TUKIA'S ISOMORPHISM THEOREM IN CAT(-1) SPACES

TUSHAR DAS, DAVID SIMMONS, AND MARIUSZ URBAŃSKI

ABSTRACT. We prove a generalization of Tukia's ('85) isomorphism theorem which states that isomorphisms between geometrically finite groups extend equivariantly to the boundary. Tukia worked in the setting of real hyperbolic spaces of finite dimension, and his theorem cannot be generalized as stated to the setting of CAT(-1) spaces. We exhibit examples of type-preserving isomorphisms of geometrically finite subgroups of finite-dimensional rank one symmetric spaces of noncompact type (ROSSONCTs) whose boundary extensions are not quasisymmetric. A sufficient condition for a type-preserving isomorphism to extend to a quasisymmetric equivariant homeomorphism between limit sets is that one of the groups in question is a lattice, and that the underlying base fields are the same, or if they are not the same then the base field of the space on which the lattice acts has the larger dimension. This in turn leads to a generalization of a rigidity theorem of Xie ('08) to the setting of finite-dimensional ROSSONCTs.

CONTENTS

1. INTRODUCTION

Tukia's isomorphism theorem [23, Theorem 3.3] states that any type-preserving isomorphism Φ between two geometrically finite subgroups of $\mathrm{Isom}(\mathbb{H}^d)$ (not necessarily the same d for both groups) extends to a quasisymmetric equivariant homeomorphism between their limit sets. In this note we prove a generalization of this theorem to the setting of CAT(-1) spaces, paying particular attention to the case of rank one symmetric spaces of noncompact type (ROSSONCTs). However, our theorem cannot be stated as the naive (word-for-word) generalization of Tukia's theorem, since such a generalization is false (cf. Example 4.8 and Remark 4.9). Instead, we establish sufficient conditions on a type-preserving isomorphism Φ between two geometrically finite subgroups of $\mathrm{Isom}(X)$, where X is a CAT(-1) space, in order for the conclusion of Tukia's theorem to hold.

Convention 1. The symbols $\lesssim$, $\gtrsim$, and $\asymp$ will denote asymptotics; a subscript of $+$ indicates that the asymptotic is additive, and a subscript of $\times$ indicates that it is multiplicative. For example, $A \lesssim_{\times,K} B$ means that there exists a constant $C > 0$ (the *implied*

constant), depending only on K, such that $A \leq CB$. Moreover, $A \lesssim_{+,\times} B$ means that there exist constants $C_1, C_2 > 0$ so that $A \leq C_1 B + C_2$.

Convention 2. The symbol $\lhd$ will be used to indicate the end of a nested proof.

Convention 3. Given a distinguished point $o \in X$, we write
$$\|x\| = d(o, x) \text{ and } \|g\| = \|g(o)\|.$$

Convention 4. We denote the maximum of two numbers A, B by $A \vee B$.

1.1. **Geometrically finite groups.** It was known for a long time that every finitely generated Fuchsian group has a finite-sided convex fundamental domain (e.g. [16, Theorem 4.6.1]). This result does not generalize beyond two dimensions (e.g. [3, 15]), but subgroups of $\mathrm{Isom}(\mathbb{H}^3)$ with finite-sided fundamental domains came to be known as *geometrically finite* groups. Several equivalent definitions of geometrical finiteness in the three-dimensional setting became known, for example Beardon and Maskit's condition that the limit set is the union of the radial limit set Λ_r with the set Λ_{bp} of bounded parabolic points [2], but the situation in higher dimensions was somewhat murky until Bowditch [5] wrote a paper which described which equivalences remain true in higher dimensions, and which do not. The condition of a finite-sided convex fundamental domain is no longer equivalent to any other conditions in higher dimensions (e.g. [1]), so a higher-dimensional Kleinian group is said to be *geometrically finite* if it satisfies any of Bowditch's five equivalent conditions (GF1)-(GF5). Four of Bowditch's equivalent conditions make sense in the more general setting of pinched Hadamard manifolds [6], but when the generality is extended further to CAT(-1) spaces, only conditions (GF1) and (GF2) remain equivalent (cf. [11, Remark 12.4.6]). Groups acting on proper CAT(-1) spaces and satisfying equivalent conditions (GF1) and (GF2) were first called geometrically finite by Roblin [21, §1F]; the authors of the current paper removed the properness assumption in [11, §12], where we gave the following definition:

Definition 1.1 ([11, Definition 12.4.1]; cf. [21, Proposition 1.10(iii)]). Let X be a CAT(-1) space. A group $G \leq \mathrm{Isom}(X)$ is said to be *strongly discrete* if for all $\rho > 0$,
$$\#\{g \in G : \|g\| \leq \rho\} < \infty.$$

A strongly discrete group G is said to be *geometrically finite* if there exists a disjoint G-invariant collection of horoballs $\mathscr{H}$ and a basepoint $o \notin \bigcup \mathscr{H}$ such that

    (I) for every $\rho > 0$, the set
$$\mathscr{H}_\rho := \{H \in \mathscr{H} : d(o, H) \leq \rho\}$$
        is finite, and

(II) there exists $\sigma > 0$ such that

$$\mathcal{C}_o \subseteq G(B(o, \sigma)) \cup \bigcup \mathscr{H},$$

where $\mathcal{C}_o$ denotes the quasiconvex core of G (with respect to the basepoint $o \in X$); cf. Definition 2.8.

1.2. **Main results.** We now introduce the terminology necessary to state our main theorem.

Definition 1.2. An isomorphism between two groups acting on CAT(-1) spaces is *type-preserving* if the image of a loxodromic (resp. parabolic, elliptic) isometry is loxodromic (resp. parabolic, elliptic). (For the definitions of loxodromic, parabolic, and elliptic isometries, see Definition 2.6 below.)

Definition 1.3. Let (Z, D) and $(\widetilde{Z}, \widetilde{D})$ be metric spaces. A homeomorphism $\phi : Z \to \widetilde{Z}$ is said to be *quasisymmetric* if there exists an increasing homeomorphism $f : (0, \infty) \to (0, \infty)$ such that

$$\frac{\widetilde{D}(\phi(z), \phi(y))}{\widetilde{D}(\phi(z), \phi(x))} \leq f\left(\frac{D(z, y)}{D(z, x)}\right) \quad \forall x, y, z \in Z.$$

Theorem 1.4 (Generalization of Tukia's isomorphism theorem)**.** *Let X, $\widetilde{X}$ be CAT(-1) spaces,[1] let $G \leq \mathrm{Isom}(X)$ and $\widetilde{G} \leq \mathrm{Isom}(\widetilde{X})$ be two geometrically finite groups, and let $\Phi : G \to \widetilde{G}$ be a type-preserving isomorphism. Let P be a complete set of inequivalent parabolic points for G.*

   (i) *If for every $p \in P$ we have*

(1.1)
$$\|\Phi(h)\| \asymp_{+, \times, p} \|h\| \quad \forall h \in G_p,$$

   then there is an equivariant homeomorphism between $\Lambda := \Lambda(G)$ and $\widetilde{\Lambda} := \Lambda(\widetilde{G})$.

   (ii) *If for every $p \in P$ there exists $\alpha_p > 0$ such that*

(1.2)
$$\|\Phi(h)\| \asymp_{+, p} \alpha_p \|h\| \quad \forall h \in G_p,$$

   then the homeomorphism of (i) *is quasisymmetric.*

Tukia's isomorphism theorem [23, Theorem 3.3] corresponds to the case of Theorem 1.4 where X and $\widetilde{X}$ are finite-dimensional real ROSSONCTs. Note that in this case, the hypothesis (1.2) always holds with $\alpha_p = 1$ (Corollary 4.6; see also [20, Theorem 5.4.3]). This is why Tukia's original theorem does not need to mention the conditions (1.1) and (1.2).

A natural question is then whether the assumptions (1.1) and/or (1.2) are really necessary. In the case of finite-dimensional nonreal ROSSONCTs, this question is partially answered by the following theorem:

Theorem 1.5. *Let X and $\widetilde{X}$ be finite-dimensional ROSSONCTs, let $G \leq \mathrm{Isom}(X)$ and $\widetilde{G} \leq \mathrm{Isom}(\widetilde{X})$ be geometrically finite groups, and let $\Phi : G \to \widetilde{G}$ be a type-preserving isomorphism. Then* (1.1) *holds. Moreover, suppose that*

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[1]Or more generally, regularly geodesic strongly hyperbolic metric spaces; cf. [11, Definitions 3.3.6 and 4.4.5].

(I) G *is a lattice, and*

(II) *if* $\mathbb{F}$ *and* $\widetilde{\mathbb{F}}$ *are the underlying base fields of* X *and* $\widetilde{X}$, *respectively, then* $\dim_{\mathbb{R}}(\mathbb{F}) \geq \dim_{\mathbb{R}}(\widetilde{\mathbb{F}})$.

Then (1.2) *holds.*

If the assumptions (I)-(II) are omitted, it is easy to construct examples of groups G, $\widetilde{G}$ satisfying the hypotheses of Theorem 1.4(i) but for which the equivariant homeomorphism is not quasisymmetric (Example 4.8 and Remark 4.9). This shows that the assumption (1.2) cannot be omitted from the second assertion of Theorem 1.4.

Using Theorems 1.4 and 1.5, we generalize a rigidity theorem of Xie [25, Theorem 3.1] to the setting of finite-dimensional ROSSONCTs:

Theorem 1.6 ([25, Theorem 3.1])**.** *Let* X, $\widetilde{X}$ *be finite-dimensional ROSSONCTs whose base fields* $\mathbb{F}$ *and* $\widetilde{\mathbb{F}}$ *satisfy* $\dim_{\mathbb{R}}(\mathbb{F}) \geq \dim_{\mathbb{R}}(\widetilde{\mathbb{F}})$, *with* $X \neq \mathbb{H}^2_{\mathbb{R}}$. *Let* $G \leq \mathrm{Isom}(X)$ *be a noncompact lattice, and let* $\widetilde{G} \leq \mathrm{Isom}(\widetilde{X})$ *be a geometrically finite group, both torsion-free. Let* $\Phi : G \to \widetilde{G}$ *be a type-preserving isomorphism. Then* $\dim_H(\Lambda(\widetilde{G})) \geq \dim_H(\Lambda(G)) = \dim(\partial X)$, *where* $\dim_H$ *denotes Hausdorff dimension. Furthermore, equality holds if and only if* $\widetilde{G}$ *stabilizes an isometric copy of* X *in* $\widetilde{X}$.

