RANDOM SHIFTS OF FINITE TYPE
WITH
WEAKLY POSITIVE TRANSFER OPERATOR

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ABSTRACT. We deal with countable alphabet random subshifts of finite type under the absence of Big Images Property and under the absence of uniform positivity of the transfer operator. We first establish the existence of random conformal measures along with good bounds for the iterates of the Perron-Frobenius operator. Then, using the technique of positive cones and proving a version of Bowen's type contraction (see [2]), we also establish a fairly complete thermodynamical formalism. This means that we prove the existence and uniqueness of fiberwise invariant measures (giving rise to a global invariant measure) equivalent to the fiberwise conformal measures. Furthermore, we establish the existence of a spectral gap for the transfer operators, which in the random context precisely means the exponential rate of convergence of the normalized iterated transfer operator. This latter property in a relatively straightforward way entails the exponential decay of correlations and the Central Limit Theorem.

1. INTRODUCTION

The thermodynamic formalism of finite and countable topological Markov shifts (subshifts of finite type) with Hölder continuous potentials is by now quite well understood. The case of finite alphabet was settled already in nineteen seventies, primarily due to the work of R. Bowen, [2], and Ruelle, [9]. For the case of countable infinite alphabet, the existence and uniqueness of conformal measures and invariant Gibbs states for finitely irreducible shifts was established in [5]. The necessity of finite irreducibility for the existence of invariant Gibbs states was shown in [10]. The spectral gap of the corresponding Perron-Frobenius (transfer) operator, and resulting from it exponential decay of correlations, the Central Limit Theorem and the Law of Iterated logarithm were established in [6]. As a matter of fact, this book contains a systematic exposition of the theory of deterministic subshifts of finite type. The work [11] is also a good reference with a different viewpoint based on the concept of so called recurrent potentials.

In the case of random dynamics, Bogenschütz and Gundlach [1] have considered the finite shift case. Then Denker, Kifer and Stadlbauer in [4], Stadlbauer [12, 13] and [8] dealt with the case of random countable topological Markov shifts. All these papers assumed some version of finite irreducibility or, to put it in different words, Big Images Property. In all these cases the Perron-Frobenius evaluated at the function identically equal to one was strictly positive, in fact its infimum over the whole space was positive.

The present paper lies on just the opposite spectrum. It is devoted to random countable subshifts of finite type but in our setting the Big Images Property does always fail. Also, strict positivity of the Perron-Frobenius does always fail, see Property (B) in Section 4. This happens however with some control as can be seen from properties (A) and (C) of the same section.

In our present paper, utilizing the concept of narrow topology and Prokhorov's Compactness Theorem, we first establish the existence of conformal measures along with good bounds for the iterates of the Perron-Frobenius operator. Then, using the technique of positive cones and proving a version of Bowen's type contraction (see [2]), by developing in the present setting our approach from a non-Markovian context of [7], we also establish a fairly complete thermodynamical formalism. First of all, we prove the existence and uniqueness of fiberwise invariant measures (giving rise to a global invariant measure) equivalent to the fiberwise conformal measures. Furthermore, we establish the existence of a spectral gap for the transfer operators, which in the random context precisely means the exponential rate of convergence of iterates of the normalized transfer operator. This latter property in a relatively straightforward way entails the exponential decay of correlations and the Central Limit Theorem

We would like to stress that the hypotheses under which we work, apart from topological mixing of the shift map, are of "the first level" type, meaning that these do not involve iterates of the random shift map, Birkhoff's sums of potentials, or iterates of the transfer operator. These involve the transfer operator itself but only very mildly as condition (C) of Section 4. It is straightforward to see that abundance of examples fulfill our hypotheses.

2. Preliminaries; Topological Aspects

Let $(X, \mathfrak{F}, m)$ be a complete probability space with $\mathfrak{F}$, a σ-algebra of subsets of X and m a complete probability measure on $\mathfrak{F}$. Let $\theta : X \to X$ be an invertible measurable transformation preserving measure m. Let E be a countable infinite set. For the ease of exposition and without loss of generality we assume throughout that
$$E = \mathbb{N} = \{0, 1, 2, \ldots\}$$
is the set of all non-negative integers. Assume that
$$X \ni x \mapsto A(x) : E \times E \to \{0, 1\}$$

is a measurable map from X into $0, 1$-matrices on E, commonly called incidence matrices. These matrices define symbol spaces as follows. For every $x \in X$ let

$$(2.1) \quad E_x^\infty = E_x^\infty(A) := \left\{ (\omega_n)_{n=0}^\infty \in E^{\mathbb{N}} : A_{\omega_i \omega_{i+1}}(\theta^i(x)) = 1 \text{ for every } i \geq 0 \right\}.$$

More generally, for every set $F \subset E$, given $n \in \mathbb{N} \cup \{\infty\}$, it is useful to introduce the sets
$$(2.2)$$
$$F_x^n = F_x^n(A) = \left\{ \omega_0 \omega_1 \ldots \omega_n \in F^{n+1} : A_{\omega_i \omega_{i+1}}(\theta^i(x)) = 1 \text{ for all } 0 \leq i \leq n-1 \right\}.$$

These are words of length $n + 1$ over the alphabet F and they will be called admissible by the matrix A, also A-admissible, or just admissible. If $m \leq n$ and if $\omega \in F_x^n$ then

$$\omega|_m := \omega_0 \omega_1 ... \omega_m \in F_x^m$$

is called the truncation, or restriction, of ω to m. The cylinder generated by a word $\omega \in E_x^n$ is defined to be

$$[\omega]_x := \left\{ \tau \in E_x^\infty : \tau|_n = \omega \right\}.$$

A straightforward extension of the notion of cylinder is this. For any $F \subset E$,

$$[F]_x := \bigcup_{j \in F} [j]_x,$$

The symbol spaces E_x^∞, $x \in X$, are naturally endowed with the subspace topology inherited from the product (Tichonov) topology on the Cartesian product $E^{\mathbb{N}}$. This latter topology, and all respective subspace topologies are generated by many natural metrics. One of them, the one we will work with, is this:

$$d(\omega, \tau) = 0 \text{ if } \omega = \tau \text{ and } d(\omega, \tau) = \exp\left(-\min\{n ; \omega_n \neq \tau_n\} \right) \text{ if } \omega \neq \tau.$$

The Cartesian product $X \times E^{\mathbb{N}}$ is further endowed with a natural measurable structure, i.e. a σ-algebra. This is the σ-algebra generated by the sets of the form $F \times B$, where F is a measurable subset of X and B is a Borel subset of $E^{\mathbb{N}}$. This σ-algebra is denoted by $\mathfrak{F} \otimes \mathcal{B}$. Let

$$(2.3) \qquad \Omega := \bigcup_{x \in X} \{x\} \times E_x^\infty \subset X \times E^{\mathbb{N}}$$

By $\mathfrak{F} \otimes |_\Omega \mathcal{B}$ we denote the restriction of $\mathfrak{F} \otimes \mathcal{B}$ to Ω, i.e. the σ-algebra consisting of the sets $Y \cap \Omega$, $Y \in \mathfrak{F} \otimes \mathcal{B}$. By $\mathcal{B}_x$, $x \in X$, we denote the Borel σ-algebra of Borel subsets of E_x^∞. The symbol σ stands for the standard shift map from $E^{\mathbb{N}}$ to $E^{\mathbb{N}}$ defined by the formula

$$\sigma\left((\omega_n)_{n=0}^\infty\right) = (\omega_{n+1})_{n=0}^\infty.$$

The spaces E_x^∞, $x \in X$, are invariant under the shift map $\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$ in the sense that

$$\sigma(E_x^\infty) \subset E_{\theta(x)}^\infty$$

for all $x \in X$. We then set

$$\sigma_x :=: \sigma\big|_{E_x^\infty} : E_x^\infty \to E_{\theta(x)}^\infty.$$

We are interested in the (fiberwise) dynamics, i. e. in the compositions (frequently referred to as iterates):

$$\sigma_x^n := \sigma_{\theta^{n-1}(x)} \circ ... \circ \sigma_x : E_x^\infty \to E_{\theta^n(x)}^\infty, \quad n \geq 0.$$

Its global version, the random shift map $\hat{\sigma}_A : \Omega \to \Omega$, defined by the formula

$$(2.4) \qquad\qquad (x,\omega) \longmapsto (\theta(x), \sigma_x(\omega)),$$

is a skew product map with base X and fibers E_x^∞. It will be frequently denoted simply by $\hat{\sigma}$. We assume this map to be mixing. The precise definition is the following.

Definition 2.1. *The random shift map $\hat{\sigma} : \Omega \to \Omega$ is called topologically mixing if for all letters $a, b \in E$ there exists $N = N_{a,b} \geq 0$ such that for every $n \geq N$ and all $x \in X$ there exists $\omega \in E_x^n$ such that the word $a\omega b$ is A-admissible.*

A more adequate name for this concept would be uniform topological mixing, but we stick to the shorter one since there will be considered in this paper no more general, or weaker, concepts of mixing.

Remark 2.2. *Notice that mixing in particular implies that no cylinders $[i]_x$ and no sets $\sigma_x^{-1}\big([i]_{\theta(x)}\big)$, $i \in E$, $x \in X$, are empty. A reformulation of the latter property is that for all $x \in X$ and all $b \in E$ there exists $a \in E$ such that $A_{ab}(x) = 1$.*

Definition 2.3. *The incidence matrix A and the random shift map $\hat{\sigma} : \Omega \to \Omega$ alike, are said to be of finite range if for every $e \in E$ there exits a finite set $D_e \subset E$ such that, for $m - a.e.$ $x \in X$,*

$$\{j \in E : A_{ej}(x) = 1\} \subset D_e$$

Then also the random shift map $\hat{\sigma}_A : \Omega \to \Omega$ is said to be of finite range.

The following lemma is a reformulation of this condition.

Lemma 2.4. *The random shift map $\hat{\sigma} : \Omega \to \Omega$ is of finite range if and only if for every $l \in E$ there exists $\hat{l} \in E$ such that, for every $e > \hat{l}$,*

$$\sigma_x^{-1}\left([e]_{\theta(x)}\right) \cap [0, ..., l]_x = \emptyset \quad \text{for } m\text{-a.e. } x \in X.$$

Definition 2.5. *The incidence matrix A and the random shift map $\hat{\sigma} : \Omega \to \Omega$ alike, are said to be of bounded access if for all $b \in E$ there exists $b^* \in E$ such that for m-a.e. $x \in X$ there exists $a = a_x \leq b^*$ such that $A_{ab}(x) = 1$ (we recall that $E = \mathbb{N}$).*

3. Preliminaries: Function Spaces and Bounded Distortion

Various function spaces will be used in the sequel. First we consider functions defined on a fiber E_x^∞. The space of bounded continuous functions on E_x^∞ will be denoted $\mathcal{C}_b(E_x^\infty)$ and will be endowed with the supremum norm $\|\cdot\|_\infty$. The α–variation of a function $g : E_x^\infty \to \mathbb{R}$ is defined to be

$$v_\alpha(g) := \inf\left\{ \frac{|g(\tau) - g(\omega)|}{d(\tau, \omega)^\alpha} \ : \ \tau \neq \omega \ , \ \ \tau_0 = \omega_0 \right\}.$$

The vector space $\mathcal{H}_\alpha(E_x^\infty)$ by definition consists of all functions $g \in \mathcal{C}_b(E_x^\infty)$ that have finite α–variation. This space will be endowed with the canonical norm

$$\|\cdot\|_\alpha = \|\cdot\|_\infty + v_\alpha(\cdot).$$

Clearly, both normed spaces $\mathcal{C}_b(E_x^\infty)$ and $\mathcal{H}_\alpha(E_x^\infty)$ are Banach.

