hence

$$\phi_\gamma\left(\overline{B}\,(o, 1/2)\right) \subset \overline{B}\,(o, 1/2)\,.$$

The functions ϕ_γ are injective contractions and

$$\|D\phi_\gamma(p)\| \approx d(\gamma, o)^{-2} \tag{5.13}$$

for every $p \in S$, in particular $\|D\phi_\gamma\|_\infty \approx d(\gamma, o)^{-2}$. To see this, first note that as in (2.17) for $p \in S$ and $\gamma \in \mathbb{G}(\mathbb{Z}) \setminus \{o\}$, $d(\gamma * p, o) \approx d(\gamma, o)$. Therefore, for every $p, q \in S$,

$$\begin{aligned} d(\phi_\gamma(p), \phi_\gamma(q)) &= d(\mathcal{J}(\gamma * p), \mathcal{J}(\gamma * q)) \\ &= \frac{d(p, q)}{d(\gamma * p, o)\, d(\gamma * q, o)} \approx d(\gamma, o)^{-2} d(p, q) \end{aligned} \tag{5.14}$$

by (5.10). Finally $\mathcal{S}_\varepsilon$ satisfies the open set condition, as one can easily check that

$$\phi_{\gamma_1}\left(B\,(o, 1/2)\right) \cap \phi_{\gamma_2}\left(B\,(o, 1/2)\right) = \emptyset \tag{5.15}$$

for all distinct $\gamma_1, \gamma_2 \in I_\varepsilon$. Indeed suppose that (5.15) fails. Then there exist distinct $\gamma_1, \gamma_2 \in I_\varepsilon$ and $p_1, p_2 \in B(o, \frac{1}{2})$ such that $\phi_{\gamma_1}(p_1) = \phi_{\gamma_2}(p_2)$ or equivalently

$$\gamma_2^{-1} * \gamma_1 * p_1 = p_2.$$

Therefore $d(\gamma_2^{-1} * \gamma_1 * p_1, o) = d(p_2, o) < \frac{1}{2}$. But by Theorem 5.2 we have

$$d(\gamma_2^{-1} * \gamma_1 * p_1, o) \geq d(\gamma_2^{-1} * \gamma_1, o) - d(p_1, o) > 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

and we have reached a contradiction.

5.4. Complex hyperbolic Kleinian groups of Schottky type

In this section we recall the concept of complex hyperbolic Kleinian groups and the subclass of Schottky groups. The main objective of this section is to associate with each finitely generated Schottky group an Iwasawa maximal conformal graph directed Markov system having the same limit set as the limit set of the group. Having done this we essentially reduce the problem of studying geometric features of complex hyperbolic Schottky groups to the task of dealing with Iwasawa conformal GDMS. We emphasize that this is the only example in this chapter where the general framework of graph directed Markov systems is needed.

Complex hyperbolic space $H_{\mathbb{C}}^{n+1}$ can be modeled as the collection of timelike vectors in complex projective space $P_{\mathbb{C}}^{n+1}$. More precisely, we equip $\mathbb{C}^{n+2}$ with an indefinite Hermitian form $\langle\langle \cdot, \cdot \rangle\rangle$ of signature $(n, 1)$, project to $P_{\mathbb{C}}^{n+1}$, and define

$$H_{\mathbb{C}}^{n+1} := \{[z] \in P_{\mathbb{C}}^{n+1} \, : \, \langle\langle z, z \rangle\rangle < 0\}.$$

Here $[z_0 : \cdots : z_{n+1}]$ denotes the equivalence class in $P_{\mathbb{C}}^{n+1}$ of a point $(z_0, \ldots, z_{n+1}) \in \mathbb{C}^{n+2}$. The space $H_{\mathbb{C}}^{n+1}$ can be identified with the unit ball of $\mathbb{C}^{n+1}$ equipped with a metric of constant negative sectional holomorphic curvature, the Bergman metric. As mentioned in Section 2.2.4, the compactified complex Heisenberg group $\overline{\mathbf{Heis}^n}$ arises as the boundary at infinity of $H_{\mathbb{C}}^{n+1}$, and conformal self-maps of $\mathbf{Heis}^n$ are the boundary values of isometries of $H_{\mathbb{C}}^{n+1}$.

A *complex hyperbolic Kleinian group* is a discrete subgroup of the isometry group $\mathrm{Isom}(H_{\mathbb{C}}^{n+1}) = \mathrm{PSU}(n, 1)$ of complex hyperbolic space. The theory of complex hyperbolic Kleinian groups is a rich and active area of research; we refer to the monograph [**16**] for more information and additional references to the literature. Here we only wish to recall the notion of complex hyperbolic Schottky group.

DEFINITION 5.4. A complex Kleinian group Γ, which we view both as a group of isometries of $H_{\mathbb{C}}^{n+1}$ and as a group of conformal transformations of $\overline{\mathbf{Heis}^n}$, is called a *Schottky group* if the following conditions are satisfied:

(i) for some integer $q \geq 2$, there exist $2q$ pairwise disjoint closed sets B_1, B_2, ..., B_q and $B_{-1}, B_{-2}, \ldots, B_{-q}$ in $\overline{\mathbf{Heis}^n}$ such that each B_i is the closure of its interior,

(ii) for every $i \in V_+ := \{1, 2, \ldots, q\}$ there exists $\gamma_i \in \Gamma$ so that

$$\gamma_i(B_i) = \overline{\overline{\mathbf{Heis}^n} \setminus B_{-i}}, \tag{5.16}$$

and

(iii) the elements $\gamma_1, \gamma_2, \ldots, \gamma_q$ generate the group Γ.

We set $V_- = -V_+ = \{-q, \ldots, -2, -1\}$ and $V := V_+ \cup V_-$. Applying $\gamma_{-i} := \gamma_i^{-1}$ to both sides of (5.16) above, we see that $\gamma_{-i}(B_{-i}) = \overline{\overline{\mathbf{Heis}^n} \setminus B_i}$. Thus (5.16) holds for all $i \in V$. It also follows from (5.16) that

$$\gamma_i(\overline{\overline{\mathbf{Heis}^n} \setminus B_i}) = B_{-i}.$$

Recall that $\Lambda(\Gamma)$, the limit set of Γ, is the set of all accumulation points of the orbit $\Gamma(x)$ for some, equivalently for any, point $x \in H_{\mathbb{C}}^{n+1}$. The limit set $\Lambda(\Gamma)$ is a non-empty closed (topologically) perfect subset of $\overline{\mathbf{Heis}}^n$ invariant under the action of Γ. See [16] for this and other properties of the limit set. Now we associate a canonical Iwasawa conformal GDMS $\mathcal{S}_\Gamma$ to Γ. The alphabet for $\mathcal{S}_\Gamma$ is the above defined set V. The edge set is $E := (V \times V) \setminus \Delta$ where $\Delta := \{(i,i) \in V \times V : i \in V\}$. The incidence matrix $A : E \times E \to \{0,1\}$ is defined by the following formula:

$$A_{(i,j),(k,l)} = \begin{cases} 1 & \text{if } j = -k \\ 0 & \text{if } j \neq -k \end{cases}.$$

Furthermore

$$t((j,k)) := k \quad \text{and} \quad i((j,k)) := -j.$$

For every $i \in V$ we set

$$X_i := B_i,$$

and for every $(i,j) \in E$,

$$\gamma_{(i,j)} := \gamma_i|_{B_j} : B_j \to B_{-i}.$$

The system

$$\mathcal{S}_\Gamma = \{V, E, A, t, i, (B_i)_{i \in V}, \{\gamma_e\}_{e \in E}\}$$

satisfies all of the requirements of an Iwasawa maximal conformal graph directed Markov system, except for the fact that the maps $\{\gamma_e\}_{e \in E}$ need not be uniform contractions. However, since the diameters of the sets $\gamma_\omega(B_{t(\omega)})$ converge to zero uniformly with respect to the length of the word ω, the bounded distortion property implies that the mappings in a sufficiently high iterate of the system $\mathcal{S}_\Gamma$ are uniformly contracting. And this is precisely what we need. We shall prove the following result establishing a close geometric connection between the complex Schottky group Γ and the associated Iwasawa conformal GDMS $\mathcal{S}_\Gamma$.

Theorem 5.5. *If Γ is a complex Schottky group, then $\Lambda(\Gamma) = J_{\mathcal{S}_\Gamma}$.*

Proof. The inclusion $J_{\mathcal{S}_\Gamma} \subset \Lambda(\Gamma)$ is obvious. In order to prove the opposite inclusion, fix a sequence $\{g_n\}_{n=1}^\infty$ of mutually distinct elements of Γ such that the limit $\lim_{n\to\infty} g_n(z)$ exists for some (equivalently, all) $z \in H_{\mathbb{C}}^{n+1}$. For an appropriate choice of indices $i_{n,k_j} \in V$ we may write

$$g_n = \gamma_{i_{n,k_n}} \circ \gamma_{i_{n,k_n-1}} \circ \ldots \circ \gamma_{i_{n,2}} \circ \gamma_{i_{n,1}}$$

in unique irreducible form. In other words, $i_{n,j+1} \neq -i_{n,j}$ for all $1 \leq j \leq k_n - 1$. Passing to a subsequence we may assume without loss of generality that $i_{n,1} = i$ for all $n \geq 1$ and for some $i \in \{1, 2, \ldots, q\}$. Fix $z \in B_{-i}$. Then

$$g_n(z) = \gamma_{(i_{n,k_n}, -i_{n,k_n-1})} \circ \gamma_{(i_{n,k_n-1}, -i_{n,k_n-2})} \circ \cdots \circ \gamma_{(i_{n,2}, -i_{n,1})} \circ \gamma_{(i_{n,1}, -i_{n,1})}(z)$$

and

$$\omega^{(n)} := (i_{n,k_n}, -i_{n,k_n-1})\,(i_{n,k_n-1}, -i_{n,k_n-2}) \cdots (i_{n,2}, -i_{n,1})(i_{n,1}, -i_{n,1})$$

is an element of $E_A^{k_n}$. Hence $g_n(z) \in g_\omega(B_{t(\omega^{(n)})})$. Since each of the sets $g_{\omega^{(n)}}(B_{t(\omega^{(n)})})$ intersects the limit set $J_\mathcal{S}$, and since the diameters of those sets converge to zero as $n \to \infty$, we conclude that $\lim_{n\to\infty} g_n(z) \in \overline{J}_{\mathcal{S}_\Gamma} = J_{\mathcal{S}_\Gamma}$, where we have written the equality sign since the system $\mathcal{S}_\Gamma$ is finite. The proof is complete. $\square$

CHAPTER 6

Countable alphabet symbolic dynamics: foundations of the thermodynamic formalism

In this chapter we recall the foundations of the thermodynamic formalism in the context of countable alphabet symbolic dynamics with complete and self-contained proofs. A more extensive exposition can be found in [**46**]. We stress that most of the results proved in this Chapter, e.g. Theorems 6.11, 6.13, 6.14, and Corollary 6.32, generalize results previously obtained in [**46**]. In [**46**] finite primitivity was frequently assumed, while we only need to assume finite irreducibility.

6.1. Subshifts of finite type and topological pressure

Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of all positive integers and let E be a countable set, either finite or infinite, called in the sequel an *alphabet*. Let

$$\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$$

be the shift map, i.e. cutting off the first coordinate. It is given by the formula

$$\sigma\big((\omega_n)_{n=1}^{\infty}\big) = \big((\omega_{n+1})_{n=1}^{\infty}\big).$$

For the reader's convenience we recall some standard notation from symbolic dynamics that we already defined in Section 4.1. For every finite word $\omega \in E^* := \cup_{n=0}^{\infty} E^n$, $|\omega|$ will denote its *length*, that is the unique integer $n \geq 0$ such that $\omega \in E^n$. We also make the standard convention that $E^0 = \{\emptyset\}$. If $\omega, \upsilon \in E^{\mathbb{N}}$, $\tau \in E^*$ and $n \geq 1$, we put

$$\begin{aligned}
\omega|_n &= \omega_1 \dots \omega_n \in E^n, \\
\tau\omega &= (\tau_1, \dots, \tau_{|\tau|}, \omega_1, \dots), \\
\omega \wedge \upsilon &= \text{longest initial block common to both } \omega \text{ and } \tau.
\end{aligned}$$

Note that $\omega \wedge \upsilon \in E^{\mathbb{N}} \cup E^*$.

All these metrics induce the same topology. A real or complex valued function defined on a subset of $E^{\mathbb{N}}$ is uniformly continuous with respect to one of these metrics if and only if it is uniformly continuous with respect to all. Also, a function is Hölder with respect to one of these metrics if and only if it is Hölder with respect to all; of course the Hölder exponent depends on the metric. If no metric is specifically mentioned, we take it to be d_1.

Now consider a $\{0, 1\}$ valued matrix $A : E \times E \to \{0, 1\}$ and let

$$E_A^{\mathbb{N}} := \{\omega \in E^{\mathbb{N}} : \ A_{\omega_i \omega_{i+1}} = 1 \text{ for all } i \in \mathbb{N}\}.$$

The words of $E_A^{\mathbb{N}}$ will be called *A-admissible*. We also define the corresponding set of finite admissible words

$$E_A^n := \{w \in E^{\mathbb{N}} : \ A_{\omega_i \omega_{i+1}} = 1 \text{ for all } 1 \leq i \leq n-1\}, \qquad n \in \mathbb{N},$$

and

$$E_A^* := \bigcup_{n=0}^{\infty} E_A^n.$$

For every $\omega \in E_A^*$, its corresponding *cylinder* is

$$[\omega] := \{\tau \in E_A^{\mathbb{N}} : \ \tau_{|_{|\omega|}} = \omega\}.$$

Recall from Section 4.1 that the metrics

$$d_\alpha(\omega, \tau) = \mathrm{e}^{-\alpha|\omega\wedge\tau|}, \quad \alpha > 0,$$

induce the same topology on $E^{\mathbb{N}}$. A real or complex valued function defined on a subset of $E^{\mathbb{N}}$ is uniformly continuous with respect to one of these metrics if and only if it is uniformly continuous with respect to all. Also, a function is Hölder with respect to one of these metrics if and only if it is Hölder with respect to all; of course the Hölder exponent depends on the metric. If no metric is specifically mentioned, $E^{\mathbb{N}}$ will be treated as a topological space with the topology defined by d_1. We also record the following obvious fact.

PROPOSITION 6.1. *The set $E_A^{\mathbb{N}}$ is a closed subset of $E^{\mathbb{N}}$, invariant under the shift map $\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$.*

Recall that a $\{0, 1\}$ valued matrix $A : E \times E \to \{0, 1\}$ is called *irreducible* if there exists $\Phi \subset E_A^*$ such that for all $i, j \in E$ there exists $\omega \in \Phi$ for which $i\omega j \in E_A^*$. If there exists a finite set Φ with the previous property, the matrix A will be called *finitely irreducible.* If in addition there exists a finite set $\Phi \subset E_A^*$ consisting of words of the same lengths such that for all $i, j \in E$ there exists $\omega \in \Phi$ such that $i\omega j \in E_A^*$, then the matrix A is called *finitely primitive.*

Given a set $F \subset E$ we put

$$F^{\mathbb{N}} := \{\omega \in E^{\mathbb{N}} : \omega_i \in F \text{ for all } \ i \in \mathbb{N}\},$$

and

$$F_A^n := E_A^n \cap F^n = \{\omega \in F^n : \ A_{\omega_i\omega_{i+1}} = 1 \text{ for all } 1 \le i \le n-1\}.$$

A sequence $(a_n)_{n=1}^{\infty}$ of real numbers is said to be *subadditive* if

$$a_{n+m} \le a_n + a_n \qquad \text{for all } m, n \ge 1.$$

We now recall the following standard lemma about subadditive sequences.

LEMMA 6.2. *If $\big(a_n\big)_{n=1}^{\infty}$ is subadditive, then $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to $\inf_{n\ge1}(a_n/n)$.*

The limit in Lemma 6.2 could be $-\infty$, but if the elements a_n are uniformly bounded below, then this limit is nonnegative.

Given $F \subset E$ and a function $f : F_A^{\mathbb{N}} \to \mathbb{R}$ we define the *n-th partition function*

$$Z_n(F, f) = \sum_{\omega\in F_A^n} \exp\left(\sup_{\tau\in[\omega]_F} \sum_{j=0}^{n-1} f(\sigma^j(\tau))\right),$$

where $[\omega]_F = \{\tau \in F_A^{\mathbb{N}} : \tau|_{|\omega|} = \omega\}$. If $F = E$, we simply write $[\omega]$ for $[\omega]_F$.

The following lemma is indispensable for the proper definition of topological pressure.

LEMMA 6.3. *The sequence $(\log Z_n(F, f))_{n=1}^{\infty}$ is subadditive.*

PROOF. We need to show that the sequence $\mathbb{N} \ni n \mapsto Z_n(F, f)$ is submultiplicative, i.e. that

$$Z_{m+n}(F, f) \le Z_m(F, f) Z_n(F, f)$$

for all $m, n \ge 1$. And indeed,

$$\begin{aligned}
Z_{m+n}(F,f) &= \sum_{\omega \in F_A^{m+n}} \exp\left(\sup_{\tau\in[\omega]_F} \sum_{j=0}^{m+n-1} f(\sigma^j(\tau))\right) \\
&= \sum_{\omega \in F_A^{m+n}} \exp\left(\sup_{\tau\in[\omega]_F} \left\{\sum_{j=0}^{m-1} f(\sigma^j(\tau)) + \sum_{j=0}^{n-1} f(\sigma^j(\sigma^m(\tau)))\right\}\right) \\
&\le \sum_{\omega \in F_A^{m+n}} \exp\left(\sup_{\tau\in[\omega]_F} \sum_{j=0}^{m-1} f(\sigma^j(\tau)) + \sup_{\tau\in[\omega]_F} \sum_{j=0}^{n-1} f(\sigma^j(\sigma^m(\tau))\right) \\
&\le \sum_{\omega \in F_A^{m}} \sum_{\rho \in F_A^{n}} \exp\left(\sup_{\tau\in[\omega]_F} \sum_{j=0}^{m-1} f(\sigma^j(\tau)) + \sup_{\gamma\in[\rho]_F} \sum_{j=0}^{n-1} f(\sigma^j(\gamma))\right) \\
&= \sum_{\omega \in F_A^{m}} \exp\left(\sup_{\tau\in[\omega]_F} \sum_{j=0}^{m-1} f(\sigma^j(\tau))\right) \cdot \sum_{\rho \in F_A^{n}} \exp\left(\sup_{\gamma\in[\rho]_F} \sum_{j=0}^{m-1} f(\sigma^j(\gamma))\right) \\
&= Z_m(F,f) Z_n(F,f).
\end{aligned}$$

□

The topological pressure of f with respect to the shift map $\sigma : F_A^{\mathbb{N}} \to F_A^{\mathbb{N}}$ is defined to be

$$\text{(6.1)} \qquad \mathrm{P}_F^\sigma(f) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(F, f) = \inf \left\{ \frac{1}{n} \log Z_n(F, f) \right\}.$$

If $F = E$ we suppress the subscript F and write simply $\mathrm{P}^\sigma(f)$ for $\mathrm{P}_E^\sigma(f)$ and $Z_n(f)$ for $Z_n(E, f)$.

DEFINITION 6.4. A uniformly continuous function $f : E^{\mathbb{N}} \to \mathbb{R}$ is called *acceptable* provided

$$\mathrm{osc}(f) := \sup_{i\in E} \{\sup(f|_{[i]}) - \inf(f|_{[i]})\} < \infty.$$

Note that if the alphabet E is infinite, then acceptable functions need not be bounded and as a matter of fact, those most important for us, giving rise to Gibbs and equilibrium states will be unbounded below.

We now introduce and briefly discuss some fundamental notions from Ergodic Theory. For more information the reader can consult e.g. the books [**53, 61**]. Let $(X, \mathcal{F}, \mu)$ be a measure space. A measurable transformation $T : X \to X$ is called *measure preserving* if

$$\mu(T^{-1}(F)) = \mu(F) \text{ for all } F \in \mathcal{F}.$$

In that case the measure μ will be called T*-invariant*. A Borel probability measure on $E_A^{\mathbb{N}}$ is called *shift-ivariant*, or simply invariant, if it is σ-invariant.

DEFINITION 6.5. Let $(X, \mathcal{F}, \mu)$ be a probability measure space. A measure preserving transformation $T : X \to X$ is called *ergodic* if for every $F \in \mathcal{F}$ such that $T^{-1}(F) = F$ either $\mu(F) = 0$ or $\mu(F) = 1$. The measure preserving transformation $T : X \to X$ is called *completely ergodic* if T^k is ergodic for all $k \in \mathbb{N}$.

Ergodic measures form fundamental blocks of measure–preserving dynamical systems. Any two distinct ergodic measures (invariant with respect to the same map T) are mutually singular. Any dynamical system T preserving a (probability) measure μ preserves an ergodic measure. In fact, more is true, if T is a continuous map of a compact metrizable space X, then T always admits a Borel probability invariant measure, this fact is known as the Bogolubov–Krylov Theorem, see [**61**, Theorem 6.9.1]. Such measures form a convex compact set and its non-empty set of extreme points coincides with Borel probability invariant ergodic measures; by the Krein–Milman Theorem the closed convex hull of the latter set is equal to the former. Even if the measurable map $T : X \to X$ (with respect to a σ-algebra $\mathcal{F}$ on X) is not continuous, the mere existence of a probability invariant measure μ on $\mathcal{F}$ frequently yields the existence of ergodic measures, and the measure μ is entirely described by them.

Indeed, let X be a Polish space, i.e. a completely metrizable separable topological space and let $\mathcal{F}$ be its σ–algebra of all Borel sets. As above $T : X \to X$ is measurable with respect to $\mathcal{F}$. Let $\mathcal{M}(T, \mathcal{F})$ denote the set of all probability T-invariant measures on $\mathcal{F}$ and let $\mathcal{E}(T, \mathcal{F})$ denote its subset consisting of all ergodic measures. The set $\mathcal{M}(T, \mathcal{F})$ is canonically endowed with a σ–algebra, namely the smallest σ–algebra for which all the maps

$$\mathcal{M}(T, \mathcal{F}) \ni \mu \longmapsto \mu(E), \ E \in \mathcal{F},$$

are measurable. It coincides with the σ-algebra of Borel sets generated by the weak*–topology on $\mathcal{M}(T, \mathcal{F})$. The set $\mathcal{E}(T, \mathcal{F})$ is then endowed with the restriction of this σ–algebra to $\mathcal{E}(T, \mathcal{F})$,

We have the following well-known theorem, for the proof see e.g. [**34**, Theorem 3.8.4]. Compare also with [**61**, p. 153] for the case of X being compact.

THEOREM 6.6 (Ergodic Decomposition). *If X is a Polish space endowed with the σ–algebra of Borel sets, $T : X \to X$ is a Borel map, and μ is a Borel probability invariant measure on $\mathcal{F}$, i.e. $\mu \in \mathcal{M}(T, \mathcal{F})$, then there exists a unique probability measure P_μ on $\mathcal{E}(T, \mathcal{F})$ such that*

$$\mu = \int_{\mathcal{E}(T,\mathcal{F})} \nu \, dP_\mu(\nu).$$

This precisely means that for every real function $g \in L^1(\mu)$, we have that

$$\int_X g \, d\mu = \int_{\mathcal{E}(T,\mathcal{F})} \Big(\int_X g \, d\nu \Big) \, dP_\mu(\nu).$$

It is immediate that in terms of the ergodic decomposition, ergodicity of the measure μ just means that the probability measure P_μ is the Dirac δ–measure supported at μ.

One of the most powerful tools yielded by probability invariant measures is Birkhoff's Ergodic Theorem, see [**53, 61**] for its proof. Here it goes.

THEOREM 6.7 (Birkhoff Ergodic Theorem). *If $T : X \to X$ is a measure preserving endomorphism of a probability space $(X, \mathcal{F}, \mu)$ and if $g : X \to \mathbb{R}$ is an integrable function, then the following limit*

$$\tilde{g}(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g \circ T^j(x)$$

exists for μ-a.e. $x \in X$. The function $\tilde{g} : X \to \mathbb{R}$ is T–invariant and $\int_X \tilde{g}\, d\mu = \int_X g\, d\mu$. If in addition T is ergodic, then the function $\tilde{g} : X \to \mathbb{R}$ is constant μ–a.e., meaning that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g \circ T^j(x) = \int g\, d\mu \quad \text{for} \quad \mu - \text{a.e. } x \in X.$$

The very last of these formulations is a particularly handy tool for applications, and we will use it throughout the book frequently. In the context of countable symbolic dynamics, which we will employ almost exclusively in the following, the ergodic version of Birkhoff's Ergodic Theorem takes the following form.

THEOREM 6.8. *Let $\tilde{\mu}$ be a Borel probability σ–invariant ergodic measure on $E_A^{\mathbb{N}}$ and let $f \in L^1(E_A^{\mathbb{N}})$. Then*

$$\lim_{n\to\infty} \frac{1}{n} S_n f(\omega) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ \sigma^j(\omega) = \int f d\tilde{\mu},$$

for $\tilde{\mu}$-a.e. $\omega \in E_A^{\mathbb{N}}$.

It is evident from the above that ergodic measures abound. To name just a few situations where they arise: all Bernoulli measures are ergodic, more generally all Markov measures generated by an irreducible stochastic matrix are ergodic, all Gibbs invariant measures (Bernoulli and Markov ones belong to them) considered later in this chapter are ergodic, in particular the celebrated Gauss measure associated with real continued fractions is ergodic, the Lebesgue measure on the unit circle is ergodic with respect to any irrational rotation, every Dirac δ–measure supported at a fixed point of T is ergodic with respect to T, and at least continuum many more measures are ergodic with respect to some dynamical system.

If $\mathcal{A}$ and $\mathcal{B}$ are two Borel partitions of X their *join* is defined as

$$\mathcal{A} \vee \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}.$$

We also set

$$\mathcal{A}^n := \mathcal{A} \vee T^{-1}(\mathcal{A}) \vee \cdots \vee T^{-(n-1)}(\mathcal{A}).$$

Let μ be a T-invariant measure on X and denote by

$$\mathrm{H}_\mu(\mathcal{A}) = -\sum_{A\in\mathcal{A}} \mu(A) \log(\mu(A)) \tag{6.2}$$

the *entropy* of the partition $\mathcal{A}$ with respect to the measure μ. The sequence

$$\left(\frac{\mathrm{H}_\mu(\mathcal{A}^n)}{n}\right)_{n=1}^{\infty}$$

is monotonically decreasing to a limit which is commonly denoted by $\mathrm{h}_\mu(T,\mathcal{A})$ and it is referred to as the *entropy of T with respect to partition $\mathcal{A}$*. The *entropy of T* is defined as

$$\mathrm{h}_\mu(T) = \sup_{\mathcal{A}}\{\mathrm{h}_\mu(T,\mathcal{A})\},$$

where the supremum is taken over all countable partitions of X with finite entropy.

It is usually very difficult to calculate $\mathrm{h}_\mu(T)$ from this definition since the supremum runs over a very big set of partitions. However if $\mathcal{A}$ is a *generating partition*, meaning that there exists a measurable set $Y \subset X$ with $\mu(Y) = 1$ such that for all $w, z \in Y$ there exists an integer $n \geq 0$ such that $T^n(w)$ and $T^n(z)$ belong to different elements (commonly referred to as atoms) of $\mathcal{A}$, and the entropy $\mathrm{H}_\mu(\mathcal{A})$ is finite, then

$$\mathrm{h}_\mu(T) = \mathrm{h}_\mu(T,\mathcal{A}). \tag{6.3}$$

Notice that if $\mathcal{A}$ is a generating partition and $\mathcal{B}$ is finer than $\mathcal{A}$, meaning that each element of $\mathcal{B}$ is contained in (exactly one) element of $\mathcal{A}$, then $\mathcal{B}$ is also generating. In particular any refinement $\mathcal{A}^n$ of $\mathcal{A}$ is then generating. Furthermore, if $\mathcal{A}$ is of finite entropy then so is too each partition $\mathcal{A}^n$. Using the Ergodic Decomposition Theorem (Theorem 6.6) the entropy of a transformation (with respect to some invariant measure) can be expressed as an average of entropies over ergodic measures. The proof of the following theorem follows as in [**53**, Theorem 2.8.11].

THEOREM 6.9. *With the hypotheses of Theorem 6.6, we have that*

$$\mathrm{h}_\mu(T) = \int_{\mathcal{E}(T,\mathcal{F})} \mathrm{h}_\nu(T)\, dP_\mu(\nu).$$

Needless to say that we will apply all these concepts, actually exclusively, to the symbolic dynamical system $\sigma : E_A^{\mathbb{N}} \to E_A^{\mathbb{N}}$. This dynamical system has canonical countable partition, namely the one demoted by us by α, consisting of initial cylinders of length 1. Precisely,

$$\alpha := \{[e] : e \in E\}$$

Of course α is a generating partition for the symbolic dynamical system $\sigma : E_A^{\mathbb{N}} \to E_A^{\mathbb{N}}$. Notice that if $q \in \mathbb{N}$ then α^q (which is the partition consisting of the cylinders of length q) is also generating and of finite entropy if the entropy of α is finite. In particular, it is easy to see (ex. [**53**, Theorem 2.8.7(b)]) that if $\mathrm{H}_\mu(\alpha^q) < \infty$ for some $q \in \mathbb{N}$, then

$$\mathrm{h}_\mu(\sigma) = \lim_{n\to\infty} \frac{\mathrm{H}_\mu(\alpha^n)}{n} = -\lim_{n\to\infty} \frac{1}{n} \sum_{A\in\alpha^n} \mu(A)\log(\mu(A)). \tag{6.4}$$

We also record the following straightforward estimate

$$\mathrm{h}_\mu(\sigma) \leq \mathrm{H}_\mu(\alpha).$$

We now examine in detail the connections between topological pressure and entropy provided by various, though related, versions of the Variational Principle.

THEOREM 6.10 (1st Variational Principle). *If $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is continuous and $\tilde{\mu}$ is a shift-invariant Borel probability measure on $E_A^{\mathbb{N}}$ such that $\int f d\tilde{\mu} > -\infty$, then*

$$\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f d\tilde{\mu} \leq \mathrm{P}^\sigma(f).$$

In addition, if $\mathrm{P}^\sigma(f) < +\infty$, then there is an integer $q \geq 1$ so that $\mathrm{H}_{\tilde{\mu}}(\alpha^q) < +\infty$.

PROOF. If $\mathrm{P}^\sigma(f)=+\infty$, there is nothing to prove. So, suppose that $\mathrm{P}^\sigma(f)<+\infty$. Then there exists $q\ge 1$ such that $Z_n(f)<+\infty$ for every $n\ge q$. Also, for every $n\ge 1$, we have

$$\sum_{|\omega|=n}\tilde\mu([\omega])\sup(S_nf|_{[\omega]})\ge\int S_nf\,d\tilde\mu=n\int f\,d\tilde\mu>-\infty,$$

where

$$S_nf=\sum_{j=0}^{n-1}f\circ\sigma^j.$$

Therefore, using concavity of the function $h(x)=-x\log x$, we obtain for every $n\ge q$,

$$\begin{aligned}\mathrm{H}_{\tilde\mu}(\alpha^n)+\int S_nf\,d\tilde\mu&\le\sum_{|\omega|=n}\tilde\mu([\omega])(\sup S_nf|_{[\omega]}-\log\tilde\mu([\omega]))\\&=Z_n(f)\sum_{|\omega|=n}Z_n(f)^{-1}e^{\sup S_nf|_{[\omega]}}h\big(\tilde\mu([\omega])e^{-\sup S_nf|_{[\omega]}}\big)\\&\le Z_n(f)h\left(\sum_{|\omega|=n}Z_n(f)^{-1}e^{\sup S_nf|_{[\omega]}}\tilde\mu([\omega])e^{-\sup S_nf|_{[\omega]}}\right)\\&=Z_n(f)h(Z_n(f)^{-1})\\&=\log Z_n(f).\end{aligned}$$

Therefore, $\mathrm{H}_{\tilde\mu}(\alpha^n)\le\log Z_n(f)+n\int(-f)d\tilde\mu<\infty$ for every $n\ge q$, and since, in addition α^q is a generator, we obtain

$$\begin{aligned}\mathrm{h}_{\tilde\mu}(\sigma)+\int f\,d\tilde\mu&\le\liminf_{n\to\infty}\left(\frac1n\left(\mathrm{H}_{\tilde\mu}(\alpha^n)+\int S_nf\,d\tilde\mu)\right)\right)\\&\le\lim_{n\to\infty}\frac1n\log Z_n(f)=\mathrm{P}^\sigma(f).\end{aligned}$$

The proof is complete. □

We will also need the following theorem, which was proved in [46] as Theorem 2.1.5.

THEOREM 6.11. *If $f:E_A^{\mathbb N}\to\mathbb R$ is acceptable and A is finitely irreducible, then*

$$\mathrm{P}^\sigma(f)=\sup\mathrm{P}_F^\sigma(f),$$

where the supremum is taken over all finite subsets F of E.

PROOF. The inequality $\mathrm{P}^\sigma(f)\ge\sup\{\mathrm{P}_F^\sigma(f)\}$ is obvious. In order to prove the converse let $\Phi\subset E_A^*$ be a set of words witnessing finite irreducibility of the matrix A. We assume first that $\mathrm{P}^\sigma(f)<+\infty$. Put

(6.5)
$$q:=\#\Phi,\ p:=\max\{|\omega|:\omega\in\Phi\},\ \text{ and }\ T:=\min\left\{\inf\sum_{j=0}^{|\omega|-1}f\circ\sigma^j|_{[\omega]}:\omega\in\Phi\right\}.$$

Fix $\varepsilon>0$. By acceptability of $f:E_A^{\mathbb N}\to\mathbb R$, we have $M:=\operatorname{osc}(f)<\infty$ and there exists $l\ge 1$ such that

$$|f(\omega)-f(\tau)|<\varepsilon \tag{6.6}$$

whenever $\omega|_l = \tau|_l$. Now, fix $k > l$. By Lemma 6.3, $\frac{1}{k}\log Z_k(f) \geq \mathrm{P}^\sigma(f)$. Therefore, there exists a finite set $F \subset E$ such that

$$\frac{1}{k}\log Z_k(F,f) > \mathrm{P}^\sigma(f) - \varepsilon. \tag{6.7}$$

We may assume that F contains Φ. Put

$$\overline{f} := \sum_{j=0}^{k-1} f \circ \sigma^j.$$

Now, for every element $\tau = \tau_1, \tau_2, \ldots, \tau_n \in F_A^k \times \cdots \times F_A^k$ (n factors) one can choose elements $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \Phi$ such that $\overline{\tau} = \tau_1\alpha_1\tau_2\alpha_2\ldots\tau_{n-1}\alpha_{n-1}\tau_n \in E_A^*$. Notice that the so defined function $\tau \mapsto \overline{\tau}$ is at most q^{n-1}-to-1 (in fact u^{n-1}-to-1, where u is the number of lengths of words composing Φ). Then for every $n \geq 1$,

$$\begin{aligned} q^{n-1}\sum_{i=kn}^{kn+p(n-1)} Z_i(F,f) &= q^{n-1}\sum_{i=kn}^{kn+p(n-1)}\sum_{\omega\in F_A^i}\exp(\sup S_i f|_{[\omega]}) \\ &\geq \sum_{\tau\in(F_A^k)^n}\exp\left(\sup_{[\overline{\tau}]}\sum_{j=0}^{|\overline{\tau}|} f\circ\sigma^j\right) \\ &\geq \sum_{\tau\in(F_A^k)^n}\exp\left(\inf_{[\overline{\tau}]}\sum_{j=0}^{|\bar{\tau}|-1} f\circ\sigma^j\right). \end{aligned} \tag{6.8}$$

Now observe that

$$\sum_{i=1}^{n}\inf \bar{f}|_{[\tau_i]} + T(n-1) \leq \inf_{[\overline{\tau}]}\sum_{j=0}^{|\overline{\tau}|-1} f\circ\sigma^j. \tag{6.9}$$

To prove (6.9) let $\omega \in [\bar{\tau}]$ then

$$\omega|_{|\bar{\tau}|} = \tau_1\alpha_1\tau_2\alpha_2\ldots\tau_{n-1}\alpha_{n-1}\tau_n$$

and

$$\begin{aligned} \sum_{j=0}^{|\bar{\tau}|-1} f(\sigma^j(\omega)) = \sum_{j=0}^{|\tau_1|-1} f\circ\sigma^j(\omega) + \sum_{j=|\tau_1|}^{|\tau_1|+|\alpha_1|-1} f\circ\sigma^j(\omega) + \sum_{j=|\tau_1|+|\alpha_1|}^{|\tau_1|+|\alpha_1|+|\tau_2|-1} f\circ\sigma^j(\omega) \\ + \cdots + \sum_{j=|\tau_1|+|\alpha_1|+\cdots+|\alpha_{n-1}|}^{|\tau_1|+|\alpha_1|+\cdots+|\alpha_{n-1}|+|\tau_n|-1} f\circ\sigma^j(\omega). \end{aligned}$$

Now note that

$$\sum_{j=0}^{|\tau_1|-1} f\circ\sigma^j(\omega) \geq \inf \bar{f}|_{[\tau_1]}.$$

and

$$\sum_{j=|\tau_1|}^{|\tau_1|+|\alpha_1|-1} f\circ\sigma^j(\omega) = \sum_{j=0}^{|\alpha_1|-1} f(\sigma^j(\sigma^{|\tau_1|}(\omega))) \geq T,$$

since $\sigma^{|\tau_1|}(\omega) \in [\alpha_1] \in \Phi$. In the same way,

$$\sum_{j=|\tau_1|+|\alpha_1|+\cdots+|\tau_{i-1}|+|\alpha_{i-1}|}^{|\tau_1|+|\alpha_1|+\cdots+|\tau_{i-1}|+|\alpha_{i-1}|+|\tau_i|-1} f\circ\sigma^j(\omega) = \sum_{j=0}^{|\tau_i|-1} f(\sigma^j(\sigma^{|\tau_1|+|\alpha_1|+\cdots+|\tau_{i-1}|+|\alpha_{i-1}|}(\omega))) \geq \inf \overline{f}|_{[\tau_i]},$$

and

$$\sum_{j=|\tau_1|+|\alpha_1|+\cdots+|\tau_i|}^{|\tau_1|+|\alpha_1|+\cdots+|\tau_i|+|\alpha_i|-1} f\circ\sigma^j(\omega) = \sum_{j=0}^{|\alpha_i|-1} f(\sigma^j(\sigma^{|\tau_1|+|\alpha_1|+\cdots+|\tau_i|}(\omega))) \geq T,$$

for $i = 2, \ldots, n$. Hence (6.9) follows. Therefore

$$\begin{aligned}\sum_{\tau\in(F_A^k)^n} \exp\left(\inf_{[\overline{\tau}]} \sum_{j=0}^{|\overline{\tau}|-1} f\circ\sigma^j\right) &\geq \sum_{\tau\in(F_A^k)^n} \exp\left(\sum_{i=1}^{n} \inf \overline{f}|_{[\tau_i]} + T(n-1)\right) \\ &= \exp(T(n-1)) \sum_{\tau\in(F_A^k)^n} \exp \sum_{i=1}^{n} \inf \overline{f}|_{[\tau_i]}.\end{aligned} \tag{6.10}$$

We will now prove that for $i = 1, \ldots, n$,

$$\sup \overline{f}|_{[\tau_i]} - \inf \overline{f}|_{[\tau_i]} \leq (k-l)\varepsilon - Ml. \tag{6.11}$$

Let $\omega_1, \omega_2 \in [\tau_i]$. Then, recalling that $k \geq l$,

$$\begin{aligned}\overline{f}(\omega_1) - \overline{f}(\omega_2) &= f(\omega_1) - f(\omega_2) + f(\sigma(\omega_1)) - f(\sigma(\omega_2)) \\ &\quad + \cdots + f(\sigma^{k-l-1}(\omega_1)) - f(\sigma^{k-l-1}(\omega_2)) \\ &\quad + \cdots + f(\sigma^{k-1}(\omega_1)) - f(\sigma^{k-1}(\omega_2)).\end{aligned} \tag{6.12}$$

Since $\omega_1|_k = \omega_2|_k$ we deduce that

$$\sigma^j(\omega_1)|_l = \sigma^j(\omega_2)|_l$$

for all $j \leq k-l-1$. Hence by (6.6),

$$|f(\sigma^j(\omega_1)) - f(\sigma^j(\omega_2))| < \varepsilon \tag{6.13}$$

for all $j \leq k-l-1$. On the other hand for all $j \leq k-1$,

$$\sigma^j(\omega_1)_1 = \sigma^j(\omega_2)_1,$$

therefore for all $k-l \leq j \leq k-1$

$$|f(\sigma^j(\omega_1)) - f(\sigma^j(\omega_2))| \leq M. \tag{6.14}$$

Hence (6.11) follows by (6.12),(6.13) and (6.14). Therefore

$$
\begin{aligned}
&\exp(T(n-1)) \sum_{\tau\in(F_A^k)^n} \exp \sum_{i=1}^{n} \inf \overline{f}|_{[\tau_i]} \\
&\geq \exp(T(n-1)) \sum_{\tau\in(F_A^k)^n} \exp\left(\sum_{i=1}^{n} (\sup \overline{f}|_{[\tau_i]} - (k-l)\varepsilon - Ml)\right) \\
&= \exp(T(n-1) - (k-l)\varepsilon n - Mln) \sum_{\tau\in(F_A^k)^n} \exp \sum_{i=1}^{n} \sup \overline{f}|_{[\tau_i]} \\
&= e^{-T} \exp\big(n(T-(k-l)\varepsilon - Ml)\big) \left(\sum_{\tau\in F_A^k} \exp(\sup \overline{f}|_{[\tau]})\right)^n .
\end{aligned}
\tag{6.15}
$$

Combining (6.8), (6.10) and (6.15) we deduce that

$$
q^{n-1} \sum_{i=kn}^{kn+p(n-1)} Z_i(F,f) \geq e^{-T} \exp\big(n(T-(k-l)\varepsilon - Ml)\big) \left(\sum_{\tau\in F_A^k} \exp(\sup \overline{f}|_{[\tau]})\right)^n .
$$

Hence, there exists $kn \leq i_n \leq (k+p)n$ such that

$$
Z_{i_n}(F,f) \geq \frac{1}{pn} e^{-T} \exp\big(n(T-(k-l)\varepsilon - Ml - \log q)\big) Z_k(F,f)^n .
$$

Note also that for k large enough

$$
\liminf_{n\to\infty} \frac{\log Z_k(F,f)}{i_n} \geq \mathrm{P}^\sigma(f) - 2\varepsilon .
$$

This follows immediately by (6.7) if $\mathrm{P}^\sigma(f) \leq 0$, actually in this case the right hand side in the previous inequality can be replaced by $\mathrm{P}^\sigma(f) - \varepsilon$. If $\mathrm{P}^\sigma(f) > 0$ then choosing k large enough such that (6.7) holds and $\frac{k}{k+p} > 1 - \frac{\varepsilon}{\mathrm{P}^\sigma(f)}$ we have that

$$
\liminf_{n\to\infty} \frac{\log Z_k(F,f)}{i_n} \geq \frac{\log Z_k(F,f)}{k} \frac{k}{k+p} > (\mathrm{P}^\sigma(f) - \varepsilon)\left(1 - \frac{\varepsilon}{\mathrm{P}^\sigma(f)}\right) > \mathrm{P}^\sigma(f) - 2\varepsilon .
$$

Hence

$$
\begin{aligned}
\mathrm{P}_F^\sigma(f) &= \lim_{n\to\infty} \frac{1}{i_n} \log Z_{i_n}(F,f) \\
&\geq \frac{-|T|}{k} - \varepsilon + \frac{l\varepsilon}{k+p} - \frac{Ml + \log q}{k} + \mathrm{P}^\sigma(f) - 2\varepsilon \\
&\geq \mathrm{P}^\sigma(f) - 5\varepsilon
\end{aligned}
$$

provided that k is large enough. Thus, letting $\varepsilon \searrow 0$, the theorem follows. The case $\mathrm{P}^\sigma(f) = +\infty$ can be treated similarly. □

REMARK 6.12. In Theorem 6.11 the supremum can be taken over all finite and irreducible sets $F \subset E$. This follows because if $\Phi \subset E_A^*$ is the set witnessing finite irreducibility for the matrix A and if

$$
G = \{e \in E : e = \omega_i \text{ for some } \omega \in \Phi, i = 1, \dots, |\omega|\},
$$

that is G is the set of letters appearing in the words of Φ, then $F \cup G$ is irreducible and $\mathrm{P}_{F\cup G}^\sigma \geq \mathrm{P}_F^\sigma$.

We say a shift-invariant Borel probability measure $\tilde{\mu}$ on $E_A^{\mathbb{N}}$ is *finitely supported* provided there exists a finite set $F \subset E$ such that $\tilde{\mu}(F_A^{\mathbb{N}}) = 1$. The well-known *variational principle* for finitely supported measures (see [**14**], [**55**], [**61**] and [**53**]) tells us that for every finite set $F \subset E$

$$\mathrm{P}_F^\sigma(f) = \sup\left\{\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f d\tilde{\mu}\right\},$$

where the supremum is taken over all shift-invariant ergodic Borel probability measures $\tilde{\mu}$ with $\tilde{\mu}(F^\infty) = 1$. Applying Theorem 6.11, we therefore obtain the following.

THEOREM 6.13 (2nd Variational Principle). *If A is finitely irreducible and if $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is acceptable, then*

$$\mathrm{P}^\sigma(f) = \sup\left\{\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f d\tilde{\mu}\right\},$$

where the supremum is taken over all shift-invariant ergodic Borel probability measures $\tilde{\mu}$ which are finitely supported.

As an immediate consequence of Theorem 6.13 and Theorem 6.10, we get the following.

THEOREM 6.14 (3rd Variational Principle). *Suppose that the incidence matrix A is finitely irreducible. If $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is acceptable, then*

$$\mathrm{P}^\sigma(f) = \sup\left\{\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f d\tilde{\mu}\right\},$$

where the supremum is taken over all shift-invariant ergodic Borel probability measures $\tilde{\mu}$ on $E_A^{\mathbb{N}}$ such that $\int f d\tilde{\mu} > -\infty$.

A *potential* is just a continuous function $f : E_A^{\mathbb{N}} \to \mathbb{R}$. We call a shift-invariant probability measure $\tilde{\mu}$ on $E_A^{\mathbb{N}}$ an *equilibrium state* of the potential f if $\int -f d\tilde{\mu} < +\infty$ and

$$\mathrm{h}_{\tilde{\mu}}(\sigma) + \int f d\tilde{\mu} = \mathrm{P}^\sigma(f). \tag{6.16}$$

We end this section with the following useful technical fact.

PROPOSITION 6.15. *If the incidence matrix A is finitely irreducible and the function $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is acceptable, then $\mathrm{P}^\sigma(f) < +\infty$ if and only if $Z_1(f) < +\infty$.*

PROOF. Let $\Phi = \{\lambda_1, \dots, \lambda_q\} \subset E_A^*$ be a set which witnesses the finite irreducibility of the incidence matrix A. Let p and T as in (6.5). For any two letters $\omega, \upsilon \in E$ we define

$$[\omega, \upsilon] = \min\{i \in \{1, \dots, q\} : \omega\lambda_i\upsilon \in E_A^*\}.$$

We then define a map

$$g : E^* \to E_A^*$$

as follows. If $\omega = (\omega_1, \dots, \omega_n) \in E^n$ we let

$$g(\omega) := \bar{\omega} := \omega_1\alpha_1\omega_2 \dots \omega_{n-1}\alpha_{n-1}\omega_n$$

where $\alpha_i = \lambda_{[\omega_i, \omega_{i+1}]}$ for $i = 1, \dots, n-1$. As in the proof of Theorem 6.11 the function $\omega \mapsto \overline{\omega}$ is at most q^{n-1}-to-1. Note also that,

$$n + (n-1) \le |\overline{\omega}| \le n + p(n-1).$$

Since f is acceptable, we therefore get

$$\begin{aligned} q^{n-1}\sum_{i=n+(n-1)}^{n+p(n-1)} Z_i(f) &= q^{n-1}\sum_{i=n+(n-1)}^{n+p(n-1)}\sum_{\omega\in E_A^i}\exp\left(\sup_{[\omega]}\left\{\sum_{j=0}^{i-1} f\circ\sigma^j\right\}\right)\\ &\geq \sum_{\omega\in E^n}\exp\left(\sup_{[\overline{\omega}]}\left\{\sum_{j=0}^{|\overline{\omega}|} f\circ\sigma^j\right\}\right)\\ &\geq \sum_{\omega\in E^n}\exp\left(\sum_{j=1}^{n}\inf\big(f|_{[\omega_j]}\big)+T(n-1)\right)\\ &\geq e^{T(n-1)}\sum_{\omega\in E^n}\exp\left(\sum_{j=1}^{n}\sup\big(f|_{[\omega_j]}\big)-\mathrm{osc}(f)n\right)\\ &= \exp\big(-T+(T-\mathrm{osc}(f))n\big)\left(\sum_{e\in E}\exp\big(\sup\big(f|_{[e]}\big)\big)\right)^n\\ &= \exp\big(-T+(T-\mathrm{osc}(f))n\big)Z_1(f)^n, \end{aligned}$$

where in order to obtain the inequalities in the third and the fourth line we argue as in (6.9) and (6.11) respectively. Hence there exists a sequence $(i_n)_{n\in\mathbb{N}}$ such that

$$n-1\leq i_n\leq p(n-1)$$

and for all $n\in\mathbb{N}$,

$$q^{n-1}((p-1)(n-1)+1)Z_{n+i_n}(f)\geq \exp\big(-T+(T-\mathrm{osc}(f))n\big)Z_1(f)^n. \tag{6.17}$$

Note that

$$\begin{aligned} \limsup_{n\to\infty}&\frac{\log(q^{n-1}((p-1)(n-1)+1)Z_{n+i_n}(f))}{n}\\ &= \log q+\limsup_{n\to\infty}\frac{\log Z_{n+i_n}(f)}{n+i_n}\frac{n+i_n}{n}. \end{aligned}$$

Since

$$1\leq\frac{n+i_n}{n}\leq p+1,$$

for all $n\in\mathbb{N}$, (6.17) implies that if $\mathrm{P}^\sigma(f)<+\infty$, then also $Z_1(f)<+\infty$. The opposite implication is obvious since $Z_n(f)\leq Z_1(f)^n$. The proof is complete. □

6.2. Gibbs states, equilibrium states and potentials

If $f: E_A^{\mathbb{N}}\to\mathbb{R}$ is a continuous function, then, following [**46**] (see also the references therein), a Borel probability measure $\tilde{m}$ on $E_A^{\mathbb{N}}$ is called a *Gibbs state* for f if there exist constants $Q_g\geq 1$ and $\mathrm{P}_{\tilde{m}}\in\mathbb{R}$ such that for every $\omega\in E_A^*$ and every $\tau\in[\omega]$

$$Q_g^{-1}\leq\frac{\tilde{m}([\omega])}{\exp\big(S_{|\omega|}f(\tau)-\mathrm{P}_{\tilde{m}}|\omega|\big)}\leq Q_g. \tag{6.18}$$

If additionally $\tilde{m}$ is shift-invariant, then $\tilde{m}$ is called an *invariant Gibbs state*.