We end this introduction by stating an open problem:

Problem 1.7. Is Theorem 1.6 true if we drop the assumption $\dim_{\mathbb{R}}(\mathbb{F}) \geq \dim_{\mathbb{R}}(\widetilde{\mathbb{F}})$?

Outline. In Section 2, we provide some background on ROSSONCTs, CAT(-1) spaces, and their geometry. In Section 3, we prove Theorem 1.4, and in Section 4 we prove Theorems 1.5 and 1.6.

2. BACKGROUND

2.1. **ROSSONCTs.** A canonical class of negatively curved manifolds is the rank one symmetric spaces of noncompact type (ROSSONCTs), which come in four flavors, corresponding to the classical division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ (quaternions), and $\mathbb{O}$ (octonions).[2] The first three division algebras have corresponding ROSSONCTs of arbitrary dimension, but there is only one ROSSONCT corresponding to the octonions; it occurs in dimension two (which corresponds to real dimension 16). The ROSSONCTs corresponding to $\mathbb{R}$ have constant negative curvature, however those corresponding to the other division algebras have variable negative curvature [19, Lemmas 2.3, 2.7, 2.11] (see also [13, Corollary of Proposition 4]). Some references for the theory of ROSSONCTs are [7, 8, 17], and we use the same notation as [11, §2].

Remark 2.1. In the remainder of the text we use the term "ROSSONCT" to refer to all ROSSONCTs *except* the the octonion ROSSONCT $\mathbb{H}^2_{\mathbb{O}}$, known as the *Cayley hyperbolic plane*,[3] in order to avoid dealing with the complicated algebra of the octonion

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[2]We denote the quaternions by $\mathbb{Q}$ in order to avoid confusion with the ROSSONCT itself, which we will denote by $\mathbb{H}$. $\mathbb{Q}$ should not be confused with the set of rational numbers.

[3]Not to be confused with the *Cayley plane*, a different mathematical object.

ROSSONCT.[4] However, it may be of interest to investigate whether our results generalize to include the Cayley hyperbolic plane (possibly after modifying the statements slightly). We leave this task to an algebraist.

Fix $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{Q}\}$ and $d \in \mathbb{N}$. Let us construct a ROSSONCT of type $\mathbb{F}$ in dimension d. In what follows we think of $\mathbb{F}^{d+1}$ as a right $\mathbb{F}$-module (i.e. scalars always act on the right).[5] Consider the skew-symmetric sesquilinear form $B_\mathcal{Q} : \mathbb{F}^{d+1} \times \mathbb{F}^{d+1} \to \mathbb{F}$ defined by

$$B_\mathcal{Q}(\mathbf{x}, \mathbf{y}) := -\overline{x}_0 y_0 + \sum_{i=1}^{d} \overline{x}_i y_i$$

and its associated quadratic form

(2.1)
$$\mathcal{Q}(\mathbf{x}) := B_\mathcal{Q}(\mathbf{x}, \mathbf{x}) = -|x_0|^2 + \sum_{i=1}^{d} |x_i|^2.$$

Let $\mathbb{P}(\mathbb{F}^{d+1})$ denote the *projectivization* of $\mathbb{F}^{d+1}$, i.e. the quotient of $\mathbb{F}^{d+1} \setminus \{\mathbf{0}\}$ under the equivalence relation $\mathbf{x} \sim \mathbf{x}a$ ($\mathbf{x} \in \mathbb{F}^{d+1} \setminus \{\mathbf{0}\}$, $a \in \mathbb{F} \setminus \{0\}$). Let

$$\mathbb{H} = \mathbb{H}_\mathbb{F}^d := \{[\mathbf{x}] \in \mathbb{P}(\mathbb{F}^{d+1}) : \mathcal{Q}(\mathbf{x}) < 0\},$$

and consider the map $d_\mathbb{H} : \mathbb{H} \times \mathbb{H} \to [0, \infty)$ defined by the equation

(2.2)
$$\cosh d_\mathbb{H}([\mathbf{x}], [\mathbf{y}]) = \frac{|B_\mathcal{Q}(\mathbf{x}, \mathbf{y})|}{\sqrt{|\mathcal{Q}(\mathbf{x})| \cdot |\mathcal{Q}(\mathbf{y})|}}, \quad [\mathbf{x}], [\mathbf{y}] \in \mathbb{H}.$$

The map $d_\mathbb{H}$ defines a metric on $\mathbb{H}$ which is compatible with the natural topology (as a subspace of the quotient space $\mathbb{P}(\mathbb{F}^{d+1})$). Moreover, for any two distinct points $[\mathbf{x}], [\mathbf{y}] \in \mathcal{H}$ there exists a unique isometric embedding $\gamma : \mathbb{R} \to \mathbb{H}$ such that $\gamma(0) = [\mathbf{x}]$ and $\gamma \circ d_\mathbb{H}([\mathbf{x}], [\mathbf{y}]) = [\mathbf{y}]$.

Definition 2.2. A *rank one symmetric space of noncompact type (ROSSONCT)* of dimension d is a pair $(\mathbb{H}_\mathbb{F}^d, d_\mathbb{H})$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{Q}\}$ and $d \in \mathbb{N}$.

This definition can be extended to infinite dimensions; cf. [11, Definition 2.2.6].

Remark 2.3. Technically, the above definition should really be a theorem (modulo the Cayley hyperbolic plane, cf. Remark 2.1), since symmetric spaces are a certain type of Riemannian manifolds, with rank and type being properties of those manifolds; the classification of rank one symmetric spaces of noncompact type then follows from the classification of symmetric spaces generally (e.g. [14, p.518]).[6] However, we prefer to define ROSSONCTs without dealing with the algebra behind symmetric spaces in general.

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[4]The complications come from the fact that the octonions are not associative, thus making it somewhat unclear what it means to say that $\mathbb{O}^3$ is a vector space "over" the octonions, since in general $(\mathbf{x}a)b \neq \mathbf{x}(ab)$.

[5]The advantage of this convention is that it allows matrices to act on the left.

[6]In the notation of [14], the spaces $\mathbb{H}_\mathbb{R}^p$, $\mathbb{H}_\mathbb{C}^p$, $\mathbb{H}_\mathbb{Q}^p$, and $\mathbb{H}_\mathbb{O}^2$ are written as $\mathrm{SO}(p, 1)/\mathrm{SO}(p)$, $\mathrm{SU}(p, 1)/\mathrm{SU}(p)$, $\mathrm{Sp}(p, 1)/\mathrm{Sp}(p)$, and $(\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$, respectively.

2.2. **Negatively curved metric spaces.** A good reference for the theory of "negative curvature" in general metric spaces is [7]. We assume that that the reader is aware of the definition of a CAT(-1) spaces, viz. these are geodesic metric spaces whose triangles are "thinner" than the corresponding triangles in two-dimensional real hyperbolic space $\mathbb{H}^2$, see [7, p.158] for details. It follows from their definition that CAT(-1) spaces are uniquely geodesic; we denote the unique geodesic segment connecting two points x, y by $[x, y]$. Any Riemannian manifold with sectional curvature bounded above by -1 is a CAT(-1) space. Since ROSSONCTs are Riemannian manifolds with sectional curvature bounded between -4 and -1, every ROSSONCT is a CAT(-1) space.

The next level of generality considers Gromov hyperbolic metric spaces. These are spaces which are "approximately $\mathbb{R}$-trees". A good reference for the basics of the theory is [24].

Definition 2.4. A metric space X is called *hyperbolic* (or *Gromov hyperbolic*) if for every four points $x, y, z, w \in X$ we have

$$(2.3) \qquad \langle x|z \rangle_w \gtrsim_+ \min(\langle x|y \rangle_w, \langle y|z \rangle_w),$$

where the expression

$$(2.4) \qquad \langle b|c \rangle_a := \frac{1}{2}[d(a,b) + d(a,c) - d(b,c)]$$

is called the *Gromov product* of b and c with respect to a. We refer to (2.3) as *Gromov's inequality*.

The *Gromov boundary* of X, denoted ∂X, is the set of Gromov sequences modulo equivalence, see [11, Definition 3.4.1] for details. The *Gromov closure* or *bordification* of X is the disjoint union $\operatorname{bord} X := X \cup \partial X$. The Gromov product can be extended in a near-continuous way to $\operatorname{bord} X$, see [11, Definition 3.4.9, Lemma 3.4.22]. For each $z \in \operatorname{bord} X$, let $\mathcal{B}_z$ denote the *Busemann function*

$$(2.5) \qquad \mathcal{B}_z(x, y) := \langle y|z \rangle_x - \langle x|z \rangle_y.$$

Note that for $z \in X$, this formula reduces to $\mathcal{B}_z(x, y) = d(z, x) - d(z, y)$.

If X is a CAT(-1) space, then unique geodesicity extends to the bordification in the sense that for all $x, y \in \operatorname{bord} X$ such that $x \neq y$, there exists a unique geodesic segment connecting x with y [11, Proposition 4.4.4]. Again, we denote this geodesic segment by $[x, y]$.

If X is a geodesic metric space, then the condition of hyperbolicity can be reformulated in several different ways, including the *thin triangles condition*.