We now define global functions $g : \Omega \to \mathbb{R}$ and bring up their basic properties. Measurability of such functions will always be with respect to the σ–algebra $\mathfrak{F} \otimes|_\Omega \mathcal{B}$ on Ω. A measurable function $g : \Omega \to \mathbb{R}$ is called *essentially continuous* if $g_x \in \mathcal{C}_b(E_x^\infty)$, $x \in X$, and if the essential supremum over X of the measurable function $x \mapsto \|g_x\|_\infty$ is finite. The vector space of such functions will be denoted by $\mathcal{C}_b(\Omega)$. It becomes a Banach space when equipped with the norm

$$|g|_\infty := \mathrm{esssup}\{\|g_x\|_\infty : x \in X\}.$$

We also have to consider various spaces of global Hölder functions especially in the context of the stochastic laws at the end of Section 5. For the moment, let us introduce *essentially α-Hölder* functions. These will be measurable functions $g : \Omega \to \mathbb{R}$ such that $g_x \in \mathcal{H}_\alpha(E_x^\infty)$, $x \in X$, and such that

$$V_\alpha(g) := \mathrm{esssup}\{v_\alpha(g_x) : x \in X\} < \infty.$$

Functions that are in the same time essentially α-Hölder and essentially continuous form a Banach space, denoted by $\mathcal{H}_\alpha(\Omega)$. The norm of a function $g \in \mathcal{H}_\alpha(\Omega)$ is

$$|g|_\alpha := \mathrm{esssup}\{\|g_x\|_\alpha : x \in X\}.$$

For every integer $n \geq 1$ and every function $g : \Omega \to \mathbb{R}$, let

$$S_n g := \sum_{j=0}^{n-1} g \circ \hat\sigma^j,$$

be the nth Birkhoff's sum of g with respect to $\hat\sigma$. We will frequently need the following technical but indispensable fact.

Lemma 3.1. *There exists a constant $C > 0$ such that*

$$(3.1) \qquad |S_n g(x, \omega) - S_n g(x, \tau)| \leq C V_\alpha(g) d(\omega, \tau)^\alpha$$

and

$$(3.2) \qquad |\exp(S_n g(x, \omega) - S_n g(x, \tau)) - 1| \leq C V_\alpha(g) d(\omega, \tau)^\alpha$$

for every essentially α-Hölder function $g : \Omega \to \mathbb{R}$, every $x \in X$, every integer $n \geq 1$, and all $\omega, \tau \in E_x^\infty$ with $\omega|_{n-1} = \tau|_{n-1}$.

The proof of this lemma is standard in hyperbolic dynamics and is left for the reader.

4. Transfer Operators and Fine Shifts

We want to associate to a function $\varphi : \Omega \to \mathbb{R}$ a family of operators $\mathcal{L}_x$, $x \in X$, by the formula

$$\mathcal{L}_x g_x(\omega) := \sum_{\substack{e \in E \\ A_{e\omega_0}(x)=1}} g_x(e\omega) e^{\varphi_x(e\omega)} \quad \text{where } g_x \in \mathcal{C}_b\big(E_x^\infty\big) \text{ and } \omega \in E_x^\infty.$$

Taking the particular function $\mathbb{1}_{[0,\dots,l]_x^c}$ where $l \in E$, then

$$\mathcal{L}_x\big(\mathbb{1}_{[0,\dots,l]_x^c}\big)(\omega) = \sum_{\substack{e > l \\ A_{e\omega_0}(x)=1}} e^{\varphi_x(e\omega)} \quad, \omega \in E_x^\infty.$$

Clearly, φ must satisfy some additional properties for these operators being well defined.

Definition 4.1. *Given $\alpha > 0$, we call an essentially α-Hölder function $\varphi : \Omega \to \mathbb{R}$ a summable potential if for every $e \in E$ there exists $0 < c_e < C_e < \infty$ such that*

$$(4.1) \qquad c_e \leq \exp \varphi_x\big|_{[e]_x} \leq C_e$$

and if the operators $\mathcal{L}_x$ are uniformy summable in the following sense:

$$(4.2) \qquad \lim_{l \to \infty} \operatorname{esssup}\left(\left|\mathcal{L}_x\left(\mathbb{1}_{[0,\dots,l]_x^c}\right)\right|_\infty : x \in X\right) = 0.$$

This summability condition deserves some comments. First of all, (4.1) and (4.2) imply that there exists $M < \infty$ such that

$$(4.3) \qquad \mathcal{L}_x \mathbb{1} \leq M \quad \text{for a.e. } x \in X.$$

Then, (4.2) is formulated in terms of the transfer operator and thus this condition does depend on the random incidence matrices. But, in general, one can use instead of the above uniform summability the following simpler condition which does only involve the function φ itself:

$$(4.4) \qquad \sum_{e \in E} \exp\big(\operatorname*{esssup}_{x \in X} \varphi_x\big|_{[e]_x}\big) < +\infty.$$

Clearly, (4.4) implies (4.2). However, for some shifts and potentials (4.4) does not hold whereas the weaker assumption (4.2) does. This is the case for the following (deterministic) example. Since we want also treat such shifts the condition (4.2) is appropriate.

Example 4.2. *The incidence matrix A here does not depend on x since we simple describe a deterministic example (which can be randomized in many ways). The one values of this matrix are defined as follows: $A_{10} = 1$ and, for every $j \geq 1$, $A_{ij} = 1$ if and only if*

$$i = (j - 1) + n^{j+1} \quad \text{for some } n \in \mathbb{N}.$$

This shift has all required properties: it is mixing, of finite range and of bounded access (every deterministic shift is of bounded access). Consider then the potential

$$\varphi(i) = -\ln(1 + i) \quad , \quad i \in \mathbb{N}.$$

Clearly φ does not satisfy the summability condition (4.4) but it is obvious that weaker summability condition (4.2) does hold.

Consider now the following three properties.

(A) *For every $e \in E$ there exists $M_e \in (0, +\infty)$ such that*
$$M_e^{-1} \leq \mathcal{L}_x \mathbb{1}\big|_{[e]_{\theta(x)}} \quad \text{for m-a.e. } x \in X.$$

(B) $\displaystyle \lim_{e \to \infty} \operatorname*{esssup}_{x \in X} \left(\mathcal{L}_x \mathbb{1}\big|_{[e]_{\theta(x)}} \right) = 0.$

(C) *There are a number $0 < \kappa < 1/4$ and a finite set $F \subset E$ such that*
$$\sup \left(\mathcal{L}_x \left(\mathbb{1}_{E_x^\infty \setminus [F]_x} \right) \right) \leq \kappa \inf \left(\mathcal{L}_x \mathbb{1}\big|_{[F]_{\theta(x)}} \right) \quad \text{for a.e. } x \in X.$$

These conditions deserve some comments. First of all, as explained in the introduction, the goal here is to consider a situation where the transfer operator is no longer strictly positive in the sense that $\inf \mathcal{L}_x \mathbb{1} > 0$. Clearly, the condition (B) is responsible for this whereas (C) gives some control of how $\mathcal{L}_x \mathbb{1}(y) \to 0$ as $y_0 \to \infty$ and (A) is a weak lower bound. An other remark is that these conditions rely only on the shift and the potential itself and so no higher iterates are involved. They are, at least for reasonable potentials, quite simple to check. Indeed, condition (C) relies on the potential but, for summable potentials, (A) and (B) totally rely on the shift, hence the incidence matrix.

Lemma 4.3. *Let $\varphi : \Omega \to \mathbb{R}$ be a summable potential. Then we have the following:*
   (1) *The shift is of bounded access if an only if (A) holds.*
   (2) *The shift is of finite range if and only if (B) holds.*

The proof of this remark is obvious.

Definition 4.4. *A random shift $\hat{\sigma} : \Omega \to \Omega$ along with a potential $\varphi : \Omega \to \Omega$ is called fine if φ is summable and if (A, B, C) hold or, equivalently, if φ is summable, the shift $\hat{\sigma}$ is of finite range and of bounded access and if (C) holds.*

Here are some more properties that will be useful in the sequel.

The condition (4.1) is just a weak bound and it implies the following immediate observation.

Lemma 4.5. *Suppose that the random shift map $\hat{\sigma} : \Omega \to \Omega$ is of finite range and that $\varphi : \Omega \to \mathbb{R}$ is essentially α-Hölder and satisfies (4.1). Then, for every $e \in E$ and every integer $n \geq 1$ there exists $c^* = c^*(e, n) > 0$ such that*

$$\exp\left(\inf\left(S_n\varphi|_{\{x\}\times[e]_x}\right)\right) \geq c^* \quad \text{for } m \text{ a.e. } x \in X.$$

For a summable potential it is clear that $\mathcal{L}_x g_x \in \mathcal{C}_b\left(E_{\theta(x)}^\infty\right)$ and so

$$\mathcal{L}_x\left(\mathcal{C}_b\left(E_x^\infty\right)\right) \subset \mathcal{C}_b\left(E_{\theta(x)}^\infty\right).$$

We record the following.

Lemma 4.6. *If $\varphi : \Omega \to \mathbb{R}$ is a summable potential, then for each $x \in X$, the linear operator*

$$\mathcal{L}_x : \mathcal{C}_b\left(E_x^\infty\right) \to \mathcal{C}_b\left(E_{\theta(x)}^\infty\right)$$

is bounded and its norm is bounded above by M. Also $\mathcal{L}_x\left(\mathcal{H}_\alpha\left(E_x^\infty\right)\right) \subset \mathcal{H}_\alpha\left(E_{\theta(x)}^\infty\right)$ and the linear operator $\mathcal{L}_x : \mathcal{H}_\alpha\left(E_x^\infty\right) \to \mathcal{H}_\alpha\left(E_{\theta(x)}^\infty\right)$ is bounded.

Proof. The first assertion of this lemma is obvious while the second results by a standard calculation. $\square$

The operators $\mathcal{L}_x$, $x \in X$, are frequently referred to as fiberwise transfer operators. They give rise to the global operators defined as follows. If $g \in \mathcal{C}_b(\Omega)$ and $x \in X$, we set

$$(\mathcal{L}g)_x := \mathcal{L}_{\theta^{-1}(x)} g_{\theta^{-1}(x)} \in \mathcal{C}_b\left(E_x^\infty\right).$$

Lemma 4.7. *If $\varphi : \Omega \to \mathbb{R}$ is a summable potential, then the function $X \ni x \mapsto \mathcal{L}_x$ is measurable in the sense that for each $g \in \mathcal{C}_b(\Omega)$ the function*

$$\Omega \ni (x, \omega) \mapsto \mathcal{L}_x g_x(\omega) \in \mathbb{R}$$

is measurable. In consequence $\mathcal{L}g \in \mathcal{C}_b(\Omega)$.

Proof. Note that, given $e \in E$, both functions $\Omega \ni (x, \omega) \mapsto g_x(e\omega)$ and $\Omega \ni (x, \omega) \mapsto e^{\varphi_x(e\omega)}$ are measurable and can be extended to measurable functions on $X \times E^\mathbb{N}$ by putting the value zero outside Ω. Since we can extend measurably in the same way the incidence matrix A, the function

$$X \times E^\mathbb{N} \ni (x, \omega) \mapsto A_{e\omega_0}(x) g_x(e\omega) e^{\varphi_x(e\omega)}$$

is measurable too. This shows measurability of the function $X \ni x \mapsto \mathcal{L}_x(g)$ since the former one can now be expresses as a convrgent series of measurable functions. $\square$

As a direct consequence of this observation and of Lemma 4.6, we get the following.

Lemma 4.8. *If $\varphi : \Omega \to \mathbb{R}$ is a summable potential, then the linear operator*

$$\mathcal{L} : \mathcal{C}_b(\Omega) \to \mathcal{C}_b(\Omega)$$

is bounded and its norm is bounded above by M. Also $\mathcal{L}\big(\mathcal{H}_\alpha(\Omega)\big) \subset \mathcal{H}_\alpha(\Omega)$ and the linear operator $\mathcal{L} : \mathcal{H}_\alpha(\Omega) \to \mathcal{H}_\alpha(\Omega)$ is bounded.

