REGULARITY OF HAUSDORFF MEASURE FUNCTION FOR CONFORMAL DYNAMICAL SYSTEMS

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Abstract. Developing the pioneering work of Lars Olsen, and the work [SUZ] we deal in the present paper with the question of continuity of numerical values of Hausdorff measures in parametrized families of linear (similarity) and conformal dynamical systems. We prove Hölder continuity of the function ascribing to a parameter the numerical value of the Hausdorff measure of the limit set, for naturally parametrized families of both conformal iterated function systems in $\mathbb{R}^k$, $k \geq 3$, and linear iterated function systems consisting of similarities in $\mathbb{R}^k$, $k \geq 1$ (in this latter case the Hölder exponent is universal equal to 1/2), both satisfying the Strong Separation Condition, and for analytic families of conformal expanding repellers in the complex plane $\mathbb{C}$. For families of naturally parametrized linear IFSs in $\mathbb{R}$, satisfying the Strong Separation Condition, this function is proved to be piecewise real–analytic. On the other hand, we also give an example of a family of linear IFS in $\mathbb{R}$ for which the Hausdorff measure function is not even differentiable at some parameters.

1. Introduction

The question of dependence on a parameter of the Hausdorff dimension of the limit (ex. Julia) set in naturally parametrized families of conformal dynamical systems has been studied intensively over the last decades. In his seminal paper [Ru2] D. Ruelle proved that if J_c is the Julia set ofthe quadratic polynomial $\hat{\mathbb{C}} \ni z \mapsto z^2 + c$, then the function $\mathbb{C} \ni c \mapsto \mathrm{HD}(J_c)$, is real-analytic if c belongs to an open disk cenetred at 0 with a sufficiently small radius; see e.g. [UZi], [UZd], [U3] or [AU] for further generalization and extensions of this result.

A more subtle question is about regularity of the function ascribing to a system, or parameter, the numerical value of the Hausdorff measure of the corresponding limit set. A breakthrough work has been done by L. Olsen in ([Ol] who proved there such continuity for finite iterated function systems consisting of similarities and satisfying the strong separation condition.

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Further continuity results (e.g. for families of conformal hyperbolic and parabolic systems, and for some families of infinite 1-dimensional iterated function systems) have been obtained in [SUZ].

In this paper, we study the modulus of continuity of the above function. We do this for several types of naturally parametrized families of systems. These are called, in the sequel, admissible families. Denoting by Λ the set of parameters here and in the sequel, for every $\lambda \in \Lambda$, J_λ denotes either the limit set of a given conformal iterated function system or the phase space of a given conformal expanding repeller, called also the limit set in the sequel, under consideration. From now onwards we denote

$$h_\lambda := \mathrm{HD}(J_\lambda).$$

We have the following results. For parametrized families of linear systems, i. e. consisting of similarities, satisfying the Strong Separation Condition we prove, in Section 6 the following.

Theorem. *Let $\Lambda' \subset \Lambda$ be an arbitrary bounded domain such that $\overline{\Lambda'} \subset \Lambda$. Let $(S_\lambda)_{\lambda\in\Lambda}$ be an admissible family of similarity iterated function systems in $\mathbb{R}^k$, $k \geq 1$. Then the function $\Lambda' \ni \lambda \mapsto \mathrm{H}_{h_\lambda}(J_\lambda)$ is Hölder continuous with Hölder exponent equal to* $1/2$.

A remarkable observation about this theorem is that the Hölder exponent (= 1/2) is unversal, i. e. does not depend on the system in question. If we however relax linearity the situation deteriorates, we still keep Hölder continuity; see the second theorem below, but the exponents worsen, do depend on the system. However, if we rather restrict the class of systems, namely requiring the ambient dimension k to be equal to 1, we obtain much better regularity of the Hausdorff measure function. Indeed, for parametrized families of linear systems acting in the real line $\mathbb{R}$ we prove, in Section 7, the following.

Theorem. *Let $(S_\lambda)_{\lambda\in\Lambda}$ be an admissible family of similarity iterated function systems in the real line $\mathbb{R}$. Then the function $\Lambda \ni \lambda \mapsto \mathrm{H}_{h_\lambda}(J_\lambda)$ is piecewise real–analytic.*

We also describe a simple example, showing that piecewise real analyticity cannot be replaced by real analyticity. It is based on the observation that the function (formula) giving the value of the Hausdorff measure can have several global minima; the phenomenon essentially already observed in [AS]. Finally, we prove in Section 8 this.

Theorem. *Let $(S_\lambda)_{\lambda\in\Lambda}$ be an admissible family either of conformal IFS in $\mathbb{R}^k$, $k \geq 3$, or conformal expanding repellers in the complex plane $\mathbb{C}$ (ex. Julia sets of expanding rational functions). Let Λ' be a bounded open set such that $\overline{\Lambda'} \subset \Lambda$.*

Then the function

$$\Lambda' \ni \lambda \mapsto \mathrm{H}_{h_\lambda}(J_\lambda)$$

is Hölder continuous with an exponent equal to $(3 + \inf\{h_\lambda : \lambda \in \Lambda'\})^{-1}$.

Our paper is organised as follows. In Section 2 we describe the class of systems we deal with in this article. In Section 3 we recall the density theorems for Hausdorff measures. These provide starting tools for our proofs. In Section 4 we develop several functional analysis techniques forming crucial ingredients of our approach. We also obtain in this section some appropriate estimates for the values of conformal measures. In Sections 5, 6, 7 and 8 we formulate and prove our results.