REMARK 6.16. Notice that the sum $S_{|\omega|}f(\tau)$ in (6.18) can be replaced by $\sup(S_{|\omega|}f|_{[\omega]})$ or by $\inf(S_{|\omega|}f|_{[\omega]})$. Also, notice that if $\tilde{m}$ is a Gibbs state and if $\tilde{\mu}$ and $\tilde{m}$ are boundedly equivalent, meaning there some $K \geq 1$ such that

$$K^{-1} \leq \tilde{\mu}([\omega])/\tilde{m}([\omega]) \leq K$$

for all $\omega \in E_A^*$, then $\tilde{\mu}$ is also a Gibbs state for the potential f. We will occasionally use these facts without explicit indication.

REMARK 6.17. We record that if a continuous function $f : E_A^{\mathbb{N}} \to \mathbb{R}$ has a Gibbs state $\tilde{m}$ then f is upper bounded. This follows immediately by (6.18) and Remark 6.16 because for all $e \in E$,

$$\sup f|_{[e]} \leq \mathrm{P}_{\tilde{m}} + \log Q_g.$$

We start with the following proposition.

PROPOSITION 6.18. *If $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is a continuous function, then the following hold:*

(i) *For every Gibbs state $\tilde{m}$ for f, $\mathrm{P}_{\tilde{m}} = \mathrm{P}^\sigma(f)$.*
(ii) *Any two Gibbs states for the function f are boundedly equivalent with Radon–Nikodym derivatives bounded away from zero and infinity.*

PROOF. We shall first prove (i). Towards this end fix $n \geq 1$ and, using Remark 6.16, sum up (6.18) over all words $\omega \in E_A^n$. Since $\sum_{|\omega|=n} \tilde{m}([\omega]) = 1$, we therefore get

$$Q_g^{-1} e^{-\mathrm{P}_{\tilde{m}} n} \sum_{|\omega|=n} \exp\bigl(\sup S_n f|_{[\omega]}\bigr) \leq 1 \leq Q_g e^{-\mathrm{P}_{\tilde{m}} n} \sum_{|\omega|=n} \exp\bigl(\sup S_n f|_{[\omega]}\bigr).$$

Applying logarithms to all three terms of this formula, dividing all the terms by n and taking the limit as $n \to \infty$, we obtain $-\mathrm{P}_{\tilde{m}} + \mathrm{P}^\sigma(f) \leq 0 \leq -\mathrm{P}_{\tilde{m}} + \mathrm{P}^\sigma(f)$, which means that $\mathrm{P}_{\tilde{m}} = \mathrm{P}^\sigma(f)$. The proof of item (i) is thus complete.

In order to prove part (ii) suppose that m and ν are two Gibbs states of the function f. Notice now that part (i) implies the existence of a constant $T \geq 1$ such that

$$T^{-1} \leq \frac{\nu([\omega])}{m([\omega])} \leq T$$

for all words $\omega \in E_A^*$. Straightforward reasoning gives now that ν and m are equivalent and $T^{-1} \leq \frac{d\nu}{dm} \leq T$. The proof is complete. □

As an immediate consequence of (6.18) and Remark 6.16 we get the following.

PROPOSITION 6.19. *Any uniformly continuous function $f : E_A^{\mathbb{N}} \to \mathbb{R}$ that has a Gibbs state is acceptable.*

For $\omega \in E_A^*$ and $n \geq 1$, let

$$E_n^\omega(A) := \{\tau \in E_A^n : A_{\tau_n \omega_1} = 1\} \quad \text{and} \quad E_*^\omega(A) := \{\tau \in E_A^* : A_{\tau_{|\tau|} \omega_1} = 1\}.$$

We shall prove the following result concerning uniqueness and some stochastic properties of Gibbs states. Recall that ergodicity and complete ergodicity were introduced in Definition 6.5. For finite primitivity of matrices see Definition 4.1.

THEOREM 6.20. *If an acceptable function $f : E_A^{\mathbb{N}} \to \mathbb{R}$ has a Gibbs state and the incidence matrix A is finitely irreducible, then f has a unique shift-invariant Gibbs state. The invariant Gibbs state is ergodic. Moreover, if A is finitely primitive, the invariant Gibbs state is completely ergodic.*

Proof. Let $\tilde{m}$ be a Gibbs state for f. Fixing $\omega \in E_A^*$, using (6.18), Remark 6.16 and Proposition 6.18(i) we get for every $n \geq 1$

(6.19)

$$\begin{aligned}\tilde{m}(\sigma^{-n}([\omega])) &= \sum_{\tau \in E_n^\omega} \tilde{m}([\tau\omega]) \\ &\leq \sum_{\tau \in E_n^\omega} Q_g \exp\big(\sup(S_{n+|\omega|} f|_{[\tau\omega]}) - \mathrm{P}^\sigma(f)(n+|\omega|)\big) \\ &\leq \sum_{\tau \in E_n^\omega} Q_g \exp\big(\sup(S_n f|_{[\tau]}) - \mathrm{P}^\sigma(f)n\big) \exp\big(\sup(S_{|\omega|} f|_{[\omega]}) - \mathrm{P}^\sigma(f)|\omega|\big) \\ &\leq \sum_{\tau \in E_n^\omega} Q_g Q_g \tilde{m}([\tau]) Q_g \tilde{m}([\omega]) \leq Q_g^3 \tilde{m}([\omega]).\end{aligned}$$

Let the finite set of words Φ witness the finite irreducibility of the incidence matrix A and let p be the maximal length of a word in Φ. Since f is acceptable, it is not difficult to show that

$$T = \min\{\inf(S_{|\alpha|} f|_{[\alpha]}) - \mathrm{P}^\sigma(f)|\alpha| : \alpha \in \Phi\} > -\infty.$$

For each $\tau, \omega \in E_A^*$, let $\alpha = \alpha(\tau, \omega) \in \Phi$ be such that $\tau\alpha\omega \in E_A^*$. Then, we have for all $\omega \in E_A^*$ and all n

(6.20)

$$\begin{aligned}\sum_{i=n}^{n+p} \tilde{m}(\sigma^{-i}([\omega])) &= \sum_{i=n}^{n+p} \sum_{\tau \in E_i^\omega} \tilde{m}([\tau\omega]) \\ &\geq \sum_{\tau \in E_A^n} \tilde{m}([\tau\alpha(\tau,\omega)\omega]) \\ &\geq \sum_{\tau \in E_A^n} Q_g^{-1} \exp\big(\inf(S_{|\tau|+|\alpha(\tau,\omega)|+|\omega|} f|_{[\tau\alpha\omega]}) - \mathrm{P}^\sigma(f)(|\tau| + |\alpha(\tau,\omega)| + |\omega|)\big) \\ &\geq Q_g^{-1} \sum_{\tau \in E_A^n} \exp\big(\inf(S_n f|_{[\tau]}) - n\,\mathrm{P}^\sigma(f) + \\ &\qquad + \inf(S_{|\alpha(\tau,\omega)|} f|_{[\alpha(\tau,\omega)]}) - |\alpha(\tau,\omega)|\,\mathrm{P}^\sigma(f) + \inf(S_{|\omega|} f|_{[\omega]}) - |\omega|\,\mathrm{P}^\sigma(f)\big) \\ &\geq Q_g^{-1} e^T \exp\big(\inf(S_{|\omega|} f|_{[\omega]}) - \mathrm{P}^\sigma(f)|\omega|\big) \sum_{\tau \in E_A^n} \exp\big(\inf(S_n f|_{[\tau]}) - n\,\mathrm{P}^\sigma(f)\big) \\ &\geq Q_g^{-2} e^T \tilde{m}([\omega]) \sum_{\tau \in E_A^n} \exp\big(\inf(S_n f|_{[\tau]}) - n\,\mathrm{P}^\sigma(f)\big) \\ &\geq Q_g^{-2} e^T \tilde{m}([\omega]) Q_g^{-1} \sum_{\tau \in E_A^n} \tilde{m}([\tau]) \\ &= Q_g^{-3} e^T \tilde{m}([\omega]).\end{aligned}$$

Recall that a (real) Banach limit is a continuous linear functional $L : \ell^\infty \to \mathbb{R}$ such that the following conditions hold,

(i) if $x = (x_n) \in \ell^\infty$ satisfies $x_n \geq 0$ for all $n \in \mathbb{N}$ then $L(x) \geq 0$,
(ii) $L(s(x)) = L(x)$ for all $x \in \ell^\infty$ where $s : \ell^\infty \to \ell^\infty$ is the shift operator, i.e. $(s(x))_n = x_{n+1}$ for all $x = (x_n) \in \ell^\infty$,
(iii) if $x = (x_n) \in \ell^\infty$ is convergent then $L(x) = \lim_{n\to\infty} x_n$.

Using the Hahn-Banach theorem one can prove the existence of Banach limits, see e.g. [23, III, Theorem 7.1]. Therefore let $L : \ell_\infty \to \mathbb{R}$ be a Banach limit. It is not difficult to check that the formula $\tilde{\mu}(B) = L\big((\tilde{m}(\sigma^{-n}(B)))_{n\geq 0}\big)$ defines a shift-invariant, finitely additive probability measure on Borel sets of $E_A^{\mathbb{N}}$ satisfying

$$\frac{Q_g^{-3}e^T}{p}\tilde{m}(B) \le \tilde{\mu}(B) \le Q_g^3\tilde{m}(B), \tag{6.21}$$

for every Borel set $B \subset E_A^{\mathbb{N}}$. Since $\tilde{m}$ is a countably additive measure, we deduce that $\tilde{\mu}$ is also countably additive.

Let us prove the ergodicity of $\tilde{\mu}$ or, equivalently of any shift-invariant Gibbs state $\tilde{m}$. Let $\omega \in E_A^n$. For each $\tau \in E^*$, as in (6.20) we find:

$$\begin{aligned}\sum_{i=n}^{n+p} \tilde{m}(\sigma^{-i}([\tau]) \cap [\omega]) &\ge \tilde{m}([\omega\alpha(\omega,\tau)\tau]) \\ &\ge Q_g^{-3}e^T\tilde{m}([\tau])\tilde{m}([\omega]).\end{aligned} \tag{6.22}$$

Take now an arbitrary Borel set $B \subset E_A^{\mathbb{N}}$ and fix $\varepsilon > 0$. Since the nested family of sets $\{[\tau] : \tau \in E_A^*\}$ generates the Borel σ-algebra on $E_A^{\mathbb{N}}$, for every $n \ge 0$ and every $\omega \in E_A^n$ we can find a subfamily Z of E_A^* consisting of mutually incomparable words such that $B \subset \bigcup\{[\tau] : \tau \in Z\}$ and for $n \le i \le n+p$,

$$\sum_{\tau\in Z} \tilde{m}(\sigma^{-i}([\tau]) \cap [\omega]) \le \tilde{m}\big([\omega] \cap \sigma^{-i}(B)\big) + \varepsilon/p.$$

Then, using (6.22) we get

$$\begin{aligned}\varepsilon + \sum_{i=n}^{n+p} \tilde{m}\big([\omega] \cap \sigma^{-I}(B)\big) &\ge \sum_{i=n}^{n+p}\sum_{\tau\in Z} \tilde{m}\big([\omega] \cap \sigma^{-i}([\tau])\big) \\ &\ge \sum_{\tau\in Z} Q_g^{-3}e^T\tilde{m}([\tau])\tilde{m}([\omega]) \\ &\ge Q_g^{-3}e^T\tilde{m}(B)\tilde{m}([\omega]).\end{aligned} \tag{6.23}$$

Hence, letting $\varepsilon \searrow 0$, we get

$$\sum_{i=n}^{n+p} \tilde{m}\big([\omega] \cap \sigma^{-(i)}(B)\big) \ge Q_g^{-3}e^T\tilde{m}(B)\tilde{m}([\omega]).$$

From this inequality we find

$$\begin{aligned}\sum_{i=n}^{n+p} \tilde{m}\big(\sigma^{-i}(E_A^{\mathbb{N}} \setminus B) \cap [\omega]\big) &= \sum_{i=n}^{n+p} \tilde{m}\big([\omega] \setminus \sigma^{-i}(B) \cap [\omega]\big) \\ &= \sum_{i=n}^{n+p} \tilde{m}([\omega]) - \tilde{m}\big(\sigma^{-i}(B) \cap [\omega]\big) \\ &\le (p - Q_g^{-3}e^T\tilde{m}(B))\tilde{m}([\omega]).\end{aligned}$$

Thus, for every Borel set $B \subset E_A^{\mathbb{N}}$, for every $n \ge 0$, and for every $\omega \in E_A^n$ we have

$$\sum_{i=n}^{n+p} \tilde{m}(\sigma^{-i}(B) \cap [\omega]\big) \le (p - Q_g^{-3}e^T(1 - \tilde{m}(B)))\tilde{m}([\omega]). \tag{6.24}$$

In order to conclude the proof of the ergodicity of σ, suppose that $\sigma^{-1}(B) = B$ with $0 < \tilde{m}(B) < 1$. Put $\gamma = 1 - Q_g^{-3}e^T(1 - \tilde{m}(B))/p$. Note that (6.21) implies that $Q_g^{-3}e^Tp^{-1} \le 1$, hence $0 < \gamma < 1$. In view of (6.24), for every $\omega \in E_A^*$ we get

$$\tilde{m}(B \cap [\omega]) = \tilde{m}(\sigma^{-i}(B) \cap [\omega]\big) \le \gamma \tilde{m}([\omega]).$$

Take now $\eta > 1$ so small that $\gamma\eta < 1$ and choose a subfamily R of E_A^* consisting of mutually incomparable words such that

$$B \subset \bigcup \{[\omega] : \omega \in R\} \text{ and } \tilde{m}\left(\bigcup_{\omega \in R} \{[\omega]\}\right) \le \eta \tilde{m}(B).$$

Then

$$\begin{aligned}\tilde{m}(B) &\le \sum_{\omega \in R} \tilde{m}(B \cap [\omega]) \le \sum_{\omega \in R} \gamma \tilde{m}([\omega]) \\ &= \gamma \tilde{m}\big(\bigcup\{[\omega] : \omega \in R\}\big) \le \gamma\eta\tilde{m}(B) < \tilde{m}(B).\end{aligned}$$

This contradiction finishes the proof of the existence part.

The uniqueness of the invariant Gibbs state follows immediately from ergodicity of any invariant Gibbs state and Proposition 6.18(ii).

Finally, let us prove the complete ergodicity of $\tilde{\mu}$ or, equivalently, of any shift-invariant Gibbs state $\tilde{m}$ in case A is finitely primitive. Essentially, we repeat the argument just given. Let Φ be a finite set of words all of length q which witness the finite primitiveness of A. Fix $r \in \mathbb{N}$. Let $\omega \in E_A^n$. For each $\tau \in E_A^*$, we find the following improvement of (6.20).

$$\begin{aligned}\tilde{m}(\sigma^{-(n+qr)}([\tau]) \cap [\omega]) &\ge \sum_{\alpha \in \Phi^r \cap E^{qr} : A_{\omega_n \alpha_1} = A_{\alpha_{qr}\tau_1} = 1} \tilde{m}([\omega\alpha\tau]) \\ &\ge Q_g^{-3}e^{rT}\tilde{m}([\tau])\tilde{m}([\omega]).\end{aligned} \tag{6.25}$$

Take now an arbitrary Borel set $B \subset E_A^{\mathbb{N}}$. Fix $\varepsilon > 0$. Since the nested family of sets $\{[\tau] : \tau \in E_A^*\}$ generates the Borel σ-algebra on $E_A^{\mathbb{N}}$, for every $n \ge 0$ and every $\omega \in E_A^n$ we can find a subfamily Z of E_A^* consisting of mutually incomparable words such that $B \subset \bigcup\{[\tau] : \tau \in Z\}$ and

$$\sum_{\tau \in Z} \tilde{m}(\sigma^{-(n+qr)}([\tau]) \cap [\omega]) \le \tilde{m}\big([\omega] \cap \sigma^{-(n+qr)}(B)\big) + \varepsilon.$$

Then, using (6.25) we get

$$\begin{aligned}\varepsilon + \tilde{m}\big([\omega] \cap \sigma^{-(n+qr)}(B)\big) &\ge \sum_{\tau \in Z} Q^{-3}e^{rT}\tilde{m}([\tau])\tilde{m}([\omega]) \\ &\ge Q_g^{-3}e^{rT}\tilde{m}(B)\tilde{m}([\omega]).\end{aligned}$$

Hence, letting $\varepsilon \searrow 0$, we get

$$\tilde{m}\big([\omega] \cap \sigma^{-(n+qr)}(B)\big) \ge \tilde{Q}(r)\tilde{m}(B)\tilde{m}([\omega]),$$

where $\tilde{Q}(r) := Q_g^{-3}\exp(rT)$. Note that it follows from this last inequality that $\tilde{Q} := \tilde{Q}(r) \le 1$. Also, from this inequality we find that

$$\begin{aligned}\tilde{m}\big(\sigma^{-(n+qr)}(E_A^{\mathbb{N}} \setminus B) \cap [\omega]\big) &= \tilde{m}\big([\omega] \setminus \sigma^{-n}(B) \cap [\omega]\big) \\ &= \tilde{m}([\omega]) - \tilde{m}\big(\sigma^{-(n+qr)}(B) \cap [\omega]\big) \\ &\le (1 - \tilde{Q}\,\tilde{m}(B))\tilde{m}([\omega]).\end{aligned}$$

Thus, for every Borel set $B \subset E_A^{\mathbb{N}}$, for every $n \geq 0$, and for every $\omega \in E_A^n$ we have

$$\tilde{m}(\sigma^{-(n+qr)}(B) \cap [\omega]) \leq \big(1 - \tilde{Q}\,(1 - \tilde{m}(B))\big)\tilde{m}([\omega]). \tag{6.26}$$

In order to conclude the proof of the complete ergodicity of σ suppose that $\sigma^{-r}(B) = B$ with $0 < \tilde{m}(B) < 1$. For $k \in \mathbb{N}$, let

$$(E_A^r)^k = \{\omega \in E_A^{rk} : \omega = v^1 \dots v^k \text{ and } v^i \in E_A^r \text{ for all } i = 1, \dots, k\},$$

and

$$(E_A^r)^* = \bigcup_{k \in \mathbb{N}} (E_A^r)^k.$$

Put $\gamma = 1 - \tilde{Q}(1 - \tilde{m}(B))$. Note that $0 < \gamma < 1$. In view of (6.26), for every $\omega \in (E_A^r)^*$ we get $\tilde{m}(B \cap [\omega]) = \tilde{m}(\sigma^{-(|\omega|+qr)}(B) \cap [\omega]\big) \leq \gamma\tilde{m}([\omega])$. Take now $\eta > 1$ so small that $\gamma\eta < 1$ and choose a subfamily R of $(E_A^r)^*$ consisting of mutually incomparable words such that $B \subset \bigcup\{[\omega] : \omega \in R\}$ and

$$\tilde{m}\big(\bigcup\{[\omega] : \omega \in R\}\big) \leq \eta\tilde{m}(B).$$

Then

$$\begin{aligned}\tilde{m}(B) &\leq \sum_{\omega \in R} \tilde{m}(B \cap [\omega]) \leq \sum_{\omega \in R} \gamma\tilde{m}([\omega]) \\ &= \gamma\tilde{m}\big(\bigcup\{[\omega] : \omega \in R\}\big) \leq \gamma\eta\tilde{m}(B) < \tilde{m}(B).\end{aligned}$$

This contradiction finishes the proof of the complete ergodicity of $\tilde{m}$. The proof is complete. □

There is a sort of converse to part of the preceding theorem which claims that finite irreducibility of the incidence matrix A is necessary for the existence of Gibbs states. This justifies well our restriction to irreducible matrices. We need the following lemma first. The next two results are due to Sarig [56].

LEMMA 6.21. *Suppose that the incidence matrix $A : E \times E \to \{0, 1\}$ is irreducible and $\tilde{m}$ is a shift-invariant Gibbs state for some acceptable function $f : E_A^{\mathbb{N}} \to \mathbb{R}$. Then there is a positive constant K such that for every $e \in E$,*

$$\min\left\{\sum_{a \in E: A_{ea}=1} \exp\big(\sup\big(f|_{[a]}\big), \sum_{a \in E: A_{ae}=1} \exp\big(\sup\big(f|_{[a]}\big)\big)\right\} \geq K.$$

PROOF. Let $\tilde{m}$ be a shift-invariant Gibbs state for f. Fix $a, b \in E$ arbitrary. Since $\tilde{m}$ is a Gibbs state for f, we have

$$\begin{aligned}Q_g^{-1} e^{-2\mathrm{P}^\sigma(f)} \exp\big(\inf\big(f|_{[a]}\big) \exp\big(\inf\big(f|_{[b]}\big)\big) &\leq \tilde{m}([ab]) \\ &\leq Q_g e^{-2\mathrm{P}^\sigma(f)} \exp\big(\sup\big(f|_{[a]} \exp\big(\sup\big(f|_{[b]}\big)\big).\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{m}([a]) &= \sum_{c \in E: A_{ac}=1} \tilde{m}([ac]) \\ &\leq Q_g e^{-2\mathrm{P}^\sigma(f)} \exp\big(\sup\big(f|_{[a]}\big)\big) \sum_{c \in E: A_{ac}=1} \exp\big(\sup\big(f|_{[c]}\big).\end{aligned}$$

Since $\tilde{m}$ is a Gibbs state for f, we have that $\tilde{m}([a]) \geq Q_g^{-1} e^{-\mathrm{P}^\sigma(f)} \exp\big(\sup\big(f|_{[a]}\big)\big)$. Hence,

$$Q_g^{-2} e^{\mathrm{P}^\sigma(f)} \leq \sum_{c \in E: A_{ac}=1} \exp\big(\sup\big(f|_{[c]}\big)\big). \tag{6.27}$$

Since $\tilde{m}$ is shift-invariant, we have that

$$\tilde{m}([a]) = \tilde{m}(\sigma^{-1}([a])) = \sum_{c \in E: A_{ca}=1} \tilde{m}([ca]).$$

But then

$$\begin{aligned} Q_g^{-1} e^{-\mathrm{P}^\sigma(f)} &\exp\big(\sup\big(f|_{[a]}\big)\big) \\ &\leq \tilde{m}(\sigma^{-1}([a])) \\ &\leq Q_g e^{-2\mathrm{P}^\sigma(f)} \sum_{c \in E: A_{ca}=1} \exp\big(\sup\big(f|_{[c]}\big)\big) \exp\big(\sup\big(f|_{[a]}\big)\big). \end{aligned}$$

Therefore,

$$Q_g^{-2} e^{\mathrm{P}^\sigma(f)} \leq \sum_{c \in E: A_{ca}=1} \exp\big(\sup\big(f|_{[c]}\big)\big).$$

Along with (6.27) this completes the proof. □

THEOREM 6.22. *Assume that an incidence matrix $A : E \times E \to \{0,1\}$ is irreducible. If an acceptable function $f : E_A^{\mathbb{N}} \to \mathbb{R}$ has an invariant Gibbs state, then the incidence matrix A is finitely irreducible.*

PROOF. Since the incidence matrix A is irreducible, it suffices to show that there exists a finite set of letters $F \subset E$ such that for every letter $e \in E$ there exist $a, b \in F$ such that

$$A_{ae} = A_{ea} = 1.$$

Without loss of generality $E = \mathbb{N}$. Since f has a Gibbs state, we have that

$$\sum_{n \in \mathbb{N}} \exp\big(\sup\big(f|_{[n]}\big)\big) \leq Q_g e^{\mathrm{P}^\sigma(f)} < +\infty.$$

So, there exists some $q \in \mathbb{N}$ such that

$$\sum_{j > q} \exp\big(\sup\big(f|_{[j]}\big)\big) < K, \tag{6.28}$$

where K is the constant coming from Lemma 6.21. Let

$$F := \{1, 2, \ldots, q\}.$$

It now follows from (6.28) and from Lemma 6.21 that every letter is followed by some element of F and every letter is preceded by some element of F. The proof is complete. □

Recall that for $\omega, \tau \in E^{\mathbb{N}}$, we defined $\omega \wedge \tau \in E^{\mathbb{N}} \cup E^*$ to be the longest initial block common for both ω and τ. Of course if both $\omega, \tau \in E_A^{\mathbb{N}}$, then also $\omega \wedge \tau \in E_A^{\mathbb{N}} \cup E_A^*$. We say that a function $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is *Hölder continuous with an exponent* $\alpha > 0$ if

$$V_\alpha(f) := \sup_{n \geq 1}\{V_{\alpha,n}(f)\} < \infty,$$

where

$$V_{\alpha,n}(f) = \sup\{|f(\omega) - f(\tau)| e^{\alpha(n-1)} : \omega, \tau \in E_A^{\mathbb{N}} \text{ and } |\omega \wedge \tau| \geq n\}.$$

Note that if $g : E_A^{\mathbb{N}} \to \mathbb{R}$ is Hölder continuous of order α and $\rho, \upsilon \in E_A^{\mathbb{N}}$, then

$$V_\alpha(g) d_\alpha(\rho, \upsilon) = V_\alpha(g) e^{-\alpha|\rho \wedge \upsilon|} \geq e^{-\alpha}|g(\rho) - g(\upsilon)|.$$

Also, note that each Hölder continuous function is acceptable.

The last theorem in this section, Theorem 6.24, shows that the assumption $\int -f\, d\tilde{\mu}_f < +\infty$ is sufficient for the appropriate Gibbs state to be a unique equilibrium state, recall (6.16) for the definition of equilibrium states. For the proof of the Theorem 6.24 we will need the following lemma.

LEMMA 6.23. *Suppose that the incidence matrix A is finitely irreducible and that the acceptable function $f : E_A^{\mathbb{N}} \to \mathbb{R}$ has a Gibbs state. Denote by $\tilde{\mu}_f$ its unique invariant Gibbs state (see Theorem 6.20). Then the following three conditions are equivalent:*

(i) $\int_{E_A^{\mathbb{N}}} -f d\tilde{\mu}_f < +\infty$.
(ii) $\sum_{e\in E} \inf(-f|_{[e]}) \exp(\inf f|_{[e]}) < +\infty$.
(iii) $\mathrm{H}_{\tilde{\mu}_f}(\alpha) < +\infty$*, where* $\alpha = \{[e] : e \in E\}$ *is the partition of* $E_A^{\mathbb{N}}$ *into initial cylinders of length* 1.

PROOF. Without loss of generality we can identify $E = \mathbb{N}$. We start with the proof of the implication (i) $\Rightarrow$ (ii). Suppose that $\int -f d\tilde{\mu}_f < \infty$. This obviously means that $\sum_{e\in E} \int_{[e]} -f d\tilde{\mu}_f < +\infty$ and consequently

$$\begin{aligned}+\infty > \sum_{i\in E} \inf(-f|_{[i]})\tilde{\mu}_f([i]) &\geq Q_g^{-1} \sum_{i\in E} \inf(-f|_{[i]}) \exp\bigl(\inf f|_{[i]} - \mathrm{P}^\sigma(f)\bigr) \\ &= Q_g^{-1} e^{-\mathrm{P}^\sigma(f)} \sum_{i\in E} \inf(-f|_{[i]}) \exp\bigl(\inf f|_{[i]}\bigr).\end{aligned}$$

We will now prove that (ii) $\Rightarrow$ (iii). By definition,

$$\begin{aligned}\mathrm{H}_{\tilde{\mu}_f}(\alpha) &= \sum_{i\in E} -\tilde{\mu}_f([i]) \log \tilde{\mu}_f([i]) \\ &\leq \sum_{i\in E} -\tilde{\mu}_f([i]) \bigl(\inf(f|_{[i]}) - \mathrm{P}^\sigma(f) - \log Q_g\bigr).\end{aligned}$$

Since $\sum_{i\in E} \tilde{\mu}_f([i])(\mathrm{P}^\sigma(f) + \log Q_g\bigr) < \infty$, it suffices to show that

$$\sum_{i\in E} -\tilde{\mu}_f([i]) \inf(f|_{[i]}) < +\infty.$$

And indeed,

$$\begin{aligned}\sum_{i\in E} -\tilde{\mu}_f([i]) \inf(f|_{[i]}) &= \sum_{i\in E} \tilde{\mu}_f([i]) \sup(-f|_{[i]}) \\ &\leq \sum_{i\in E} \tilde{\mu}_f([i]) \bigl(\inf(-f|_{[i]}) + \mathrm{osc}(f)\bigr).\end{aligned}$$

Since $\sum_{i\in E} \tilde{\mu}_f([i])\mathrm{osc}(f) = \mathrm{osc}(f)$, it is enough to show that

$$\sum_{i\in E} \tilde{\mu}_f([i]) \inf(-f|_{[i]}) < +\infty.$$

Since $\tilde{\mu}_f$ is a probability measure, $\lim_{i\to\infty} \tilde{\mu}_f([i]) = 0$. Therefore, it follows from (6.18) that $\lim_{i\to\infty}\bigl(\sup(f|_{[i]}) - \mathrm{P}^\sigma(f)\bigr) = -\infty$. Thus, for all i sufficiently large, say

$i \ge k$, $\sup(f|_{[i]}) < 0$. Hence, for all $i \ge k$, $\inf(-f|_{[i]}) = -\sup(f|_{[i]}) > 0$. So, using (6.18) again, we get

$$\sum_{i\ge k} \tilde\mu_f([i]) \inf(-f|_{[i]}) \le \sum_{i\ge k} Q_g \exp\big(\inf(f|_{[i]}) - \mathrm{P}^\sigma(f)\big) \inf(-f|_{[i]})$$
$$= Q_g e^{-\mathrm{P}^\sigma(f)} \sum_{i\ge k} \exp\big(\inf(f|_{[i]})\big) \inf(-f|_{[i]})$$

which is finite due to our assumption. Hence we have shown that $\mathrm{H}_{\tilde\mu_f}(\alpha) < +\infty$, and the proof of the implication (ii) ⇒ (iii) is complete.

Finally we will show that (iii) ⇒ (i). Suppose that $\mathrm{H}_{\tilde\mu_f}(\alpha) < +\infty$. We need to show that $\int -f\, d\tilde\mu_f < +\infty$. We have

$$\infty > \mathrm{H}_{\tilde\mu_f}(\alpha) = \sum_{i\in E} -\tilde\mu_f([i]) \log\big(\tilde\mu_f([i])\big)$$
$$\ge \sum_{i\in E} -\tilde\mu_f([i])\big(\inf(f|_{[i]}) - \mathrm{P}^\sigma(f) + \log Q_g\big).$$

Hence, $\sum_{i\in E} -\tilde\mu_f([i]) \inf(f|_{[i]}) < +\infty$ and therefore

$$\int -f d\tilde\mu_f = \sum_{i\in E} \int_{[i]} -f d\tilde\mu_f$$
$$\le \sum_{i\in E} \sup(-f|_{[i]})\tilde\mu_f([i]) = \sum_{i\in E} -\inf(f|_{[i]})\tilde\mu_f([i]) < +\infty.$$

The proof is complete. □

For the next theorem recall (6.16) for the definition of equilibrium states.

Theorem 6.24. *Suppose that the incidence matrix A is finitely irreducible. Suppose also that $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is a Hölder continuous function which has a Gibbs state and*

$$\int -f d\tilde\mu_f < +\infty,$$

where $\tilde\mu_f$ is the unique invariant Gibbs state for the potential f (see Theorem 6.20). Then $\tilde\mu_f$ is the unique equilibrium state for the potential f.

Proof. In order to show that $\tilde\mu_f$ is an equilibrium state of the potential f consider $\alpha = \{[e] : e \in E\}$, the partition of $E_A^{\mathbb{N}}$ into initial cylinders of length one. By Lemma 6.23, $\mathrm{H}_{\tilde\mu_f}(\alpha) < +\infty$. Applying the Breiman–Shannon–McMillan Theorem [53, Theorem 2.5.4], Birkhoff's Ergodic Theorem, and (6.18), we get for $\tilde\mu_f$-a.e. $\omega \in E_A^{\mathbb{N}}$

$$\mathrm{h}_{\tilde\mu_f}(\sigma) = \lim_{n\to\infty} -\frac{1}{n} \log \tilde\mu_f([\omega|_n])$$
$$\ge \lim_{n\to\infty} -\frac{1}{n}\big(\log Q_g + S_n f(\omega) - n\,\mathrm{P}^\sigma(f)\big) = \lim_{n\to\infty} \frac{-1}{n} S_n f(\omega) + \mathrm{P}^\sigma(f)$$
$$= \int -f d\tilde\mu_f + \mathrm{P}^\sigma(f)$$

which, in view of Theorem 6.13, implies that $\tilde\mu_f$ is an equilibrium state for the potential f.

In order to prove uniqueness of equilibrium states we follow the reasoning taken from the proof of Theorem 1 in [28]. So, suppose that $\tilde\nu \ne \tilde\mu_f$ is an equilibrium state for the potential $f : E_A^{\mathbb{N}} \to \mathbb{R}$. Assume first that $\tilde{\nu}$ is ergodic. Then, using (6.18), we have for every $n \geq 1$:

$$\begin{aligned}
0 = n(\mathrm{h}_{\tilde{\nu}}(\sigma) + \int (f - \mathrm{P}^\sigma(f))d\tilde{\nu}) &\leq \mathrm{H}_{\tilde{\nu}}(\alpha^n) + \int (S_n f - \mathrm{P}^\sigma(f)n)d\tilde{\nu} \\
&= -\sum_{|\omega|=n} \tilde{\nu}([\omega]) \left(\log \tilde{\nu}([\omega]) - \frac{1}{\tilde{\nu}([\omega])} \int_{[\omega]} (S_n f - \mathrm{P}^\sigma(f)n)d\tilde{\nu} \right) \\
&\leq -\sum_{|\omega|=n} \tilde{\nu}([\omega]) \left(\log \tilde{\nu}([\omega]) - (S_n f(\tau_\omega) - \mathrm{P}^\sigma(f)n)\right) \text{ for a suitable } \tau_\omega \in [\omega] \\
&= -\sum_{|\omega|=n} \tilde{\nu}([\omega]) \big(\log[\tilde{\nu}([\omega]) \exp\big(\mathrm{P}^\sigma(f)n - S_n f(\tau_\omega)\big)]\big) \\
&\leq -\sum_{|\omega|=n} \tilde{\nu}([\omega]) \big(\log \tilde{\nu}([\omega])(\mu_f([\omega])Q_g)^{-1}\big) \\
&= \log Q_g - \sum_{|\omega|=n} \tilde{\nu}([\omega]) \log \left(\frac{\tilde{\nu}([\omega])}{\tilde{\mu}_f([\omega])} \right).
\end{aligned}$$

Therefore, in order to show that $\tilde{\nu} = \mu_f$, it suffices to show that

$$\lim_{n\to\infty} \left(-\sum_{|\omega|=n} \tilde{\nu}([\omega]) \log \left(\frac{\tilde{\nu}([\omega])}{\tilde{\mu}_f([\omega])} \right) \right) = -\infty.$$

Since both measures $\tilde{\nu}$ and $\tilde{\mu}_f$ are ergodic and $\tilde{\nu} \neq \tilde{\mu}_f$, the measures $\tilde{\nu}$ and $\tilde{\mu}_f$ must be mutually singular. In particular,

$$\lim_{n\to\infty} \tilde{\nu} \left(\left\{ \omega \in E_A^{\mathbb{N}} : \frac{\tilde{\nu}([\omega|_n])}{\tilde{\mu}_f([\omega|_n])} \leq S \right\} \right) = 0$$

for every $S > 0$. For every $j \in \mathbb{Z}$ and every $n \geq 1$, set

$$F_{n,j} = \left\{ \omega \in E_A^{\mathbb{N}} : e^{-j} \leq \frac{\tilde{\nu}([\omega|_n])}{\tilde{\mu}_f([\omega|_n])} < e^{-j+1} \right\}.$$

Then

$$\tilde{\nu}(F_{n,j}) = \int_{F_{n,j}} \frac{\tilde{\nu}([\omega|_n])}{\tilde{\mu}_f([\omega|_n])} d\tilde{\mu}_f(\omega) \leq e^{-j+1} \tilde{\mu}_f(F_{n,j}) \leq e^{-j+1}.$$

Notice

$$-\sum_{|\omega|=n} \tilde{\nu}([\omega]) \log \left(\frac{\tilde{\nu}([\omega|_n])}{\tilde{\mu}_f([\omega|_n])} \right) = -\int \log \left(\frac{\tilde{\nu}([\omega|n])}{\tilde{\mu}_f([\omega|n])} \right) d\tilde{\nu}(\omega) \leq \sum_{j\in\mathbb{Z}} j\tilde{\nu}(F_{n,j}).$$

Now, for each $k = -1, -2, -3, \dots$ we have

$$\begin{aligned}
-\sum_{|\omega|=n} \tilde{\nu}([\omega]) \log \left(\frac{\tilde{\nu}([\omega|_n])}{\tilde{\mu}_f([\omega|_n])} \right) &\leq k \sum_{j\leq k} \tilde{\nu}(F_{n,j}) + \sum_{j\geq 1} j e^{-j+1} \\
&= k\tilde{\nu} \left(\left\{ \omega \in E_A^{\mathbb{N}} : \frac{\tilde{\nu}([\omega|_n])}{\tilde{\mu}_f([\omega|_n])} \geq e^{-k} \right\} \right) + \sum_{j\geq 1} j e^{-j+1}.
\end{aligned}$$

Thus, we have for each negative integer k,

$$\limsup_{n\to\infty} \left(-\sum_{|\omega|=n} \tilde{\nu}([\omega]) \log \left(\frac{\tilde{\nu}([\omega])}{\tilde{\mu}_f([\omega])} \right) \right) \leq k + \sum_{j\geq 1} j e^{-j+1}.$$

Now, letting k go to $-\infty$ completes the proof that $\tilde{\nu} = \mu_f$ in the case when $\tilde{\nu}$ is ergodic.

Now let $\tilde{\nu}$ be an arbitrary equilibrium state for f, then, as $E_A^{\mathbb{N}}$ is a Polish space, it follows from Theorem 6.6 and Theorem 6.9 that

$$\mathrm{P}^\sigma(f) = \mathrm{h}_{\tilde{\nu}}(\sigma) + \int_{E_A^{\mathbb{N}}} f\, d\tilde{\nu} = \int_{\mathcal{E}(\sigma,\mathcal{B})} \Big(\mathrm{h}_\nu(\sigma) + \int_{E_A^{\mathbb{N}}} f\, d\nu\Big)\, dP_{\tilde{\nu}}(\nu), \tag{6.29}$$

where $\mathcal{B}$ is the Borel σ-algebra on $E_A^{\mathbb{N}}$. Note also that again by Theorem 6.6

$$-\infty < \int_{E_A^{\mathbb{N}}} f d\tilde{\nu} = \int_{\mathcal{E}(\sigma,\mathcal{B})} \Big(\int_{E_A^{\mathbb{N}}} f\, d\nu\Big)\, dP_{\tilde{\nu}}(\nu),$$

hence Remark 6.17 implies that for $P_{\tilde{\nu}}$-a.e. $\nu \in \mathcal{E}(\sigma,\mathcal{B})$ we have that

$$\int_{E_A^{\mathbb{N}}} f d\nu > -\infty. \tag{6.30}$$

Now by (6.30) and the 1st Variational Principle (Theorem 6.10) we deduce that the integrand on the rightmost part of this formula is always bounded above by $\mathrm{P}^\sigma(f)$. Therefore, by the first part of the proof, it is equal to $\mathrm{P}^\sigma(f)$ if and only if $\nu = \mu_f$. But, if $\tilde{\nu} \neq \mu_f$ then $P_{\tilde{\nu}}$ is not the Dirac-δ measure supported on μ_f and, in consequence, the rightmost integral in (6.29) is smaller than $\mathrm{P}^\sigma(f)$. This contradiction finishes the proof. □

6.3. Perron-Frobenius Operator

In this section we define the appropriate Perron-Frobenius operators and collect some basic properties of them. These operators are primarily applied (see Theorem 6.31) to prove the existence of Gibbs states. We start with the following technical result usually referred to as a bounded distortion lemma.

Lemma 6.25. *If $g : E_A^{\mathbb{N}} \to \mathbb{C}$ and $V_\alpha(g) < \infty$, then for all $n \geq 1$, for all $\omega, \tau \in E_A^{\mathbb{N}}$ with $\omega_1 = \tau_1$, and all $\rho \in E_A^n$ with $A_{\rho_n\omega_1} = A_{\rho_n\tau_1} = 1$ we have*

$$\big|S_n g(\rho\omega) - S_n g(\rho\tau)\big| \leq \frac{V_\alpha(g)}{e^\alpha - 1} d_\alpha(\omega,\tau).$$

Proof. We have

$$\begin{aligned}
\big|S_n g(\rho\omega) - S_n g(\rho\tau)\big| &\leq \sum_{i=0}^{n-1} |g(\sigma^i(\rho\omega)) - g(\sigma^i(\tau\omega))| \\
&\leq \sum_{i=0}^{n-1} e^\alpha V_\alpha(g) d_\alpha(\sigma^i(\rho\omega), \sigma^i(\tau\omega)) \\
&\leq e^\alpha V_\alpha(g) \sum_{i=0}^{n-1} e^{-\alpha(n-i)} d_\alpha(\omega,\tau) \\
&\leq V_\alpha(g) \frac{e^{-2\alpha}}{1-e^{-\alpha}} d_\alpha(\omega,\tau) \\
&\leq \frac{V_\alpha(g)}{e^\alpha - 1} d_\alpha(\omega,\tau).
\end{aligned}$$

The proof is complete. □

We set

$$T(g) = \exp\left(\frac{V_\alpha(g)}{e^\alpha - 1}\right).$$

From now throughout this section $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is assumed to be a Hölder continuous function with an exponent $\beta > 0$, and it is assumed to satisfy the following requirement

$$\sum_{e\in E} \exp(\sup(f|_{[e]})) < \infty. \tag{6.31}$$

Functions f satisfying this condition will be called in the sequel *summable*. We note that if f has a Gibbs state, then f is summable. This requirement allows us to define the *Perron–Frobenius operator* $\mathcal{L}_f : C_b(E_A^{\mathbb{N}}) \to C_b(E_A^{\mathbb{N}})$, acting on the space of bounded continuous functions $C_b(E_A^{\mathbb{N}})$ endowed with $||\cdot||_\infty$, the supremum norm, as follows:

$$\mathcal{L}_f(g)(\omega) = \sum_{e\in E: A_{e\omega_1}=1} \exp(f(e\omega))g(e\omega).$$

Then $||\mathcal{L}_f||_\infty \le \sum_{e\in E} \exp(\sup(f|_{[e]})) < +\infty$ and for every $n \ge 1$

$$\mathcal{L}_f^n(g)(\omega) = \sum_{\tau\in E_A^n: A_{\tau_n\omega_1}=1} \exp\big(S_nf(\tau\omega)\big)g(\tau\omega).$$

The conjugate operator $\mathcal{L}_f^*$ acting on the space $C_b^*(E_A^{\mathbb{N}})$ has the following form:

$$\mathcal{L}_f^*(\mu)(g) = \mu(\mathcal{L}_f(g)) = \int \mathcal{L}_f(g)\, d\mu.$$

Our first goal now is to study eigenmeasures of the conjugate operator $\mathcal{L}_f^*$, precisely to show that these eigenmeasures coincide with Gibbs states of f. We next show the existence of eigenmeasures in the case of finite alphabet, and then, using the two above facts, to prove the the existence of eigenmeasures of $\mathcal{L}_f^*$ for any countable alphabet.

We now assume that $\tilde{m}$ is an eigenmeasure of the conjugate operator $\mathcal{L}_f^* : C_b^*(E_A^{\mathbb{N}}) \to C_b^*(E_A^{\mathbb{N}})$. The corresponding eigenvalue is denoted by λ. Since $\mathcal{L}_f$ is a positive operator, we have that $\lambda \ge 0$. Obviously

$$\mathcal{L}_f^{*n}(\tilde{m}) = \lambda^n \tilde{m}$$

for every integer $n \ge 0$. The integral version of this equality takes on the following form

$$\int_{E_A^{\mathbb{N}}} \sum_{\tau\in E^n: A_{\tau_n\omega_1}=1} \exp\big(S_nf(\tau\omega)\big)g(\tau\omega)\, d\tilde{m}(\omega) = \lambda^n \int_{E_A^{\mathbb{N}}} g\, d\tilde{m}, \tag{6.32}$$

for every function $g \in C_b(E_A^{\mathbb{N}})$. In fact this equality extends to the space of all bounded Borel functions on $E_A^{\mathbb{N}}$. In particular, taking $\omega \in E_A^*$, say $\omega \in E_A^n$, a Borel set $A \subset E_A^{\mathbb{N}}$ such that $A_{\omega_n\tau_1} = 1$, for every $\tau \in A$, and $g = 1\!\!1_{[\omega A]}$, we obtain from

(6.32)
$$\lambda^n\tilde{m}([\omega A]) = \int \sum_{\tau\in E^n : A_{\tau_n\rho_1}=1} \exp\big(S_nf(\tau\rho)\big)\mathbb{1}_{[\omega A]}(\tau\rho)\,d\tilde{m}(\rho)$$

(6.33)
$$= \int_{\{\rho\in A : A_{\omega_n\rho_1}=1\}} \exp\big(S_nf(\omega\rho)\big)\,d\tilde{m}(\rho)$$
$$= \int_A \exp\big(S_nf(\omega\rho)\big)\,d\tilde{m}(\rho)$$

Remark 6.26. Note that if (6.33) holds, then by representing a Borel set $B\subset E_A^{\mathbb{N}}$ as a union $\bigcup_{\omega\in E^n}[\omega B_\omega]$, where $B_\omega = \{\alpha\in E_A^{\mathbb{N}} : A_{\omega_n\alpha_1}=1 \text{ and } \omega\alpha\in B\}$, a straightforward calculation based on (6.33) demonstrates that (6.32) is satisfied for the characteristic function $\mathbb{1}_B$ of the set B. Next, it follows from standard approximation arguments, that (6.32) is satisfied for all $\tilde{m}$-integrable functions g. Finally, we note that $\tilde{m}$ is an eigenmeasure of the conjugate operator $\mathcal{L}_f^*$ if and only if formula (6.33) is satisfied.

Theorem 6.27. *If the incidence matrix A is finitely irreducible and $f : E_A^{\mathbb{N}}\to\mathbb{R}$ is a summable Hölder continuous function, then the eigenmeasure $\tilde{m}$ is a Gibbs state for f. In addition, its corresponding eigenvalue is equal to $e^{\mathrm{P}^\sigma(f)}$.*

Proof. It immediately follows from (6.33) and Lemma 6.25 that for every $\omega\in E_A^*$ and every $\tau\in[\omega]$

(6.34)
$$\tilde{m}([\omega]) \le \lambda^{-n}T(f)\exp\big(S_nf(\tau)\big) = T(f)\exp\big(S_nf(\tau) - n\log\lambda\big),$$

where $n=|\omega|$. On the other hand, let Φ be a minimal set which witnesses the finite irreducibility of A. For every $\alpha\in\Phi$, let

$$E_a = \{\tau\in E_A^{\mathbb{N}} : \omega\alpha\tau\in E_A^{\mathbb{N}}\}.$$

By the definition of Φ, $\bigcup_{\alpha\in\Phi}E_a = E_A^{\mathbb{N}}$. Hence, there exists $\gamma\in\Phi$ such that $\tilde{m}(E_\gamma)\ge(\#\Phi)^{-1}$. Writing $p=|\gamma|$ we therefore have

(6.35)
$$\begin{aligned}
\tilde{m}([\omega]) &\ge \tilde{m}([\omega\gamma]) = \lambda^{-(n+p)}\int_{\rho\in E_A^{\mathbb{N}} : A_{\gamma_p\rho_1}=1} \exp\big(S_{n+p}f(\omega\gamma\rho)\big)d\tilde{m}(\rho)\\
&= \lambda^{-(n+p)}\int_{\rho\in E^\infty : A_{\gamma_p\rho_1}=1} \exp\big(S_nf(\omega\gamma\rho)\big)\exp\big(S_pf(\gamma\rho)\big)d\tilde{m}(\rho)\\
&\ge \lambda^{-n}\exp\big(\min\{\inf(S_{|\alpha|}f|_{[\alpha]}) : \alpha\in\Phi\} - p\log\lambda\big)\int_{\rho\in E_A^{\mathbb{N}} : A_{\gamma_p\rho_1}=1}\exp\big(S_nf(\omega\gamma\rho)\big)d\tilde{m}(\rho)\\
&= C\lambda^{-n}\int_{E_\gamma}\exp\big(S_nf(\omega\gamma\rho)\big)d\tilde{m}(\rho) \ge CT(f)^{-1}\lambda^{-n}m(E_\gamma)\exp\big(S_nf(\tau)\big)\\
&\ge CT(f)^{-1}(\#\Phi)^{-1}\exp\big(S_nf(\tau) - n\log\lambda\big),
\end{aligned}$$

where $C = \exp\big(\min\{\inf(S_{|\alpha|}f|_{[\alpha]}) : \alpha\in\Phi\} - p\log\lambda\big)$. Thus $\tilde{m}$ is a Gibbs state for f. The equality $\lambda = e^{\mathrm{P}^\sigma(f)}$ follows now immediately from Proposition 6.18. The proof is complete. □

Theorem 6.28. *If the incidence matrix A is finitely irreducible and $f : E_A^{\mathbb{N}}\to\mathbb{R}$ is a summable Hölder continuous function, then the conjugate operator $e^{-\mathrm{P}^\sigma(f)}\mathcal{L}_f^*$ fixes at most one Borel probability measure.*

PROOF. Suppose that $\tilde{m}$ and $\tilde{m}_1$ are such two fixed points. In view of Proposition 6.18(b) and Theorem 6.27, the measures $\tilde{m}$ and $\tilde{m}_1$ are equivalent. Consider the Radon–Nikodym derivative $\rho = \frac{d\tilde{m}_1}{d\tilde{m}}$. Temporarily fix $\omega \in E_A^*$, say $\omega \in E_A^n$. It then follows from (6.33) and Theorem 6.27 that

$$
\begin{aligned}
&\tilde{m}([\omega]) = \\
&= \int_{\tau\in E^\infty : A_{\omega_n\tau_1}=1} \exp\big(S_n f(\omega\tau) - \mathrm{P}^\sigma(f)n\big)\, d\tilde{m}(\tau) \\
&= \int_{\tau\in E^\infty : A_{\omega_n\tau_1}=1} \exp\big(S_n f(\sigma(\omega\tau))\big) - \mathrm{P}^\sigma(f)(n-1)\big) \exp\big(f(\omega\tau) - \mathrm{P}^\sigma(f)\big)\, d\tilde{m}(\tau) \\
&= \int_{\tau\in E^\infty : A_{(\sigma(\omega))_{n-1}\tau_1}=1} \exp\big(S_n f(\sigma(\omega\tau))\big) - \mathrm{P}^\sigma(f)(n-1)\big) \exp\big(f(\omega\tau) - \mathrm{P}^\sigma(f)\big)\, d\tilde{m}(\tau).
\end{aligned}
$$

Thus,

$$
\inf\big(\exp\big(f|_{[\omega]} - \mathrm{P}^\sigma(f)\big)\big)\tilde{m}([\sigma\omega]) \le \tilde{m}([\omega]) \le \sup\big(\exp\big(f|_{[\omega]} - \mathrm{P}^\sigma(f)\big)\big)\tilde{m}([\sigma\omega]).
$$

Since $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is Hölder continuous, we therefore conclude that for every $\omega \in E_A^{\mathbb{N}}$

$$
\lim_{n\to\infty} \frac{\tilde{m}\big([\omega|_n]\big)}{\tilde{m}\big([\sigma(\omega)|_{n-1}]\big)} = \exp\big(f(\omega) - \mathrm{P}^\sigma(f)\big) \tag{6.36}
$$

and the same formula is true with $\tilde{m}$ replaced by $\tilde{m}_1$. Using Theorem 6.27 and Theorem 6.20, there exists a set of points $\omega \in E_A^{\mathbb{N}}$ with $\tilde{m}$ measure 1 for which the Radon–Nikodym derivatives $\rho(\omega)$ and $\rho(\sigma(\omega))$ are both defined. Let $\omega \in E_A^{\mathbb{N}}$ be such a point. Then from (6.36) and its version for $\tilde{m}_1$ we obtain

$$
\begin{aligned}
\rho(\omega) &= \lim_{n\to\infty}\left(\frac{\tilde{m}_1\big([\omega|_n]\big)}{\tilde{m}\big([\omega|_n]\big)}\right) \\
&= \lim_{n\to\infty}\left(\frac{\tilde{m}_1\big([\omega|_n]\big)}{\tilde{m}_1\big([\sigma(\omega)|_{n-1}]\big)} \cdot \frac{\tilde{m}_1\big([\sigma(\omega)|_{n-1}]\big)}{\tilde{m}\big([\sigma(\omega)|_{n-1}]\big)} \cdot \frac{\tilde{m}\big([\sigma(\omega)|_{n-1}]\big)}{\tilde{m}\big([\omega|_n]\big)}\right) \\
&= \exp\big(f(\omega) - \mathrm{P}^\sigma(f)\big)\rho(\sigma(\omega)) \exp\big(\mathrm{P}^\sigma(f) - f(\omega)\big) = \rho(\sigma(\omega)).
\end{aligned}
$$

But according to Theorem 6.20, $\sigma : E_A^{\mathbb{N}} \to E_A^{\mathbb{N}}$ is ergodic with respect to a shift-invariant measure equivalent with $\tilde{m}$, we conclude that ρ is $\tilde{m}$-almost everywhere constant. Since $\tilde{m}_1$ and $\tilde{m}$ are both probability measures, $\tilde{m}_1 = \tilde{m}$. The proof is complete. □

We now, for a brief moment, deal with the case of finite alphabet. We shall prove the following.