For $n \geq 1$, we define the iterated operator

$$\mathcal{L}_x^n := \mathcal{L}_{\theta^{n-1}(x)} \circ ... \circ \mathcal{L}_x : \mathcal{C}_b\big(E_x^\infty\big) \to \mathcal{C}_b\big(E_{\theta^n(x)}^\infty\big)$$

Of course $\mathcal{L}_x^n\big(\mathcal{H}_\alpha\big(E_x^\infty\big)\big) \subset \mathcal{H}_\alpha(E_{\theta^n(x)}^\infty)$. A standard straightforward inductive calculation shows that

$$\mathcal{L}_x^n(g)(\omega) = \sum_{\substack{\tau \in E_x^n \\ A_{\tau_{n-1}\omega_o(x)}=1}} \exp\big(S_n\varphi(x, \tau\omega)\big)g(\tau\omega).$$

As an immediate consequence of Lemma 3.1 we get the following.

Lemma 4.9. *For every $n \geq 1$, every $x \in X$, and all $\omega, \tau \in E_x^\infty$, with $\omega_0 = \tau_0$,*

$$\frac{\mathcal{L}_x^n \mathbb{1}(\omega)}{\mathcal{L}_x^n \mathbb{1}(\tau)} \leq 1 + CV_\alpha(\varphi)d(\omega, \tau)^\alpha .$$

In particular,

$$\mathcal{L}_x^n \mathbb{1}(\omega) \leq K\mathcal{L}_x^n \mathbb{1}(\tau),$$

where $K = 1 + CV_\alpha(\varphi)$.

Our goal now is three-folded: to prove the existence of conformal measures, of their invariant versions and finally to obtain exponential rate of convergence for the iterated normalized operator along with stochastic laws. This will be done for random shifts and potentials that satisfy the conditions formulated in the next section.

5. Random Measures and Main Theorems

The first thing we want to do in this section is to recall the concept of random measures and to bring up some of its basic properties. We do this in our context of the measurable space Ω and measure m on X.

Definition 5.1. *A measure ν on $\big(\Omega, \mathfrak{F} \otimes |_\Omega \mathcal{B}\big)$ with marginal m, i.e. such that*

$$\nu \circ \pi_X^{-1} = m,$$

is called a random measure if $\nu(\Omega) = 1$, i.e. if it is a probability measure on Ω. Equivalently, if its disintegrations ν_x, $x \in X$, respectively belong to the spaces $\mathcal{P}(E_x^\infty)$ of probability measures on $\big(E_x^\infty, \mathcal{B}_x\big)$. The space of all random measures (on Ω with marginal m) will be denoted by $\mathcal{P}_m(\Omega)$.

According to this definition, for every random measure ν, i. e. belonging to $\mathcal{P}_m(\Omega)$, and for every every function $g \in \mathcal{C}_b(\Omega)$, we have

$$(5.1) \qquad \nu(g) := \int_\Omega g \, d\nu = \int_X \int_{E_x^\infty} g_x \, d\nu_x \, dm(x) \le |g|_\infty.$$

This formula naturally introduces the space $g \in L^1(\nu)$, i.e. the space of all real-valued measurable functions on Ω integrable with respect to ν. Precisely, $g \in L^1(\nu)$ if and only if

$$\int_X \int_{E_x^\infty} |g_x| \, d\nu_x \, dm(x) < +\infty \,.$$

We would like to mention that random measures, as defined in Crauel's book [3], are all probability measures on the set $X \times E^\mathbb{N}$ with marginal m. Denote them just by $\mathcal{P}_m$. But in this paper we actually are interested only in the class $\mathcal{P}_m(\Omega)$, i.e. the one which consists of all random measures ν on $X \times E^\mathbb{N}$ with support in Ω, i.e. such that $\nu(\Omega) = 1$.

The key concept pertaining to random measures is that of narrow topology (a version of weak convergence). Namely, if Λ is a directed set, then a net $(\nu^\alpha)_{\alpha \in \Lambda}$ in $\mathcal{P}_m$ is said to converge to a random measure $\nu \in \mathcal{P}_m$ if

$$\lim_{\alpha \in \Lambda} \nu^\alpha(g) = \nu(g) \quad \text{for every } g \in \mathcal{C}_b(X \times E^\mathbb{N}).$$

This concept of convergence defines a topology on $\mathcal{P}_m$ called in [3] the narrow topology. The narrow topology on $\mathcal{P}_m(\Omega)$ is the one inhereted from the narrow topology on $\mathcal{P}_m$. Since $\mathbb{1}_\Omega$, the characteristic function of Ω in $X \times E^\mathbb{N}$, belongs to $\mathcal{C}_b(X \times E^\mathbb{N})$, we have that

$$\nu(\mathbb{1}_\Omega) = \lim_{\alpha \in \Lambda} \nu^\alpha(\mathbb{1}_\Omega) = 1$$

for any net $(\nu^\alpha)_{\alpha \in \Lambda}$ in $\mathcal{P}_m(\Omega)$ converging to a random measure $\nu \in \mathcal{P}_m$. This means that then $\nu \in \mathcal{P}_m(\Omega)$, leading to the following.

Proposition 5.2. $\mathcal{P}_m(\Omega)$ *is a closed subset of* $\mathcal{P}_m$ *with respect to the narrow topology on* $\mathcal{P}_m$.

Recall from [3] that a subset Γ of $\mathcal{P}_m$ is called tight if its projection $\pi_{E^\mathbb{N}}^{-1}(\Gamma)$ on $E^\mathbb{N}$ is a tight subset of Borel probability measures on $E^\mathbb{N}$, the latter (commonly) meaning that for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset E^\mathbb{N}$ such that $\nu \circ \pi_{E^\mathbb{N}}^{-1}(K_\varepsilon) \ge 1 - \varepsilon$ for all $\nu \in \Gamma$. For us, the crucial property of narrow topology is that of Prohorov's Compactness Theorem (Theorem 4.4 in [3]) which asserts that a subset $\mathcal{M} \subset \mathcal{P}_m$ is relatively compact with respect to the narrow topology if and only if it is tight. Along with Proposition 5.2, this entails the following.

Theorem 5.3. *A subset* $\Gamma \subset \mathcal{P}_m(\Omega)$ *is relatively compact with respect to the narrow topology if and only if it is tight. Furthermore, it is compact if and only if it is tight and closed.*

We will use tightness heavily in the next section and for this we will need Proposition 4.3 from [3]), which provides convenient sufficient conditions for tightness to hold. For this in turn, we will need concepts of random closed and compact sets. Indeed, following Definition 2.1 in [3]) we say that a function

$$X \ni x \mapsto C_x,$$

ascribing to each point $x \in X$ a closed subset C_x of $E^{\mathbb{N}}$, is called a random closed set if for each $\omega \in E^{\mathbb{N}}$ the function

$$X \ni x \mapsto \text{dist}(\omega, C_x) \in \mathbb{R}$$

is measurable. Since the probability measure m on X is assumed to be complete, being a closed random set precisely means (see Proposition 2.4 in [3]) that the union

$$\bigcup_{x \in X} \{x\} \times C_x \quad \text{(all sets } C_x \text{ are assumed to be closed)}$$

is a measurable subset of $X \times E^{\mathbb{N}}$. A random closed set $X \ni x \mapsto C_x$ is called a random compact set if all sets C_x, $x \in X$, are compact (in $E^{\mathbb{N}}$). A function $X \ni x \mapsto V_x$ is called a random open set if the function $X \ni x \mapsto E^{\mathbb{N}} \setminus V_x$ is a closed random set. The theorems about tightness announced above are these (see Proposition 4.3 in [3]).

Proposition 5.4. *A subset $\Gamma \subset \mathcal{P}_m(\Omega)$ is tight if and only if for every $\varepsilon > 0$ there exists a random compact set $X \ni x \mapsto K_x$ such that $K_x \subset E_x^\infty$ for all $x \in X$ and*

$$\int_X \nu_x(K_x)\, dm(x) \geq 1 - \varepsilon$$

for all $\nu \in \Gamma$.

As an immediate consequence of this proposition, we get the following.

Corollary 5.5. *Let Γ be a subset of $\mathcal{P}_m(\Omega)$. Suppose that for every $\varepsilon > 0$ there exists a random compact set $X \ni x \mapsto K_x$ such that $K_x \subset E_x^\infty$ for all $x \in X$ and*

$$\nu_x(K_x) \geq 1 - \varepsilon$$

for all $\nu \in \Gamma$ and for m-a.e. $x \in X$, then Γ is tight.

Passing to dynamics, i. e. to the fine random shift $\hat{\sigma} : \Omega \to \Omega$ and fine potential $\varphi : \Omega \to \mathbb{R}$, as indicated in the introduction, the measures we are looking for are defined in dynamical terms and form some special random measures on Ω.

Definition 5.6. *A random conformal measure is a measure $\nu = (\nu_x)_{x \in X} \in \mathcal{P}_m(\Omega)$ for which there exists a measurable function $X \ni x \mapsto \lambda_x \in (0, +\infty)$ such that*

$$(5.2) \qquad\qquad \mathcal{L}_x^* \nu_{\theta(x)} = \lambda_x \nu_x \quad \text{for m-a.e. } x \in X\,.$$

We can now present the main results of this paper.

Theorem 5.7. *Let the random shift $\hat{\sigma} : \Omega \to \Omega$ and the potential $\varphi : \Omega \to \mathbb{R}$ be fine. Then there exists a random conformal measure, i.e. a measure $(\nu_x)_{x \in X}$ that satisfies (5.2) for some measurable function $X \ni x \mapsto \lambda_x \in (0, +\infty)$. In addition,*

$$\|\log \lambda\|_\infty < \infty \,.$$

Our second main theorem is this.

Theorem 5.8. *Let the random shift $\hat{\sigma} : \Omega \to \Omega$ and the potential $\varphi : \Omega \to \mathbb{R}$ be fine. Then*

(1) *There exists a unique positive function $\rho \in \mathcal{H}_\alpha(\Omega)$ such that $\hat{\mathcal{L}}\rho = \rho$, which fiberwise means that*

$$\hat{\mathcal{L}}_x \rho_x = \rho_{\theta(x)} \quad \text{for } m\text{-a.e. } x \in X.$$

(2) *There exist constants $B > 0$ and $\vartheta \in (0, 1)$ such that for m-a.e. $x \in X$*

$$\left\| \hat{\mathcal{L}}_\omega^n g - \nu_\omega(g) \rho_{\theta^n(\omega)} \right\|_\alpha \leq B\vartheta^n \quad \text{for every } g \in \mathcal{H}_\alpha(X_\omega) \,.$$

As an immediate consequence of part (1) of this theorem, we get the following.

Corollary 5.9. *Let the random shift $\hat{\sigma} : \Omega \to \Omega$ and the potential $\varphi : \Omega \to \mathbb{R}$ be fine. Let $\rho \in \mathcal{H}_\alpha(\Omega)$ come from Theorem 5.8. Then for the fiber measures $\mu_x = \rho_x \nu_x \in \mathcal{P}(E_x^\infty)$, $x \in X$, we have that*

$$\mu_x \circ \sigma_x^{-1} = \mu_{\theta(x)},$$

and μ, the random measure on Ω with disintegration $(\mu_x)_{x \in X}$, is the unique $\hat{\sigma}$-invariant measure on Ω absolutely continuous with respect to ν. In addition, the two measures μ and ν are equivalent.

We want to conclude this section with two striking stochastic consequences of Theorem 5.7 and Theorem 5.8, particularly its item (2). We mean exponential decay of correlations and the Central Limit Theorem. These follow from Theorems 5.7 and 5.8 in an analogous way as corresponding theorems in [7] followed from its respective counterparts of Theorems 5.7 and 5.8. We therefore only formulate the exponential decay of correlations and the Central Limit Theorem inviting the interested readers to look for proofs to [7]. In order to formulate these theorems we need some new, more general, function spaces.