Proposition 2.5 ([7, §III.H.1]). *Assume that X is a geodesic hyperbolic metric space. Let $[x, y]$ denote the geodesic segment connecting two points $x, y \in X$.*

(i) *For all $x, y, z \in X$,*
$$d(z, [x, y]) \asymp_+ \langle x|y \rangle_z.$$

(ii) *(Rips' thin triangles condition) For all $x, y_1, y_2 \in X$ and $z \in [y_1, y_2]$, we have*
$$\min_{i=1}^{2} d(z, [x, y_i]) \asymp_+ 0.$$

In fact, the thin triangles condition is equivalent to hyperbolicity; see e.g. [7, Proposition III.H.1.22].

Convention 5. In the remainder of this text, X denotes a CAT(-1) space, and $o \in X$ denotes a distinguished point.

In particular, X is a geodesic Gromov hyperbolic metric space. However, X is not necessarily proper. We keep in mind the special case where X is a ROSSONCT (finite- or infinite-dimensional).

We now recall various definitions and theorems from [11].

2.3. **Classification of isometries.** For $g \in \mathrm{Isom}(X)$, let $\mathrm{Fix}(g) := \{x \in \mathrm{bord}\, X : g(x) = x\}$. Given $\xi \in \mathrm{Fix}(g) \cap \partial X$, ξ is a *neutral* or *indifferent* fixed point if $g'(\xi) = 1$, an *attracting* fixed point if $g'(\xi) < 1$, and a *repelling* fixed point if $g'(\xi) > 1$. Here, $g'(\xi)$ denotes the metric derivative of g at ξ (see [11, §4.2.2]).

Definition 2.6. An isometry $g \in \mathrm{Isom}(X)$ is called
- *elliptic* if the orbit $\{g^n(o) : n \in \mathbb{N}\}$ is bounded,
- *parabolic* if it is not elliptic and has a unique fixed point, which is neutral, and
- *loxodromic* if it has exactly two fixed points, one of which is attracting and the other of which is repelling.

The categories of elliptic, parabolic, and loxodromic are clearly mutually exclusive. Conversely, any isometry is either elliptic, parabolic, or loxodromic (e.g. [11, Theorem 6.1.4]).

2.4. **The limit set.** An important invariant of a group $G \leq \mathrm{Isom}(X)$ is its *limit set* $\Lambda \subseteq \partial X$, defined as the intersection of the closure of $G(o)$ with ∂X. The limit set Λ is both closed and G-invariant, and conversely

Proposition 2.7 ([9, Théorème 5.1] or [11, Theorem 7.4.1]). *Fix $G \leq \mathrm{Isom}(X)$. Then any closed G-invariant subset of ∂X containing at least two points contains Λ.*

2.5. **The quasiconvex core.**

Definition 2.8. The *quasiconvex hull* of a set $S \subseteq \mathrm{bord}\, X$ is the set Given $S \subseteq \mathrm{bord}\, X$, let

$$\mathrm{Hull}_1(S) := \bigcup_{x,y \in S} [x, y].$$

The *quasiconvex core* of a group $G \leq \mathrm{Isom}(X)$ is the set

$$\mathcal{C}_o := X \cap \overline{\mathrm{Hull}_1(G(o))}.$$

Although the quasiconvex core depends on the distinguished point o, for any $x, y \in X$ the Hausdorff distance between the sets $\mathcal{C}_x$ and $\mathcal{C}_y$ is finite [11, Proposition 7.5.9].

Remark 2.9. The quasiconvex hull of a set $S \subseteq \operatorname{bord} X$ is in general smaller than the *convex hull*, which is by definition the smallest convex subset of $\operatorname{bord} X$ which contains S. However, for a general CAT(-1) space the operation of taking the convex hull may be quite badly behaved; cf. [11, Remark 7.5.6] for a more detailed discussion with references.

2.6. Horoballs.

Definition 2.10. A *horoball* is a set of the form
$$H_{\xi,t} = \{x \in X : \mathcal{B}_\xi(o, x) > t\},$$
where $\xi \in \partial X$ and $t \in \mathbb{R}$. The point ξ is called the *center* of a horoball $H_{\xi,t}$, and will be denoted $\operatorname{center}(H_{\xi,t})$. Note that for any horoball H, we have
$$\overline{H} \cap \partial X = \{\operatorname{center}(H)\}.$$

2.7. Parabolic fixed points.

Definition 2.11. A group $G \leq \operatorname{Isom}(X)$ is *parabolic* if $G(o)$ is an unbounded set and G has a global fixed point $\xi \in \operatorname{Fix}(G)$ such that
$$g'(\xi) = 1 \ \ \forall g \in G,$$
i.e. ξ is neutral with respect to every element of G.

Definition 2.12. Let $G \leq \operatorname{Isom}(X)$. A point $\xi \in \partial X$ is a *parabolic fixed point* of G if the semigroup
$$G_\xi := \operatorname{Stab}(G; \xi) = \{g \in G : g(\xi) = \xi\}$$
is a parabolic group. If in addition there exists a set $S \subseteq X$ whose closure does not contain ξ such that
$$G(o) \subseteq G_\xi(S),$$
then ξ is called a *bounded parabolic point*. We denote the set of bounded parabolic poitns by $\Lambda_{\mathrm{bp}}(G)$.

Lemma 2.13 ([11, Lemma 12.3.6])**.** *Let ξ be a bounded parabolic limit point of G, and let H be a horoball centered at ξ satisfying $G(o) \cap H = \varnothing$. Then there exists $\rho > 0$ such that*

(2.6) $$\mathcal{C}_o \cap \partial H \subseteq G_\xi(B(o, \rho)).$$

2.8. Quasi-isometric embeddings of G into X.

Definition 2.14. Let X, Y be metric spaces. A map $f : X \to Y$ is called a *quasi-isometric embedding* if for all $x_1, x_2 \in X$,
$$d_Y(f(x_1), f(x_2)) \asymp_{+,\times} d_X(x_1, x_2).$$

A set $S \subseteq Y$ is *cobounded* if there exists $\rho > 0$ such that for all $y_1 \in Y$, there exists $y_2 \in S$ such that $d_Y(y_1, y_2) \leq \rho$. A quasi-isometric embedding whose image is cobounded is called a *quasi-isometry*.

If $G \leq \text{Isom}(X)$ is a geometrically finite group without parabolic points, then by the Milnor–Schwarz lemma [7, Proposition I.8.19], G is finitely generated, and for any Cayley graph of G, the orbit map $g \mapsto g(o)$ is a quasi-isometric embedding. If G is geometrically finite with parabolic points, then in general neither of these things is true.[7] Nevertheless, by considering a certain *weighted* Cayley metric with infinitely many generators, we can recover the rough metric structure of the orbit $G(o)$.

Definition 2.15. Let Γ be a group, let $E_0 \subseteq \Gamma$ be a generating set, and let $\ell_0 : E_0 \to (0, \infty)$. Assume that for all $g \in E_0$, we have $g^{-1} \in E_0$ and $\ell_0(g^{-1}) = \ell_0(g)$. Then the *weighted Cayley metric* on Γ corresponding to the generating set E_0 and the weight function ℓ_0 is the metric d_Γ given by the formula

$$d_\Gamma(g_1, g_2) := \inf_{\substack{(h_i)_1^n \in E_0^n \\ g_1 = g_2 h_1 \cdots h_n}} \sum_{i=1}^n \ell_0(h_i).$$

Let $G \leq \text{Isom}(X)$ be a geometrically finite group. In what follows, we describe a generating set and a weight function whose weighted Cayley metric recovers the rough metric structure of $G(o)$. Let P be a complete set of inequivalent parabolic points of G, and consider the set

$$E := \bigcup_{p \in P} G_p,$$

and for each $h \in G$ let

(2.7) $$\ell_0(h) := 1 \vee \|h\|.$$

When G is endowed with its weighted Cayley metric corresponding to the generating set $E \cup F$ (where F is a sufficiently large finite set) and the weight function ℓ_0, then the orbit map will be a quasi-isometric embedding:

Theorem 2.16 ([11, Theorem 12.4.14]). *If $G \leq \text{Isom}(X)$ is geometrically finite (cf. Definition 1.1), then*

  (i) *There exists a finite set F such that G is generated by $E \cup F$.*
  (ii) *The orbit map $g \mapsto g(o)$ is a quasi-isometric embedding with respect to the weighted Cayley metric corresponding to the generating set $E \cup F$ and the weight function* (2.7).

3. PROOF OF THEOREM 1.4

In this section, the notation and assumptions will be as in Theorem 1.4.

It follows e.g from [11, Theorem 6.2.3] that a subgroup of G is parabolic if and only if it is infinite and consists only of parabolic and elliptic elements. Since Φ is type-preserving, it follows that Φ preserves the class of parabolic subgroups, and also the class of maximal parabolic subgroups. But all maximal parabolic subgroups of G are of the form G_ξ, where ξ is a parabolic fixed point of G. It follows that there is a bijection

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[7]For examples of infinitely generated strongly discrete parabolic groups, see [11, Examples 11.2.18 and 11.2.20]; these examples can be extended to nonelementary examples by taking a Schottky product with a lineal group. [11, Theorem 11.2.6] guarantees that the orbit map of a parabolic group is never a quasi-isometric embedding.

$\phi : \Lambda_{\mathrm{bp}}(G) \to \Lambda_{\mathrm{bp}}(\widetilde{G})$ such that $\Phi(G_\xi) = \widetilde{G}_{\phi(\xi)}$ for all $\xi \in \Lambda_{\mathrm{bp}}(G)$. The equivariance of ϕ implies that $\widetilde{P} := \phi(P)$ is a complete set of inequivalent parabolic points for $\widetilde{G}$.

Let d_G and $d_{\widetilde{G}}$ denote the weighted Cayley metrics on G and $\widetilde{G}$, respectively, with respect to the generating set and weight function of Theorem 2.16.

Lemma 3.1. $d_G \asymp_\times d_{\widetilde{G}} \circ \Phi$.

