RANDOM COUNTABLE ALPHABET CONFORMAL ITERATED FUNCTUION SYSTEMS SATISFYING THE TRANSVERSALITY CONDITION

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ABSTRACT. Dealing with with countable (finite and infinite alike) alphabet random conformal iterated function systems with overlaps, we formulate appropriate Transversality Conditions and then prove the relevant, in such context, Moran-Bowen's formula which determines the Hausdorff dimension of random limit sets in dynamical terms. We also provide large classes of examples of such random systems satisfying the Transversality Condition.

1. INTRODUCTION

In this paper we consider conformal iterated function systems with countable alphabet. The main task in such context is to come up with a dynamical formula determining the Hausdorff dimension of the corresponding limit sets. The class of conformal iterated function systems naturally divides into the deterministic systems and the random ones on the the one hand and those satisfying the Open Set Condition against those with overlaps on the other hand. Similarity systems form subclasses of all of them.

Sticking to the class of deterministic systems, the Open Set Condition is important because it guarantees a dynamical formula for the Hausdorff dimension to hold. It is commonly referred to as the Moran-Bowen's formula. Open Set condition is also frequently naturally seen from the definition of the IFS in question to hold. However, a much bigger collection of IFSs fail to satisfy the Moran-Bowen's formula. Notably amongst them are the ones coming up from Bernoulli convolutions, see [10], [8], [14], and the references therein. Although there are some attempts to establish this formula for some single conformal IFSs with overlaps, there is no uniform approach to this issue. Nevertheless, when one considers a family of conformal IFSs parametrized by some bounded open subset of an Euclidean space, then the famous Transversality Condition due to Peres, Pollicott, Simon and Solomyak, (see [14], [8], [9]) guarantees the Moran-Bowen's formula to hold for Lebesgue almost every member of such family.

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From a different perspective, the Moran-Bowen's formula has been proved in [12], under the the assumption of the Open Set Condition, for random conformal iterated function systems with countable alphabet. In the present paper we consider the most general case of random conformal IFSs with overlaps. We formulate appropriate Transversality Conditions and then prove the relevant, in such context, Moran-Bowen's formula. This is done in Theorem 5.1, which is also formulated now:

Theorem. Let $X \subset \mathbb{R}^d$ be a compact domain. Suppose that $\mathcal{S} = \{\phi_e^\lambda : X \to X : e \in E, \lambda \in \Lambda\}$ is a conformal random iterated function system with a base map $\theta : \Lambda \to \Lambda$ preserving a probability measure m. Suppose further that $\mathcal{S}$ is either strongly regular or it is evenly varying. If the system $\mathcal{S}$ satisfies the Random Transversality Condition of Finite Type with respect to measure m, then

(a)
$$\mathrm{HD}(J_\lambda) = \min\{b(\mathcal{S}), d\}$$
for m-a.e. $\lambda \in \Lambda$.

(b) If $b(\mathcal{S}) > d$, then $\ell_d(J_\lambda) > 0$ for m-a.e. $\lambda \in \Lambda$. In particular $\mathrm{HD}(J_\lambda) = d$.

As an immediate consequence of this theorem, we get the following corollary, which is also new, in the case of a finite alphabet.

Corollary. Let $X \subset \mathbb{R}^d$ be a compact domain. Suppose that $\mathcal{S} = \{\phi_e^\lambda : X \to X : e \in E, \lambda \in \Lambda\}$ is a conformal random iterated function system with finite alphabet E and a base map $\theta : \Lambda \to \Lambda$ preserving a probability measure m. If the system $\mathcal{S}$ satisfies the Random Transversality Condition then

(a)
$$\mathrm{HD}(J_\lambda) = \min\{b(\mathcal{S}), d\}$$
for m-a.e. $\lambda \in \Lambda$.

(b) If $b(\mathcal{S}) > d$, then $\ell_d(J_\lambda) > 0$ for m-a.e. $\lambda \in \Lambda$. In particular $\mathrm{HD}(J_\lambda) = d$.

This corollary is also stated as Corollary 5.4 in Section 5.

Remark 1.1. As already mentioned in this introduction, the iterated function systems, deterministic and random alike, consisting of similarities form a subclass respectively of deterministic and random conformal iterated function systems. In particular, the above theorem and corollary apply to the case of similarities.

Remark 1.2. We would like to remark that the above theorem and corollary would remain true if instead of dealing with iterated function systems, we would deal with graph directed Markov systems (see [5] modeled on a finitely irreducible subshift of finite type rather than the full shift (i. e. the case of IFSs). The proofs would require only cosmetic, mainly notational, changes.

In Section 6, named Examples, we provide two large classes of examples of sub-random conformal IFSs satisfying the Random Transversality Condition of Finite Type. As explained in Remark 6.7 these systems easily give rise to abundance of random conformal IFSs satisfying the Random Transversality Condition of Finite Type.

The central idea in this paper, as actually always when dealing with the issue of Hausdorff dimension in the context of Transversality Condition is to use the capacity (energy functional) characterization of Hausdorff dimension. The appropriate measures on the random limit sets are obtained as projections of natural Gibbs/equilibrium states on the symbol space. The general line of estimates in the proof of Theorem 5.1 stems from those in [13] In the proof of the current paper we make an extensive use of Birkhoff's Ergodic Theorem rather than comparing the values of iterated contractions for nearby parameters, the measure we project from the symbol space is different than in [13] in a three-folded way. Firstly, it emerges from random rather than deterministic thermodynamic formalism, secondly, it corresponds to a different parameter determining the Gibbs measure, and thirdly, this Gibbs measure is now restricted to a set of positive measure, "good" from the perspective Birkhoff's Ergodic Theorem. Many further technical steps are different.

2. Preliminaries on Symbol Random Dynamical Systems

Let $(\Lambda, \mathfrak{F}, m)$ be a complete probability space with a σ-algebra $\mathfrak{F}$ and probability measure m. Let E be a countable set, either finite or infinite. It will be called an alphabet in the sequel. The symbol space $E^{\mathbb{N}}$ is naturally endowed with the product (Tichonov) topology. This topology is generated by many natural metrics. One class of them is defined as follows. If $s \in (0,1)$, then

$$d_s(\omega, \tau) := s^{\min\{n\geq 1: \omega_n \neq \tau_n\}},$$

with the common convention that $s^\infty = 0$. All these metrics are Hölderly equivalent in the sense that the identity map is, with respect to them, Hölder continuous. Denote by $\mathfrak{B}$ the σ-algebra of Borel subsets of $E^{\mathbb{N}}$. Let

$$\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$$

be the shift map, i. e.

$$\sigma\big((\omega_n)_{n=1}^\infty\big) = (\omega_{n+1})_{n=1}^\infty.$$

Of course σ is an open continuous map. The Cartesian product $\Lambda \times E^{\mathbb{N}}$ becomes a measurable space when equipped with the product σ-algebra $\mathfrak{F} \otimes \mathfrak{B}$, i.e. the σ-algebra generated by countable unions of Cartesian products of the form $F \times B$ with $B \in \mathfrak{B}$ and $F \in \mathfrak{F}$. Let

$$p_{E^{\mathbb{N}}} : \Lambda \times E^{\mathbb{N}} \to E^{\mathbb{N}} \ \text{ and } \ p_\Lambda : \Lambda \times E^{\mathbb{N}} \to \Lambda$$

be the canonical projections onto $E^{\mathbb{N}}$ and Λ, respectively, i.e.

$$p_{E^{\mathbb{N}}}(\lambda, \omega) = \omega \ \text{ and } \ p_\Lambda(\lambda, \omega) = \lambda.$$

Both projections are trivially measurable. In fact, $\mathfrak{B} \otimes \mathfrak{F}$ is the smallest σ-algebra with respect to which both projections are measurable. The product map

$$\theta \times \sigma : \Lambda \times E^{\mathbb{N}} \to \Lambda \times E^{\mathbb{N}},$$

defined as

$$(\theta \times \sigma)(\lambda, \omega) := \big(\theta(\lambda), \sigma(\omega)\big),$$

is obviously measurable. Denote by $C(E^{\mathbb{N}})$ the space of all continuous real-valued functions on $E^{\mathbb{N}}$ and by $C^b(E^{\mathbb{N}})$ its vector subspace consisting of all bounded continuous functions, i.e. those $g \in C(E^{\mathbb{N}})$ for which

$$\|g\|_\infty := \sup\{|g(\omega)| : \omega \in E^{\mathbb{N}}\} < \infty.$$

With this norm $C^b(E^{\mathbb{N}})$ becomes a Banach space. Let $0 < s < 1$. For every $g \in C((E^{\mathbb{N}})$, set

$$v_{s,k}(g) := \inf_{C \geq 0}\big\{|g(\omega) - g(\tau)| \leq Cs^k : \omega, \tau \in F_A^\infty \text{ such that } |\omega \wedge \tau| \geq k\}$$

and

$$v_s(g) := \sup\big\{v_{s,k}(g) : k \in \mathbb{N}\big\}.$$

A function $g \in C(E^{\mathbb{N}})$ is called Hölder continuous with exponent s if $v_s(g) < \infty$. The constant $v_s(g)$ is the smallest Hölder constant such a g admits. We shall denote by $H_s(E^{\mathbb{N}})$ the vector space of all Hölder continuous functions with exponent s. This is precisely the class of all real-valued functions on $E^{\mathbb{N}}$ that are Hölder continuous with respect to the metric d_s. We shall further denote by $H_s^b(E^{\mathbb{N}})$ the vector subspace of all Hölder continuous functions with exponent s which are bounded, i.e. $H_s^b(E^{\mathbb{N}}) := H_s(E^{\mathbb{N}}) \cap C^b(E^{\mathbb{N}})$. Endowed with the norm

$$\|g\|_s := \|g\|_\infty + v_s(g),$$

the space $H_s^b(E^{\mathbb{N}})$ becomes a Banach space. We now turn our attention to spaces of random functions.

Definition 2.1. A function $f : \Lambda \times E^{\mathbb{N}} \to \mathbb{R}$ is said to be a random continuous function if

- for all $\omega \in E^{\mathbb{N}}$ the ω-section $\Lambda \ni \lambda \mapsto f_\omega(\lambda) := f(\lambda, \omega)$ is measurable; and
- for all $\lambda \in \Lambda$ the λ-section $E^{\mathbb{N}} \ni \omega \mapsto f_\lambda(\omega) := f(\lambda, \omega)$ is continuous on E^N.

We shall denote the vector space of all random continuous functions on $E^{\mathbb{N}}$ by $C_\Lambda(E^{\mathbb{N}})$. Note that by Lemma 1.1 in [3], any random continuous function f is jointly measurable. It is then natural to make the following definitions.

Definition 2.2. A random continuous function $f \in C_{\mathcal{L}}(E^{\mathbb{N}})$ is said to be bounded if

$$\|f_\lambda\|_\infty < \infty \ \ \forall \lambda \in \Lambda \qquad \text{and} \qquad \|f\|_\infty := \operatorname{ess\,sup}\{\|f_\lambda\|_\infty : \lambda \in \Lambda\} < \infty.$$

Bounded random continuous functions, as defined above, are random continuous in the sense of Crauel [3] (cf. Definition 3.9). The space of all bounded random continuous functions on $E^{\mathbb{N}}$ will be denoted by $C_\Lambda^b(E^{\mathbb{N}})$. When equipped with the norm $\|f\|_\infty$, this space is Banach.

Definition 2.3. A random continuous function $f \in C_\Lambda(E^{\mathbb{N}})$ is said to be Hölder with exponent s if

$$v_s(f_\lambda) < \infty,\ \forall \lambda \in \Lambda \qquad \text{and} \qquad v_s(f) := \operatorname{ess\,sup}\{v_s(f_\lambda) : \lambda \in \Lambda\} < \infty.$$

We shall denote by $H_{s,\Lambda}(E^{\mathbb{N}})$ the vector space of all random Hölder continuous functions with exponent s and by $H^b_{s,\Lambda}(E^{\mathbb{N}})$ the subspace of all bounded random Hölder continuous functions with exponent s. Endowed with the norm

$$\|f\|_s := \|f\|_\infty + v_s(f),$$

the space $H^b_{s,\Lambda}(E^{\mathbb{N}})$ becomes a Banach space. We now introduce the concept of summability for random continuous functions.

Definition 2.4. A random continuous function $f \in C_\Lambda(E^{\mathbb{N}})$ is called summable if

$$M_f := \sum_{e\in E} \exp\Big(\operatorname{ess\,sup}\{\sup(f_\lambda|_{[e]}) : \lambda \in \Lambda\}\Big) < \infty. \tag{2.1}$$

Note that no bounded random continuous function on $E^{\mathbb{N}}$ is summable if E is infinite. Henceforth, we shall denote by a superscript Σ spaces of summable functions. For instance, the vector space of all summable random Hölder continuous functions with exponent s will be denoted by $H^\Sigma_{s,\Lambda}(E^{\mathbb{N}})$.