LEMMA 6.29. *If the alphabet E is finite and the incidence matrix A is irreducible, then there exists an eigenmeasure $\tilde{m}$ of the conjugate operator $\mathcal{L}_f^*$.*

PROOF. By our assumptions, primarily by irreducibility of the incidence matrix A, the operator $\mathcal{L}_f$ is strictly positive (in the sense that it maps strictly positive functions into strictly positive functions). In particular the following formula

$$
\nu \mapsto \frac{\mathcal{L}_f^*(\nu)}{\mathcal{L}_f^*(\nu)(1\!\!1)}
$$

defines a continuous self-map of $M_A(\sigma)$, the space of Borel probability measures on $E_A^{\mathbb{N}}$ endowed with the topology of weak convergence. Since $E_A^{\mathbb{N}}$ is a compact metrizable space, $M_A(\sigma)$ is a convex compact subset of $C^*(E_A^{\mathbb{N}})$ which itself is a locally convex topological vector space when endowed with the weak topology. The Schauder–Tychonoff Theorem [**29**, V.10.5 Theorem 5] thus applies, and as its consequence, we conclude that the map defined above has a fixed point, say $\tilde{m}$. Then $\mathcal{L}_f^*(\tilde{m}) = \lambda\tilde{m}$, where $\lambda = \mathcal{L}_f^*(\tilde{m})(\mathbb{1})$, and the proof is complete. $\square$

For the proof of Theorem 6.31, which is actually the main result of this section, we will need a simple fact about irreducible matrices. We will provide its short proof for the sake of completeness and convenience of the reader. It is more natural and convenient to formulate it in the language of directed graphs. Let us recall that a directed graph is said to be strongly connected if and only if its incidence matrix is irreducible. In other words, it means that every two vertices can be joined by a path of admissible edges.

LEMMA 6.30. *If $\Gamma =< E, V >$ is a strongly connected directed graph, then there exists a sequence of strongly connected subgraphs $< E_n, V_n >$ of Γ such that all the vertices $V_n \subset V$ and all the edges E_n are finite, $\{V_n\}_{n=1}^\infty$ is an increasing sequence of vertices, $\{E_n\}_{n=1}^\infty$ is an increasing sequence of edges, $\bigcup_{n=1}^\infty V_n = V$ and $\bigcup_{n=1}^\infty E_n = E$.*

PROOF. Indeed, let $V = \{v_n : n \geq 1\}$ be a sequence of all vertices of Γ. and let $E = \{e_n : n \geq 1\}$ be a sequence of edges of Γ. We will proceed inductively to construct the sequences $\{V_n\}_{n=1}^\infty$ and $\{E_n\}_{n=1}^\infty$. In order to construct $< E_1, V_1 >$ let α be a path joining v_1 and v_2 ($i(\alpha) = v_1$, $t(\alpha) = v_2$) and let β be a path joining v_2 and v_1 ($i(\beta) = v_2$, $t(\beta) = v_1$). These paths exist since Γ is strongly connected. We define $V_1 \subset V$ to be the set of all vertices of paths α and β and $E_1 \subset E$ to be the set of all edges from α and β enlarged by e_1 if this edge is among all the edges joining the vertices of V_1. Obviously $< E_1, V_1 >$ is strongly connected and the first step of inductive procedure is complete. Suppose now that a strongly connected graph $< E_n, V_n >$ has been constructed. If $v_{n+1} \in V_n$, we set $V_{n+1} = V_n$ and E_{n+1} is then defined to be the union of E_n and all the edges from $\{e_1, e_2, \dots, e_n, e_{n+1}\}$ that are among all the edges joining the vertices of V_n. If $v_{n+1} \notin V_n$, let α_n be a path joining v_n and v_{n+1} and let β_n be a path joining v_{n+1} and v_n. We define V_{n+1} to be the union of V_n and the set of all vertices of α_n and β_n. E_{n+1} is then defined to be the union of E_n, all the edges building the paths α_n and β_n and all the edges from $\{e_1, e_2, \dots, e_n, e_{n+1}\}$ that are among all the edges joining the vertices of V_{n+1}. Since $< E_n, V_n >$ was strongly connected, so is $< E_{n+1}, V_{n+1} >$. The inductive procedure is complete. It immediately follows from the construction that $V_n \subset V_{n+1}$, $E_n \subset E_{n+1}$. $\bigcup_{n=1}^\infty V_n = V$ and $\bigcup_{n=1}^\infty E_n = E$. We are done. $\square$

Our first main result is the following.

THEOREM 6.31. *Suppose that $A : E \times E \to \{0, 1\}$ is an irreducible incidence matrix and that $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is a Hölder continuous function such that*

$$\sum_{e \in E} \exp(\sup(f|_{[e]})) < +\infty.$$

Then there exists a Borel probability measure $\tilde{m}$ on $E_A^{\mathbb{N}}$ which is an eigenvector of the conjugate operator $\mathcal{L}_f^* : C_b^*(E_A^{\mathbb{N}}) \to C_b^*(E_A^{\mathbb{N}})$.

PROOF. Without loss of generality we may assume that $E = \mathbb{N}$. Since the incidence matrix A is irreducible, it follows from Lemma 6.30 that we can reorder the set $\mathbb{N}$ such that there exists an increasing to infinity sequence $\big(l_n\big)_{n\geq 1}$ such that for every $n \geq 1$ the matrix $A|_{\{1,\dots,l_n\}\times\{1,\dots,l_n\}}$ is irreducible. Then, in view of Lemma 6.29, there exists an eigenmeasure $\tilde{m}_n$ of the operator $\mathcal{L}_n^*$, conjugate to the Perron–Frobenius operator

$$\mathcal{L}_n : C(E_{l_n}^{\mathbb{N}}) \to C(E_{l_n}^{\mathbb{N}})$$

associated to the potential $f|_{E_{l_n}^{\mathbb{N}}}$, where, for any $q \geq 1$,

$$E_q^{\mathbb{N}} := E_A^{\mathbb{N}} \cap \{1,\dots,q\}^{\mathbb{N}} = \{(e_k)_{k\geq 1} : 1 \leq e_k \leq q \text{ and } A_{e_k e_{k+1}} = 1 \text{ for all } k \geq 1\}.$$

We will also use the following notation for $q, n \geq 1$,

$$E_q^n := E_A^n \cap \{1,\dots,q\}^{\mathbb{N}} = \{(e_k)_{k\geq 1} : 1 \leq e_k \leq q \text{ and } A_{e_k e_{k+1}} = 1 \text{ for } 1 \leq k \leq n-1\}.$$

A family of Borel probability measures $\mathcal{M}$ in a topological space X is called *tight*, or *uniformly tight* if for every $\varepsilon > 0$ there exists a compact set K_ε such that $\mu(K_\varepsilon) > 1-\varepsilon$ for all $\mu \in \mathcal{M}$. If X is a complete metric space and $\mathcal{M}$ is a tight family of Borel probability measures then Prohorov's Theorem, see e.g. [**11**, Book II, Theorem 8.6.2], asserts that every sequence in $\mathcal{M}$ contains a weakly convergent subsequence.

We will show that the sequence $\{\tilde{m}_n\}_{n\geq 1}$ is tight (where all $\tilde{m}_n$, $n \geq 1$, are treated here as Borel probability measures on $E_A^{\mathbb{N}}$) hence Prohorov's theorem will allow us to extract a weakly convergent subsequence from $\{\tilde{m}_n\}_{n\geq 1}$. Let $\mathrm{P}_n = \mathrm{P}^{\sigma|_{E_{l_n}^{\mathbb{N}}}}(f|_{E_{l_n}^{\mathbb{N}}})$. Obviously $\mathrm{P}_n \geq \mathrm{P}_1$ for all $n \geq 1$. For every $k \geq 1$ let $\pi_k : E_A^{\mathbb{N}} \to \mathbb{N}$ be the projection onto the k-th coordinate, i.e. $\pi(\{(\tau_u)_{u\geq 1}\}) = \tau_k$. By Theorem 6.27, e^{P_n} is the eigenvalue of $\mathcal{L}_n^*$ corresponding to the eigenmeasure $\tilde{m}_n$. Therefore, applying (6.33), we obtain for every $n \geq 1$, every $k \geq 1$, and every $s \in \mathbb{N}$ that

$$\begin{aligned}
\tilde{m}_n(\pi_k^{-1}(s)) = \sum_{\omega\in E_{l_n}^k : \omega_k = s} \tilde{m}_n([\omega]) &\leq \sum_{\omega\in E_{l_n}^k : \omega_k = s} \exp\big(\sup(S_k f|_{[\omega]}) - \mathrm{P}_n k\big) \\
&\leq e^{-\mathrm{P}_n k} \sum_{\omega\in E_{l_n}^k : \omega_k = e} \exp\big(\sup(S_{k-1} f|_{[\omega]}) + \sup(f|_{[s]})\big) \\
&\leq e^{-\mathrm{P}_1 k}\left(\sum_{i\in\mathbb{N}} e^{\sup(f|_{[i]})}\right)^{k-1} e^{\sup(f|_{[s]})}.
\end{aligned}$$

Therefore

$$\tilde{m}_n(\pi_k^{-1}([s+1,\infty))) \leq e^{-\mathrm{P}_1 k}\left(\sum_{i\in\mathbb{N}} e^{\sup(f|_{[i]})}\right)^{k-1} \sum_{j>s} s e^{\sup(f|_{[j]})}.$$

Fix now $\varepsilon > 0$ and for every $k \geq 1$ choose an integer $n_k \geq 1$ so large that

$$e^{-\mathrm{P}_1 k}\left(\sum_{i\in\mathbb{N}} e^{\sup(f|_{[i]})}\right)^{k-1} \sum_{j>n_k} e^{\sup(f|_{[j]})} \leq \frac{\varepsilon}{2^k}.$$

Then, for every $n \geq 1$ and every $k \geq 1$, $\tilde{m}_n(\pi_k^{-1}([n_k+1,\infty))) \leq \varepsilon/2^k$. Hence

$$\tilde{m}_n\left(E_A^{\mathbb{N}} \cap \prod_{k\geq 1}[1,n_k]\right) \geq 1 - \sum_{k\geq 1} \tilde{m}_n(\pi_k^{-1}([n_k+1,\infty))) \geq 1 - \sum_{k\geq 1}\frac{\varepsilon}{2^k} = 1-\varepsilon.$$

Since $E_A^{\mathbb{N}} \cap \prod_{k\geq 1}[1, n_k]$ is a compact subset of $E_A^{\mathbb{N}}$, the tightness of the sequence $\{\tilde{m}_n\}_{n\geq 1}$ is therefore proved. Thus, in view of Prohorov's Theorem there exists $\tilde{m}$, a Borel probability measure on $E_A^{\mathbb{N}}$ which is a weak-limit point of the sequence $\{\tilde{m}_n\}_{n\geq 1}$. Passing to a subsequence, we may assume that the sequence $\{\tilde{m}_n\}_{n\geq 1}$ itself converges weakly to the measure $\tilde{m}$. Let

$$\mathcal{L}_{0,n} = e^{-\mathrm{P}_n}\mathcal{L}_n \quad \text{and} \quad \mathcal{L}_0 = e^{-\mathrm{P}^\sigma(f)}\mathcal{L}$$

be the corresponding normalized operators. Fix $g \in C_b(E_A^{\mathbb{N}})$ and $\varepsilon > 0$. Let us now consider an integer $n \geq 1$ so large that the following requirements are satisfied.

$$\sum_{i>n} ||g||_\infty \exp\big(\sup(f|_{[i]}) - \mathrm{P}^\sigma(f)\big) \leq \frac{\varepsilon}{6}, \tag{6.37}$$

$$\sum_{i\leq n} ||g||_\infty \exp\big(\sup(f|_{[i]})\big)\left|e^{-\mathrm{P}^\sigma(f)} - e^{-\mathrm{P}_n}\right| \leq \frac{\varepsilon}{6}, \tag{6.38}$$

$$|\tilde{m}_n(g) - \tilde{m}(g)| \leq \frac{\varepsilon}{3}, \tag{6.39}$$

and

$$\left|\int \mathcal{L}_0(g)d\tilde{m} - \int \mathcal{L}_0(g)d\tilde{m}_n\right| \leq \frac{\varepsilon}{3}. \tag{6.40}$$

It is possible to make condition (6.38) satisfied since, due to Theorem 6.11, $\lim_{n\to\infty} \mathrm{P}_n = \mathrm{P}^\sigma(f)$. Let $g_n := g|_{E_{l_n}^\infty}$. The first two observations are the following.

$$\begin{aligned}
\mathcal{L}_{0,n}^*\tilde{m}_n(g) &= \int_{E_A^{\mathbb{N}}} \sum_{i\leq n: A_{i\omega_n}=1} g(i\omega)\exp(f(i\omega) - \mathrm{P}_n)d\tilde{m}_n(\omega) \\
&= \int_{E_{l_n}^{\mathbb{N}}} \sum_{i\leq n: A_{i\omega_n}=1} g(i\omega)\exp(f(i\omega) - \mathrm{P}_n)d\tilde{m}_n(\omega) \\
&= \int_{E_{l_n}^{\mathbb{N}}} \sum_{i\leq n: A_{i\omega_n}=1} g_n(i\omega)\exp(f(i\omega) - \mathrm{P}_n)d\tilde{m}_n(\omega) \\
&= \mathcal{L}_{0,n}^*\tilde{m}_n(g_n) = \tilde{m}_n(g_n),
\end{aligned} \tag{6.41}$$

and

$$\tilde{m}_n(g_n) - \tilde{m}_n(g) = \int_{E_{l_n}^{\mathbb{N}}} (g_n - g)d\tilde{m}_n = \int_{E_{l_n}^{\mathbb{N}}} 0 d\tilde{m}_n = 0. \tag{6.42}$$

Using the triangle inequality we get the following.

(6.43)
$$\begin{aligned}
\left|\mathcal{L}_0^*\tilde{m}(g) - \tilde{m}(g)\right| &\leq |\mathcal{L}_0^*\tilde{m}(g) - \mathcal{L}_0^*\tilde{m}_n(g)| + |\mathcal{L}_0^*\tilde{m}_n(g) - \mathcal{L}_{0,n}^*\tilde{m}_n(g)| + \\
&+ |\mathcal{L}_{0,n}^*\tilde{m}_n(g) - \tilde{m}_n(g_n)| + |\tilde{m}_n(g_n) - \tilde{m}_n(g)| + |\tilde{m}_n(g) - \tilde{m}(g)|
\end{aligned}$$

Let us look first at the second summand. Applying (6.38) and (6.37) we get

(6.44)

$$
\begin{aligned}
&|\mathcal{L}_0^*\tilde{m}_n(g) - \mathcal{L}_{0,n}^*\tilde{m}_n(g)| = \\
&\quad = \Big| \int_{E_A^{\mathbb{N}}} \sum_{i\le n: A_{i\omega_n}=1} g(i\omega)\big(\exp(f(i\omega) - \mathrm{P}^\sigma(f)) - \exp(f(i\omega) - \mathrm{P}_n)\big) d\tilde{m}_n(\omega) \\
&\quad + \int_{E_A^{\mathbb{N}}} \sum_{i>n: A_{i\omega_n}=1} g(i\omega)\exp\big(f(i\omega) - \mathrm{P}^\sigma(f)\big) d\tilde{m}_n(\omega) \Big| \\
&\quad \le \sum_{i\le n} ||g||_\infty \exp\big(\sup(f|_{[i]})\big)\big|e^{-\mathrm{P}^\sigma(f)} - e^{-\mathrm{P}_n}\big| + \\
&\qquad\qquad + \sum_{i>n} ||g||_\infty \exp\big(\sup(f|_{[i]} - \mathrm{P}^\sigma(f)\big) \\
&\quad \le \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.
\end{aligned}
$$

Combining now in turn (6.40), (6.44), (6.41), (6.42) and (6.39) we get from (6.43) that

$$|\mathcal{L}_0^*\tilde{m}(g) - \tilde{m}(g)| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Letting $\varepsilon \searrow 0$ we therefore get $\mathcal{L}_0^*\tilde{m}(g) = \tilde{m}(g)$ or $\mathcal{L}_f^*\tilde{m}(g) = e^{\mathrm{P}^\sigma(f)}\tilde{m}(g)$. Hence $\mathcal{L}_f^*\tilde{m} = e^{\mathrm{P}(f)}\tilde{m}$ and the proof is complete. □

As an immediate consequence of Theorem 6.31, Theorem 6.28, Theorem 6.27, Theorem 6.20, and Theorem 6.24, we get the following result summarizing what we did about the thermodynamic formalism.

Corollary 6.32. *Suppose that $f : E_A^{\mathbb{N}} \to \mathbb{R}$ is a Hölder continuous function such that*

$$\sum_{e\in E} \exp(\sup(f|_{[e]})) < +\infty$$

and the incidence matrix A is finitely irreducible. Then

(i) *There exists a unique eigenmeasure $\tilde{m}_f$ of the conjugate Perron–Frobenius operator $\mathcal{L}_f^*$ and the corresponding eigenvalue is equal to $e^{\mathrm{P}^\sigma(f)}$.*

(ii) *The eigenmeasure $\tilde{m}_f$ is a Gibbs state for f.*

(iii) *The function $f : E_A^{\mathbb{N}} \to \mathbb{R}$ has a unique shift-invariant Gibbs state $\tilde{\mu}_f$.*

(iv) *The measure $\tilde{\mu}_f$ is ergodic, equivalent to $\tilde{m}_f$ and $\log(d\tilde{\mu}_f/d\tilde{m}_f)$ is uniformly bounded.*

(v) *If $\int -f\, d\tilde{\mu}_f < +\infty$, then the shift-invariant Gibbs state $\tilde{\mu}_f$ is the unique equilibrium state for the potential f.*

(vi) *The Gibbs state $\tilde{\mu}_f$ is ergodic, and in case the incidence matrix A is finitely primitive, it is completely ergodic.*

CHAPTER 7

Hausdorff dimension of limit sets

In this chapter we employ the thermodynamic formalism from Chapter 6 as well as the distortion theorems from Chapter 4 to study dimensions of limit sets of conformal GDMS in general Carnot groups. In Section 7.1 we revisit topological pressure in the setting of weakly conformal GDMS $\mathcal{S}$ and we define the θ-number of $\mathcal{S}$, as well as Bowen's parameter. We use the pressure function to define regular, strongly regular, and co-finitely regular systems. In Section 7.2 we introduce conformal measures and we prove a dynamical formula for the Hausdorff dimension of the limit set of a finitely irreducible Carnot conformal GDMS. This formula traces back to the fundamental work of Rufus Bowen [**13**] and is in spirit closest to an analogous formula in [**46**] in the context of Euclidean spaces. Section 7.3 contains a characterization of strongly regular systems. Finally in Section 7.4 we prove that if $\mathcal{S}$ is a Carnot conformal IFS and $t \in (0, \theta)$, there exists a subsystem of $\mathcal{S}$ with Hausdorff dimension t.

7.1. Topological pressure, θ-number, and Bowen's parameter

Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a finitely irreducible Carnot conformal GDMS. For $t \geq 0$, $n \in \mathbb{N}$ and $F \subset E$ let

$$Z_n(F, t) = \sum_{\omega \in F_A^n} ||D\phi_\omega||_\infty^t.$$

When $F = E$, we just write $Z_n(t)$ instead of $Z_n(E, t)$. By (4.6) we easily see that

$$Z_{m+n}(t) \leq Z_m(t) Z_n(t), \tag{7.1}$$

and consequently, the sequence $(\log Z_n(t))_{n=1}^\infty$ is subadditive. Thus, the limit

$$\lim_{n\to\infty} \frac{\log Z_n(t)}{n}$$

exists and equals $\inf_{n\in\mathbb{N}}(\log Z_n(t)/n)$. The value of the limit is denoted by $\mathrm{P}(t)$ or, if we want to be more precise, by $\mathrm{P}_E(t)$ or $\mathrm{P}_{\mathcal{S}}(t)$. It is called the *topological pressure* of the system $\mathcal{S}$ evaluated at the parameter t.

Let $\zeta : E_A^{\mathbb{N}} \to \mathbb{R}$ be defined by the formula

$$\zeta(\omega) = \log \|D\phi_{\omega_1}(\pi(\sigma(\omega))\|, \tag{7.2}$$

where the coding map π was defined in (4.2). Using Lemma 4.16 we get easily (see [**46**, Proposition 3.1.4] for complete details) the following.

Lemma 7.1. *For $t \geq 0$ the function $t\zeta : E_A^{\mathbb{N}} \to \mathbb{R}$ is Hölder continuous and* $\mathrm{P}^\sigma(t\zeta) = \mathrm{P}(t)$.

DEFINITION 7.2. We say that a nonnegative real number t belongs to $\mathrm{Fin}(\mathcal{S})$ if

$$\sum_{e\in E}||D\phi_e||_\infty^t<+\infty. \tag{7.3}$$

Let us record the following immediate observation.

REMARK 7.3. A nonnegative real number t belongs to $\mathrm{Fin}(\mathcal{S})$ if and only if the Hölder continuous potential $t\zeta : E_A^{\mathbb{N}} \to \mathbb{R}$ is summable.

We now fix some $t \in \mathrm{Fin}(\mathcal{S})$. Remark 7.3 along with Chapter 6, especially Section 6.3, allow us to consider the bounded linear operator $\mathcal{L}_t := \mathcal{L}_{t\zeta}$ acting on $C_b(E_A^{\mathbb{N}})$, which is, we recall, the Banach space of all real-valued bounded continuous functions on $E_A^{\mathbb{N}}$ endowed with the supremum norm $||\cdot||_\infty$. Immediately from the definition of $\mathcal{L}_{t\zeta}$ we get that

$$\mathcal{L}_t g(\omega)=\sum_{i:\,A_{i\omega_1}=1} g(i\omega)\|D\phi_i(\pi(\omega))\|^t, \qquad \text{for } \omega\in E_A^{\mathbb{N}}. \tag{7.4}$$

A straightforward inductive calculation gives

$$\mathcal{L}_t^n g(\omega)=\sum_{\tau\in E_A^n:\tau\omega\in E_A^{\mathbb{N}}} g(\tau\omega)\|D\phi_\tau(\pi(\omega))\|^t \tag{7.5}$$

for all $n \in \mathbb{N}$. Note that formulas (7.4) and (7.5) clearly extend to all Borel bounded functions $g : E_A^{\mathbb{N}} \to \mathbb{R}$.

The following theorem is simply Corollary 6.32 applied to the functions $t\zeta$. Recall that $\mathcal{L}_t^* : C^*(E_A^{\mathbb{N}}) \to C^*(E_A^{\mathbb{N}})$ is the dual operator for $\mathcal{L}_t$.

THEOREM 7.4. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible Carnot conformal GDMS and $t \in \mathrm{Fin}(\mathcal{S})$. Then*

(i) There exists a unique eigenmeasure $\tilde{m}_t$ of the conjugate Perron-Frobenius operator $\mathcal{L}_t^*$ and the corresponding eigenvalue is equal to $e^{\mathrm{P}(t)}$.
(ii) The eigenmeasure $\tilde{m}_t$ is a Gibbs state for $t\zeta$.
(iii) The function $t\zeta : E_A^{\mathbb{N}} \to \mathbb{R}$ has a unique shift-invariant Gibbs state $\tilde{\mu}_t$.
(iv) The measure $\tilde{\mu}_t$ is ergodic, equivalent to $\tilde{m}_t$ and $\log(d\tilde{\mu}_t/d\tilde{m}_t)$ is uniformly bounded.
(v) If $\int \zeta\, d\tilde{\mu}_t > -\infty$, then the shift-invariant Gibbs state $\tilde{\mu}_t$ is the unique equilibrium state for the potential $t\zeta$.
(vi) The Gibbs state $\tilde{\mu}_t$ is ergodic, and in case the system $\mathcal{S}$ is finitely primitive, it is completely ergodic.

Theorem 7.4 (i) means that

$$\mathcal{L}_t^*\tilde{m}_t=e^{\mathrm{P}(t)}\tilde{m}_t. \tag{7.6}$$

Based on (7.6) and standard approximation arguments we obtain that

$$\mathcal{L}_t^{*n}\tilde{m}_t(g)=e^{\mathrm{P}(t)}\tilde{m}_t(g) \tag{7.7}$$

for all Borel bounded functions $g : E_A^{\mathbb{N}} \to \mathbb{R}$.

Observe also that given $t \in \mathrm{Fin}(\mathcal{S})$ it immediately follows from (6.34) and (6.35) that

$$c_t^{-1}e^{-\mathrm{P}(t)|\omega|}||D\phi_\omega||_\infty^t\le\tilde{m}_t([\omega])\le c_te^{-\mathrm{P}(t)|\omega|}||D\phi_\omega||_\infty^t \tag{7.8}$$

for all $\omega \in E_A^*$, where $c_t \geq 1$ denotes some constant.

We can now introduce the *θ-number* and *Bowen's parameter* , some of the most fundamental notions of thermodynamic formalism. For any set $F \subset E$ let

$$\theta_F = \theta_{\mathcal{S}_F} := \inf \mathrm{Fin}(\mathcal{S}_F) = \inf\{t \geq 0 : Z_1(F,t) < +\infty\},$$

where $\mathcal{S}_F = \{\phi_e\}_{e \in F}$. When $F = E$ we simply denote

$$\theta = \theta_{\mathcal{S}} := \inf \mathrm{Fin}(\mathcal{S}).$$

The number

$$h = h_{\mathcal{S}} := \inf\{t \geq 0 : \mathrm{P}(t) \leq 0\}$$

is called *Bowen's parameter* of the system $\mathcal{S}$. It will soon turn out to coincide with the Hausdorff dimension of the limit set $J_{\mathcal{S}}$.

The following facts follow easily from Proposition 6.15, Lemma 7.1, the bounded distortion property (Lemma 4.9) and the uniform contractivity of all generators of the system $\mathcal{S}$.

Proposition 7.5. *Let $\mathcal{S}$ be a finitely irreducible Carnot conformal GDMS. Then the following conclusions hold.*

(i) $\mathrm{Fin}(\mathcal{S}) = \{t \geq 0 : \mathrm{P}(t) < +\infty\}$.
(ii) $\theta = \inf\{t \geq 0 : \mathrm{P}(t) < +\infty\}$.
(iii) The topological pressure P *is strictly decreasing on* $[0, +\infty)$ *with* $\mathrm{P}(t) \to -\infty$ *as* $t \to +\infty$. *Moreover, the function* P *is convex and continuous on the closure of* $\mathrm{Fin}(\mathcal{S})$.
(iv) $\mathrm{P}(0) = +\infty$ *if and only if E is infinite.*

Note that Proposition 7.5 (ii) immediately implies that $h \leq \theta$. Moreover by Proposition 7.5 (iii) we have the following.

Remark 7.6. If $\mathcal{S}$ is a finitely irreducible Carnot conformal GDMS, then $h \in \mathrm{Fin}(\mathcal{S})$ and $\mathrm{P}(h) \leq 0$.

For some of our results we will need to assume extra regularity for the Carnot conformal GDMS in terms of the behavior of the pressure function.

Definition 7.7. A finitely irreducible Carnot conformal GDMS $\mathcal{S}$ is

- *regular* if $\mathrm{P}(h) = 0$,
- *strongly regular* if there exists $t \geq 0$ such that $0 < \mathrm{P}(t) < +\infty$, and
- *co-finitely regular* if $\mathrm{P}(\theta) = +\infty$.

The previous notions of regularity are related to each other, as the following simple proposition asserts.

Proposition 7.8. *Every co-finitely regular finitely irreducible Carnot conformal GDMS is strongly regular and every strongly regular finitely irreducible Carnot conformal GDMS is regular. That is,*

$$\textit{co-finitely regular} \Rightarrow \textit{strongly regular} \Rightarrow \textit{regular}.$$

Proof. Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a finitely irreducible Carnot conformal GDMS. If $\mathcal{S}$ is strongly regular then it is also regular by Proposition 7.5 (iii). Now assume that $\mathcal{S}$ is co-finitely regular. By Theorem 6.11, Remark 6.12 and Lemma 7.1 there exists a finite and irreducible $F \subset E$ such that $P_F(\theta_{\mathcal{S}}) > 1$. By Proposition 7.5

(iii), the mapping $t \mapsto P_F(t)$ is continuous, therefore there exists an $\varepsilon > 0$ such that $P_F(\theta_{\mathcal{S}} + \varepsilon) > 1$. Therefore,

$$P_E(\theta_{\mathcal{S}} + \varepsilon) \geq P_F(\theta_{\mathcal{S}} + \varepsilon) > 1.$$

Hence $P_F(\theta_{\mathcal{S}} + \varepsilon) \in (0, +\infty)$, and the proof is complete. □

As an immediate consequence of Proposition 7.5 we get some extra information regarding regular systems.

Remark 7.9. A finitely irreducible Carnot conformal GDMS $\mathcal{S}$ is regular if and only if $\mathrm{P}(t) = 0$ for some $t \geq 0$. Moreover, if $\mathrm{P}(t) = 0$ for some $t \geq 0$, then $t = h_{\mathcal{S}}$.

Let us also record the following obvious fact which provides a simple mechanism for obtaining lower bounds on Bowen's parameter $h_{\mathcal{S}}$ (which, as we mentioned earlier and we will shortly see, coincides with the Hausdorff dimension of $J_{\mathcal{S}}$ in many cases).

Proposition 7.10. *If $\mathcal{S}$ is strongly regular, in particular, if $\mathcal{S}$ is co-finitely regular, then $\theta_{\mathcal{S}} < h_{\mathcal{S}}$.*

When the GDMS is finite, regularity comes for free.

Remark 7.11. Each finite irreducible Carnot conformal GDMS $\mathcal{S}$ is regular.

In fact more is true; since we are always assuming that E contains at least two elements (see Section 4.1) then $\mathcal{S}$ is strongly regular. This is proven in the following proposition.

Proposition 7.12. *If $\mathcal{S} = \{\phi_e\}_{e\in E}$ is a finite and irreducible weakly conformal GDMS then $\mathrm{P}(0) > 0$. In particular $\mathcal{S}$ is strongly regular.*

Proof. Let $\Phi \subset E_A^*$ be the finite set witnessing irreducibility. We can also also assume that Φ consists of words with minimal length, in the sense that if $a, b \in E$, $\omega \in \Phi$, and $a\omega b \in E_A^*$ then if $a\tau b \in E_A^*$ for some $\tau \in E_A^*$ then $|\omega| \leq |\tau|$.

Recall that we always assume that E contains at least two elements. Let $a, b \in E$, $a \neq b$. By irreducibility there exist $\omega, \tau, \in \Phi$ such that

$$\rho^1 := a\omega a \in E_A^*,$$
$$\rho^2 := a\tau b \in E_A^*.$$

Observe that the words $\rho^i, i = 1, 2$, are mutually incomparable. Assume by way of contradiction that ρ^1 and ρ^2 are comparable, and without loss of generality also assume that $|\rho^2| > |\rho^1|$. Then there exists some $\upsilon \in E_A^*$ such that

$$\rho^2 = \rho^1 \upsilon b = a\omega a\upsilon b \in E_A^*.$$

Hence $\tau = \omega a \upsilon$, $a\upsilon b \in E_A^*$ and $|\upsilon| < |\tau|$. But this violates the minimality of τ as $\tau \in \Phi$. By irreducibility there exist $\tilde{\rho}^i \in E_A^{\mathbb{N}}, i = 1, 2$, such that

$$\tilde{\rho}^i|_{|\rho^i|} = \rho^i, \text{ for } i = 1, \dots, 2.$$

For example if $\gamma \in \Phi$ such that $b\gamma a \in E_A^*$ one could take $\tilde{\rho}^1 = a\omega a\omega a \dots$ and $\tilde{\rho}^2 = a\tau b\gamma\tilde{\rho}^1$. Let

$$\mathtt{q} = \max\{|\omega| : \omega \in \Phi\} + 2. \tag{7.9}$$

Then the words $\tilde{\rho}^i|_q, i = 1, 2$, are mutually incomparable.

Therefore we have shown that for every $e \in E$ there exist incomparable words $\omega^1(e), \omega^2(e) \in E_A^{\mathtt{q}}$ such that $e = \omega^1(e)_1 = \omega^2(e)_1$. This implies that for all $n \in \mathbb{N}$

$$\sharp E_A^{n(\mathtt{q}-1)+1} \geq 2^n. \tag{7.10}$$

We can now finish the proof of the proposition. Recall that for $n \in \mathbb{N}$,

$$Z_n(0) = \sum_{\omega \in E_A^n} 1 = \sharp E_A^n.$$

Hence by (7.10),

$$\begin{aligned} \mathrm{P}(0) &= \lim_{n\to\infty} \frac{\log Z_n(0)}{n} = \lim_{n\to\infty} \frac{\log Z_{n(\mathtt{q}-1)+1}(0)}{n(\mathtt{q}-1)+1} \\ &= \lim_{n\to\infty} \frac{\log \sharp E_A^{n(\mathtt{q}-1)+1}}{n(\mathtt{q}-1)+1} \geq \limsup_{n\to\infty} \frac{\log 2^n}{\mathtt{q}n} = \frac{\log 2}{\mathtt{q}}. \end{aligned}$$

The proof is complete. □

Note also that (7.8) and Proposition 7.5 (i) imply that:

REMARK 7.13. If $t \in \mathrm{Int}(\mathrm{Fin}(\mathcal{S}))$, then $\int \zeta \, d\tilde{\mu}_t > -\infty$.

Obviously, if the system $\mathcal{S}$ is strongly regular, then $h = h_{\mathcal{S}} \in \mathrm{Int}(\mathrm{Fin}(\mathcal{S}))$. Remark 7.13 then entails the following.

REMARK 7.14. If the system $\mathcal{S}$ is strongly regular, then $\int \zeta \, d\tilde{\mu}_h > -\infty$.

7.2. Hausdorff dimension and Bowen's formula

In this section we exhibit a dynamical formula for the Hausdorff dimension of the limit set of a finitely irreducible Carnot conformal GDMS. Due to its correspondence to Bowen's work [13], we refer to it as *Bowen's formula*.

We will denote by $\mathcal{H}^s$, resp. $\mathcal{P}^s$, the Hausdorff, resp. packing measure of dimension s in $(\mathbb{G}, d)$. We will also denote by $\mathcal{S}^s$ the s-dimensional spherical Hausdorff measure. The Hausdorff and packing dimensions of $S \subset (\mathbb{G}, d)$ will be denoted by $\dim_{\mathcal{H}}(S)$ and $\dim_{\mathcal{P}}(S)$ respectively. See [42] for the exact definitions.

We begin with the following simple observation following from the open set condition.

LEMMA 7.15. *Let $\mathcal{S}$ be an Carnot conformal GDMS. For all $0 < \kappa_1 < \kappa_2 < \infty$, for all $r > 0$, and for all $p \in \mathbb{G}$, the cardinality of any collection of mutually incomparable words $\omega \in E_A^*$ that satisfy the conditions $B(p,r) \cap \phi_\omega(X_{t(\omega)}) \neq \emptyset$ and $\kappa_1 r \leq \mathrm{diam}(\phi_\omega(X_{t(\omega)})) \leq \kappa_2 r$ is bounded above by*

$$\mathtt{m}_{\kappa_1,\kappa_2} := \left(\frac{(1+\kappa_2) M \Lambda K \, C}{\tilde{R}_{\mathcal{S}} \kappa_1} \right)^Q, \tag{7.11}$$

where Q is the homogeneous dimension of $\mathbb{G}$, $\tilde{R}_{\mathcal{S}}$ was defined in Lemma 4.13, C is the quasiconvexity constant from Corollary 3.12, and K and Λ denote the constants from Lemmas 4.11 and 4.9 respectively, and $M = \mathrm{diam}\, X$.

PROOF. Recall that $|\cdot|$ denotes the Haar measure on $\mathbb{G}$, and $c_0 = |B(o,1)|$. For every $v \in V$ let p_v be the center of a ball with radius $\tilde{R}_{\mathcal{S}}$ contained in $\mathrm{Int}(X_v)$. Let

F be any collection of A-admissible words satisfying the hypotheses of the lemma. Then

$$\phi_\omega(X_{t(\omega)}) \subset B(p, r + \mathrm{diam}(\phi_\omega(X_{t(\omega)}))) \subset B(p, (1+\kappa_2)r)$$

for every $\omega \in F$. Since, by the open set condition, the sets $\{\phi_\omega(\mathrm{Int} X_{t(\omega)}) : \omega \in F\}$ are mutually disjoint, applying (1.9) along with (4.9) and (4.13) yields

$$\begin{aligned}
c_0(1+\kappa_2)^Q r^Q &= |B(p,(1+\kappa_2)r)| \\
&\geq \left| \bigcup_{\omega \in F} \phi_\omega(X_{t(\omega)}) \right| \\
&= \sum_{\omega \in F} |\phi_\omega(\mathrm{Int}(X_{t(\omega)}))| \\
&\geq \sum_{\omega \in F} |B(\phi_\omega(p_{t(\omega)}), (KC)^{-1} \tilde{R}_{\mathcal{S}} ||D\phi_\omega||_\infty)| \\
&\geq \sum_{\omega \in F} |B(\phi_\omega(p_{t(\omega)}), (M\Lambda KC)^{-1} \tilde{R}_{\mathcal{S}} \, \mathrm{diam}(\phi_\omega(X_{t(\omega)}))| \\
&\geq \sum_{\omega \in F} |B(\phi_\omega(p_{t(\omega)}), (M\Lambda KC)^{-1} \tilde{R}_{\mathcal{S}} \kappa_1 r)| \\
&= c_0(\sharp F)((M\Lambda KC)^{-1} \tilde{R}_{\mathcal{S}} \kappa_1)^Q r^Q.
\end{aligned}$$

Hence $\sharp F$ is bounded above by $\mathtt{m}_{\kappa_1,\kappa_2}$ and we are done. □

In the following proposition we prove that if a GDMS $\mathcal{S}$ is conformal or if $J_{\mathcal{S}}$ has positive h-Hausdorff measure, then for all $v \in V$, $J_{\mathcal{S}} \cap X_v$ is an infinite set. This proposition will be essential for Chapter 11, where we will use the fact that the limit set does not collapse to a point inside any $X_v, v \in V$.

Proposition 7.16. *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a finitely irreducible weakly Carnot conformal GDMS.*

(i) If $\mathcal{S}$ is Carnot conformal then $\sharp(J_{\mathcal{S}} \cap X_v) = \infty$ for all $v \in V$. In fact $\overline{J_{\mathcal{S}}} \cap X_v$ is a non-empty compact perfect set and $J_{\mathcal{S}} \cap X_v$ is of cardinality $\mathfrak{c}$.
(ii) If $\mathcal{H}^h(J_{\mathcal{S}}) > 0$ then $\sharp(J_{\mathcal{S}} \cap X_v) = \infty$ for all $v \in V$. In fact $\overline{J_{\mathcal{S}}} \cap X_v$ is a non-empty compact perfect set and $J_{\mathcal{S}} \cap X_v$ is of cardinality $\mathfrak{c}$.

Proof. We will first prove (i). Observe that it is enough to prove (i) for irreducible finite systems. To verify that such assertion is indeed enough, let $\mathcal{S}$ be a finitely irreducible infinite system and let $\Phi \subset E_A^*$ be the set witnessing finite irreducibility for the matrix A. Let

$$F = \{e \in E : e = \omega_i \text{ for some } \omega \in \Phi, i = 1, \dots, |\omega|\},$$

i.e. F consists of the letters from E appearing in the words of Φ. Also for any $v \in V$ choose one $e_v \in E$ such that $i(e_v) = v$. Let $\tilde{F} = F \cup \{e_v : v \in V\}$. Then $\mathcal{S}_{\tilde{F}}$ is an irreducible finite subsystem of $\mathcal{S}$, and $J_{\mathcal{S}} \supset J_{\mathcal{S}_{\tilde{F}}}$. Hence if (i) holds for $\mathcal{S}_{\tilde{F}}$ then it will also hold for $\mathcal{S}$. Therefore we can assume that E is finite. Fix $v \in V$ and $\xi \in J_{\mathcal{S}} \cap X_v$. For $r > 0$ define,

$$\mathcal{F}_\xi(r) = \{\omega \in E_A^* : \xi \in \phi_\omega(X_{t(\omega)}), \|D\phi_\omega\|_\infty \leq r, \text{ and } \|D\phi_{\omega|_{|\omega|-1}}\|_\infty > r\}.$$

If $\omega \in \mathcal{F}_\xi(r)$, (4.9) implies that

$$\mathrm{diam}(\phi_\omega(X_{t(\omega)})) \leq \Lambda M r. \tag{7.12}$$

We now define

$$\mathtt{D}_0 := \min\{\|D\phi_e\|_\infty : e \in E\} > 0. \tag{7.13}$$

Hence if $\omega \in \mathcal{F}_\xi(r)$ by (4.6) and (4.14),

$$\operatorname{diam}(\phi_\omega(X_{t(\omega)})) \geq 2(K^2C)^{-1}\mathtt{D}_0\tilde{R}_{\mathcal{S}}r. \tag{7.14}$$

Therefore by Lemma 7.15, (7.12) and (7.14) we deduce that

$$\sharp\mathcal{F}_\xi(r) \leq \mathtt{m}_{\kappa_1,\kappa_2}, \tag{7.15}$$

where $\kappa_1 = 2(K^2C)^{-1}\mathtt{D}_0\tilde{R}_{\mathcal{S}}$ and $\kappa_2 = \Lambda M$. Observe also that $\mathcal{F}_\xi(r)$ consists of mutually incomparable words.

We will show that

$$\sharp\{\tau \in E_A^{\mathbb{N}} : \pi(\tau) = \xi\} \leq \mathtt{m}_{\kappa_1,\kappa_2}. \tag{7.16}$$

Let I be an index set such that $\{\tau \in E_A^N : \pi(\tau) = \xi\} = (\tau^i)_{i\in I}$ and the words τ^i are distinct. Since E is finite,

$$r := \min\{\|D\phi_{\tau_1^i}\|_\infty : i \in I\} \geq \mathtt{D}_0 > 0.$$

Therefore for all $i \in I$, there exists some $k(i) \in \mathbb{N}$ such that

$$\|D\phi_{\tau^i|_{k(i)}}\|_\infty \leq r/2 \text{ and } \|D\phi_{\tau^i|_{k(i)-1}}\|_\infty > r/2.$$

Hence $\{\tau^i|_{k(i)} : i \in \mathbb{N}\} \subset \mathcal{F}_\xi(r/2)$, and (7.16) follows by (7.15). Recalling, (7.10), there exist infinitely many words $\omega \in E_A^N$ such that $\omega_1 = e_v$. Equivalently there exist infinitely many words $\omega \in E_A^N$, such that $\pi(\omega) \in J_{\mathcal{S}} \cap X_v$. Hence (i) follows by (7.16).

We now move to the proof of (ii). First, we record that Proposition 7.12 implies that $\mathcal{S}$ is strongly regular and $h > 0$. If $\mathcal{H}^h(J_{\mathcal{S}}) > 0$ then there exists some v_0 such that $J_{\mathcal{S}} \cap X_{v_0}$ has the cardinality of the continuum. Let

$$E_0 = \{e \in E : i(e) = v_0\},$$

and for $e \in E_0$, let

$$W_e = \{x \in J_{\mathcal{S}} \cap X_{v_0} : x = \pi(\omega) \text{ for some } \omega \in E_A^{\mathbb{N}} \text{ such that } \omega_1 = e\}.$$

Then $J_{\mathcal{S}} \cap X_{v_0} = \cup_{e\in E} W_e$. Since $J_{\mathcal{S}} \cap X_{v_0}$ has the cardinality of the continuum there exists some $e_0 \in E_0$ such that $\sharp W_{e_0} = \infty$. Therefore there exists a sequence of distinct words $(\omega^i)_{i\in\mathbb{N}}$ such that $\pi(\omega^i) \neq \pi(\omega^j)$ for $i \neq j$, and $\omega_1^i = e_0$ for all $i \in \mathbb{N}$.

Now let $v \in V$ and let $e_v \in E$ such that $i(e_v) = v$. By irreducibility of E there exists some $\rho \in \Phi$ such that $e_v\rho e_0 \in E_A^*$. We consider the sequence $(x_i)_{i\in\mathbb{N}}$ where $x_i = \pi(e_v\rho\omega^i)$. Notice that $x_i \in J_{\mathcal{S}} \cap X_v$ for all $i \in \mathbb{N}$. In order to finish the proof of (ii) it is enough to show that $x_i \neq x_j$ for $i \neq j$. This follows because if $i \neq j$, $x_i = \phi_{e_v\rho}(\pi(\omega^i))$ and $x_j = \phi_{e_v\rho}(\pi(\omega^j))$ and $\pi(\omega^i) \neq \pi(\omega^j)$. The proof of (ii) is complete. □

As an immediate consequence of the definition of regularity of a conformal GDMS and of Theorem 7.4, we get the following.

Proposition 7.17. *If $\mathcal{S}$ is a finitely irreducible weakly Carnot conformal GDMS, then the following conditions are equivalent:*

(i) The system $\mathcal{S}$ is regular.

(ii) There exist $t \in \mathrm{Fin}(\mathcal{S})$ and a Borel probability measure $\hat{m}$ on $E_A^{\mathbb{N}}$ such that $\mathcal{L}_t^*\hat{m} = \hat{m}$. Then necessarily $t = h$ and $\hat{m} = \tilde{m}_h$.

(iii) $\mathcal{L}_h^*\tilde{m}_h = \tilde{m}_h$.

For all $t \in \mathrm{Fin}(\mathcal{S})$ we will denote

$$m_t := \tilde{m}_t \circ \pi^{-1} \quad \text{and} \quad \mu_t := \tilde{\mu}_t \circ \pi^{-1}. \tag{7.17}$$

If the system $\mathcal{S}$ is regular, then formula (7.8) for $t = h$ yields that for every $\omega \in E_A^*$

$$c_h^{-1}||D\phi_\omega||_\infty^h \leq \tilde{m}_h([\omega]) \leq c_h||D\phi_\omega||_\infty^h, \tag{7.18}$$

where $c_h \geq 1$ is some finite constant. Throughout the manuscript we will call $\tilde{m}_h$ the $h_{\mathcal{S}}$-*conformal measure* for $E_A^{\mathbb{N}}$, and its pullback

$$m_h = \tilde{m}_h \circ \pi^{-1},$$

which is supported on $J_{\mathcal{S}}$, will be called the $h_{\mathcal{S}}$-*conformal measure* for $\mathcal{S}$.

In the following theorem we prove that in the case when $\mathcal{S}$ is finite, the conformal measure m_h is Ahlfors h-regular, thus obtaining information about the Hausdorff and packing measures of $J_{\mathcal{S}}$.

THEOREM 7.18. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finite irreducible Carnot conformal GDMS. Then the measure m_h is an Ahlfors h-regular measure on $(J_{\mathcal{S}}, d)$, i.e., there exists a constant $c_{\mathcal{S}} \geq 1$ such that*

$$c_{\mathcal{S}}^{-1}r^h \leq m_h(B(p,r)) \leq c_{\mathcal{S}}r^h \tag{7.19}$$

for all $p \in J_{\mathcal{S}}$ and all $0 < r \leq 1$. In particular, both the Hausdorff and the packing measures of $J_{\mathcal{S}}$, $\mathcal{H}^h(J_{\mathcal{S}})$ and $\mathcal{P}^h(J_{\mathcal{S}})$, are positive and finite, and $(J_{\mathcal{S}}, d)$ has Hausdorff dimension equal to h.

PROOF. Without loss of generality we can assume that E contains at least two elements. Proposition 7.12 implies that $\mathcal{S}$ is strongly regular and $h > 0$. Recalling (7.13), $\mathtt{D}_0 := \min\{||D\phi_e||_\infty : e \in E\} > 0$.