Proof. Let E and F be as in Theorem 2.16, and let $\widetilde{E}$ and $\widetilde{F}$ be the corresponding sets for $\widetilde{G}$. Since $\widetilde{P} = \phi(P)$, we have $\widetilde{E} = \Phi(E)$. On the other hand, for all $h \in E$, we have $\ell_0(h) \asymp_\times \ell_0(\Phi(h))$ by (1.1). Thus, edges in the weighted Cayley graph of G have roughly (multiplicatively asymptotically) the same weight as their corresponding edges in the weighted Cayley graph of $\widetilde{G}$. (The sets F and $\widetilde{F}$ are both finite, and so their edges are essentially irrelevant.) The lemma follows. $\qquad\square$

Thus, the map $\Phi(g(o)) := \Phi(g)(o)$ is a quasi-isometry between $G(o)$ and $\widetilde{G}(o)$. At this point, we would like to extend Φ to an equivariant homeomorphism between Λ and $\widetilde{\Lambda}$. However, all known theorems which give such extensions, e.g. [4, Theorem 6.5], require the spaces in question to be geodesic or at least roughly geodesic – for the good reason that the extension theorems are false without this hypothesis[8] – but the spaces $G(o)$ and $\widetilde{G}(o)$ are not roughly geodesic. They are, however, embedded in the roughly geodesic metric spaces $\mathcal{C}_o$ and $\widetilde{\mathcal{C}}_o$, which suggests the strategy of extending the map Φ to a quasi-isometry between $\mathcal{C}_o$ and $\widetilde{\mathcal{C}}_o$. It turns out that this strategy works if we assume (1.2), and thus proves the existence of a *quasisymmetric* equivariant homeomorphism between Λ and $\widetilde{\Lambda}$ in that case. Since we know that the equivariant homeomorphism is not necessarily quasisymmetric if (1.2) fails (Example 4.8 and Remark 4.9), this strategy can't be used to prove part (i) of Theorem 1.4. Thus the proof splits into two parts at this point, depending on whether we have the stronger assumption (1.2) which guarantees quasisymmetry, or only the weaker assumption (1.1).

3.1. **Completion of the proof assuming** (1.2).

Lemma 3.2. *Fix $p \in P$ and let $\widetilde{p} = \phi(p)$. Let*

$$A = A(p) = \bigcup_{h \in G_p} [h(o), p],$$

---

[8]A counterexample is given by letting

$$X_1 = X_2 = \mathbb{R}, \qquad d_1(x, y) = \log(1 + |y - x|), \qquad d_2(x, y) = \begin{cases} d_1(x, y) & xy \geq 0 \\ d_1(0, x) + d_1(0, y) & xy \leq 0 \end{cases},$$

and letting $\Phi : X_1 \to X_2$ be the identity map – since $\#(\partial X_1) = 1 < 2 = \#(\partial X_2)$, Φ cannot be extended to a homeomorphism between ∂X_1 and ∂X_2. On the other hand, if one of the spaces in question is geodesic, then the extension theorem can be proven by isometrically embedding the other space into a geodesic hyperbolic metric space via [4, Theorem 4.1] – a fact which however has no relevance to the present situation.

and define a bijection $\psi = \psi_p : A \to \widetilde{A} := A(\widetilde{p})$ by

$$\psi([h(o), p]_t) = [\Phi(h)(o), \widetilde{p}]_{\alpha_p t}.$$

Then ψ is a quasi-isometry.

Proof. Fix two points $x_i = [h_i(o), p]_{t_i} \in A$, $i = 1, 2$. Write $y_i = h_i(o)$, $i = 1, 2$. Then

(3.1) $$d(x_1, x_2) \asymp_+ |t_2 - t_1| \vee (d(y_1, y_2) - t_1 - t_2).$$

(This can be seen e.g. by repeated application of Proposition 2.5(ii).) On the other hand, if we write $\widetilde{y}_i = \Phi(h_i)(o)$, $\widetilde{t}_i = \alpha_p t_i$, and $\widetilde{x}_i = [\widetilde{y}_i, \widetilde{p}]_{\widetilde{t}_i}$, then by (1.2) we have $d(\widetilde{y}_1, \widetilde{y}_2) \asymp_+ \alpha_p d(y_1, y_2)$; applying (3.1) along with its tilded version, we see that $d(\widetilde{x}_1, \widetilde{x}_2) \asymp_+ \alpha_p d(x_1, x_2)$. $\quad\square$

For $g(p) \in G(P) = \Lambda_{\mathrm{bp}}(G)$, write $A_{g(p)} = g(A_p)$ and $\psi_{g(p)} = \Phi(g) \circ \psi_p \circ g^{-1}$; then $\psi_{g(p)} : A_{g(p)} \to A_{\phi(g(p))}$ is a quasi-isometry, and the implied constants are independent of $g(p)$. Let

$$S = S(G) = \bigcup_{\xi \in \Lambda_{\mathrm{bp}}(G)} A_\xi \supseteq G(o),$$

and define $\psi : S \to \widetilde{S} := S(\widetilde{G})$ by letting

$$\psi(x) = \psi_\xi(x) \ \ \forall \xi \in \Lambda_{\mathrm{bp}}(G) \ \ \forall x \in A_\xi.$$

Note that for $g \in G$, $\psi(g(o)) = \Phi(g)(o)$.

Lemma 3.3. *ψ is a quasi-isometry.*

Proof. Fix two points $x_1, x_2 \in S$. For each $i = 1, 2$, write $x_i \in A_{g_i(p_i)}$ for some $g_i(p_i) \in \Lambda_{\mathrm{bp}}(G_i)$. If $g_1(p_1) = g_2(p_2)$, then $d(\psi(x_1), \psi(x_2)) \asymp_+ d(x_1, x_2)$ by Lemma 3.2. Otherwise, let $t > 0$ be large enough so that the collection $\mathscr{H} = \{H_{g(p)} := g(H_{p,t}) : g \in G, \ p \in P\}$ is disjoint. Then $y_i := [x_i, g_i(p_i)]_t \in H_{g_i(p_i)}$. It follows that the geodesic $[y_1, y_2]$ intersects both $\partial H_{g_1(p_1)}$ and $\partial H_{g_2(p_2)}$ (cf. Figure 3.1), say in the points z_1, z_2. By Lemma 2.13, there exist points $w_i \in g_i G_{p_i}(o)$ such that $d(z_i, w_i) \asymp_+ 0$. To summarize, we have

$$d(x_1, x_2) \asymp_{+,t} d(y_1, y_2) = d(z_1, z_2) + \sum_{i=1}^{2} d(y_i, z_i) \asymp_{+,t} d(w_1, w_2) + \sum_{i=1}^{2} d(x_i, w_i).$$

As $x_i, w_i \in A_{g_i(p_i)}$, we have $d(\psi(x_i), \psi(w_i)) \asymp_+ d(x_i, w_i)$ by Lemma 3.2. On the other hand, since $w_1, w_2 \in G(o)$, we have $d(\widetilde{w}_1, \widetilde{w}_2) \asymp_{+,\times} d(w_1, w_2)$ by Lemma 3.1 and Theorem 2.16(ii). (Here $\widetilde{x} = \psi(x)$.) Thus,

$$d(x_1, x_2) \asymp_{+,\times} d(\widetilde{w}_1, \widetilde{w}_2) + \sum_{i=1}^{2} d(\widetilde{x}_i, \widetilde{w}_i) \geq d(\widetilde{x}_1, \widetilde{x}_2).$$

Since the situation is symmetric, the reverse inequality holds as well. $\quad\square$

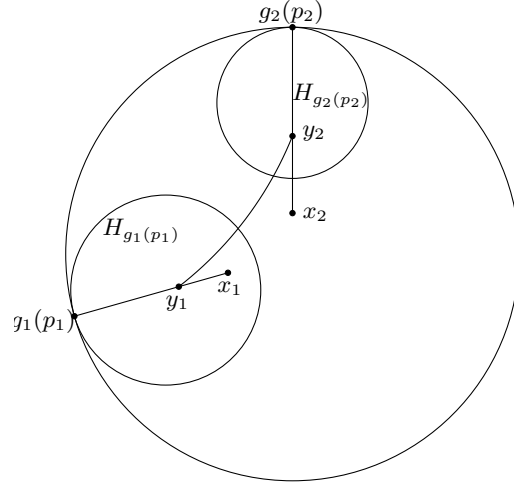**Lemma 3.4.** *S is cobounded in $C = \mathcal{C}_o$.*

FIGURE 3.1. The proof of Lemma 3.3. The distance between y_1 and y_2 is broken up into three segments, each of which is coarsely asymptotically preserved upon applying ψ.
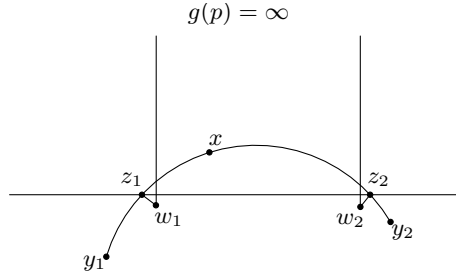

FIGURE 3.2. The proof of Lemma 3.4, in the upper half-space model (cf. e.g. [11, §2.5.2]). The thin triangles condition guarantees that x is close to one of the geodesics $[w_1, g(p)]$, $[w_2, g(p)]$, both of which are contained in S.

Proof. Fix $x \in \mathcal{C}_o$. If $x \notin \bigcup \mathscr{H}$, then $d(x, S) \leq d(x, G(o)) \asymp_+ 0$. So suppose $x \in H = H_{g(p)}$ for some $g \in G$, $p \in P$. Write $x \in [y_1, y_2]$ for some $y_1, y_2 \in G(o)$. Then there exist $z_1, z_2 \in [y_1, y_2] \cap \partial H$ such that $x \in [z_1, z_2]$. By Lemma 2.13, there exist $w_1, w_2 \in gG_p(o)$ such that $d(z_i, w_i) \asymp_+ 0$. It follows that $\langle w_1 | w_2 \rangle_x \asymp_+ 0$. By Proposition 2.5, we have

$$d(x, S) \leq d(x, S_{g(p)}) \leq d(x, [w_1, g(p)] \cup [w_2, g(p)]) \asymp_+ 0$$

(cf. Figure 3.2). This completes the proof.   □

Thus, the embedding map from S to C is an equivariant quasi-isometry. Thus S, C, $\widetilde{S}$, and $\widetilde{C}$ are all equivariantly quasi-isometric. By [4, Theorem 6.5], the quasi-isometry between C and $\widetilde{C}$ extends to a quasisymmetric homeomorphism between $\partial C = \Lambda$ and $\partial \widetilde{C} = \widetilde{\Lambda}$. This completes the proof of Theorem 1.4(ii).