Now we shall define and describe some properties of random measures which will play a crucial role later. Denote by $P_\Lambda(m)$ the space of all probability measures ν on $(\Lambda\times E^{\mathbb{N}}, \mathfrak{F}\otimes\mathfrak{B})$ whose marginal is ν, i.e. all probability measures ν such that

$$\nu \circ p_\Lambda^{-1} = m.$$

The members of $P_\Lambda(m)$ are called the random measures with base m. By Propositions 3.3(ii) and 3.6 in [3], this space is canonically isomorphic to the space of all functions

$$\Lambda \times E^{\mathbb{N}} \ni (\lambda, B) \mapsto \nu_\lambda(B) \in [0,1]$$

such that

- for every $B \in \mathfrak{B}$, the function $\Lambda \ni \lambda \mapsto \nu_\lambda(B)$ is measurable; and
- for m-a.e. $\lambda \in \Lambda$, the function $\mathfrak{B} \ni B \mapsto \nu_\lambda(B)$ is a Borel probability measure.

The defining relation between these two concepts is that the formula

$$\nu(B) = \int_\Lambda \int_{E^{\mathbb{N}}} \nu_\lambda(B)\, dm(\lambda)$$

We shortly write this relation as

$$\nu = m \otimes \nu_\lambda.$$

3. Preliminaries from Random Countable Alphabet Conformal Iterated Function Systems

Let X be compact connected subset of $\mathbb{R}^d$, $d \geq 1$, with $X = \overline{\mathrm{Int}(X)}$. Let $(\Lambda, \mathfrak{F}, m)$ and E be as in the previous section. A random conformal iterated function system

$$\mathcal{S} = (\theta : \Lambda \to \Lambda, \{\Lambda \ni \lambda \mapsto \varphi_e^\lambda\}_{e \in E})$$

is generated by an invertible ergodic measure-preserving map

$$\theta : (\theta, \mathfrak{F}, m) \to (\Lambda, \mathfrak{F}, m)$$

of the complete probability space $(\Lambda, \mathfrak{F}, m)$ and one-to-one conformal contractions

$$\varphi_e^\lambda : X \to X, \quad e \in E, \ \lambda \in \Lambda,$$

with Lipschitz constants not exceeding a common number $0 < s < 1$. We in fact assume that there exists a bounded open connected set $W \subset \mathbb{R}^d$ containing X such that all maps $\phi_e^\lambda : X \to X$ extend conformally to (injective) maps from W to W. All the maps $\varphi_e^\lambda : W \to W$ are assumed to satisfy the following:

Property 3.1 (BDP1)**.** *There exist $\alpha > 0$ and a function $K : [0, 1) \to [1, +\infty)$ such that $\lim_{t \searrow 0} K(t) = K(0) = 1$, and*

$$\left| \frac{\left|(\phi_\omega^\lambda)'(y)\right|}{\left|(\phi_\omega^\lambda)'(x)\right|} - 1 \right| \leq K(t)||y - x||^\alpha$$

for all $\omega \in E^*$, all $\lambda \in \Lambda$, all $x \in X$ and all $y \in W$ with $||y - x|| \leq t\mathrm{dist}(x, \mathbb{R}^d \setminus W)$.

With a possibly larger $K(t)$, as an immediate consequence of this property, we get the following.

Property 3.2 (BDP2)**.** *There exists a function $K : [0, 1) \to [1, +\infty)$ such that $\lim_{t \searrow 0} K(t) = K(0) = 1$, and*

$$\sup \left\{ \frac{\left|(\phi_\omega^\lambda)'(y)\right|}{\left|(\phi_\omega^\lambda)'(x)\right|} : \omega \in E^*, \, \lambda \in \Lambda, \, x \in X, \, ||y - x|| \leq t\mathrm{dist}(x, \mathbb{R}^d \setminus W) \right\} \leq K(t).$$

Remark 3.3. According to Proposition 4.2.1 in [5], Property 3.1 and Property 3.2 are automatically satisfied with $\alpha = 1$ when $d \geq 2$. This condition is also fulfilled if $d = 1$, the alphabet E is finite and all the φ_e^λ's are of class $C^{1+\varepsilon}$ for some $\varepsilon > 0$.

We also require some common measurability conditions for the system $\mathcal{S}$ to satisfy. Precisely, we assume that for every $e \in E$ and every $x \in X$ the map

$$\Lambda \ni \lambda \mapsto \varphi_e^\lambda(x)$$

is measurable. According to Lemma 1.1 in [3], this implies that all the maps, for all $e \in E$,

$$\Lambda \times X \ni (\lambda, x) \mapsto \varphi_e(x, \lambda) := \varphi_e^\lambda(x)$$

are (jointly) measurable. For every $\omega \in E^*$, set

$$\Lambda \ni \lambda \mapsto \varphi_\omega^\lambda := \varphi_{\omega_1}^\lambda \circ \varphi_{\omega_2}^{\theta(\lambda)} \circ \ldots \circ \varphi_{\omega_{|\omega|}}^{\theta^{|\omega|-1}(\lambda)}.$$

This formula clearly exhibits the random aspect of our iteration: we choose consecutive generators $\varphi_{\omega_1}, \varphi_{\omega_2}, \ldots, \varphi_{\omega_n}$ according random process governed by the ergodic map $\theta : \Lambda \to \Lambda$. This random aspect is particularly striking if θ is a Bernoulli shift; it then means that we choose our maps ϕ_e^λ in an identically distributed independent way. Given $\omega \in E^{\mathbb{N}}$, analogously to the deterministic case, we note that the intersection

$$\bigcap_{n=1}^{\infty} \varphi_{\omega|_n}^\lambda (X_{t(\omega_n)})$$

is a singleton, and we denote its only element by $\pi_\lambda(\omega)$. Thus we have a map

$$\pi_\lambda : E^{\mathbb{N}} \to X.$$

It is straightforward to see that this map is continuous; in fact it is Hölder continuous with respect to every metric d_u and Lipschitz continuous with respect to d_s. Define

$$J_\lambda := \pi_\lambda(E^\infty) \subset X.$$

The set J_λ is called the limit set corresponding to the parameter λ.

Definition 3.4. The system $\mathcal{S}$ is called evenly varying if

$$\Delta := \sup_{e \in E} \frac{\operatorname{ess\,sup}\{\|(\varphi_e^\lambda)'\| : \lambda \in \Lambda\}}{\operatorname{ess\,inf}\{\|(\varphi_e^\lambda)'\| : \lambda \in \Lambda\}} < \infty.$$

Now we want to introduce some technical but useful terminology.

Definition 3.5. With the above notation:

- We call the collection $\mathcal{S} = \{\phi_e^\lambda : e \in E, \lambda \in \Lambda\}$ a pre-random conformal function system if no map $\theta : \Lambda \to \Lambda$ nor any measure m on Λ are taken into consideration (need not be specified).
- We call the collection $\mathcal{S} = \{\phi_e^\lambda : e \in E, \lambda \in \Lambda\}$ a sub-random conformal iterated function system if also a measurable map $\theta : \Lambda \to \Lambda$ is given but no θ-invariant measure m on Λ is taken into consideration (need not be specified).
- We recall that if $\mathcal{S}$ is a sub-random conformal function system and an ergodic θ-invariant measure m on Λ is given, then $\mathcal{S}$ is called a random conformal iterated function system.
- If Λ is a singleton the system $\mathcal{S}$ is called deterministic or, changeably, autonomous.

Keep now $\mathcal{S}$ a random conformal iterated function system. We shall describe a symbol dynamics and appropriate thermodynamic formalism induced by the system $\mathcal{S}$. Define the potential $\zeta : \Lambda \times E^{\mathbb{N}} \to \mathbb{R}$ as follows:

$$\zeta(\omega, \lambda) = \log\left|(\varphi_{\omega_1}^\lambda)'(\pi_{\theta(\lambda)}(\sigma\omega))\right|.$$

The map $E^{\mathbb{N}} \ni \omega \mapsto \zeta(\omega,\lambda)$ is continuous for each $\lambda \in \Lambda$, while the map $\Lambda \ni \lambda \mapsto \zeta(\omega,\lambda)$ is measurable for each $\omega \in E_A^\infty$. Thus, the map ζ is jointly measurable, i. e. it is a random continuous function.

Definition 3.6. We say that $t \in \mathcal{F}in(\mathcal{S})$ if $t \in [0,+\infty)$ and

$$M_t := \sum_{e\in E} \operatorname{ess\,sup}\{\|(\varphi_e^\lambda)'\|^t : \lambda \in \Lambda\} < \infty.$$

Note that the potential $t\zeta$ is summable if and only if $t \in \mathcal{F}in(\mathcal{S})$. In fact, $t\zeta \in H_{s,\Lambda}^\Sigma(E^{\mathbb{N}})$ and $t\zeta$ is bounded over finite subalphabets for every $t > 0$. Therefore, the thermodynamic formalism for random dynamical systems (see [2] and [7] if E is finite; see Theorem 2.12 in [12] with $f = t\zeta$ when E is infinite) gives the following:

If $t \in \mathcal{F}in(\mathcal{S})$, then for m-a.e. $\lambda \in \Lambda$ there are a unique bounded measurable function

$$\Lambda \ni \lambda \mapsto \mathrm{P}_\lambda(t) := \mathrm{P}_\lambda(t\zeta)$$

and a unique random probability measure $\nu^t \in P_\Lambda(m)$ such that

$$\nu_\lambda^t(\omega A) = e^{-\mathrm{P}_\lambda(t)|\omega|} \int_A \left|\left(\varphi_\omega^{\theta^{|\omega|}(\lambda)}\right)'\left(\pi_{\theta^{|\omega|}(\lambda)}(\tau)\right)\right|^t d\nu_{\theta^{|\omega|}(\lambda)}^t(\tau). \tag{3.1}$$

for m-a.e. $\lambda \in \Lambda$, all $\omega \in E^*$, and all Borel sets $A \subset E^{\mathbb{N}}$. Furthermore, there exists a unique non-negative $q^t \in C_\Lambda^b(E^{\mathbb{N}})$ with the following properties:

- (a) $\int_{E^{\mathbb{N}}} q_\lambda^t(\omega)\, d\nu_\lambda^t(\omega) = 1$ for m-a.e. $\lambda \in \Lambda$;
- (b) $0 < C(t)^{-1} \le \inf\{q_\lambda^t(\omega) : \omega \in E^{\mathbb{N}}, \lambda \in \Lambda\} \le \sup\{q_\lambda^t(\omega) : \omega \in E^{\mathbb{N}}, \lambda \in \Lambda\} \le C(t) < \infty$ for some constant $C(t) \ge 1$;
- (c) $(q_\lambda^t\nu_\lambda^t)\circ\sigma^{-1} = q_{\theta(\lambda)}^t\nu_{\theta(\lambda)}^t$ for m-a.e. $\lambda \in \Lambda$;
- (d) $(m\otimes q_\lambda^t\nu_\lambda^t)\circ(\theta\times\sigma)^{-1} = (m\otimes q_\lambda^t\nu_\lambda^t)$, that is, the measure $m\otimes q_\lambda^t\nu_\lambda^t$ is $(\theta\times\sigma)$-invariant.

Letting

$$\mu_\lambda^t := q_\lambda^t\nu_\lambda^t,$$

we can rewrite (c) and (d) in the more compact form

$$\mu_\lambda^t \circ \sigma^{-1} = \mu_{\theta(\lambda)}^t, \quad \nu\text{-a.e. } \lambda \in \Lambda \tag{3.2}$$

and

$$(m \otimes \mu_\lambda^t)\circ(\theta\times\sigma)^{-1} = m \otimes \mu_\lambda^t. \tag{3.3}$$

Put

$$\mu^t := m \otimes \mu_\lambda^t$$

Property (3.3) then says the following.

Proposition 3.7.

$$\mu^t \circ (\theta \times \sigma)^{-1} = \mu^t,$$

i.e. the random probability measure μ^t *is* $(\theta \times \sigma)$*-invariant. Moreover,*

$$\mu^t \circ p_\Lambda^{-1} = m,$$

where, we recall, $p_\Lambda : E^{\mathbb{N}} \times \Lambda \to \Lambda$ *is the canonical projection onto* Λ*. That is,* $\mu^t \in P_\Lambda(m)$*.*

The measure μ^t is called the equilibrium/Gibbs state for the potential $t\zeta$ relative to the measure m.

Remark 3.8. We would like to emphasize that for sole technical purposes of the current paper we will need formulas (3.1) – (3.3), items (a)-(d), and Proposition 3.7 only in the case when the alphabet under consideration (usually denoted F in the sequel) is finite. These results then directly follow from [7] and do not need quite involved considerations of [12].

Since all maps ϕ_e^λ are bi-Lipschitz, it immediately follows from Birkhoff's Ergodic Theorem that m-almost all limit sets J_λ have the same Hausdorff dimension. Denote this common value by $h_{\mathcal{S}}$ or simpler, by just by h. It was proved in Proposition 3.12 of [12] that for every $t \in \mathcal{F}in(\mathcal{S})$ the function $\Lambda \ni \lambda \mapsto \mathrm{P}_\lambda(t) \in \mathbb{R}$ is integrable, and we denote

$$\mathcal{E}\mathrm{P}(t) := \int_\Lambda \mathrm{P}_\lambda(t)\, dm(\lambda).$$

We call this number the expected pressure value of the parameter t. Let

$$t_{\mathcal{S}} := \inf\big(\mathcal{F}in(\mathcal{S})\big)$$

and

$$b(\mathcal{S}) := \inf\{t \in \mathcal{F}in(\mathcal{S}) : \mathcal{E}\mathrm{P}(t) \le 0\}.$$

Of course $t_{\mathcal{S}} \le b(\mathcal{S})$. Call the latter number, i. e. $b(\mathcal{S})$, the Bowen's parameter of the system $\mathcal{S}$. The following relation, without making use of the Open Set Condition (!), between $h_{\mathcal{S}}$ and $b(\mathcal{S})$ has been established in [12] as Lemma 3.19.