Fix $p \in J_{\mathcal{S}}$ and $0 < r < \frac{1}{2}\min\{\mathrm{diam}(X_v) : v \in V\}$. Then $p = \pi(\tau)$ for some $\tau \in E_A^{\mathbb{N}}$. Let $n = n(\tau) \geq 0$ be the least integer such that $\phi_{\tau|_n}(X_{t(\tau_n)}) \subset B(p,r)$. By (7.18) and (4.5) we have

$$\begin{aligned} m_h(B(p,r)) &\geq m_h\big(\phi_{\tau|_n}(X_{t(\tau_n)})\big) \\ &\geq \tilde{m}_h([\tau|_n])) \\ &\geq c_h||D\phi_{\tau|_n}||_\infty^h \\ &\geq \mathtt{D}_0\, c_h K^{-h}||D\phi_{\tau|_{(n-1)}}||_\infty^h. \end{aligned}$$

By the definition of n and Lemma 4.11, we have

$$r \leq \mathrm{diam}(\phi_{\tau_{|n-1}}(X_{t(\tau_{n-1})})) \leq M\Lambda||D\phi_{\tau|_{(n-1)}}||_\infty$$

and hence

$$m_h(B(p,r)) \geq \mathtt{D}_0\, c_h(M\Lambda K)^{-h}r^h. \tag{7.20}$$

To prove the opposite inequality, let Z be the family of all minimal length words $\omega \in E_A^*$ such that

$$\phi_\omega(X_{t(\omega)}) \cap B(p,r) \neq \emptyset \qquad \text{and} \qquad \phi_\omega(X_{t(\omega)}) \subset B(p,2r). \tag{7.21}$$

Consider an arbitrary $\omega \in Z$ with $|\omega| = n$. Then

$$\mathrm{diam}\big(\phi_\omega(X_{t(\omega)})\big) \leq 4r \tag{7.22}$$

and

$$\operatorname{diam}(\phi_{\omega|_{(n-1)}}(X_{t(\omega|_{(n-1)})})) \geq r. \tag{7.23}$$

To prove (7.23) first notice that as $\omega \in Z$, $\phi_{\omega|_{(n-1)}}(X_{t(\omega|_{(n-1)})}) \cap B(p,r) \neq \emptyset$. Moreover because $\omega|_{(n-1)} \notin Z$ we also have that $\phi_{\omega|_{(n-1)}}(X_{t(\omega|_{(n-1)})}) \not\subset B(p,2r)$. Therefore (7.23) follows. Making use of (4.9), (4.14), (4.5) and (7.23) we get

$$\begin{aligned}
\operatorname{diam}\big(\phi_\omega(X_{t(\omega)})\big) &\geq 2(KC)^{-1}\tilde{R}_{\mathcal{S}}||D\phi_\omega||_\infty \\
&\geq 2(K^2C)^{-1}\tilde{R}_{\mathcal{S}}||D\phi_{\omega|_{(n-1)}}||_\infty \cdot ||D\phi_{\omega_n}||_\infty \\
&\geq 2\mathtt{D}_0(K^2C)^{-1}\tilde{R}_{\mathcal{S}}(M\Lambda)^{-1}\operatorname{diam}\big(\phi_{\omega|_{(n-1)}}(X_{t(\omega|_{(n-1)})})\big) \\
&\geq 2\mathtt{D}_0\tilde{R}_{\mathcal{S}}(M\Lambda K^2C)^{-1}r.
\end{aligned} \tag{7.24}$$

Since the family Z consists of mutually incomparable words, Lemma 7.15 along with (7.22) and (7.24) imply that

$$\sharp Z \leq \Gamma := \left(\frac{5M^2\Lambda^2K^3C^2}{2\mathtt{D}_0\tilde{R}_{\mathcal{S}}^2}\right)^Q. \tag{7.25}$$

Since $\pi^{-1}(B(p,r)) \subset \bigcup_{\omega\in Z}[\omega]$, we get from (7.18), (4.14), (7.22), and (7.25) that

$$\begin{aligned}
m_h(B(p,r)) &= \tilde{m}_h \circ \pi^{-1}(B(p,r)) \\
&\leq \tilde{m}_h\left(\bigcup_{\omega\in Z}[\omega]\right) = \sum_{\omega\in Z}\tilde{m}_h([\omega]) \\
&\leq \sum_{\omega\in Z}||D\phi_\omega||_\infty^h \leq \sum_{\omega\in Z}\left(KC(2\tilde{R}_{\mathcal{S}})^{-1}\operatorname{diam}(\phi_\omega(X_{t(\omega)}))\right)^h \\
&\leq (2KC\tilde{R}_{\mathcal{S}}^{-1})^h\sum_{\omega\in Z}r^h = (2KC\tilde{R}_{\mathcal{S}}^{-1})^h(\sharp Z)r^h \leq (2KC\tilde{R}_{\mathcal{S}}^{-1})^h\Gamma r^h.
\end{aligned}$$

Along with (7.20) this completes the proof of (7.19). The remaining conclusions are easy consequences of the h-regularity of $(J_{\mathcal{S}}, d)$, see for example [**42**, Theorem 5.7]. □

The following is the main theorem of this section. Note that we do not assume in this theorem that the edge set E is a finite set. Recall also that Bowen's parameter $h_{\mathcal{S}}$ is defined to be $h_{\mathcal{S}} = \inf\{t \geq 0: \ \mathrm{P}(t) \leq 0\}$.

THEOREM 7.19. *If $\mathcal{S}$ is a finitely irreducible Carnot conformal GDMS, then*

$$h_{\mathcal{S}} = \dim_{\mathcal{H}}(J_{\mathcal{S}}) = \sup\{\dim_{\mathcal{H}}(J_F) : F \subset E \text{ finite}\}.$$

PROOF. Put $h_\infty = \sup\{\dim_{\mathcal{H}}(J_F) : F \subset E \text{ finite}\}$ and $H = \dim_{\mathcal{H}}(J_{\mathcal{S}})$. Fix $t > h_{\mathcal{S}}$. Then $\mathrm{P}(t) < 0$ and for all $n \in \mathbb{N}$ large enough, we have

$$Z_n(t) = \sum_{\omega\in E_A^n}||D\phi_\omega||_\infty^t \leq \exp\left(\frac{1}{2}\mathrm{P}(t)n\right).$$

Hence by (4.9)

$$\sum_{\omega\in E_A^n}\big(\operatorname{diam}(\phi_\omega(X_{t(\omega)})\big)^t \leq (\Lambda M)^t\sum_{\omega\in E_A^n}||D\phi_\omega||_\infty^t \leq (\Lambda M)^t\exp\left(\frac{1}{2}\mathrm{P}(t)n\right).$$

Since the family $\{\phi_\omega(X_{t(\omega)})\}_{\omega\in E_A^n}$ covers $J_{\mathcal{S}}$, by [42, Lemma 4.6], we obtain that $\mathcal{H}^t(J_{\mathcal{S}}) = 0$ upon letting $n \to \infty$. This implies that $t \geq H$, and consequently, $h_{\mathcal{S}} \geq H$. Since obviously $h_\infty \leq H$, we thus have

$$h_\infty \leq H \leq h_{\mathcal{S}}.$$

We are left to show that $h_{\mathcal{S}} \leq h_\infty$. If F is a finite and irreducible subset of E, then in virtue of Theorem 7.18, $h_F \leq h_\infty$, and in particular $\mathrm{P}_F(h_\infty) \leq 0$. So, by Theorem 6.11, Remark 6.12 and Lemma 7.1, we have

$$\mathrm{P}(h_\infty) = \sup\{P_F(h_\infty) : F \subset E \text{ finite and irreducible}\} \leq 0.$$

Hence $h_\infty \geq h_{\mathcal{S}}$ and the proof is complete. □

In the particular case when $\mathcal{S}$ is a Carnot IFS consisting of metric similarities we get the following conclusion. Recall that the open set condition is a standing assumption in our definition of conformal GDMS. Note also that any IFS $\mathcal{S} = \{\phi_e\}_{e\in E}$ is a finitely irreducible GDMS because any finite subset of E witness irreducibility for E.

Corollary 7.20. *Let $(\mathbb{G}, d)$ be an arbitrary Carnot group equipped with a homogeneous metric d. Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a Carnot iterated function system consisting of metric similarities, i.e. the contractions ϕ_e satisfy the equation*

$$d(\phi_e(p), \phi_e(q)) = r_{\phi_e} d(p, q)$$

for all $p, q \in W_{t(e)}$, where $r_{\phi_e} = \|D\phi_e\|_\infty$ is the scaling factor of ϕ_e. Then

$$h = \dim_{\mathcal{H}}(J_{\mathcal{S}}) = \inf\left\{t \geq 0 : \sum_{e\in E} \|D\phi_e\|_\infty^t < 1\right\}.$$

7.3. Characterizations of co-finitely regular and strongly regular Carnot conformal IFS

If $F \subset E$ is co-finite, that is the set $E\backslash F$ is finite, we will say that $\mathcal{S}_F = \{\phi_e\}_{e\in F}$ is a *co-finite subsystem* of $\mathcal{S} = \{\phi_e\}_{e\in E}$. We also record the following lemma which will turn out to be useful in the following. It is an immediate corollary of Proposition 7.5 (i).

Lemma 7.21. *Let $\mathcal{S}$ be a finitely irreducible Carnot conformal CIFS. The following conditions are equivalent.*

(i) $Z_1(t) < \infty$.
(ii) There exists a co-finite subsystem $\mathcal{S}_F$ of $\mathcal{S}$ such that $Z_1(F, t) < \infty$.
(iii) For every co-finite subsystem $\mathcal{S}_F$ of $\mathcal{S}$ it holds that $Z_1(F, t) < \infty$.
(iv) $\mathrm{P}(t) < \infty$.
(v) There exists a co-finite subsystem $\mathcal{S}_F$ of $\mathcal{S}$ such that $\mathrm{P}_F(t) < \infty$.
(vi) For every co-finite subsystem $\mathcal{S}_F$ of $\mathcal{S}$ it holds that $\mathrm{P}_F(t) < \infty$.

The following proposition provides a useful characterization of co-finitely regular systems.

Proposition 7.22. *A Carnot conformal IFS $\mathcal{S}$ is co-finitely regular if and only if every co-finite subsystem $\mathcal{S}_F$ is regular.*

Proof. First note that by Lemma 7.21 $\theta_F = \theta_{\mathcal{S}} = \theta$ for every co-finite subset F of E, because trivially E and all its subsets are finitely irreducible. Suppose now that $\mathcal{S}$ is co-finitely regular. In view of Proposition 7.5(i) this means that $Z_1(\theta) = +\infty$. But then again by Lemma 7.21 $Z_1(F, \theta) = +\infty$ for every co-finite subset F of E. A second application of Proposition 7.5(i) ensures that each such system $\mathcal{S}_F$ is co-finitely regular, thus regular.

For the converse suppose that the system $\mathcal{S}$ is not co-finitely regular. A third application of Proposition 7.5(i) yields that $Z_1(\theta) < +\infty$. Note that there exists a co-finite subset F of E such that $Z_1(F, \theta) < 1$. Hence, by the definition of topological pressure,

$$\mathrm{P}_F(\theta) \leq \limsup_{n \to \infty} \frac{1}{n} \log Z_1(F, \theta)^n < 0.$$

Hence F is not regular and we have reached a contradiction. □

We will now provide another characterization of the θ_S number.

Theorem 7.23. *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a finitely irreducible Carnot conformal CIFS. Then*

$$\theta_{\mathcal{S}} = \inf\{h_{E \setminus T} : T \textit{ finite subset of } E\}.$$

Proof. By Lemma 7.21 $\theta_{\mathcal{S}} = \theta_F$ for every co-finite set $F \subset E$. Moreover $\theta_F \leq h_F$ for every $F \subset E$, therefore,

$$\theta_{\mathcal{S}} \leq \inf\{h_{E \setminus T} : T \text{ finite subset of } E\}.$$

For the other direction let $t > \theta_{\mathcal{S}}$. Then $\sum_{e \in E} \|D\phi_e\|^t < \infty$, therefore there exists some finite set $F \subset E$ such that

$$\sum_{e \in E \setminus F} \|D\phi_e\|^t < 1.$$

Thus for every finite set T such that $F \subset T \subset E$,

$$Z_1(E \setminus T, t) \leq Z_1(E \setminus F, t) < 1.$$

Now as in the proof of Proposition 7.22 we deduce that

$$\mathrm{P}_{E \setminus T}(t) \leq \log Z_1(E \setminus T, t) < 0.$$

Hence $t \geq h_{E \setminus T}$, for such finite sets T and in particular

$$\inf\{h_{E \setminus T} : T \text{ finite subset of } E\} \leq t.$$

Therefore $\inf\{h_{E \setminus T} : T \text{ finite subset of } E\} \leq \theta_{\mathcal{S}}$ and the proof is complete. □

We will now prove the following characterization of strongly regular Carnot conformal IFS via their subsystems. This characterization will be employed in the study of dimension of Iwasawa continued fractions in Section 9.2.

Theorem 7.24. *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a finitely irreducible Carnot conformal CIFS. Then the following conditions are equivalent.*

(i) *$\mathcal{S}$ is strongly regular.*
(ii) *$h_{\mathcal{S}} > \theta_{\mathcal{S}}$.*
(iii) *There exists a proper co-finite subsystem $\mathcal{S}' \subset \mathcal{S}$ such that $h_{\mathcal{S}'} < h_{\mathcal{S}}$.*
(iv) *For every proper subsystem $\mathcal{S}' \subset \mathcal{S}$ it holds that $h_{\mathcal{S}'} < h_{\mathcal{S}}$.*

PROOF. The implications (iv)⇒(iii) and (ii)⇒(i) are immediate. In order to prove the implication (iii)⇒(ii) suppose by way of contradiction that $h_{\mathcal{S}} = \theta_{\mathcal{S}}$. Let $\mathcal{S}'$ be a co-finite subsystem of $\mathcal{S}$. By Theorem 7.23 we deduce that $h_{\mathcal{S}'} \geq \theta_S$, hence by our assumption

$$h_{S'} \geq \theta_{\mathcal{S}} = h_{\mathcal{S}}.$$

Therefore $h_{\mathcal{S}'} = h_{\mathcal{S}}$ for every co-finite subsystem $\mathcal{S}' \subset \mathcal{S}$, which contradicts (iii).

For the remaining implication (i)⇒(iv) let $E' \subset E$ and consider the corresponding proper subsystem of $\mathcal{S}$, $\mathcal{S}' = \{\phi_e\}_{e \in E'}$. If $\mathcal{S}'$ is not regular then by Remark 7.6 $P_{\mathcal{S}'}(h_{\mathcal{S}'}) < 0$ and by Proposition 7.5 we deduce that $P_{\mathcal{S}'}(\theta_{\mathcal{S}'}) < 0$. Since $\mathcal{S}$ is strongly regular, Proposition 7.10 implies that there exists $\alpha \in (\theta_{\mathcal{S}}, h_{\mathcal{S}})$. Therefore since $\alpha > \theta_{\mathcal{S}} \geq \theta_{\mathcal{S}'}$ and the pressure function is strictly decreasing we deduce that $P_{\mathcal{S}'}(\alpha) < 0$. Thus by the definition of the parameter $h_{\mathcal{S}'}$, we get that $h_{\mathcal{S}'} \leq \alpha < h_{\mathcal{S}}$ and we are done in the case when $\mathcal{S}'$ is not regular.

Now by way of contradiction assume that $\mathcal{S}'$ is regular and

$$h_{\mathcal{S}} = h_{\mathcal{S}'} := h.$$

By Theorem 7.4 there exist unique measures $\tilde{\mu}_h$ on $E^{\mathbb{N}}$ and $\tilde{\mu}'_h$ on ${E'}^{\mathbb{N}}$, which are ergodic and shift-invariant with respect to $\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$ and $\sigma' : {E'}^{\mathbb{N}} \to {E'}^{\mathbb{N}}$ respectively. Moreover, again by Theorem 7.4 we have that

$$m_h \ll \tilde{\mu}_h \ll m_h \text{ and } m'_h \ll \tilde{\mu}'_h \ll m'_h, \tag{7.26}$$

where m'_h stands for the h-conformal measure corresponding to $\mathcal{S}'$. Notice also that if $\omega \in {E'}^*$, then by (7.18)

$$m'_h([\omega]) \approx \|D\phi_\omega\|_\infty^h \approx m_h([\omega]),$$

therefore

$$\tilde{\mu}_h([\omega]) \approx \tilde{\mu}'_h([\omega]). \tag{7.27}$$

Now in the obvious way we can extend $\tilde{\mu}_h$ to a Borel measure in $E^{\mathbb{N}}$, defined by

$$\tilde{\nu}_h(B) := \tilde{\mu}'_h(B \cap {E'}^{\mathbb{N}})$$

for $B \subset E_A^{\mathbb{N}}$. By (7.27) we deduce that $\tilde{\nu}_h$ is absolutely continuous with respect to $\tilde{\mu}_h$.

Using standard arguments one can show that $\tilde{\nu}_h$ is shift-invariant with respect to $\sigma : E_A^{\mathbb{N}} \to E_A^{\mathbb{N}}$.

We will now show that $\tilde{\nu}_h$ is ergodic with respect to σ. To do so by way of contradiction suppose that there exists some Borel subset F of $E^{\mathbb{N}}$ such that $\sigma^{-1}(F) = F$ and $0 < \tilde{\nu}_h(F) < 1$. Let

$$F_1 = F \cap {E'}^{\mathbb{N}} \text{ and } F_2 = F \setminus F_1.$$

Since ${\sigma'}^{-1}(F_1) \subset \sigma^{-1}(F_1) \subset F_1 \cup F_2$ and ${\sigma'}^{-1}(F_1) \cap F_2 = \emptyset$ we deduce that

$${\sigma'}^{-1}(F_1) \subset F_1. \tag{7.28}$$

Moreover,

$$F_1 \subset \sigma^{-1}(F) \subset \bigcup_{j \in E'} \{jf : f \in F\} \cup \bigcup_{j \in E \setminus E'} \{jf : f \in F\}.$$

Therefore

$$F_1 \subset \bigcup_{j \in E'} \{jf : f \in F_1\} \cap E'^{\mathbb{N}} = \sigma'^{-1}(F_1),$$

which combined with (7.28) implies that

$$F_1 = \sigma'^{-1}(F_1). \tag{7.29}$$

But since $\tilde{\mu}'_h$ is ergodic with respect to σ' we deduce that either $\tilde{\mu}'_h(F_1) = 0$ or $\tilde{\mu}'_h(F_1) = 1$. Therefore, since $\tilde{\nu}_h(F) = \tilde{\mu}_h(F_1)$,

$$\tilde{\nu}_h(F) = 0 \text{ or } \tilde{\nu}_h(F) = 1,$$

and we have reached a contradiction. Thus $\tilde{\nu}_h$ is ergodic with respect to σ.

Hence we have shown that there exist two probability Borel measures on E_A^n, $\tilde{\mu}_h$ and $\tilde{\nu}_h$, which are shift-invariant and ergodic with respect to σ and they are absolutely continuous with respect to m_h. Now Theorem 7.4 implies that

$$\tilde{\mu}_h \equiv \tilde{\nu}_h. \tag{7.30}$$

If $j \in E \setminus E'$, then $\tilde{\nu}_h([j]) = 0$. On the other hand, because $\tilde{\mu}_h$ is equivalent to m_h, by (7.18) $\tilde{\mu}_h([j]) > 0$. Therefore (7.30) cannot hold and we have reached a contradiction. The proof is complete. □

7.4. Dimension spectrum for subsystems of Carnot conformal IFS

In this section we show that when $\mathcal{S}$ is a Carnot conformal IFS, the spectrum of the Hausdorff dimensions of its subsystems is at least $(0, \theta_{\mathcal{S}})$. This theorem will be applied when we will revisit continued fractions on Iwasawa groups (see Section 9.2).

We start with a lemma.

LEMMA 7.25. *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a Carnot conformal IFS and let F be a subset of E such that $E \setminus F$ is infinite. Then for every $\varepsilon > 0$ there exists some $e \in E \setminus F$ such that*

$$\dim_{\mathcal{H}}(J_{F \cup \{e\}}) \leq \dim_{\mathcal{H}}(J_F) + \varepsilon.$$

PROOF. Without loss of generality assume that $E = \mathbb{N}$. Let $\varepsilon > 0$ and set $h = \dim_{\mathcal{H}}(J_F)$. We record now that $P_F(h + \varepsilon) < 0$. We will now show that there exists some $\alpha \in (0,1)$ ans some $j_0 \in \mathbb{N}$ such that

$$Z_j(F, h + \varepsilon) < \alpha^j \text{ for all } j \geq j_0. \tag{7.31}$$

If (7.31) does not hold then for every $\alpha \in (0,1)$ there exists a sequence of natural numbers $(j_m)_{m \in \mathbb{N}}$ such that $Z_{j_m}(F, h + \varepsilon) \geq \alpha^{j_m}$ for all $m \in \mathbb{N}$. But this implies that for every $\alpha \in (0,1)$,

$$P_F(h + \varepsilon) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(F, h + \varepsilon) \geq \limsup_{m \to \infty} \frac{1}{j_m} \log Z_{j_m}(F, h + \varepsilon) \geq \log \alpha.$$

Hence $P_F(h + \varepsilon) \geq 0$ and we have reached a contradiction, thus (7.31) holds.

We now wish to estimate $Z_n(F \cup \{e\}, h + \varepsilon)$ for $e \in \mathbb{N} \setminus F$. We have that

$$Z_n(F \cup \{e\}, h + \varepsilon) = \sum_{\omega \in (F \cup \{e\})^n} \|D\phi_\omega\|_\infty^{h+\varepsilon} = \sum_{j=0}^{n} \sum_{\omega \in F_j} \|D\phi_\omega\|_\infty^{h+\varepsilon}, \tag{7.32}$$

where

$$F_j = \{\omega \in (F \cup \{e\})^n : \omega_i = e \text{ for } n - j \text{ } i\text{'s}\}.$$

Now by (4.6) and (7.31) we get that for $n > j_0$

$$\begin{aligned}\sum_{j=0}^{n}\sum_{\omega\in F_j}\|D\phi_\omega\|_\infty^{h+\varepsilon} &\leq \|D\phi_e\|_\infty^{n(h+\varepsilon)} + \sum_{j=1}^{n}\binom{n}{j}(K\|D\phi_e\|_\infty)^{(n-j)(h+\varepsilon)}\sum_{\omega\in F^j}\|D\phi_\omega\|_\infty^{h+\varepsilon}\\ &= \|D\phi_e\|_\infty^{n(h+\varepsilon)} + \sum_{j=1}^{n}\binom{n}{j}(K\|D\phi_e\|_\infty)^{(n-j)(h+\varepsilon)}Z_j(F,h+\varepsilon)\\ &\leq \|D\phi_e\|_\infty^{n(h+\varepsilon)} + \sum_{j=1}^{j_0}\binom{n}{j}(K\|D\phi_e\|_\infty)^{(n-j)(h+\varepsilon)}Z_j(F,h+\varepsilon)\\ &\quad + \sum_{j=j_0+1}^{n}\binom{n}{j}(K\|D\phi_e\|_\infty)^{(n-j)(h+\varepsilon)}\alpha^j\\ &= \|D\phi_e\|_\infty^{n(h+\varepsilon)} + I_1 + I_2,\end{aligned}$$

where the last identity serves also as the definition of I_1 and I_2. Therefore by (7.32) we get that for $n > j_0$,

$$Z_n(F\cup\{e\},h+\varepsilon) \leq \|D\phi_e\|_\infty^{n(h+\varepsilon)} + I_1 + I_2. \tag{7.33}$$

For I_2 we have that,

$$I_2 \leq (\alpha + (K\|D\phi_e\|)^{h+\varepsilon})^n. \tag{7.34}$$

For I_1 we estimate,

$$I_1 \leq \|D\phi_e\|_\infty^{(n-j_0)(h+\varepsilon)} n^{j_0} K^{n(h+\varepsilon)} \sum_{j=0}^{j_0} Z_j(F,h+\varepsilon).$$

Since $P_F(h+\varepsilon) < \infty$, by Proposition 7.5 and (7.1) we deduce that $Z_j(F,h+\varepsilon)$ is finite for all $j \in \mathbb{N}$. Therefore,

$$I_1 \leq \|D\phi_e\|_\infty^{(n-j_0)(h+\varepsilon)}\, n^{j_0}\, K^{n(h+\varepsilon)}\, j_0 \max_{1\leq j\leq j_0} Z_j(F,h+\varepsilon). \tag{7.35}$$

Now notice that if e and n are chosen big enough, by Lemma 4.18, (7.33), (7.34) and (7.35) we get that $Z_n(F\cup\{e\},h+\varepsilon) < 1$. Therefore $P_{F\cup\{e\}}(h+\varepsilon) \leq 0$, and consequently by Theorem 7.19

$$\dim_{\mathcal{H}}(J_{F\cup\{e\}}) \leq h+\varepsilon.$$

The proof of the lemma is complete. □

THEOREM 7.26. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a Carnot conformal IFS. Then for every $t \in (0,\theta_{\mathcal{S}})$ there exists a proper subsystem $\mathcal{S}_t$ of $\mathcal{S}$ such that $\dim_{\mathcal{H}}(J_{\mathcal{S}_t}) = t$.*

PROOF. Let $t \in (0,\theta_{\mathcal{S}})$. Again without loss of generality assume that $E = \mathbb{N}$. Let $E_1 = \{1\}$, then trivially $\dim_{\mathcal{H}}(J_{E_1}) < t$. Now using Lemma 7.25 we can inductively construct a sequence of sets $\{E_n\}_{n\in\mathbb{N}}$ such that $\dim_{\mathcal{H}}(J_{E_n}) < t$ for all $n \in \mathbb{N}$. Suppose that E_n has been constructed then by Lemma 7.25 choose the minimal $k_n \in \mathbb{N}$ such that $k_n > \max\{e : e \in E_n\}$ and $\dim_{\mathcal{H}}(J_{E_n\cup\{k_n\}}) < t$. Then we choose $E_{n+1} = E_n \cup \{k_n\}$. Now let,

$$E_t = \bigcup_{n=1}^{\infty} E_n.$$

Notice that E_t is infinite and by Theorem 7.19,

$$\dim_{\mathcal{H}}(J_{E_t}) = \sup_{n\in\mathbb{N}}\{\dim_{\mathcal{H}}(J_{E_n})\} \le t. \tag{7.36}$$

Now notice that $\mathbb{N}\setminus E_t$ is infinite. Because if not, Theorem 7.23 and Theorem 7.19 imply that

$$\dim_{\mathcal{H}}(J_{E_t}) \ge \theta_{\mathcal{S}} > t,$$

and this contradicts (7.36). Now if $\dim_{\mathcal{H}}(J_{E_t}) = t$ we are done. If not, since $\mathbb{N}\setminus E_t$ is infinite, we can apply Lemma 7.25 once more in order to find a value $q \in \mathbb{N}\cap(k_n, k_{n+1})$ for some n such that

$$\dim_{\mathcal{H}}(J_{E_t\cup\{q\}}) < t.$$

But in this case we also have that $\dim_{\mathcal{H}}(J_{E_{n+1}\cup\{q\}}) < t$ and this contradicts the minimality of k_{n+1}. Therefore we have reached a contradiction and the proof of the theorem follows. $\square$

CHAPTER 8

Conformal measures and regularity of domains

In this chapter, under suitable additional hypotheses on the underlying GDMS, we establish fundamental estimates for conformal measures which will play a crucial role in the subsequent chapters. Using these estimates, we show that under some mild assumptions the Hausdorff dimension of a conformal GDMS in a Carnot group $(\mathbb{G}, d)$ is strictly less than the homogeneous dimension of $\mathbb{G}$.

8.1. Bounding coding type and null boundary

As before, let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a Carnot conformal GDMS. Recall that any measure supported on $J_\mathcal{S}$ which satisfies

$$m_h := \tilde{m}_h \circ \pi^{-1}$$

is called h-conformal. Moreover recalling Definition 7.2 we say that $t \in \mathrm{Fin}(\mathcal{S})$ if $t \geq 0$ and $\sum_{e\in E} ||D\phi_e||_\infty^t < +\infty$.

Definition 8.1. Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS. If $t \in \mathrm{Fin}(\mathcal{S})$, then the measure $m_t = \tilde{m}_t \circ \pi^{-1}$ is said to be of *null boundary* if

$$m_t\big(\phi_\omega(X_{t(\omega)}) \cap \phi_\tau(X_{t(\tau)})\big) = 0 \tag{8.1}$$

whenever ω and τ are two different A-admissible words of the same length.

Let us record the following immediate consequence of being of null boundary. We record that any $\omega \in E_A^\mathbb{N}$ will be also called a *code*. If $x = \pi(\omega)$, then we will say that ω is a code of x.

Remark 8.2. If a finitely irreducible Carnot conformal GDMS is of null boundary, then for every $t \in \mathrm{Fin}(\mathcal{S})$, m_t almost every point in $J_\mathcal{S}$ has a unique code.

Remark 8.3. If m_t is of null boundary, then (8.1) holds for all (not necessarily of the same length) incomparable A-admissible words ω and τ.

For the following definition recall that the matrix $\hat{A}$ was introduced in Definition 4.3. Any infinite word $\omega \in E_{\hat{A}}^\mathbb{N}$ will be called a *pseudocode* (from the vantage point of the original system $\mathcal{S}$); frequently also all elements of $E_{\hat{A}}^*$ will be called pseudocodes. In particular each element of $E_A^* \cup E_A^\mathbb{N}$ is a pseudocode.

Definition 8.4. Let $\mathcal{S}$ be a graph directed (not necessarily conformal) Markov system. Given $q \geq 1$, we say that two different words $\rho, \tau \in E_{\hat{A}}^*$, i.e two different pseudocodes, of the same length, say $n \geq q$, form a *pair of q-pseudocodes* at a point $x \in X$ if

$$x \in \phi_\rho(X_{t(\rho)}) \cap \phi_\tau(X_{t(\tau)})$$

and

$$\rho|_{n-q} = \tau|_{n-q}.$$

The graph directed Markov systems $\mathcal{S}$ is said to be of *bounded coding type* if for every $q \geq 1$ there is no point in X (or equivalently in $J_{\mathcal{S}}$) with arbitrarily long pairs of q-pseudocodes.

It will turn out that each conformal system with a mild boundary regularity condition is of bounded coding type.

THEOREM 8.5. *Suppose that a finitely irreducible Carnot conformal GDMS $\mathcal{S} = \{\phi_e\}_{e \in E}$ is of bounded coding type. If $t \in \mathrm{Fin}(\mathcal{S})$, then the measure m_t is of null boundary.*

PROOF. Suppose on the contrary that

$$m_t(\phi_\rho(X_{t(\rho)}) \cap \phi_\tau(X_{t(\tau)})) > 0$$

for two different words $\rho, \tau \in E_A^*$ of the same length, say $q \geq 1$, for which $i(\rho) = i(\tau) = v$ with some $v \in V$. This equivalently means that

$$\mu_t(\phi_\rho(X_{t(\rho)}) \cap \phi_\tau(X_{t(\tau)})) > 0,$$

where μ_t was defined in (7.17). Let $E := \phi_\rho(X_{t(\rho)}) \cap \phi_\tau(X_{t(\tau)})$, and, for every $n \in \mathbb{N}$, let $E_n := \sigma^{-n}(\pi^{-1}(E))$. Note that each element of $\pi(E_n)$ admits at least two different q-pseudocodes of length $n+q$. To see this let $x = \pi(\omega) \in \pi(E_n)$, then $\sigma^n(\omega) \in \pi^{-1}(E)$ hence

$$x \in \phi_{\omega|_n \rho}(X_{t(\rho)}) \cap \phi_{\omega|_n \tau}(X_{t(\tau)}).$$

Since $|\omega|_n \rho| = |\omega|_n \tau| = n + q$, x admits at least two q-pseudocodes of length $n+q$. Therefore since $\mathcal{S}$ is of bounded coding type we conclude that

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \pi(E_n) = \emptyset.$$

Hence

$$E_\infty := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \subset \pi^{-1}\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \pi(E_n)\right) = \emptyset$$

and

$$\tilde{\mu}_t(E_\infty) = 0.$$

On the other hand, by Theorem 7.4 $\tilde{\mu}_t$ is shift-invariant, thus we have $\tilde{\mu}_t(E_n) = \tilde{\mu}_t(\pi^{-1}(E)) = \mu_t(E)$ and

$$\tilde{\mu}_t(E_\infty) \geq \mu_t(E) > 0.$$

This contradiction finishes the proof. □

As an immediate consequence of Theorem 8.5 and Remark 8.2, we get the following.

COROLLARY 8.6. *If $\mathcal{S} = \{\phi_e\}_{e \in E}$ is a finitely irreducible Carnot conformal GDMS of bounded coding type, then for every $t \in \mathrm{Fin}(\mathcal{S})$, m_t almost every point in $J_{\mathcal{S}}$ has a unique code.*

8.2. Regularity properties of domains in Carnot groups

In this section, we indicate geometric conditions on domains in Iwasawa groups which suffice for the application of the results in prior sections.

DEFINITION 8.7. An open subset Ω of a Carnot group $\mathbb{G}$ satisfies the *weak corkscrew condition (WCC)* if there exists $\alpha \in (0,1]$ such that for every $p \in \partial\Omega$ and every $r > 0$ sufficiently small (smallness possibly depending on p), there exists $q \in \mathbb{G}$ such that

$$B(q, \alpha r) \subset \Omega \cap B(p, r).$$

We say that Ω satisfies the *corkscrew condition (CC)* if a common $\alpha > 0$ can be taken for all sufficiently small $r > 0$ and all $p \in \partial\Omega$ (smallness of r independent of p). We also say that $\overline{\Omega}$, the closure of Ω, satisfies the weak corkscrew condition or the corkscrew condition respectively if Ω does.

EXAMPLE 8.8. Every $C^{1,1}$ domain in an Iwasawa group $\mathbb{G}$ satisfies the corkscrew condition. This is proved in [21, Theorem 14] for the Carnot–Carathéodory metric on general step two Carnot groups; its validity for the gauge metric follows from the comparability of these two metrics.

Any number $\alpha \in (0,1]$ satisfying the property in Definition 8.7 is called a *corkscrew constant* of Ω. We denote by α_Ω the supremum of all such constants. As a consequence of Corollary 3.12 we obtain the following.

PROPOSITION 8.9. *Let U be an open connected subset in a Carnot group $\mathbb{G}$. Let Ω be a bounded open subset of U satisfying the weak corkscrew condition. Assume that $\operatorname{dist}(\Omega, \partial U) > 0$ and let S be a compact set such that $\Omega \subset S \subset U$. Let $K \geq 1$ be the distortion constant associated to S as in Lemma 3.6. If $\phi : U \to \ \Omega$ is a conformal homeomorphism, then $\phi(\Omega)$ also satisfies the weak corkscrew condition, moreover,*

$$\alpha_{\phi(\Omega)} \geq K^{-1} C^{-2} \alpha_\Omega.$$

PROOF. Let $p' = \phi(p) \in \Omega$. Set $R_0 = \sup\{R > 0 : B(p', R) \subset \phi(\Omega)\}$. Since Ω satisfies the WCC, there exists some $r_0(p)$ such that for all $r < r_0(p)$, there exists some $q_{r,p}$ such that $B(q_{r,p}, \alpha_\Omega r) \subset B(p, r) \cap \Omega$. Now set

$$r_1(p) = \min\left\{ \frac{\operatorname{dist}(\Omega, \partial U)}{4L}, \frac{r_0(p)}{2}, \frac{R_0}{C\|D\phi\|_\infty} \right\},$$

and

$$R_1(p) = C\|D\phi\|_\infty r_1(p).$$

Using the WCC for p and $r \leq r_1(p)$, and applying Corollary 3.12 twice we get,

$$\begin{aligned} B(\phi(q), \alpha_\Omega K^{-1} C^{-1} \|Df\|_\infty r) &\subset \phi(B(q, \alpha_\Omega r)) \\ &\subset \phi(B(p, r)) \\ &\subset B(\phi(p), C\|D\phi_\omega\|_\infty r) \\ &\subset B(\phi(p), R_0) \subset \phi(\Omega). \end{aligned} \tag{8.2}$$

Let $R \leq R_1(p)$. Then there exists some $r \leq r_1(p)$ such that $R = C\|D\phi\|_\infty r$. Hence by (8.2),

$$B(\phi(q), \alpha_\Omega K^{-1} C^{-2} R) = B(\phi(q), \alpha_\Omega K^{-1} C^{-1} \|Df\|_\infty r) \subset B(\phi(q), R) \subset \phi(\Omega).$$

Therefore $\phi(\Omega)$ satisfies the WCC and $\alpha_{\phi(\Omega)} \geq \alpha_\Omega K^{-1} C^{-2}$. The proof of the proposition is finished. □

DEFINITION 8.10. A Carnot conformal GDMS $\mathcal{S} = \{\phi_e\}_{e\in E}$ satisfies the *weak corkscrew condition (WCC)* if every set X_v, $v \in V$, satisfies such condition. We then put

$$\alpha_{\mathcal{S}} := \min\{\alpha_{X_v} : v \in V\}.$$

COROLLARY 8.11. *If $\mathcal{S}$ is a Carnot conformal GDMS satisfying the weak corkscrew condition, then for each point $x \in X$ there are at most $K^Q C^{2Q} a_{\mathcal{S}}^{-Q}$ pseudocodes of x.*

PROOF. Suppose $p \in X$ admits $N > 1$ pseudocodes. Then there exist mutually incomparable words $\tau_1, \tau_2, \ldots, \tau_N \in E^*_{\hat{A}}$. Then for each $1 \le j \le N$, $x \in \phi_{\tau_j}\big(X_{t(\tau_j)}\big)$. Since $N > 1$, we have $p \in \partial X$, and therefore for each $1 \le j \le N$ there exists a point $p_j \in \partial X_{t(\tau_j)}$ such that

$$p = \phi_{\tau_j}(p_j).$$

By our hypotheses and by Proposition 8.9, each set $\phi_{\tau_j}\big(X_{t(\tau_j)}\big)$ satisfies the weak corkscrew condition and

$$\alpha_{\phi_{\tau_j}(\mathrm{Int}(X_{t(\tau_j)}))} \ge K^{-1} C^{-2} \alpha_{\mathcal{S}}.$$

Hence for all $1 \le j \le N$ and all sufficiently small $r > 0$, each set $\phi_{\tau_j}(\mathrm{Int}(X_{t(\tau_j)})) \cap B(p, r)$ contains a ball $B(q_j, K^{-1}C^{-2}\alpha_{\mathcal{S}} r)$. Due to the mutual incomparability of the words $\tau_1, \tau_2, \ldots, \tau_N$, all of these balls are pairwise disjoint. Thus

$$c_0 r^Q = |B(p,r)| \ge \sum_{j=1}^{N} |B(q_j, K^{-1}C^{-2}\alpha_{\mathcal{S}} r)| \ge N c_0 (K^{-1}C^{-2}\alpha_{\mathcal{S}} r)^Q$$

and the proof is complete. □

Arguing exactly as in Proposition 7.16 we can show that if a conformal GDMS is finite and irreducible then the conclusion of Corollary 8.11 holds without the weak corkscrew condition.

REMARK 8.12. If $\mathcal{S}$ is a finite and irreducible Carnot conformal GDMS, then each point $x \in X$ has at most $\mathtt{m}_{\kappa_1,\kappa_2}$ pseudocodes. See Lemma 7.15 and the proof of Proposition 7.16, especially (7.15), for the definition of $\mathtt{m}_{\kappa_1,\kappa_2}$.

The following proposition is the main result of this section.

PROPOSITION 8.13. *Every Carnot conformal GDMS $\mathcal{S}$ satisfying the weak corkscrew condition is of bounded coding type.*

PROOF. Suppose to the contrary that there exists a point $p \in X$ having, for some $q \ge 1$, arbitrarily long pairs of q-pseudocodes. This means that for each $k \in \mathbb{N}$ there exist finite words $\omega^{(k)} \in E^*_A$ and $\tau^{(k)}, \rho^{(k)} \in E^q_A$, such that $\tau^{(k)}$ and $\rho^{(k)}$ are different,

$$\lim_{k\to\infty} |\omega^{(k)}| = \infty, \tag{8.3}$$

and

$$p \in \phi_{\omega^{(k)}} \circ \phi_{\tau^{(k)}}\big(X_{t(\tau^{(k)})}\big) \cap \phi_{\omega^{(k)}} \circ \phi_{\rho^{(k)}}\big(X_{t(\rho^{(k)})}\big)$$

for all $k \in \mathbb{N}$. We now construct by induction for each $n \in \mathbb{N}$ a set C_n which contains at least $n+1$ mutually incomparable pseudocodes of p. The existence of such a set for large n will contradict the statement of Corollary 8.11 and hence will complete the proof.

Set

$$C_1 := \{\omega^{(1)}\tau^{(1)}, \omega^{(1)}\rho^{(1)}\}$$

and suppose that the set C_n has been defined for some $n \in \mathbb{N}$. In view of (8.3), there exists $k_n \in \mathbb{N}$ such that

$$|\omega^{(k_n)}| > \max\{|\xi| : \xi \in C_n\}. \tag{8.4}$$

If $\omega^{(k_n)}\rho^{(k_n)}$ does not extend any word from C_n, then by (8.4) the word $\omega^{(k_n)}\rho^{(k_n)}$ is not comparable with any element of C_n. We obtain C_{n+1} from C_n by adding the word $\omega^{(k_n)}\rho^{(k_n)}$ to C_n. Similarly, if $\omega^{(k_n)}\tau^{(k_n)}$ does not extend any word from C_n, then C_{n+1} is formed by adding $\omega^{(k_n)}\tau^{(k_n)}$ to C_n. On the other hand, if $\omega^{(k_n)}\rho^{(k_n)}$ extends an element $\alpha \in C_n$ and $\omega^{(k_n)}\tau^{(k_n)}$ extends an element $\beta \in C_n$, then we obtain from (8.4) that $\alpha = \omega^{(k_n)}|_{|\alpha|}$ and $\beta = \omega^{(k_n)}|_{|\beta|}$. Since C_n consists of mutually incomparable words, this implies that $\alpha = \beta$. Now, form C_{n+1} by first removing $\alpha = \beta$ from C_n and then adding both $\omega^{(k_n)}\rho^{(k_n)}$ and $\omega^{(k_n)}\tau^{(k_n)}$. Note that no element $\gamma \in C_n \setminus \{\alpha\}$ is comparable with $\omega^{(k_n)}\rho^{(k_n)}$ or $\omega^{(k_n)}\tau^{(k_n)}$, since otherwise $\gamma = \omega^{(k_n)}|_{|\gamma|}$, and consequently, γ would be comparable with α. Since also $\omega^{(k_n)}\rho^{(k_n)}$ and $\omega^{(k_n)}\tau^{(k_n)}$ are incomparable, it follows that C_{n+1} consists of mutually incomparable pseudocodes of p. This completes the inductive construction, and hence finishes the proof. □

As an immediate consequence of this proposition and respectively Theorem 8.5 and Corollary 8.6, we get the following results.

THEOREM 8.14. *If $\mathcal{S} = \{\phi_e\}_{e\in E}$ is a finitely irreducible Carnot conformal GDMS satisfying the weak corkscrew condition, then for every $t \in \mathrm{Fin}(\mathcal{S})$, the measure m_t is of null boundary.*

COROLLARY 8.15. *If $\mathcal{S} = \{\phi_e\}_{e\in E}$ is a finitely irreducible Carnot conformal GDMS satisfying the weak corkscrew condition, then for every $t \in \mathrm{Fin}(\mathcal{S})$, m_t almost every point in $J_{\mathcal{S}}$ has a unique code.*

8.3. Conformal measure estimates

In this section we prove several estimates for the conformal measure of a finitely irreducible Carnot conformal GDMS $\mathcal{S}$ when it satisfies some of the conditions discussed in the previous sections.

LEMMA 8.16. *Let $\mathcal{S} = \{\phi_e : e \in E\}$ be a finitely irreducible Carnot conformal GDMS. Let also let $t \in \mathrm{Fin}(\mathcal{S})$ and $\omega \in E_A^*$. Then for every Borel set $F \subset \mathrm{Int}(X_{t(\omega)})$*

$$\pi^{-1}(\phi_\omega(F)) = \{\tau \in E_A^{\mathbb{N}} : \tau|_{|\omega|} = \omega \text{ and } \pi(\sigma^{|\omega|}(\tau)) \in F\}. \tag{8.5}$$

If moreover m_t is of null boundary, then for every Borel set $F \subset X_{t(\omega)}$

$$\tilde{m}_t(\pi^{-1}(\phi_\omega(F))) = \tilde{m}_t(\{\tau \in E_A^{\mathbb{N}} : \tau|_{|\omega|} = \omega \text{ and } \pi(\sigma^{|\omega|}(\tau)) \in F\}). \tag{8.6}$$

PROOF. For ease of notation let

$$A_\omega := \pi^{-1}(\phi_\omega(F)) = \{\tau \in E_A^{\mathbb{N}} : \pi(\tau) \in \phi_\omega(F)\}$$

and

$$B_\omega = \{\tau \in E_A^{\mathbb{N}} : \tau|_{|\omega|} = \omega \text{ and } \pi(\sigma^{|\omega|}(\tau)) \in F\}.$$

First notice that since for every $\tau \in B_\omega$, $\pi(\tau) = \phi_\omega(\pi(\sigma^{|\omega|}(\tau)))$ and $\pi(\sigma^{|\omega|}(\tau)) \in F$ we have that $B_\omega \subset A_\omega$. Moreover

$$\{\tau \in E_A^{\mathbb{N}} : \pi(\tau) \in \phi_\omega(F), \tau|_{|\omega|} = \omega \text{ and } \pi(\sigma^{|\omega|}(\tau)) \notin F\} = \emptyset,$$

hence

$$A_\omega \setminus B_\omega \subset \{\tau \in E_A^{\mathbb{N}} : \pi(\tau) \in \phi_\omega(F) \text{ and } \tau|_{|\omega|} \neq \omega\} := C_\omega. \tag{8.7}$$

Now if $\tau \in C_\omega$ then $\pi(\sigma^{|\omega|}(\tau)) \in X_{t(\tau_{|\omega|})}$, so $\pi(\tau) \in \phi_{\tau|_{|\omega|}}(X_{t(\tau_{|\omega|})})$ and $\tau|_{|\omega|} \neq \omega$. Therefore

$$\pi(C_\omega) \subset \bigcup_{\{\tau \in E_A^{|\omega|} : \tau \neq \omega\}} \phi_\tau(X_{t(\tau)}) \cap \phi_\omega(F), \tag{8.8}$$

and if m_t is of null boundary we deduce that $\tilde{m}_t(C_\omega) = 0$ and (8.6) follows.

Let $F \subset \text{Int}(X_{t(\omega)})$. Then as in (8.8),

$$\pi(C_\omega) \subset \bigcup_{\{\tau \in E_A^{|\omega|} : \tau \neq \omega\}} \phi_\tau(X_{t(\tau)}) \cap \phi_\omega(\text{Int}(X_{t(\omega)})). \tag{8.9}$$

By the open set condition, and recalling that the maps ϕ_e are homeomorphisms, we get that for all $\tau \in E_A^{|\omega|}$, $\tau \neq \omega$,

$$\phi_\tau(X_{t(\tau)}) \cap \phi_\omega(\text{Int}(X_{t(\omega)})) = \overline{\phi_\tau(\text{Int}(X_{\tau(\tau)}))} \cap \phi_\omega(\text{Int}(X_{t(\omega)})) = \emptyset. \tag{8.10}$$

Combining (8.9) and (8.10) we get that $C_\omega = \emptyset$, which implies (8.5). □

PROPOSITION 8.17. *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a finitely irreducible Carnot conformal GDMS. If $t \in \text{Fin}(\mathcal{S})$ and m_t is of null boundary then for every $\omega \in E_A^*$ and every Borel set $F \subset X_{t(\omega)}$ we have*

$$m_t(\phi_\omega(F)) \leq e^{-\text{P}(t)|\omega|} ||D\phi_\omega||_\infty^t m_t(F). \tag{8.11}$$

PROOF. Fix $\omega \in E_A^*$ and a Borel set $F \subset X_{t(\omega)}$. Denote $k := |\omega|$. Then, by Theorem 8.14 and Lemma 8.16, in particular (8.6), we have

$$\begin{aligned} m_t(\phi_\omega(F)) &= \tilde{m}_t \circ \pi^{-1}(\phi_\omega(F)) \\ &= \tilde{m}_t(\{\tau \in E_A^{\mathbb{N}} : \pi(\tau) \in \phi_\omega(F)\}) \\ &= \tilde{m}_t(\{\tau \in E_A^{\mathbb{N}} : \tau|_{|\omega|} = \omega \text{ and } \pi(\sigma^{|\omega|}(\tau)) \in F\}) \\ &= \tilde{m}_t\big([\omega] \cap \sigma^{-k}(\pi^{-1}(F))\big). \end{aligned} \tag{8.12}$$

Hence by (7.7) and (7.6)

$$\begin{aligned} m_t(\phi_\omega(F)) &= e^{-\text{P}(t)k} \mathcal{L}_t^{*k} \tilde{m}_t(\mathbb{1}_{[\omega] \cap \sigma^{-k}(\pi^{-1}(F))}) \\ &= e^{-\text{P}(t)k} \int_{E_A^{\mathbb{N}}} \sum_{\tau \in E_A^k :\, A_{\tau_k \rho_1} = 1} ||D\phi_\tau(\pi(\rho))||^t \mathbb{1}_{[\omega] \cap \sigma^{-k}(\pi^{-1}(F)}(\tau\rho) \, d\tilde{m}_t(\rho) \\ &= e^{-\text{P}(t)k} \int_{\{\rho \in \pi^{-1}(F) : A_{\omega_k \rho_1} = 1\}} ||D\phi_\omega(\pi(\rho))||^t \, d\tilde{m}_t(\rho) \\ &\leq e^{-\text{P}(t)k} ||D\phi_\omega||_\infty^t \tilde{m}_t\big(\{\rho \in \pi^{-1}(F) : A_{\omega_k \rho_1} = 1\}\big) \\ &\leq e^{-\text{P}(t)k} ||D\phi_\omega||_\infty^t \tilde{m}_t\big(\pi^{-1}(F)\big) = e^{-\text{P}(t)k} ||D\phi_\omega||_\infty^t m_t(F). \end{aligned} \tag{8.13}$$

The proof is complete. □

Note that in the previous proof the fact that m_t has null boundary was only used in the third equality of (8.12). Therefore if $F \subset \mathrm{Int}(X_{t(\omega)})$ using (8.5) we can prove (8.11) without assuming the weak corkscrew condition. We state this observation in the following remark.

REMARK 8.18. Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible Carnot conformal GDMS. If $t \in \mathrm{Fin}(\mathcal{S})$, then for every $\omega \in E_A^*$, (8.11) holds for every Borel set $F \subset \mathrm{Int}(X_{t(\omega)})$.

A lower bound corresponding to (8.11) is given in Proposition 8.22. First, we need the following lemma and some new definitions.