3.2. **Completion of the proof assuming only** (1.1)**.** We begin by recalling the Morse lemma:

Definition 3.5. A path $\gamma : [a, b] \to X$ is a *K-quasigeodesic* if for all $a \leq t_1 < t_2 \leq b$,

$$\frac{1}{K}(t_2 - t_1) - K \leq d(\gamma(t_1), \gamma(t_2)) \leq K(t_2 - t_1) + K.$$

(In other words, γ is a K-quasigeodesic if $d(\gamma(t_1), \gamma(t_2)) \asymp_{+,\times} t_2 - t_1$, and the implied constants are both equal to K.)

Lemma 3.6 (Morse Lemma, [12, Theorem 9.38])**.** *For every $K > 0$, there exists $K_2 > 0$ such that the Hausdorff distance between any K-quasigeodesic γ and the geodesic $[\gamma(a), \gamma(b)]$ is at most K_2.*

Lemma 3.7. *Fix $h_1, \ldots, h_n \in E \cup F$, let $g_k = h_1 \cdots h_k$ and $x_k = g_k(o)$ for all $k = 0, \ldots, n$, and suppose that*

$$(3.2) \qquad d(x_k, x_\ell) \asymp_\times \sum_{i=k+1}^{\ell} \ell_0(h_i) \ \ \forall 0 \leq k < \ell \leq n.$$

Then the path $\gamma = \bigcup_{k=0}^{n-1}[x_k, x_{k+1}]$ is a K-quasigeodesic, where $K > 0$ is independent of $h_1, \ldots, h_n$.

Proof. Fix $0 < k \leq \ell < n$ and points $z \in [x_{k-1}, x_k]$, $w \in [x_\ell, x_{\ell+1}]$. To show that γ is a quasigeodesic, it suffices to show that

$$(3.3) \qquad d(z, w) \gtrsim_{+,\times} d(z, x_k) + d(x_k, x_\ell) + d(x_\ell, w).$$

Claim 3.8. $d(z, w) \gtrsim_+ \min(d(z, x_{k-1}), d(z, x_k))$.

Proof. If $h_k \in F$, then $d(z, x_k) \leq d(x_{k-1}, x_k) \asymp_+ 0$, so $d(z, w) \gtrsim_+ d(z, x_k)$. Thus, suppose that $h_k \in E$; then $h_k \in G_p$ for some $p \in P$. Let $g = g_{k-1}$; since $g^{-1}(z) \in [o, h_k(o)]$, by Proposition 2.5(i) we have $d(g^{-1}(z), [y, p]) \asymp_+ 0$, where either $y = o$ or $y = h_k(o)$.

Subclaim 3.9. *There exists $t > 0$ independent of $h_1, \ldots, h_n$ such that $g^{-1}(w) \notin H_{p,t}$.*

(Cf. Figure 3.3.)

Proof. If $h_{\ell+1} \in F$, then $d(g^{-1}(w), g^{-1}(x_{\ell+1})) \asymp_+ 0$, in which case the subclaim follows from the fact that p is a bounded parabolic point. Thus suppose $h_{\ell+1} \in E$; then $h_{\ell+1} \in G_\eta$ for some $\eta \in P$. Let $k = g_\ell$; since $k^{-1}(w) \in [o, h_{\ell+1}(o)]$, by Proposition 2.5(i) we have $d(k^{-1}(w), [p, \eta]) \asymp_+ 0$, where either $p = o$ or $p = h_k(o)$. In particular $\mathcal{B}_\eta(o, k^{-1}(w)) \gtrsim_+ 0$, so by the disjointness of the family $\mathscr{H}$, there exists $t > 0$ such that $k^{-1}(w) \notin j(H_{q,t})$ for all $q \in P$ and $j \in G$ such that $j(q) \neq \eta$. In particular, letting $j = k^{-1}g = (h_k \cdots h_\ell)^{-1}$ and $q = p$, we have $g^{-1}(w) \notin H_{p,t}$ unless $j(p) = \eta$. But if $j(p) = \eta$, then $j = \text{id}$ due to the minimality P, and this contradicts (3.2). ◁

It follows that

$$d(z, w) \geq \mathcal{B}_p(g^{-1}(w), g^{-1}(z)) = \mathcal{B}_p(o, g^{-1}(z)) - \mathcal{B}_p(o, g^{-1}(w))$$
$$= d(y, g^{-1}(z)) - \mathcal{B}_p(o, g^{-1}(w)) \gtrsim_+ d(y, g^{-1}(z)) - t.$$

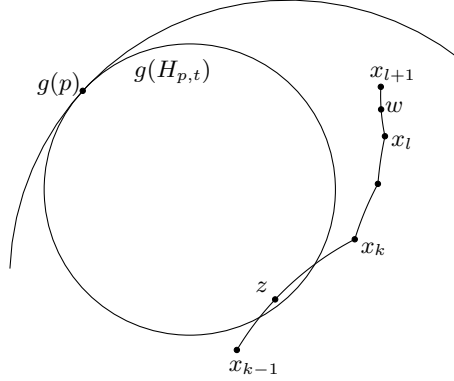Applying g to both sides finishes the proof of Claim 3.8. ◁

FIGURE 3.3. In Subclaim 3.9, the geodesics $[x_{k-1}, x_k]$ and $[x_\ell, x_{\ell+1}]$ cannot penetrate the same cusp, thus guaranteeing some distance between z and w.

A similar argument shows that $d(z, w) \gtrsim_+ \min(d(w, x_\ell), d(w, x_{\ell+1}))$. Now let $y_1 \in \{x_{k-1}, x_k\}$ and $y_2 \in \{x_\ell, x_{\ell+1}\}$ be such that

(3.4) $$d(z, w) \gtrsim_+ d(z, y_1) \text{ and } d(z, w) \gtrsim_+ d(w, y_2).$$

Then the triangle inequality gives $d(z, w) \gtrsim_{+,\times} d(y_1, y_2)$. On the other hand, (3.2) implies that $d(y_1, y_2) \gtrsim_{+,\times} d(y_1, x_k) + d(x_k, x_\ell) + d(x_\ell, y_2)$. Combining with (3.4) and using the triangle inequality gives (3.3). □

Lemma 3.10. *For all $x, y, z \in G(o)$,*

$$\langle \widetilde{x} | \widetilde{y} \rangle_{\widetilde{z}} \asymp_{+,\times} \langle x | y \rangle_z.$$

Proof. Fix $g_1, g_2 \in G$, and we will show that

(3.5) $$\langle \widetilde{g}_1(o) | \widetilde{g}_2(o) \rangle_o \lesssim_{+,\times} \langle g_1(o) | g_2(o) \rangle_o.$$

The reverse inequality will then follow by symmetry. By Theorem 2.16(ii), there exists a sequence $h_1, \ldots, h_n \in E \cup F$ such that $g_2 = g_1 h_1 \cdots h_n$ and satisfying (3.2). By Lemma 3.1, the sequence $\widetilde{h}_1, \ldots, \widetilde{h}_n \in \widetilde{E} \cup \widetilde{F}$ also satisfies (3.2). Let $x_k = g_1 h_1 \cdots h_k(o)$. By Lemma 3.7, the paths

$$\gamma = \bigcup_{k=0}^{n-1} [x_k, x_{k+1}]$$

$$\widetilde{\gamma} = \bigcup_{k=0}^{n-1} [\widetilde{x}_k, \widetilde{x}_{k+1}]$$

are quasigeodesics. So by Lemma 3.6, γ and $\widetilde{\gamma}$ lie within a bounded Hausdorff distance of the geodesics they represent, namely $[x_0, x_n]$ and $[\widetilde{x}_0, \widetilde{x}_n]$. Combining with Proposition

2.5(i), we have

$$\langle g_1(o)|g_2(o)\rangle_o = \langle x_0|x_n\rangle_o \asymp_+ d(o,[x_0,x_n]) \asymp_+ d(o,\gamma),$$

and similarly for $\widetilde{\gamma}$. So to prove (3.5), we need to show that $d(o,\widetilde{\gamma}) \lesssim_{+,\times} d(o,\gamma)$.