2. Preliminaries

We now describe the class of analytic families of dynamical systems which we deal with in this paper. As we have already indicated in the introduction these systems will be referred to as admissible systems.

2.1. **Similarity IFSs satisfying Strong Separation Condition.** Let φ be a similarity contraction in $\mathbb{R}^k$, $k \geq 1$. Then φ can be uniquely written in the form

$$\varphi(x) = \eta \cdot Ax + b, \tag{2.1}$$

where $0 < \eta < 1$, A is an orthogonal matrixs, i. e. a linear isometry map, and $b \in \mathbb{R}^k$.
Now given an integer $N \geq 1$, let

$$\varphi^i(x) = \eta^i \cdot A^{(i)}x + b^i, \quad i = 1, \dots, N,$$

be a family of similarity contractions in $\mathbb{R}^k$. It is well-known that there exists a unique compact subset of $\mathbb{R}^k$ such that

$$J(S) = \bigcup_{i=1}^{N} \varphi^i\left(J(S)\right).$$

It is called the limit set, or attractor, of the system S, and will be denoted in the sequel by $J = J(S)$. We say that the linear IFS satisfies the Strong Separation Condition (SSC) if, for $i \neq j$,

$$\varphi^i(J(S)) \cap \varphi^j(J(S)) = \emptyset. \tag{2.2}$$

Remark 2.1. *Note that the Strong Separation Condition equivalently means that there exists a neighbourhood* V *of* $J(S)$ *such that*

$$\overline{\varphi^i(V)} \subset V \quad \textit{and} \quad \overline{\varphi^i(V)} \cap \overline{\varphi^j(V)} = \emptyset$$

whenever in the latter $i \neq j$. The limit set of such a system is a topological Cantor set, i. e. compact, perfect and totally disconnected.

We denote

$$\Sigma_n := \{1, \dots N\}^n \ \text{ and } \ \Sigma_\infty := \{1, 2 \dots, N\}^{\mathbb{N}}.$$

The space Σ_∞ is equipped with the metric d_∞ defined as $d_\infty(\mathbf{i}, \mathbf{j}) = \frac{1}{2^n}$ where n is the least integer for which $i_n \neq j_n$. The topology induced by this metric coincides with the product (Tichonov) topology on Σ_∞.

For an arbitrary sequence $\mathbf{i} \in \Sigma_n$, $\mathbf{i} = (i_1, i_2, \dots, i_n)$ we denote by $\varphi^{\mathbf{i}}$ the composition

$$\varphi^{\mathbf{i}} = \varphi^{i_1} \circ \cdots \circ \varphi^{i_n}. \tag{2.3}$$

For $\mathbf{i} \in \Sigma_\infty$ we denote by $\mathbf{i}_{|n}$ the initial segment of first n terms of $\mathbf{i}$. Then the infinite intersection $\bigcap_{n=1}^\infty \varphi^{\mathbf{i}_n}(\overline{V})$ is a singleton, and one can define a natural coding $\pi : \Sigma_\infty \to J(S)$ by

$$\{\pi(\mathbf{i})\} := \bigcap_{n=1}^\infty \varphi^{\mathbf{i}_{|n}}(\overline{V}) \tag{2.4}$$

Note that both maps $\pi : \Sigma_\infty \to J(S)$ and $\pi^{-1} : J(S) \to \Sigma_\infty$ are Hölder continuous homeomorphisms. The limit set $J(S)$ can be now equivalently characterized as

$$J(S) = \bigcap_{n=1}^\infty \bigcup_{\mathbf{i} \in \Sigma_n} \varphi^{\mathbf{i}}(\overline{V}). \tag{2.5}$$

As a particular case, we will consider real linear IFSs, i. e. the systems acting on some closed bounded interval in the real line $\mathbb{R}$.

2.2. **Conformal Iterated Function Systems in $\mathbb{R}^k$, $k \geq 3$, satisfying Strong Separation Condition.**

It is well known (Liouville's Theorem) that every conformal map defined in an open connected subset of $\mathbb{R}^k$, $k \geq 3$, is a restriction of a Möbius transformation. More precisely, every C^1 conformal homeomorphism defined on an open connected subset of $\mathbb{R}^k$ is a restriction of some map of the form

$$\varphi = \eta A \circ i + b, \tag{2.6}$$

where $\eta > 0$ is a positive scalar, A is a linear isometry in $\mathbb{R}^k$, i is either the inversion with respect to some sphere $S(a, 1)$ (with center $a \in \mathbb{R}^k$ and radius 1), or the identity map. In the latter case, φ is just a similarity map.

Definition 2.2. *Let $S = \{\varphi^i\}_{i=1,\dots N}$ be a finite collection of conformal maps. We assume that there exists an open and connected set $V \subset \mathbb{R}^k$ such that, for every $i = 1, \dots, N$*

$$\overline{\varphi^i(V)} \subset V$$

and that all the maps $\varphi^i_{|V}$ are contractions, with Lipschitz constants $s_i \leq s < 1$. The colection S is then called a Conformal Iterated Function System (CIFS).

The limit set $J(S)$ of the system S is then defined, similarly as in the previous section, as the unique compact subset of V such that

$$J(S) = \bigcup_{i=1}^{N} \varphi^i(J(S)).$$

We keep the notation $\varphi^{\mathbf{i}}$, introduced in (2.3). The natural coding map $\pi : \Sigma \to J(S)$ is defined analogously as in (2.4). Like for similarity IFSs, this map is Hölder continuous. The Strong Separation Condition (SSC) has the same formulation as in the previous section, see (2.2). If it holds, then then also the inverse map $\pi^{-1} : J(S) \to \Sigma$ is a Hölder continuous too. Briefly, the projection $\pi : \Sigma \to J(S)$ is then a Hölder continuous homeomorphism. As in the previous linear case, the formula (2.5) is true.