Lemma 3.9. *If* $\mathcal{S}$ *is a random conformal iterated function system, then*

$$h_{\mathcal{S}} \le \min\{b(\mathcal{S}), d\}.$$

Definition 3.10. We call a random conformal iterated function system $\mathcal{S}$ strongly regular if $t_{\mathcal{S}} < b(\mathcal{S}) < +\infty$.

Because of Proposition 3.13 in [12] (part (c) holds if $\mathcal{F}in(\mathcal{S}) \ne \emptyset$) we immediately get the following.

Observation 3.11. *A random conformal iterated function system* $\mathcal{S}$ *is strongly regular if and only if there exists* $t \in \mathcal{F}in(\mathcal{S})$ *such that*

$$0 < \mathcal{E}\mathrm{P}(t) < +\infty.$$

It is the main objective of this paper to obtain the inequality reversed to that of Lemma 3.9. Towards this end we will need some versions of the celebrated transversality condition due to Peres, Pollicott, Simon and Solomyak, see [14], [8], [9] and consult also the more recent literature. We formulate them in the next section.

4. Transversality Conditions

We now will formulate appropriate transversality conditions. We start with the following.

Definition 4.1 (**Random Transversality Condition of Finite Type).** A sub-random conformal iterated function system $\mathcal{S}$, acting on a seed set $X \subset \mathbb{R}^d$, is said to satisfy the random transversality condition of finite type with respect to a finite measure m on Λ if for every finite subset F of E there exists a constant $C_F \in [0, +\infty)$ such that for all $\omega, \tau \in F^{\mathbb{N}}$ with $\omega_1 \neq \tau_1$, we have

$$m\big(\{\lambda \in \Lambda : ||\pi_\lambda(\omega) - \pi_\lambda(\tau)|| \le r\}\big) \le C_F r^d.$$

Furthermore:

Definition 4.2 (**Random Transversality Condition).** A sub-random conformal iterated function system $\mathcal{S}$, acting on a seed set $X \subset \mathbb{R}^d$, is said to satisfy the random transversality condition with respect to a finite measure m on Λ if there exists a function $C : E \times E \setminus \{(e, e) : e \in E\}$ such that for all $\omega, \tau \in E^{\mathbb{N}}$ with $\omega_1 \neq \tau_1$, we have

$$m\big(\{\lambda \in \Lambda : ||\pi_\lambda(\omega) - \pi_\lambda(\tau)|| \le r\}\big) \le C(\omega_1, \tau_1) r^d.$$

Of course the Random Transversality Condition entails the Random Transversality Condition of Finite Type but it is the latter one which we really need. We shall prove the following.

Lemma 4.3. *Suppose that $\mathcal{S}$ is a sub-random conformal iterated function system satisfying the random transversality condition of finite type with respect to a finite measure m on Λ. Then for every finite subset F of E and every number $\alpha \in (0, d)$ there exists a constant $C_F(\alpha) \in (0, \infty)$ such that*

$$\int_\Lambda \frac{dm(\lambda)}{\|\pi_\lambda(\omega) - \pi_\lambda(\tau)\|^\alpha} \le C_F(\alpha).$$

Proof. To shorten long formulas, put in this proof $|X| := \text{diam}(X)$. In view of our transversality condition, we can estimate as follows:

$$
\begin{aligned}
\int_\Lambda \frac{d\lambda}{\|\pi_\lambda(\omega)-\pi_\lambda(\tau)\|^\alpha} &= \int_0^\infty m\left\{\lambda\in\Lambda:\ \frac{1}{\|\pi_\lambda(\omega)-\pi_\lambda(\tau)\|^\alpha}\ge x\right\}dx \\
&= \int_0^\infty m\left\{\lambda\in\Lambda: \|\pi_\lambda(\omega)-\pi_\lambda(\tau)\|\le r\right\}r^{-\alpha-1}dr \\
&= \int_0^{|X|} m\left\{\lambda\in\Lambda: \|\pi_\lambda(\omega)-\pi_\lambda(\tau)\|\le r\right\}r^{-\alpha-1}dr+ \\
&\qquad\qquad + \int_{|X|}^\infty m\left\{\lambda\in\Lambda:\ |\pi_\lambda(\omega)-\pi_\lambda(\tau)|\le r\right\}r^{-\alpha-1}dr \\
&\le C_F\int_0^{|X|} r^{d-\alpha-1}dr + \int_{|X|}^\infty m(\Lambda)r^{-\alpha-1}dr \\
&\le C_F(d-\alpha)^{-1}\text{diam}^{d-\alpha}(X)+\alpha^{-1}\text{diam}^{-\alpha}(X).
\end{aligned}
$$

Thus the lemma is proved. □

Remark 4.4. Of course, in the case when the alphabet E is finite both the Random Transversality Condition and the Random Transversality Condition of Finite Type coincide.

Now we assume that Λ is a q-dimensional Riemannian manifold endowed with a Borel probability measure λ_q equivalent to Riemannian volume on Λ. In fact, we assume more, namely that

$$C^{-1}r^q \le \lambda_q(B(x,r)) \le Cr^q$$

for all $r\in[0,1]$ with some constant $C\ge 1$.

Definition 4.5 (Strong Random Transversality Condition). A sub-random conformal iterated function system $\mathcal{S}$ is said to satisfy the Strong Random Transversality Condition with respect to a finite measure λ_q on Λ if there exists a function $C: E\times E\setminus\{(e,e): e\in E\}$ such that for all $\omega,\tau\in E^{\mathbb{N}}$ with $\omega_1\ne\tau_1$, we have

$$N_r\big(\{\lambda\in\Lambda: ||\pi_\lambda(\omega)-\pi_l(\tau)||\}\big)\le C(\omega_1,\tau_1)r^{d-q}$$

where $N_r(F)$ is the minimal number of balls with radii r needed to cover the set F

Of course:

Observation 4.6. *The Strong Random Transversality Condition entails the Random Transversality Condition.*

Similarly as the Random Transversality Condition of Finite Type we can introduce the Strong Random Transversality Condition of Finite Type. We then have:

Observation 4.7. *The Strong Random Transversality Condition of Finite Type entails the Random Transversality Condition of Finite Type .*

In order to come up with a bunch of examples satisfying the Strong Random Transversality Condition in the phase space of dimension $d = 1$, we quote the folowing lemma, stated and proved in [13] as Lemma 7.3.

Lemma 4.8. *Let $U \subset \mathbb{R}^q$ be an open, bounded set with smooth boundary. Suppose that f is a C^1 real-valued function in a neighborhood of $\overline{U}$ such that for some $i \in \{1, \dots, q\}$ there exists $\eta > 0$ satisfying*

$$\mathbf{t} \in U, \quad |f(\mathbf{t})| \le \eta \implies \frac{\partial f(\mathbf{t})}{\partial t_i} \ge \eta. \tag{4.1}$$

Then there exists $C = C(\eta)$ such that

$$N_r(\{\mathbf{t} \in U : \ |f(\mathbf{t})| \le r\}) \le C r^{1-q}, \quad \forall\, r > 0. \tag{4.2}$$

5. Moran-Bowen's Formula

This section is entirely devoted to prove the following main theorem of our paper along with its finite alphabet consequence.

Theorem 5.1. *Let $X \subset \mathbb{R}^d$ be a compact domain. Suppose that $\mathcal{S} = \{\phi_e^\lambda : X \to X : e \in E,\ \lambda \in \Lambda\}$ is a conformal random iterated function system with a base map $\theta : \Lambda \to \Lambda$ preserving a probability measure m. Suppose further that $\mathcal{S}$ is either strongly regular or it is evenly varying. If the system $\mathcal{S}$ satisfies the Random Transversality Condition of Finite Type with respect to measure m, then*

(a)
$$\mathrm{HD}(J_\lambda) = \min\{b(\mathcal{S}), d\}$$
for m-a.e. $\lambda \in \Lambda$.

(b) *If $b(\mathcal{S}) > d$, then $\ell_d(J_\lambda) > 0$ for m-a.e. $\lambda \in \Lambda$. In particular* $\mathrm{HD}(J_\lambda) = d$.

Proof. We shall first prove item (a). Because of part (b), which will be independently proved in the next step, we may assume that

$$b(\mathcal{S}) \le d.$$

If the system $\mathcal{S}$ is strongly regular, then fix any

$$t \in (t_{\mathcal{S}}, b(\mathcal{S})).$$

Then $0 < \mathcal{E}\mathrm{P}(t) < +\infty$. So, by Proposition 3.12 in [12] there exists a finite set $F \subset E$ such that

$$\mathcal{E}\mathrm{P}_F(t) > 0. \tag{5.1}$$

If $\mathcal{S}$ is not strongly regular, then $b(\mathcal{S}) = t_{\mathcal{S}}$, and also, by our hypotheses, $\mathcal{S}$ is evenly varying. Have then t any number in $(0, b(\mathcal{S}) = t_{\mathcal{S}})$. The proof of Lemma 3.17 in [12] (read with θ replaced by t) then produces a finite set $F \subset E$ such that (5.1) holds. From now on the (same) proof runs in either case. Choose then $\varepsilon > 0$ arbitrary so small that

$$(1 + 2t)\varepsilon < \mathcal{E}\mathrm{P}_F(t). \tag{5.2}$$

Let $\mu \in P_\Lambda(m)$ be the equilibrium/Gibbs state for the potential $t\zeta|_{\Lambda\times F^{\mathbb{N}}}$. By Birkhoff's Ergodic Theorem then there exists an integer $N \geq 1$ and $\Lambda_\varepsilon \subset \Lambda$, a measurable set such that

$$m(\Lambda_\varepsilon) > 1 - \varepsilon, \tag{5.3}$$

$$\left|\frac{1}{n}\mathrm{P}^n_{F,\lambda}(t) - \mathcal{E}\mathrm{P}_F(t)\right| < \varepsilon, \tag{5.4}$$

for all $\lambda \in \Lambda_\varepsilon$ and all integers $n \geq N$, for every $\lambda \in \Lambda_\varepsilon$ there exists a measurable set $\Gamma_\varepsilon(\lambda) \subset F^{\mathbb{N}}$ such that

$$\Gamma_\varepsilon := \bigcup_{\lambda\in\Lambda_\varepsilon} \{\lambda\} \times \Gamma_\varepsilon(\lambda) \tag{5.5}$$

is measurable in $\Lambda \times F^{\mathbb{N}}$,

$$\mu_\lambda\big(\Gamma_\varepsilon(\lambda)\big) > 0, \tag{5.6}$$

and

$$\left|\frac{1}{n}\log\left|\left|\left(\phi_{\omega|_n}\right)'\right|\right| + \chi\right| < \varepsilon \tag{5.7}$$

for all $\omega \in \Gamma_\varepsilon(\lambda)$ and all $n \geq n$, where

$$\chi = \chi_{T,F} := -\int_{\Lambda\times F^{\mathbb{N}}} \zeta\, d\mu > 0.$$

Assume also $\varepsilon > 0$ to be so small that

$$(1 + t)\varepsilon < \mathrm{h}_\mu((\theta \times \sigma)|\theta), \tag{5.8}$$

where the latter is the entropy of $(\theta \times \sigma)$ with respect to μ relative to base θ. Define a finite Borel measure $\hat{\mu}$ on $E^{\mathbb{N}}$ by the following formula:

$$\hat{\mu} := \mu|_{\Gamma_\varepsilon} \circ p^{-1}_{F^{\mathbb{N}}}, \tag{5.9}$$

where $p_{F^{\mathbb{N}}} : \Lambda \times F^{\mathbb{N}} \to F^{\mathbb{N}}$ is the canonical projection onto the second coordinate. For every $\lambda \in \Lambda$, in particular for every $\lambda \in \Lambda_\varepsilon$,

$$\hat{\mu}_\lambda := \hat{\mu} \circ \pi_\lambda^{-1}$$

is a Borel finite non-vanishing measure on $J_{F,\lambda} \subset J_\lambda$. Our main technical goal is to show that the following integral is finite.

$$R := \int_{\Lambda_\varepsilon} \int_{J_{F,\lambda}\times J_{F,\lambda}} \frac{d\hat{\mu}_\lambda(x)\, d\hat{\mu}_\lambda(y)}{||x - y||^t}\, dm(\lambda). \tag{5.10}$$

Directly from the definition of measures $\hat\mu_\lambda$ we see that

$$R=\int_{\Lambda_\varepsilon}\int_{(F^{\mathbb N})^2}\frac{d\hat\mu(\omega)\,d\hat\mu(\tau)}{||\pi_\lambda(\omega)-\pi_\lambda(\tau)||^t}\,dm(\lambda). \tag{5.11}$$

In order to further estimate this integral we first need the following.