Given a number $0 < p \leq \infty$ let $\mathcal{H}_\alpha^p(\Omega)$ be the space of functions $g : \Omega \to \mathbb{R}$ with Hölder fibers $g_x \in \mathcal{H}_\alpha(E_x^\infty)$ and such that $\|g_x\|_\alpha \in L^p(m)$ for m-a.e. $x \in X$. The canonical norm on this spaces is

$$|g|_{\alpha,p} = \left(\int_X \|g_x\|_\alpha^p \, dm(x) \right)^{\frac{1}{p}} \,.$$

It makes $\mathcal{H}_\alpha^p(\Omega)$ a Banach space. Replacing in this definition the α–Hölder condition on the fiber E_x^∞ by a $L^1(\nu_x)$ condition leads to a space of functions that will be denoted by $L_\nu^{1,p}(\Omega)$. The natural norm is in this case

$$|g|_\nu^{1,p} = \left(\int_X \|g_x\|_{L^1(\nu_x)}^p \, dm(x) \right)^{\frac{1}{p}}.$$

Clearly, if $p = 1$ then $L_\nu^{1,1}(\Omega) = L^1(\nu)$. In both cases we also consider $p = \infty$ and then the L^p norms are understood as the sup–norms.

Theorem 5.10 (Exponential Decay of Correlations). *Let the random shift $\hat{\sigma} : \Omega \to \Omega$ and the potential $\varphi : \Omega \to \mathbb{R}$ be fine. Let μ be the corresponding $\hat{\sigma}$-invariant measure on Ω produced in Theorem 5.8. Let $p, q \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for every $g \in L_\nu^{1,p}(\Omega)$, $h \in \mathcal{H}_\alpha^q(\Omega)$ with $\int_{X_x^\infty} h_x \, d\mu_x = 0$ and for every $n \geq 1$, we have*

$$\left| \int_\Omega (g \circ \hat{\sigma}^n) \, h \, d\mu \right| = \left| \int_X \int_{E_x^\infty} (g_{\theta^n(x)} \circ \hat{\sigma}_x^n) \, h_x \, d\mu_x \, dm(x) \right| \leq b \vartheta^n \, |g|_\nu^{1,p} \, |h|_{\beta,q}$$

for some positive constant b and some $\vartheta \in (0, 1)$.

Theorem 5.11 (Central Limit Theorem). *Let the random shift $\hat{\sigma} : \Omega \to \Omega$ and the potential $\varphi : \Omega \to \mathbb{R}$ be fine. Let μ be the corresponding $\hat{\sigma}$-invariant measure on Ω produced in Theorem 5.8. Let $\psi \in \mathcal{H}_\alpha(\Omega)$ such that $\int_{E_x^\infty} \psi_x d\mu_x = 0$, $x \in X$. If ψ is not cohomologous to 0, then there exists $\sigma > 0$ such that, for every $t \in \mathbb{R}$,*

$$\mu\left(\left\{ z \in \mathcal{J} \,;\, \frac{1}{\sqrt{n}} S_n \psi(z) \leq t \right\} \right) \to \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^t exp(-u^2 / 2\sigma^2) \, du$$

6. The Existence of Conformal Measures

This section is devoted to prove Theorem 5.7. It follows from the invariance relation (5.2) that

$$\lambda_x = \int \mathcal{L}_x \mathbb{1} \, d\nu_{\theta(x)},$$

and so our task is to look for random measures $(\nu_x)_{x \in X}$ that are invariant under the map $\Phi : \mathcal{P}_m(\Omega) \to \mathcal{P}_m(\Omega)$ that is fiberwise defined as follows:

$$\Phi(\nu)_x := \frac{\mathcal{L}_x^* \nu_{\theta(x)}}{\mathcal{L}_x^* \nu_{\theta(x)}(\mathbb{1})}.$$

We want to obtain these measures in the usual way by employing Schauder–Tychonov Fixed Point Theorem. But, since the sets E_x^∞ need not be compact, this can be done only if a convex compact and Φ–invariant subset $\mathcal{M}$ of $\mathcal{P}_m(\Omega)$ can found, and if in addition the map Φ, restricted to this set, is continuous. Such a subspace will now be found with, in particular, the help of the assumption (C).

Remember that this assumption involves a finite union of cylinders F. Reenumerating $E = \mathbb{N}$ if necessary, we may assume that $F = \{0, 1, ..., q\}$ for some $q \geq 0$. We shall prove the following.

Lemma 6.1. *Let the random shift $\hat{\sigma} : \Omega \to \Omega$ and the potential $\varphi : \Omega \to \mathbb{R}$ be fine. Then, there exists $(\gamma_n)_{n=0}^{\infty}$, a sequence of measurable positive functions defined on X, with the following properties:*

(1) $\gamma_0(x) \equiv \frac{1}{2}$,

(2) $\lim_{n \to \infty} \gamma_n(x) = 0$ *for m-a.e. $x \in X$,*

(3) *there exists an ascending sequence of compact random sets, whose fibers we denote by $K_{n,x}$, $x \in X$, and a measurable point $x \mapsto \xi(x)$ such that*

(3.1) $\xi(x) \in K_{0,x} \subset [F]_x$ *for m-a.e. $x \in X$, and*

(3.2) *the set*

$$\mathcal{M} := \left\{ \nu \in \mathcal{P}_m(\Omega) \, , \ \nu_x(K_{n,x}^c) \leq \gamma_n(x) \ \text{for all} \ n \geq 0 \ \text{and a.e.} \ x \in X \right\}$$

is Φ–invariant, precisely meaning that $\Phi(\mathcal{M}) \subset \mathcal{M}$.

Proof. First of all, notice that $\mathcal{F} = \bigcup_{x \in X} \{x\} \times [F]_x$ is a closed random set with non-empty fibers $[F]_x$ (see Remark 2.2). Therefore, the Selection Theorem (Theorem 2.6 in [3]) implies that there exists a measurable point $\xi \in \mathcal{F}$, i.e. a measurable map $x \mapsto \xi(x) \in [F]_x$.

We will now define inductively a required sequence $(\gamma_n)_{n=0}^{\infty}$. During this inductive process we will also define an auxiliary sequence $\left(X \ni x \mapsto N_n(x) \in \mathbb{N} \right)_{n=0}^{\infty}$ of integer-valued functions meeting the following properties:

(1) the functions $N_n(x)$ are measurable for all integers $n \geq 0$.

(2) $\left(N_n(x) \right)_{n=0}^{\infty}$ is an increasing sequence for every $x \in X$.

(3) $N_0 \equiv q$ and $N_n(x) \geq \xi_n(x)$ for all integres $n \geq 0$ and all $x \in X$,

where $\xi_n(x)$ is the n–th coordinate of the measurable function ξ define just above.

Assuming that such a sequence $N_n(x)$ is given, define further

$$N_{j,n-j}(x) := N_n(x)$$

for all $j = 0, 1, ..., n$, and define also

(6.1) $$K_{n,x} := \left\{ \omega \in E_x^{\infty} : \omega_i \leq N_{i,n}(\theta^i(x)) \ \text{for all} \ i \geq 0 \right\}.$$

Notice that these sets $K_{n,\omega}$ are compact subsets of E_x^{∞} and that, due to the measurability of the functions N_n, the family $(K_{n,x})_{x \in X}$ forms a compact random set. Moreover,

$$\xi(x) \in K_{0,x} \subset [F]_x \quad \text{and} \quad K_{n,x} \subset K_{n+1,x}$$

for all integres $n \geq 0$ and all $x \in X$. In order to demonstrate Φ–invariance of the set $\mathcal{M}$, let us first make the following observation. Let $\tau \in K_{n+1,\theta(x)}$ and $j \in E$ be

such that $j\tau \in E_x^\infty$ or, equivalently, such that $A_{j\tau_0}(x) = 1$. Then $\mathbb{1}_{K_{n,x}^c}(j\tau) = 1$ if and only if either

$$j > N_n(x) \quad \text{or else} \quad \tau_i > N_{i+1,n}(\theta^{i+1}(x)) = N_{n+i+1}(\theta^{i+1}(x)) \quad \text{for some } i \geq 0 \,.$$

On the other hand, $\tau_i \leq N_{i,n+1}(\theta^i(\theta(x))) = N_{n+i+1}(\theta^{i+1}(x))$ since $\tau \in K_{n+1,\theta(x)}$. Consequently, $\tau \in K_{n+1,\theta(x)}$ and $\mathbb{1}_{K_{n,x}^c}(j\tau) = 1$ if and only if $j > N_n(x)$. This allows us to make the following estimation valid for all $\tau \in K_{n+1,\theta(x)}$:

$$(6.2) \qquad \mathcal{L}_x \mathbb{1}_{K_{n,x}^c}(\tau) = \sum_{\substack{j \in E \\ j\tau \in E_x^\infty}} e^{\varphi_x(j\tau)} \mathbb{1}_{K_{n,x}^c}(j\tau) = \sum_{\substack{j > N_n(x) \\ j\tau \in E_x^\infty}} e^{\varphi_x(j\tau)} \leq Z_x(N_n(x))$$

where, for every $k \geq 0$ and every $x \in X$,

$$Z_x(k) := \sup\left(\mathcal{L}_x \mathbb{1}_{[0,1,\dots,k]_v^c}\right) = \sup_{\tau \in E_{\theta(x)}} \left(\sum_{\substack{j > k \\ j\tau \in E_x^\infty}} e^{\varphi_x(j\tau)}\right).$$

Summability of the potential φ (see Definition 4.1) implies that

$$(6.3) \qquad\qquad\qquad \lim_{k \to \infty} Z_x(k) = 0 \quad \text{for} \quad m - a.e. \ x \in X.$$

Take now an arbitrary $\nu \in \mathcal{M}$. Since

$$\mathcal{L}_x^* \nu_{\theta(x)}(K_{n,\omega}^c) = \int \mathcal{L}_x \mathbb{1}_{K_{n,x}^c} d\nu_{\theta(x)}$$

$$= \int_{K_{n+1,\theta(x)}^c} \mathcal{L}_x \mathbb{1}_{K_{n,x}^c} d\nu_{\theta(x)} + \int_{K_{n+1,\theta(x)}} \mathcal{L}_x \mathbb{1}_{K_{n,x}^c} d\nu_{\theta(x)} \,,$$

it follows from summability of φ and from (6.2) that

$$(6.4) \qquad \begin{aligned} \mathcal{L}_x^* \nu_{\theta(x)}(K_{n,x}^c) &\leq \|\mathcal{L}_x \mathbb{1}\|_\infty \nu_{\theta(x)}(K_{n+1,\theta(x)}^c) + Z_x(N_n(x))\nu_{\theta(x)}(K_{n+1,\theta(x)}) \\ &\leq M\gamma_{n+1}(\theta(x)) + Z_x(N_n(x)). \end{aligned}$$

On the other hand, since $\nu \in \mathcal{M}$, we have

$$\mathcal{L}_x^* \nu_{\theta(x)}(\mathbb{1}) \geq \int_{K_{0,\theta(x)}} \mathcal{L}_x \mathbb{1} d\nu_{\theta(x)} \geq \nu_{\theta(x)}(K_{0,\theta(x)}) \inf_{K_{0,\theta(x)}} \left(\mathcal{L}_x \mathbb{1}\right)$$

$$\geq \frac{1}{2} \inf_{K_{0,\theta(x)}} \left(\mathcal{L}_x \mathbb{1}\right)$$

$$\geq \frac{1}{2} \inf_{[F]_{\theta(x)}} \left(\mathcal{L}_x \mathbb{1}\right) := c > 0$$

For some constant $c > 0$ because of (A). Together with (6.4) this leads to

$$\Phi(\nu)_x \left(K_{n,x}^c\right) \leq \frac{1}{c}\left(M\gamma_{n+1}(\theta(x)) + Z_x(N_n(x))\right).$$

Therefore, assuming that a measurable function γ_n, is given, if we can find measurable functions γ_{n+1} and N_n such that

$$(6.5) \qquad M\gamma_{n+1}(\theta(x)) + Z_x(N_n(x)) \leq c\gamma_n(x),$$

then the proof of our lemma will be complete.

Let us first consider the case of $n = 0$. Since we have put $\gamma_0 = 1/2$ and $N_0 = q$, formula (6.5) then takes on the form

$$M\gamma_1(\theta(x)) + Z_x(q) \leq \frac{1}{4} \inf_{[F]_{\theta(x)}} \left(\mathcal{L}_x \mathbb{1} \right).$$

Therefore, condition (C) yields that it is sufficient to take measurable $\gamma_1(\theta(x))$ such that

$$M\gamma_1(\theta(x)) < \left(\frac{1}{4} - \kappa \right) \inf_{[F]_{\theta(x)}} \left(\mathcal{L}_x \mathbb{1} \right).$$

This is possible as $\kappa < 1/4$.