2.3. **Conformal Expanding Repellers in $\mathbb{C}$.**

Below, we recall the definition of a conformal expanding repeller in $\mathbb{C}$:

Definition 2.3. *Let U be an open subset of $\mathbb{C}$. Let J be a compact subset of U. Let $T : U \to \mathbb{C}$ be a conformal map and note that T is not required to be one-to-one. The map T is called a conformal expanding repeller if the following conditions hold:*

(1) *$T(J) = J$,*

(2) *$|T'|_{|J} > 1$,*

(3) *there exists an open set V such that $\overline{V} \subset U$ and*

$$J = \bigcap_{k=0}^{\infty} T^{-n}(V).$$

(4) *$T_{|J}$ is topologically transitive.*

Abusing slightly notaion we frequently refer also to the sets J alone as a conformal expanding repeller. In order to use a uniform terminology we also call J the limit set of T.

Typical examples of conformal expanding repellers are provided by the following.

Proposition 2.4. *If f is a rational map in $\mathbb{C}$ of degree $d \geq 2$, such that the map f restricted to the Julia set $J(f)$ is expanding, then $J(f)$ is a conformal expanding repeller.*

A conformal IFS in $\mathbb{C}$ satisfying the Strong Separation Condition is a very special example of expanding repeller:

Proposition 2.5. *For every conformal iterated function system S in $\mathbb{C}$ satisfying the Strong Separation Condition, the limit set $J(S)$ is a conformal expanding repeller; see* (2.7) *below.*

2.4. **Admissible systems; Dynamics.** In the sequel, we denote by J the limit set of one of three types of dynamics described in Section 2: the limit set $J(S)$ of a similarity IFS in $\mathbb{R}^k$, $k \geq 1$, satisfying SSC, the limit set $J(S)$ of a conformal IFS in $\mathbb{R}^k$, $k \geq 3$, satisfying SSC, or a conformal expanding repeller in $\mathbb{C}$, defined in Definition 2.3. Recall that for all these systems the set J is invariant under the expanding map defined in a neighbourhood of J, and denoted in the sequel by T. For IFSs the map T is defined as

$$T(x) = \left(\varphi^i\right)^{-1}(x) \text{ for } x \in \varphi^i(V). \tag{2.7}$$

For conformal expanding repellers, the map T is given by its definition. The systems described above will be referred to as admissible systems. The set J will be referred to as the limit set of an admissible system.

2.5. **Distortion estimates.**

We will use the notation $|\varphi'(x)|$ to denote the norm of the derivative of the map ϕ at the point x. Observe that for a similarity map φ, the number $|\varphi'(x)|$ is just the value $|\eta|$ in the representation (2.1). For a conformal map in $\mathbb{R}^k$, $k \geq 3$, $|\varphi'(x)|$ is equal to $|\eta| \cdot |i'(x)|$, according to the representation (2.6). We need the standard distortion estimates in the three types of dynamical systems described above. In the conformal case in $\mathbb{R}^k$, $k \geq 3$, we use the following estimate: (see e.g [SUZ]).

Proposition 2.6. *Suppose V is a non-empty open connected subset of $\mathbb{R}^k$, where $k \geq 3$ and $F \subset V$ is a bounded set such that $\overline{F} \subset V$. If $\varphi : V \to \mathbb{R}^k$ is a conformal map (implying in particuar that $\varphi^{-1}(\infty) \notin V$), then*

$$\frac{|\varphi'(x)|}{|\varphi'(y)|} \leq \left(1 + \frac{\operatorname{diam}(F)}{\operatorname{dist}(F, V^c)}\right)^2$$

for all $x, y \in F$.

Corollary 2.7. *If $S = \{\varphi^i\}_{i=1}^N$ is a conformal IFS satisfying the Strong Separation Condition, then the above estimate in Proposition* 2.6 *applies to an arbitrary map $\varphi^{\mathbf{i}}$, $\mathbf{i} \in \Sigma_n$, and for any set $F \subset J(S)$.*

Proof. Indeed, it is enough to notice that $\delta = \mathrm{dist}(J(S), \partial V) > 0$, and that for every $\mathbf{i}$, $\varphi^{\mathbf{i}}(V) \subset V$, so that $V \subset (\varphi^{\mathbf{i}})^{-1}(V)$. As $\infty \notin V$, this yields $(\varphi^{\mathbf{i}})^{-1}(\infty) \notin V$. $\square$

For the case of conformal expanding repellers and conformal IFS satisfying Strong Separation Condition in $\mathbb{C}$, we use the following classical Koebe Distortion Theorem:

Theorem 2.8. *If $g : B(0,1) \to \mathbb{C}$ is a univalent holomorphic function, the for all $z \in B(0,R)$:*

$$\frac{1-|z|}{1+|z|^3} \le \frac{|g'(z)|}{|g'(0)|} \le \frac{1+|z|}{(1-|z|)^3}$$

Since every conformal map in the plane is either holomorphic or antiholomorphic, and since the complex conjugation is an isometry, we get immediately:

Corollary 2.9. *The statement of Theorem* 2.8 *holds true for any univalent conformal function $g : B(0,1) \to \mathbb{C}$.*

3. Hausdorff dimension and Hausdorff Measure

In this section we collect some well-known general density theorems. We start with the following density theorem for Hausdorff measures (see [Ma] for example).