Fix $z \in \gamma$, and we will show that $\|z\| \gtrsim_{+,\times} d(o,\widetilde{\gamma})$. Write $z \in [x_{k-1}, x_k]$ for some $k = 1, \ldots, n$. By Proposition 2.5(i), we have

$$d(o,\widetilde{\gamma}) \leq d(o,[\widetilde{x}_{k-1}, \widetilde{x}_k]) \asymp_+ \langle \widetilde{x}_{k-1}|\widetilde{x}_k\rangle_o$$
$$\|z\| \geq d(o,[x_{k-1}, x_k]) \asymp_+ \langle x_{k-1}|x_k\rangle_o,$$

so to complete the proof of Lemma 3.10 it suffices to show that

(3.6) $$\langle \widetilde{x}_{k-1}|\widetilde{x}_k\rangle_o \lesssim_{+,\times} \langle x_{k-1}|x_k\rangle_o.$$

Now, if $h_k \in F$, then $d(x_{k-1}, x_k) \asymp_+ d(\widetilde{x}_{k-1}, \widetilde{x}_k) \asymp_+ 0$, so (3.6) follows from Theorem 2.16(ii). Thus, suppose that $h_k \in E$, and write $h_k \in G_p$ for some $p \in P$. Use the notations $g = g_1 h_1 \cdots h_{k-1}$ and $h = h_k$, so that $x_{k-1} = g(o)$ and $x_k = gh(o)$. Then for $y = o, h(o)$ we have (see [11, Corollary 3.4.12]) that

$$\langle y|p\rangle_{g^{-1}(o)} \asymp_+ \frac{1}{2}[d(g^{-1}(o), y) - \mathcal{B}_p(g^{-1}(o), y)] \gtrsim_+ \frac{1}{2} d(g^{-1}(o), y) \asymp_\times \|g(y)\|,$$

so by Gromov's inequality,

$$\langle x_{k-1}|x_k\rangle_o = \langle g(o)|gh(o)\rangle_o = \langle o|h(o)\rangle_{g^{-1}(o)} \gtrsim_{+,\times} \|g(y)\| \asymp_{+,\times} \|\widetilde{g}(\widetilde{y})\| \geq \langle \widetilde{x}_{k-1}|\widetilde{x}_k\rangle_o.$$

This demonstrates (3.6) and completes the proof of Lemma 3.10. $\square$

It follows that the map Φ sends Gromov sequences to Gromov sequences, so it induces an equivariant homeomorphism $\partial\Phi : \Lambda \to \widetilde{\Lambda}$. This completes the proof of Theorem 1.4(i).

4. APPLICATIONS TO FINITE-DIMENSIONAL ROSSONCTs

A particularly interesting case of Theorem 1.4 is when X and $\widetilde{X}$ are both finite-dimensional ROSSONCTs. In this case, (1.1) always holds, but (1.2) does not; nevertheless, there is a reasonable sufficient condition for (1.2) to hold. Specifically, we have the following:

Proposition 4.1. *Let X and $\widetilde{X}$ be finite-dimensional ROSSONCTs, let $G \leq \mathrm{Isom}(X)$ and $\widetilde{G} \leq \mathrm{Isom}(\widetilde{X})$ be geometrically finite groups, and let $\Phi : G \to \widetilde{G}$ be a type-preserving isomorphism. Fix $p \in P$, and let $\widetilde{p} = \phi(p) \in \Lambda_{\mathrm{bp}}(\widetilde{G})$ be the unique point such that $\Phi(G_p) = \widetilde{G}_{\widetilde{p}}$. Then*
  (i) *(1.1) holds.*
  (ii) *Let $H \leq G_p$ be a nilpotent subgroup of finite index. If the underlying base fields of X and $\widetilde{X}$ are the same, say $\mathbb{F}$, and if $\mathrm{rank}([H, H]) = \dim_\mathbb{R}(\mathbb{F}) - 1$, then (1.2) holds.*

Before we begin the proof of Proposition 4.1, it will be necessary to understand the structure of a parabolic subgroup of $\mathrm{Isom}(X)$.

Let $X = \mathbb{H} = \mathbb{H}_\mathbb{F}^d$ be a finite-dimensional ROSSONCT, let $p = [(1, 1, \mathbf{0})]$, and let $J_p = \mathrm{Stab}(\mathrm{Isom}(X); p)$. Note that J_p is a parabolic group in the sense of Lie theory, while it is a focal group according to the classification of [11, §6] (and in particular not parabolic

in the sense of Definition 2.11). To study the group J_p, we use the coordinate system generated by the basis

$$\mathbf{f}_0 = (\mathbf{e}_0 + \mathbf{e}_1)/2, \quad \mathbf{f}_1 = \mathbf{e}_1 - \mathbf{e}_0, \quad \mathbf{f}_i = \mathbf{e}_i \quad (i = 2, \ldots, d).$$

In this coordinate system, the sesquilinear form $B_{\mathcal{Q}}$ takes the form

$$B_{\mathcal{Q}}(\mathbf{x}, \mathbf{y}) = \overline{x}_0 y_1 + \overline{x}_1 y_0 + \sum_{i=2}^{d} \overline{x}_i y_i,$$

the point p takes the form $p = [\mathbf{f}_0]$, and the group J_p can be written (cf. [11, Theorem 2.3.3]) as

$$J_p = \left\{ h_{\lambda,a,\mathbf{v},\mathbf{w},m,\sigma} := \begin{bmatrix} \lambda & a & \mathbf{w}^\dagger \\ & \lambda^{-1} & \\ & \mathbf{v} & m \end{bmatrix} \sigma^{d+1} : \begin{array}{l} \lambda > 0, \ a \in \mathbb{F}, \ \mathbf{v}, \mathbf{w} \in \mathbb{F}^{d-1}, \\ m \in \mathrm{SO}(\mathbb{F}^{d-1}; \mathcal{E}), \ \sigma \in \mathrm{Aut}(\mathbb{F}) \end{array} \right\} \cap \mathrm{Isom}(X),$$

where $\mathcal{E}$ denotes the Euclidean quadratic form on $\mathbb{F}^{d-1}$. Given $\lambda, a, \mathbf{v}, \mathbf{w}, m$, it is readily verified that $h_{\lambda,a,\mathbf{v},\mathbf{w},m} \in \mathrm{Isom}(X)$ if and only if

$$2\lambda^{-1} \mathrm{Re}(a) + \|\mathbf{v}\|^2 = 0 \text{ and } \lambda^{-1}\mathbf{w}^\dagger + \mathbf{v}^\dagger m = \mathbf{0}.$$

Consequently, it makes sense to rewrite J_p as

$$J_p = \left\{ h_{\lambda,a,\mathbf{v},m,\sigma} := \begin{bmatrix} \lambda & a - \lambda\|\mathbf{v}\|^2/2 & -\lambda\mathbf{v}^\dagger m \\ & \lambda^{-1} & \\ & \mathbf{v} & m \end{bmatrix} \sigma^{d+1} : \begin{array}{l} \lambda > 0, \ a \in \mathrm{Im}(\mathbb{F}), \ \mathbf{v} \in \mathbb{F}^{d-1}, \\ m \in \mathrm{SO}(\mathbb{F}^{d-1}; \mathcal{E}), \ \sigma \in \mathrm{Aut}(\mathbb{F}) \end{array} \right\}.$$

We can now define the *Langlands decomposition* of J_p:

$$M_p = \{h_{1,0,\mathbf{0},m,\sigma} : m \in \mathrm{SO}(\mathbb{F}^{d-1}; \mathcal{E}), \ \sigma \in \mathrm{Aut}(\mathbb{F})\}$$
$$A_p = \{h_{\lambda,0,\mathbf{0},I_{d-1},e} : \lambda > 0\}$$
$$N_p = \{n(a, \mathbf{v}) := h_{1,a,\mathbf{v},I_{d-1},e} : a \in \mathrm{Im}(\mathbb{F}), \ \mathbf{v} \in \mathbb{F}^{d-1}\}$$
$$J_p = M_p A_p N_p.$$

We observe the following facts about the Langlands decomposition: the groups M_p and A_p commute with each other and normalize N_p, which is nilpotent of order at most 2. Moreover, the subgroup $M_p N_p$ is exactly the kernel of the homomorphism $J_p \ni h \mapsto h'(p)$, where h' denotes the metric derivative. Equivalently, $M_p N_p$ is the largest parabolic subgroup of J_p, where "parabolic" is interpreted in the sense of Definition 2.11.

Let's look a bit more closely at the internal structure of N_p. The composition law is given by

(4.1) $$n(a_1, \mathbf{v}_1)n(a_2, \mathbf{v}_2) = n(a_1 + a_2 + \mathrm{Im}\, B_{\mathcal{E}}(\mathbf{v}_2, \mathbf{v}_1), \mathbf{v}_1 + \mathbf{v}_2),$$

confirming that N_p is nilpotent of order at most two, and that its commutator is given by

$$Z_p = \{n(a, \mathbf{0}) : a \in \mathrm{Im}(\mathbb{F})\}.$$

Moreover, the map $\pi : n(a, \mathbf{v}) \mapsto \mathbf{v} \in \mathbb{F}^{d-1}$ is a homomorphism whose kernel is Z_p.

Now let $H \leq M_p N_p$ be a discrete parabolic subgroup. By Margulis's lemma, H is almost nilpotent, and so by [10, Lemma 3.4], there exist a finite index subgroup $H_2 \subseteq H$

and a homomorphism $\psi : H_2 \to N_p$ such that $\psi(h)(o) = h(o)$ for all $h \in H_2$. (Here $o = [\mathbf{e}_0] = [2\mathbf{f}_0 - \mathbf{f}_1]$ as usual.) We then let $H_3 = \psi(H_2) \leq N_p$.

Definition 4.2. The group H is *regular* if $\pi(H_3)$ is a discrete subgroup of $\mathbb{F}^{d-1}$. If H is regular, we define its *quasi-commutator* to be the subgroup

$$Z = Z(H) = \psi^{-1}(Z_p) = \mathrm{Ker}(\pi \circ \psi) \leq H.$$

Note that in general, the quasi-commutator of H cannot be determined from its algebraic structure; cf. Example 4.8. Nevertheless, since $\mathbb{F}^{d-1}$ is abelian, the quasi-commutator of H always contains the commutator of H_2.

In general, if $H \leq \mathrm{Isom}(X)$ is a discrete parabolic subgroup, we can conjugate the fixed point of H to $[(1,1,\mathbf{0})]$, apply the above construction, and then conjugate back to get a subgroup $Z(H) \leq H$.