Suppose finally that, for some $n \geq 0$, a measurable function γ_n, is given. Then, because of (6.3), one can find a measurable function $N_n \geq \xi_n$ such that $Z_x(N_n(x)) \leq \frac{1}{2}c\gamma_n(x)$. So, setting then

$$\gamma_{n+1}(x) := M^{-1}\big(c\gamma_n(\theta^{-1}(x)) - Z_{\theta^{-1}(x)}(N_n(\theta^{-1}(x)))\big)$$

defines a measurable function from X to $(0, +\infty)$ for which (6.5) holds. The proof is complete. □

The set of measures $\mathcal{M}$ produced in this lemma is not only Φ-invariant but it has all the required properties.

Lemma 6.2. *The set $\mathcal{M}$ is non-empty, convex, and compact.*

Proof. With the same notation as in Lemma 6.1, we have $\xi(x) \in K_{0,x} \subset K_{n,x}$ for every $n \geq 0$ and a.e. $x \in X$. Consider then the measure ν defined fiberwise by $\nu_x = \delta_{\xi(x)}$, the Dirac δ-measure supported at $\xi(x)$, $x \in X$. Then obviously $\nu_x(K_{n,x}^c) = 0 \leq \gamma_n(x)$, which shows that $\nu \in \mathcal{M}$ and thus $\mathcal{M} \neq \emptyset$. Convexity of $\mathcal{M}$ is obvious. In order to prove compactness of $\mathcal{M}$, we shall show that $\mathcal{M}$ is closed and tight.

Tightness first. Fix $\varepsilon > 0$ arbitrary. Since, by item (2) of Lemma 6.1, the sequence $(\gamma_n)_n$ converges pointwise to 0, there exists an integer $k = k(\varepsilon) \geq 0$ such that $m\big(\gamma_k^{-1}([\varepsilon/2, +\infty))\big) < \varepsilon/2$. Then, for every $\nu \in \mathcal{M}$ we have that

$$\int_X \nu_\omega \left(K_{k,x}^c \right) dm(x) = \int_{\gamma_k^{-1}([\varepsilon/2,\infty))} \nu_\omega \left(K_{k,x}^c \right) dm(x) + \int_{\gamma_k^{-1}([0,\varepsilon/2))} \nu_\omega \left(K_{k,x}^c \right) dm(x)$$

$$\leq m(\gamma_k^{-1}([\varepsilon/2, \infty))) + \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, the set $\mathcal{M}$ is tight (see Proposition 5.4).

To see that $\mathcal{M}$ is closed let Λ be a directed set and $(\nu^\alpha)_{\alpha \in \Lambda}$ a net in $\mathcal{M}$ converging to a measure $\nu \in \mathcal{P}_m(\Omega)$ in the narrow topology. Let H be an arbitrary measurable subset of X. Then, for every integer $n \geq 0$, the function

$$X \ni x \mapsto \begin{cases} K_{n,x}^c & \text{if } x \in H \\ \emptyset & \text{if } x \notin H \end{cases}$$

defines a random open set. It then follows from Portmenteau's Theorem (see Theorem 3.14 (iv) in [3]) that

$$\int_H \nu_x\left(K_{n,x}^c\right) dm(x) = \nu\left(\bigcup_{x \in H} \{x\} \times K_{n,x}^c\right) \leq \varliminf_{\alpha \in \Lambda} \nu_\alpha\left(\bigcup_{x \in H} \{x\} \times K_{n,x}^c\right)$$

$$= \varliminf_{\alpha \in \Lambda} \int_H \nu_{\alpha,x}\left(K_{n,x}^c\right) dm(x)$$

$$\leq \int_H \gamma_n(x)\, dm(x).$$

Hence, arbitrariness of H yields $\nu_x\left(K_{n,x}^c\right) \leq \gamma_n(x)$ for m-a.e. $x \in X$. Thus, $\nu \in \mathcal{M}$, yielding closeness of $\mathcal{M}$.

So, since the set $\mathcal{M}$ is closed and tight, its compactness follows from Prohorov's Compactness Theorem (Theorem 4.4 in [3]). The proof is complete. $\square$

The last step in the proof of Theorem 5.7 is the following.

Proposition 6.3. *Let $\mathcal{M}$ be the invariant set of random measures coming from Lemma 6.1. Then the map $\Phi : \mathcal{M} \to \mathcal{M}$ is continuous with respect to the narrow topology.*

Proof. Suppose that Λ is a directed set and $(\nu^\alpha)_{\alpha \in \Lambda}$ is a net in $\mathcal{P}_m(\Omega)$ converging to some measure $\nu \in \mathcal{M}$ in the narrow topology. If $h_{\theta(x),\alpha} := 1/\nu_{\theta(x)}^\alpha(\mathcal{L}_x \mathbb{1})$, then

$$\Phi(\nu^\alpha)_x = \mathcal{L}_x^*\left(\frac{1}{\nu_{\theta(x)}^\alpha(\mathcal{L}_x \mathbb{1})} \nu_{\theta(x)}^\alpha\right) = \mathcal{L}_x^*\left(h_{\theta(x),\alpha}\, \nu_{\theta(x)}^\alpha\right).$$

We have to estimate $h_{\theta(x),\alpha}$. Since $K_{0,\theta(x)} \subset [F]_x$ and since $\nu^\alpha \in \mathcal{M}$, we have

$$\nu_{\theta(x)}^\alpha(\mathcal{L}_x \mathbb{1}) \geq \int_{K_{0,\theta(x)}} \mathcal{L}_\omega \mathbb{1}\, d\nu_{\theta(x)}^\alpha \geq \inf\left(\left.(\mathcal{L}_x \mathbb{1})\right|_{[F]_x}\right) \nu_{\theta(x)}^\alpha(K_{0,\theta(x)}) \geq \frac{1}{2c_F}$$

where $c_F^{-1} = \operatorname{essinf}\left(\inf\left(\left.(\mathcal{L}_x \mathbb{1})\right|_{[F]_x}\right) : x \in X\right) > 0$ by (A) since the set F is finite. Therefore,

(6.6) $$1/M \leq h_{\theta(x),\alpha} \leq 2c_F.$$

The measures $h_\alpha \nu^\alpha$, $\alpha \in \Lambda$, need not be random probability measures but, thanks to (6.6), their fiber total values $h_{\theta(x),\alpha} \nu_{\theta(x)}^\alpha(E_x^\infty)$ are uniformly bounded away from

zero and ∞. Since $\mathcal{M}$ is compact, it follows that $\{h_\alpha \nu^\alpha : \alpha \in \Lambda\}$ is a tight family. Let μ be an arbitrary accumulation point of this net. It has been shown (as a matter of fact for sequences but the same argument works for all nets) in Lemma 2.9 of [8] that then

$$\mu = h\nu$$

for some measurable function $h : X \to (0, \infty)$. Since the dual operator $\mathcal{L}^* : \mathcal{C}_b^*(\Omega) \to \mathcal{C}_b^*(\Omega)$ is continuous, the measure $\mathcal{L}^*\mu$ is an accumulation point of the net $\big(\mathcal{L}^*\big(h_\alpha \nu^\alpha\big) : \alpha \in \Lambda\big)$. But all elements of this net belong to $\mathcal{P}_m(\Omega)$, whence $\mathcal{L}^*\mu \in \mathcal{P}_m(\Omega)$. This means that the disintegrations of this measure

$$\mathcal{L}_x^*\mu_{\theta(x)} = h_{\theta(x)}\mathcal{L}_x^*(\nu_{\theta(x)}), \ x \in X,$$

are all (Borel) probability measures in respective spaces E_x^∞. Therefore,

$$1 = \mathcal{L}_x^*\mu_{\theta(x)}(\mathbb{1}) = h_{\theta(x)}\mathcal{L}_x^*(\nu_{\theta(x)})(\mathbb{1}),$$

which implies that

$$\mu_{\theta(x)} = \frac{1}{\mathcal{L}_x^*(\nu_{\theta(x)})(\mathbb{1})}\nu_{\theta(x)} \ , \quad x \in X \,.$$

This means that $\mathcal{L}^*\mu = \Phi(\nu)$. Since also $\big(\mathcal{L}^*\big(h_\alpha \nu^\alpha\big) : \alpha \in \Lambda\big) = \big(\Phi(\nu^\alpha) : \alpha \in \Lambda\big)$, we thus conclude that the net $\big(\Phi(\nu^\alpha) : \alpha \in \Lambda\big)$ converges to $\Phi(\nu)$. The proof of continuity of Φ is complete. $\hspace{1cm}\square$

Proof of Theorem 5.7. Consider the vector space $C_b^*(\Omega)$ but now endowed with the narrow (weak convergence) topology analogously as for $\mathcal{P}_m(X)$. It is standard to see that $C_b^*(\Omega)$ becomes then a locally convex topological vector space. Since, by Lemma 6.2, $\mathcal{M}$ is a non-empty convex compact subset of $C_b^*(\Omega)$, since, by Lemma 6.1, $\mathcal{M}$ is Φ–invariant and since, by Proposition 6.3, Φ continuous, the Schauder-Tichonov Fixed Point Theorem applies and yields a fixed point ν of Φ in $\mathcal{M}$. But being such a fixed point $\nu \in \mathcal{M}$ precisely means being a random conformal measure. Finally, $\lambda_x = \mathcal{L}_x^*\nu_{\theta(x)}(E_x^\infty)$, and thus $\|\log \lambda\|_\infty < \infty$ results from the estimate (6.6). $\hspace{1cm}\square$

Here and in the sequel $\nu \in \mathcal{M}$ is a conformal measure obtained in Lemma (6.1). We will frequently need the following useful estimate.

Lemma 6.4. *Suppose that $\hat{\sigma} : \Omega \to \Omega$ and $\varphi : \Omega \to \mathbb{R}$ are fine. Then, for every finite set $D \subset E$ and for every integer $n \geq 0$ there exists a constant $\beta = \beta_{D,n} > 0$ such that*

$$\nu_x([\omega]_x) \geq \beta$$

for every integer $0 \leq k \leq n$, every tuple $\omega \in D_x^n$ and m-a.e. $x \in X$.

Proof. It is enough to show this statement for $k = n$. We keep the same notation as in Lemma 6.1 and its proof. Since $\nu \in \mathcal{M}$, by Lemma (6.1) (1) and (3.2), we have $\nu_x(K_{0,x}) \geq 1 - \gamma_0(x) = 1/2$ for m-a.e $x \in X$. Along with item (3.1) of

Lemma (6.1), this yields $\nu_x([F]_x) \geq 1/2$ for m-a.e. $x \in X$. In consequence, for each such x there exists $b_x \in F$ such that

(6.7) $$\nu_x([b_x]_x) \geq \frac{1}{2q}, \quad \text{where} \quad q = \#F \,.$$

Since both sets D and F are finite, by the topological mixing property of the map $\hat{\sigma} : \Omega \to \Omega$, there exists an integer $N \geq 0$ such that, for every $x \in X$, every $a \in D$ and every $b \in F$ can be connected by a word $\tau = \tau(a, b) \in E^N_{\theta(x)}$, i.e. $axb \in E^{N+2}_x$. We will use this fact in what follows with $b = b_x$, the point defined just above, see (6.7).