LEMMA 8.19. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible Carnot conformal GDMS of bounded coding type (for example satisfying the weak corkscrew condition). If $t \in \mathrm{Fin}(\mathcal{S})$, then*

$$\tilde{m}_t(\pi^{-1}(A) \cap B) = m_t\big(A \cap \pi(B)\big)$$

for every Borel set $A \subset X$ and every Borel set $B \subset E_A^{\mathbb{N}}$. In particular (taking $A = J_{\mathcal{S}}$),

$$\tilde{m}_t(B) = m_t(\pi(B)).$$

PROOF. Assume that $\omega \in \pi^{-1}(\pi(B)) \setminus B$. Then there must exist $\tau \in B$ such that $\pi(\tau) = \pi(\omega)$. Hence $\tau \neq \omega$, and let $k \in \mathbb{N}$ be the least integer such that $\tau_k \neq \omega_k$. So,

$$\omega \in \pi^{-1}\big(\phi_{\omega|_k}(X_{t(\omega_k)}) \cap \phi_{\tau|_k}(X_{t(\tau_k)})\big).$$

In conclusion,

$$\pi^{-1}(\pi(B)) \setminus B \subset \pi^{-1}\left(\bigcup_{k=1}^{\infty} \bigcup_{\substack{|\omega|=|\tau|=k \\ \omega\neq\tau}} \phi_\omega(X_{t(\omega_k)}) \cap \phi_\tau(X_{t(\tau_k)})\right).$$

Since m_t is of null boundary (see Theorem 8.5), we get

$$\tilde{m}_t\big(\pi^{-1}(\pi(B)) \setminus B\big) = 0.$$

Since also $B \subset \pi^{-1}(\pi(B))$, we therefore get

$$m_t\big(A\cap\pi(B)\big) = \tilde{m}_t\big(\pi^{-1}(A\cap\pi(B))\big) = \tilde{m}_t\big(\pi^{-1}(A)\cap\pi^{-1}(\pi(B))\big) = \tilde{m}_t\big(\pi^{-1}(A)\cap B\big)$$

and the proof is complete. □

For each $a \in E$, let $E_a^- := \{b \in E : A_{ab} = 1\}$ and $E_a^\infty := \{\omega \in E_A^{\mathbb{N}} : \omega_1 \in E_a^-\}$.

DEFINITION 8.20. A Carnot conformal GDMS $\mathcal{S} = \{\phi_e : e \in E\}$ satisfies the *strong separation condition* (SSC) if $\overline{\pi(E_e^\infty)} \cap \overline{\pi(E_i^\infty)} = \emptyset$ for all $e \neq i \in E$.

REMARK 8.21. Each system satisfying the strong separation condition is of bounded coding type.

PROPOSITION 8.22. *Suppose that $\mathcal{S} = \{\phi_e\}_{e\in E}$ is a finitely irreducible Carnot conformal GDMS of bounded coding type (for example satisfying the weak corkscrew condition). If $t \in \mathrm{Fin}(\mathcal{S})$, then for every $\omega \in E_A^*$ and every Borel set $F \subset X_{t(\omega)}$, we have*

$$m_t(\phi_\omega(F)) \geq K^{-t} e^{-\mathrm{P}(t)|\omega|} ||D\phi_\omega||_\infty^t m_t\big(F \cap \pi\big(E_{\omega_{|\omega|}}^\infty\big)\big).$$

Proof. Put $k := |\omega|$. Starting with the third line of (8.13) and using Lemma 8.19 we continue as follows:

$$\begin{aligned} m_t(\phi_\omega(F)) &= e^{-\mathrm{P}(t)k}\int_{\{\rho\in\pi^{-1}(F):A_{\omega_k\rho_1}=1\}} ||D\phi_\omega(\pi(\rho))||^t\,d\tilde{m}_t \\ &\ge K^{-t}e^{-\mathrm{P}(t)k}||D\phi_\omega||_\infty^t\tilde{m}_t\big(\{\rho\in\pi^{-1}(F):A_{\omega_k\rho_1}=1\}\big) \\ &= K^{-t}e^{-\mathrm{P}(t)k}||D\phi_\omega||_\infty^t\tilde{m}_t\big(\pi^{-1}(F)\cap E_{\omega_k}^\infty\big) \\ &= K^{-t}e^{-\mathrm{P}(t)k}||D\phi_\omega||_\infty^t m_t\big(F\cap\pi(E_{\omega_k}^\infty)\big). \end{aligned}$$

The proof is complete. □

Proposition 8.23. *Suppose that $\mathcal{S} = \{\phi_e\}_{e\in E}$ is a finitely irreducible Carnot conformal GDMS of bounded coding type (for example satisfying the weak corkscrew condition). If $t \in \mathrm{Fin}(\mathcal{S})$, then $M_t := \inf\{m_t(\pi(E_e^\infty)) : e \in E\} > 0$.*

Proof. Let $\Phi \subset E_A^*$ be a finite set witnessing finite irreducibility of the incidence matrix A. Let

$$\Phi_1 := \{\omega_1 : \omega \in F\}.$$

By (7.18), we have

$$\gamma = \min\{\tilde{m}_t([e]) : e \in \Phi_1\} > 0.$$

By the definition of Φ, for every $e \in E$ there exists $b \in \Phi_1$ such that $A_{eb} = 1$, which means that $b \in E_e^-$. Hence, $E_e^\infty \supset [b]$. Thus, using Lemma 8.19, we get

$$m_t\big(\pi(E_e^\infty)\big) = \tilde{m}_t(E_e^\infty) \ge \gamma > 0.$$

The proof is complete. □

Corollary 8.24. *Suppose that $\mathcal{S} = \{\phi_e\}_{e\in E}$ is a finitely irreducible maximal Carnot conformal GDMS of bounded coding type (for example satisfying the weak corkscrew condition). If $t \in \mathrm{Fin}(\mathcal{S})$, then for every $\omega \in E_A^*$ and every Borel set $F \subset J_{\mathcal{S}} \cap X_{t(\omega)}$, we have*

$$m_t(\phi_\omega(F)) \ge K^{-t}e^{-\mathrm{P}(t)|\omega|}||D\phi_\omega||_\infty^t m_t(F).$$

Proof. In view Proposition 8.22 it suffices to show that for every $\omega \in E_A^*$ and every Borel set $F \subset J_{\mathcal{S}} \cap X_{t(\omega)}$

$$F \cap \pi(J^-_{\omega_{|\omega|}}) = F. \tag{8.14}$$

By way of contradiction suppose that there exists some $x \in F \setminus \pi(J^-_{\omega|_{|\omega|}})$. By maximality,

$$\begin{aligned} J^-_{\omega|_{|\omega|}} &= \{\tau \in E_A^{\mathbb{N}} : A_{\omega_{|\omega|}\tau_1} = 1\} = \{\tau \in E_A^{\mathbb{N}} : t(\omega_{|\omega|}) = i(\tau_1)\} \\ &= \{\tau \in E_A^{\mathbb{N}} : \phi_{\tau_1}(X_{t(\tau_1)}) \subset X_{t(\omega)}\}. \end{aligned} \tag{8.15}$$

If $x = \pi(\tau)$ for some $\tau \in E_A^{\mathbb{N}}$, by (8.15)

$$\phi_{\tau_1}(X_{t(\tau_1)}) \cap X_{t(\omega)} = \emptyset,$$

and we have reached a contradiction because $x \in F \subset X_{t(\omega)}$ and trivially $x \in \phi_{\tau_1}(X_{t(\tau_1)})$. The proof of the corollary is complete. □

REMARK 8.25. Notice that for $\mathcal{S}$ and t as above, if $\omega \in E_A^*$, $x \in X_{t(\omega)}$ and $r < \eta_{\mathcal{S}}$ then

$$\begin{aligned} m_t(\phi_\omega(B(x,r))) &\geq m_t(\phi_\omega(B(x,r) \cap J_{\mathcal{S}} \cap X_{t(\omega)})) \\ &\geq K^{-t} e^{-\mathrm{P}(t)|\omega|} ||D\phi_\omega||_\infty^t m_t(B(x,r) \cap J_{\mathcal{S}} \cap X_{t(\omega)}) \\ &= K^{-t} e^{-\mathrm{P}(t)|\omega|} ||D\phi_\omega||_\infty^t m_t(B(x,r)), \end{aligned}$$

where in the last equality we also used that the sets $S_v, v \in V$, are disjoint.

Before proving the next theorem which gives some very general bounds on the size of the limit set we need to introduce some notation. If $\mathcal{S} = \{\phi_e\}_{e \in E}$ is a GDMS, for $n \in \mathbb{N}$ let

$$X^n = \bigcup_{\omega \in E_A^n} \phi_\omega(X_t(\omega)).$$

There are exotic examples of conformal GDMSs, even IFSs, for which

$$|J_{\mathcal{S}}| = 0 \ \text{ and } \ \dim_{\mathcal{H}}(J_{\mathcal{S}}) = Q.$$

There are such examples which in addition are of bounded coding type and for which $|\mathrm{Int}X \setminus X^1| > 0$. Indeed, the space X can be taken to be the interval $[0,1]$. Example 5.2.4 of [**46**], originally appearing as Example 4.5 of [**44**], has all such features. This system is (necessarily) irregular. If however we merely assume that $\mathcal{S}$ is regular, then, as the following theorem (formula (8.17)) shows, the picture changes dramatically.

THEOREM 8.26. *If $\mathcal{S} = \{\phi_e\}_{e \in E}$ is a finitely irreducible Carnot conformal GDMS such that $|\mathrm{Int}X \setminus X^1| > 0$ then*

$$|J_{\mathcal{S}}| = 0. \tag{8.16}$$

If moreover $\mathcal{S}$ is regular and of bounded coding type then

$$\dim_{\mathcal{H}} J_{\mathcal{S}} < Q. \tag{8.17}$$

PROOF. For $n \in \mathbb{N}$ and $v \in V$, let

$$X_v^n := X^n \cap X_v = \bigcup_{\omega \in E_A^n : i(\omega) = v} \phi_\omega(X_t(\omega)).$$

Since $|\mathrm{Int}X \setminus X^1| > 0$ there exists some $v_0 \in V$ such that

$$|\mathrm{Int}X_{v_0} \setminus X_{v_0}^1| > 0. \tag{8.18}$$

We will first show that there exists some $n_0 \in \mathbb{N}$ such that

$$|\mathrm{Int}X_v \setminus X_v^{n_0}| > 0, \tag{8.19}$$

for all $v \in V$.

Let $v \in V$ and let $e, e_0 \in E$ such that $i(e) = v$ and $t(e_0) = v_0$. Since $\mathcal{S}$ is finitely irreducible there exists some $\omega \in E_A^*$, with $|\omega| \leq m_0$ for some $m_0 \in \mathbb{N}$ depending only on $\mathcal{S}$, such that $\omega' := e\omega e_0 \in E_A^*$. Therefore,

$$|\phi_{\omega'}(\mathrm{Int}X_{v_0}) \setminus \phi_{\omega'}(X_{v_0}^1)| > 0. \tag{8.20}$$

Now if $|\omega'| = k$ notice that

$$X_v^{k+1} \cap \phi_{\omega'}(\mathrm{Int}X_{v_0}) \subset \phi_{\omega'}(X_{v_0}^1). \tag{8.21}$$

First observe that (8.21) makes sense because $v \in \tilde{V}$, hence $X^1_{v_0} \neq \emptyset$. Moreover,

$$\begin{aligned} X_v^{k+1} &\subset \phi_{\omega'}\left(\bigcup_{e\in E: t(e)=v} \phi_e(X_{t(e)})\right) \cup \bigcup_{\tau\in I_v} \phi_\tau(X_{t(\tau)}) \\ &= \phi_{\omega'}(X^1_{v_0}) \cup \bigcup_{\tau\in I_v} \phi_\tau(X_{t(\tau)}), \end{aligned}$$

where $I_v = \{\tau \in E_A^{k+1} : i(\tau) = v \text{ and } \tau|_k \neq \omega'\}$. But for every $\tau \in I_v$ by the open set condition

$$\phi_\tau(X_{t(\tau)}) \cap \phi_{\omega'}(\mathrm{Int}X_{v_0}) \subset \phi_{\tau|_k}(X_{t(\tau_k)}) \cap \phi_{\omega'}(\mathrm{Int}X_{v_0}) = \emptyset,$$

and (8.21) follows. Now using (8.20) and (8.21) one can show that

$$|\mathrm{Int}X_v \setminus X_v^{k+1}| \geq |\phi_{\omega'}(\mathrm{Int}X_{v_0}) \setminus \phi_{\omega'}(X^1_{v_0})| > 0. \tag{8.22}$$

To see (8.22) notice that

$$\phi_{\omega'}(\mathrm{Int}(X_{t(\omega')})) = \phi_{\omega'}(\mathrm{Int}(X_{t(e_0)})) \subset \mathrm{Int}X_{i(e)} = \mathrm{Int}X_v.$$

Hence by (8.21)

$$\begin{aligned} \mathrm{Int}X_v \setminus X_v^{k+1} &\supset \phi_{\omega'}(\mathrm{Int}X_{v_0}) \setminus X_v^{k+1} \\ &\supset \phi_{\omega'}(\mathrm{Int}X_{v_0}) \setminus \phi_{\omega'}(X^1_{v_0}) = \phi_{\omega'}(\mathrm{Int}X_{v_0} \setminus X^1_{v_0}), \end{aligned}$$

and (8.22) follows by (8.20). Now (8.22) implies (8.19) because $k = |\omega'| \leq m_0 + 2$.

For n_0 as in (8.19) set $G_v = \mathrm{Int}X_v \setminus X_v^{n_0}$ for $v \in V$. Then by Theorem 3.4 for every $\omega \in E_A^*$,

$$\frac{|\phi_\omega(G_{t(\omega)})|}{|\phi_\omega(X_{t(\omega)})|} \geq K^{-Q}\frac{|G_{t(\omega)}|}{|X_{t(\omega)}|} \geq K^{-Q}\gamma, \tag{8.23}$$

where $\gamma := \min_{v\in V} \frac{|G_v|}{|X_v|} > 0.$ because by (8.19), $|G_v| > 0$ for all $v \in V$.

Let $n \in \mathbb{N}$, then

$$\begin{aligned} X^{n+n_0} &\subset \bigcup_{\omega\in E_A^n} \phi_\omega(X^{n_0}_{t(\omega)}) \subset \bigcup_{\omega\in E_A^n} \phi_\omega(X_{t(\omega)} \setminus G_{t(\omega)}) \\ &= \bigcup_{\omega\in E_A^n} \phi_\omega(X_{t(\omega)}) \setminus \bigcup_{\omega\in E_A^n} \phi_\omega(G_{t(\omega)}) \\ &= X^n \setminus \bigcup_{\omega\in E_A^n} \phi_\omega(G_{t(\omega)}). \end{aligned}$$

Therefore by the open set condition and (8.23) we have for every $n \in \mathbb{N}$

$$\begin{aligned} |X^{n+n_0}| &\leq |X^n| - \left|\bigcup_{\omega\in E_A^n} \phi_\omega(G_{t(\omega)})\right| \\ &= |X^n| - \sum_{\omega\in E_A^n} |\phi_\omega(G_{t(\omega)})| \\ &= |X^n| - K^{-Q}\gamma \sum_{\omega\in E_A^n} |\phi_\omega(X_{t(\omega)})| \\ &\leq |X^n| - K^{-Q}\gamma|X^n| = (1 - K^{-Q}\gamma)|X^n|. \end{aligned} \tag{8.24}$$

Since X^n is a decreasing sequence of sets, (8.24) implies that

$$\lim_{n\to\infty} |X^n| = 0, \tag{8.25}$$

hence we also deduce that $|J_{\mathcal{S}}| = 0$, because $J_{\mathcal{S}} \subset X^n$ for all $n \in \mathbb{N}$. Thus the first part of the theorem is proven.

We will now prove the second part of the theorem by way of contradiction. To this end suppose that $\dim_{\mathcal{H}}(J_{\mathcal{S}}) = Q$. Since $\mathcal{S}$ is regular Theorem 7.19 implies that $\mathrm{P}(Q) = 0$. Hence by Proposition 7.5 $Q \in \mathrm{Fin}(\mathcal{S})$. Therefore for every $\omega \in E_A^*$ and every Borel set $A \subset X_{t(\omega)}$ such that $|A| > 0$, using Proposition 8.17, Theorem 8.5 and Theorem 3.4 we have,

$$\begin{aligned} m_Q(\phi_\omega(A)) &\leq \|D\phi_\omega\|^Q m_Q(A) = K^Q K^{-Q} \|D\phi_\omega\|^Q |A| \frac{m_Q(A)}{|A|} \\ &\leq K^Q |\phi_\omega(A)| \frac{m_Q(A)}{|A|}. \end{aligned} \tag{8.26}$$

For $\omega \in E_A^n$ we denote

$$Y_\omega = \phi_\omega(X_{t(\omega)}) \cap \bigcup_{\tau \in E_A^n \setminus \{\omega\}} \phi_\tau(X_{t(\tau)}).$$

Since $\mathcal{S}$ is of bounded coding type, by Theorem 8.5 we deduce that m_Q is of null boundary hence,

$$m_Q(Y_\omega) = 0. \tag{8.27}$$

Notice also that by the open set condition it is not difficult to see that

$$\phi_\omega^{-1}(Y_\omega) \subset \partial X. \tag{8.28}$$

Using (8.27) we get

$$\begin{aligned} m_Q(X^n) &\leq \sum_{\omega \in E_A^n} m_Q(\phi_\omega(X_{t(\omega)})) = \sum_{\omega \in E_A^n} m_Q(\phi_\omega(X_{t(\omega)}) \setminus Y_\omega) \\ &= \sum_{\omega \in E_A^n} m_Q(\phi_\omega(X_{t(\omega)} \setminus \phi_\omega^{-1}(Y_\omega))). \end{aligned}$$

Notice that (8.28) implies that $|X_{t(\omega)} \setminus \phi_\omega^{-1}(Y_\omega)| > 0$. Therefore by (8.26) and (8.28)

$$\begin{aligned} m_Q(X^n) &\leq \sum_{\omega \in E_A^n} m_Q(\phi_\omega(X_{t(\omega)} \setminus \phi_\omega^{-1}(Y_\omega))) \\ &\leq K^Q \sum_{\omega \in E_A^n} \frac{m_Q(X_{t(\omega)} \setminus \phi_\omega^{-1}(Y_\omega))}{|X_{t(\omega)} \setminus \phi_\omega^{-1}(Y_\omega)|} |\phi_\omega(X_{t(\omega)} \setminus \phi_\omega^{-1}(Y_\omega))| \\ &\leq K^Q \frac{m_Q(X)}{\min\{|\mathrm{Int} X_v| : v \in V\}} \sum_{\omega \in E_A^n} |\phi_\omega(X_{t(\omega)} \setminus \phi_\omega^{-1}(Y_\omega))|. \end{aligned}$$

Observe that

$$\phi_\omega(X_{t(\omega)} \setminus \phi_\omega^{-1}(Y_\omega)) \cap \phi_\tau(X_{t(\tau)} \setminus \phi_\tau^{-1}(Y_\tau)) = \emptyset$$

for $\omega, \tau \in E_A^n$ with $\omega \neq \tau$. Therefore, setting $\delta = K^Q \frac{m_Q(X)}{\min\{|\mathrm{Int} X_v| : v \in V\}}$, we have

$$
\begin{aligned}
m_Q(X^n) &\leq \delta \sum_{\omega \in E_A^n} |\phi_\omega(X_{t(\omega)} \setminus \phi_\omega^{-1}(Y_\omega))| = \delta \left| \bigcup_{\omega \in E_A^n} \phi_\omega(X_{t(\omega)} \setminus \phi_\omega^{-1}(Y_\omega)) \right| \\
&\leq \delta \left| \bigcup_{\omega \in E_A^n} \phi_\omega(X_{t(\omega)}) \right| = \delta |X^n|,
\end{aligned} \tag{8.29}
$$

for all $n \in \mathbb{N}$. Using (8.29) and (8.25) we deduce that $m_Q(J_{\mathcal{S}}) = 0$. Hence we have reached a contradiction, because $\mathcal{S}$ is regular and for example by (7.18), $m_Q(J_{\mathcal{S}}) > 0$. The proof of the theorem is complete. □

CHAPTER 9

Examples revisited

In this chapter we return to the examples described in Chapter 5. We illustrate the results of the previous chapters by discussing their implications for the invariant sets of the iterated function systems and graph directed Markov systems of that chapter. We provide computations of and estimates for two different Hausdorff dimensions of such invariant sets: the dimension with respect to the sub-Riemannian metric and the dimension with respect to the underlying Euclidean metric. Note that these invariant sets are defined by iteration of mappings which, while they are conformal in the sub-Riemannian sense, are no longer conformal in the Euclidean sense. From the Euclidean perspective the maps in question are quite general non-linear C^1 mappings.

Recall from section 1.3 that the Dimension Comparison Problem in sub-Riemannian Carnot groups asks for sharp comparison estimates relating the Hausdorff dimensions of sets with respect to the aforementioned two bi-Lipschitz inequivalent metrics. In the following sections we point out the computations and estimates for dimensions of invariant sets arising from the results of the previous chapters together with the Dimension Comparison Theorem.

9.1. Infinite self-similar iterated function systems

Let $\mathbb{G}$ be an arbitrary Carnot group and let $\mathcal{S} = \{\phi_e : \mathbb{G} \to \mathbb{G}\}_{e \in E}$ be a self-similar IFS with countably infinite alphabet E as in section 5.1. Assume that the open set condition is satisfied. Then

$$\dim_{\mathcal{H}} J_{\mathcal{S}} = \dim_{\mathcal{P}} J_{\mathcal{S}} = h,$$

where h denotes the similarity dimension for $\mathcal{S}$, i.e.,

$$h = \inf \left\{ t \geq 0 \, : \, \sum_{e \in E} r_e^t < 1 \right\}.$$

Here r_e denotes the contraction ratio for the similarity ϕ_e. In view of the Dimension Comparison Theorem, we further obtain the estimates

$$(\beta_+)^{-1}(h) \leq \dim_{\mathcal{H},E} J_{\mathcal{S}} \leq (\beta_-)^{-1}(h)$$

for the Euclidean Hausdorff dimension of $J_{\mathcal{S}}$. Here β_+ and β_- denote the dimension comparison functions for the Carnot group $\mathbb{G}$. These results were obtained in [9, Section 4] for finite alphabet self-similar IFS in Carnot groups; our primary contribution here is to extend these formulas and estimates to the case of countably infinite alphabets.

9.2. Continued fractions in groups of Iwasawa type

In this section we apply the results we obtained in Chapters 7 and 8 to the continued fractions systems that we introduced in Section 5.3. We remind the reader that we are studying continued fractions as limit sets of dynamical systems in Iwasawa groups. In particular for $\varepsilon \geq 0$ we consider the conformal iterated function systems

$$\mathcal{S}_\varepsilon = \left\{\phi_\gamma : \overline{B}\,(o,1/2) \to \overline{B}\,(o,1/2)\right\}_{\gamma\in I_\varepsilon}$$

where

- $I_\varepsilon = \mathbb{G}(\mathbb{Z}) \cap B(o,\Delta_\varepsilon)^c$,
- $\Delta_\varepsilon = \frac{5}{2} + \varepsilon$, and
- $\phi_\gamma = \mathcal{J} \circ \ell_\gamma$.

In our first theorem we calculate the θ-number for such systems.

Theorem 9.1. *Let $\mathbb{G}$ be a Carnot group of Iwasawa type and for $\varepsilon \geq 0$ let $\mathcal{S}_\varepsilon = \{\phi_\gamma\}_{\gamma\in I_\varepsilon}$ be the continued fraction conformal dynamical system. Then for all $\varepsilon \geq 0$,*

$$\theta_{\mathcal{S}_\varepsilon} = \frac{Q}{2}$$

and $\mathcal{S}_\varepsilon$ is co-finitely regular.

Proof. Fix $\varepsilon \geq 0$, and for simplifying notation set $I = I_\varepsilon$ and $\Delta = \Delta_\varepsilon$. Moreover without loss of generality we can assume that A_2 from Theorem 5.2 satisfies $A_2 \geq 1$. Let

$$I_0 = \{\gamma \in I : \Delta \leq d(\gamma,o) < \Delta + D\}$$

and for $k \in \mathbb{N}$, let

$$I_k = \{\gamma \in I : \Delta + D^k \leq d(\gamma,o) < \Delta + D^{k+1}\}$$

where $D = \max\{2\Delta, 10A_2\}$. Using (5.13) for $t \geq 0$,

$$Z_1(t) = \sum_{\gamma\in I} \|D\phi_\gamma\|_\infty^t \approx \sum_{k\geq 0}\sum_{\gamma\in I_k} d(\gamma,0)^{-2t} \approx \sum_{k\geq 0}\sum_{\gamma\in I_k} D^{-2kt}. \tag{9.1}$$

We will now fix some $k \in \mathbb{N}$, and we will estimate the cardinality of the sets I_k. First note that

$$\sharp I_k \leq \sharp\{\mathbb{G}(\mathbb{Z}) \cap B(o,\Delta + D^{k+1})\} := \sharp J_k.$$

By Theorem 5.2(ii) the balls $\{B(\gamma,1/2)\}_{\gamma\in\mathbb{G}(Z)}$ are pairwise disjoint, hence

$$\begin{aligned}
\sharp I_k\, 2^{-Q} &\leq \sharp J_k\, 2^{-Q} \\
&= \frac{1}{c_0}\sum_{\gamma\in J_k} |B(\gamma,1/2)| \\
&= \frac{1}{c_0}\left|\bigcup_{\gamma\in J_k} B(\gamma,1/2)\right| \\
&\leq \frac{1}{c_0}|B(o,\Delta + D^{k+1} + 1/2)| \approx D^{kQ},
\end{aligned}$$

where we remind the reader that $c_0 = |B(o,1)|$. Therefore

$$\sharp I_k \lesssim D^{kQ}. \tag{9.2}$$

We now turn our attention to the lower bound of $\sharp I_k$. Set

$$S_k = B(o, \Delta + D^{k+1} - A_2) \setminus \overline{B}(o, \Delta + D^k + A_2).$$

Recalling Theorem 5.2 it follows easily that

$$S_k \subset \bigcup_{\gamma \in I_k} B(\gamma, A_2).$$

Therefore $|S_k| \leq c_0 \sharp I_k \, (A_2)^Q$ and since $|S_k| \approx D^{kQ}$ we conclude that

$$\sharp I_k \gtrsim D^{kQ}. \tag{9.3}$$

Hence by (9.1), (9.2) and (9.3),

$$Z_1(t) \approx \sum_{k=0}^{\infty} (D^{Q-2t})^k,$$

and in view of Proposition 7.5(i), $\theta = Q/2$. Moreover $Z_1(Q/2) = \infty$ and recalling Definition 7.7 we conclude that the system is co-finitely regular. The proof is complete. □

We can now provide size estimates, on the level of Hausdorff dimension, for the limit sets of continued fraction conformal dynamical systems.

THEOREM 9.2. *Let $\mathbb{G}$ be a Carnot group of Iwasawa type and for $\varepsilon \geq 0$ let $\mathcal{S}_\varepsilon = \{\phi_\gamma\}_{\gamma \in I_\varepsilon}$ be the continued fraction conformal dynamical system. Then*

(1) for all $\varepsilon \geq 0$, $\dim_{\mathcal{H}} J_{\mathcal{S}_\varepsilon} > Q/2$,
(2) for all $\varepsilon > 0$, $\dim_{\mathcal{H}} J_{\mathcal{S}_\varepsilon} < Q$.

PROOF. Theorem 9.1 implies that for all $\varepsilon \geq 0$ the system $\mathcal{S}_\varepsilon$ is co-finitely regular. Therefore by Proposition 7.10, Theorem 7.19 and Theorem 9.1 we conclude that

$$\dim_{\mathcal{H}} J_{\mathcal{S}_\varepsilon} = h_{\mathcal{S}_\varepsilon} > \theta_{\mathcal{S}_\varepsilon} = \frac{Q}{2}.$$

For the proof of (2) it suffices to show that the systems $\mathcal{S}_\varepsilon$ satisfy the assumptions of Theorem 8.26 for $\varepsilon > 0$. Notice that for all $\varepsilon \geq 0$ by Example 8.8 the systems $\mathcal{S}_\varepsilon$ satisfy the weak corkscrew condition. Hence by Proposition 8.13 the systems $\mathcal{S}_\varepsilon$ are of bounding coding type. Moreover by Theorem 9.1 the systems $\mathcal{S}_\varepsilon$ are co-finitely regular, hence regular. Therefore in order to prove (2) it suffices to show that for $\varepsilon > 0$

$$\left| \overline{B}\,(o, 1/2) \setminus \bigcup_{\gamma \in I_\varepsilon} \phi_\gamma \left(\overline{B}\,(o, 1/2) \right) \right| > 0. \tag{9.4}$$

For $\gamma \in I_\varepsilon$ arguing as in (5.12) we have that

$$\phi_\gamma \left(\overline{B}\,(o, 1/2) \right) \subset \overline{B} \left(o, \frac{1}{2+\varepsilon} \right).$$

Hence we deduce that

$$\bigcup_{\gamma \in I_\varepsilon} \phi_\gamma \left(\overline{B}\,(o, 1/2) \right) \subset \overline{B} \left(o, \frac{1}{2+\varepsilon} \right),$$

and (9.4) follows. The proof of the theorem is complete. □

The following theorem concerns the spectrum of Hausdorff dimensions of the limit sets of subsystems of continued fractions conformal dynamical systems.

THEOREM 9.3. *Let $\mathbb{G}$ be a Carnot group of Iwasawa type and for $\varepsilon \geq 0$ let $\mathcal{S}_\varepsilon = \{\phi_\gamma\}_{\gamma \in I_\varepsilon}$ be the continued fraction conformal dynamical system. For every $t \in (0, Q/2)$ there exists a proper subsystem $\mathcal{S}_{\varepsilon,t}$ of $\mathcal{S}_\varepsilon$ such that*

$$\dim_{\mathcal{H}} J_{\mathcal{S}_{\varepsilon,t}} = t.$$

PROOF. The proof is a direct application of Theorem 7.26 and Theorem 9.2. □

We can also prove that in any Iwasawa group G there exist continued fractions conformal dynamical systems whose limit set has Hausdorff dimension arbitrarily close to $Q/2$.

THEOREM 9.4. *Let $\mathbb{G}$ be a Carnot group of Iwasawa type. Then there exists an increasing sequence $(R_n)_{n=1}^\infty$ with $R_n \to \infty$ such that*

$$\lim_{n \to \infty} \dim_{\mathcal{H}} J_{\mathcal{S}_{R_n}} = \frac{Q}{2}.$$

Here $\mathcal{S}_{R_n} = \{\phi_\gamma\}_{\gamma \in I_{R_n}}$ corresponds to the continued fractions conformal system as in (5.11).

PROOF. To simplify notation let $I_0 = I$ and if $T \subset I$ let $\mathcal{S}_{I \setminus T} = \{\phi_\gamma\}_{\gamma \in I \setminus T}$. By Theorem 7.23, Theorem 7.19 and Theorem 9.1 we have that

$$\frac{Q}{2} = \inf\{\dim_{\mathcal{H}} J_{\mathcal{S}_{I \setminus T}} : T \subset I \text{ finite}\}. \tag{9.5}$$

Therefore there exists an increasing sequence $(T_n)_{n=1}^\infty$ of finite subsets of I so that

$$\dim_{\mathcal{H}} J_{\mathcal{S}_{I \setminus T_n}} \to Q/2. \tag{9.6}$$

Moreover for every $n \in \mathbb{N}$ there exists some R_n such that $T_n \subset B(o, R_n)$, recalling Subsection 5.3.2 this implies that $T_n \subset B(o, \Delta_{R_n})$. Since $I_{R_n} = I \setminus T$ for some finite set $T \subset I$, (9.5) implies

$$\dim_{\mathcal{H}} J_{\mathcal{S}_{I \setminus T_n}} \geq \dim_{\mathcal{H}} J_{\mathcal{S}_{R_n}} \geq \frac{Q}{2}.$$

By (9.6) we deduce that $\dim_{\mathcal{H}} J_{\mathcal{S}_{R_n}} \to Q/2$ and the proof is complete. □

In view of the Dimension Comparison Theorem 1.2, Theorems 9.2, 9.3 and 9.4 have obvious Euclidean consequences. For instance, the following corollary is obtained by applying the estimates in Theorem 1.2 in connection with Theorem 9.2.

COROLLARY 9.5. *Let $\mathbb{G}$ be a Carnot group of Iwasawa type and for $\varepsilon \geq 0$ let $\mathcal{S}_\varepsilon = \{\phi_\gamma\}_{\gamma \in I_\varepsilon}$ be the continued fraction conformal dynamical system. Identify $\mathbb{G}$ with the Euclidean space $\mathbb{R}^N$ equipped with the Euclidean metric d_E. Then*

(1) for all $\varepsilon \geq 0$, $\dim_{\mathcal{H},E} J_{\mathcal{S}_\varepsilon} > (\beta_+)^{-1}(Q/2)$,
(2) for all $\varepsilon > 0$, $\dim_{\mathcal{H},E} J_{\mathcal{S}_\varepsilon} < N$.

Note that the value of $(\beta_+)^{-1}(Q/2)$ depends on the particular Iwasawa group $\mathbb{G}$. In fact, the choice of which expression in (1.12) is relevant is different depending on the group $\mathbb{G}$. In the complex Heisenberg groups $\mathbf{Heis}^n = \mathbf{Heis}^n_{\mathbb{C}}$ we always have

$$(\beta_+)^{-1}(\tfrac{Q}{2}) = \tfrac{Q}{2} - 1 = n,$$

while in the first octonionic Heisenberg group $\mathbf{Heis}^1_{\mathbb{O}}$ we have

$$(\beta_+)^{-1}(\tfrac{Q}{2}) = \tfrac{Q}{4} = \tfrac{11}{2}.$$

In the quaternionic Heisenberg groups $\mathbf{Heis}^n_{\mathbb{H}}$ the answer depends on the value of n:

$$(\beta_+)^{-1}(\tfrac{Q}{2}) = \begin{cases} \frac{Q}{2} - 1 = 4 & \text{in } \mathbf{Heis}^1_{\mathbb{H}}, \\ \frac{Q}{4} = n + \frac{3}{2} & \text{in } \mathbf{Heis}^n_{\mathbb{H}},\ n \geq 2. \end{cases}$$

REMARK 9.6. Estimates for the Euclidean dimensions of Carnot–Carathéodory self-similar sets were previously obtained in [**7**] and [**9**].

9.3. Iwasawa conformal Cantor sets

In this section we return to the conformal Cantor sets that where introduced in Section 5.2. Recall that conformal Cantor sets are limit sets of conformal iterated function systems

$$\mathcal{S} = \{\phi_n : \overline{G} \to \overline{G}\}_{n \in \mathbb{N}}$$

where

$$\phi_n = \ell_{p_n} \circ \delta_{r_n} \circ \ell_{\mathcal{J}(p_n)^{-1}} \circ \mathcal{J},$$

and

- G is an open set, $\overline{G}$ is compact and $o \notin G$,
- $P = (p_n)_{n=1}^{\infty}$ is a discrete sequence of points in G,
- $d_n = \inf_{m \neq n} d(p_n, p_m)$ and $\lim_{n\to\infty} d_n = 0$,
- $\operatorname{dist}(P, \partial G) > \sup_{n\in\mathbb{N}} d_n$,
- $\mathrm{d}_0 := \operatorname{diam} \mathcal{J}(\mathrm{G})$, and $r_n < \min\{s, \frac{d_n}{2\mathrm{d}_0}\}$ for some $s < 1$.

For all $n \in \mathbb{N}$ and $p \in G$, by (3.6) and (3.5)

$$\|D\phi_n(p)\| = r_n \|D\mathcal{J}(p)\| = \frac{r_n}{d(p,o)^2}.$$

Hence $\|D\phi_n\|_\infty \approx r_n$ where the constant depends only on the open set G. Therefore if $\mathcal{S}$ is a conformal iterated function system as above, by Proposition 7.5

$$\theta_{\mathcal{S}} = \inf \left\{ t \geq 0 : \sum_{n\in\mathbb{N}} r_n^t < \infty \right\}. \tag{9.7}$$

We now describe a specific type of conformal Cantor sets in Iwasawa groups and provide estimates for their Hausdorff dimensions.

Let $\varepsilon > 1$ and for $n \in \mathbb{N}$ set

$$\Sigma_n = \partial B \left(o, \sum_{j=1}^{n} \frac{1}{j^\varepsilon} \right).$$

Denote by Π_n the maximal collection of points in Σ_n with mutual distances at least $\left(\frac{1}{n+2}\right)^{\varepsilon}$. It is well known, see e.g. [**22**] and [**27**], that the $(Q-1)$-dimensional spherical Hausdorff measure $\mathcal{S}^{Q-1}$ restricted to the boundaries of Korányi balls is

Ahlfors $(Q-1)$-regular, recall (7.19) for the definition of Ahlfors regularity. Since for all $n \in \mathbb{N}$ the restriction of $\mathcal{S}^{Q-1}$ on Σ_n is Ahlfors $(Q-1)$-regular one can show that

$$\sharp \Pi_n \approx \left(\frac{1}{n+2}\right)^{\varepsilon(1-Q)}. \tag{9.8}$$

Let $\Pi = \bigcup_{n \in \mathbb{N}} \Pi_n$. We will now show that if $p \in \Pi_n$ then

$$\left(\frac{1}{n+2}\right)^{\varepsilon} \leq \inf_{q \in \Pi \setminus \{p\}} d(p,q) \leq 4 \left(\frac{1}{n+2}\right)^{\varepsilon}. \tag{9.9}$$

For the right hand side inequality first notice that

$$\Sigma_n \subset \bigcup_{q \in \Pi_n} B\left(q, 2\left(\frac{1}{n+2}\right)^{\varepsilon}\right).$$

Now fix some $p \in \Pi_n$. Since Σ_n is a connected metric space with the relative topology of d it follows that $B\left(p, 2\left(\frac{1}{n+2}\right)^{\varepsilon}\right)$ intersects some set from the family $\left\{\Sigma_n \cap B\left(q, 2\left(\frac{1}{n+2}\right)^{\varepsilon}\right)\right\}_{q \in \Pi_n \setminus \{p\}}$ of (relative) open sets. Therefore there exists $q \in \Pi_n$ such that

$$\Sigma_n \cap B\left(p, 2\left(\frac{1}{n+2}\right)^{\varepsilon}\right) \cap B\left(q, 2\left(\frac{1}{n+2}\right)^{\varepsilon}\right) \neq \emptyset.$$

In particular

$$d(p,q) \leq 4\left(\frac{1}{n+2}\right)^{\varepsilon}$$

and the desired inequality follows.

For the remaining inequality we will first show that for all $n \in \mathbb{N}$

$$\operatorname{dist}(\Sigma_n, \Sigma_{n+1}) \geq \left(\frac{1}{n+1}\right)^{\varepsilon}. \tag{9.10}$$

To prove (9.10) suppose by way of contradiction that there exist $x_n \in \Sigma_n$ and $x_{n+1} \in \Sigma_{n+1}$ such that $d(x_n, x_{n+1}) < \left(\frac{1}{n+1}\right)^{\varepsilon}$. Then

$$d(x_{n+1}, o) \leq d(x_n, x_{n+1}) + d(x_n, o) < \sum_{j=1}^{n} \frac{1}{j^{\varepsilon}} + \left(\frac{1}{n+1}\right)^{\varepsilon} = \sum_{j=1}^{n+1} \frac{1}{j^{\varepsilon}}$$

and we have reached a contradiction since $x_{n+1} \in \Sigma_{n+1}$. Therefore by (9.10)

$$\inf_{q \in \Pi \setminus \{p\}} d(p,q) = \inf_{q \in \Pi_n \setminus \{p\}} d(p,q) \geq \left(\frac{1}{n+2}\right)^{\varepsilon}$$

and (9.9) follows.

We remark that if we write $\Pi = (p_k)_{k=1}^{\infty}$ and set

$$d_k = \inf_{l \neq k} d(p_k, p_l)$$

then (9.9) implies that

$$\lim_{k \to \infty} d_k = 0.$$

We can now define the CIFS as in Section 5.2 . Let

$$G_\varepsilon = \left\{ B\left(o, \sum_{n\in\mathbb{N}} \frac{1}{n^\varepsilon} + 1\right) \setminus \overline{B}\left(o, \frac{1}{2}\right) \right\}.$$

For $k \in \mathbb{N}$ we define the conformal maps $\phi_k : \overline{G_\varepsilon} \to \overline{G_\varepsilon}$ by

$$\phi_k = \ell_{p_k} \circ \delta_{d_k/(10\mathrm{d}_0)} \circ \ell_{\mathcal{J}(p_k)^{-1}} \circ \mathcal{J}, \tag{9.11}$$

where as before $\mathrm{d}_0 = \operatorname{diam} \mathcal{J}(\mathrm{G})$, and we set $\mathcal{S}_\varepsilon = \{\phi_k : \overline{G_\varepsilon} \to \overline{G_\varepsilon}\}_{k\in\mathbb{N}}$.

We will now verify that $\mathcal{S}_\varepsilon$ satisfies the open set condition. Let $p_k, p_l \in \Pi$, $p_k \neq p_l$, and let $n_l, n_k \in \mathbb{N}$ such that $p_k \in \Pi_{n_k}$ and $p_l \in \Pi_{n_l}$. If $n_l \neq n_k$, assume without loss of generality that $n_l > n_k$ and notice that

$$d(p_k, p_l) \geq \operatorname{dist}(\Sigma_{n_k}, \Sigma_{n_l}) \geq \operatorname{dist}(\Sigma_{n_k}, \Sigma_{n_k+1}) \geq \left(\frac{1}{n_k+1}\right)^\varepsilon.$$

But by (9.9)

$$\operatorname{diam}(\phi_k(G_\varepsilon)) = \frac{d_k}{10\mathrm{d}_0} \operatorname{diam}(\mathcal{J}(G_\varepsilon)) \leq \frac{4}{10}\left(\frac{1}{n_k+2}\right)^\varepsilon. \tag{9.12}$$

Moreover $\operatorname{diam}(\phi_l(\overline{G_\varepsilon})) < \operatorname{diam}(\phi_k(\overline{G_\varepsilon}))$ because $n_l > n_k$. Hence by (9.12) we deduce that $\phi_k(G_\varepsilon) \cap \phi_l(G_\varepsilon) = \emptyset$. If $n_l = n_k := n$ then

$$d(p_l, p_k) \geq \left(\frac{1}{n+2}\right)^\varepsilon.$$

As in (9.12) we have that

$$\operatorname{diam}(\phi_k(G_\varepsilon)) + \operatorname{diam}(\phi_l(G_\varepsilon)) \leq \frac{8}{10}\left(\frac{1}{n+2}\right)^\varepsilon.$$

Hence for all $k, l \in \mathbb{N}, k \neq l$,

$$\phi_k(G_\varepsilon) \cap \phi_l(G_\varepsilon) = \emptyset.$$

To finish the proof of the open set condition it remains to show that for all $k \in \mathbb{N}$, $\phi_k(G_\varepsilon) \subset G_\varepsilon$. First notice that by (9.12) for all $k \in \mathbb{N}$, $\operatorname{diam}(\phi_k(G_\varepsilon)) < 1/2$ and there exist $n_k \in \mathbb{N}$ such that $p_k \in \Sigma_{n_k}$, hence for all $k \in \mathbb{N}$,

$$1 \leq d(p_k, o) < \sum_{n\in\mathbb{N}} \frac{1}{n^\varepsilon}.$$

Therefore if $x \in \phi_k(G_\varepsilon)$, then

$$d(x, o) \leq d(x, p_k) + d(p_k, o) < \sum_{n\in\mathbb{N}} \frac{1}{n^\varepsilon} + 1/2,$$

and in the same way

$$d(x, o) \geq d(p_k, o) - d(x, p_k) > 1/2.$$

Therefore the conformal iterated function system $\mathcal{S}_\varepsilon$ satisfies the open set condition.

The following theorem gathers information about the Hausdorff dimension of the Cantor sets $J_{\mathcal{S}_\varepsilon}$. Note that by part (iv) of the following theorem, it follows that every value $t \in (0, Q)$ arises as the Hausdorff dimension of the invariant set of some subsystem of a GDMS associated to a conformal Cantor set.

THEOREM 9.7. *Let $\mathbb{G}$ be a Carnot group of Iwasawa type. Let $\varepsilon > 1$ and consider the conformal iterated function system $\mathcal{S}_\varepsilon$ defined by the conformal maps ϕ_k as in (9.11). Then*

(i) $\theta_{\mathcal{S}_\varepsilon} = Q - \frac{\varepsilon-1}{\varepsilon}$,
(ii) the system $\mathcal{S}_\varepsilon$ is co-finitely regular,
(iii) $\dim_{\mathcal{H}} J_{\mathcal{S}_\varepsilon} > Q - \frac{\varepsilon-1}{\varepsilon}$,
(iv) for all $t \in (0, Q - \frac{\varepsilon-1}{\varepsilon})$ there exists a proper subsystem $\mathcal{S}_{\varepsilon,t}$ of $\mathcal{S}_\varepsilon$ such that

$$\dim_{\mathcal{H}} J_{\mathcal{S}_{\varepsilon,t}} = t.$$

PROOF. First observe that once we have shown (i) and (ii) then the remaining statements of the theorem follow easily. Indeed, (iii) follows by (i), (ii), Proposition 7.10, Theorem 7.19 and Theorem 9.1, while (iv) follows from (i) and Theorem 7.26.

It remains to show (i) and (ii). Recalling (9.7) and the way that the maps ϕ_k were defined, we have

$$\theta_{\mathcal{S}_\varepsilon} = \inf \left\{ t \geq 0 : \sum_{k \in \mathbb{N}} \left(\frac{d_k}{10 \mathrm{d}_0} \right)^t < \infty \right\}.$$

By (9.8) and (9.9)

$$\begin{aligned} \sum_{k \in \mathbb{N}} \left(\frac{d_k}{10 \mathrm{d}_0} \right)^t &\approx \sum_{n \in \mathbb{N}} \sum_{k \in \Pi_n} d_k^t \approx \sum_{n \in \mathbb{N}} \sum_{k \in \Pi_n} \left(\frac{1}{n+2} \right)^{\varepsilon t} \\ &\approx \sum_{n \in \mathbb{N}} \left(\frac{1}{n+2} \right)^{\varepsilon(1-Q)} \left(\frac{1}{n+2} \right)^{\varepsilon t} \\ &= \sum_{n \in \mathbb{N}} \left(\frac{1}{n+2} \right)^{\varepsilon(1-Q+t)}. \end{aligned} \tag{9.13}$$

Hence in view of Proposition 7.5(i), $\theta_{\mathcal{S}_\varepsilon} = Q - \frac{\varepsilon-1}{\varepsilon}$. Moreover by (9.13), we easily see that

$$Z_1 \left(Q - \frac{\varepsilon - 1}{\varepsilon} \right) = \infty.$$

Recalling Definition 7.7 we conclude that the system $\mathcal{S}_\varepsilon$ is co-finitely regular. The proof is complete. □

REMARK 9.8. Notice that the Hausdorff dimension of the conformal Cantor sets $J_{\mathcal{S}_\varepsilon}$ can be arbitrarily close to Q, since

$$\lim_{\varepsilon \to 1} \dim_{\mathcal{H}} J_{\mathcal{S}_\varepsilon} = Q$$

by Theorem 9.7.

The following corollary of Theorem 9.7 can be established using the Dimension Comparison Theorem 1.2.

COROLLARY 9.9. *Let $\mathbb{G}$ be a Carnot group of Iwasawa type. Let $\varepsilon > 1$ and consider the conformal iterated function system $\mathcal{S}_\varepsilon$ defined by the conformal maps ϕ_k as in (9.11). Then the Euclidean Hausdorff dimension of the invariant set of $\mathcal{S}_\varepsilon$ satisfies*

$$\dim_{\mathcal{H},E} J_{\mathcal{S}_\varepsilon} > N - \frac{\varepsilon - 1}{\varepsilon}.$$

CHAPTER 10

Finer properties of limit sets: Hausdorff, packing and invariant measures

In this chapter we investigate finer properties of limit sets of Carnot conformal GDMS. In particular we are concerned with the positivity of Hausdorff and packing measures of limit sets, as well as the Hausdorff dimension of invariant measures. Under some mild assumptions on the GDMS we can prove that the h-Hausdorff measure of the limit set is finite and under some standard separation condition we can show that the h-packing measure is positive. This is performed in Section 10.1. Deciding whether the h-Hausdorff measure is positive or if the h-packing measure is finite are subtler questions. In Section 10.2 we provide necessary and sufficient conditions for the Hausdorff measure of the limit set to be positive and for its packing measure to be finite. In Section 10.3 we use results from Section 10.2 to further explore the Iwasawa continued fractions that were introduced in Section 5.3. Finally in Section 10.4 we obtain a dynamical formula for the Hausdorff dimension of invariant measures of limit sets of Carnot conformal GDMS.

10.1. Finiteness of Hausdorff measure and positivity of packing measure

The next theorem shows that under some general conditions the h-Hausdorff measure of the limit set of a Carnot conformal GDMS is finite.

THEOREM 10.1. *Let $\mathcal{S}$ be a finitely irreducible weakly Carnot conformal GDMS.*

(i) If the system is regular, then
  - *(a) the restriction of the Hausdorff measure to $J_{\mathcal{S}}$, i.e. $\mathcal{H}^h\lfloor J_{\mathcal{S}}$, is absolutely continuous with respect to the conformal measure m_h, and*
  - *(b) $\|d(\mathcal{H}^h\lfloor J_{\mathcal{S}})/dm_h\|_\infty < \infty$.*
  - *(c) In particular, $\mathcal{H}^h(J_{\mathcal{S}}) < \infty$.*

(ii) If the system is not regular then $\mathcal{H}^h(J_{\mathcal{S}}) = 0$.