Fact 3.1. *Let X be a metric space, with $\mathrm{HD}(X) = h$, such that $\mathrm{H}_h(X) < +\infty$. Then (see p. 91 in* [Ma]*),*

$$\lim_{r\to 0} \sup \left\{ \frac{\mathrm{H}_h(F)}{\mathrm{diam}^h(F)} : x \in F,\ \overline{F} = F,\ \mathrm{diam}(F) \le r \right\} = 1$$

for H_h–a.e. $x \in X$.

As an immediate consequence of this, we get the following fundamental fact, which was extensively explored in [Ol] and [SUZ].

Theorem 3.2. *Let X be a metric space and $0 < \mathrm{H}_h(X) < +\infty$. Denote by H_h^1 the normalized h-dimensional Hausdorff measure on X, i.e $\mathrm{H}_h^1(F) = \mathrm{H}_h(F)/\mathrm{H}_h(X)$ for all subsets F of X. Then*

$$\mathrm{H}_h(X) = \lim_{r\to 0} \inf \left\{ \frac{\mathrm{diam}^h(F)}{\mathrm{H}_h^1(F)} : x \in F,\ \overline{F} = F,\ \mathrm{diam}(F) \le r \right\} \tag{3.1}$$

for H_h^1–a.e. $x \in X$.

Clearly, we then have the following.

Corollary 3.3. *If X is a subset of a Euclidean metric space $\mathbb{R}^d$ and $0 < \mathrm{H}_h(X) < +\infty$, then*
(3.2)
$$\mathrm{H}_h(X) = \lim_{r\to 0}\inf\left\{\frac{\operatorname{diam}^h(F)}{\mathrm{H}^1_h(F)} : x \in F,\ F \subset \mathbb{R}^d \text{ is closed, convex, } \operatorname{diam}(F) \le r\right\}$$
for H^1_h–a.e. $x \in X$.

For subsets of the real line $\mathbb{R}$ we can write a more specific formula:

Corollary 3.4. *If X is a subset of an interval $\Delta \subset \mathbb{R}$ and $0 < \mathrm{H}_h(X) < +\infty$, then*
$$\mathrm{H}_h(X) = \lim_{r\to 0}\inf\left\{\frac{\operatorname{diam}^h(F)}{\mathrm{H}^1_h(F)} : x \in F,\ F \subset \Delta \text{ is a closed interval, } \operatorname{diam}(F) \le r\right\}$$
for H^1_h–a.e. $x \in X$.

4. Functional Analysis tools

In subsections 4.1– 4.5 we recall, in a suitable form the basic facts about dependence of the Perron-Frobenius operator on a parameter. We use them in subsection 4.6 to obtain the required estimates for conformal measures.

4.1. **Perron-Frobenius operator.** Let J be the limit set of an admissible system S. Fix some $\alpha \in (0,1]$. Let $\mathcal{H}_\alpha(J)$ be the space of all complex valued Hölder continuous functions, defined in J. As usually, we denote
$$v_\alpha(\psi) = \inf\left\{L \ge 0 : |\psi(x) - \psi(y)| \le L||x-y||^\alpha \text{ for all } x, y \in J\right\}.$$
The vectorr space $\mathcal{H}_\alpha(J)$, endowed with the norm
$$||\psi||_\alpha := \max\{||\psi||_\infty, v_\alpha(\psi)\}$$
becomes a Banach space. We denote by $\mathcal{H}^*_\alpha(J)$ the dual spaces to $\mathcal{H}_\alpha(J)$, i. e. the space of all complex valued continuous functionals in $\mathcal{H}_\alpha(J)$, endowed with the norm topology. It is evident that every probability measure, being a continuous functional in the space $C(J)$, of all continuous functions from J to $\mathbb{C}$, is also an element of the space $\mathcal{H}^*_\alpha(J)$. Denote by $|||\cdot|||_\alpha$ the norm in the space $\mathcal{H}^*_\alpha(J)$. Also, denote by $L(\mathcal{H}_\alpha(J))$ the space of all bounded linear operators from $\mathcal{H}_\alpha(J)$ to $\mathcal{H}_\alpha(J)$.

Definition 4.1. *For every $\alpha \in (0,1]$ the Perron-Frobenius operator $\mathcal{L}_t : \mathcal{H}_\alpha(J) \to \mathcal{H}_\alpha(J)$ is defined as:*
$$\mathcal{L}_t(g)(x) = \sum_{y\in T^{-1}(x)} \frac{1}{|T'(y)|^t}\cdot g(y).$$

Remark 4.2. *Note that, in the case of Iterated Function Systems, the formula defining $\mathcal{L}_t$ can be rewritten as*

$$\mathcal{L}_t(g)(x) = \sum_{i=1}^{N} |(\varphi^i)'(x)|^t \cdot g(\varphi^i(x)).$$

We denote by

$$\mathrm{P}(t) := \mathrm{P}(-t \log |T'|)$$

the topological pressure for the potential $-t \log |T'|$ and the dynamical system $T : J \to J$. Its definition and most transparent properties can be found for example in [Wa1], [Wa2] or [PU]. The following fact (see for example [PU] for its proof) is crucial for our considerations.