Lemma 5.2.

$$\lim_{n\to\infty}\sup\{\hat\mu([\omega]):\omega\in F^n\}=0.$$

Proof. Since μ is the relative Gibbs state for the potential $t\zeta|_{\Lambda\times F^{\mathbb N}}$, it follows from (5.9) that if $|\omega|\ge N$, then

$$\begin{aligned}\hat\mu([\omega])=\mu|_{\Gamma_\varepsilon}(\Lambda\times[\omega])&=\int_{\Lambda_\varepsilon}\mu_\lambda([\omega]\times\Gamma_\varepsilon(\lambda))\,dm(\lambda)\preceq\int_{\Lambda_\varepsilon}e^{-\mathrm{P}_\lambda^n(t)}||(\phi_\omega^\lambda)'||^t\delta_{\omega,\lambda}\,dm(\lambda)\\ &\le\int_{\Lambda_\varepsilon}\exp\big(-(\mathcal E\mathrm P(t)+t\chi)n\big)\exp\big(\varepsilon(1+t)n\big)\,dm(\lambda)\\ &=\exp\big(-(\mathcal E\mathrm P(t)+t\chi)n\big)\exp\big(\varepsilon(1+t)n\big)m(\Lambda_\varepsilon)\\ &=\exp\big(-\mathrm h_\mu((\theta\times\sigma)|\theta)n\big)\exp\big(\varepsilon(1+t)n\big)m(\Lambda_\varepsilon),\end{aligned}$$

where $\delta_{\omega,\lambda}$ is equal to 0 or 1 respectively when the intersection $[\omega]\times\Gamma_\varepsilon(\lambda)$ is empty or not. The proof is now concluded by invoking (5.8). □

For every $\rho\in F^*$ put

$$A_\rho:=\big\{(\omega,\tau)\in F^{\mathbb N}\times F^{\mathbb N}:\omega|_{|\rho|}=\tau|_{|\rho|}\text{ and }\omega_{|\rho|+1}\ne\tau_{|\rho|+1}\big\}\subset[\rho]\times[\rho],$$

and let

$$\Delta:=\big\{(\omega,\omega)\in F^{\mathbb N}\times F^{\mathbb N}\big\}$$

be the diagonal of $F^{\mathbb N}\times F^{\mathbb N}$. We shall prove the following.

Lemma 5.3.

$$\hat\mu\otimes\hat\mu(\Delta)=0.$$

Proof. For every integer $n\ge1$ we have

$$\begin{aligned}\hat\mu\otimes\hat\mu(\Delta)\le\sum_{\rho\in F^n}\hat\mu\otimes\hat\mu(A_\rho)&\le\sum_{\rho\in F^n}\hat\mu\otimes\hat\mu([\rho]\times[\rho])=\sum_{\rho\in F^n}\hat\mu([\rho])\hat\mu([\rho])\\ &\le\sup\{\hat\mu([\omega]):\omega\in F^n\}\sum_{\rho\in F^n}\hat\mu([\rho])\\ &\le\sup\{\hat\mu([\omega]):\omega\in F^n\},\end{aligned}$$

and the proof is completed by invoking Lemma 5.2. □

Having this lemma, we can rewrite the integral (5.11) in the following form:

$$R=\sum_{n=0}^\infty\sum_{|\rho|=n}\int_{\Lambda_\varepsilon}\int_{A_\rho\cap\Gamma_\lambda^2(\varepsilon)}\frac{d\hat\mu(\omega)\,d\hat\mu(\tau)}{||\pi_\lambda(\omega)-\pi_\lambda(\tau)||^t}\,dm(\lambda). \tag{5.12}$$

Now fix $n \geq 0$ and $\rho \in F^n$. Define

$$\Lambda_\varepsilon(\rho) := \{\lambda \in \Lambda_\varepsilon : [\rho] \cap \Gamma_\varepsilon(\lambda) \neq \emptyset\}. \tag{5.13}$$

Write further

$$M_\rho := \inf\left\{||(\phi_\rho^\gamma)'|| : \gamma \in \Lambda_\varepsilon(\rho)\right\} > 0. \tag{5.14}$$

Then using Lemma 4.3 (which applies since $t < d$) and finiteness of the set F, we get
(5.15)

$$\begin{aligned}
R_\rho :&= \int_{\Lambda_\varepsilon} \int_{A_\rho \cap \Gamma_\lambda^2(\varepsilon)} \frac{d\hat{\mu}(\omega)\, d\hat{\mu}(\tau)}{||\pi_\lambda(\omega) - \pi_\lambda(\tau)||^t}\, dm(\lambda) = \int_{\Lambda_\varepsilon(\rho)} \int_{A_\rho \cap \Gamma_\lambda^2(\varepsilon)} \frac{d\hat{\mu}(\omega)\, d\hat{\mu}(\tau)}{||\pi_\lambda(\omega) - \pi_\lambda(\tau)||^t}\, dm(\lambda) \\
&= \int_{\Lambda_\varepsilon(\rho)} \int_{A_\rho \cap \Gamma_\lambda^2(\varepsilon)} \frac{d\hat{\mu}(\omega)\, d\hat{\mu}(\tau)}{\big|\big|\phi_\rho^\lambda\big(\pi_{\theta^n(\lambda)}(\sigma^n(\omega))\big) - \phi_\rho^\lambda\big(\pi_{\theta^n(\lambda)}(\sigma^n(\tau))\big)\big|\big|^t}\, dm(\lambda) \\
&\leq K^t \int_{\Lambda_\varepsilon(\rho)} \int_{A_\rho \cap \Gamma_\lambda^2(\varepsilon)} \frac{d\hat{\mu}(\omega)\, d\hat{\mu}(\tau)}{\big|\big|\big(\phi_\rho^\lambda\big)'\big|\big|^t \big|\big|\pi_{\theta^n(\lambda)}(\sigma^n(\omega)) - \pi_{\theta^n(\lambda)}(\sigma^n(\tau))\big|\big|^t}\, dm(\lambda) \\
&\leq K^t \int_{\Lambda_\varepsilon(\rho)} M_\rho^{-t} \int_{A_\rho \cap \Gamma_\lambda^2(\varepsilon)} \frac{d\hat{\mu}(\omega)\, d\hat{\mu}(\tau)}{\big|\big|\pi_{\theta^n(\lambda)}(\sigma^n(\omega)) - \pi_{\theta^n(\lambda)}(\sigma^n(\tau))\big|\big|^t}\, dm(\lambda) \\
&\leq K^t M_\rho^{-t} \int_{\Lambda_\varepsilon(\rho)} \int_{A_\rho} \frac{d\hat{\mu}(\omega)\, d\hat{\mu}(\tau)}{\big|\big|\pi_{\theta^n(\lambda)}(\sigma^n(\omega)) - \pi_{\theta^n(\lambda)}(\sigma^n(\tau))\big|\big|^t}\, dm(\lambda) \\
&= K^t M_\rho^{-t} \int_{A_\rho} \int_{\Lambda_\varepsilon(\rho)} \frac{dm(\lambda)}{\big|\big|\pi_{\theta^n(\lambda)}(\sigma^n(\omega)) - \pi_{\theta^n(\lambda)}(\sigma^n(\tau))\big|\big|^t}\, d\hat{\mu}(\omega)\, d\hat{\mu}(\tau) \\
&\leq K^t M_\rho^{-t} \int_{A_\rho} \int_{\Lambda} \frac{dm(\lambda)}{\big|\big|\pi_{\theta^n(\lambda)}(\sigma^n(\omega)) - \pi_{\theta^n(\lambda)}(\sigma^n(\tau))\big|\big|^t}\, d\hat{\mu}(\omega)\, d\hat{\mu}(\tau) \\
&= K^t M_\rho^{-t} \int_{A_\rho} \int_{\Lambda} \frac{dm(\lambda)}{\big|\big|\pi_\lambda(\sigma^n(\omega)) - \pi_\lambda(\sigma^n(\tau))\big|\big|^t}\, d\hat{\mu}(\omega)\, d\hat{\mu}(\tau) \\
&\leq C_F K^t M_\rho^{-t} \int_{A_\rho} d\hat{\mu}(\omega)\, d\hat{\mu}(\tau) \\
&= C_F K^t M_\rho^{-t} \hat{\mu} \otimes \hat{\mu}(A_\rho) < +\infty,
\end{aligned}$$

and in addition,

$$R_\rho = 0$$

whenever $\Lambda_\varepsilon(\rho) = \emptyset$. If $\Lambda_\varepsilon(\rho) \neq \emptyset$ and $n = |\rho| \geq N$, then it follows from (5.7) that

$$M_\rho \geq \exp\big(-(\chi + \varepsilon)n\big) \tag{5.16}$$

Consequently, making also use of (5.4) and directly of (5.7) as well as (3.1) and item (b) below it (both generating the constant $C_2 > 0$ below), we then can continue (5.15) to get

$$\begin{aligned}
R_\rho &\le C_F K^t \exp\big((\chi+\varepsilon)tn\big)\hat\mu\otimes\hat\mu(A_\rho) \le C_F K^t \exp\big((\chi+\varepsilon)tn\big)\hat\mu\otimes\hat\mu([\rho]\times[\rho]) \\
&= C_F K^t \exp\big((\chi+\varepsilon)tn\big)\hat\mu([\rho])\hat\mu([\rho]) \\
&= C_F K^t \exp\big((\chi+\varepsilon)tn\big)\hat\mu([\rho])\mu|_{\Gamma_\varepsilon}\big(p_{F^{\mathbb N}}^{-1}([\rho])\big) \\
&= C_F K^t \exp\big((\chi+\varepsilon)tn\big)\hat\mu([\rho])\mu\big(\Gamma_\varepsilon \cap p_{F^{\mathbb N}}^{-1}([\rho])\big) \\
&= C_F K^t \exp\big((\chi+\varepsilon)tn\big)\hat\mu([\rho])\int_{\Lambda_\varepsilon}\mu_\lambda\big(\Gamma_\varepsilon(\lambda)\cap[\rho]\big)\,dm(\lambda) \\
&= C_F K^t \exp\big((\chi+\varepsilon)tn\big)\hat\mu([\rho])\int_{\Lambda_\varepsilon(\rho)}\mu_\lambda\big(\Gamma_\varepsilon(\lambda)\cap[\rho]\big)\,dm(\lambda) \\
&\le C_F K^t \exp\big((\chi+\varepsilon)tn\big)\hat\mu([\rho])\int_{\Lambda_\varepsilon(\rho)}\mu_\lambda\big([\rho]\big)\,dm(\lambda) \\
&\le C_F C_2 K^t \exp\big((\chi+\varepsilon)tn\big)\hat\mu([\rho])\int_{\Lambda_\varepsilon(\rho)}\exp\big(-\mathrm{P}_\lambda^n(t)\big)||\big(\phi_\rho^\lambda\big)'||^t\,dm(\lambda) \\
&\le C_F C_2 K^t \exp\big((\chi+\varepsilon)tn\big)\hat\mu([\rho])\exp\big((-\mathcal{E}\mathrm{P}(t)+\varepsilon)n\big)\exp\big((-\chi+\varepsilon)tn\big)m(\Lambda_\varepsilon(\rho)) \\
&= C_F C_2 K^t \exp\big((-\mathcal{E}\mathrm{P}(t)+(1+2t)\varepsilon)n\big)\hat\mu([\rho])m(\Lambda_\varepsilon(\rho)) \\
&\le C_F C_2 K^t \exp\big(-(\mathcal{E}\mathrm{P}(t)-(1+2t)\varepsilon)n\big)\hat\mu([\rho]).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n=N}^\infty\sum_{|\rho|=n}R_\rho &\le C_F C_2 K^t\sum_{n=N}^\infty \exp\big(-(\mathcal{E}\mathrm{P}(t)-(1+2t)\varepsilon)n\big)\sum_{|\rho|=n}\hat\mu([\rho]) \\
&\le C_F C_2 K^t\sum_{n=N}^\infty \exp\big(-(\mathcal{E}\mathrm{P}(t)-(1+2t)\varepsilon)n\big)\hat\mu(F^{\mathbb N}) \\
&\le C_F C_2 K^t\sum_{n=N}^\infty \exp\big(-(\mathcal{E}\mathrm{P}(t)-(1+2t)\varepsilon)n\big) < +\infty,
\end{aligned}$$

where the last inequality sign was written due to (5.2). Since the set F is finite, it follows from this estimate, (5.15), and (5.12) that

$$R = \sum_{n=0}^{N-1}\sum_{|\rho|=n}R_\rho + \sum_{n=N}^\infty\sum_{|\rho|=n}R_\rho < +\infty. \tag{5.17}$$

Now the conclusion of the proof of Theorem 5.1 is straightforward. First, (5.10) and (5.17) imply that

$$\int_{J_{F,\lambda}\times J_{F,\lambda}}\frac{d\hat\mu_\lambda(x)\,d\hat\mu_\lambda(y)}{||x-y||^t} < +\infty$$

for all λ in some measurable set

$$\Lambda_{t,\varepsilon}' \subset \Lambda_\varepsilon \ \text{ with } \ m(\Lambda_\varepsilon\setminus\Lambda_{t,\varepsilon}') = 0. \tag{5.18}$$

The well-known potential-theoretic characterization of Hausdorff dimension (see [4], [6]) tells us now that

$$\mathrm{HD}(J_\lambda) \geq \mathrm{HD}(J_{F,\lambda}) \geq t \tag{5.19}$$

for every $\lambda \in \Lambda'_{t,\varepsilon}$. For every $n > 1/h$ let $\varepsilon + n > 0$ be chosen so that (5.2) and (5.8) hold with $t = h - \frac{1}{n}$. Define

$$\Lambda_* := \bigcap_{n>1/h} \bigcup_{k>1/\varepsilon_n} \Lambda_{h-\frac{1}{n},1/k}$$

By (5.3) and (5.18), $m(\Lambda_*) = 1$, and by (5.19), $\mathrm{HD}(J_\lambda) \geq h$ for all $\lambda \in \Lambda_*$. Since, also by Lemma 3.9, $\mathrm{HD}(J_\lambda) \leq h$ for m-a.e. $\lambda \in \Lambda$, the proof of Theorem 5.1 (a) is complete.