Let $\omega \in D^n_x$ be a word of length $n+1$ over the alphabet D and let $b := b_{\theta^{n+N}(x)}$. Let also $\tau := \tau(\omega_n, b) \in E^N_{\theta^{n+1}(x)}$. Then, by conformality of ν,

$$\nu_x([\omega]_x) \geq \nu([\omega\tau b]_x) \geq \lambda_x^{-(n+N)} \exp\left(\inf\left(S_{n+N}\varphi_{\big|[\omega\tau b]_x}\right)\right) \nu_{\theta^{n+N}(x)}[b_{\theta^{n+N}(x)}] \,.$$

Notice that

$$Q_n := \inf \left\{ \exp\left(S_{n+N}\varphi_{\big|[\omega\tau b]_x}\right) : \ x \in X, \ \omega \in D^n_x \right\}$$

is positive because of (4.1) since D is finite and $b_{\theta^{n+N}(\omega)} \in F$ varies in a finite set too. Therefore,

$$\nu_x([\omega]_x) \geq \frac{1}{2q} \exp\left(-(n+N)\|\log \lambda\|_\infty\right) Q_n > 0 \,.$$

The proof is complete. □

7. UNIFORM BOUNDS AND TWO-NORM INEQUALITY

In this section we establish uniform bounds for the iterated normalized operators

$$\hat{\mathcal{L}}^n_x = \lambda_x^{-n}\mathcal{L}^n_x \quad \text{where} \quad \lambda^n_x = \lambda_x...\lambda_{\theta^{n-1}(x)} \,.$$

We will then conclude from these estimates the existence of a random invariant measure equivalent to the random conformal measure produced in the previous section. From now on, the map $\hat{\sigma} : \Omega \to \Omega$ and the potential $\varphi : \Omega \to \mathbb{R}$ are assumed again to be fine. We start with the following.

Proposition 7.1. *There exisits a constant $\hat{M} < +\infty$ such that*

$$\|\hat{\mathcal{L}}^n_x\|_\infty = \|\hat{\mathcal{L}}^n_x \mathbb{1}\|_\infty \leq \hat{M} \quad \text{for every } n \geq 0 \text{ and every } x \in X.$$

Proof. The first equality is obvious. Conformality of ν and the distortion result, Lemma 4.9, yield, for every $\omega \in E^\infty_{\theta^n(x)}$,

$$1 = \int_{E^\infty_{\theta^n(x)}} \hat{\mathcal{L}}^n_x \mathbb{1} \, d\nu_{\theta^n(x)} \geq \frac{1}{K}\hat{\mathcal{L}}^n_x \mathbb{1}(\omega) \nu_{\theta^n(x)}\left([\omega_0]_{\theta^n(x))}\right)$$

Given $l \geq 1$, let $\beta := \beta_{\{0,1,\dots,l\},0}$ come from Lemma 6.4. Then $\nu_{\theta^n(x)}\big([\omega_0]_{\theta^n(x)}\big) \geq \beta$ if $\omega_0 \in \{0,\dots,l\}$. For such points ω we therefore get

$$(7.1) \qquad\qquad \hat{\mathcal{L}}_\omega^n \mathbb{1}(\omega) \leq K/\beta\,.$$

By the property (B) and since $\log \lambda \in L^\infty$, we can adjust (increasing if necessary) the integer l such that

$$(7.2) \qquad\qquad \hat{\mathcal{L}}\mathbb{1}_x(\omega) \leq 1$$

for m-a.e. $x \in X$ if $\omega \in E_{\theta(x)}^\infty$ and $\omega_0 > l$. We shall prove by induction that

$$(7.3) \qquad\qquad \|\hat{\mathcal{L}}_x\mathbb{1}\|_\infty \leq K/\beta$$

for m-a.e. $x \in X$. For $n = 1$ this directly results from (7.1) and (7.2). For the inductive step, assume that (7.3) holds for some $n \geq 1$. For all points $\omega \in E_{\theta^{n+1}(x)}^\infty$ with $\omega_0 \leq l$ everything is fine due to (7.1). So, suppose that $\omega_0 > l$. Then (7.2) along with the inductive hypothesis yield,

$$\hat{\mathcal{L}}_x^{n+1}\mathbb{1}(\omega) = \hat{\mathcal{L}}_{\theta^n(x)}\left(\hat{\mathcal{L}}_x^n\mathbb{1}\right)(\omega) \leq (K/\beta)\hat{\mathcal{L}}_{\theta^n(x)}\mathbb{1}(\omega) \leq K/\beta\,.$$

The proof is complete by taking $\hat{M} := K/b$. $\qquad\qquad\qquad\qquad\qquad\square$

Having this uniform bound and the distortion Lemma 3.1, we can now obtain the following two-norm inequality.

Lemma 7.2. *There exists a constant $S < \infty$ such that*

$$v_\alpha(\hat{\mathcal{L}}_x^n g) \leq S\Big(\|g\|_\infty + e^{-\alpha n}v_\alpha(g)\Big)\,.$$

for every $x \in X$, every integer $n \geq 0$ and every $g \in \mathcal{H}_\alpha\big(E_x^\infty\big)$.

Proof. Let $x \in x$, $n \geq 0$, $g \in \mathcal{H}_\alpha\big(E_x^\infty\big)$ and let $\omega, \tau \in E_{\theta^n(x)}^\infty$ with $\omega_0 = \tau_0 \in E$. Then,

$$|\hat{\mathcal{L}}_x^n g(\omega) - \hat{\mathcal{L}}_x^n g(\tau)| \leq \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_2 := \lambda_x^{-n} \sum_{\substack{\gamma \in E_x^n \\ \gamma\omega_0 \in E_x^{n+1}}} \exp\big(S_n\varphi(x,\gamma\tau)\big)|g(\gamma\omega) - g(\gamma\tau)| \leq \hat{\mathcal{L}}_x^n\mathbb{1}(\tau)v_\alpha(g)e^{-\alpha n}d^\alpha(\omega,\tau)^\alpha$$

and

$$\Sigma_1 := \lambda_x^{-n} \sum_{\substack{\gamma \in E_x^n \\ \gamma\omega_0 \in E_x^{n+1}}} \Big|\exp\big(S_n\varphi(x,\gamma\omega)\big) - \exp\big(S_n\varphi(x,\gamma\tau)\big)\Big||g(\gamma\omega)|$$

$$\leq \|g\|_\infty \lambda_x^{-n} \sum_{\substack{\gamma \in E_x^n \\ \gamma\omega_0 \in E_x^{n+1}}} \exp\big(S_n\varphi(x,\gamma\omega)\big)\left|1 - \frac{\exp\big(S_n\varphi(x,\gamma\tau)\big)}{\exp\big(S_n\varphi(x,\gamma\omega)\big)}\right|$$

$$\leq \|g\|_\infty \|\hat{\mathcal{L}}_x^n\mathbb{1}\|_\infty CV_\alpha(\varphi)d(\omega,\tau)^\alpha,$$

where the last inequality results from Lemma 3.1 and the fact that $\varphi : \Omega \to \mathbb{R}$ is essentially α-Hölder. It suffices now to combine the above estimates of both terms Σ_1 and Σ_2 along with the uniform bound from Proposition 7.1. $\qquad\square$

As an immediate consequence of the preceding lemma and of Proposition 7.1, we get the following.

Proposition 7.3. *For every integer $n \geq 0$ and m-a.e. $x \in X$, we have that*
$$\|\hat{\mathcal{L}}_x^n\|_\alpha \leq \hat{M},$$
where here $\hat{M}$ is the maximum of S and the former $\hat{M}$.

8. Invariant positive cones, Bowen's contraction and the spectral gap

From the uniform bounds obtained in the previous section one can construct immediately invariant densities and measures by bare hands. However, in order to get further, more sophisticated, properties, especially a spectral gap, additional investigations are needed. Since the phase spaces E_x^∞ are not compact, we are here in a situation similar to that of random iterations of transcendental functions considered in [7]. Therefore, we will now introduce, apply and develop analogues of these tools worked out in the context of transcendental situation. Particularly, appropriated invariant positive cones and employment of a Bowen's type contraction. Doing this, the invariant densities, hence the invariant measures equivalent to conformal measures, will emerge in a sense for free.

8.1. **Invariant cones.** Consider the following cones:
(8.1)
$$\mathcal{C}_x := \left\{ g : E_x^\infty \to [0, +\infty) \ : \ \|g\|_\infty \leq \mathcal{A} \int g \, d\nu_x < \infty \ \text{ and } \ v_\alpha(g) \leq H \int g \, d\nu_x \right\}.$$

$$(8.2) \qquad \mathcal{C}_{x,0} := \left\{ g \in \mathcal{C}_x \ : \ g \leq 2\hat{M}\mathcal{A} \left(\int g \, d\nu_x \right) \hat{\mathcal{L}}_{\theta^{-1}(x)} \mathbb{1} \right\}.$$

Since we are primarily interested in the projective features of these cones, it is convenient to use the following slices
$$(8.3) \qquad \Lambda_x = \{ g \in \mathcal{C}_x , \ \nu_x(g) = 1 \} \quad \text{and} \quad \Lambda_{x,0} = \Lambda_x \cap \mathcal{C}_{x,0} \ , \ \ x \in X \, .$$

The constant $\hat{M}$, which is here and in the sequel, still forms a common upper bound of Proposition 7.1 and Proposition 7.3. Both type of cones do depend on the constants $\alpha \leq 1$, $\mathcal{A} > 0$, $H > 0$. Those must be chosen carefully in the sequel in order to obtain invariant cones.

We continue to write C for the distortion constant appearing in Lemma 3.1 and Lemma 4.9. First of all, fix an integer $k \geq 1$ so large that
$$(8.4) \qquad\qquad\qquad \frac{1}{2} + \left(2\hat{M} + 4\right)e^{-\alpha k} \leq 1 \, .$$

Then, property (B) and the fact that $\|\log\lambda\|_\infty < \infty$ yield the existence of an integer $l_0 \geq 0$ such that

$$(8.5) \qquad 2\hat{M}\hat{\mathcal{L}}_x \mathbb{1}(x) \leq 1 \quad \text{for all } \omega \in E^\infty_{\theta(x)} \text{ with } \omega_0 \geq l_0$$

for m-a.e. $x \in X$. Let $\hat{l}_0 \geq 0$ be the integer associated to l_0 by Lemma 2.4. Iterating this procedure define further

$$l_1 := \hat{l}_0\,, \quad l_2 := \hat{l}_1 \ ... \ l_k := \hat{l}_{k-1}\,.$$

Set

$$(8.6) \qquad \mathcal{K}_x := \{\omega \in E^\infty_x \ : \ \omega_j \leq l_j \ \text{ for all } \ j = 0, 1, \ldots, k\}\,, \quad x \in X\,.$$

Then, for every $\omega \in E^\infty_x \setminus \mathcal{K}_x$, we have that $\omega_0 > l_0$.

Now, if $D = \{0, ..., \max_{0 \leq j \leq k} l_j\}$ and $\beta = \beta_{D,k}$ is from Lemma 6.4, define

$$(8.7) \qquad \mathcal{A} := 2\max\{1, \hat{M}, \beta^{-1}\} \quad \text{and} \quad H := 2\hat{M}\mathcal{A} + 4\,.$$

Notice that $\mathcal{A} \geq 1$. This ensures that the constant function $\mathbb{1} \in \mathcal{C}_x$, $x \in X$. Let finally $N_0 \geq 1$ be such that

$$(8.8) \qquad \hat{M}He^{-\alpha N_0} \leq 1\,.$$

Proposition 8.1. *With the above choice of constants and for every $n \geq N_0$, we have*

$$\hat{\mathcal{L}}^n_x(\mathcal{C}_x) \subset \mathcal{C}_{\theta^n(x),0} \subset \mathcal{C}_{\theta^n(x)}\,, \quad x \in X\,.$$

Proof. Let $g \in \mathcal{C}_x$. We may assume without loss of generality that $\int g d\nu_x = 1$. Fix $n \geq N_0$ arbitrarily. We will show that $\hat{\mathcal{L}}^n_x g \in \mathcal{C}_{\theta^n(x),0}$. Clearly $\hat{\mathcal{L}}^n_x g \geq 0$. From the two-norm type inequality in Lemma 7.2 and from the definition of the cones, we get that

$$(8.9) \qquad v_\alpha(\hat{\mathcal{L}}^n_x g) \leq \hat{M}\left(\mathcal{A} + e^{-\alpha n}H\right) \leq \hat{M}\mathcal{A} + 1 \leq H,$$

where the last two inequalities result from the choice of N_0, i.e. (8.8), and from the definition of H. Since $\nu_{\theta^n(x)}\left(\hat{\mathcal{L}}^n_x g\right) = \nu_x(g) = 1$, the second condition for belonging to $\mathcal{C}_{\theta^n(x)}$ is thus satisfied by the function $\hat{\mathcal{L}}^n_x g$. Also, by Proposition 7.1,

$$(8.10) \qquad \hat{\mathcal{L}}^n_x g = \hat{\mathcal{L}}_{\theta^{n-1}(x)}\left(\hat{\mathcal{L}}^{n-1}_x g\right) \leq \hat{M}\|g\|_\infty \hat{\mathcal{L}}_{\theta^{n-1}(x)}\mathbb{1} \leq \hat{M}\mathcal{A}\hat{\mathcal{L}}_{\theta^{n-1}(x)}\mathbb{1}\,.$$