PROOF. Let A be an arbitrary closed subset of $J_{\mathcal{S}}$. For every $n \in \mathbb{N}$ put

$$A_n := \{\omega \in E_A^n : \phi_\omega(J_{\mathcal{S}}) \cap A \neq \emptyset\}.$$

Then the sequence of sets

$$\left(\bigcup_{\omega \in A_n} \phi_\omega(X_{t(\omega)})\right)_{n=1}^\infty$$

is descending and

$$\bigcap_{n \geq 1} \Big(\bigcup_{\omega \in A_n} \phi_\omega(X_{t(\omega)})\Big) = A.$$

Notice also that since for every $\omega \in E_A^*$ $\pi([\omega]) \subset \phi_\omega(X_{t(\omega)})$,

$$\begin{aligned}\sum_{\omega\in A_n} \tilde{m}_h([\omega]) &= \tilde{m}_h\big(\bigcup_{\omega\in A_n}[\omega]\big) = m_h\big(\bigcup_{\omega\in A_n}\pi^{-1}([\omega])\big) \\ &\leq m_h\big(\bigcup_{\omega\in A_n}\phi_\omega(X_{t(\omega)})\big).\end{aligned} \tag{10.1}$$

Using (4.9), (7.8) and (10.1) we obtain

$$\begin{aligned}\mathcal{H}^h(A) &\leq \liminf_{n\to\infty}\sum_{\omega\in A_n}\big(\mathrm{diam}(\phi_\omega(X_{t(\omega)}))\big)^h \\ &\leq (M\Lambda)^h \liminf_{n\to\infty}\sum_{\omega\in A_n}||D\phi_\omega||_\infty^h \\ &\leq c_h(M\Lambda)^h \liminf_{n\to\infty} e^{n\mathrm{P}(h)}\sum_{\omega\in A_n}\tilde{m}_h([\omega]) \\ &\leq c_h(M\Lambda)^h \liminf_{n\to\infty} e^{n\mathrm{P}(h)}\left(m_h\big(\bigcup_{\omega\in A_n}\phi_\omega(X_{t(\omega)})\big)\right).\end{aligned} \tag{10.2}$$

If $\mathcal{S}$ is not regular then $\mathrm{P}(h) < 0$ and by (10.2) we get that $\mathcal{H}^h(J_\mathcal{S}) = 0$. On the other hand if $\mathcal{S}$ is regular we have that $\mathrm{P}(h) = 0$ hence (10.2) gives that

$$\mathcal{H}^h(A) \leq (M\Lambda)^h c_h m_h(A).$$

Since the metric space $J_\mathcal{S}$ is separable, the measure m_h is regular and consequently the inequality $\mathcal{H}^h(A) \leq (M\Lambda)^h c_h^{-1} m_h(A)$ extends to all Borel subsets A of $J_\mathcal{S}$. Thus the proof is finished. □

Definition 10.2. We say that the Carnot conformal GDMS $\mathcal{S}$ satisfies the *strong open set condition* (SOSC) if

$$J_\mathcal{S} \cap \mathrm{Int}(X) \neq \emptyset. \tag{10.3}$$

Recall that the set X was defined in (4.4).

Assuming the strong open set condition we can show in a straightforward manner that the limit set of a finitely irreducible regular conformal GDMS has positive h-packing measure.

Proposition 10.3. *If $\mathcal{S}$ is a finitely irreducible, regular Carnot conformal GDMS satisfying the strong open set condition, then $\mathcal{P}^h(J_\mathcal{S}) > 0$.*

Proof. Let $v \in V$ such that $J_\mathcal{S} \cap \mathrm{Int}(X_v) \neq \emptyset$. Then there exists some $\tau \in E_A^q$ for some $q \in \mathbb{N}$ such that $i(\tau) = v$ and $\phi_\tau(X_{t(\tau)}) \subset \mathrm{Int}(X_v)$. Set

$$\gamma = \min\{\mathrm{dist}(\phi_\tau(X_{t(\tau)}), \partial X), \eta_\mathcal{S}\},$$

where $\eta_\mathcal{S}$ was defined in (4.10). Let

$$\mathcal{W} = \{\omega \in E_A^\mathbb{N} : \omega|_{[n+1,n+q]} = \tau \text{ for infinitely many } n's\}$$

and let $\mathcal{W}_0$ be the subset of $E_A^\mathbb{N}$ whose elements do not contain τ as a subword. Since $[\tau] \cap \mathcal{W}_0 = \emptyset$ we conclude that $\tilde{\mu}_h(\mathcal{W}_0) < 1$. Recall that by Theorem 7.4 $\tilde{\mu}_h$ is ergodic with respect to σ. Ergodicity implies that for any Borel set $S \subset E_A^\mathbb{N}$ such that $\sigma^{-1}(S) \subset S$, $\tilde{\mu}_h(S) \in \{0,1\}$. To see this, let S be a Borel subset of $E_A^\mathbb{N}$ such that $\sigma^{-1}(S) \subset S$. Since σ is $\tilde{\mu}_h$-measure preserving, $\tilde{\mu}_h(\sigma^{-1}(S)) = \tilde{\mu}_h(S)$. Hence

$$\tilde{\mu}_h(\sigma^{-1}(S)\triangle S) = \tilde{\mu}_h(S) - \tilde{\mu}_h(T^{-1}(S)) = 0,$$

and by the ergodicity of $\tilde{\mu}_h$ we conclude that $\tilde{\mu}_h(S) \in \{0,1\}$. Now notice that $\sigma^{-1}(E_A^{\mathbb{N}} \setminus \mathcal{W}_0) \subset E_A^{\mathbb{N}} \setminus \mathcal{W}_0$. Hence by the previous observation and the fact that $\tilde{\mu}_h(E_A^{\mathbb{N}} \setminus \mathcal{W}_0) > 0$ we deduce that $\tilde{\mu}_h(\mathcal{W}_0) = 0$. It is also easy to see that $E_A^{\mathbb{N}} \setminus \mathcal{W} = \cup_{n\in\mathbb{N}}\sigma^{-n}(\mathcal{W}_0)$. Therefore, since $\tilde{\mu}_h$ is shift-invariant, we deduce that $\tilde{\mu}_h(E_A^{\mathbb{N}} \setminus \mathcal{W}) = 0$ and consequently

$$\mu_h(J_{\mathcal{S}} \setminus \pi(\mathcal{W})) = \tilde{\mu}_h \circ \pi^{-1}(J_{\mathcal{S}} \setminus \pi(\mathcal{W})) \leq \tilde{\mu}_h(E_A^{\mathbb{N}} \setminus \mathcal{W}) = 0.$$

Now for any $\omega \in \mathcal{W}$ and $n \in \mathbb{N}$ such that $\omega|_{[n+1,n+q]} = \tau$ by (4.7) we have that

$$B(\pi(\omega),(KC)^{-1}\|D\phi_{\omega|_n}\|_\infty\,\gamma) \subset \phi_{\omega|_n}(B(\pi(\sigma^n(\omega)),\gamma)).$$

Moreover by the choice of γ,

$$B(\pi(\sigma^n(\omega)),\gamma) \subset B(\phi_t(X_{t(\tau)}),\gamma) \subset \mathrm{Int}(X_{i(\tau)}) = \mathrm{Int}(X_{t(\omega|_n)}).$$

Hence by Remark 8.18;

$$\begin{aligned} m_h(B(\pi(\omega),(KC)^{-1}\|D\phi_{\omega|_n}\|_\infty\,\gamma)) &\leq \|D\phi_{\omega|_n}\|_\infty^h m_h(B(\pi(\sigma^n(\omega)),\gamma)) \\ &\leq (KC\gamma^{-1})^h\,((KC)^{-1}\|D\phi_{\omega|_n}\|_\infty\gamma)^h. \end{aligned}$$

Since by Theorem 7.4 m_h and μ_h are equivalent, the proof is concluded by invoking Theorem [**46**, A2.0.13(1)]. □

10.2. Positivity of Hausdorff measure and finiteness of packing measure

Let (X,ρ) be a metric space. Let ν be a finite Borel measure on X. Fix $s>0$. We say that the measure ν is *upper geometric* with exponent s if

$$\nu(B(x,r)) \leq c_\nu\, r^s \tag{10.4}$$

for all $x \in X$, all radii $r \geq 0$ and some constant $c_\nu \in [0,+\infty)$. Likewise, we say that the measure ν is *lower geometric* with exponent s if

$$\nu(B(x,r)) \geq c_\nu\, r^s \tag{10.5}$$

for all $x \in X$, all radii $r \in [0,1]$ and some constant $c_\nu \in (0,+\infty]$. If ν is both upper geometric and lower geometric with the same exponent $s>0$, it is called *geometric* with exponent s. Geometric measures with exponent s are also frequently referred to as Ahlfors s-regular measures. If we do not care at a moment about the particular value of the exponent s, we simply refer to the aforementioned measures as upper geometric, lower geometric, geometric, or Ahlfors regular measures.

Definition 10.4. A set $X \subset \mathbb{G}$ is said to satisfy the *boundary regularity condition* if there exists a constant $\gamma_X \in (0,1]$ such that

$$|\mathrm{Int}(X) \cap B(x,r)| \geq \gamma_X |B(x,r)|$$

for all $x \in X$ and all radii $r \in (0, \mathrm{diam}(X))$.

Alternatively, for all $t>0$ there exists $\gamma_{X,t}$ such that

$$|\mathrm{Int}(X) \cap B(x,r)| \geq \gamma_{X,t} |B(x,r)|$$

for all $x \in X$ and all radii $r \in (0,t)$.

Definition 10.5. A Carnot conformal GDMS $\mathcal{S} = \{\phi_e\}_{e\in E}$ is said to be *boundary regular* if each set X_v, $v \in V$, satisfies the boundary regularity condition. We put $\gamma := \min\{\gamma_{X_v} : v \in V\}$ and $\gamma_t := \min\{\gamma_{X_v,t} : v \in V\}$ for all $t>0$.

REMARK 10.6. Every Carnot conformal GDMS satisfying the corkscrew condition is boundary regular.

For boundary regular Carnot conformal GDMS we shall prove now the following improvement of Lemma 7.15.

LEMMA 10.7. *If $\mathcal{S}$ is a boundary regular Carnot conformal GDMS, then for all $\kappa > 0$, for all $r > 0$ and for all $x \in X$, the cardinality of any collection of mutually incomparable words $\omega \in E_A^*$ that satisfy the conditions*

$$B(x,r) \cap \phi_\omega(X_{t(\omega)}) \neq \emptyset \tag{10.6}$$

and

$$\operatorname{diam}(\phi_\omega(X_{t(\omega)})) \geq \kappa r, \tag{10.7}$$

is bounded above by $(3\Lambda K)^Q \gamma^{-1}_{\Lambda M \kappa^{-1}}$.

PROOF. Let F be any collection of A-admissible words satisfying the hypotheses of our lemma. By (10.7) and (4.9) we get that

$$||D\phi_\omega||_\infty^{-1} r \leq \Lambda M \kappa^{-1}$$

for every $\omega \in F$. Again by hypotheses, for every $\omega \in F$ there exists $p_\omega \in \operatorname{Int}(X_{t(\omega)})$ such that $\phi_\omega(p_\omega) \in B(x,r)$. By the open set condition all the sets

$$\phi_\omega\big(B(p_\omega, ||D\phi_\omega||_\infty^{-1} r) \cap \operatorname{Int}(X_{t(\omega)})\big),\ \omega \in F,$$

are mutually disjoint. Also by (4.8)

$$\operatorname{diam}\big(\phi_\omega\big(B(p_\omega, ||D\phi_\omega||_\infty^{-1} r) \cap \operatorname{Int}(X_{t(\omega)})\big)\big) \leq 2\Lambda r.$$

Therefore, since $\phi_\omega(B(p_\omega)) \in B(x,r)$,

$$\phi_\omega\big(B(p_\omega, ||D\phi_\omega||_\infty^{-1} r) \cap \operatorname{Int}(X_{t(\omega)})\big) \subset B(x, 3\Lambda r).$$

Therefore, by Theorem 3.4

$$\begin{aligned}
c_0(3\Lambda r)^Q = |B(x,3\Lambda r)| &\geq \left| \bigcup_{\omega \in F} \phi_\omega\big(B(p_\omega, ||D\phi_\omega||_\infty^{-1} r) \cap \operatorname{Int}(X_{t(\omega)})\big) \right| \\
&= \sum_{\omega \in F} |\phi_\omega\big(B(p_\omega), ||D\phi_\omega||_\infty^{-1} r) \cap \operatorname{Int}(X_{t(\omega)})| \\
&\geq \sum_{\omega \in F} K^{-Q} ||D\phi_\omega||_\infty^Q \, |B(p_\omega, ||D\phi_\omega||_\infty^{-1} r) \cap \operatorname{Int}(X_{t(\omega)})| \\
&\geq \sum_{\omega \in F} K^{-Q} \gamma_{\Lambda M \kappa^{-1}} ||D\phi_\omega||_\infty^Q \, |B(p_\omega, ||D\phi_\omega||_\infty^{-1} r)| \\
&= c_0\, K^{-Q} \gamma_{\Lambda M \kappa^{-1}} \sum_{\omega \in F} ||D\phi_\omega||_\infty^Q (||D\phi_\omega||_\infty^{-1} r)^Q \\
&= c_0 K^{-Q} \gamma_{\Lambda M \kappa^{-1}} \# F r^Q
\end{aligned}$$

Therefore $\# F \leq (3\Lambda K)^Q \gamma^{-1}_{\Lambda M \kappa^{-1}}$ and the proof is complete. ☐

The first main result of this section gives necessary and sufficient conditions for the Hausdorff measure of $J_\mathcal{S}$ to be positive. In particular it shows that $\mathcal{H}^h(J_\mathcal{S})$ is positive if and only if the conformal measure m_h is upper geometric with exponent h.

THEOREM 10.8. *If $\mathcal{S}$ is a maximal regular Carnot conformal GDMS satisfying the corkscrew condition, then the following conditions are equivalent.*

(i) $\mathcal{H}_h(J_{\mathcal{S}}) > 0$.

(ii) *There exists $H > 0$ such that*

$$m_h(B(y,r)) \leq Hr^h$$

for every $e \in E$, every radius $r \geq \operatorname{diam}(\phi_e(X_{t(e)}))$, and every $y \in \phi_e(X_{t(e)})$.

(iii) *There exist $H > 0$ and $\gamma \geq 1$ such that for every $e \in E$ and every radius $r \geq \gamma \operatorname{diam}(\phi_e(X_{t(e)}))$ there exists $y \in \phi_e(X_{t(e)})$ such that*

$$m_h(B(y,r)) \leq Hr^h.$$

(iv) *The measure m_h is upper geometric with exponent h. More precisely, there exists $c_{\mathcal{S}} > 0$ such that*

$$m_h(B(x,r)) \leq c_{\mathcal{S}}\, r^h$$

for every $x \in J_{\mathcal{S}}$ and all $r \in [0, +\infty)$.

PROOF. (i) $\Rightarrow$ (ii). In order to prove this implication suppose that (ii) fails. Then for every $H > \eta_{\mathcal{S}}^{-h}$, where $\eta_{\mathcal{S}}$ is as in (4.10), there exists $j \in E$ such that

$$m_h(B(x,r)) > Hr^h$$

for some $x \in \phi_j(X_{t(j)})$ and some $r \geq \operatorname{diam}(\phi_j(X_{t(j)}))$. Let $E_A^{\mathbb{N}}(\infty)$ be the set of words in $E_A^{\mathbb{N}}$ that contain each element of E infinitely often. Let

$$J_1 := \pi(E_A^{\mathbb{N}}(\infty)).$$

Fix $\omega \in E_A^{\mathbb{N}}(\mathbb{N})$ and put $z := \pi(\omega)$. Then $\omega_{n+1} = j$ for some $n \in \mathbb{N}$. Set $z_n = \pi(\sigma^n(\omega))$. So, $z = \phi_{\omega|_n}(z_n)$ and . Therefore $z_n, x \in \phi_j(X_{t(j)}) \subset X_{t(\omega_n)}$ and $z_n \in \overline{B}(x,r)$, hence by (4.8)

$$d(\phi_{\omega|_n}(z_n), \phi_{\omega|_n}(x)) \leq \Lambda ||D\phi_{\omega|_n}||_\infty\, r.$$

Moreover notice that $r \leq 1/H^{1/h} \leq \eta_{\mathcal{S}}$, hence by Corollary 4.10 we get

$$B(\phi_{\omega|_n}(x), C\, ||D\phi_{\omega|_n}||_\infty\, r) \supset \phi_{\omega|_n}(B(x,r)).$$

Thus,

$$B\big(z, (C+\Lambda)||D\phi_{\omega|_n}||_\infty\, r\big) \supset \phi_{\omega|_n}(B(x,r)). \tag{10.8}$$

Since $r \leq \eta_{\mathcal{S}}$ and $\mathcal{S}$ is maximal by (10.8) and Remark 8.25 we get

$$\begin{aligned} m_h\big(B(z,(C+\Lambda)||D\phi_{\omega|_n}||_\infty\, r)\big) &\geq K^{-h}||D\phi_{\omega|_n}||_\infty^h m_h(B(x,r)) \\ &> K^{-h}H||D\phi_{\omega|_n}||_\infty^h r^h \\ &= \frac{H}{((C+\Lambda)K)^h}\big((C+\Lambda)||D\phi_{\omega|_n}||_\infty\, r\big)^h. \end{aligned}$$

Hence, by [**46**, Theorem A2.0.12],

$$\mathcal{H}_h(J_1) \lesssim H^{-1},$$

with constants independent of H. Now, letting $H \to \infty$ we conclude that $\mathcal{H}_h(J_1) = 0$. By Theorem 7.4, Birkhoff's Ergodic Theorem, and (7.18), using a standard argument it follows that $m_h(J \setminus J_1) = 0$. This in turn, in view of Theorem 10.1, shows that $\mathcal{H}_h(J \setminus J_1) = 0$. Thus, $\mathcal{H}_h(J) = 0$ and therefore the proof of the implication (i)$\Rightarrow$(ii) is finished.

The implication (ii)⇒(iii) is obvious.

(iii)⇒(iv). Increasing Λ or M if necessary, we may assume that $\Lambda M\tilde{R}_{\mathcal{S}}^{-1} \geq 1$, where $\tilde{R}_{\mathcal{S}}$ was defined in Lemma 4.13. Take an arbitrary point $x \in J_{\mathcal{S}}$ and radius $r > 0$. Set

$$\tilde{r} = 2KC\Lambda M\, \eta_{\mathcal{S}}^{-1} r,$$

where $\eta_{\mathcal{S}}$ is as in (4.10). Notice that increasing again Λ or M we can also assume that $\tilde{r} \geq r$.

For every $z \in B(x,r) \cap J_{\mathcal{S}}$ consider a shortest word $\omega = \omega(z)$ such that $z \in \pi([\omega])$ and $\phi_\omega(X_{t(\omega)}) \subset B(z,\tilde{r})$. Then

$$\operatorname{diam}(\phi_{\omega|_{|\omega|-1}}(X_{t(\omega_{|\omega|-1})}) \geq \tilde{r}. \tag{10.9}$$

Let

$$\mathcal{W} := \{\omega(z)|_{|\omega(z)|-1} : z \in J_{\mathcal{S}} \cap B(x,r)\}.$$

Since $\lim_{e\in E} \operatorname{diam}(\phi_e(X_{t(e)})) = 0$ and $\lim_{n\to\infty} \sup\{\operatorname{diam}(\phi_\omega(X_{t(\omega)})) : \omega \in E_A^n\} = 0$ the set $\mathcal{W}$ is finite. In particular this implies that there exists a finite set $\{z_1, z_2, \ldots, z_k\} \subset J_{\mathcal{S}} \cap B(x,r)$ such that all the words $\omega(z_j)|_{|\omega(z_j)|-1}$, $j = 1, 2, \ldots, k$, are mutually incomparable and the collection

$$\mathcal{W}^* = \{\pi([\omega(z_j)|_{|\omega(z_j)|-1}]) : j = 1, 2, \ldots, k\}$$

covers the set $J_{\mathcal{S}} \cap B(x,r)$. Notice that since

$$\operatorname{diam}(\phi_{\omega(z_j)|_{|\omega(z_j)|-1}}(\operatorname{Int}(X_{t(\omega(z_j)|_{|\omega(z_j)|-1})}))) \geq \tilde{r}$$

for all $j = 1, \ldots, k$ it follows from Lemma 10.7 and Remark 10.6 that $k \leq (3\Lambda K)^Q \gamma^{-1}_{\Lambda M\kappa^{-1}}$. where $\kappa = 2KC\Lambda M\eta_{\mathcal{S}}^{-1}$. Therefore

$$k \leq (3\Lambda K)^Q \gamma_{\eta_{\mathcal{S}}(2KC)^{-1}}. \tag{10.10}$$

Now temporarily fix an element $z \in \{z_1, z_2, \ldots, z_k\}$, set $\omega = \omega(z)$, $q := |\omega|$, and $\psi := \phi_{\omega|_{q-1}}$. By the choice of ω, (4.5) and (4.13) imply

$$2\tilde{r} \geq \operatorname{diam}(\phi_\omega(X_{t(\omega)})) \geq 2(KC)^{-1}\|D\phi_\omega\|_\infty \tilde{R}_{\mathcal{S}} \geq 2K^{-2}C^{-1}\|D\phi_{\omega_q}\|_\infty \|D\psi\|_\infty \tilde{R}_{\mathcal{S}}.$$

Therefore using (4.9) we deduce that

$$\operatorname{diam}(\phi_{\omega_q}(X_{t(\omega_q)})) \leq \Lambda M\|D\phi_{\omega_q}\|_\infty \leq \Lambda MK^2C\tilde{R}_{\mathcal{S}}^{-1}\, \|D\psi\|_\infty^{-1}\tilde{r}. \tag{10.11}$$

By the assumption (iii) there exists some $y \in \phi_{\omega_q}(X_{t(\omega_q)})$, corresponding to the radius $\gamma 3\Lambda MK^2C\tilde{R}_{\mathcal{S}}^{-1}||D\psi||_\infty^{-1}\tilde{r} \geq \gamma \operatorname{diam}(\phi_{\omega_q}(X_{t(\omega_q)}))$, such that

$$m_h(B(y, 3\gamma\Lambda MK^2C\tilde{R}_{\mathcal{S}}^{-1}||D\psi||_\infty^{-1}\tilde{r})) \leq H(3\gamma\Lambda MK^2C\tilde{R}_{\mathcal{S}}^{-1}||D\psi||_\infty^{-1}\tilde{r})^h. \tag{10.12}$$

Now notice that by (10.9) and (4.9)

$$\tilde{r} \leq \Lambda M\|D\psi\|_\infty, \tag{10.13}$$

therefore

$$2KC\|D\psi\|_\infty^{-1}\, r \leq 2KC\Lambda M\tilde{r}^{-1}r = \eta_{\mathcal{S}}. \tag{10.14}$$

Since $z \in \pi([\omega])$, we have that $\psi^{-1}(z) \in \phi_{\omega_q}(X_{t(\omega_q)})$. Hence by Corollary 4.10 and (10.14),

$$\psi\big(B(\psi^{-1}(z), 2rKC||D\psi||_\infty^{-1})\big) \supset B(z, 2r) \supset B(x,r). \tag{10.15}$$

Noting that $y, \psi^{-1}(z) \in \phi_{\omega_q}(X_{t(\omega_q)})$ and using (10.11) we have that

$$\begin{aligned} B(\psi^{-1}(z), 2KC\|D\psi\|_\infty^{-1} r) &\subset B(y, 2KC\|D\psi\|_\infty^{-1} r + \operatorname{diam}(\phi_{\omega_q}(X_{t(\omega_q)}))) \\ &\subset B(y, (2KC + \Lambda M K^2 C \tilde{R}_{\mathcal{S}}^{-1})||D\psi||_\infty^{-1} \tilde{r}). \end{aligned}$$

Therefore using (10.15) and recalling that $\tilde{r} \geq r$ we obtain

$$B(x,r) \subset \psi\big(B(y, 3\Lambda M K^2 C \tilde{R}_{\mathcal{S}}^{-1}||D\psi||_\infty^{-1}\tilde{r}\big), \tag{10.16}$$

where we also used the fact that $\Lambda M \tilde{R}_{\mathcal{S}}^{-1} \geq 1$. So, employing Proposition 8.17, (10.16) , (10.12) and recalling that $\gamma \geq 1$ we get

(10.17)

$$\begin{aligned} m_h\big(B(x,r) \cap \psi(X_{t(\omega_{q-1})})\big) &\leq m_h\big(\psi(X_{t(\omega_{q-1})}) \cap \psi(B(y, 3\Lambda M K^2 C \tilde{R}_{\mathcal{S}}^{-1}||D\psi||_\infty^{-1}\tilde{r}))\big) \\ &= m_h\big(\psi\big(X_{t(\omega_{q-1})} \cap B(y, 3\Lambda M K^2 C \tilde{R}_{\mathcal{S}}^{-1}||D\psi||_\infty^{-1}\tilde{r})\big)\big) \\ &\leq ||D\psi||_\infty^h m_h\big(X_{t(\omega_{q-1})} \cap B(y, 3\Lambda M K^2 C \tilde{R}_{\mathcal{S}}^{-1}||D\psi||_\infty^{-1}\tilde{r})\big) \\ &\leq ||D\psi||_\infty^h m_h\big(B(y, 3\gamma\Lambda M K^2 C \tilde{R}_{\mathcal{S}}^{-1}||D\psi||_\infty^{-1}\tilde{r})\big) \\ &\leq ||D\psi||_\infty^h H\big(3\gamma\Lambda M K^2 C \tilde{R}_{\mathcal{S}}^{-1}||D\psi||_\infty^{-1}\tilde{r}\big)^h \\ &= H\left(\frac{6\gamma\Lambda^2 M^2 K^3 C^2}{\tilde{R}_{\mathcal{S}}\eta_{\mathcal{S}}}\right)^h r^h. \end{aligned}$$

By the definition of $\mathcal{W}^*$ we see that

$$\{\phi_{\omega(z_j)|_{|\omega(z_j)-1|}}(X_{t(\omega(z_j)|_{|\omega(z_j)-1|})})\}_{j=1}^k$$

covers the set $J_{\mathcal{S}} \cap B(x,r)$. Finally by (10.17), (10.10), and Remark 10.6, since the words $\omega(z_j)|_{|\omega(z_j)-1|}$ are mutually incomparable, we get

$$\begin{aligned} m_h(B(x,r)) &\leq \sum_{j=1}^k m_h(B(x,r) \cap \phi_{\omega(z_j)|_{|\omega(z_j)-1|}}(X_{t(\omega(z_j)|_{|\omega(z_j)-1|})})) \\ &\leq \#\mathcal{W}^*\, H\left(\frac{6\gamma\Lambda^2 M^2 K^3 C^2}{R_{\mathcal{S}}\eta_{\mathcal{S}}}\right)^h r^h \\ &\leq H(3\Lambda K)^Q \gamma_{\eta_{\mathcal{S}}(2KC)^{-1}} \left(\frac{6\gamma\Lambda^2 M^2 K^3 C^2}{R_{\mathcal{S}}\eta_{\mathcal{S}}}\right)^h r^h \\ &:= c_{\mathcal{S}} r^h, \end{aligned}$$

and (iv) is proved.

The implication (iv)⇒(i) is an immediate consequence of Frostman's Lemma (see for example [**43**] or [**46**]). Thus the whole theorem has been proved. □

REMARK 10.9. It is obvious that it suffices for the above proof that conditions (ii) and (iii) of Theorem 10.8 be satisfied for a cofinite subset of I.

REMARK 10.10. Notice that in the proof of Theorem 10.8 the corkscrew condition was only needed to establish the implication (iii)⇒(iv).

REMARK 10.11. If condition (iv) of Theorem 10.8 holds, then the stated inequality holds in fact for all $x \in \overline{J}_{\mathcal{S}}$.

We now move to the second main result of this section, which provides necessary and sufficient conditions for the packing measure of $J_{\mathcal{S}}$ to be finite. In particular we prove that under certain conditions $\mathcal{P}^h(J_{\mathcal{S}})$ is finite if and only if the conformal measure m_h is lower geometric with exponent h. Before proving the theorem we make the following observation.

REMARK 10.12. If $\mathcal{S} = \{\phi_e : e \in E\}$ is a maximal, finitely irreducible Carnot conformal GDMS satisfying the strong open set condition, then

$$J_{\mathcal{S}} \cap \text{Int}(\phi_e(X_{t(e)})) \neq \emptyset$$

for all $e \in E$.

PROOF. Let $e \in E$. By the strong open set condition there exist $v \in V$ and $e_0 \in E$ such that $X_v = X_{t(e_0)}$ and $\text{Int}(X_{t(e_0)}) \cap J_{\mathcal{S}} \neq \emptyset$. Since $\mathcal{S}$ is finitely irreducible there exists $\omega \in E_A^*$ such that $e\omega e_0 \in E_A^*$. Let $x_0 = \pi(v) \in J_{\mathcal{S}} \cap \text{Int}(X_{t(e_0)})$. Then $x_0 \in X_{t(e_0)} \cap X_{i(v_1)}$, hence $t(e_0) = i(v_1)$ and by the maximality of $\mathcal{S}$ we deduce that $A_{e_0 v_1} = 1$. In particular $\phi_{e_0}(x_0) \in J_{\mathcal{S}}$, and

$$\phi_{e\omega e_0}(x_0) \in J_{\mathcal{S}}. \tag{10.18}$$

Moreover,

$$\phi_{e\omega e_0}(x_0) \in \phi_e(\text{Int}(\phi_{\omega e_0}(X_t(e_0)))) \subset \text{Int}(\phi_e(X_{i(\omega_1)})) = \text{Int}(\phi_e(X_{t(e)})). \tag{10.19}$$

The proof follows by (10.18) and (10.19). □

DEFINITION 10.13. We say that a Carnot conformal GDMS $\mathcal{S} = \{\phi_e\}_{e\in E}$ has *thin boundary* if there exists some $\varepsilon > 0$ such that

$$\sharp\{e \in E : \phi_e(X_{t(e)}) \cap B(\partial X, \varepsilon) \neq \emptyset\} < \infty.$$

THEOREM 10.14. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a maximal regular Carnot conformal GDMS with thin boundary which satisfies the strong open set condition and the weak corkscrew condition. Then the following conditions are equivalent:*

(i) $\mathcal{P}_h(J_{\mathcal{S}}) < +\infty$.

(ii) There are three constants $H > 0$, $\xi > 0$, and $\gamma > 1$ such that for every $e \in E$ and every $r \in [\gamma \,\text{diam}(\phi_e(X_{t(e)})), \xi]$ there exists $y \in \phi_e(X_{t(e)})$ such that $m_h(B(y,r)) \geq Hr^h$.

(iii) The measure m_h is lower-geometric with exponent h. This precisely means that there exists a constant $c_{\mathcal{S}} \in (0, +\infty)$ such that $m_h(B(x,r)) \geq c_{\mathcal{S}} r^h$ for every $x \in J_{\mathcal{S}}$ and all $r \in [0, \text{diam}(X)]$.

PROOF. By Lemma 10.12 for every $e \in E$ there exists some $x_e \in J_{\mathcal{S}} \cap \text{Int}(\phi_e(X_{t(e)}))$. For $e \in E$ set $d_e = d(x_e, \partial X)$. Since $\mathcal{S}$ has thin boundary there exists some $\varepsilon > 0$ such that the set $E_\varepsilon = \{e \in E : \phi_e(X_{t(e)}) \cap B(\partial X, \varepsilon) \neq \emptyset\}$ is finite. Therefore if $\delta_\varepsilon := \frac{1}{2}\min\{d_e : e \in E_e\}$ we get that $\min_{e\in E} d_\varepsilon > \delta_\varepsilon$.

We will now prove that (i)⇒(ii). By way of contradiction assume that (ii) fails. Fix $H > 0$, $\xi \in (0, \min\{\eta_S, \delta_\varepsilon\})$ and $\gamma > 1$. Then there exist $e \in E$ and a radius r with $\text{diam}(\phi_e(X_{t(e)})) < r \leq \xi$ such that for every $y \in \phi_e(X_{t(e)})$, we have

$$m_h(B(y,r)) \leq H\, r^h.$$

Notice that

$$B(x_e, r/2) \subset \text{Int} X_{i(e)} \setminus B(\partial X, \delta_\varepsilon)$$

because $d(x_e, \partial X) > \delta_\varepsilon$ and $r \le \xi < \delta_\varepsilon$. Hence if $x_e = \pi(\omega^e)$ for some $\omega^e \in E_A^{\mathbb{N}}$ then there exists some $n_0 \in \mathbb{N}$ such that $\pi([\omega^e]) \subset B(x_e, r/2)$. By (7.18) and Theorem 7.4 we deduce that $\tilde{\mu}_h(\pi^{-1}(B(x_e, r/2))) > 0$. Therefore Birkhoff's Ergodic Theorem implies that if

$$A = \{\omega \in E_A^{\mathbb{N}} : \sigma^n(\omega) \in \pi^{-1}(B(x_e, r/2)) \text{ for infinitely many } n\}$$

then $\tilde{\mu}_h(A) = 1$. Hence by Theorem 7.4 $\tilde{m}_h(A) = 1$, and if $B = \pi(A) \subset J_{\mathcal{S}}$ then $m_h(B) = 1$ and for every $z = \pi(\omega) \in B, \omega \in A$,

$$\pi(\sigma^n(\omega)) \in B(x_e, r/2)$$

for infinitely many n's. Notice that for such a point z and such an integer $n \ge 1$, $\pi(\sigma^n(\omega)) \in X_{t(\omega_n)} = X_{i(e)}$ and $d(\pi(\sigma^n(\omega)), \partial X_{i(e)}) > \delta_\varepsilon$ therefore

$$B(\pi(\sigma^n(\omega)), \delta_\varepsilon) \subset \mathrm{Int}(X_{i(e)}). \tag{10.20}$$

Hence by Remark 8.18, recalling that $r \le \xi < \delta_\varepsilon$,

$$\begin{aligned} m_h(\phi_{\omega|_n}(B(\pi(\sigma^n(\omega)), r/2))) &\le \|D\phi_{\omega|n}\|_\infty^h m_h(B(\pi(\sigma^n(\omega)), r/2) \\ &\le \|D\phi_{\omega|_n}\|_\infty^h m_h(B(x_e, r)) \le \|D\phi_{\omega|_n}\|_\infty^h H\, r^h. \end{aligned}$$

By (4.7), since $z = \phi_{\omega|_k}(\pi(\sigma^n(\omega)))$,

$$\phi_{\omega|_n}\big(B(\pi(\sigma^n(\omega)), r/2)\big) \supset B(z, \|D\phi_{\omega|_n}\|_\infty (KC)^{-1} r/2).$$

So,

$$m_h\big(B(z, (KC)^{-1}\|D\phi_{\omega|_n}\|_\infty r/2)\big) \le H\|D\phi_{\omega|_n}\|_\infty^h r^h.$$

We can now apply Frostman's Lemma (as in [**43**], or as in [**46**, Theorem A.2.013]) and obtain

$$\mathcal{P}^h(J_{\mathcal{S}}) \gtrsim (2KC)^{-h}\, H^{-1}.$$

So letting $H \searrow 0$, we get $\mathcal{P}^h(J_{\mathcal{S}}) = \infty$. This finishes the contrapositive proof of the implication (i)⇒(ii).

(ii)⇒(iii). First notice that by (4.7) for all $\omega \in E_A^*$ and $p \in X_{t(\omega)}$,

$$\phi_\omega(N_{t(\omega)}) \supset \phi_\omega(B(p, 4^{-1}\eta_{\mathcal{S}})) \supset B(\phi_\omega(p), (4KC)^{-1}\|D\phi_\omega\|_\infty \eta_{\mathcal{S}}), \tag{10.21}$$

where the sets $N_v = B(X_v, \mathrm{dist}(X_v, \partial S_v)/2), v \in V$ where defined in Remark 4.17.

First notice that decreasing H if necessary, the assumption of the lemma continues to be fulfilled if the number ξ is replaced by any other positive number, for example by $\eta_{\mathcal{S}}/4$. Fix $0 < r < \xi$, $x = \pi(\omega) \in J_{\mathcal{S}}$, and take maximal $k \in \mathbb{N}$ such that

$$\phi_{\omega|_k}(N_{t(\omega_k)}) \supset B(x, (4KC\Lambda_0 M_0)^{-1}\eta_{\mathcal{S}} r), \tag{10.22}$$

where $M_0 = \mathrm{diam}(\cup_{v\in V} N_v)$. By (10.21), since $x = \phi_{\omega|_{k+1}}(\pi(\sigma^{k+1}(\omega)))$ and $\pi(\sigma^{k+1}(\omega)) \in X_{t(\omega_{k+1})}$,

$$\phi_{\omega|_{k+1}}(N_{t(\omega_{k+1})}) \supset B(x, (4KC)^{-1}\|D\phi_{\omega|_{k+1}}\|_\infty \eta_{\mathcal{S}}).$$

By the maximality of k, $\phi_{\omega|_{k+1}}(N_{t(\omega_{k+1})})$ does not contain $B(x, (4KC\Lambda_0 M_0)^{-1}\eta_{\mathcal{S}} r)$, hence $(4KC\Lambda_0 M_0)^{-1}\eta_{\mathcal{S}} r > (4KC)^{-1}\|D\phi_{\omega|_{k+1}}\|_\infty \eta_{\mathcal{S}}$, or equivalently

$$r > \Lambda_0 M_0 \|D\phi_{\omega|_{k+1}}\|_\infty. \tag{10.23}$$

Hence, using (4.9), we get

$$B(x, r) \supset B(x, \Lambda_0 M_0 \|D\phi_{\omega|_{k+1}}\|_\infty) \supset \phi_{\omega|_{k+1}}(X_{t(\omega_{k+1})}).$$

Hence by Propositions 8.22 and 8.23, we get

$$(10.24)\qquad \begin{aligned} m_h(B(x,r)) &\geq K^{-h}||D\phi_{\omega|_{k+1}}||_\infty^h\, m_h\big(X_{t(\omega_{k+1})}\cap \pi(J^-_{\omega|_{k+1}})\big) \\ &\geq M_h K^{-h}||D\phi_{\omega|_{k+1}}||_\infty^h, \end{aligned}$$

where as in Proposition 8.23, $M_h = \inf\{m_h(\pi(J_e^-)) : e\in E\}$. Now notice that by (4.18)

$$(10.25)\qquad \phi_{\omega|_k}(N_{t(\omega_k)}) \subset B(x, \Lambda_0 M_0\|D\phi_{\omega|_k}\|_\infty),$$

therefore, using (10.22), $M_0\Lambda_0\|D\phi_{\omega|_k}\|_\infty > (4KC\Lambda_0M_0)^{-1}\eta_{\mathcal{S}} r$ or equivalently,

$$(10.26)\qquad \frac{r}{\|D\phi_{\omega|_k}\|_\infty} < \frac{4(\Lambda_0M_0)^2KC}{\eta_{\mathcal{S}}}.$$

Put

$$\alpha := \min\left\{\frac{\eta_{\mathcal{S}}^2}{8(\Lambda_0M_0)^3K^2C}, \frac{1}{2\Lambda MCK}\right\}.$$

Notice that by the choice of α and (10.26), we have that

$$(10.27)\qquad 2\Lambda_0M_0K\alpha\|D\phi_{\omega|_k}\|_\infty^{-1} r < \eta_{\mathcal{S}}.$$

We now consider two cases. First, if $\gamma||D\phi_{\omega|_{k+1}}||_\infty \geq \alpha r$, then by (10.24)

$$m_h(B(x,r)) \geq (\alpha(\gamma K)^{-1})^h M_h r^h,$$

and we are done. Otherwise, i.e. the second case,

$$(10.28)\qquad \gamma||D\phi_{\omega|_{k+1}}||_\infty < \alpha r.$$

By (4.5) and (4.9) we have that

$$\operatorname{diam}(\phi_{\omega_{k+1}}(X_{t(\omega_{k+1})})) \leq \Lambda MK\|D\phi_{\omega|_{k+1}}\|_\infty\|D\phi_{\omega|_k}\|_\infty^{-1}.$$

Hence, using also (10.28), if $y\in\phi_{\omega_{k+1}}(X_{t(\omega_{k+1})})$,

$$(10.29)\qquad B(y, \Lambda MK\alpha\|D\phi_{\omega|_k}\|^{-1}r) \subset B(\pi(\sigma^k(\omega)), 2\Lambda MK\alpha\|D\phi_{\omega|_k}\|_\infty^{-1}r).$$

Now by (4.7) and (10.27)

$$(10.30)\quad \phi_{\omega|_k}(B(\pi(\sigma^k(\omega)), 2\Lambda MK\alpha\|D\phi_{\omega|_k}\|_\infty^{-1}r)) \subset B(x, 2C\Lambda MK\alpha r) \subset B(x,r).$$

Now notice that by (10.28), (4.5) and (4.9)

$$(10.31)\qquad \Lambda MK\alpha\|D\phi_{\omega|_k}\|_\infty^{-1}r \geq \Lambda M\gamma\|D\phi_{\omega|_{k+1}}\|_\infty \geq \gamma\operatorname{diam}(\phi_{\omega_{k+1}}(X_{t(\omega_{k+1})})).$$

By (10.29), (10.27) and Remark 8.25,

$$m_h(\phi_{\omega|_k}(B(\pi(\sigma^k(\omega)), 2\Lambda MK\alpha\|D\phi_{\omega|_k}\|_\infty^{-1}r))) \geq \|D\phi_{\omega|_k}\|_\infty^h m_h(B(y, \Lambda MK\alpha\|D\phi_{\omega|_k}\|^{-1}r)).$$

Also by (10.31) and (ii)

$$m_h(B(y, \Lambda MK\alpha\|D\phi_{\omega|_k}\|_\infty^{-1}r)) \geq H\,(\Lambda MK\alpha\|D\phi_{\omega|_k}\|_\infty^{-1}r)^h.$$

Hence

$$m_h(\phi_{\omega|_k}(B(\pi(\sigma^k(\omega)), 2\Lambda MK\alpha\|D\phi_{\omega|_k}\|_\infty^{-1}r))) \geq H(\Lambda MK\alpha)^h\, r^h$$

and by (10.30)

$$m_h(B(x,r)) \geq H(\Lambda MK\alpha)^h\, r^h.$$

The implication (iii)⇒(i) is an immediate consequence of Frostman's Lemma (see for example [43] or [46]). Thus the whole theorem has been proved. □

We close this section with a few remarks and observations concerning Theorem 10.14.

REMARK 10.15. It is obvious that in Theorem 10.14 it suffices for conditions (ii) and (iii) to be satisfied for a cofinite subset of E.

REMARK 10.16. We do not need to assume that $\mathcal{S}$ has thin boundary and satisfies the SOSC in order to prove the implications (ii)⇒(iii)⇒(i).

Following the reasoning in the proof of the implication (i) ⇒ (ii), we can prove the following weaker statement where we do not need to assume the thin boundary and weak corkscrew conditions.

PROPOSITION 10.17. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a maximal regular Carnot conformal GDMS which satisfies the SOSC. If $\mathcal{P}_h(J_{\mathcal{S}}) < +\infty$ then there are two constants $H > 0$ and $\gamma > 1$ such that for every $e \in E$, every $r \in [\gamma \operatorname{diam}(\phi_e(X_{t(e)})), \eta_{\mathcal{S}}]$ and every $x \in \phi_e(X_{t(e)})$ such that $B(x, r) \subset X_{i(e)}$, $m_h(B(x, r)) \geq Hr^h$.*

10.3. Hausdorff and packing measures for continued fraction systems in groups of Iwasawa type

We apply the general theorems about Hausdorff and packing measures, proved in the previous section, to the class of continued fraction systems introduced in Section 5.3 and further developed in Section 9.2. Our main theorem in this section reads as follows.

THEOREM 10.18. *Let $\mathbb{G}$ be a Carnot group of Iwasawa type and let $\varepsilon > 0$. Let $\mathcal{S}_\varepsilon = \{\phi_\gamma\}_{\gamma\in I_\varepsilon}$ be the corresponding continued fraction conformal iterated function system. Then*

$$\mathcal{H}_{h_\varepsilon}(J_{\mathcal{S}_\varepsilon}) = 0 \quad \text{and} \quad 0 < \mathcal{P}_{h_\varepsilon}(J_{\mathcal{S}_\varepsilon}) < +\infty,$$

where, we recall, $h_\varepsilon := \dim_{\mathcal{H}} J_{\mathcal{S}_\varepsilon}$ is the Hausdorff dimension of the limit set $J_{\mathcal{S}_\varepsilon}$.

PROOF. We want to apply Theorem 10.8 and Theorem 10.14 respectively to prove the first and the second assertion of our current theorem. The hypotheses of thin boundary, the strong open set condition, and the weak corkscrew condition, required to apply these two theorems, are immediate from the definition of the system $\mathcal{S}_\varepsilon$.

We know from Theorem 9.1 that the system $\mathcal{S}_\varepsilon$ is co-finitely regular. Let then m_ε be the corresponding h_ε-conformal measure. As always, put

$$X := \overline{B}(o, 1/2).$$

Formula (5.13) yields

$$\|D\phi_\gamma\|_\infty \approx d(\gamma, o)^{-2} \tag{10.32}$$

for all $\gamma \in I_\varepsilon$. It also immediately follows from (5.13), in fact the calculation of (5.14) does it, that

$$\operatorname{diam}(\phi_\gamma(X)) \approx d(\gamma, o)^{-2} \tag{10.33}$$

for all $\gamma \in I_\varepsilon$. It furthermore follows from (5.12) that

$$\phi_\gamma(X) \subset \overline{B}\left(o, K_\varepsilon d(\gamma, o)^{-1}\right) \tag{10.34}$$

with some $K_\varepsilon > 0$ for all $\gamma \in I_\varepsilon$. Now for every $r > 0$ let

$$I(r) := \{\gamma \in I_\varepsilon : r/2 < K_\varepsilon d(\gamma, o)^{-1} < r\} = \{\gamma \in I_\varepsilon : K_\varepsilon r^{-1} < d(\gamma, o) < 2K_\varepsilon r^{-1}\}.$$

Improving in a straightforward way the arguments leading to (9.2) and (9.3), we get that there exists some $\beta_\varepsilon > 1$ such that

$$\beta_\varepsilon^{-1} r^{-Q} \le \sharp I(r) \le \beta_\varepsilon r^{-Q} \tag{10.35}$$

for all $\gamma \in I_\varepsilon$ and every $r \in (0,1)$ small enough. Therefore, using Theorem 8.14, Proposition 8.17 and (10.32)-(10.35), we get

$$\begin{aligned} m_\varepsilon(B(o,r)) &\ge \sum_{\gamma \in I(r)} m_\varepsilon(\phi_\gamma(X)) \approx \sum_{\gamma \in I(r)} \|D\phi_\gamma\|_\infty^{h_\varepsilon} \approx \sum_{\gamma \in I(r)} d(\gamma,o)^{-2h_\varepsilon} \\ &\approx \sharp I(r) r^{2h_\varepsilon} \approx r^{2h_\varepsilon - Q}. \end{aligned} \tag{10.36}$$

Therefore, by virtue of Theorem 9.2, we get that

$$\limsup_{r\to 0} \frac{m_\varepsilon(B(o,r))}{r^{h_\varepsilon}} \gtrsim \limsup_{r \to 0} r^{h_\varepsilon - Q} = +\infty.$$

This in turn, in conjunction with Theorem 10.8 and Remark 10.11 entails that $\mathcal{H}_{h_\varepsilon}(J_{\mathcal{S}_\varepsilon}) = 0$, and the first part of our theorem is thus proved.

Passing to the second part, note that $\mathcal{P}_{h_\varepsilon}(J_{\mathcal{S}}) > 0$ follows from Proposition 10.3. In order to prove that $\mathcal{P}_{h_\varepsilon}(J_{\mathcal{S}}) < +\infty$ we will check that condition (ii) of Theorem 10.14 holds. So, let $\gamma \in I_\varepsilon$ be arbitrary and let $x \in \phi_\gamma(X)$. Fix a radius

$$r \in \big(\rho \operatorname{diam}(\phi_\gamma(X)), 1\big) \tag{10.37}$$

with a sufficiently large constant $\rho > 0$, independent of γ, x, and r, to be determined in the course of the proof. We will consider three cases. Assume first that

$$r \le d(\mathcal{J}(\gamma), 0). \tag{10.38}$$

It then follows from (2.16) and (2.17) that

$$\begin{aligned} \inf\{d(\mathcal{J}(x),\gamma) :&x \in \partial B(\mathcal{J}(\gamma), r)\} = \\ &= \inf\{d(\mathcal{J}(x), \mathcal{J}(\mathcal{J}(\gamma))) : x \in \partial B(\mathcal{J}(\gamma), r)\} \\ &= \inf\left\{\frac{d(x,\mathcal{J}(\gamma))}{d(x,o)d(\mathcal{J}(\gamma),o)} : x \in \partial B(\mathcal{J}(\gamma), r)\right\} \\ &= r d(\gamma, o) \inf\left\{\frac{1}{d(x,o)} : x \in \partial B(\mathcal{J}(\gamma), r)\right\}. \end{aligned}$$

But

$$d(x,o) \le d(x, \mathcal{J}(\gamma)) + d(\mathcal{J}(\gamma), o) = r + \frac{1}{d(\gamma,o)} \le \frac{2}{d(\gamma,o)}.$$

Therefore,

$$\inf\{d(\mathcal{J}(x),\gamma) : x \in \partial B(\mathcal{J}(\gamma), r)\} \ge \frac{1}{2} d(\gamma,o)^2 r.$$

Hence,

$$\mathcal{J}(B(\mathcal{J}(\gamma), r)) \supset B\left(\gamma, \frac{1}{2} d(\gamma,o)^2 r\right).$$

Now let

$$I_\gamma(r) := \{g \in I_\varepsilon : \overline{B}(g, 1/2) \subset B\left(\gamma, \frac{1}{2} d(\gamma,o)^2 r\right).$$

With considerations analogous to those leading to (9.2) and (9.3) (see also (10.35)), we get

$$\sharp I_\gamma(r) \approx \sharp\left\{g \in \mathbb{G}(\mathbb{Z}) : \overline{B}(g,1/2) \subset B\left(o, \frac{1}{2} d(\gamma,o)^2 r\right)\right\} \approx d(\gamma,o)^{2Q} r^Q. \tag{10.39}$$

Observe that for every $g \in I_\gamma(r)$ we have

$$\begin{aligned} \phi_g(X) &= \mathcal{J} \circ \ell_g(X)(\overline{B}(g,1/2)) \subset \mathcal{J}\left(B\left(\gamma, \frac{1}{2}d(\gamma,o)^2 r\right)\right) \\ &\subset \mathcal{J}(\mathcal{J}(B(\mathcal{J}(\gamma),r))) \\ &= B(\mathcal{J}(\gamma),r). \end{aligned} \tag{10.40}$$

Therefore by (10.40), Theorem 8.14, Corollary 8.24 and (10.32), we obtain

$$\begin{aligned} m_\varepsilon(B(\phi_\gamma(o),r)) = m_\varepsilon((B(\mathcal{J}(\gamma),r)) &\geq \sum_{g\in I_\gamma(r)} m_\varepsilon(\phi_g(X)) \\ &\approx \sum_{g\in I_\gamma(r)} ||D\phi_g||_\infty^{h_\varepsilon} \\ &\approx \sum_{g\in I_\gamma(r)} d(g,o)^{-2h_\varepsilon}. \end{aligned}$$

But, for every $g \in I_\gamma(r)$,

$$\begin{aligned} d(g,o) \leq g(g,\gamma) + d(\gamma,o) &\leq \frac{1}{2}d(\gamma,o)^2 r + d(\gamma,o) \\ &\leq \frac{1}{2}d(\gamma,o) + d(\gamma,o) \\ &\leq 2d(\gamma,o). \end{aligned}$$

Notice also that by (10.33) and the fact that $\operatorname{diam}(\phi_\gamma(X)) \geq \rho^{-1} r$ we deduce that

$$d(\gamma,0)^2 \gtrsim r^{-1}. \tag{10.41}$$

Therefore, using (10.39) along with (10.37) and (10.41), we get that

$$\begin{aligned} m_\varepsilon(B(\phi_\gamma(o),r)) &\gtrsim \sharp I_\gamma(r) d^{-2h_\varepsilon}(\gamma,o) \approx d(\gamma,o)^{2(Q-h_\varepsilon)} r^Q \\ &\gtrsim r^{h_\varepsilon - Q} r^Q \\ &= r^{h_\varepsilon}, \end{aligned}$$

and we are done in this case. As for the second case, suppose that $d(J(\gamma),o) < r \leq 2d(J(\gamma),o)$. Then $d(J(\gamma),o) \geq r/2$, and so, using what we have obtained in the first case,we get

$$m_\varepsilon(B(\phi_\gamma(o),r)) \geq m_\varepsilon(B(\phi_\gamma(o),r/2)) \gtrsim (r/2)^{h_\varepsilon} = 2^{-h_\varepsilon} r^{h_\varepsilon},$$

and we are done in this case too. Finally, if $r \geq 2d(J(\gamma),o)$, then $B(\phi_\gamma(o),r) \supset B(o,r/2)$ and, in view of (10.36), along with the already noted fact that $h_\varepsilon < Q$, we get

$$m_\varepsilon(B(\phi_\gamma(o),r)) \geq m_\varepsilon(B(o,r/2)) \gtrsim (r/2)^{2h_\varepsilon - Q} = 2^{Q-2h_\varepsilon} r^{h_\varepsilon - Q} r^{h_\varepsilon} \gtrsim r^{h_\varepsilon},$$

and we are done in this case as well. The proof is complete. □

10.4. Hausdorff dimensions of invariant measures

In this section we establish a formula for the Hausdorff dimension of the projection of a shift-invariant measure onto the limit set of a Carnot GDMS. We assume throughout the section that $\mathcal{S}$ is a finitely irreducible Carnot conformal GDMS with countable alphabet. Let us observe first that the same argument as at the beginning of the proof of Theorem 8.5 gives the following fact which can be called a measure theoretic open set condition.