Theorem 4.3. *For every $\alpha \in (0, 1]$ the linear operator $\mathcal{L}_t$ is bounded as an operator acting on the space $\mathcal{H}_\alpha(J)$. The value* $\exp(\mathrm{P}(t))$ *is a simple eigenvalue of this operator and an isolated point of the spectrum $\sigma(\mathcal{L}_t)$.*

4.2. **Conformal measures.**

Let S be an admissible system. For every $t \in \mathbb{R}$ the t–conformal measure m_t is defined as a Borel probability measure on the limit set satisfying the following:

$$\mathcal{L}_t^*(m_t) = \exp(\mathrm{P}(t))m_t,$$

or, equivalently,

$$m_t(T(A)) = \exp(\mathrm{P}(t)) \int_A |T'| dm_t$$

for every Borel set $A \in J$ such that $T_{|A}$ is injective.

If S is an admissible system then, for every $t \in \mathbb{R}$, the measure m_t exists and is unique; see [PU] and [MU]. There is also a unique normalized, i. e. $\int_J \rho_t dm_t = 1$, Hölder continuous eigenfunction ρ_t of $\mathcal{L}_t$, corresponding to the for the eigenvalue $\exp(\mathrm{P}(t))$. In addition, the function ρ_t is strictly positive.

The well-known Bowen's type formula asserts that the Hausdorff dimension of the limit set J can be expressed in terms of the pressure function as the only value $h = h_J$ for which

$$\mathrm{P}(h) = 0.$$

The following proposition is also well-known; see ex. [PU], [MU] and the references therein.

Proposition 4.4. *If S is an admissible system and $h = \mathrm{HD}(J)$, then the Hausdorff measure $\mathrm{H}_{h|J}$ is positive and finite. Therefore, m_h is then equal to the normalized Hausdorff measure H^1_h.*

4.3. **Parametrized families of maps. Admissible families.**

For S, a family of similarity IFSs in $\mathbb{R}^k$, $k \geq 1$, the representation (2.1) provides a natural parametrization by some open subset of $\mathbb{R}^M$, where

$$M_k(S) = N\left(\frac{k(k+1)}{2} + 1\right). \tag{4.1}$$

Similarly, for conformal IFSs in $\mathbb{R}^k$, $k \geq 3$, using (2.1) and (2.6) we have a natural local parametrization by an open subset of $\mathbb{R}^M$. Here,

$$M = M_k(S) = N_1\left(\frac{k(k+3)}{2} + 1\right) + N_2\left(\frac{k(k+1)}{2} + 1\right), \tag{4.2}$$

where N_1 is the number of maps φ^i for which i is an inversion, and N_2 is the number of maps for which i is the identity map. Of course if $k \geq 3$, then the case of similarity IFSs is included in conformal IFSs, just $N_1 = 0$. In addition, the set of parameters for which the SSC is satisfied is open. This justifies the following.

Definition 4.5. *Let $S_\lambda = \{\varphi^i_\lambda\}_{i=1}^N$, $\lambda \in \Lambda \subset \mathbb{R}^M$, be a system of linear (similarities) or conformal Iterated Function Systems, with their natural parametrizations. The family $\{S_\lambda\}_{\lambda\in\Lambda}$ is called admissible if for every $\lambda \in \Lambda$ the system S_λ is admissible. In particular, all systems S_λ, $\lambda \in \Lambda$, satisfy SSC, the Strong Separation Condition.*

Let S_λ be an admissible family of IFS. Denote by J_λ the limit set of the system S_λ. We recall that the natural coding $\pi_\lambda : \Sigma_\infty \to J_\lambda$ introduced in (2.4), gives, for every $\lambda \in \Lambda$, a Hölder continuous homeomorphism between the coding space Σ_∞ and the limit set J_λ; the inverse map is also Hölder continuous. Thus, fixing some $\lambda_0 \in \Lambda$, we can define, for $\lambda \in \Lambda$, a homeomorphism $\tau_\lambda : J_{\lambda_0} \to J_\lambda$, given by

$$\tau_\lambda = \pi_\lambda \circ \pi_{\lambda_0}^{-1}$$

The function $\tau_\lambda : J_{\lambda_0} \to J_\lambda$ is Hölder continuous, with Hölder exponent converging to 1 as $\lambda \to \lambda_0$. Note that τ_λ conjugates the dynamics:

$$\tau_\lambda \circ \varphi^i_{\lambda_0} = \varphi^i_\lambda \circ \tau_\lambda.$$

In the case of conformal expanding repellers, we consider analytic families of maps, defined as follows.

Definition 4.6. *Let $\hat{\Lambda}$ be an open subset of $\mathbb{C}$. A family $T_\lambda : U \to U$, $\lambda \in \hat{\Lambda}$, of admissible systems in $\mathbb{C}$, is called analytic if the function*

$$\Lambda \times U \ni (\lambda, z) \mapsto T_\lambda(z) \in \mathbb{C}$$

is analytic.

Observe that because of Hartog's Theorem we would not gain any bigger generality if we considered $\hat{\Lambda}$ as an open subset of some multidimensional space $\mathbb{C}^n$.

Denote by J_λ the limit set of T_λ. The conjugating homeomorphism $\tau_\lambda : J_{\lambda_0} \to J_\lambda$ is provided (see [PU] for ex.) by the following.