Hence, in order to conclude that $\hat{\mathcal{L}}^n_x g \in \mathcal{C}_{\theta^n(x),0}$ it remains to estimate $\|\hat{\mathcal{L}}^n_x g\|_\infty$ from above by $\mathcal{A}$. Since we already have (8.9), we obtain, for every $\omega \in \mathcal{K}_{\theta^n(x)}$, the following:

$$\begin{aligned}
1 = \int \hat{\mathcal{L}}^n_x g \, d\nu_{\theta^n(x)} &\geq \int_{[\omega|_k]_{\theta^n(x)}} \hat{\mathcal{L}}^n_x g \, d\nu_{\theta^n(x)} \\
&\geq \left(\hat{\mathcal{L}}^n_x g(\omega) - He^{-\alpha k}\right)\nu_{\theta^n(x)}([\omega|_k]_{\theta^n(x)}) \\
&\geq \left(\hat{\mathcal{L}}^n_x g(\omega) - He^{-\alpha k}\right)\beta\,,
\end{aligned}$$

where the last inequality was written due to Lemma 6.4. Therefore,

$$\hat{\mathcal{L}}_x^n g(\omega) \leq \beta^{-1} + He^{-\alpha k} \leq \mathcal{A}\left(\frac{1}{2} + (2\hat{M}+4)e^{-\alpha k}\right) \leq \mathcal{A},$$

by (8.7) and (8.4). If, on the other hand, $\omega \notin \mathcal{K}_{\theta^n(x)}$, then $\omega_0 \geq l_0$ and it follows from (8.10) and (8.5) that

$$\hat{\mathcal{L}}_x^n g(\omega) \leq \hat{M}\mathcal{A}\hat{\mathcal{L}}_{\theta^{n-1}(x)} 1\!\!1(\omega) \leq \mathcal{A}.$$

The proof is complete. $\qquad\qquad\square$

8.2. Cone Contraction via Bowen's Lemma.

This part is based on a slightly stronger version of (8.5). Increasing $l_0 \geq 1$ if necessary we may assume that

$$(8.11) \qquad 2\mathcal{A}\hat{M}\hat{\mathcal{L}}_x 1\!\!1(\omega) < 1 \quad \text{for all } \omega \in E_{\theta(x)}^\infty \text{ with } \omega_0 \geq l_0$$

for m-a.e. $x \in X$. We shall prove the following.

Lemma 8.2. *For every $l \geq l_0$ there are $N = N_l \geq N_0$ and $a = a_l > 0$ such that*

$$\hat{\mathcal{L}}_x^N g\big|_{[0,...,l]_{\theta^n(x)}} \geq a \quad \text{for every } g \in \Lambda_{x,0} \text{ and } x \in X.$$

Proof. Let $g \in \Lambda_{x,0}$. Since $\int g d\nu_x = 1$, we have that $\|g\|_\infty \geq 1$. On the other hand, the choice of l_0 implies that

$$g \leq 2\hat{M}\mathcal{A}\,\hat{\mathcal{L}}_{\theta^{-1}(x)} 1\!\!1 \leq 1 \quad \text{in} \quad \{\omega \in E_x^\infty : \omega_0 \geq l_0\}.$$

Thus, there exists $\hat{\omega} \in [0,...,l_0]_x$ with $g(\hat{\omega}) \geq 1$. Let $q \geq 1$ be so large that $He^{-\alpha q} \leq 1/2$. Since our random shift $\hat{\sigma} : \Omega \to \Omega$ is fine, hence topologically mixing, there exists $N = N_l \geq N_0$ such that $\hat{\sigma}^N\big([\hat{\omega}|_q]_x\big) \supset [0,...,l_0]_{\theta^N(x)}$. So, fixing an arbitrary $\tau \in [0,...,l_0]_{\theta^N(x)}$, there exists $\hat{\tau} \in [\hat{\omega}|_q]_x \cap \hat{\sigma}^{-N}(\tau)$. We therefore get that

$$\hat{\mathcal{L}}_x^N g(\tau) \geq \lambda_x^{-N} e^{S_N \varphi_x(\hat{\tau})} g(\hat{\tau}).$$

Now, $g(\hat{\tau}) \geq g(\hat{\omega}) - He^{-\alpha q} \geq 1 - (1/2) = 1/2$ and $\lambda_x^{-N} e^{S_N \varphi_x(\hat{\tau})}$ is bounded away from zero because of Lemma 4.5 and inequality $\|\log \lambda\|_\infty < +\infty$ which is a part of Theorem 5.7. This shows the existence of a required constant $a > 0$ and the proof is complete. $\qquad\qquad\square$

Notice that there is no way to get a global version of Lemma 8.2, valid on the whole shift space Ω. This is why we have to work with the following truncated functions:

$$\chi_{l,x} := 1\!\!1\big|_{[0,...,l]_x}\hat{\mathcal{L}}_{\theta^{-1}(x)} 1\!\!1 \,, \quad l \geq 0 \text{ and } x \in X.$$

Then,

$$(8.12) \qquad 0 \leq \chi_{l,x} \leq \hat{\mathcal{L}}_{\theta^{-1}(x)} 1\!\!1 \quad \text{and} \quad v_\alpha(\chi_{l,x}) \leq v_\alpha(\hat{\mathcal{L}}_{\theta^{-1}(x)} 1\!\!1).$$

Moreover, we may suppose that $l_0 \geq 1$ is sufficiently large so that

$$\chi_{l,x} \in \mathcal{C}_{x,0} \quad \text{for all} \quad l \geq l_0 \text{ and } m\text{-a.e. } x \in X.$$

We will now obtain a version of Bowen's contraction lemma [2, Lemma 1.9]. In order to do so, we have to define one more constant: let $\eta > 0$ be such that

$$(8.13) \qquad 0 < \eta \leq \min\left\{ \frac{1}{3}, \ \frac{1}{H}, \ \frac{1}{2}\frac{a}{\hat{M}} \right\} .$$

Lemma 8.3. *For every $l \geq l_0$ and with $N = N_l$ given by Lemma 8.2,*

$$\hat{\mathcal{L}}_x^N g - \eta\chi_{l,\theta^N(x)} \in \mathcal{C}_{\theta^N(x),0} \quad \text{for every} \quad g \in \Lambda_{x,0} .$$

Proof. Let $x \in X$, let $g \in \Lambda_{x,0}$, and let $l \geq l_0$. Lemma 8.2 and Proposition 7.1 (applied with $n = 1$) show that for $0 < \eta < 2/(a\hat{M})$,

$$\hat{\mathcal{L}}_x^N g - \eta\chi_{l,\theta^N(x)} \leq \frac{a}{2} > 0 \quad \text{on} \quad [0, \ldots, l]_{\theta^N(x)} .$$

Set

$$(8.14) \qquad g' := \frac{\hat{\mathcal{L}}_x^N g - \eta\chi_{l,\theta^N(x)}}{1 - \eta_{l,\theta^N(x)}} , \quad \text{where} \quad \eta_{l,\theta^N(x)} := \eta \int_{E_{\theta^N(x)}^\infty} \chi_{l,\theta^N(x)} \, d\nu_{\theta^N(x)} .$$

Then $\int g' d\nu_{\theta^N(x)} = 1$ and $g' > 0$. Hence, by Proposition 7.1, and since $g \in \Lambda_x$, we have,

$$\begin{aligned}(1 - \eta_{l,\theta^N(x)}) \, g' &\leq \hat{\mathcal{L}}_x^N(|g|) + \eta\chi_{l,\theta^N(x)} \leq \hat{M}\|g\|_\infty \hat{\mathcal{L}}_{\theta^{(N-1)}(x)}\mathbb{1} + \eta\hat{\mathcal{L}}_{\theta^{(N-1)}(x)}\mathbb{1} \\ &\leq (\hat{M}\mathcal{A} + \eta)\hat{\mathcal{L}}_{\theta^{(N-1)}(x)}\mathbb{1} .\end{aligned}$$

But as $0 < \eta_{l,\theta^N(x)} \leq \eta \leq 1/3$, and $\eta \leq 1/H$, the inequality above along with the definition of H yield $g' \leq 2\hat{M}\mathcal{A}\hat{\mathcal{L}}_{\theta^{(N-1)}(x)}\mathbb{1}$. In conclusion, the function $g' \in \Lambda_{\theta^N(x),0}$ provided that we can show that $g' \in \mathcal{C}_{\theta^N(x),0}$.

In order to estimate the α-variation of g' we use the two-norm type inequality, i. e. Lemma 7.2:

$$v_\alpha(g') \leq \frac{1}{1 - \eta_{l,\theta^N(x)}} \left(\hat{M}\|g\|_\infty + \hat{M}v_\alpha(g)e^{-\alpha N} + \eta v_\alpha(\chi_{l,\theta^N(x)}) \right) .$$

Remember that $g, \chi_{l,\theta^N(x)} \in \mathcal{C}_x$, that $\eta \leq \min\{1/3, 1/H\}$, and that we have (8.8). Therefore,

$$v_\alpha(g') \leq 2(\hat{M}\mathcal{A} + 1 + 1) = 2M\mathcal{A} + 4 = H .$$

It remains to estimate $\|g'\|_\infty$. If $\omega \in [0, \ldots, l_0]_{\theta^N(x)}$, then

$$1 = \int_{E_{\theta^N(x)}^\infty} g' d\nu_{\theta^N(x)} \geq \int_{[\omega|_k]_{\theta^N(x)}} g' d\nu_{\theta^N(x)} \geq (g'(\omega) - He^{-\alpha k})\nu_{\theta^N(x)}([\omega|_k]_{\theta^N(x)}) .$$

Using Lemma 6.4 and the choice of k in (8.4), we thus obtain

$$g'(\omega) \leq \beta - 1 + He^{-\alpha k} \leq \frac{\mathcal{A}}{2} + (2\hat{M}\mathcal{A} + 4)e^{-\alpha k} \leq \mathcal{A}\left(\frac{1}{2} + (2\hat{M} + 4)e^{-\alpha k}\right) \leq \mathcal{A} .$$

If $\omega \in E^{\infty}_{\theta^N(x)} \setminus [0, \ldots, l_0]_{\theta^N(x)}$, then $g'(\omega) \leq 2\hat{M}\mathcal{A}\hat{\mathcal{L}}_{\theta^{(N-1)}(x)}\mathbb{1}(\omega) \leq \mathcal{A}$ by the choice of l_0 (see (8.11)). Thus $\|g'\|_{\infty} \leq \mathcal{A}$. Consequently $g' \in \Lambda_{\theta^N(x),0}$, implying that $g \in \mathcal{C}_{\theta^N(x),0}$. The proof is complete. $\qquad\square$

Applying repeatedly Lemma 8.3 we now shall prove the desired contraction of Perron-Frobenius operators.