Let us now prove item (b) of Theorem 5.1. So, we assume that $b(\mathcal{S}) > d$. Fix then an arbitrary $t \in (d, b(\mathcal{S}))$. In exactly the same was as in the proof of item (b), we get a finite set $F \subset E$ such that (5.1) holds. The first part of the proof of (b) is now borrowed verbatim from the proof of item (a). We do it until the definition of the measures $\hat{\mu}_\lambda$ (included). For every $x \in \mathbb{R}^d$ define

$$\underline{D}(\hat{\mu}_\lambda, x) := \varliminf_{r\to 0} \frac{\hat{\mu}_\lambda(B(x,r))}{\ell_d(B(x,r))},$$

where ℓ_d denotes d-dimensional Lebesgue measure on $\mathbb{R}^d$. Now we define

$$\mathcal{I} := \int_{\Lambda_\varepsilon} \int_{\mathbb{R}^d} \underline{D}(\hat{\mu}_\lambda, x)\, d\hat{\mu}_\lambda(x) dm(\lambda). \tag{5.20}$$

Applying Fatou's Lemma yields

$$\begin{aligned}
\mathcal{I} &\leq \varliminf_{r\to 0} \int_{\Lambda_\varepsilon} \int_{\mathbb{R}^d} \frac{\hat{\mu}_\lambda(B(x,r))}{\ell_d(B(x,r))}\, d\hat{\mu}_\lambda(x) dm(\lambda) \\
&= \ell_d^{-1}(B(0,1)) \varliminf_{r\to 0} \int_{\Lambda_\varepsilon} \int_{\mathbb{R}^d} \frac{\hat{\mu}_\lambda(B(x,r))}{r^d}\, d\hat{\mu}_\lambda(x) dm(\lambda) \\
&= \ell_d^{-1}(B(0,1)) \varliminf_{r\to 0} r^{-d} \int_{\Lambda_\varepsilon} \int_{\mathbb{R}^d} \hat{\mu}_\lambda(B(x,r))\, d\hat{\mu}_\lambda(x) dm(\lambda)
\end{aligned}$$

Now,

$$\begin{aligned}
\int_{\mathbb{R}^d} \hat{\mu}_\lambda(B(x,r))\, d\hat{\mu}_\lambda(x) &= \int_{J_{F,\lambda}} \hat{\mu}_\lambda(B(x,r))\, d\hat{\mu}_\lambda(x) = \int_{J_{F,\lambda}} \hat{\mu} \circ \pi_\lambda^{-1}(B(x,r))\, d\hat{\mu} \circ \pi_\lambda^{-1}(x) \\
&= \int_{F^{\mathbb{N}}} \hat{\mu} \circ \pi_\lambda^{-1}\big(B(\pi_\lambda(\omega), r)\big)\, d\hat{\mu}(\omega) \\
&= \int_{F^{\mathbb{N}}} \int_{\mathbb{R}^d} 1\!\!1_{B(\pi_\lambda(\omega),r)}\, d\hat{\mu} \circ \pi_\lambda^{-1}\, d\hat{\mu}(\omega) \\
&= \int_{F^{\mathbb{N}}} \int_{F^{\mathbb{N}}} 1\!\!1_{B(\pi_\lambda(\omega),r)} \circ \pi_\lambda(\tau)\, d\hat{\mu}(\tau) d\hat{\mu}(\omega) \\
&= \int_{F^{\mathbb{N}}} \int_{F^{\mathbb{N}}} 1\!\!1_{\{\tau\in F^{\mathbb{N}}: ||\pi_\lambda(\tau)-\pi_\lambda(\omega)||<r\}}\, d\hat{\mu}(\tau) d\hat{\mu}(\omega).
\end{aligned}$$

Therefore, using Lemma 5.3 we get

(5.21)

$$\begin{aligned}
\mathcal{I} &\le \ell_d^{-1}(B(0,1)) \varliminf_{r\to 0} r^{-d}\int_{\Lambda_\varepsilon}\int_{F^{\mathbb{N}}}\int_{F^{\mathbb{N}}} 1\!\!1_{\{\tau\in F^{\mathbb{N}}:||\pi_\lambda(\tau)-\pi_\lambda(\omega)||<r\}}\, d\hat\mu(\tau)d\hat\mu(\omega)dm(\lambda) \\
&= \ell_d^{-1}(B(0,1)) \varliminf_{r\to 0} r^{-d}\int_{F^{\mathbb{N}}}\int_{F^{\mathbb{N}}}\int_{\Lambda_\varepsilon} 1\!\!1_{\{(\lambda,\omega,\tau)\in\Lambda_e\times F^{\mathbb{N}}\times F^{\mathbb{N}}:||\pi_\lambda(\tau)-\pi_\lambda(\omega)||<r\}}\, dm(\lambda)d\hat\mu(\omega)d\hat\mu(\tau) \\
&= \ell_d^{-1}(B(0,1)) \varliminf_{r\to 0} r^{-d}\int_{F^{\mathbb{N}}}\int_{F^{\mathbb{N}}}\int_{\Lambda_\varepsilon} 1\!\!1_{\{\lambda\in\Lambda_e:||\pi_\lambda(\tau)-\pi_\lambda(\omega)||<r\}}\, dm(\lambda)d\hat\mu(\omega)d\hat\mu(\tau) \\
&= \ell_d^{-1}(B(0,1)) \varliminf_{r\to 0} r^{-d}\int_{F^{\mathbb{N}}}\int_{F^{\mathbb{N}}} m\big(\{\lambda\in\Lambda_e : ||\pi_\lambda(\tau)-\pi_\lambda(\omega)||<r\}\big)\, d\hat\mu(\omega)d\hat\mu(\tau) \\
&= \ell_d^{-1}(B(0,1)) \varliminf_{r\to 0} r^{-d}\sum_{n=0}\sum_{\rho\in F^n}\int_{A(\rho)} m\big(\{\lambda\in\Lambda_e : ||\pi_\lambda(\tau)-\pi_\lambda(\omega)||<r\}\big)\, d\hat\mu(\omega)d\hat\mu(\tau).
\end{aligned}$$

Recalling the definition of M_ρ from (5.14) and making use of the Random Transversality Condition of Finite Type (Definition 4.1) as well as θ-invariance of measure m, we can now estimate, for $\omega,\tau\in A(\rho)$ with $|\rho|=n$, as follows:

$$\begin{aligned}
m\big(\{\lambda\in\Lambda_e :&||\pi_\lambda(\tau)-\pi_\lambda(\omega)||<r\}\big) = \\
&= m\big(\{\lambda\in\Lambda_e : \big\|\phi_\rho^\lambda\big(\pi_{\theta^n(\lambda)}(\sigma^n(\tau))\big)-\phi_\rho^\lambda\big(\pi_{\theta^n(\lambda)}(\sigma^n(\omega))\big)\big\|<r\}\big) \\
&\le m\big(\{\lambda\in\Lambda : \big\|\pi_{\theta^n(\lambda)}(\sigma^n(\tau))-\pi_{\theta^n(\lambda)}(\sigma^n(\omega))\big)\big\|<rM_\rho^{-1}\}\big) \\
&= m\big(\{\lambda\in\Lambda : \big\|\pi_\lambda(\sigma^n(\tau))-\pi_\lambda(\sigma^n(\omega))\big)\big\|<rM_\rho^{-1}\}\big) \\
&\le C_F M_\rho^{-d} r^d.
\end{aligned} \tag{5.22}$$

Therefore,

$$\begin{aligned}
\int_{A(\rho)} m\big(\{\lambda\in\Lambda_e :&||\pi_\lambda(\tau)-\pi_\lambda(\omega)||<r\}\big)\, d\hat\mu(\omega)d\hat\mu(\tau) \le \\
&\le C_F M_\rho^{-d} r^d \hat\mu\otimes\hat\mu(A_\rho) \\
&\le C_F M_\rho^{-d} r^d \hat\mu\otimes\hat\mu([\rho]\times[\rho]) \\
&= C_F M_\rho^{-d} r^d \hat\mu^2([\rho]).
\end{aligned} \tag{5.23}$$

But assuming that $|\rho| = n \geq N$, recalling the definition of $\Lambda_\varepsilon(\rho)$ from (5.13), and making use of (5.16), (c3) and (b), we get

$$\begin{aligned}
\hat{\mu}([\rho]) &= \mu|_{\Gamma_\varepsilon}\big(p_{E^{\mathbb{N}}}^{-1}([\rho])\big) = \mu\big(\Gamma_\varepsilon \cap p_{E^{\mathbb{N}}}^{-1}([\rho])\big) = \int_{\Lambda_\varepsilon} \mu_\lambda(\Gamma_e(\lambda) \cap [\rho])\, dm(\lambda) \\
&= \int_{\Lambda_\varepsilon(\rho)} \mu_\lambda(\Gamma_e(\lambda) \cap [\rho])\, dm(\lambda) \\
&\leq \int_{\Lambda_\varepsilon(\rho)} \mu_\lambda([\rho])\, dm(\lambda) \\
&\preceq \int_{\Lambda_\varepsilon(\rho)} \exp\big(-\mathrm{P}_\lambda^n(t)\big)||(\phi_\rho^\lambda)'||^t\, dm(\lambda) \\
&\leq \exp\big((-\mathcal{E}\mathrm{P}(t)+\varepsilon)n\big)\exp\big((-\chi+\varepsilon)tn\big)m\big(\Lambda_\varepsilon(\rho)\big) \\
&\preceq \exp\big((-\mathcal{E}\mathrm{P}(t)+\chi t)n + (1+t)\varepsilon n\big),
\end{aligned}$$

and moreover $\hat{\mu}([\rho]) = 0$ if $\Lambda_\varepsilon(\rho) = \emptyset$. Inserting this and (5.16) to (5.23), we get

$$\begin{aligned}
\int_{A(\rho)} & m\big(\{\lambda \in \Lambda_e : ||\pi_\lambda(\tau) - \pi_\lambda(\omega)|| < r\}\big) \leq \\
&\leq C_F r^d \mu([\rho]) \exp\big((\chi+\varepsilon)dn\big)\exp\big((-\mathcal{E}\mathrm{P}(t)+\chi t)n + (1+t)\varepsilon n\big) \\
&= C_F r^d \exp\big((-\mathcal{E}\mathrm{P}(t) - (1+t+d)\varepsilon n)\big)\exp\big((d-t)\chi n\big)\mu([\rho]) \\
&\leq C_F r^d \exp\big((-\mathcal{E}\mathrm{P}(t) - (1+2t)\varepsilon n)\big)\mu([\rho]).
\end{aligned}$$

Inserting in turn this and (5.23) (for $n < N$) to (5.21), we get, with the use of (5.2), that

$$\begin{aligned}
\mathcal{I} &\leq \overline{\lim_{r\to 0}} \sum_{n=0}^{N-1} C_F \hat{\mu}^2([\rho]) + \underline{\lim_{r\to 0}} \sum_{n=N}^{\infty}\sum_{\rho\in F^n} C_F \exp\big((-\mathcal{E}\mathrm{P}(t) - (1+2t)\varepsilon n)\big)\mu([\rho]) \\
&\preceq C_F N + C_F \sum_{n=N}^{\infty} \exp\big((-\mathcal{E}\mathrm{P}(t) - (1+2t)\varepsilon n)\big) < +\infty.
\end{aligned} \tag{5.24}$$

Now, the conclusion of the proof of item (b) of Theorem 5.1 is straightforward. First, (5.20) (5.24) imply that

$$\int_{J_{F,\lambda}} \underline{D}(\hat{\mu}_\lambda, x)\, d\hat{\mu}_\lambda(x) < +\infty$$

for all λ in some measurable set $\Lambda'_\varepsilon \subset \Lambda_\varepsilon$ with $m(\Lambda_\varepsilon \setminus \Lambda'_\varepsilon) = 0$. In view of Theorem 2.12 (3) in [6] this tells us that $\hat{\mu}_\lambda$ is absolutely continuous with respect to d-dimensional Lebesgue measure ℓ_d on $\mathbb{R}^d$ for all $\lambda \in \Lambda'_\varepsilon$. Since $\hat{\mu}_\lambda$ is a non-zero measure on $J_{F,\lambda}$ for all $\lambda \in \Lambda_\varepsilon$, we hence conclude that $\ell_d(J_\lambda) > 0$ for all $\lambda \in \Lambda'_\varepsilon$. Therefore,

$$\ell_d(J_\lambda) > 0$$

for all $\lambda \in \Lambda' := \bigcup_{n=k}^{\infty} \Lambda'_{1/n}$, where $k \geq 1$ is so large that $\varepsilon = 1/k$ satisfies (5.2) and (5.8). Since $m\big(\Lambda'_{1/n}\big) > 1 - \frac{1}{n}$, we have that $m(\Lambda') = 1$, and the proof of Theorem 5.1 (b) is complete. $\square$

As an immediate consequence of this theorem along with Remark 4.4, we get the following corollary, which is also new, in the case of a finite alphabet.