Proposition 4.7. *Let $\left(T_\lambda\right)_{\lambda\in\hat{\Lambda}}$ be an analytic family of conformal maps from U to $\mathbb{C}$, where U is a bounded open subset of $\mathbb{C}$. Assume that, for some $\lambda_0 \in \hat{\Lambda}$, T_{λ_0} is a conformal expanding repeller with the limit set J_{λ_0}. Then there exists an open subset $\Lambda \subset \hat{\Lambda}$ containing λ_0 such that for every $\lambda \in \Lambda$ sufficiently close to λ_0, there exists a conformal expanding repeller T_λ, whose limit set J_λ is homeomorphic to J_{λ_0}, and a homeomorphism $\tau_\lambda : J_{\lambda_0} \to J_\lambda$, being a holomorphic motion (see* [MSS], *comp* [Mi] *for a more contemporary state of arts), such that, for every fixed z the map*

$$\Lambda \ni \lambda \mapsto \tau_\lambda(z)$$

is holomorphic, and for every $\lambda \in \Lambda$ the map

$$z \mapsto \tau_\lambda(z)$$

is Hölder continuous, with Hölder exponent converging to 1 as $|\lambda - \lambda_0| \to 0$. The homeomorphism τ_λ conjugates T_{λ_0} and T_λ:

$$T_\lambda \circ \tau_\lambda = \tau_\lambda \circ T_{\lambda_0}.$$

Definition 4.8. *The family T_λ, restricted to the set of parameters Λ described in Proposition 4.7 will be called an admissible family of conformal expanding repellers.*

Notation. We shall frequently use the common notation S_λ and the common name admissible family of systems, to denote a family satisfying Definition 4.5 or Definition 4.8. Let S_λ be an admissible family of systems. For every $\lambda \in \Lambda$ and every $t \in \mathbb{R}$ we denote by $m_{\lambda,t}$ the t-conformal measure for the system S_λ. We denote by h_λ Hausdorff dimension of the set J_λ. The conformal measure m_{λ,h_λ}, which is just the normalized Hausdorff measure $\mathrm{H}_{h_\lambda | J_\lambda}$, will be denoted by m_λ. The Perron-Frobenius operator for the system S_λ will be denoted by $\mathcal{L}_{\lambda,t}$. We will need the following common lower bound for the conformal measure:

Lemma 4.9. *Let S_λ, $\lambda \in \Lambda$, be an admissible system. Let $\Lambda' \subset \Lambda$ be a bounded subset such that $\overline{\Lambda'} \subset \Lambda$. Then there exists a constant $c > 0$ such that, for every $\lambda \in \Lambda'$, every $x \in J_\lambda$ and every $0 \le r \le \operatorname{diam}(J_\lambda)$,*

$$m_\lambda(B(x,r)) \ge c r^{h_\lambda}. \tag{4.3}$$

Proof. The proof is based on standard arguments. We outline it briefly. Put $M = \sup_{\lambda\in\Lambda'} ||T_\lambda'||_\infty$. Next, fix some $A > 0$ such that, for every $\Lambda \in \Lambda'$ and for every $y \in J_\lambda$, the map T_λ is defined, and univalent in $B(x, 2A)$. Since the conformal measure m_λ depends continuously, in weak-* topology, on λ, there exists another positive constant D such that, for every $\lambda \in \Lambda'$, $y \in J_\lambda$, we have that

$$m_\lambda(B(y, A)) > D.$$

Now, take some $\lambda \in \Lambda'$, and an arbitrary point $x \in J_\lambda$. Let n be the largest integer for which $\mathrm{diam} T_\lambda^n(B(x, r)) \le A$. Then $\mathrm{diam}(T_\lambda^{n+1}(B(x, r))) > A$, and $\mathrm{diam} T_\lambda^n(B(x, r)) > A/M$. The standard distortion estimates give that for every $z \in B(x, r)$ we have $C^{-1}A|(T_\lambda^n)'(z)|r \le CA$, where C is a constant independent of x and r. Thus,

$$m_\lambda(B(x, r)) \ge \frac{D}{\sup_{z\in B(x,r)} |(T_\lambda^n)'(z)|^{h_\lambda}} \ge \frac{Dr^{h_\lambda}}{C^{h_\lambda}A^{h_\lambda}} \ge cr^{h_\lambda}$$

for another constant c. Thus, the estimate (4.3) is established for every $0 < r < A$. Modifying the constant c, and using the fact that $\sup_{\lambda\in\Lambda'} \mathrm{diam} J_\lambda$ is finite, we get the required estimate (4.3) for every $r \le \mathrm{diam}(J_\lambda)$.

□

4.4. **Analytic perturbations of Perron–Frobenius operators; the case of Similarity and Conformal IFSs.**

Let $\{S_\lambda\}_{\lambda\in\Lambda}$ be a an admissible family of either similarity or conformal IFSs in $\mathbb{R}^k$ with $k \ge 1$ or $k \ge 3$ depending on whether the former or the latter case is considered. We recall the following fact proved, in a more general setting, in [RU].

Proposition 4.10. *Let $S = \{S_\lambda\}_{\lambda\in\Lambda}$ be an admissible family of either similarity or conformal IFSs in $\mathbb{R}^k$ respectively with $k \ge 1$ or with $k \ge 3$. Let $\lambda_0 \in \Lambda$. Then, for every $\alpha \in (0, 1)$ there exists a neighbourhood Λ^* of λ_0 in $\mathbb{C}^{M_k(S)}$ ($M_k(S)$ defined respectively either by (4.1) or by (4.2)) such that, for every $\lambda \in \Lambda^*$, the maps φ_λ^i, $i = 1, \dots N$, of S_λ form a contracting holomorphic, though usually not conformal, IFS in $\mathbb{C}^k$ satisfying the SSC. We denote it also by S_λ. For every $\lambda \in \Lambda^* \cap \mathbb{R}^{M_k(S)}$ the system S_λ acts in $\mathbb{R}^k$ as a conformal IFS. Keeping the notation J_λ for the the limit set of systems S_λ, $\lambda \in \Lambda^*$, we have the following:*
For every $\mathbf{i} \in \Sigma_\infty$ the projection

$$\Lambda^* \ni \lambda \mapsto \pi_\lambda(\mathbf{i})$$

is a holomorphic function. Similarly, for every $z \in J_{\lambda_0}$ the function

$$\lambda \mapsto \tau_\lambda(z) = \pi_\lambda \circ \pi_{\lambda_0}^{-1}(z)$$

is holomorphic, and both functions $\tau_\lambda : J_{\lambda_0} \to J_\lambda$ *and* $\tau_\lambda^{-1} : J_\lambda \to J_{\lambda_0}$ *are Hölder continuous with exponent* α*. The proof of the following theorem can be found in* [RU].