Proposition 8.4. *For every $\varepsilon > 0$ there exists $n_\varepsilon \geq 1$ such that for every $n \geq n_\varepsilon$ and m–a.e. $x \in X$,*

$$(8.15) \qquad \left\|\hat{\mathcal{L}}^n_x g_x - \hat{\mathcal{L}}^n_x h_x\right\|_\alpha \leq \varepsilon \quad \text{for all} \quad g_x, h_x \in \Lambda_{x,0} \, .$$

Proof. Let $l \geq l_0$ and $N = N_l \geq N_0$ be the same as in Lemma 8.3. Let $g = g^{(0)}_x \in \Lambda_{x,0}$. With the notation of the previous proof, and in particular with the numbers $\eta_{l,\theta^N(x)}$ defined in (8.14), we get from Lemma 8.3 that

$$\hat{\mathcal{L}}^N_x g = \eta\chi_{l,\theta^N(x)} + (1 - \eta_{l,\theta^N(x)})g^{(1)}_{\theta^N(x)}$$

for some $g^{(1)}_{\theta^N(x)} \in \Lambda_{\theta^N(x),0}$. Applying $\hat{\mathcal{L}}^N_{\theta^N(x)}$ to this equation and using once more Lemma 8.3, gives

$$\hat{\mathcal{L}}^{2N}_x g = \eta\hat{\mathcal{L}}^N_{\theta^N(x)}\chi_{l,\theta^N(x)} + (1 - \eta_{l,\theta^N(x)})\eta\chi_{l,\theta^{2N}(x)} + (1 - \eta_{l,\theta^N(x)})(1 - \eta_{l,\theta^{2N}(x)})g^{(2)}_{\theta^{2N}(x)}$$

for some $g^{(2)}_{\theta^{2N}(x)} \in \Lambda_{\theta^{2N}(x),0}$. Inductively, it follows that for every $k \geq 1$ there is a function $g^{(k)}_{\theta^{kN}(x)} \in \Lambda_{\theta^{kN}(x),0}$ such that

$$\hat{\mathcal{L}}^{kN}_x g = \eta\sum_{j=1}^{k}\left(\prod_{i=1}^{j-1}(1 - \eta_{l,\theta^{iN}(x)})\right)\hat{\mathcal{L}}^{(k-j)N}_{\theta^{jN}(x)}\chi_{l,\theta^{jN}(x)} + \prod_{i=1}^{k}(1 - \eta_{l,\theta^{iN}(x)})\,g^{(k)}_{\theta^{kN}(x)} \, .$$

Observe that the first of these two terms does not depend on g. Therefore, for every two $g, h \in \Lambda_{x,0}$ there are $g^{(k)}_{\theta^{kN}(x)}, h^{(k)}_{\theta^{kN}(x)} \in \Lambda_{\theta^{kN}(x),0}$ such that

$$(8.16) \qquad \hat{\mathcal{L}}^{kN}_x g - \hat{\mathcal{L}}^{kN}_x h = \prod_{i=1}^{k}(1 - \eta_{l,\theta^{iN}(x)})\left(g^{(k)}_{\theta^{kN}(x)} - h^{(k)}_{\theta^{kN}(x)}\right) \, .$$

But, by property (A),

$$Q_{l_0} := \operatorname{ess\,inf}\left(\hat{\mathcal{L}}_x\mathbb{1}|_{[0,\ldots,l_0]_x}, x \in X\right) > 0.$$

Therefore, we have for all $l \geq l_0$ and all $x \in X$ that

$$\eta_{l,x} = \eta\int_{E^{\infty}_x}\chi_{l,x}d\nu_x \geq \eta\int_{[0,\ldots,l_0]_x}\hat{\mathcal{L}}_{\theta^{-1}(x)}\mathbb{1}d\nu_x \geq \eta Q_{l_0}\nu_x([0,\ldots,l_0]_x) \geq \eta Q_{l_0}\beta_{l_0} > 0,$$

where $\beta_{l_0} := \beta_{D,l_0} > 0$, comes from Lemma 6.4. Thus, writing, $\tilde{\eta} := \eta Q_{l_0}\beta_{l_0} > 0$, we have that

$$1 - \eta_{l,x} \leq 1 - \tilde{\eta} < 1.$$

Along with (8.16), this allows us to deduce the uniform bound of Proposition 8.4, with some $n_{1,\varepsilon} \geq 1$ sufficiently large, for the supremum norm rather than the Hölder one. In fact, in what follows we want $n_{1,\varepsilon}$ to witness $\varepsilon/\max\{2, 4\hat{M}\}$ rather than merely ε. In order to get the appropriate estimate for the α–variation, we need once more the two-norm type inequality, i. e. Lemma 7.2. Write $n = m + n_{2,\varepsilon} + n_{1,\varepsilon}$ with some integer $n_{2,\varepsilon} \geq 0$ to be determined in a moment and some integer $m \geq 0$. Then for all $g, h \in \Lambda_{x,0}$, we have

$$
\begin{aligned}
v_\alpha \left(\hat{\mathcal{L}}_x^n g - \hat{\mathcal{L}}_x^n h \right) &= v_\alpha \left(\hat{\mathcal{L}}_{\theta^{n_{1,\varepsilon}}(x)}^{m+n_{2,\varepsilon}} \left(\hat{\mathcal{L}}_x^{n_{1,\varepsilon}} (g-h) \right) \right) \\
&\leq \hat{M} \left(\left\| \hat{\mathcal{L}}_x^{n_{1,\varepsilon}} (g-h) \right\|_\infty + e^{-\alpha(m+n_{2,\varepsilon})} v_\alpha \left(\hat{\mathcal{L}}_x^{n_{1,\varepsilon}} (g-h) \right) \right) \\
&\leq \hat{M} \frac{\varepsilon}{4\hat{M}} + e^{-\alpha n_{2,\varepsilon}} H \\
&= \frac{\varepsilon}{4} + 2e^{-\alpha n_{2,\varepsilon}} H,
\end{aligned}
$$

where the second summand in the second last inequality was written due to the fact that $\hat{\mathcal{L}}_x^{n_{1,\varepsilon}} g, \hat{\mathcal{L}}_x^{n_{1,\varepsilon}} h \in \Lambda_{\theta^{n_{1,\varepsilon}}(x),0}$. It suffices now to choose the integer $n_{2,\varepsilon} \geq 0$ sufficiently large so that $2He^{-\alpha n_{2,\varepsilon}} \leq \varepsilon/4$. Then $v_\alpha \left(\hat{\mathcal{L}}_x^n g - \hat{\mathcal{L}}_x^n h \right) \leq \varepsilon/2$, and in consequence

$$
\|\hat{\mathcal{L}}_x^n g - \hat{\mathcal{L}}_x^n h\|_\alpha \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
$$

The proof is complete. $\square$

Proof of Theorem 5.8 (1). For every integer $k \geq 0$ set $\rho^{(k)} := \hat{\mathcal{L}}^k \mathbb{1}$, i. e. $\rho_x^{(k)} = \hat{\mathcal{L}}_{\theta^{-k}(x)}^k \mathbb{1}$. First of all, Proposition 8.1 implies that $\rho_x^{(k)} \in \Lambda_{x,0}$ for every $k \geq N_0$. Given $\varepsilon > 0$ arbitrary, put $q := \max\{N_0, n_{\varepsilon/2}\}$, the latter coming from Proposition 8.4. Proposition 8.1 also implies that $\hat{\mathcal{L}}_{\theta^{-n}(x)}^{n-q} \mathbb{1} \in \Lambda_{\theta^{-q}(x),0}$ for every integer $n \geq q + N_0$. Hence Proposition 8.4 applies to give

$$
\left\| \rho_x^q - \rho_x^n \right\|_\alpha = \left\| \hat{\mathcal{L}}_{\theta^{-q}(x)}^q \mathbb{1} - \hat{\mathcal{L}}_{\theta^{-n}(x)}^n \mathbb{1} \right\| = \left\| \hat{\mathcal{L}}_{\theta^{-k}(x)}^q \mathbb{1} - \hat{\mathcal{L}}_{\theta^{-q}(x)}^k \left(\hat{\mathcal{L}}_{\theta^{-n}(x)}^{n-q} \mathbb{1} \right) \right\| \leq \varepsilon/2
$$

for all $x \in X$. Therefore, if $k, l \geq q + N_0$, then

$$
\left\| \rho_x^l - \rho_x^k \right\|_\alpha \leq \left\| \rho_x^l - \rho_x^q \right\|_\alpha + \left\| \rho_x^q - \rho_x^k \right\|_\alpha \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
$$

for all $x \in X$. This shows that $(\rho_x^{(n)})_{n=0}^\infty$ is a Cauchy sequence in $(\mathcal{H}_\alpha(E_x^\infty), \|.\|_\beta)$ uniformly with respect to $x \in X$. Hence, this sequence has a limit, call it ρ_x, in $\mathcal{H}_\alpha(E_x^\infty)$, and $\rho_x \in \Lambda_{x,0}$. Since also

$$
\hat{\mathcal{L}}_x \rho_x^{(n)} = \hat{\mathcal{L}}_{\theta^{-(n+1)}(\theta(x))}^{n+1} \mathbb{1} = \rho_{\theta(x)}^{(n+1)},
$$

and since $\hat{\mathcal{L}}_x : \mathcal{H}_\alpha(E_x^\infty) \to \mathcal{H}_\alpha(E_{\theta(x)}^\infty)$ is a continuous operator, we conclude that $\hat{\mathcal{L}}\rho_x = \rho_{\theta(x)}$ for m-a.e. $x \in X$. Uniqueness of this function follows immediately from the contraction formula (8.15) of Proposition 8.4. $\square$

Proof of Theorem 5.8 (2). Since $\mathcal{A}, H \geq 2$, we have that

$$\left\{ \mathbb{1} + h_x : \ \|h_x\|_\alpha < 1/4 \right\} \subset \mathcal{C}_x,$$

for all $x \in X$. Let $g \in \mathcal{H}_\alpha(E_x^\infty)$, $g \not\equiv 0$, be arbitrary. Then

$$h := \frac{g}{8\|g\|_\alpha} = (h + \mathbb{1}) - \mathbb{1}$$

is a difference of two functions from $\mathcal{C}_x$. If $\varepsilon > 0$ and if $n = n_\varepsilon$ is given by Proposition 8.4, then applying this proposition we get:

$$\left\| \hat{\mathcal{L}}_x^n \left(h - \left(\int h d\nu_x \right) \rho_{\theta^n(x)} \right) \right\|_\alpha \leq \left\| \hat{\mathcal{L}}_x^n h - \left(\int h d\nu_x \right) \rho_{\theta^n(x)} \right\|_\alpha \leq$$

$$\leq \left\| \hat{\mathcal{L}}_x^n (\mathbb{1} + h) - \int (\mathbb{1} + h) \, d\nu_x \, \rho_{\theta^n(x)} \right\|_\alpha + \left\| \hat{\mathcal{L}}_x^n \mathbb{1} - \rho_{\theta^n(x)} \right\|_\alpha$$

$$\leq \varepsilon \int (\mathbb{1} + h) \, d\nu_x + \varepsilon$$

$$\leq \frac{17}{8}\varepsilon.$$

This shows that for every $\varepsilon > 0$ there exists $N = N_\varepsilon$ such that

$$\left\| \hat{\mathcal{L}}_x^N \left(g - \left(\int g d\nu_x \right) \rho_{\theta^N(x)} \right) \right\|_\alpha \leq \varepsilon \|g\|_\alpha = \varepsilon \left\| g - \left(\int g d\nu_x \right) \right\|_\alpha$$

for every $\in \mathcal{H}_\alpha(\mathcal{J}_x)$. Fix $\varepsilon := 1/2$ and let $N = N_{1/2}$. Write any integer $n \geq 0$ in a unique form as $n = kN + m$, where $k \geq 0$ and $m \in \{0, ..., N - 1\}$. Then, using also Proposition 7.3, for every $g \in \mathcal{H}_\alpha(\mathcal{J}_x)$, we have

$$\left\| \hat{\mathcal{L}}_x^n g - \int g d\nu_x \rho_{\theta^n(x)} \right\|_\alpha = \left\| \hat{\mathcal{L}}_{\theta^{kN}(x)}^m \left(\hat{\mathcal{L}}_x^{kN} \left(g - \int g d\nu_x \rho_x \right) \right) \right\|_\alpha$$

$$\leq \hat{M} \left(\frac{1}{2} \right)^k \left\| g - \int g d\nu_x \rho_x \right\|_\alpha$$

$$\leq 2\hat{M} \left(\frac{1}{2^{1/N}} \right)^n (1 + \|\rho_x\|_\alpha) \|g\|_\alpha .$$

This completes the proof of Theorem 5.8 (2). $\qquad\qquad\qquad\qquad\square$

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