Corollary 5.4. *Let $X \subset \mathbb{R}^d$ be a compact domain. Suppose that $\mathcal{S} = \{\phi_e^\lambda : X \to X : e \in E,\ \lambda \in \Lambda\}$ is a conformal random iterated function system with finite alphabet E and a base map $\theta : \Lambda \to \Lambda$ preserving a probability measure m. If the system $\mathcal{S}$ satisfies the Random Transversality Condition then*

(a)
$$\mathrm{HD}(J_\lambda) = \min\{b(\mathcal{S}), d\}$$
for m-a.e. $\lambda \in \Lambda$.

(b) *If $b(\mathcal{S}) > d$, then $\ell_d(J_\lambda) > 0$ for m-a.e. $\lambda \in \Lambda$. In particular* $\mathrm{HD}(J_\lambda) = d$.

Remark 5.5. As already mentioned in this introduction, the iterated function systems, deterministic and random alike, consisting of similarities form a subclass respectively of deterministic and random conformal iterated function systems. In particular, the above theorems and corollaries proved and stated in this section, apply to the case of similarities.

Remark 5.6. We would like to remark that the theorems and corollaries, proved and stated in this section, would remain true if instead of dealing with iterated function systems, we would deal with graph directed Markov systems (see [5] modeled on a finitely irreducible subshift of finite type rather than the full shift (i. e. the case of IFSs). The proofs would require only cosmetic, mainly notational, changes.

6. Examples

In this section we provide two large classes of examples of sub-random conformal IFSs satisfying the Random Transversality Condition of Finite Type. As explained in Remark 6.7 these systems easily give rise to abundance of random conformal IFSs satisfying the Random Transversality Condition of Finite Type.

Let E be a countable set and let $\mathcal{S}$ be a deterministic conformal IFS acting on some compact domain $X \subset \mathbb{R}^d$ and a bounded convex open set $V \subset \mathbb{R}^d$ containing X. Let

$$s = s(\mathcal{S}) := \sup\{||\phi_e'|| : e \in E\} < 1$$

be the corresponding contraction parameter of the system $\mathcal{S}$. Fix

$$0 < \eta < \mathrm{dist}(X, \mathbb{R}^d \setminus V).$$

Then

$$\phi_e(\overline{B}(X, \eta) \subset \overline{B}(X, s\eta)$$

for all $e \in E$. Let F be an arbitrary subset of E. Fix $1 \le p \le +\infty$ and let $\ell_p(\mathbb{R}^d)^F$ be the corresponding Banach space, i.e.

$$\ell_p(\mathbb{R}^d)^F := \left\{ x \in (\mathbb{R}^d)^F : ||x||_p := \left(\sum_{e \in F} ||x_e||^p \right)^{1/p} < +\infty \right\}$$

if $p < +\infty$ and

$$\ell_\infty(\mathbb{R}^d)^F := \big\{ x \in (\mathbb{R}^d)^F : ||x||_\infty := \sup\{||x_e|| : e \in F\big\} < +\infty.$$

Fix a set

$$\Lambda \subset B_{\mathbb{R}^d}(0,R)^F \cap \ell_p(\mathbb{R}^d)^F \tag{6.1}$$

with some $R \in (0,(1-s)\eta)$. For every $\lambda \in \Lambda$ and every $e \in E$ define the map $\phi_e^\lambda : \overline{B}(X,\eta) \to \mathbb{R}^d$ by the formula

$$\phi_e^\lambda(x) = \begin{cases} \phi_e(x) + \lambda_e & \text{if } e \in F \\ \phi_e(x) & \text{if } e \in E \setminus F. \end{cases}$$

Then

$$\phi_e^\lambda(\overline{B}(X,\eta) \subset \overline{B}(X,\eta)$$

and

$$\mathcal{S}^{(F,\eta)} := \big\{ \phi_e^\lambda : \overline{B}(X,\eta) \to \overline{B}(X,\eta) : e \in E, \lambda \in \Lambda \big\} \tag{6.2}$$

is a a pre-random conformal IFS. With having in addition any measurable map $\theta : \Lambda \to \Lambda$ it becomes a sub-random conformal IFS. We shall prove the following.

Proposition 6.1. *Fix $p \in [1,+\infty]$. Suppose that Λ is an open subset of $B_{\mathbb{R}^d}(0.R)^F \cap \ell_p(\mathbb{R}^d)^F$ and that $\theta : \Lambda \to \Lambda$ is a C^1-map. If $||\theta'|| < +\infty$ and $s||\theta'|| < 1$, then for every $\omega \in E^{\mathbb{N}}$, the map*

$$\Lambda \ni \lambda \mapsto \pi_\lambda(\omega) = \pi_{\mathcal{S}^{(F,\eta)}\lambda}(\omega) \in \mathbb{R}^d$$

is differentiable, in fact C^1, and

$$||D|_{(\cdot)}\pi(\omega)||_\infty := \sup\{||D|_\lambda \pi(\omega)||_\infty : \lambda \in \Lambda\} \le (1 - s||\theta'||_\infty)^{-1}.$$

Proof. Fix $z \in X$ and put

$$\pi_\lambda^n(\omega) = \phi_{\omega|_n}^\lambda(z)$$

for all $\omega \in E^{\mathbb{N}}$, all $\lambda \in \Lambda$, and all $n \ge 0$. Then $\big(\Lambda \ni \lambda \mapsto \pi_\lambda^n(\omega)\big)_0^\infty$ is a sequence of C^1 maps converging uniformly (exponentially fast) to $\pi_\lambda(\omega)$. Thus the function $\Lambda \ni \lambda \mapsto \pi_\lambda(\omega) \in \overline{B}(X,\eta)$ is continuous. Our goal is to show that the sequence $\big(D\pi_\lambda^n(\omega)\big)_0^\infty$ of derivatives of the elements of the sequence $\big(\pi_\lambda^n(\omega)\big)_0^\infty$ also converges uniformly. We declare that $\lambda_e = 0$ for all $e \in E \setminus F$, and we can then uniformly write

$$\phi_e^\lambda(x) = \phi_e(x) + \lambda_e. \tag{6.3}$$

Since

$$\pi_\lambda^{n+1}(\omega) = \phi_{\omega_1}^\lambda(\pi_{\theta(\lambda)}^n(\sigma(\omega))) = \lambda_{\omega_1} + \phi_{\omega_1}(\pi_{\theta(\lambda)}^n(\sigma(\omega)),$$

we get by the Chain rule that

$$D|_\lambda\pi^{n+1}(\omega)=\Delta(\omega_1)+\phi'_{\omega_1}\big(\pi^n_{\theta(\lambda)}(\sigma(\omega))\big)D|_{\theta(\lambda)}\pi^n(\sigma(\omega))\theta'(\lambda), \tag{6.4}$$

where $\Delta(\omega_1)$ is represented in the natural coordinate basis by the matrix (function from $(\mathbb{R}^d)^F\times\mathbb{R}^d$ to $\mathbb{R}^d$) consisting of 0 matrices $d\times d$ for every element $e\in F\setminus\{\omega_1\}$ and the identity matrix $d\times d$ corresponding to ω_1 if the latter belongs to F. In any case

$$||\Delta(\omega_1)||\le 1. \tag{6.5}$$

We shall prove the following.

Claim 1: We have

$$\sup\big\{||D|_\lambda\pi^n(\omega)||:\omega\in E^{\mathbb{N}},\lambda\in\Lambda,\, n\ge 0\big\}\le(1-s||\theta'||_\infty)^{-1}.$$

Proof. For $n=0$ this is immediate as the supremum of zeros is zero. Supposing that the claim is true for some $n\ge 0$, we get from (6.4) and (6.5) that

$$\begin{aligned}||D|_\lambda\pi^{n+1}(\omega)||&\le 1+||\phi'_{\omega_1}||\cdot||D|_{\theta(\lambda)}\pi(\sigma(\omega))||\cdot||\theta'(\lambda)||\le 1+s||\theta'||_\infty(1-s||\theta'||_\infty)^{-1}\\ &=\frac{1-s||\theta'||_\infty+s||\theta'||_\infty}{(1-s||\theta'||_\infty)}\\ &=(1-s||\theta'||_\infty)^{-1}.\end{aligned}$$

The Claim 1 is proved. □

Now we will show that the sequence $\big(D|_\lambda\pi^n(\omega)\big)_{n=0}^\infty$ is uniformly Cauchy with respect to $\omega\in E^{\mathbb{N}}$ and $\lambda\in\Lambda$. Fix two integers $n\ge k\ge 0$. By virtue of (6.4) again, by the Bounded Distortion Property (Property 3.2) for the system $\mathcal{S}$, and also by Claim 1, we get

$$\begin{aligned}&\big\|D|_\lambda\pi^{n+1}(\omega)-D|_\lambda\pi^{k+1}(\omega)\big\|=\\ &\quad=\big\|\phi'_{\omega_1}\big(\pi_{\theta^n(\lambda)}(\sigma(\omega))\big)D|_{\theta(\lambda)}\pi^n(\sigma(\omega))\theta'(\lambda)-\phi'_{\omega_1}\big(\pi_{\theta^k(\lambda)}(\sigma(\omega))\big)D|_{\theta(\lambda)}\pi^k(\sigma(\omega))\theta'(\lambda)\big\|\\ &\quad\le\big\|\phi'_{\omega_1}\big(\pi_{\theta^n(\lambda)}(\sigma(\omega))\big)-\phi'_{\omega_1}\big(\pi_{\theta^k(\lambda)}(\sigma(\omega))\big)\big\|\cdot\big\|D|_{\theta(\lambda)}\pi^n(\sigma(\omega))\big\|\cdot||\theta'(\lambda)||+\\ &\qquad\quad+\big\|\phi'_{\omega_1}\big(\pi_{\theta^k(\lambda)}(\sigma(\omega))\big)\big\|\cdot\big\|D|_{\theta(\lambda)}\pi^n(\sigma(\omega))-D|_{\theta(\lambda)}\pi^k(\sigma(\omega))\big\|\cdot||\theta'(\lambda)||\\ &\quad\le H_{\mathcal{S}}\big\|\pi_{\theta^n(\lambda)}(\sigma(\omega))-\pi_{\theta^k(\lambda)}(\sigma(\omega))\big\|^\alpha||\theta'||_\infty(1-s||\theta'||_\infty)^{-1}+\\ &\qquad\quad+s||\theta'||_\infty\cdot\big\|D|_{\theta(\lambda)}\pi^n(\sigma(\omega))-D|_{\theta(\lambda)}\pi^k(\sigma(\omega))\big\|\\ &\quad\le H_{\mathcal{S}}||\theta'||_\infty(1-s||\theta'||_\infty)^{-1}s^{\alpha k}+s||\theta'||_\infty\cdot\big\|D|_{\theta(\lambda)}\pi^n(\sigma(\omega))-D|_{\theta(\lambda)}\pi^k(\sigma(\omega))\big\|,\end{aligned} \tag{6.6}$$

where the constant $H_{\mathcal{S}}$ comes from the Bounded Distortion Property for the system $\mathcal{S}$. Fix now $\varepsilon>0$ and fix then an integer $l\ge 0$ so large that

$$H_{\mathcal{S}}||\theta'||_\infty(1-s||\theta'||_\infty)^{-1}s^{\alpha l}<\frac{1-s||\theta'||_\infty}{2}\frac{\varepsilon}{2}.$$

Assume $n\ge k\ge l$. Then

$$H_{\mathcal{S}}||\theta'||_\infty(1-s||\theta'||_\infty)^{-1}s^{\alpha k}<\frac{1-s||\theta'||_\infty}{2}\frac{\varepsilon}{2}.$$

Denote$D|_\lambda \pi^j(\tau)$ by $a_j(\tau,\lambda)$. Hence, if $||a_n(\sigma(\omega),\theta(\lambda))-a_k(\sigma(\omega),\theta(\lambda))||<\varepsilon/2$, then it follows from (6.6) that

$$||a_{n+1}(\omega,\lambda)-a_{k+1}(\omega,\lambda)||<\frac{1-s||\theta'||_\infty}{2}\frac{\varepsilon}{2}+s||\theta'||_\infty\frac{\varepsilon}{2}<\varepsilon. \tag{6.7}$$

So, assume that

$$||a_n(\sigma(\omega),\theta(\lambda))-a_k(\sigma(\omega),\theta(\lambda))||\geq\varepsilon/2.$$

Then

$$H_{\mathcal{S}}||\theta'||_\infty(1-s||\theta'||_\infty)^{-1}s^{\alpha k}<\frac{1-s||\theta'||_\infty}{2}||a_n(\sigma(\omega),\theta(\lambda))-a_k(\sigma(\omega),\theta(\lambda))||,$$

and therefore, we get from (6.6) that

$$||a_{n+1}(\sigma(\omega),\lambda)-a_{k+1}(\omega,\lambda)||\leq\frac{1+s||\theta'||_\infty}{2}||a_n(\sigma(\omega),\theta(\lambda))-a_k(\sigma(\omega),\theta(\lambda))||.$$