Theorem 4.11. *For every* $z \in J_{\lambda_0}$ *the real-valued function*

$$(\Lambda^* \cap \mathbb{R}^{M_k(S)}) \times \mathbb{R} \ni (\lambda, t) \mapsto t \log |(\varphi_\lambda^i)'|(\tau_\lambda(z))$$

admits a holomorphic extension

$$\Lambda^* \times \mathbb{C} \ni (\lambda, t) \mapsto t\zeta_z(\lambda) \in \mathbb{C}.$$

The function $(\Lambda^* \cap \mathbb{R}^{M_k(S)}) \times \mathbb{R} \ni (\lambda, t) \mapsto \mathcal{L}^0_{\lambda,t} \in L\left(\mathcal{H}_\alpha(J_{\lambda_0})\right)$ *defined by:*

$$\mathcal{L}^0_{\lambda,t} g(z) = \sum_{i=1}^{N} |(\varphi_\lambda^i)'|^t(\tau_\lambda(z)) g(\varphi^i_{\lambda_0}(z)),$$

extends to a holomorphic function $\Lambda^* \times \mathbb{C} \ni (\lambda, t) \to L_{\lambda,t} \in L(\mathcal{H}_\alpha(J_{\lambda_0}))$*:*

$$L_{\lambda,t} g(z) = \sum_{i=1}^{N} \exp(t\zeta_z(\lambda)) g(\varphi^i_{\lambda_0}(z)).$$

4.5. **Analytic perturbations of the Perron–Frobenius operator; the case of conformal expanding repellers in $\mathbb{C}$.**

In this section we consider an admissible family of systems formed by conformal expanding repellers, see Definition 4.8.

Fix $\alpha \in (0, 1)$. Let $\lambda_0 \in \Lambda$. Fix a positive r such that for all $|\lambda - \lambda_0| \le r$, the minimal Hölder exponent of the maps τ_λ and τ_λ^{-1} in $|\lambda - \lambda_0| \le r$ is not smaller than α . Then for every $t \in \mathbb{R}$ the operator $\mathcal{L}^0_{\lambda,t}$ defined as the Perron–Frobenius operator induced by the weight function $|T'_\lambda \circ \tau_\lambda|^{-t} : J_{\lambda_0} \to \mathbb{R}$:, i.e.

$$\mathcal{L}^0_{\lambda,t} g(z) = \sum_{w \in T^{-1}_{\lambda_0}(z)} |T'_\lambda(\tau_\lambda(w))|^{-t} g(w),$$

act continuously as a linear operator from $\mathcal{H}_\alpha(J_{\lambda_0})$ to $\mathcal{H}_\alpha(J_{\lambda_0})$. Note that, for $\lambda = \lambda_0$ the operator $\mathcal{L}^0_{\lambda_0,t}$ is just the operator $\mathcal{L}_t : \mathcal{H}_\alpha(J_{\lambda_0}) \to \mathcal{H}_\alpha(J_{\lambda_0})$ introduced in Definition 4.1. So, $\mathcal{L}^0_{\lambda_0,t}$ is a family of perturbations of $\mathcal{L}_t$. A parameter $\lambda \in \mathbb{C}$ can be naturally embedded into $\mathbb{C}^2$ by identifying $\lambda = \lambda_1 + i\lambda_2$ with the point $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \subset \mathbb{C}^2$. Similarly, a real parameter t can be treated as an element of $\mathbb{C}$. The following theorem goes back to [Ru1], its proof can be also found in [PU].

Theorem 4.12. *Fix* $\lambda_0 \in \Lambda$*. Then there exists a neighbourhood* Λ^* *of* λ_0 *in* $\mathbb{C}^2$*, and a holomorphic function* $L : \Lambda^* \times \mathbb{C} \to L\left(\mathcal{H}_\alpha(J_{\lambda_0})\right)$*,* $(\lambda, t) \mapsto L_{\lambda,t}$*, such that, for every* $(\lambda, t) \in (\Lambda^* \times \mathbb{C}) \cap (\mathbb{C} \times \mathbb{R})$*,*

$$L_{\lambda,t} = \mathcal{L}^0_{\lambda,t}.$$

4.6. **Application of the perturbation theory of linear operators.** In this section we consider together all kinds of families of admissible systems, linear (similarity) IFSs in $\mathbb{R}^k$, $k \ge 1$, conformal IFSs in $\mathbb{R}^k$, $k \ge 3$, and conformal expanding repellers in the complex plane $\mathbb{C}$. We do it simultaneously without discerning which of these cases holds. We keep the notation of Theorem 4.10 and Theorem 4.12. We denote by M either the number $M_k(S)$ coming from (4.1) or from (4.2) or just the number 2, depending on the, one of the three, kinds of admissible systems under consideration. Fix some $(\lambda_0, t_0) \in \Lambda \times \mathbb{R}$, and denote, following Theorem 4.3, $\gamma_{\lambda_0,t_0} = \exp\bigl(\mathrm{P}(-t_0 \log|T'_{\lambda_0}|)\bigr)$. Now, the analytic perturbation theory of linear operators (see [Zi] for example) together with Theorem 4.3 both guarantee the existence of a holomorphic function