If H is regular, then the quasi-commutator $Z \leq H$ can be used to give an algebraic description of the function $h \mapsto \|h\|$. Specifically, we have the following:

Lemma 4.3. *Let d_H and d_Z be any Cayley metrics on H and Z, respectively.*

(i)

(4.2) $$\|h\| \asymp_{+,\times} 0 \vee \log d_H(e, h).$$

(ii) *If H is regular, then*

(4.3) $$\|h\| \asymp_+ \min_{z \in Z} \left(0 \vee 2 \log d_H(z, h) \vee \log d_Z(e, z) \right) \quad \forall h \in H.$$

Proof. Let $F \subseteq H$ be a finite set so that $H_2 F = H$, and let $H_3 = \psi(H_2)$. Then for all $h \in H$, we can write $h = h_2 f$ for some $h_2 \in H_2$ and $f \in F$, and then

$$\|h\| \asymp_+ \|h_2\| = \|\psi(h_2)\|$$
$$d_H(z, h) \asymp_+ d_H(z, h_2) \asymp_\times d_{H_2}(z, h_2) = d_{H_3}(\psi(z), \psi(h_2))$$
$$\min_{z \in Z} \left(0 \vee 2 \log d_H(z, h) \vee \log d_Z(e, z) \right) \asymp_+ \min_{z \in \psi(Z)} \left(0 \vee 2 \log d_{H_3}(z, \psi(h_2)) \vee \log d_{\psi(Z)}(e, z) \right).$$

Thus, we may without loss of generality assume that $H = H_3$, i.e. that $H \leq N_p$ and $Z_H = H \cap Z_p$. We can also without loss of generality assume that $p = [(1,1,\mathbf{0})]$.

The following formula regarding the function $n(a, \mathbf{v})$ can be verified by direct computation (cf. [10, (3.5)]):

(4.4) $$\|n(a, \mathbf{v})\| \asymp_+ 0 \vee 2 \log \|\mathbf{v}\| \vee \log |a|$$

On the other hand, iterating (4.1) gives

(4.5)
$$\|\mathbf{v}\| \lesssim_\times d_H(e, n(a, \mathbf{v}))$$
$$|a| \lesssim_\times d_H(e, n(a, \mathbf{v}))^2$$
$$|a| \lesssim_\times d_Z(e, n(a, \mathbf{0})).$$

These formulas make it easy to verify the $\lesssim$ direction of (4.3): given $h = n(a, \mathbf{v}) \in H$ and $z = n(b, \mathbf{0}) \in Z$, we have

$$
\begin{aligned}
0 \vee 2 \log d_H(z, h) \vee \log d_Z(e, z) = & \; 0 \vee 2 \log d_H(e, n(a - b, \mathbf{v})) \vee \log d_Z(e, n(b, \mathbf{0})) \\
\geq & \; 0 \vee 2 \log \left(\|\mathbf{v}\| \vee \sqrt{|a - b|} \right) \vee \log |b| \\
= & \; 0 \vee 2 \log \|\mathbf{v}\| \vee \log |a - b| \vee \log |b| \\
\gtrsim_+ & \; 0 \vee 2 \log \|\mathbf{v}\| \vee \log |a| \asymp_+ \|n(a, \mathbf{v})\| = \|h\|.
\end{aligned}
$$

Setting $z = e$ yields the $\lesssim$ direction of (4.2).

To prove the $\gtrsim$ directions, we will need the following easily verified fact:

Fact 4.4. If V is a finite-dimensional vector space, $\Lambda \leq V$ is a discrete subgroup, and d_Λ is a Cayley metric on Λ, then $d_\Lambda(\mathbf{0}, \mathbf{v}) \asymp_\times \|\mathbf{v}\|$ for all $\mathbf{v} \in \Lambda$. Here $\|\cdot\|$ denotes any norm on V.

To prove the $\gtrsim$ direction of (4.3), assume that H is regular, fix $h = n(a, \mathbf{v}) \in H$, and let F be a finite generating set for H. Since H is regular, the group $\Lambda = \pi(H) \leq \mathbb{F}^{d-1}$ is discrete. Since $\mathbb{F}^{d-1}$ is a finite-dimensional vector space, Fact 4.4 guarantees the existence of a sequence $f_1, \ldots, f_n \in F$ such that $\pi(f_1 \cdots f_n) = \pi(h)$ and $n \lesssim_\times \|\mathbf{v}\|$. Let $f = f_1 \cdots f_n$ and let $z = h f^{-1} \in \pi^{-1}(0) = Z$, say $z = n(b, \mathbf{0})$. Applying (4.1) and the second equation of (4.5), we see that $|b| \lesssim_\times |a| \vee \|\mathbf{v}\|^2 \vee n^2 \lesssim_\times |a| \vee \|\mathbf{v}\|^2$. On the other hand, applying Fact 4.4 to Z_p gives $d_Z(e, z) \lesssim_\times |b|$. Thus

$$
\begin{aligned}
0 \vee 2 \log d_H(z, h) \vee \log d_Z(e, z) = & \; 0 \vee 2 \log d_H(e, f) \vee \log d_Z(e, z) \\
\lesssim_+ & \; 0 \vee 2 \log(n) \vee \log |b| \\
\lesssim_+ & \; 0 \vee 2 \log \|\mathbf{v}\| \vee \log(|a| \vee \|\mathbf{v}\|^2) \\
= & \; 0 \vee 2 \log \|\mathbf{v}\| \vee \log |a| = \|h\|.
\end{aligned}
$$

This completes the proof of (4.3).

To prove the $\gtrsim$ direction of (4.2), let $\overline{H}$ and $\overline{Z}$ be the Zariski closures of H and Z in N_p, respectively. Then $\overline{H}/\overline{Z}$ and $\overline{Z}$ are abelian Lie groups, and therefore isomorphic to finite-dimensional vector spaces. Let $\widetilde{\pi} : \overline{H} \to \overline{H}/\overline{Z}$ be the projection map. Note that $\|\widetilde{\pi}(n(a, \mathbf{v}))\| \lesssim_\times |a| \vee \|\mathbf{v}\|$ for all $n(a, \mathbf{v}) \in H$. Here $\|\cdot\|$ denotes any norm on $\overline{H}/\overline{Z}$.

Since $\overline{Z}$ is a vector space, the fact that Z is Zariski dense in $\overline{Z}$ simply means that Z is a lattice in $\overline{Z}$. In particular, Z is cocompact in $\overline{Z}$, which implies that $\widetilde{\pi}(H)$ is discrete. Fix $h = n(a, \mathbf{v}) \in H$, and let F be a finite generating set for H. Then by Fact 4.4, there exists a sequence $f_1, \ldots, f_n \in F$ such that $\widetilde{\pi}(f_1) \cdots \widetilde{\pi}(f_n) = \widetilde{\pi}(h)$ and $n \lesssim_\times \|\widetilde{\pi}(h)\| \lesssim_\times |a| \vee \|\mathbf{v}\|$. Let $f = f_1 \cdots f_n$ and let $z = h f^{-1} \in H \cap \widetilde{\pi}^{-1}(0) = H \cap \overline{Z} = Z$, say $z = n(b, \mathbf{0})$. Applying (4.1) and the second equation of (4.5), we see that $|b| \lesssim_\times |a| \vee \|\mathbf{v}\|^2 \vee n^2 \lesssim_\times |a|^2 \vee \|\mathbf{v}\|^2$. On the other hand, applying Fact 4.4 to $\overline{Z}$ gives $d_Z(e, z) \lesssim_\times |b|$. Thus

$$
\begin{aligned}
0 \vee \log d_H(e, h) \leq & \; 0 \vee \log d_H(e, f) \vee \log d_Z(e, z) \\
\lesssim_+ & \; 0 \vee \log(n) \vee \log |b| \\
\lesssim_+ & \; 0 \vee \log(|a| \vee \|\mathbf{v}\|) \vee \log(|a|^2 \vee \|\mathbf{v}\|^2) \\
\asymp_\times & \; 0 \vee 2 \log \|\mathbf{v}\| \vee \log |a| = \|h\|.
\end{aligned}
$$

This completes the proof of (4.2). □

Corollary 4.5. *Let X and $\widetilde{X}$ be finite-dimensional ROSSONCTs, let $H \leq \mathrm{Isom}(X)$ and $\widetilde{H} \leq \mathrm{Isom}(\widetilde{X})$ be parabolic groups with fixed points p and $\widetilde{p}$, respectively, and let $\Phi : H \to \widetilde{H}$ be an isomorphism. Then*

   (i) *(1.1) holds.*

   (ii) *If H and $\widetilde{H}$ are regular, then (1.2) holds if and only if $\Phi(Z)$ is commensurable to $\widetilde{Z}$. Here $Z = Z(H)$ and $\widetilde{Z} = Z(\widetilde{H})$.*

Proof. (1.1) follows immediately from (4.2). Suppose that H and $\widetilde{H}$ are regular and that $\Phi(Z)$ is commensurable to $\widetilde{Z}$. Since the right hand side of (4.3) depends on both h and Z, let us write it as a function $R(h, Z)$. We then have

$$\|h\| \asymp_+ R(h, Z) = R(\widetilde{h}, \Phi(Z)) \asymp_+ R(\widetilde{h}, \widetilde{Z}) \asymp_+ \|\widetilde{h}\|.$$

On the other hand, suppose that $\Phi(Z)$ and $\widetilde{Z}$ are not commensurable. Without loss of generality, suppose that the index of $\Phi(Z) \cap \widetilde{Z}$ in $\Phi(Z)$ is infinite. Since $\Phi(Z)$ is a finitely generated abelian group, it follows that there exists $\widetilde{h} = \Phi(h) \in \Phi(Z)$ such that $\widetilde{h}^n \notin \widetilde{Z}$ for all $n \in \mathbb{Z} \setminus \{0\}$. Without loss of generality, suppose that $\widetilde{h} \in \widetilde{H}_2$; otherwise replace h by an appropriate power. Then (4.4) implies that

$$\|h^n\| \asymp_{+,h} \log(n) \text{ but } \|\widetilde{h}^n\| \asymp_{+,h} 2\log(n).$$

Thus (1.2) fails along the sequence $(h_n)_1^\infty$. □

Corollary 4.6. *In the context of Corollary 4.5, if X and $\widetilde{X}$ are both real ROSSONCTs, then (1.2) holds.*

Proof. Since $\mathrm{Im}(\mathbb{R}) = \{0\}$, the group Z_p is trivial and thus Z and $\widetilde{Z}$ are trivial as well; moreover, every discrete parabolic group is regular. □

Corollary 4.7. *In the context of Corollary 4.5, if we assume both that*

   (I) *H is a lattice in $M_p N_p$, and that*

   (II) *the underlying base fields $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ of X and $\widetilde{X}$ satisfy $\dim_{\mathbb{R}}(\mathbb{F}) \geq \dim_{\mathbb{R}}(\widetilde{\mathbb{F}})$,*

then (1.2) holds.