Combining this with (6.7) and putting

$$\kappa:=\frac{1+s||\theta'||_\infty}{2},$$

we get that

$$||a_{n+1}(\sigma(\omega),\lambda)-a_{k+1}(\omega,\lambda)||\leq\max\{\varepsilon,\kappa||a_n(\sigma(\omega),\theta(\lambda))-a_k(\sigma(\omega),\theta(\lambda))||\}.$$

For any $j\geq i\geq 0$ put

$$A_{j,i}:=\sup\left\{||a_j(\tau,\gamma)-a_i(\tau,\gamma)||:\tau\in E^{\mathbb{N}},\ \gamma\in\Lambda\right\}.$$

The last formula implies then for all $n\geq k\geq l$ that

$$||a_{n+1}(\sigma(\omega),\lambda)-a_{k+1}(\omega,\lambda)||\leq\max\{\varepsilon,\kappa A_{n,k}\}.$$

Hence, taking the suprema:

$$A_{n+1,k+1}\leq\max\{\varepsilon,\kappa A_{n,k}\}. \tag{6.8}$$

We shall show by induction that given $n\geq k\geq l$ we have

$$A_{n+j,k+j}\leq\max\{\varepsilon,\kappa^j A_{n,k}\} \tag{6.9}$$

for all $j\geq 0$. Indeed, for $j=0$ this is true as $\kappa^0=1$. So, suppose that (6.9) holds for some $j\geq 0$. Using also (6.8) we then get

$$A_{n+j+1,k+j+1}\leq\max\left\{\varepsilon,\kappa A_{n+j,k+j}\right\}\leq\max\{\varepsilon,\max\{\kappa\varepsilon,\kappa^{j+1}A_{n,k}\}\}=\max\{\varepsilon,\kappa^{j+1}A_{n,k}\}.$$

Formula (6.9) is thus proved. This formula readily implies that the sequence $\left(D|_\lambda\pi^n(\omega)\right)_{n=0}^\infty$ is uniformly Cauchy. Thus, the sequence $\left((\lambda,\omega)\mapsto\pi^n_\lambda(\omega)\right)_{n=0}^\infty$ converges uniformly (with respect to $\omega\in E^{\mathbb{N}}$ and $\lambda\in\Lambda$) to the C^1-function $\Lambda\ni\lambda\mapsto\pi_\lambda(\omega)$ for all $\omega\in E^{\mathbb{N}}$. Because of Claim 1, $||D|_\lambda\pi(\omega)||_\infty\leq(1-s||\theta'||_\infty)^{-1}$. The proof of Proposition 6.1 is complete. □

Keeping F an arbitrary subset of E we say that the map $\theta : \Lambda \to \Lambda$ is of finite character with respect to F if $p = +\infty$,

$$\Lambda = \prod_{e \in F} U_e \tag{6.10}$$

where U_e, $e \in E$, are open subsets of $B_{\mathbb{R}^d}(0, R)$ with smooth boundary,

$$F = \bigcup_{j=1}^{\infty} F_j$$

is a disjoint union of finite sets (possibly empty or consisting of just one element if E is finite) and

$$\theta = \prod_{j=1}^{\infty} \theta_j, \tag{6.11}$$

where, for each $j \geq 1$, $\theta_j : \prod_{e \in F_j} U_e \to \prod_{e \in F_j} U_e$ is a C^1-diffeomorphism. Now we can prove the following.

Theorem 6.2. *Let E be a countable set and let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be an autonomous (deterministic) system acting on some compact domain $X \subset \mathbb{R}$ and a bounded open interval $V \subset \mathbb{R}$. Let E_* be an arbitrary subset of E. Assume that*

(a) *$\phi_a(X) \cap \phi_b(X) = \emptyset$ for all $a, b \in E \setminus E_*$ with $a \neq b$,*
(b) *$s||\theta'||_\infty < 1$ and $s||\theta'||_\infty(1 - s||\theta'||_\infty)^{-1} < 1/2$.*

Let Λ be given by (6.1) *with $p = \infty$ and let $\theta : \Lambda \to \Lambda$ be a C^1-diffeomorphism of finite character with respect to E_*. Then the sub-random conformal IFS $\mathcal{S}^{(E_*, \eta)}$ defined by* (6.2) *satisfies the Random Transversality Condition of Finite Type with respect to the product measure $\otimes_{e \in E_*} \ell_e$ on $\prod_{e \in E_*} U_e$, where ℓ_e is the normalized d-dimensional Lebesgue measure on U_e.*

Proof. Fix $k \geq 1$. Let $F_k := \bigcup_{j=1}^{k} E_{*,j}$ and let F be an arbitrary finite set such that

$$F_k \subset F \subset F_k \cup (E \setminus E_*). \tag{6.12}$$

Since the set $F_k \setminus E_*$ is finite and all sets $\phi_E(X)$, $e \in E \setminus E_*$ are compact, it follows from (a) that if $\eta \in (0, \mathrm{dist}(X, V^c))$ is small enough, then

$$\inf \left\{ \mathrm{dist}\big(\phi_a^\lambda(X), \phi_b^\lambda(X)\big) : \lambda \in \Lambda \ \text{ and } \ a, b \in F \setminus E_* \ \text{ with } \ a \neq b \right\} > 0.$$

Fix $\omega, \tau \in F^{\mathbb{N}}$ with $\omega_1 \neq \tau_1$. Our first goal is to verfy the assumptions of Lemma 4.8 for the function

$$\Lambda \ni \lambda \mapsto f(\lambda) := \pi_\lambda(\omega) - \pi_\lambda(\tau).$$

Indeed, if both ω_1 and τ_1 are in $F \setminus E_*$, then the formula (4.1) in Lemma 4.8 is vacuously satisfied. So, we may assume that at least one of the two elements ω_1 or τ_1 is in F_k. Assume without loss of generality that $\omega_1 \in F_k$. Then

$$f(\lambda) = \pi_\lambda(\omega) - \pi_\lambda(\tau) = \lambda_{\omega_1} - \lambda_{\omega_2} + \phi_{\omega_1}(\pi_{\theta(\lambda)}(\sigma(\omega))) - \phi_{\tau_1}(\pi_{\theta(\lambda)}(\sigma(\tau))).$$

We know from Proposition 6.1 that all the functions $\pi_\gamma(\beta)$ are differentiable. Since $\tau_1 \neq \omega_1$, we thus get

$$\frac{\partial f}{\partial \lambda_{\omega_1}} = 1 + \phi'_{\omega_1}(\pi_{\theta(\lambda)}(\sigma(\omega)))D|_{\theta(\lambda)}\pi(\sigma(\omega))\frac{\partial \theta}{\partial \lambda_{\omega_1}} - \phi'_{\tau_1}(\pi_{\theta(\lambda)}(\sigma(\tau)))D|_{\theta(\lambda)}\pi(\sigma(\tau))\frac{\partial \theta}{\partial \lambda_{\omega_1}}.$$

Hence, by virtue of the last assertion of Proposition 6.1 and of assumption (b). we get

$$\begin{aligned}\frac{\partial f}{\partial \lambda_{\omega_1}} &\geq 1 - \left(||\phi'_{\omega_1}|| \cdot \left\|D|_{\theta(\lambda)}\pi(\sigma(\omega))\right\| + ||\phi'_{\tau_1}|| \cdot \left\|D|_{\theta(\lambda)}\pi(\sigma(\tau))\right|\right) \left\|\frac{\partial \theta}{\partial \lambda_{\omega_1}}\right\|_\infty \\ &\geq 1 - 2s||\theta'||_\infty(1 - s||\theta'||_\infty)^{-1} > 0.\end{aligned}$$

So, the implication (4.1) of Lemma 4.8 holds and the hypotheses of this lemma have been verified. Now notice that since both $\omega, \tau \in F_k \cup (E \setminus E_*)$, and since $\theta : \Lambda \to \Lambda$ is of finite character with respect to E_*, all the maps $\pi_{\theta^i(\lambda)}(\sigma^i(\omega))$ and $\pi_{\theta^i(\lambda)}(\sigma^i(\tau))$ depend only on $(\lambda_e)_{e \in F_k}$ and on the map

$$\prod_{j=1}^k \theta_j : \prod_{j=1}^k \prod_{e \in E_{*,j}} U_e \longrightarrow \prod_{j=1}^k \prod_{e \in E_{*,j}} U_e,$$

the coordinates of λ beyond F_k and the maps θ_j, $j \geq k+1$, do not matter. Therefore, (by Lemma 4.8) we get

$$\begin{aligned}\bigotimes_{e \in E_*} \ell_e\big(\{\lambda \in \Lambda : |\pi_\lambda(\omega) - \pi_\lambda(\tau)| \leq r\}\big) &= \\ &= \bigotimes_{e \in F_k} \ell_e\left(\left\{\lambda \in \prod_{e \in F_k} U_e : |\pi_\lambda(\omega) - \pi_\lambda(\tau)| \leq r\right\}\right) \\ &\leq C_F r^{1-u} r^u = C_F r,\end{aligned}$$

where $u = d\sum_{j=1}^k \#E_{*,j}$. The proof is complete. □

Remark 6.3. We now would like to indicate how easy it is to create random conformal IFSs satisfying all the hypotheses of Theorem 6.2. Indeed, the only issue requiring perhaps to be addressed, is condition (b); (a) can be always achieved easily by taking $E_* = E$ for example, and C^1 maps θ of finite character abound and are independent of actual IFSs. And condition (b) can be always achieved by taking sufficiently large iterate $\mathcal{S}^k = \{\phi_\omega : \omega \in E^k\}$ of an original system $\mathcal{S} = \{\phi_e : e \in E\}$, and then to use $\mathcal{S}^k$ to form a random system $\mathcal{S}^{k(F,\eta)}$ as given by formula (6.2).

Now we want to describe one more class of examples of sub-random conformal iterated function systems satisfying the Random transversality Condition of Finite Type. Again, let E be a countable set and let $\mathcal{S} = \{\phi_e : e \in E\}$ be an autonomous (deterministic) conformal IFS acting on some compact domain $X \subset \mathbb{R}^d$ and a bounded convex open set $V \subset \mathbb{R}^d$ containing X. Let $0 < \eta < \mathrm{dist}(X, \mathbb{R}^d \setminus V)$. Then

$$\phi_e(\overline{B}(X, \eta)) \subset B(X, s\eta)$$

for all $e \in E$. Let F be an arbitrary subset of E. Fix $1 \le p \le +\infty$ and with some $R \in \big(0, \min\{\frac{1-s}{s}\eta, \text{dist}(X, \mathbb{R}^d \setminus V) - \eta\}\big)$ let

$$\Lambda \subset B_{\mathbb{R}^d}(0, R)^F \cap \ell_l((\mathbb{R}^d)^F))$$

be a C^1-submanifold of $\ell_l((\mathbb{R}^d)^F))$. For every $e \in E \setminus F$ set $\lambda_e = 0$. For every $e \in E$ and every $\lambda \in \Lambda$ define the map $\phi_e^\lambda : \overline{B}(X, \eta) \to \overline{B}(X, \eta)$ by the formula

$$\phi_e^\lambda(x) = \phi_e(x + \lambda_e).$$

Then

$$\phi_e^\lambda(\overline{B}(X, \eta)) \subset \overline{B}(X, (R + \eta)s)$$

and $(R + \eta)s < \eta$. Define

$$\mathcal{S}^{(F,\eta,R)} = \big\{\phi_e^\lambda : \overline{B}(X, \eta)) \to \overline{B}(X, (R + \eta) : e \in E, \lambda \in \Lambda\big\}. \tag{6.13}$$

With any map $\theta : \Lambda \to \Lambda$, the pre-random conformal system $\mathcal{S}^{(F,\eta,R)}$ becomes then a sub-random conformal IFS. We shall prove the following.

Proposition 6.4. *If $\theta : \Lambda \to \Lambda$ is a C^1-map, $||\theta'||_\infty < +\infty$, and moreover $s||\theta'||_\infty < 1$, then the map*

$$\Lambda \ni \lambda \mapsto \pi_\lambda(\omega) = \pi_{\mathcal{S}^{(F,R,\eta)}\lambda}(\omega) \in \mathbb{R}^d$$

is differentiable, in fact C^1, and

$$||D|_{(\cdot)}\pi(\omega)||_\infty := \sup\{||D|_\lambda \pi(\omega)||_\infty : \lambda \in \Lambda\} \le (1 - s||\theta'||_\infty)^{-1}.$$

Proof. Fix $z \in X$ and put

$$\pi_\lambda^n(\omega) = \phi_{\omega|_n}^\lambda(z)$$

for all $\omega \in E^{\mathbb{N}}$, all $\lambda \in \Lambda$, and all $n \ge 0$. Then $\big(\Lambda \ni \lambda \mapsto \pi_\lambda^n(\omega)\big)_0^\infty$ is a sequence of C^1 maps converging uniformly (exponentially fast) to $\pi_\lambda(\omega)$. Thus the function $\Lambda \ni \lambda \mapsto \pi_\lambda(\omega) \in \overline{B}(X, \eta)$ is continuous. Our goal is to show that the sequence $\big(D\pi_\lambda^n(\omega)\big)_0^\infty$ of derivatives of the elements of the sequence $\big(\pi_\lambda^n(\omega)\big)_0^\infty$ also converges uniformly. Since

$$\pi_\lambda^{n+1}(\omega) = \phi_{\omega_1}^\lambda(\pi_{\theta(\lambda)}^n(\sigma(\omega))) = \phi_{\omega_1}(\lambda_{\omega_1} + \pi_{\theta(\lambda)}^n(\sigma(\omega))),$$

we get by the Chain Rule that

$$D|_\lambda \pi^{n+1}(\omega) = \phi'_{\omega_1}(\lambda_{\omega_1} + \pi_{\theta(\lambda)}^n(\sigma(\omega)))\big(\Delta(\omega_1) + D|_{\theta(\lambda)}\pi^n(\sigma(\omega))\theta'(\lambda)\big), \tag{6.14}$$

where $\Delta(\omega_1)$ has the same meaning as in the formula (6.4); in particular formula (6.5) holds for it. We shall prove the following.