THEOREM 10.19. *Let $\mathcal{S}$ be a Carnot conformal GDMS of bounded coding type. If μ is a Borel probability shift-invariant ergodic measure on $E_A^{\mathbb{N}}$, then*

$$\mu \circ \pi^{-1}\big(\phi_\omega(X_{t(\omega)}) \cap \phi_\tau\big(X_{t(\tau)}\big)\big) = 0 \tag{10.42}$$

for all incomparable words $\omega, \tau \in E_A^*$.

Recall that if ν is a finite Borel measure on a metric space X, then $\dim_{\mathcal{H}}(\nu)$, the Hausdorff dimension of ν, is the minimum of the Hausdorff dimensions of sets of full ν measure. Now we want to establish a dynamical formula for the Hausdorff dimension of the projection measures $\mu \circ \pi^{-1}$, frequently referred to as volume lemma. We will do in fact more: we will prove exact dimensionality of such measures and will provide a dynamical formula for the local Hausdorff dimension. We start off with the precise definitions of the concepts involved.

DEFINITION 10.20. Let μ be a non-zero Borel measure on a metric space (X, ρ). We define

$$\underline{\dim}_{\mathcal{H}}(\mu) := \inf\{\dim_{\mathcal{H}}(Y) : \mu(Y) > 0\},$$

and

$$\dim_{\mathcal{H}}(\mu) = \inf\{\dim_{\mathcal{H}}(Y) : \mu(X \setminus Y) = 0\}.$$

and call $\dim_{\mathcal{H}}(\mu)$ the *Hausdorff dimension* of the measure μ and $\underline{\dim}_{\mathcal{H}}(\mu)$ the *lower Hausdorff dimension* of μ.

Of course $\underline{\dim}_{\mathcal{H}}(\mu) \le \dim_{\mathcal{H}}(\mu)$. We will see soon that in the context of conformal GDMS these quantities are frequently equal.

Analogously $\underline{\dim}_{\mathcal{P}}(\mu)$ and $\dim_{\mathcal{P}}(\mu)$ will respectively denote the *lower packing dimension* and the *packing dimension* of the measure μ.

The next definition introduces concepts that form effective tools to calculate the dimensions introduced above.

DEFINITION 10.21. Let μ be a Borel probability measure on a metric space (X, ρ). For every point $x \in X$ we define the *lower and upper pointwise dimension* of the measure μ at the point $x \in X$ respectively as

$$\underline{d}_\mu(x) := \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) := \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

In the case when both numbers $\underline{d}_\mu(x)$ and $\overline{d}_\mu(x)$ are equal, we denote their common value by $d_\mu(x)$. We then obviously have

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

and we call $d_\mu(x)$ the *pointwise dimension* of the measure μ at the point $x \in X$. If in addition the function $X \ni x \mapsto d_\mu(x)$ is μ-a.e. constant (referred to as d_μ in the sequel), we call the measure μ *dimensional exact*.

The following theorem about Hausdorff and packing dimensions of Borel measures is well known. Its proof can be found, for example, in [53].

THEOREM 10.22. *If μ is a Borel probability measure on (X, ρ), then* $\underline{\dim}_{\mathcal{H}}(\mu) = \operatorname{ess\,inf} \underline{d}_\mu$, $\dim_{\mathcal{H}}(\mu) = \operatorname{ess\,sup} \underline{d}_\mu$, $\underline{\dim}_{\mathcal{P}}(\mu) = \operatorname{ess\,inf} \overline{d}_\mu$, *and* $\dim_{\mathcal{P}}(\mu) = \operatorname{ess\,sup} \overline{d}_\mu$.

As an immediate consequence of Theorem 10.22 we get the following.

COROLLARY 10.23. *If μ is a dimensional exact Borel probability measure on a metric space (X,ρ), then all its dimensions are equal:*

$$\underline{\dim}_{\mathcal{H}}(\mu)=\dim_{\mathcal{H}}(\mu)=\underline{\dim}_{\mathcal{P}}(\mu)=\dim_{\mathcal{P}}(\mu)=d_\mu.$$

We now shall prove the following main result of this section, versions of which have been established in many contexts of conformal dynamics. Recall that the *characteristic Lyapunov exponent* with respect to μ and σ is defined as

$$\chi_\mu(\sigma)=-\int_{E_A^{\mathbb{N}}}\zeta\,d\mu>0$$

where $\zeta: E_A^{\mathbb{N}}\to\mathbb{R}$ is the function defined in (7.2). Recall also (6.2) and (6.4) for the definitions of the entropies $\mathrm{H}_\mu(\alpha)$ and $\mathrm{h}_\mu(\sigma)$.

THEOREM 10.24. *Let $\mathcal{S}$ be a boundary regular Carnot conformal GDMS. Suppose that μ is a Borel probability shift-invariant ergodic measure on $E_A^{\mathbb{N}}$ such that at least one of the numbers $\mathrm{H}_\mu(\alpha)$ or $\chi_\mu(\sigma)$ is finite. Then the measure $\mu\circ\pi^{-1}$ is dimensional exact and*

$$\begin{aligned}\underline{\dim}_{\mathcal{H}}(\mu\circ\pi^{-1})&=\dim_{\mathcal{H}}(\mu\circ\pi^{-1})=d_{\mu\circ\pi^{-1}}\\&=\frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}=\underline{\dim}_{\mathcal{P}}(\mu\circ\pi^{-1})=\dim_{\mathcal{P}}(\mu\circ\pi^{-1}).\end{aligned}$$

PROOF. By virtue of Corollary 10.23 it only suffices to prove the equality

$$d_{\mu\circ\pi^{-1}}=\frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}. \tag{10.43}$$

Suppose first that $\mathrm{H}_\mu(\alpha)<+\infty$. Since $\mathrm{H}_\mu(\alpha)<\infty$ and since α is a generating partition, the entropy $\mathrm{h}_\mu(\sigma)=\mathrm{h}_\mu(\sigma,\alpha)\le\mathrm{H}_\mu(\alpha)$ is finite. Thus, Birkhoff's Ergodic Theorem and the Breiman–Shannon–McMillan Theorem imply that there exists a set $Z\subset E_A^{\mathbb{N}}$ such that $\mu(Z)=1$,

$$\lim_{n\to\infty}-\frac{1}{n}\sum_{j=0}^{n-1}\zeta\circ\sigma^j(\omega)=\chi_\mu(\sigma) \tag{10.44}$$

and

$$\lim_{n\to\infty}\frac{-\log(\mu([\omega|_n]))}{n}=\mathrm{h}_\mu(\sigma) \tag{10.45}$$

for all $\omega\in Z$. Note that (10.44) holds even if $\chi_\mu(\sigma)=+\infty$ since the function $-\zeta: E_A^{\mathbb{N}}\to\mathbb{R}$ is everywhere positive. Fix now $\omega\in Z$ and $\eta>0$. For $r>0$ let $n=n(\omega,r)\ge0$ be the least integer such that $\phi_{\omega|_n}(X_{t(\omega_n)})\subset B(\pi(\omega),r)$. Then

$$\begin{aligned}\log\big(\mu\circ\pi^{-1}(B(\pi(\omega),r))\big)&\ge\log\big(\mu\circ\pi^{-1}(\phi_{\omega|_n}(X_{t(\omega_n)}))\big)\\&\ge\log(\mu([\omega|_n]))\ge-(\mathrm{h}_\mu(\sigma)+\eta)n\end{aligned}$$

for r small enough (i.e., for $n=n(\omega,r)$ is large enough), moreover,

$$\operatorname{diam}\big(\phi_{\omega|_{n-1}}(X_{t(\omega_{n-1})})\big)\ge r.$$

Using (4.9) and Lemma 4.9 the latter inequality implies that

$$\log r\le\log\big(\operatorname{diam}(\phi_{\omega|_{n-1}}(X_{t(\omega_{n-1})}))\big)\le\log\big(\Lambda MK|\phi'_{\omega|_{n-1}}(\pi(\sigma^{n-1}(\omega)))|\big).$$

Recalling that $\zeta(\omega) = \log \|D\phi_{\omega_1}(\pi(\sigma(\omega)))\|$, it follows from (10.44) that for arbitrary $N > 0$ and sufficiently large n,

$$-\frac{1}{n-1}\sum_{j=0}^{n-2}\log\|D\phi_{\omega_{j+1}}(\pi(\sigma^{j+1}(\omega)))\| = -\frac{1}{n-1}\sum_{j=1}^{n-1}\log\|D\phi_{\omega_j}(\pi(\sigma^j(\omega)))\| \geq \chi'_\mu(\sigma) - \eta$$

where $\chi'_\mu(\sigma) = \min\{N, \chi_\mu(\sigma)\}$. Therefore

$$\begin{aligned}\log r &\leq \log(\Lambda M K) + \sum_{j=1}^{n-1}\log|D\phi'_{\omega_j}(\pi(\sigma^j(\omega)))| \\ &\leq \log(\Lambda M K) - (n-1)(\chi'_\mu(\sigma) - \eta)\end{aligned}$$

for all $r > 0$ small enough. Hence

$$\begin{aligned}\frac{\log\big(\mu\circ\pi^{-1}(B(\pi(\omega), r))\big)}{\log r} &\leq \frac{-(\mathrm{h}_\mu(\sigma)+\eta)n}{\log(\Lambda M K) - (n-1)(\chi'_\mu(\sigma)-\eta)} \\ &= \frac{\mathrm{h}_\mu(\sigma)+\eta}{\frac{-\log(\Lambda M K)}{n} + \frac{n-1}{n}(\chi'_\mu(\sigma)-\eta)}\end{aligned}$$

for such r. Letting $r \to 0$ (and consequently $n \to \infty$), we obtain

$$\limsup_{r\to 0}\frac{\log\big(\mu\circ\pi^{-1}(B(\pi(\omega), r))\big)}{\log r} \leq \frac{\mathrm{h}_\mu(\sigma)+\eta}{\chi'_\mu(\sigma)-\eta}.$$

Since $\eta > 0$ was arbitrary, we have that

$$\limsup_{r\to 0}\frac{\log\big(\mu\circ\pi^{-1}(B(\pi(\omega), r))\big)}{\log r} \leq \frac{\mathrm{h}_\mu(\sigma)}{\chi'_\mu(\sigma)}$$

for all $\omega \in Z$. Letting $M \to +\infty$, we finally obtain

$$\overline{d}_{\mu\circ\pi^{-1}}(\pi(\omega)) = \limsup_{r\to 0}\frac{\log\big(\mu\circ\pi^{-1}(B(\pi(\omega), r))\big)}{\log r} \leq \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}$$

for all $\omega \in Z$. Therefore, as $\mu\circ\pi^{-1}(\pi(Z)) = 1$, we get that

$$\operatorname{ess\,sup}(\overline{d}_{\mu\circ\pi^{-1}}) \leq \mathrm{h}_\mu(\sigma)/\chi_\mu(\sigma). \tag{10.46}$$

Let us now prove the opposite counterpart of this inequality. If $\chi_\mu(\sigma) = +\infty$, then $\mathrm{h}_\mu(\sigma)/\chi_\mu(\sigma) = 0$ and we are done. For the rest of the proof, we assume that $\chi_\mu(\sigma) < +\infty$. Let $J_1 \subset J_{\mathcal{S}}$ be an arbitrary Borel set such that $\mu\circ\pi^{-1}(J_1) > 0$. Fix $\eta > 0$. In view of (10.45) and Egorov's Theorem there exist $n_0 \geq 1$ and a Borel set $\tilde{J}_2 \subset \pi^{-1}(J_1)$ such that $\mu(\tilde{J}_2) > \mu(\pi^{-1}(J_1))/2 > 0$,

$$\mu([\omega|_n]) \leq \exp\big((-\mathrm{h}_\mu(\sigma)+\eta)n\big) \tag{10.47}$$

and

$$\|D\phi_{\omega|_n}(\pi(\sigma^n(\omega))\| \geq \exp\big((-\chi_\mu(\sigma)-\eta)n\big)$$

for all $n \geq n_0$ and all $\omega \in \tilde{J}_2$. In view of (4.14), the final inequality implies that there exists $n_1 \geq n_0$ such that

$$\begin{aligned}\operatorname{diam}\big(\phi_{\omega|_n}(X_{t(\omega_n)})\big) &\geq (\Lambda M K)^{-1}\exp\big((-\chi_\mu(\sigma)-\eta)n\big) \\ &\geq \exp\big(-(\chi_\mu(\sigma)+2\eta)n\big)\end{aligned} \tag{10.48}$$

for all $n \ge n_1$ and all $\omega \in \tilde{J}_2$. Given $0 < r < \exp\bigl(-(\chi_\mu(\sigma) + 2\eta)n_1\bigr)$ and $\omega \in \tilde{J}_2$, let $n(\omega, r)$ be the least number n such that

$$\operatorname{diam}\bigl(\phi_{\omega|_{n+1}}(X_{t(\omega_{n+1})})\bigr) < r. \tag{10.49}$$

Using (10.48) we deduce that $n(\omega, r) + 1 > n_1$, hence $n(\omega, r) \ge n_1$ and

$$\operatorname{diam}\bigl(\phi_{\omega|_{n(\omega,r)}}(X_{t(\omega_{n(\omega,r)})})\bigr) \ge r.$$

Using Lemma 10.7 we find a universal constant $\Gamma \ge 1$ such that for every $\omega \in \tilde{J}_2$ and $0 < r < \exp\bigl(-(\chi_\mu(\sigma) + 2\eta)n_1\bigr)$ there exist $k \le \Gamma$ points $\omega^{(1)}, \dots, \omega^{(k)} \in \tilde{J}_2$ such that

$$\pi(\tilde{J}_2) \cap B(\pi(\omega), r) \subset \bigcup_{j=1}^{k} \phi_{\omega^{(j)}|_{n(\omega^{(j)},r)}} \left(X_{t\left(\omega^{(j)}_{n(\omega^{(j)},r)}\right)} \right).$$

Let $\tilde{\mu} = \mu|_{\tilde{J}_2}$ be the restriction of the measure μ to the set $\tilde{J}_2$. Using (10.42), (10.47), (10.48) and (10.49) we get

$$\begin{aligned}
\tilde{\mu} \circ \pi^{-1}(B(\pi(\omega), r)) &\le \sum_{j=1}^{k} \mu \circ \pi^{-1}\bigl(\phi_{\omega^{(j)}|_{n(\omega^{(j)},r)}}(X_{t(\omega^{(j)}_{n(\omega^{(j)},r)})})\bigr) \\
&= \sum_{j=1}^{k} \mu\bigl([\omega^{(j)}|_{n(\omega^{(j)},r)}]\bigr) \le \sum_{j=1}^{k} \exp\bigl((-\mathrm{h}_\mu(\sigma) + \eta)n(\omega^{(j)}, r)\bigr) \\
&= \sum_{j=1}^{k} \Bigl(\exp\bigl(-(\chi_\mu(\sigma) + 2\eta)(n(\omega^{(j)}, r) + 1)\bigr)\Bigr)^{\frac{n(\omega^{(j)},r)}{n(\omega^{(j)},r)+1} \cdot \frac{-\mathrm{h}_\mu(\sigma)+\eta}{-(\chi_\mu(\sigma)+2\eta)}} \\
&\le \sum_{j=1}^{k} \operatorname{diam}\left(\phi_{\omega^{(j)}|_{n(\omega^{(j)},r)+1}}\left(X_{t\left(\omega^{(j)}_{n(\omega^{(j)},r)+1}\right)}\right)\right)^{\frac{n(\omega^{(j)},r)}{n(\omega^{(j)},r)+1} \cdot \frac{\mathrm{h}_\mu(\sigma)-\eta}{\chi_\mu(\sigma)+2\eta}} \\
&\le \sum_{j=1}^{k} r^{\frac{n(\omega^{(j)},r)}{n(\omega^{(j)},r)+1} \cdot \frac{\mathrm{h}_\mu(\sigma)-\eta}{\chi_\mu(\sigma)+2\eta}} \le L r^{\frac{\mathrm{h}_\mu(\sigma)-2\eta}{\chi_\mu(\sigma)+2\eta}},
\end{aligned}$$

where the last inequality is valid provided n_1 is chosen so large that

$$\frac{n_1}{n_1 + 1} \cdot \frac{\mathrm{h}_\mu(\sigma) - \eta}{\chi_\mu(\sigma) + 2\eta} \ge \frac{\mathrm{h}_\mu(\sigma) - 2\eta}{\chi_\mu(\sigma) + 2\eta}.$$

Hence $\dim_{\mathcal{H}}(J_1) \ge \dim_{\mathcal{H}}(\pi(\tilde{J}_2)) \ge \frac{\mathrm{h}_\mu(\sigma)-2\eta}{\chi_\mu(\sigma)+2\eta}$. Since η was arbitrary, $\dim_{\mathcal{H}}(J_1) \ge \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}$. Thus $\underline{\dim}_{\mathcal{H}}(\mu \circ \pi^{-1}) \ge \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}$. By virtue of Theorem 10.22 this means that

$$\operatorname{ess\,inf} \underline{d}_{\mu \circ \pi^{-1}} \ge \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)},$$

and along with (10.46) this completes the proof of (10.43) in the case of finite entropy.

If $\mathrm{H}_\mu(\alpha) = \infty$ but $\chi_\mu(\sigma)$ is finite, then it is not difficult to see that there exists a set $Z \subset E_A^{\mathbb{N}}$ such that $\mu(Z) = 1$ and

$$\lim_{n \to \infty} \frac{-\log(\mu([\omega|_n]))}{n} = +\infty$$

for all $\omega \in Z$. Therefore the above considerations would imply that $\dim_{\mathcal{H}}(\mu) = +\infty$ which is impossible, and the proof is finished. $\square$

REMARK 10.25. Observe that the property $\mu([\omega]) = \mu\circ\pi^{-1}(\phi_\omega(X_{t(\omega)}))$, which is equivalent with (10.42), was not used in the proof of the inequality $\dim_{\mathcal{H}}(\mu\circ\pi^{-1}) \le \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}$.

REMARK 10.26. It is worth noting that $\mathrm{H}_\mu(\alpha) < \infty$ if and only if $\mathrm{H}_\mu(\alpha^q) < \infty$ for some $q \ge 1$, and therefore it suffices to assume, in Theorem 10.24, that $\mathrm{H}_\mu(\alpha^q) < \infty$ for some $q \ge 1$.

COROLLARY 10.27. *If the boundary regular Carnot conformal GDMS* $S = \{\phi_e\}_{e\in E}$ *is strongly regular, or more generally if it is regular and* $\mathrm{H}_{\tilde\mu_h}(\alpha) < \infty$, *then*

$$\dim_{\mathcal{H}}(m_h) = \dim_{\mathcal{H}}(\mu_h) = \dim_{\mathcal{H}}(J_{\mathcal{S}}).$$

Recall that μ_h is the shift-invariant version of the h-conformal measure m_h.

PROOF. We remark first that for each strongly regular system $\mathcal{S}$, $\mathrm{H}_{\tilde\mu_h}(\alpha) < \infty$. Indeed, since $\mathcal{S}$ is strongly regular, there exists $\eta > 0$ such that $Z_1(h-\eta) < \infty$, which means that $\sum_{e\in E} ||D\phi_e||_\infty^{h-\eta} < \infty$. Since by Lemma 4.18 $\lim_{e\in E} ||D\phi_e||_\infty = 0$ we have that $||D\phi_e||_\infty^{-\eta} \ge -h\log(||D\phi_e||_\infty)$ for all but finitely many $e \in E$, and therefore the series $\sum_{e\in E} -h\log(||D\phi_e||_\infty)||D\phi_e||_\infty^h$ converges. Hence, by virtue of (7.18),

$$\sum_{e\in E} -\log(\tilde\mu_h([e])\tilde\mu_h([e]) < \infty$$

which means that $\mathrm{H}_{\tilde\mu_h}(\alpha) < \infty$.

Suppose that $\mathcal{S}$ is regular and $\mathrm{H}_{\tilde\mu_h}(\alpha) < \infty$. Since m_h and μ_h are equivalent, $\dim_{\mathcal{H}}(\mu_h) = \dim_{\mathcal{H}}(m_h)$, and hence by Theorem 7.19 it suffices to show that $\dim_{\mathcal{H}}(\mu_h) = h$. Notice that for $\omega \in E_A^{\mathbb{N}}$,

$$\begin{aligned}\sum_{j=0}^{n-1} -\zeta\circ\sigma^j(\omega) &= -\sum_{j=0}^{n-1}\log\|D\phi_{\omega_{j+1}}(\pi(\sigma^{j+1}(\omega)))\| \\ &= -\log\left(\|D\phi_{\omega_1}(\pi(\sigma^1(\omega)))\|\,\|D\phi_{\omega_2}(\pi(\sigma^2(\omega)))\|\cdots\|D\phi_{\omega_n}(\pi(\sigma^n(\omega)))\|\right).\end{aligned}$$

Using (3.6) and the fact that $\pi(\sigma^k(\omega)) = \phi_{\omega_{k+1}}\circ\cdots\circ\phi_{\omega_n}(\pi(\sigma^n(\omega)))$ for $1 \le k \le n$ we deduce that

$$\sum_{j=0}^{n-1} -\zeta\circ\sigma^j(\omega) = -\log\|D\phi_{\omega|_n}(\pi(\sigma^n(\omega)))\|. \tag{10.50}$$

Since $\tilde\mu_h$ and $\tilde m_h$ are equivalent, by (7.18) there exists $\mathtt{const} > 0$ such that

$$-\log(\mathtt{const}^{-1}\,\tilde\mu_h([\omega|_n])^{1/h}) \le -\log\|D\phi_{\omega|_n}(\pi(\sigma^n(\omega)))\| \le -\log(\mathtt{const}\,\tilde\mu_h([\omega|_n])^{1/h}).$$

Therefore,

$$\lim_{n\to\infty}\frac{-\log(\tilde\mu_h([\omega|_n]))}{n} = h\lim_{n\to\infty}\frac{-\log\|D\phi_{\omega|_n}(\pi(\sigma^n(\omega)))\|}{n}. \tag{10.51}$$

Therefore by (10.50), (10.51), Birkhoff's Ergodic Theorem and the Breiman–Shannon–McMillan Theorem we get that

$$\mathrm{h}_{\tilde\mu_h}(\sigma) = \lim_{n\to\infty}\frac{-\log(\tilde\mu_h([\omega|_n]))}{n} = h\lim_{n\to\infty}\frac{-\sum_{j=0}^{n-1}\zeta\circ\sigma^j(\omega)}{n} = h\chi_{\tilde\mu_h}(\sigma).$$

for a.e. $\omega \in E_A^{\mathbb{N}}$. The proof is now completed by invoking Theorem 10.24. $\square$

We end this chapter by showing that $\tilde{\mu}_h$ is essentially the only invariant measure on $E_A^{\mathbb{N}}$ whose projection onto $J_{\mathcal{S}}$ has maximal dimension $\dim_{\mathcal{H}}(J_{\mathcal{S}})$.

THEOREM 10.28. *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a finitely irreducible and boundary regular conformal Carnot GDMS. Let μ be a Borel ergodic probability shift-invariant measure on $E_A^{\mathbb{N}}$ such that $\mathrm{H}_\mu(\alpha) < +\infty$. If $\dim_{\mathcal{H}}(\mu \circ \pi^{-1}) = h := \dim_{\mathcal{H}}(J_{\mathcal{S}})$, then $\mathcal{S}$ is regular and $\mu = \tilde{\mu}_h$.*

PROOF. If $\chi_\mu(\sigma) = +\infty$, then Theorem 10.24 implies that $h = \dim_{\mathcal{H}}(\mu \circ \pi^{-1}) = 0$ which is a contradiction. Thus $\chi_\mu(\sigma) < +\infty$ and it follows from Theorem 10.24 that $\mathrm{h}_\mu(\sigma) - h\chi_\mu(\sigma) = 0$. In view of the Second Variational Principle (Theorem 6.13), $\mathrm{P}(h) \geq 0$. Observation 7.6 then entails $\mathrm{P}(h) = 0$, meaning that $\mathcal{S}$ is regular. In consequence both shift-invariant measures μ and $\tilde{\mu}_h$ are equilibrium states of the potential $h\zeta : E_A^{\mathbb{N}} \to \mathbb{R}$. Observation 7.14 and Theorem 7.4(e) guarantee that $\mu = \tilde{\mu}_h$. The proof is complete. □

CHAPTER 11

Equivalent separation conditions for finite GDMS

In this chapter we consider the problem of finding equivalent separation conditions for finite graph directed Markov systems (GDMS). We record that the topic of equivalent separation conditions for iterated function systems (IFS) has attracted considerable attention and has been investigated from various viewpoints, see [**57**], [**58**], [**51**], [**38**], [**35**], [**6**], [**54**]. In the following we prove that for a finite irreducible weakly Carnot conformal GDMS $\mathcal{S}$, the open set condition, the strong open set condition and the positivity of the h-dimensional Hausdorff measure of the limit set $J_{\mathcal{S}}$, are equivalent conditions.

For self-similar Euclidean iterated function systems this equivalence is a celebrated result of Schief [**57**]. Peres, Rams, Simon and Solomyak [**51**] provided a beautiful proof of the aforementioned equivalence for conformal Euclidean iterated function systems. Our main result in this Chapter, namely Theorem 11.6, extends the result of Peres, Rams, Simon and Solomyak in a twofold manner. Our theorem is valid for the broader class of conformal graph directed Markov systems (IFS are examples of GDMS) and it also holds on general Carnot groups. We stress that the equivalences proved in Theorem 11.6 involve GDMS and they are new even in Euclidean spaces. Although our proof follows the scheme of Peres, Rams, Simon and Solomyak from [**51**], many nontrivial modifications are needed, partly because of the sub-Riemannian structure of $\mathbb{G}$ and partly because we work with GDMS.

If $\mathcal{S} = \{\phi_e\}_{e\in E}$ is a weakly Carnot conformal GDMS we will use the notation

$$J_\omega := J_{\mathcal{S},\omega} := \phi_\omega(J_{\mathcal{S}} \cap X_{t(\omega)}),$$

for $\omega \in E_A^{\mathbb{N}}$.

Definition 11.1. Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a weakly Carnot conformal GDMS on $(\mathbb{G}, d)$ and let $\varepsilon > 0$. The finite words $\omega, \tau \in E_A^*$ are *ε-relatively close* if $t(\omega) = t(\tau)$ and for every $p \in J_{\mathcal{S}} \cap X_{t(\omega)}$

$$d(\phi_\omega(p), \phi_\tau(p)) < \varepsilon \min\{\operatorname{diam}(J_\omega), \operatorname{diam}(J_\tau)\}.$$

Remark 11.2. Notice that if ω, τ are ε-relatively close and

$$\varepsilon < \frac{\min\{\operatorname{dist}(X_{v_1}, X_{v_2}) : v_1, v_2 \in V\}}{\max\{\operatorname{diam}(X_v) : v \in V\}} := \lambda_{\mathcal{S}}$$

then $i(\omega) = i(\tau)$.

The following condition was introduced by Bandt and Graf in [**10**].

Definition 11.3. A weakly Carnot conformal GDMS $\mathcal{S} = \{\phi_e\}_{e\in E}$ on $(\mathbb{G}, d)$ satisfies the *Bandt–Graf condition* if $\sharp(J_{\mathcal{S}} \cap X_v) > 1$ for all $v \in V$ and there exists some $\varepsilon > 0$ such that for every $\omega, \tau \in E_A^*$ with $t(\omega) = t(\tau)$ the maps ϕ_ω and ϕ_τ are not ε-relatively close.

DEFINITION 11.4. Two weakly Carnot conformal GDMS $\mathcal{S}$ and $\mathcal{S}'$ in $(\mathbb{G}, d)$ are called *equivalent* if:

(i) they share the same associated directed multigraph (E, V),
(ii) they have the same incidence matrix A and the same functions $i, t : E \to V$,
(iii) they are defined by the same set of conformal maps $\{\phi_e : W_{t(e)} \to W_{i(e)}\}$, where W_v are open connected sets, and for every $v \in V$, $X_v \cup X'_v \subset W_v$.

We state the following straightforward observation as a separate remark.

REMARK 11.5. If $\mathcal{S}$ and $\mathcal{S}'$ are two equivalent weakly Carnot conformal GDMS then $J_{\mathcal{S}} = J_{\mathcal{S}'}$.

Our main result in this chapter reads as follows.

THEOREM 11.6. *Let $\mathcal{S}$ be a finite, irreducible and maximal weakly Carnot conformal GDMS on $(\mathbb{G}, d)$. Then the following conditions are equivalent.*

(i) *There exists a Carnot conformal GDMS $\mathcal{S}'$ which is equivalent to $\mathcal{S}$.*
(ii) $\mathcal{H}^h(J_{\mathcal{S}}) > 0$ *where h is Bowen's parameter.*
(iii) *There exists a weakly Carnot conformal $\mathcal{S}'$ equivalent to $\mathcal{S}$ which satisfies the Bandt–Graf condition.*
(iv) *There exists a Carnot conformal GDMS $\mathcal{S}'$ equivalent to $\mathcal{S}$ which satisfies the strong open set condition.*

Recall that the strong open set condition (SOSC) was introduced in Definition 10.2.

REMARK 11.7. If $\mathcal{S}$ is any finite and irreducible GDMS then by Proposition 4.5 there exists another finite and irreducible GDMS $\hat{\mathcal{S}}$ which is maximal and $J_{\mathcal{S}} = J_{\hat{\mathcal{S}}}$. In several instances in the proof of Theorem 11.6 we will use the fact that the sets X_v are disjoint, while recalling the proof of Proposition 4.5, the sets $\hat{X}_v, v \in \hat{V}$, are not necessarily disjoint. Nevertheless this can be rectified using formal lifts of GDMS as it was described in Remark 4.20. Therefore the maximality assumption in Theorem 11.6 is not essential.

PROOF. The implication (i)⇒(ii) was proved in Theorem 7.18, and the implication (iv)⇒(i) is obvious, hence we only need to show that (ii)⇒(iii)⇒(iv). Before giving the proof of (ii)⇒(iii) we need several auxiliary propositions. The first one is an immediate corollary of Propositions 4.15 and 7.16.

COROLLARY 11.8. *Let $S = \{\phi_e\}_{e \in E}$ be a weakly Carnot conformal GDMS. If $\mathcal{S}$ is conformal or if $\mathcal{H}^h(J_{\mathcal{S}}) > 0$, then for every $\omega \in E_A^*$,*

$$\operatorname{diam}(\phi_\omega(J_{\mathcal{S}} \cap X_{t(\omega)})) \geq (2L^2K)^{-1}\kappa_0\mu_0\|D\phi_\omega\|_\infty,$$

where κ_0 is as in Lemma 4.14 and $\mu_0 = \min\{\operatorname{diam}(J_{\mathcal{S}} \cap X_v) : v \in V\}$.

We will also need the following two propositions involving properties of ε-relatively close words.

PROPOSITION 11.9. *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a maximal and finitely irreducible weakly Carnot conformal GDMS on $(\mathbb{G}, d)$ such that $\sharp(J_{\mathcal{S}} \cap X_v) > 1$.*

(a) *If* τ *and* τ' *are* ε*-relatively close and* $\omega \in E_A^*$ *is such that both* $\omega\tau, \omega\tau' \in E_A^*$ *then* $\omega\tau$ *and* $\omega\tau'$ *are* $c_1\varepsilon$*-relatively close, where*

$$c_1 = \frac{2(\Lambda K L)^2 M}{\kappa_0 \mu_0}.$$

(b) *If* τ *and* τ' *are* ε*-relatively close and* $\omega \in E_A^*$ *is such that both* $\tau\omega, \tau'\omega \in E_A^*$ *then* $\tau\omega$ *and* $\tau'\omega$ *are* $c_2\|D\phi_\omega\|_\infty^{-1}\varepsilon$*-relatively close, where*

$$c_2 = \frac{2(KL)^2 \Lambda M}{\kappa_0 \mu_0}.$$

(c) *If* ω *and* τ *are* ε*-relatively close then*

$$\operatorname{diam}(J_\omega) \leq (1+2\varepsilon)\operatorname{diam}(J_\tau).$$

(d) *If the words* ω, τ *are* ε_1*-relatively close and the words* τ, ρ*, are* ε_2*-relatively close words, then* ω *and* ρ *are* $(\varepsilon_1 + \varepsilon_2 + 4\varepsilon_1\varepsilon_2)$*-relatively close.*

PROOF. (a) Let $x \in J_{\mathcal{S}} \cap X_{t(\tau)}$. Then by (4.8),

$$\begin{aligned} d(\phi_{\omega\tau}(x), \phi_{\omega\tau'}(x)) &\leq \Lambda \|D\phi_\omega\|_\infty d(\phi_\tau(x), \phi_{\tau'}(x)) \\ &\leq \Lambda \|D\phi_\omega\|_\infty \varepsilon \min\{\operatorname{diam}(J_\tau), \operatorname{diam}(J_{\tau'})\}. \end{aligned} \tag{11.1}$$

Moreover by (4.6), (4.9) and Proposition 4.15 we obtain

$$\begin{aligned} \|D\phi_\omega\|_\infty \operatorname{diam}(J_\tau) &\leq \Lambda M \|D\phi_\omega\|_\infty \|D\phi_\tau\|_\infty \\ &\leq \Lambda M K \|D\phi_{\omega\tau}\|_\infty \\ &\leq \frac{2\Lambda M L^2 K^2}{\kappa_0\mu_0} \operatorname{diam}(J_{\omega\tau}), \end{aligned} \tag{11.2}$$

and in an identical manner

$$\|D\phi_\omega\|_\infty \operatorname{diam}(J_{\tau'}) \leq \frac{2\Lambda M L^2 K^2}{\kappa_0\mu_0} \operatorname{diam}(J_{\omega\tau'}). \tag{11.3}$$

Hence (a) follows after combining (11.1), (11.2) and (11.3).

(b) Let $x \in J_{\mathcal{S}} \cap X_{t(\omega)}$, then

$$\phi_\omega(x) \in J_{\mathcal{S}} \cap \phi_\omega(X_{t(\omega)}) \subset J_{\mathcal{S}} \cap X_{i(\omega)} = J_{\mathcal{S}} \cap X_{t(\tau)} = J_{\mathcal{S}} \cap X_{t(\tau')}.$$

Hence, since τ, τ' are ε-relatively close,

$$d(\phi_{\tau\omega}(x), \phi_{\tau'\omega}(x)) \leq \varepsilon \min\{\operatorname{diam}(J_\tau), \operatorname{diam}(J_{\tau'})\}. \tag{11.4}$$

Notice also that by (4.6), (4.9) and Proposition 4.15

$$\begin{aligned} \operatorname{diam}(J_\tau) \leq \Lambda M \|D\phi_\tau\|_\infty &\leq \Lambda M K \|D\phi_\omega\|_\infty^{-1} \|D\phi_{\tau\omega}\|_\infty \\ &\leq \frac{2\Lambda M (LK)^2}{\kappa_0\mu_0} \|D\phi_\omega\|_\infty^{-1} \operatorname{diam}(J_{\tau\omega}), \end{aligned}$$

and an identical inequality holds if we replace τ by τ'. These, combined with (11.4), imply (b).

(c) Since ω and τ are ε-relatively close, $X_{t(\omega)} = X_{t(\tau)}$. Then if $x, y \in J_{\mathcal{S}} \cap X_{t(\omega)}$

$$\begin{aligned} d(\phi_\omega(x), \phi_\omega(y)) &\leq d(\phi_\omega(x), \phi_\tau(x)) + d(\phi_\tau(x), \phi_\tau(y)) + d(\phi_\omega(y), \phi_\tau(y)) \\ &\leq (1+2\varepsilon)\operatorname{diam}(J_\tau). \end{aligned}$$

(d) First notice that $X_{t(\omega)} = X_{t(\tau)} = X_{t(\rho)}$. Then if $x \in J_{\mathcal{S}} \cap X_{t(\omega)}$,

$$\begin{aligned} d(\phi_\omega(x), \phi_\rho(x)) &\leq d(\phi_\omega(x), \phi_\tau(x)) + d(\phi_\rho(x), \phi_\tau(x)) \\ &\leq \varepsilon_1 \min\{\operatorname{diam}(J_\omega), \operatorname{diam}(J_\tau)\} \\ &\quad + \varepsilon_2 \min\{\operatorname{diam}(J_\rho), \operatorname{diam}(J_\tau)\}. \end{aligned} \tag{11.5}$$

If $\operatorname{diam}(J_\omega) \leq \operatorname{diam}(J_\rho)$ then trivially,

$$\min\{\operatorname{diam}(J_\omega), \operatorname{diam}(J_\tau)\} \leq \operatorname{diam}(J_\omega) = \min\{\operatorname{diam}(J_\omega), \operatorname{diam}(J_\rho)\}.$$

On the other hand if $\operatorname{diam}(J_\omega) \geq \operatorname{diam}(J_\rho)$ then by (c)

$$\begin{aligned} \operatorname{diam}(J_\tau) &\leq (1 + 2\varepsilon_2) \operatorname{diam}(J_\rho) \\ &= (1 + 2\varepsilon_2) \min\{\operatorname{diam}(J_\omega), \operatorname{diam}(J_\rho)\}. \end{aligned}$$

Thus,

$$\min\{\operatorname{diam}(J_\omega), \operatorname{diam}(J_\tau)\} \leq (1 + 2\varepsilon_2) \min\{\operatorname{diam}(J_\omega), \operatorname{diam}(J_\rho)\}. \tag{11.6}$$

In the same manner we also obtain,

$$\min\{\operatorname{diam}(J_\rho), \operatorname{diam}(J_\tau)\} \leq (1 + 2\varepsilon_1) \min\{\operatorname{diam}(J_\omega), \operatorname{diam}(J_\rho)\}. \tag{11.7}$$

Hence by (11.5), (11.6), and (11.7) for every $x \in J_{\mathcal{S}} \cap X_{t(\omega)}$

$$\begin{aligned} d(\phi_\omega(x), \phi_\rho(x)) &\leq \varepsilon_1 (1 + 2\varepsilon_2) \min\{\operatorname{diam}(J_\omega), \operatorname{diam}(J_\rho)\} \\ &\quad + \varepsilon_2 (1 + 2\varepsilon_1) \min\{\operatorname{diam}(J_\omega), \operatorname{diam}(J_\rho)\} \\ &= (\varepsilon_1 + \varepsilon_2 + 4\varepsilon_1\varepsilon_2) \min\{\operatorname{diam}(J_\omega), \operatorname{diam}(J_\rho)\} \end{aligned}$$

and the words ω and ρ are $(\varepsilon_1 + \varepsilon_2 + 4\varepsilon_1\varepsilon_2)$-relatively close. □

PROPOSITION 11.10. *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a maximal and finitely irreducible weakly Carnot conformal GDMS on $(\mathbb{G}, d)$ such that $\sharp(J_{\mathcal{S}} \cap X_v) > 1$. If for every $\varepsilon \in (0, \lambda_{\mathcal{S}})$ there exist $\omega^1, \omega^2 \in E_A^*$ which are ε-relatively close, then for every $N \in \mathbb{N}, N \geq 2$, and for every $\varepsilon \in (0, \lambda_{\mathcal{S}})$ there exist distinct words $\rho^1, \ldots, \rho^N \in E_A^*$ which are pairwise ε-relatively close.*

PROOF. It is enough to show that if for every $\varepsilon > 0$ there exist $N \geq 2$ distinct words in E_A^* which are pairwise ε-relatively close then for every ε we can find $2N$ distinct words in E_A^* which are ε-relatively close.

For the rest of the proof c_1 and c_2 will be as in the proof of Proposition 11.9. Let $\varepsilon \in (0, \lambda_{\mathcal{S}})$ and let $\rho^1, \ldots, \rho^N \in E_A^*$ be distinct words which are pairwise ε_1-relatively close, where

$$\varepsilon_1 = \min\left\{1, \lambda_{\mathcal{S}}, \frac{\varepsilon}{10c_1}\right\}.$$

Let $\Phi \subset E_A^*$ be the set of words witnessing finite irreducibility for $\mathcal{S}$. We set

$$\bar{m} = \min\{\|D\phi_{\rho^i}\|_\infty : i = 1, \ldots, N\} \cdot \min\{\|D\phi_\tau\|_\infty : \tau \in \Phi\}. \tag{11.8}$$

Let also $\omega^1, \omega^2 \in E_A^*$ which are ε_2-relatively close and

$$\varepsilon_2 = \min\left\{\lambda_{\mathcal{S}}, \frac{\varepsilon\,\bar{m}}{10\,K\,c_2}\right\}.$$

Notice that by the definition of relatively close words, $t(\omega_1) = t(\omega_2)$ and $i(\rho^i) = i(\rho^j)$ for all $i, j = 1, \dots, N$. Since $\mathcal{S}$ is maximal and finitely irreducible there exists $\tau \in E_A^*$ such

$$\omega^1 \tau \rho^i \in E_A^* \text{ and } \omega^2 \tau \rho^i \in E_A^*,$$

for all $i = 1, \dots, N$. Let

$$\mathcal{W} = \{\omega \in E_A^* : \omega = \omega^1 \tau \rho^i \text{ or } \omega = \omega^2 \tau \rho^i \text{ for some } i = 1, \dots, N\}.$$

Since $\sharp \mathcal{W} = 2N$, in order to finish the proof of the proposition it suffices to show that if $\omega, \omega' \in \mathcal{W}$ then they are ε-relatively close. We are going to prove this assertion by considering several cases.

By Proposition 11.9 (a) the words $\omega^1 \tau \rho^i$ and $\omega^1 \tau \rho^j, i, j = 1, \dots, N$, are

$$c_1 \varepsilon_1\text{-relatively close.}$$

For the same reason the words $\omega^2 \tau \rho^i$ and $\omega^2 \tau \rho^j, i, j = 1, \dots, N$, are also

$$c_1 \varepsilon_1\text{-relatively close.}$$

Moreover by Proposition 11.9 (b) we deduce that the words $\omega^1 \tau \rho^i$ and $\omega^2 \tau \rho^i$ are

$$c_2 \|D\phi_{\tau\rho}\|_\infty^{-1} \varepsilon_2\text{-relatively close.}$$

By (4.6) and recalling (11.8)

$$\|D\phi_{\tau\rho}\|_\infty \geq K^{-1} \|D\phi_\tau\|_\infty \|D\phi_{\rho_i}\|_\infty \geq K^{-1} \bar{m},$$

hence the words $\omega^1 \tau \rho^i$ and $\omega^2 \tau \rho^i$ are

$$c_2 K \bar{m}^{-1} \varepsilon_2\text{-relatively close.}$$

Finally if $i, j = 1, \dots, N$ by Proposition 11.9(d) the words $\omega^1 \tau \rho^i$ and $\omega^2 \tau \rho^j$ are

$$(c_1 \varepsilon_1 + K\, c_2 \bar{m}^{-1} \varepsilon_2 + 4K\, c_1 c_2 \bar{m}^{-1} \varepsilon_1 \varepsilon_2)\text{-relatively close.}$$

By the choice of ε_1 and ε_2 we deduce that any two words from $\mathcal{W}$ are ε-relatively close. The proof is complete. $\square$

(ii)$\Rightarrow$(iii): First of all Proposition 7.16 implies that $\sharp(J_\mathcal{S} \cap X_v) > 1$ for all $v \in V$. We fix some $N \in \mathbb{N}$. By Proposition 11.10 there exist $\rho^1, \dots, \rho^N$ which are $\lambda_\mathcal{S}/2$-relatively close. Recalling Theorem 7.4, let $\tilde{\mu}_h$ be the unique shift-invariant probability measure on $E_A^\mathbb{N}$. By (7.18) $\tilde{\mu}_h([\rho^1]) > 0$, hence by Birkhoff's Ergodic Theorem if

$$A = \{\omega \in E_A^\mathbb{N} : \sigma^n(\omega) \in [\rho^1] \text{ for infinitely many } n \in \mathbb{N}\},$$

then $\tilde{\mu}_h(A) = 1$. So by Theorem 7.4 $\tilde{m}_h(A) = \tilde{\mu}_h(A) = 1$.

Let $\omega \in A$ and let $n \in \mathbb{N}$ such that $\sigma^n(\omega) \in [\rho^1]$. Then $t(\omega_n) = i(\rho^1_1)$ and since $\mathcal{S}$ is maximal and the words $\rho^1, \dots, \rho^N$ are $\lambda_\mathcal{S}/2$-relatively close we deduce by Remark 11.2 that $t(\omega_n) = i(\rho^i_1)$ for all $i = 1, \dots N$ hence

$$\omega|_n \rho^i \in E_A^*,$$

for all $i = 1, \dots, N$. Hence Proposition 11.9 (a) implies that the words $\omega|_n \rho^1, \dots, \omega|_n \rho^N$ are $c_1 \lambda_\mathcal{S}/2$-relatively close. Now notice that if $p, q \in J_\mathcal{S} \cap X_{t(\rho^i)}$ and $i, j = 1, \dots, N$ then

$$\begin{aligned} d(\phi_{\omega|_n \rho^i}(p), \phi_{\omega|_n \rho^i}(q)) &\leq d(\phi_{\omega|_n \rho^i}(p), \phi_{\omega|_n \rho^j}(p)) + d(\phi_{\omega|_n \rho^j}(p), \phi_{\omega|_n \rho^j}(q)) \\ &\quad + d(\phi_{\omega|_n \rho^j}(q), \phi_{\omega|_n \rho^i}(q)) \\ &\leq (1 + c_1 \lambda_\mathcal{S}) \operatorname{diam}(J_{\omega|_n \rho^j}). \end{aligned}$$

Hence we deduce that

$$\max_{i=1,\dots,N} \operatorname{diam}(J_{\omega|_n\rho^i}) \le (1+c_1\lambda_{\mathcal{S}}) \min_{i=1,\dots,N} \operatorname{diam}(J_{\omega|_n\rho^i}). \tag{11.9}$$

We will now show that if $x=\pi(\omega), x\in A$ then

$$\bigcup_{i=1}^{N} J_{\omega|_n\rho^i} \subset B(x,r), \tag{11.10}$$

where

$$r=\left(1+\frac{c_1\lambda_{\mathcal{S}}}{2}\right)\max_{i=1,\dots,N}\{\operatorname{diam}(J_{\omega|_n\rho^i})\}. \tag{11.11}$$

First notice that since the words ρ^i are $\lambda_{\mathcal{S}}/2$-relatively close, Proposition 11.9(a) implies that for any $i=1,\dots,N$ and any $p\in J_{\mathcal{S}}\cap X_{t(\rho^1)}$ (recall that $t(\rho^1)=t(\rho^i)$ for all $i=1,\dots,N$) we have

$$d(\phi_{\omega|_n\rho^1}(p),\phi_{\omega|_n\rho^i}(p)) \le \frac{c_1\lambda_{\mathcal{S}}}{2}\min\{\operatorname{diam}(J_{\omega|_n\rho^1}),\operatorname{diam}(J_{\omega|_n\rho^i})\}. \tag{11.12}$$

Since $x=\pi(\omega)$ and $\sigma^n(\omega)\in[\rho^1]$, x can be written as

$$x=\phi_{\omega|_n\rho^1}(\pi(\sigma^{n+|\rho^1|}(\omega))):=\phi_{\omega|_n\rho^1}(x_0),$$

where $x_0\in J_{\mathcal{S}}\cap X_{t(\rho^1)}$. Therefore, for any $i=2,\dots,\mathbb{N}$, and for any $q=\phi_{\omega|_n\rho^i}(p), p\in J_{\mathcal{S}}\cap X_{t(\rho^i)}$,

$$\begin{aligned} d(x,q) &= d(\phi_{\omega|_n\rho^1}(x_0),\phi_{\omega|_n\rho^i}(p))\\ &\le d(\phi_{\omega|_n\rho^1}(x_0),\phi_{\omega|_n\rho^1}(p))+d(\phi_{\omega|_n\rho^1}(p),\phi_{\omega|_n\rho^i}(p))\\ &\le \operatorname{diam}(J_{\omega|_n\rho^1})+\frac{c_1\lambda_{\mathcal{S}}}{2}\max_{i=1,\dots,N}\{\operatorname{diam}(J_{\omega|_n\rho^i})\}\\ &\le \left(1+\frac{c_1\lambda_{\mathcal{S}}}{2}\right)\max_{i=1,\dots,N}\operatorname{diam}(J_{\omega|_n\rho^i})\}, \end{aligned}$$

and (11.10) follows.