Proof. Let H_2, ψ, H_3, and $Z = Z(H)$ be as on page 17. Without loss of generality, we may assume that $H = H_3$ and $\widetilde{H} = \widetilde{H}_3$. Then H is a lattice in N_p and $\widetilde{H} \leq \widetilde{N}_{\widetilde{p}}$.

Since H is a lattice in N_p, H is Zariski dense in N_p; this implies that $[H, H]$ is Zariski dense in $Z_p = [N_p, N_p]$. Thus, the rank of $[H, H]$ (and also of $\Phi([H, H]) = [\widetilde{H}, \widetilde{H}]$) is equal to $\dim_{\mathbb{R}}(\mathrm{Im}(\mathbb{F})) = \dim_{\mathbb{R}}(\mathbb{F}) - 1$. Thus $\dim_{\mathbb{R}}(\widetilde{\mathbb{F}}) - 1 = \mathrm{rank}([H, H]) \leq \dim(Z_p) = \dim_{\mathbb{R}}(\widetilde{\mathbb{F}}) - 1$. Since by assumption $\dim_{\mathbb{R}}(\mathbb{F}) \geq \dim_{\mathbb{R}}(\widetilde{\mathbb{F}})$, equality holds. Thus Z is a lattice in Z_p and is commensurable to $[H, H]$. Similarly, $\widetilde{Z}$ is a lattice in $\widetilde{Z}_{\widetilde{p}}$ and is commensurable to $[\widetilde{H}, \widetilde{H}]$. Thus, the groups H and $\widetilde{H}$ are regular. Finally, $\widetilde{Z}$ is commensurable to $[\widetilde{H}, \widetilde{H}] = \Phi([H, H])$ which is commensurable to $\Phi(Z)$, so Corollary 4.5 finishes the proof. □

We can now prove Theorems 1.5 and 1.6 from the introduction:

Proof of Theorem 1.5. If $G \leq \mathrm{Isom}(X)$ is a lattice, then every parabolic subgroup G_p satisfies (I). Thus, combining Corollaries 4.5 and 4.7 proves Theorem 1.5.   □

Proof of Theorem 1.6. Xie has observed that the main result of his paper generalizes to ROSSONCTs once one verifies that Tukia's isomorphism theorem and the Global Measure Formula both generalize to that setting (cf. [25, p.1]). We have just shown that Tukia's isomorphism theorem generalizes (to the present setting at least), and the Global Measure Formula has been shown to generalize by Schapira [22, Théorème 3.2].

Actually, we should mention a minor change that needs to be made to Xie's proof in the setting of ROSSONCTs: Since the Hausdorff and topological dimensions of the boundary of a nonreal ROSSONCT are not equal, at the top of [25, p.252] one should use Pansu's lemma [18, Proposition 6.5], [25, Lemma 2.3(a)] to deduce the lower bound on the Hausdorff dimension of $\Lambda(G_2)$ (i.e. [25, p.252, line 4]) rather than using Szpilrajn's inequality between Hausdorff and topological dimensions (cf. [25, p.252, lines 2-3]).   □

Note that in Xie's proof, quasisymmetry is used in an essential way due to his use of Pansu's lemma [18, Corollary 7.2], [25, Lemma 2.3]. Thus, the fact that the stronger asymptotic (1.2) holds in the context of Corollary 4.7 is essential to the proof of Theorem 1.6. It remains to be answered whether Theorem 1.6 holds if we drop the assumption of identical base fields.

We end this section by giving an example of groups for which (1.2) fails.

Example 4.8. Let $\mathbb{H} = \mathbb{H}_{\mathbb{C}}^3$, let $p = [(1,1,\mathbf{0})]$, and define a homomorphism $\theta : \mathbb{R}^3 \to N_p$ by $\theta(x, y, z) = n(xi, (y, z))$, where $i = \sqrt{-1}$. Consider the parabolic groups $H, H', H'' \leq N_p$ defined by

$$H = \theta(\mathbb{Z} \times \mathbb{Z} \times \{0\})$$
$$H' = \theta(\Lambda \times \{0\})$$
$$H'' = \theta(\{0\} \times \mathbb{Z} \times \mathbb{Z}).$$

In the middle equation, Λ denotes a lattice in $\mathbb{R}^2$ which does not intersect the axes. Then the groups H, H', H'' are all isomorphic, but we will show below that (1.2) cannot hold for any isomorphisms between them. This is accounted for in Corollary 4.5 as follows: The group H' is irregular, so Corollary 4.5 does not apply; The groups $Z(H)$ and $Z(H'')$ are not almost isomorphic (the former is isomorphic to $\mathbb{Z}$ while the latter is isomorphic to $\{0\}$), so Corollary 4.5 does not apply.

Proof. Note that the function $\| \cdot \|$ is described on $\theta(\mathbb{R}^3)$ by

$$\|\theta(x, y, z)\| \asymp_+ 0 \vee \log|x| \vee 2\log(|y| \vee |z|)$$

(cf. (4.4)). Now let $h_1 = \theta((1, 0, 0)) \in H$, $h_2 = \theta((0, 1, 0)) \in H$. Then

$$\|h_i^n\| \asymp_+ i \log(n);$$

but if Φ is an isomorphism from H to either H' or H'', then

$$\|\Phi(h_i)^n\| \asymp_+ 2\log(n).$$

This demonstrates the failure of (1.2), as setting $h = h_1^n$ gives $\alpha_p = 2$ while setting $h = h_2^n$ gives $\alpha_p = 1$.

Next, let $d_{H'}$ and $d_{H''}$ be Cayley metrics on H' and H'', respectively. Then for all $R \geq 1$,

$$\sup_{d_{H'}(e,h') \leq R} \|h'\| \asymp_+ 2\log(R) > \log(R) \asymp_+ \inf_{d_{H'}(e,h') > R} \|h'\|.$$

but

$$\sup_{d_{H''}(e,h'') \leq R} \|h''\| \asymp_+ \inf_{d_{H''}(e,h'') > R} \|h''\| \asymp_+ 2\log(R)$$

This demonstrates the failure of (1.2) for any isomorphism between H' and H'', as taking the supremum over a ball in the Cayley metric gives $\alpha_p = 1$, while taking the infimum over the complement of a ball in the Cayley metric gives $\alpha_p = 2$. □

Remark 4.9. The above proof actually shows more; namely, it shows that if $\Phi : G \to \widetilde{G}$ is a type-preserving isomorphism so that for some $p \in \Lambda_{\mathrm{bp}}$, G_p and $\widetilde{G}_{\widetilde{p}}$ are distinct elements of $\{H, H', H''\}$, then the equivariant boundary extension of Φ is not quasisymmetric.

Proof. By contradiction suppose that the equivariant boundary extension $\phi : \Lambda \to \widetilde{\Lambda}$ is quasisymmetric. Fix $\zeta \in \Lambda \setminus \{p\}$, and let $\widetilde{\zeta} = \phi(\zeta)$. Then by equivariance, for each $h \in G_p$ we have

$$\phi(h(\zeta)) = \widetilde{h}(\widetilde{\zeta}).$$

Let $f : (0, \infty) \to (0, \infty)$ be as in Definition 1.3, so that for all $\xi, \eta_1, \eta_2 \in \Lambda$,

$$\frac{\widetilde{D}(\widetilde{\xi}, \widetilde{\eta}_2)}{\widetilde{D}(\widetilde{\xi}, \widetilde{\eta}_1)} \leq f\left(\frac{D(\xi, \eta_2)}{D(\xi, \eta_1)}\right).$$

Letting $\xi = p$ and $\eta_i = h_i(\zeta)$ gives

$$\frac{\widetilde{D}(\widetilde{p}, \widetilde{h}_2(\widetilde{\zeta}))}{\widetilde{D}(\widetilde{p}, \widetilde{h}_1(\widetilde{\zeta}))} \leq f\left(\frac{D(p, h_2(\zeta))}{D(p, h_1(\zeta))}\right).$$

But $D(p, h_i(\zeta)) \asymp_{\times, \zeta} D(p, h_i(o)) = e^{(1/2)\|h_i\|}$; thus

$$\exp\left(\frac{1}{2}\left[\|\widetilde{h}_2\| - \|\widetilde{h}_1\|\right]\right) \leq f_2 \exp\left(\frac{1}{2}\left[\|h_2\| - \|h_1\|\right]\right),$$

where $f_2(t) = Cf(Ct)$ for some constant $C > 0$. Letting $f_3(t) = 2\log f_2(e^{(1/2)t})$ gives

$$\|\widetilde{h}_2\| - \|\widetilde{h}_1\| \leq f_3(\|h_2\| - \|h_1\|).$$

But this is readily seen to contradict the proof of Example 4.8. □

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UNIVERSITY OF WISCONSIN – LA CROSSE, DEPARTMENT OF MATHEMATICS, 1725 STATE STREET, LA CROSSE, WI 54601, USA

E-mail address: tdas@uwlax.edu

URL: https://sites.google.com/a/uwlax.edu/tdas/

UNIVERSITY OF YORK, DEPARTMENT OF MATHEMATICS, HESLINGTON, YORK YO10 5DD, UK

E-mail address: simmons.465@osu.edu

URL: https://sites.google.com/site/davidsimmonsmath/

UNIVERSITY OF NORTH TEXAS, DEPARTMENT OF MATHEMATICS, 1155 UNION CIRCLE #311430, DENTON, TX 76203-5017, USA

E-mail address: urbanski@unt.edu

URL: http://www.urbanskimath.com/