Claim 1: We have

$$\sup\big\{||D|_\lambda \pi^n(\omega)|| : \omega \in E^{\mathbb{N}}, \lambda \in \Lambda, \, n \ge 0\big\} \le s(1 - s||\theta'||_\infty)^{-1}.$$

Proof. For $n = 0$ this is immediate as the supremum of zeros is zero. Supposing that the claim is true for some $n \geq 0$, we get from (6.14) and (6.5) that

$$\begin{aligned}\big\|D|_\lambda \pi^{n+1}(\omega)\big\| &\leq ||\phi'_{\omega_1}||\big(1 + \big\|D|_{\theta(\lambda)}\pi^n(\sigma(\omega))\big\| \cdot ||\theta'(\lambda)||\big)\\ &\leq s\big(1 + s||\theta'||_\infty(1 - s||\theta'||_\infty)^{-1}\big)\\ &= s\frac{1 - s||\theta'||_\infty + s||\theta'||}{1 - s||\theta'||_\infty} = s(1 - s||\theta'||_\infty)^{-1}.\end{aligned}$$

Claim 1 is proved. □

Now we will show that the sequence $\big(D|_\lambda \pi^n(\omega)\big)_{n=0}^\infty$ is uniformly Cauchy with respect to $\omega \in E^{\mathbb{N}}$ and $\lambda \in \Lambda$. Fix two integers $n \geq k \geq 0$. By virtue of (6.14) (6.5) again, by the Bounded Distortion Property for the system $\mathcal{S}$, and also by Claim 1, we get

$$\begin{aligned}&\big\|D|_\lambda \pi^{n+1}(\omega) - D|_\lambda \pi^{k+1}(\omega)\big\| =\\ &\quad= \big\|\phi'_{\omega_1}(\lambda_{\omega_1} + \pi^n_{\theta(\lambda)}(\sigma(\omega)))\big(\Delta(\omega_1) + D|_{\theta(\lambda)}\pi^n(\sigma(\omega))\theta'(\lambda)\big) -\\ &\qquad\quad - \phi'_{\omega_1}(\lambda_{\omega_1} + \pi^k_{\theta(\lambda)}(\sigma(\omega)))\big(\Delta(\omega_1) + D|_{\theta(\lambda)}\pi^k(\sigma(\omega))\theta'(\lambda)\big)\big\|\\ &\quad\leq \big\|\big(\phi'_{\omega_1}(\lambda_{\omega_1} + \pi^n_{\theta(\lambda)}(\sigma(\omega))) - \phi'_{\omega_1}(\lambda_{\omega_1} + \pi^k_{\theta(\lambda)}(\sigma(\omega)))\big)\Delta(\omega_1)\big\| +\\ &\qquad\quad + \big\|\phi'_{\omega_1}(\lambda_{\omega_1} + \pi^n_{\theta(\lambda)}(\sigma(\omega)))\big(D|_{\theta(\lambda)}\pi^n(\sigma(\omega)) - D|_{\theta(\lambda)}\pi^k(\sigma(\omega))\big)\theta'(\lambda)\big)\big\| +\\ &\qquad\quad + \big\|\big(\phi'_{\omega_1}(\lambda_{\omega_1} + \pi^n_{\theta(\lambda)}(\sigma(\omega))) - \phi'_{\omega_1}(\lambda_{\omega_1} + \pi^k_{\theta(\lambda)}(\sigma(\omega)))\big)D|_{\theta(\lambda)}\pi^k(\sigma(\omega))\theta'(\lambda)\big\|\\ &\quad\leq \big\|\big(\phi'_{\omega_1}(\lambda_{\omega_1} + \pi^n_{\theta(\lambda)}(\sigma(\omega))) - \phi'_{\omega_1}(\lambda_{\omega_1} + \pi^k_{\theta(\lambda)}(\sigma(\omega)))\big)\big\| +\\ &\qquad\quad + \big\|\phi'_{\omega_1}(\lambda_{\omega_1} + \pi^n_{\theta(\lambda)}(\sigma(\omega)))\big(D|_{\theta(\lambda)}\pi^n(\sigma(\omega)) - D|_{\theta(\lambda)}\pi^k(\sigma(\omega))\big)\theta'(\lambda)\big)\big\| +\\ &\qquad\quad + \big\|\big(\phi'_{\omega_1}(\lambda_{\omega_1} + \pi^n_{\theta(\lambda)}(\sigma(\omega))) - \phi'_{\omega_1}(\lambda_{\omega_1} + \pi^k_{\theta(\lambda)}(\sigma(\omega)))\big)D|_{\theta(\lambda)}\pi^k(\sigma(\omega))\theta'(\lambda)\big\|\\ &\quad\leq H_{\mathcal{S}}\big\|\pi_{\theta^n(\lambda)}(\sigma(\omega)) - \pi_{\theta^k(\lambda)}(\sigma(\omega))\big\|^\alpha + s\big\|D|_{\theta(\lambda)}\pi^n(\sigma(\omega)) - D|_{\theta(\lambda)}\pi^k(\sigma(\omega))\big\| \cdot ||\theta'(\lambda)|| +\\ &\qquad\quad + H_{\mathcal{S}}\big\|D|_{\theta(\lambda)}\pi^k(\sigma(\omega))\big\| \cdot ||\theta'(\lambda)|| \cdot \big\|\pi_{\theta^n(\lambda)}(\sigma(\omega)) - \pi_{\theta^k(\lambda)}(\sigma(\omega))\big\|^\alpha\\ &\quad\leq H_{\mathcal{S}}s^{\alpha k} + s||\theta'||_\infty\|D|_{\theta(\lambda)}\pi^n(\sigma(\omega)) - D|_{\theta(\lambda)}\pi^k(\sigma(\omega))\| + sH_{\mathcal{S}}||\theta'||_\infty(1 - s||\theta'||_\infty)^{-1}s^{\alpha k}\\ &\quad= H_{\mathcal{S}}||\theta'||_\infty\big(1 + (1 - s||\theta'||_\infty)^{-1}\big)s^{\alpha k} + s||\theta'||_\infty\|D|_{\theta(\lambda)}\pi^n(\sigma(\omega)) - D|_{\theta(\lambda)}\pi^k(\sigma(\omega))\|.\end{aligned}$$

So, we got exactly the same inequality as (6.3) except for the additional term 1 in the first summand. The remainder of the proof is the same as the proof of Proposition 6.1. □

Now we can prove the following.

Theorem 6.5. *Let E be a countable set and let $\mathcal{S} = \{\phi_e : e \in E\}$ be an autonomous (deterministic) conformal IFS acting on some compact interval $X \subset \mathbb{R}$ and a bounded open interval $V \subset \mathbb{R}$. Let E_* be an arbitrary subset of E. Assume that*

(a) *$\phi_a(X) \cap \phi_b(X) = \emptyset$ for all $a, b \in E \setminus E_*$ with $a \neq b$,*
(b) *$s||\theta'||_\infty < 1$ and $s||\theta'||_\infty(1 - s||\theta'||_\infty)^{-1} < 1/2$.*

Let Λ be given by (6.1) with $p = \infty$ and let $\theta : \Lambda \to \Lambda$ be a C^1-diffeomorphism of finite character with respect to E_*. Then the sub-random conformal IFS $\mathcal{S}^{(E_*,\eta,R)}$ defined by (6.13) *satisfies the Random Transversality Condition of Finite Type with respect to the product measure $\otimes_{e\in E_*}\ell_e$ on $\prod_{e\in E_*} U_e$, where, we recall, ℓ_e is the normalized d-dimensional Lebesgue measure on U_e.*

Proof. Fix $k \geq 1$. Let $F_k := \bigcup_{j=1}^k E_{*,j}$ and let F be an arbitrary finite set such that

$$F_k \subset F \subset F_k \cup (E \setminus E_*). \tag{6.15}$$

Put

$$u_k := \min\{||\phi_e'|| : e \in F_k\} < +\infty.$$

Since the set $F_k \setminus E_*$ is finite and all the sets $\phi_E(X)$, $e \in E \setminus E_*$ are compact, it follows from (a) that if $\eta + R$ is small enough, then

$$\xi := \inf\left\{\text{dist}\big(\phi_a^\lambda(X), \phi_b^\lambda(X)\big) : \lambda \in \Lambda \ \text{ and } \ a, b \in F \setminus E_* \ \text{ with } \ a \neq b\right\} > 0.$$

Fix $\omega, \tau \in F^{\mathbb{N}}$ with $\omega_1 \neq \tau_1$. if both ω_1 and τ_1 are in $F \setminus E_*$, then for every $r \in (0, \xi)$

$$\{\lambda \in \Lambda : |\pi_\lambda(\omega) - \pi_\lambda(\tau)| \leq r\} = \emptyset,$$

whence trivially,

$$\bigotimes_{e \in E_*} \ell_e\big(\{\lambda \in \Lambda : |\pi_\lambda(\omega) - \pi_\lambda(\tau)| \leq r\}\big) \leq C_F r,$$

for all $r > 0$. So, we may assume that at least one of the two elements ω_1 or τ_1 is in F_k. Assume without loss of generality that $\omega_1 \in F_k$. Then

$$\begin{aligned} \pi_\lambda(\omega) - \pi_\lambda(\tau) &= \phi_{\omega_1}(\lambda_{\omega_1} + \pi_{\theta(\lambda)}(\sigma(\omega))) - \phi_{\tau_1}(\lambda_{\tau_1} + \pi_{\theta(\lambda)}(\sigma(\tau))) \\ &= \phi_{\omega_1}'(c)\big(\pi_{\theta(\lambda)}(\sigma(\omega))) - \pi_{\theta(\lambda)}(\sigma(\tau))) + \lambda_{\omega_1} - \lambda_{\tau_1}\big) \end{aligned} \tag{6.16}$$

with some c belonging to the closed interval whose endpoints are $\lambda_{\omega_1} + \pi_{\theta(\lambda)}(\sigma(\omega))$ and $\lambda_{\tau_1} + \pi_{\theta(\lambda)}(\sigma(\tau))$. Put

$$f(\lambda) := \pi_{\theta(\lambda)}(\sigma(\omega)) - \pi_{\theta(\lambda)}(\sigma(\tau)) + \lambda_{\omega_1} - \lambda_{\tau_1}.$$

Then by (6.16),

$$\{\lambda \in \Lambda : |\pi_\lambda(\omega) - \pi_\lambda(\tau)| \leq r\} \subset \{\lambda \in \Lambda : |f(\lambda)| \leq u_k^{-1} r\},$$

and therefore it suffices (see the very last, starting with "Notice that", part of the proof of Theorem 6.2) to prove Lemma 4.8. We have

$$\begin{aligned} \frac{\partial f}{\partial \lambda_{\omega_1}} &= 1 + D|_{\theta(\lambda)}\pi(\sigma(\omega))\frac{\partial \theta}{\partial \lambda_{\omega_1}} - D|_{\theta(\lambda)}\pi(\sigma(\tau))\frac{\partial \theta}{\partial \lambda_{\omega_1}} \\ &= 1 + \big(D|_{\theta(\lambda)}\pi(\sigma(\omega)) - D|_{\theta(\lambda)}\pi(\sigma(\tau))\big)\frac{\partial \theta}{\partial \lambda_{\omega_1}}. \end{aligned}$$

Hence, using also the last assertion of Proposition 6.4, we get

$$\frac{\partial f}{\partial \lambda_{\omega_1}} \geq 1 - \left|\frac{\partial \theta}{\partial \lambda_{\omega_1}}\right|\big(\big\|D|_{\theta(\lambda)}\pi(\sigma(\omega))\big\| + \big\|D|_{\theta(\lambda)}\pi(\sigma(\tau))\big\|\big) \geq 1 - 2s||\theta'||_\infty(1 - s||\theta'||_\infty)^{-1}.$$

As by (b), $1 - 2s||\theta'||_\infty(1 - s||\theta'||_\infty)^{-1} > 0$, the hypotheses of Lemma 4.8 are thus verified and the proof is complete. $\square$

Remark 6.6. We can remark exactly the same in regard to the systems described in Theorem 6.5 as we did in Remark 6.3 with respect the systems described in Theorem 6.2.

Remark 6.7. If the map $\theta : \Lambda \to \Lambda$ described by (6.11) and (6.10) and involved in both Theorem 6.2 and Theorem 6.5, preserves a Borel probability measure m absolutely continuous with respect to the measure $\bigotimes_{e \in E_*} \ell_e$ with the corresponding Radon-Nikodym derivative bounded above, then the sub-random IFSs $\mathcal{S}^{(E_*,\eta)}$ and $\mathcal{S}^{(E_*,\eta,R)}$ become random conformal IFSs with respect to the measure m (and satisfy the Random Transversality Condition of Finite Type with respect to m).

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