Now notice that $[\omega|_n\rho^i]\subset\pi^{-1}(J_{\omega|_n\rho^i})$. Hence by (11.10), (7.18) and (4.9) we deduce that if $x=\pi(\omega)$ and $\omega\in A$,

$$\begin{aligned} m_h(B(x,r)) &= \tilde{m}_h(\pi^{-1}(B(x,r)))\\ &\ge m_h\left(\bigcup_{i=1}^{N}\pi^{-1}(J_{\omega|_n\rho^i})\right)\\ &\ge \tilde{m}_h\left(\bigcup_{i=1}^{N}[\omega|_n\rho^i]\right)=\sum_{i=1}^{N}\tilde{m}_h([\omega|_n\rho^i])\\ &\ge c_h\sum_{i=1}^{N}\|D\phi_{\omega|_n\rho^i}\|_\infty^h\\ &\ge c_h(\Lambda M)^{-1}\sum_{i=1}^{N}\operatorname{diam}(J_{\omega|_n\rho^i})^h\\ &\ge c_h(\Lambda M)^{-1}N\min_{i=1,\dots,N}\{\operatorname{diam}(J_{\omega|_n\rho^i})^h\}. \end{aligned}$$

Therefore by (11.9) and (11.11)

$$\frac{m_h(B(x,r))}{r^h} \geq \frac{c_h}{\Lambda M(1+c_1\lambda_{\mathcal{S}})^{2h}} N. \tag{11.13}$$

Since $r \to 0$ as $n \to \infty$, (11.13) implies that

$$\limsup_{s\to 0} \frac{m_h(B(x,s))}{s^h} \geq \frac{c_h}{\Lambda M(1+c_1\lambda_{\mathcal{S}})^{2h}} N,$$

for $x \in \pi(A)$. Hence by [**46**, Theorem A2.0.12] we deduce that

$$\mathcal{H}^h(\pi(A)) \lesssim N^{-1} m_h(\pi(A))$$

where the constant only depends on the system $\mathcal{S}$. Moreover,

$$m_h(\pi(A)) = \tilde{m}_h(\pi^{-1}(\pi(A))) \geq \tilde{m}_h(A) = 1,$$

hence $m_h(J_{\mathcal{S}} \setminus \pi(A)) = 0$ and by Theorem 10.1 we also get that $\mathcal{H}^h(J_{\mathcal{S}} \setminus \pi(A)) = 0$. Therefore

$$\mathcal{H}^h(J_{\mathcal{S}}) \lesssim N^{-1} m_h(J_{\mathcal{S}}).$$

Since N can be taken arbitrarily large we deduce that $\mathcal{H}^h(J_{\mathcal{S}}) = 0$ and we have reached a contradiction. The proof of the implication (ii)⇒(iii) is complete.

For the proof of the implication (iii)⇒(iv) we will use several lemmas. Following [**51**] for any $\omega \in E_A^*$ and any $\alpha > 0$ and $T \geq 1$ we define

$$W_{\alpha,T}(\omega) = \Big\{\rho \in E_A^* : i(\omega) = i(\rho), \quad T^{-1} \leq \frac{\operatorname{diam}(J_\rho)}{\operatorname{diam}(J_\omega)} \leq T, \\ \text{and } \operatorname{dist}(J_\rho, J_\omega) \leq \alpha \operatorname{diam}(J_\omega)\Big\}.$$

In the following lemma we show that the Bandt–Graf condition implies that the cardinality of $W_{\alpha,T}(\omega)$ is bounded for all $\omega \in E_A^*$ and the bounds only depend on α and T.

Lemma 11.11. *Let $\mathcal{S}$ be a finite, irreducible, and maximal weakly Carnot conformal GDMS on $(\mathbb{G}, d)$. If $\mathcal{S}$ satisfies the Bandt–Graf condition, then for all $\alpha > 0$ and $T \geq 1$, there exist positive constants $C(\alpha,T)$ such that for all $\omega \in E_A^*$,*

$$\sharp W_{\alpha,T}(\omega) \leq C(\alpha,T).$$

Proof. Fix $\omega \in E_A^*$ and $\alpha > 0, T \geq 1$. For $v \in V$ let

$$W^v_{\alpha,T}(\omega) = \{\rho \in W_{\alpha,T} : t(\rho) = v\}.$$

Notice that it is enough to show that for all $v \in V$ there exist positive constants $C^v(\alpha,T)$ such that

$$\sharp W^v_{\alpha,T}(\omega) \leq C^v(\alpha,T). \tag{11.14}$$

To this end, fix some $v \in V$ and let

$$r_\varepsilon = \min\Big\{\eta_{\mathcal{S}}, \frac{\kappa_0 \mu_0}{8\Lambda^3 K T^2}\varepsilon\Big\}$$

where all the appearing constants are as in Section 4.2, and ε comes from the Bandt–Graf condition. Since $\mathcal{S}$ is finite, the limit set $J_{\mathcal{S}}$ is compact. Hence there exists some $n_0 \in \mathbb{N}$ and $\{x_1, \dots, x_{n_0}\} \in J_{\mathcal{S}} \cap X_v$ such that

$$J_{\mathcal{S}} \cap X_v \subset \bigcup_{i=1}^{n_0} B(x_i, r_\varepsilon).$$

By the Bandt–Graf condition there exists $x_v \in J_{\mathcal{S}} \cap X_v$ such that for all $\tau, \rho \in W^v_{\alpha,T}$,

$$d(\phi_\tau(x_v), \phi_\rho(x_v)) \geq \varepsilon \min\{\mathrm{diam}(J_\tau), \mathrm{diam}(J_\rho)\}. \tag{11.15}$$

If $\tilde{x} \in \mathbb{G}$ and $d(x_v, \tilde{x}) \leq r_\varepsilon$ then $\tilde{x} \in S_v$. Hence by the choice of r_e, (11.15), (4.11) and Corollary 4.16 imply that

$$\begin{aligned} d(\phi_\tau(\tilde{x}), \phi_\rho(\tilde{x})) &\geq d(\phi_\tau(x_v), \phi_\rho(x_v)) - d(\phi_\rho(\tilde{x}), \phi_\rho(x_v)) - d(\phi_\tau(\tilde{x}), \phi_\tau(x_v)) \\ &\geq \varepsilon \min\{\mathrm{diam}(J_\tau), \mathrm{diam}(J_\rho)\} - \Lambda d(x_v, \tilde{x})(\|D\phi_\rho\|_\infty + \|D\phi_\tau\|_\infty) \\ &\geq \varepsilon \min\{\mathrm{diam}(J_\tau), \mathrm{diam}(J_\rho)\} - \Lambda r_\varepsilon(\|D\phi_\rho\|_\infty + \|D\phi_\tau\|_\infty) \\ &\geq \varepsilon \min\{\mathrm{diam}(J_\tau), \mathrm{diam}(J_\rho)\} - \frac{2\Lambda^3 K}{\kappa_0 \mu_0} r_\varepsilon(\mathrm{diam}(J_\tau) + \mathrm{diam}(J_\rho)). \end{aligned}$$

Since $\rho, \tau \in W_{\alpha,T}(\omega)$,

$$d(\phi_\tau(\tilde{x}), \phi_\rho(\tilde{x})) \geq \left(\varepsilon T^{-1} - \frac{4\Lambda^3 K}{\kappa_0 \mu_0} T\, r_\varepsilon\right) \mathrm{diam}(J_\omega),$$

hence by the choice of r_ε,

$$d(\phi_\tau(\tilde{x}), \phi_\rho(\tilde{x})) \geq \frac{\varepsilon}{2T} \mathrm{diam}(J_\omega). \tag{11.16}$$

Using the standard method for constructing product measures, see e.g. [30, Section 2.5], there exists a Borel measure ν on $\mathbb{G}^{n_0}$, which is identified with $(\mathbb{R}^N)^{n_0}$, such that if $\{A_i\}_{i=1}^{n_0}$ are Borel subsets of $\mathbb{G}$ then

$$\nu\left(\prod_{i=1}^{n_0} A_i\right) = |A_i|^{n_0}. \tag{11.17}$$

For $\Xi = (\xi_1, \dots, x_{n_0}) \in \mathbb{G}^{n_0}$ and $r > 0$ let

$$\mathbb{B}(\Xi, r) := \{Y = (y_1, \dots, y_{n_0}) \in \mathbb{G}^{n_0} : d(y_i, \xi_i) < r\}.$$

For any $\rho \in W^v_{\alpha,T}(\omega)$, let

$$\Xi_\rho = (\phi_\rho(x_1), \dots, \phi_\rho(x_{n_0})) \in \mathbb{G}^{n_0}.$$

Notice that Ξ_ρ is well defined because for all $i = 1, \dots, n_0$, $x_i \in X_v$, $t(\rho) = v$ and and $\mathcal{S}$ is maximal, hence $\phi_\rho(x_i)$ makes sense for $i = 1, \dots, n_0$. We also define,

$$\mathbb{B}_\rho := \mathbb{B}\left(\Xi_\rho, \frac{\mathrm{diam}(J_\omega)\varepsilon}{4T}\right).$$

We are now going to prove that

$$\mathbb{B}_\rho \cap \mathbb{B}_\tau = \emptyset, \text{ for } \tau \neq \rho, \tau, \rho \in W^v_{\alpha,T}(\omega). \tag{11.18}$$

By way of contradiction assume that (11.18) fails. Then there exists some $Y = (y_1, \dots, y_{n_0}) \in \mathbb{B}_\rho \cap \mathbb{B}_\tau$ such that

$$d(y_i, \phi_\rho(x_i)) < \frac{\mathrm{diam}(J_\omega)\varepsilon}{4T} \text{ and } d(y_i, \phi_\tau(x_i)) < \frac{\mathrm{diam}(J_\omega)\varepsilon}{4T},$$

for all $i = 1, \dots, n_0$. Hence

$$d(\phi_\rho(x_i), \phi_\tau(x_i)) < \frac{\mathrm{diam}(J_\omega)\varepsilon}{2T}. \tag{11.19}$$

Since $J_{\mathcal{S}} \cap X_v \subset \cup_{i=1}^{n_0} B(x_i, r_\varepsilon)$ there exists some $i_0 = 1, \dots, n_0$, such that $d(x_v, x_{i_0}) < r_\varepsilon$. Hence by (11.16)

$$d(\phi_\rho(x_{i_0}), \phi_\tau(x_{i_0})) \geq \frac{\operatorname{diam}(J_\omega)\varepsilon}{2T},$$

which contradicts (11.19). Therefore (11.18) follows.

For every $x \in J_{\mathcal{S}} \cap X_v$ and for every $\rho, \tau \in W^v_{\alpha,T}(\omega)$,

$$\begin{aligned} d(\phi_\rho(x), \phi_\tau(x)) &\leq \operatorname{diam}(J_\tau) + \operatorname{dist}(J_\tau, J_\omega) + \operatorname{diam}(J_\omega) \\ &\qquad + \operatorname{dist}(J_\omega, J_\rho) + \operatorname{diam}(J_\rho) \\ &\leq (2(\alpha + T) + 1) \operatorname{diam}(J_\omega). \end{aligned} \tag{11.20}$$

Fix some $\rho \in W^v_{\alpha,T}(\omega)$ and let

$$\mathbb{B}_0 = \mathbb{B}\big(\Xi_\rho, (1 + 2(\alpha + T) + \varepsilon T^{-1}) \operatorname{diam}(J_\omega)\big).$$

We are now going to show that

$$\mathbb{B}_\tau \subset \mathbb{B}_0 \tag{11.21}$$

for every $\tau \in W^v_{\alpha,T}(\omega)$. Let $\tau \in W^v_{\alpha,T}(\omega)$ and let $Y = (y_1, \dots, y_{n_0}) \in \mathbb{B}_\tau$. Then by (11.20) for all $i = 1, \dots, n_0$

$$\begin{aligned} d(y_i, \phi_\rho(x_i)) &\leq d(y_i, \phi_\tau(x_i)) + d(\phi_\tau(x_i), \phi_\rho(x_i)) \\ &\leq \left(\frac{1}{4T}\varepsilon + 2(\alpha + T) + 1 \right) \operatorname{diam}(J_\omega), \end{aligned}$$

and (11.21) follows.

Notice that by (1.9)

$$\begin{aligned} \nu(\mathbb{B}_0) &= \nu \left(\prod_{i=1}^{n_0} B(\phi_\rho(x_i), (1 + 2(\alpha + T) + \varepsilon T^{-1}) \operatorname{diam}(J_\omega)) \right) \\ &= \prod_{i=1}^{n_0} \big| B(\phi_\rho(x_i), (1 + 2(\alpha + T) + \varepsilon T^{-1}) \operatorname{diam}(J_\omega)) \big| \\ &= c_0 \left((1 + 2(\alpha + T) + \varepsilon T^{-1}) \operatorname{diam}(J_\omega) \right)^{n_0 Q}. \end{aligned}$$

Moreover by (11.18) and (11.21)

$$\nu(\mathbb{B}_0) \geq \sum_{\tau \in W^v_{\alpha,T}(\omega)} \nu(\mathbb{B}_\tau) = \sharp W^v_{\alpha,T}(\omega) \left(\frac{\operatorname{diam}(J_\omega)}{4T} \varepsilon \right)^{n_0 Q}.$$

Hence,

$$\sharp W^v_{\alpha,T}(\omega) \leq \left(\frac{4T(1 + 2(\alpha + T) + \varepsilon T^{-1})}{\varepsilon} \right)^{n_0 Q},$$

and the proof is complete. □

REMARK 11.12. For any $\omega \in E_A^*$ and any $\alpha > 0, T \geq 1$ let

$$\begin{aligned} W^{cc}_{\alpha,T}(\omega) = \Big\{ \rho \in E_A^* : i(\omega) = i(\rho), \quad T^{-1} \leq \frac{\operatorname{diam}_{cc}(J_\rho)}{\operatorname{diam}_{cc}(J_\omega)} \leq T, \\ \text{and } \operatorname{dist}_{cc}(J_\rho, J_\omega) \leq \alpha \operatorname{diam}_{cc}(J_\omega) \Big\}. \end{aligned}$$

It follows immediately by Lemma 11.11 and (1.7) that there exist positive constants $C^{cc}(\alpha, T)$ such that

$$\sharp W^{cc}_{\alpha,T}(\omega) \leq C^{cc}(\alpha, T). \tag{11.22}$$

In fact $C^{cc}(\alpha, T) = C(L\alpha, LT)$.

LEMMA 11.13. *Let $\mathcal{S}$ be a weakly Carnot conformal GDMS on $(\mathbb{G}, d)$. If $\tau \in E_A^*$ satisfies $\operatorname{diam}_{cc}(J_\tau) < \frac{\eta_{\mathcal{S}}}{LKC}$ then for all $\omega \in E_A^*$ such that $\omega\tau \in E_A^*$ and for all $y \in \mathbb{G}$ such that $\operatorname{dist}_{cc}(y, J_\tau) \leq \operatorname{diam}_{cc}(J_\tau)$,*

$$\exp(-\tilde{C} \operatorname{diam}_{cc}(J_\tau)) \leq \frac{\operatorname{diam}_{cc}(J_{\omega\tau})}{\|D\phi_\omega(y)\| \operatorname{diam}_{cc}(J_\tau)} \leq \exp(\tilde{C} \operatorname{diam}_{cc}(J_\tau)), \tag{11.23}$$

where $\tilde{C} = \frac{\Lambda_0 K(2+LKC)}{1-s}$.

PROOF. We will first establish the right hand inequality. Notice that

$$\begin{aligned} \operatorname{dist}_{cc}(y, X_{i(\tau)}) &\leq \operatorname{dist}_{cc}(y, \phi_\tau(X_t(\tau))) \leq \operatorname{dist}_{cc}(y, J_\tau) \\ &\leq \operatorname{diam}_{cc}(J_\tau) < \eta_{\mathcal{S}}. \end{aligned} \tag{11.24}$$

Since $\omega\tau \in E_A^*$, we know that $i(\tau) = t(\omega)$ hence by (11.24) $y \in S_{t(\omega)}$. Therefore $\phi_\omega(y)$ and $\|D\phi_\omega(y)\|$ are well defined.

For $p, q \in \mathbb{G}$, if $\gamma_{p,q} : [0, T] \to \mathbb{G}$ is the horizontal geodesic curve joining p and q, we will denote its arc by

$$[\gamma_{p,q}] = \{\gamma_{p,q}(t) : t \in [0, T]\}.$$

Let $p, q \in J_{\mathcal{S}} \cap X_{t(\tau)}$. Then by the segment property [12, Corollary 5.15.6],

$$\begin{aligned} [\gamma_{\phi_\tau(p),\phi_\tau(q)}] &\subset B_{cc}(\phi_\tau(p), d_{cc}(\phi_\tau(p), \phi_\tau(q))) \\ &\subset B_{cc}(\phi_\tau(p), \operatorname{diam}_{cc}(J_\tau)) \\ &\subset B_{cc}(\phi_\tau(p), \eta_{\mathcal{S}}) \subset N_{t(\omega)}. \end{aligned} \tag{11.25}$$

By Lemma 3.8 and (11.25) there exists some $z \in \mathbb{G}$ such that,

$$\operatorname{dist}_{cc}(z, J_\tau) < \eta_{\mathcal{S}}, \tag{11.26}$$

$$\|D\phi_\omega(z)\| = \max\{\|D\phi_\omega(\zeta)\| : \zeta \in [\gamma_{\phi_\tau(p),\phi_\tau(q)}]\}, \tag{11.27}$$

and

$$d_{cc}(\phi_{\omega\tau}(p), \phi_{\omega\tau}(q)) \leq \|D\phi_\omega(z)\| d_{cc}(\phi_\tau(p), \phi_\tau(q)). \tag{11.28}$$

If $y \in \mathbb{G}$ satisfies $\operatorname{dist}_{cc}(y, J_\tau) < \operatorname{diam}_{cc}(J_\tau)$, then $y \in N_{t(\omega)}$ and by Remark 4.17, since also $z \in N_{t(\omega)}$ by (11.25),

$$\frac{\|D\phi_\omega(z)\|}{\|D\phi_\omega(y)\|} \leq \exp\left(\frac{\Lambda_0 K}{1-s} d_{cc}(y, z)\right). \tag{11.29}$$

Moreover by (11.26),

$$\begin{aligned} d_{cc}(y, z) &\leq \operatorname{dist}_{cc}(y, J_\tau) + \operatorname{dist}_{cc}(z, J_\tau) + \operatorname{diam}_{cc}(J_\tau) \\ &\leq 3 \operatorname{diam}_{cc}(J_\tau), \end{aligned}$$

hence by (11.29)

$$\frac{\|D\phi_\omega(z)\|}{\|D\phi_\omega(y)\|} \leq \exp\left(\frac{3\Lambda_0 K}{1-s} \operatorname{diam}_{cc}(J_\tau)\right). \tag{11.30}$$

By (11.28) and (11.30) we deduce that

$$\mathrm{diam}_{\mathrm{cc}}(J_{\omega\tau}) \leq \|D\phi_\omega(y)\| \exp\left(\frac{3\Lambda_0 K}{1-s}\,\mathrm{diam}_{\mathrm{cc}}(J_\tau)\right)\mathrm{diam}_{\mathrm{cc}}(J_\tau),$$

and the right hand inequality has been proven.

We now move to the left hand inequality. First notice that by (11.28) and Corollary 4.10 if $p \in J_{\mathcal{S}} \cap X_{t(\tau)}$ then

$$\begin{aligned} B_{cc}(\phi_{\omega\tau}(p), \mathrm{diam}_{\mathrm{cc}}(J_{\omega\tau})) &\subset B_{cc}(\phi_{\omega\tau}(p), \|D\phi_\omega\|_\infty \,\mathrm{diam}_{\mathrm{cc}}(J_\tau)) \\ &\subset \phi_\omega(B_{cc}(\phi_\tau(p), LKC\,\mathrm{diam}_{\mathrm{cc}}(J_\tau))). \end{aligned}$$

Hence if $\xi \in B_{cc}(\phi_{\omega\tau}(p), \mathrm{diam}_{\mathrm{cc}}(J_{\omega\tau}))$ then

$$\phi_\omega^{-1}(\xi) \in B_{cc}(\phi_\tau(p), LKC\,\mathrm{diam}_{\mathrm{cc}}(J_\tau)) \subset B_{cc}(\phi_\tau(p), \eta_{\mathcal{S}}). \tag{11.31}$$

In particular,

$$\mathrm{dist}_{\mathrm{cc}}(\phi^{-1}(\xi), J_\tau) \leq LKC\,\mathrm{diam}_{\mathrm{cc}}(J_\tau). \tag{11.32}$$

If $p, q \in J_{\mathcal{S}} \cap X_{t(\tau)}$ then by (11.28),

$$d_{cc}(\phi_{\omega\tau}(p), \phi_{\omega\tau}(q)) \leq \|D\phi_\omega\|_\infty \,\mathrm{diam}_{\mathrm{cc}}(J_\tau).$$

Hence if $\gamma_{\phi_{\omega\tau}(p),\phi_{\omega\tau}(q)}$ is the horizontal geodesic joining $\phi_{\omega\tau}(p)$ and $\phi_{\omega\tau}(q)$,

$$[\gamma_{\phi_{\omega\tau}(p),\phi_{\omega\tau}(q)}] \subset B_{cc}(\phi_{\omega\tau}(p), d_{cc}(\phi_{\omega\tau}(p), \phi_{\omega\tau}(q))) \subset N_{i(\omega)}.$$

Therefore by Lemma 3.8 there exists some $z \in B_{cc}(\phi_{\omega\tau}(p), d_{cc}(\phi_{\omega\tau}(p), \phi_{\omega\tau}(q)))$ such that

$$\begin{aligned} d_{cc}(\phi_\tau(p), \phi_\tau(q)) &= d_{cc}(\phi_\omega^{-1}(\phi_{\omega\tau}(p)), \phi_\omega^{-1}(\phi_{\omega\tau}(p))) \\ &\leq \|D\phi_\omega^{-1}(z)\|\, d_{cc}(\phi_{\omega\tau}(p), \phi_{\omega\tau}(q)) \\ &= \|D\phi_\omega(\phi_\omega^{-1}(z))\|^{-1} d_{cc}(\phi_{\omega\tau}(p), \phi_{\omega\tau}(q)). \end{aligned} \tag{11.33}$$

If y satisfies $\mathrm{dist}_{cc}(y, J_\tau) \leq \mathrm{diam}_{\mathrm{cc}}(J_\tau)$, then by (11.32)

$$d_{cc}(y, \phi_\omega^{-1}(z)) \leq (2 + LKC)\,\mathrm{diam}_{\mathrm{cc}}(J_\tau).$$

Now by Remark 4.17, since by (11.31) $\phi_\omega^{-1}(z) \subset B_{cc}(\phi_\tau(p), \eta_{\mathcal{S}}) \subset N_{t(\omega)}$,

$$\frac{\|D\phi_\omega(y)\|}{\|D\phi_\omega(\phi_\omega^{-1}(z))\|} \leq \exp\left(\frac{(2+LKC)\Lambda_0 K}{1-s}\,\mathrm{diam}_{\mathrm{cc}}(J_\tau)\right). \tag{11.34}$$

The left hand side inequality follows after combining (11.33) and (11.34). The proof of the lemma is complete. □

LEMMA 11.14. *Let $\mathcal{S}$ be a finite, maximal, weakly Carnot conformal GDMS on $(\mathbb{G}, d)$. Let $T_0 \geq 1$ and $\varepsilon > 0$. Then there exists some $\mathtt{d}_{T_0,\varepsilon}$ such that for every $\tau \in E_A^*$ such that $\mathrm{diam}_{\mathrm{cc}}(J_\tau) \leq \mathtt{d}_{T_0,\varepsilon}$, every $\alpha \in [0,1]$, every $T \in [T_0, 2T_0]$ and every pair of words, $\omega, \rho \in E_A^*$ such that $\rho \in W^{cc}_{\alpha,T}(\tau)$ and $\omega\rho \in E_A^*$,*

$$\omega\rho \in W^{cc}_{\alpha(1+\varepsilon),T(1+\varepsilon)}(\omega\tau). \tag{11.35}$$

PROOF. Recalling the definition of $W^{cc}_{\alpha,T}(\tau)$, see Remark 11.12, we deduce that $i(\rho) = i(\tau)$. Since $\omega\rho \in E_A^*$, $i(\rho) = t(\omega)$; hence $t(\omega) = i(\tau)$ and by the maximality of $\mathcal{S}$, $\omega\tau \in E_A^*$. We will assume that $\mathtt{d}_{T_0,\varepsilon} < \frac{\eta_{\mathcal{S}}}{2T_0 LKC}$. Since trivially $i(\omega\tau) = i(\omega\rho)$, in order to establish (11.35) it suffices to show that

$$\frac{1}{T(1+\varepsilon)} \leq \frac{\mathrm{diam}_{\mathrm{cc}}(J_{\omega\tau})}{\mathrm{diam}_{\mathrm{cc}}(J_{\omega\rho})} \leq T(1+\varepsilon), \tag{11.36}$$

and

$$\operatorname{dist}_{cc}(J_{\omega\tau}, J_{\omega\rho}) \le \alpha(1+\varepsilon)\operatorname{diam}_{cc}(J_{\omega\tau}). \tag{11.37}$$

We will first check (11.36). Since $\rho \in W^{cc}_{\alpha,T}(\tau)$,

$$\operatorname{dist}_{cc}(J_\tau, J_\rho) \le \alpha \operatorname{diam}_{cc}(J_\tau). \tag{11.38}$$

Thus, there exists some $z \in J_\rho$ such that

$$d_{cc}(z, J_\tau) \le \alpha \operatorname{diam}_{cc}(J_\tau) \le \mathtt{d}_{T_0,\varepsilon} < \frac{\eta_{\mathcal{S}}}{LKC}.$$

Therefore we can apply Lemma 11.13 and obtain,

$$\operatorname{diam}_{cc}(J_{\omega\tau}) \le \|D\phi_\omega(z)\| \exp(\tilde{C}\operatorname{diam}_{cc}(J_\tau))\operatorname{diam}_{cc}(J_\tau). \tag{11.39}$$

Notice that since $\rho \in W^{cc}_{\alpha,T}(\tau)$,

$$\operatorname{diam}_{cc}(J_\rho) \le T\operatorname{diam}_{cc}(J_\tau) \le 2T_0\mathtt{d}_{T_0,\varepsilon} < \frac{\eta_{\mathcal{S}}}{LKC}.$$

Therefore, since also $z \in J_\rho$, we can apply Lemma 11.13 once more and get

$$\operatorname{diam}_{cc}(J_{\omega\rho}) \ge \|D\phi_\omega(z)\| \exp(-\tilde{C}\operatorname{diam}_{cc}(J_\rho))\operatorname{diam}_{cc}(J_\rho). \tag{11.40}$$

Combining (11.39) and (11.40) we obtain

$$\begin{aligned}\frac{\operatorname{diam}_{cc}(J_{\omega\tau})}{\operatorname{diam}_{cc}(J_{\omega\rho})} &\le \frac{\operatorname{diam}_{cc}(J_\tau)}{\operatorname{diam}_{cc}(J_\rho))}\exp(\tilde{C}\mathtt{d}_{T_0,\varepsilon}(1+2T_0)) \\ &\le T\exp(\tilde{C}\mathtt{d}_{T_0,\varepsilon}(1+2T_0)).\end{aligned}$$

Choosing $\mathtt{d}_{T_0,\varepsilon}$ small enough, we obtain the right hand inequality in (11.36). The proof of the remaining inequality in (11.36) is very similar and we omit it.

We will now establish (11.37). By (11.38) there exist $p \in J_{\mathcal{S}} \cap X_{t(\tau)}$ and $q \in J_{\mathcal{S}} \cap X_{t(\rho)}$ such that

$$\operatorname{dist}_{cc}(J_\tau, J_\rho) = d_{cc}(\phi_\tau(p), \phi_\rho(q)) \le \alpha \operatorname{diam}_{cc}(J_\tau). \tag{11.41}$$

Hence, if $\gamma_{\phi_\tau(p),\phi_\rho(q)}$ is the horizontal geodesic curve joining $\phi_\tau(p)$ and $\phi_\rho(q)$, arguing as in (11.25) and using (11.41) we deduce that

$$\begin{aligned}[\gamma_{\phi_\tau(p),\phi_\rho(q)}] &\subset B_{cc}(\phi_\tau(p), d_{cc}(\phi_\tau(p), \phi_\tau(q)) \\ &\subset B_{cc}(\phi_\tau(p), \alpha\operatorname{diam}_{cc}(J_\tau)) \\ &\subset B_{cc}(\phi_\tau(p), \mathtt{d}_{T_0,\varepsilon}) \subset N_{i(\omega)}.\end{aligned}$$

Therefore by Lemma 3.8 there exists some $\zeta \in N_{i(\omega)}$ such that

$$\operatorname{dist}_{cc}(\zeta, J_\tau) \le \operatorname{diam}_{cc}(J_\tau) \le \mathtt{d}_{T_0,\varepsilon}$$

and

$$d_{cc}(\phi_{\omega\tau}(p), \phi_{\omega\rho}(q)) \le \|D\phi_\omega(\zeta)\| d_{cc}(\phi_\tau(p), \phi_\rho(q)).$$

Thus,

$$\operatorname{dist}_{cc}(J_{\omega\tau}, J_{\omega\rho}) \le \alpha\|D\phi_\omega(\zeta)\|\ \operatorname{diam}_{cc}(J_\tau). \tag{11.42}$$

By Lemma 11.13,

$$\|D\phi_\omega(\zeta)\| \le \frac{\operatorname{diam}_{cc}(J_{\omega\tau})}{\operatorname{diam}_{cc}(J_\tau)}\exp(\tilde{C}\operatorname{diam}_{cc}(J_\tau)). \tag{11.43}$$

Combining (11.42) and (11.43),

$$\begin{aligned}\operatorname{dist}_{cc}(J_{\omega\tau}, J_{\omega\rho}) &\leq \alpha \operatorname{diam}_{cc}(J_{\omega\tau}) \exp(\tilde{C}\mathtt{d}_{T_0,\varepsilon})) \\ &\leq \alpha(1+\varepsilon) \operatorname{diam}_{cc}(J_{\omega\tau}),\end{aligned}$$

assuming that $\mathtt{d}_{T_0,\varepsilon}$ is taken small enough. Therefore (11.37) has been proven and the proof of the lemma is complete. □

PROPOSITION 11.15. *Let $\mathcal{S}$ be a finite, irreducible and maximal weakly Carnot conformal GDMS on $(\mathbb{G}, d)$. If $\mathcal{S}$ satisfies the Bandt–Graf condition, then there exist open sets $O_v, v \in V$, such that,*

(1) $J_{\mathcal{S}} \cap O_v \neq \emptyset$ for all $v \in V$,
(2) $O_v \subset \operatorname{Int}(N_v) \subset W_v$, for all $v \in V$,
(3) if $O := \cup_{v\in V} O_v$ then for all $\tau, \rho \in E_A^*, \tau \neq \rho$,

$$\phi_\tau(O) \subset O \text{ and } \phi_\tau(O) \cap \phi_\rho(O) = \emptyset.$$

PROOF. Recall that the sets N_v where defined in Remark 4.17. Since E is finite, there exists T_0 large enough such that for all $e \in E$ and for all $\tau \in E_A^*$ such that $\tau e \in E_A^*$,

$$\operatorname{diam}_{cc}(J_\tau) \leq T_0^2 \operatorname{diam}_{cc}(J_{\tau e}) \tag{11.44}$$

and

$$T_0 \operatorname{diam}_{cc}(J_e) \geq 1. \tag{11.45}$$

This is possible because by (1.7), (4.9), (4.6), and Corollary 4.15,

$$\begin{aligned}\operatorname{diam}_{cc}(J_\tau) &\leq L \operatorname{diam}(\phi_\tau(J_{\mathcal{S}} \cap X_{t(\tau)})) \\ &\leq LM\|D\phi_\tau\|_\infty \|D\phi_e\|_\infty \frac{1}{\|D\phi_e\|_\infty} \\ &\leq \frac{LMK}{\min_{e\in E}\{\|D\phi_e\|_\infty\}} \|D\phi_{\tau e}\|_\infty \\ &\leq \frac{C(L,M,K)}{\kappa_0\mu_0 \min_{e\in E}\{\|D\phi_e\|_\infty\}} \operatorname{diam}_{cc}(J_{\tau e}).\end{aligned}$$

We will now show that if $r \in (0,1]$, then for any $\omega \in E_A^*$ such that $\operatorname{diam}_{cc}(J_\omega) < r$ there exists some $k = 1, \dots, |\omega|$, such that

$$T_0^{-1} \leq \frac{\operatorname{diam}_{cc}(J_{\omega|_k})}{r} \leq T_0. \tag{11.46}$$

By (11.45), there exists some $k = 1, \dots, |\omega|$, such that

$$\operatorname{diam}_{cc}(J_{\omega|_k}) \geq \frac{r}{T_0}. \tag{11.47}$$

Let k_0 be the minimal $k \in \mathbb{N}$ satisfying (11.47). Since $\operatorname{diam}_{cc}(J_\omega) < r$, $k_0 < |\omega|$. By the minimality of k_0,

$$\operatorname{diam}_{cc}(J_{\omega|_{k_0+1}}) < \frac{r}{T_0}. \tag{11.48}$$

By (11.44) and (11.48),

$$\operatorname{diam}_{cc}(J_{\omega|_{k_0}}) \leq T_0^2 \operatorname{diam}_{cc}(J_{\omega|_{k_0+1}}) \leq T_0 r.$$

Hence (11.45) follows.

We now introduce some notation following [51]. Recalling Remark 11.12, for $\alpha \in [0,1]$ and $\omega \in E_A^*$ let

$$W_\alpha(\omega) = W^{cc}_{\alpha,(1+\alpha)T_0}(\omega) \text{ and } M_\alpha(\omega) = \sharp W^{cc}_\alpha(\omega).$$

Then by (11.22),

$$M_\alpha(\omega) \le C^{cc}(1, 2T_0). \tag{11.49}$$

Let $\overline{C} = \lceil C^{cc}(1, 2T_0)\rceil + 1$, where $\lceil x \rceil = \min\{k \in \mathbb{N} : k \ge x\}$, for $x \ge 0$.

For fixed $\omega \in E_A^*$ consider the function $f_\omega : [0,1] \to \mathbb{N}$ defined by $f_\omega(\alpha) = M_\alpha(\omega)$. By the definition of the sets $W_\alpha(\omega)$ we deduce that the function f_ω is increasing. For $r \ge 0$, let

$$\widetilde{M_\alpha}(r) = \sup\{M_\alpha(\omega) : \omega \in E_A^*, \mathrm{diam}_{\mathrm{cc}}(J_\omega) \le r\}. \tag{11.50}$$

Notice that the function $\widetilde{f}_r : [0,1] \to \mathbb{N}$ defined by $\widetilde{f}_r(\alpha) = \widetilde{M_\alpha}(r)$ has the following properties,

(1) it is increasing,
(2) it is bounded by $C^{cc}(1, 2T_0)$,
(3) there exist $\alpha_1, \alpha_2, \in [0,1]$ such that $\alpha_2 - \alpha_1 \ge \frac{1}{\overline{C}}$ and $\widetilde{f}_r$ is constant on $[a_1, a_2]$.

The first property follows by the definition of $W^{cc}_\alpha(\omega)$ and the second property follows by (11.49). To show that (iii) holds, suppose on the contrary that it fails. Subdivide $[0,1]$ in $\overline{C}$ subintervals of length $1/\overline{C}$. Since $\widetilde{f}_r$ is integer valued and increasing, if $[a,b]$ is any of the aforementioned subintervals, $\widetilde{f}_r(b) - \widetilde{f}_r(a) > 1$. Hence $\widetilde{f}_r(1) \ge \overline{C} > C^{cc}(1, 2T_0)$, which contradicts (11.49).

Observe that the maximum is attained in (11.50). That is there exists some $\overline{\omega} \in E_A^*$ such that ,

$$\mathrm{diam}_{\mathrm{cc}}(J_{\overline{\omega}}) \le r$$

and

$$\widetilde{M_{\alpha_1}}(r) = M_{\alpha_1}(\overline{\omega}).$$

Since $f_{\overline{\omega}}$ is increasing,

$$M_{\alpha_2}(\overline{\omega}) \ge M_{\alpha_1}(\overline{\omega}) = \widetilde{M_{\alpha_1}}(r) = \widetilde{M_{\alpha_2}}(r) \ge M_{\alpha_2}(\overline{\omega}), \tag{11.51}$$

in particular

$$M_{\alpha_1}(\overline{\omega}) = M_{\alpha_2}(\overline{\omega}). \tag{11.52}$$

Observe that since $a_1 \le a_2$, $W_{\alpha_1}(\overline{\omega}) \subset W_{\alpha_2}(\overline{\omega})$. Thus by (11.52), $W_{\alpha_1}(\overline{\omega}) = W_{\alpha_2}(\overline{\omega})$.

Now let $\varepsilon = \frac{1}{2\overline{C}}$ and let $r = \min\{1, \mathtt{d}_{T_0,\varepsilon}\}$, where $\mathtt{d}_{T_0,\varepsilon}$ is as in Lemma 11.14. We will first show that if $\tau \in W_{\alpha_1}(\overline{\omega})$ and $\rho \in E_A^*$ such that $\rho\overline{\omega} \in E_A^*$ then

$$\rho\tau \in W_{\alpha_2}(\rho\overline{\omega}). \tag{11.53}$$

Notice that since $\rho\overline{\omega} \in E_A^*$, $t(\rho) = i(\overline{\omega})$. By the definition of the sets $W_{\alpha_1}(\overline{\omega})$, if $\tau \in W_{\alpha_1}(\overline{\omega})$ then $i(\tau) = i(\overline{\omega})$. Hence $i(\tau) = t(\rho)$, and by the maximality of $\mathcal{S}$ we deduce that $\rho\tau \in E_A^*$. Observe that since $\tau \in W_{\alpha_1}(\overline{\omega}) = W^{cc}_{\alpha_1,(1+\alpha_1)T_0}, \rho\tau \in E_A^*$ and $\mathrm{diam}_{\mathrm{cc}}(J_{\overline{\omega}}) < \mathtt{d}_{T_0,\varepsilon}$, Lemma 11.14 implies that,

$$\rho\tau \in W^{cc}_{\alpha_1(1+\varepsilon),(1+\varepsilon)(1+\alpha_1)T_0}(\rho\overline{\omega}).$$

Hence (11.53) will follow if we show

$$\alpha_1(1+\varepsilon) \le \alpha_2, \tag{11.54}$$

and

$$(1+\alpha_1)(1+\varepsilon) \le (1+\alpha_2). \tag{11.55}$$

Since $\alpha_2 - \alpha_1 \ge \frac{1}{\overline{C}}$, and $\alpha_1 \le 1$,

$$\alpha_2 \ge \alpha_1 + \frac{\alpha_1}{\overline{C}} > \alpha_1\left(1+\frac{1}{2\overline{C}}\right) = \alpha_1(1+\varepsilon),$$

hence (11.54) follows. Moreover

$$\begin{aligned}(1+\alpha_1)(1+\varepsilon) &\le (1+\alpha_1)\left(1+\frac{1}{2\overline{C}}\right) = 1+\alpha_1+\frac{1}{2\overline{C}}+\frac{\alpha_1}{2\overline{C}}\\ &\le 1+\alpha_1+\frac{1}{\overline{C}} \le 1+\alpha_2,\end{aligned}$$

hence (11.55) follows. Therefore (11.53) follows as well.

We will now show that if $\rho\overline{\omega} \in E_A^*$ then

$$M_{\alpha_2}(\rho\overline{\omega}) = M_{\alpha_2}(\overline{\omega}). \tag{11.56}$$

We will first prove that if $\rho\overline{\omega} \in E_A^*$ then

$$M_{\alpha_2}(\rho\overline{\omega}) \ge M_{\alpha_1}(\overline{\omega}). \tag{11.57}$$

As we have remarked already in the proof of (11.53), if $\tau \in W_{\alpha_1}(\overline{\omega})$ then $\rho\tau \in E_A^*$. Hence by (11.53), $\rho\tau \in W_{\alpha_2}(\rho\overline{\omega})$. This implies that

$$M_{\alpha_2}(\rho\overline{\omega}) = \sharp W_{\alpha_2}(\rho\overline{\omega}) \ge \sharp W_{\alpha_1}(\overline{\omega}) = M_{\alpha_1(\overline{\omega})},$$

and thus (11.57) holds.

Recall that $r \le \mathtt{d}_{T_0,\varepsilon} < \eta_{\mathcal{S}}$, hence since $\operatorname{diam}_{\mathrm{cc}}(J_{\overline{\omega}}) \le r$ using Lemma 3.8 as in Lemma 11.13 we deduce that

$$\operatorname{diam}_{\mathrm{cc}}(J_{\rho\overline{\omega}}) \le \|D\phi_\rho\|_\infty \operatorname{diam}_{\mathrm{cc}}(J_{\overline{\omega}}) < \operatorname{diam}_{\mathrm{cc}}(J_{\overline{\omega}}) < r.$$

Hence $M_{\alpha_2}(\rho\overline{\omega}) \le \widetilde{M_{\alpha_2}}(r)$ and by (11.51) we deduce that

$$M_{\alpha_2}(\rho\overline{\omega}) \le M_{\alpha_1}(\overline{\omega}). \tag{11.58}$$

By (11.57) and (11.58), we deduce that

$$M_{\alpha_2}(\rho\overline{\omega}) = M_{\alpha_1}(\overline{\omega}). \tag{11.59}$$

Now (11.56) follows by (11.59) and (11.52). Thus (11.53) and (11.56) imply that

$$W_{\alpha_2}(\rho\overline{\omega}) = \{\rho\tau : \tau \in W_{\alpha_2}(\overline{\omega})\}. \tag{11.60}$$

Let

$$O = \bigcup_{\tau\in E_A^*:\tau\overline{\omega}\in E_A^*} \phi_\tau(B_{cc}(J_{\overline{\omega}}, \mathtt{d}_1)),$$

where

$$\mathtt{d}_1 = \min\left\{\frac{\eta_{\mathcal{S}}}{10}, \frac{\min\{\lambda_{\mathcal{S}}^{cc}, \alpha_2\}\,\kappa_0\mu_0\|D\phi_{\overline{\omega}}\|_\infty}{5K^2\Lambda_0}\right\},$$

and recalling Remark 11.2 $\lambda_{\mathcal{S}}^{cc}$ is just $\lambda_{\mathcal{S}}$ with respect to the d_{cc} metric. Observe that if we set $O_v = W_v \cap O$ for $v \in V$ then $O = \cup_{v\in V} O_v$,

$$O_v \cap J_{\mathcal{S}} \ne \emptyset \text{ and } O_v \subset \operatorname{Int}(N_v).$$

Therefore in order to finish the proof of the proposition it is enough to show that for all $\rho, \tau \in E_A^*, \rho \neq \tau$, $\phi_\tau(O) \subset O$ and

$$\phi_\rho(O) \cap \phi_\tau(O) = \emptyset. \tag{11.61}$$

First notice that $\phi_\tau(O)$ is well defined for all $\tau \in E_A^*$, because by the irreducibility of $\mathcal{S}$ for all $\tau \in E$ there exists some $\upsilon \in E_A^*$ such that $\tau\upsilon\overline{\omega} \in E_A^*$. Moreover by the definition of the set O it readily follows that $\phi_\tau(O) \subset O$. Hence in order to prove (11.61) it is enough to show that for all $e, e' \in E$ and for all $\tau, \tau' \in E_A^*$ such that $e\tau\overline{\omega} \in E_A^*$ and $e'\tau'\overline{\omega} \in E_A^*$,

$$\phi_{e\tau}(B_{cc}(J_{\overline{\omega}}, \mathtt{d}_1)) \cap \phi_{e'\tau'}(B_{cc}(J_{\overline{\omega}}, \mathtt{d}_1)) = \emptyset. \tag{11.62}$$

For $p \in B_{cc}(J_{\overline{\omega}}, \mathtt{d}_1)$ and $q \in J_{\overline{\omega}}$ such that $d_{cc}(p,q) < \mathtt{d}_1$, using (1.7), (4.18), (4.6) and Proposition 4.15 we obtain

$$\begin{aligned}
\operatorname{dist}_{cc}(\phi_{e\tau}(p), J_{e\tau\overline{\omega}}) &\leq d_{cc}(\phi_{e\tau}(p), \phi_{e\tau}(q)) \\
&\leq L\Lambda_0 \|D\phi_{e\tau}\|_\infty d_{cc}(p,q) \\
&\leq L\Lambda_0 \|D\phi_{e\tau}\|_\infty \mathtt{d}_1 \\
&\leq L\Lambda_0 \|D\phi_{e\tau\overline{\omega}}\|_\infty \frac{\|D\phi_{e\tau}\|_\infty}{\|D\phi_{e\tau\overline{\omega}}\|_\infty} \mathtt{d}_1 \\
&\leq KL\Lambda_0 \|D\phi_{e\tau\overline{\omega}}\|_\infty \frac{\|D\phi_{e\tau}\|_\infty}{\|D\phi_{e\tau}\|_\infty \|D\phi_{\overline{\omega}}\|_\infty} \mathtt{d}_1 \\
&= \frac{KL\Lambda_0 \mathtt{d}_1}{\|D\phi_{\overline{\omega}}\|_\infty} \|D\phi_{e\tau\overline{\omega}}\|_\infty \\
&\leq \frac{2K^2 L\Lambda_0 \mathtt{d}_1}{\kappa_0 \mu_0 \|D\phi_{\overline{\omega}}\|_\infty} \operatorname{diam}_{\mathrm{cc}}(J_{e\tau\overline{\omega}}).
\end{aligned}$$

Hence,

$$\phi_{e\tau}(B_{cc}(J_{\overline{\omega}}, \mathtt{d}_1)) \subset B_{cc}(J_{e\tau\overline{\omega}}, \mathtt{d}_2 \operatorname{diam}_{\mathrm{cc}}(J_{e\tau\overline{\omega}})), \tag{11.63}$$

where

$$\mathtt{d}_2 = \frac{2K^2 L\Lambda_0}{\kappa_0 \mu_0 \|D\phi_{\overline{\omega}}\|_\infty} \mathtt{d}_1.$$

In the same manner

$$\phi_{e'\tau'}(B_{cc}(J_{\overline{\omega}}, \mathtt{d}_1)) \subset B_{cc}(J_{e'\tau'\overline{\omega}}, \mathtt{d}_2 \operatorname{diam}_{\mathrm{cc}}(J_{e'\tau'\overline{\omega}})). \tag{11.64}$$

Notice that by by (11.63) and (11.64), in order to prove (11.62) it is enough to show that

$$B_{cc}(J_{e\tau\overline{\omega}}, \mathtt{d}_2 \operatorname{diam}_{\mathrm{cc}}(J_{e\tau\overline{\omega}})) \cap B_{cc}(J_{e'\tau'\overline{\omega}}, \mathtt{d}_2 \operatorname{diam}_{\mathrm{cc}}(J_{e'\tau'\overline{\omega}})) = \emptyset. \tag{11.65}$$

If $i(e) \neq i(e')$, then $X_{i(e)} \cap X_{i(e')} = \emptyset$. We will prove (11.65) by contradiction. If (11.65) fails, then using that $\mathtt{d}_2 < \frac{\lambda_{\mathcal{S}}^{cc}}{2}$,

$$\begin{aligned}
\operatorname{dist}_{\mathrm{cc}}(X_{i(e)}, X_{i(e')}) &\leq \operatorname{dist}_{\mathrm{cc}}(J_{e\tau\overline{\omega}}, J_{e'\tau'\overline{\omega}}) \\
&\leq 2\mathtt{d}_2 \max\{\operatorname{diam}_{\mathrm{cc}}(X_v), v \in V\} \\
&< \lambda_{\mathcal{S}}^{cc} \max\{\operatorname{diam}_{\mathrm{cc}}(X_v), v \in V\} \\
&= \min\{\operatorname{dist}_{\mathrm{cc}}(X_{v_1}, X_{v_2}) : v_1, v_2 \in V\},
\end{aligned}$$

which is impossible. Therefore in the case where $i(e) \neq i(e')$, (11.65) holds, and hence (11.62) follows.

Now suppose that $i(e) = i(e')$. Without loss of generality we can assume that

$$\operatorname{diam}_{cc}(J_{e\tau\overline{\omega}}) \geq \operatorname{diam}_{cc}(J_{e'\tau'\overline{\omega}}).$$

Let $\overline{\omega}' = \tau'\overline{\omega}$. By (11.46) there exists some $m = 1, \ldots, |\tau'| + |\overline{\omega}|$, such that

$$T_0^{-1} \leq \frac{\operatorname{diam}_{cc}(J_{e'\overline{\omega}'|_m})}{\operatorname{diam}_{cc}(J_{e\tau\overline{\omega}})} \leq T_0. \tag{11.66}$$

We are going to prove that

$$\operatorname{dist}_{cc}(J_{e\tau\overline{\omega}}, J_{e'\overline{\omega}'|_m}) > \alpha_2 \operatorname{diam}_{cc}(J_{e\tau\overline{\omega}}). \tag{11.67}$$

Recall that

$$\begin{aligned} W^{cc}_{\alpha_2,T_0}(e\tau\overline{\omega}) = \Big\{\rho \in E_A^* :\ & i(e\tau\overline{\omega}) = i(\rho), \\ & T_0^{-1} \leq \frac{\operatorname{diam}_{cc}(J_\rho)}{\operatorname{diam}_{cc}(J_{e\tau\overline{\omega}})} \leq T_0, \\ & \text{and } \operatorname{dist}_{cc}(J_\rho, J_{e\tau\overline{\omega}}) \leq \alpha_2 \operatorname{diam}_{cc}(J_{e\tau\overline{\omega}})\Big\}. \end{aligned}$$

Suppose that (11.67) fails. Hence, by (11.66) and the fact that $i(e) = i(e')$ we deduce that

$$e'\overline{\omega}'|_m \in W^{cc}_{\alpha_2,T_0}(e\tau\overline{\omega}) \subset W^{cc}_{\alpha_2,(1+\alpha_2)T_0}(e\tau\overline{\omega}) = W_{\alpha_2}(e\tau\overline{\omega}). \tag{11.68}$$

But this is not possible, because by (11.60)

$$W_{\alpha_2}(e\tau\overline{\omega}) = \{e\tau\upsilon : \upsilon \in W_{\alpha_2}(\overline{\omega})\},$$

and this contradicts (11.68) because $e'\overline{\omega}'|_m \notin W_{\alpha_2}(e\tau\overline{\omega})$. Hence (11.67) holds and in particular it implies that

$$\operatorname{dist}_{cc}(J_{e\tau\overline{\omega}}, J_{e'\tau'\overline{\omega}}) > \alpha_2 \operatorname{diam}_{cc}(J_{e\tau\overline{\omega}}). \tag{11.69}$$

Since $\mathtt{d}_2 \leq \alpha_2/2$, (11.69) implies (11.65). Therefore (11.62) follows. □

We can now complete the proof of Theorem 11.6 by proving the remaining implication (iii)⇒ (iv). For $v \in V$ let $X'_v = \overline{O_v}$, where the sets O_v are as in Proposition 11.15. By the same proposition the GDMS $\mathcal{S}' = \{\phi_\varepsilon : X'_{t(e)} \to X'_{i(e)}\}$ satisfies the strong open set condition and it is equivalent to $\mathcal{S}$. The proof is complete. □

Remark 11.16. In [**54**], Rajala and Vilppolainen study finite weakly controlled Moran contractions in metric spaces. We remark that due to our bounded distortion theorems proved earlier in Chapters 3 and 4, it follows immediately that finite conformal IFS on a Carnot group $(\mathbb{G}, d)$ are properly semiconformal, see [**54**] for the exact definition. In particular if $\mathcal{S}$ is a finite conformal IFS the equivalence (i)⇔(ii)⇔(iv) of Theorem 11.6 follows also from the results of Rajala and Vilppolainen, see [**54**, Theorem 4.9].

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