$$\gamma : \Lambda^* \times \mathbb{D}_{\mathbb{C}}(t_0, R) \to \mathbb{C}$$

(with possibly smaller open set $\Lambda^* \ni \lambda_0$ and some $R > 0$) such that $\gamma(\lambda_0, t_0) = \gamma_{\lambda_0,t_0}$, and $\gamma(\lambda, t)$ is an isolated simple eigenvalue of the operator $L_{\lambda,t}$. So, in particular, for every $(\lambda, t) \in (\Lambda^* \times \mathbb{D}_{\mathbb{C}}(t_0, R)) \cap (\Lambda \times \mathbb{R})$ we have

$$\gamma(\lambda, t) = \exp\bigl(\mathrm{P}(-t \log|T'_\lambda|)\bigr).$$

Proposition 4.13. *For every $(\lambda_0, t) \in \Lambda \times \mathbb{R}$ there exists a neighbourhood $\Lambda^* \times D_{\mathbb{C}}(t_0, R) \subset \mathbb{C}^M \times \mathbb{C}$ of the point (λ_0, t_0) in $\mathbb{C}^M \times \mathbb{C}$ and a holomorphic function*

$$\nu : \Lambda^* \times \mathbb{D}_{\mathbb{C}}(t_0, R) \to \mathcal{H}^*_\alpha(J_{\lambda_0})$$

such that for every $(\lambda, t) \in \Lambda^* \times \mathbb{D}_{\mathbb{C}}(t_0, R) \cap (\Lambda \times \mathbb{R})$ and every $g \in \mathcal{H}_1(J_\lambda)$ we have

$$\nu_{\lambda,t}(g \circ \tau_\lambda) = m_{\lambda,t}(g).$$

In particular, $\nu_{\lambda_0,t}(g) = m_{\lambda_0,t}(g)$.

Proof. Let $\varepsilon > 0$ be so small that with $G := D(\gamma(\lambda_0, t_0), \varepsilon)$, the point $\gamma(\lambda_0, t_0)$ is the only element of the spectrum of L_{λ_0,t_0} in the disk $2G := D(\gamma(\lambda_0, t_0), 2\varepsilon)$. Choosing the neighbourhood Λ^* and the value R small enough, we may assume that for all $(\lambda, t) \in \Lambda^* \times D_{\mathbb{C}}(t_0, R)$, $\gamma(\lambda, t)$ is the only element of the spectrum of the operator $L_{\lambda,t}$ in $2G$. Denote by $1\!\!1_G(L_{\lambda,t})$ the projection onto the eigenspace corresponding to this eigenvalue:

$$1\!\!1_G(L_{\lambda,t}) = \frac{1}{2\pi i} \int_{\partial G} (zI - L_{\lambda,t})^{-1} \, dz.$$

Then the function

$$\Lambda^* \times D_{\mathbb{C}}(t_0, R) \ni (\lambda, t) \mapsto 1\!\!1_G(L_{\lambda,t})$$

is analytic.

Let S be an arbitrary element of $\mathcal{H}_\alpha^*(J_{\lambda_0})$, not vanishing on some eigenvector of L_{λ_0,t_0} corresponding to the eigenvalue $\gamma(\lambda_0,t_0)$. Then, shrinking the neighbourhood $\Lambda^*\times D_{\mathbb{C}}(t_0,R)$ if necessary, we may assume that for every $(\lambda,t)\in\Lambda^*\times D_{\mathbb{C}}(t_0,R)$, the functional S does not vanish on some eigenvector of $L_{\lambda,t}$ corresponding to the eigenvalue $\gamma(\lambda,t)$.

Consider the vector $v=1\!\!1_G(L_{\lambda_0,t_0})(1\!\!1)$. We claim that $v\neq 0$. Indeed, writing $L_{\lambda_0,t_0}=1\!\!1_G(L_{\lambda_0,t_0})+(L_{\lambda_0,t_0}-1\!\!1_G(L_{\lambda_0,t_0}))=L_1+L_2$ we have $L_1\circ L_2=L_2\circ L_1=0$, $L_j\circ L_j=L_j$ $(j=1,2)$, and $\mathcal{H}_\alpha(J_{\lambda_0})=E_1\oplus E_2$, where $E_j=L_j(\mathcal{H}_\alpha(J_{\lambda_0}))$. Both E_1 and E_2 are invariant subspaces, and the spectral radius of $L_{\lambda_0,t_0}:E_2\to E_2$ is strictly smaller than $\gamma(\lambda_0,t_0)$. On the other hand, see [PU], we have

$$\lim_{n\to\infty}\gamma(\lambda_0,t_0)^{-n}L^n_{\lambda_0,t_0}(1\!\!1)\to\rho_{\lambda_0,t_0},$$

and the convergence is uniform. This shows that $1\!\!1\notin E_2$ and $1\!\!1_G(L_{\lambda_0,t_0})(1\!\!1)\neq 0$. Thus, shrinking the neighbourhood $\Lambda^*\times D_{\mathbb{C}}(t_0,R)$ again, if necessary, we can assume that $1\!\!1_G(L_{\lambda,t})(1\!\!1)\neq 0$ for all $(\lambda,t)\in\Lambda^*\times D_{\mathbb{C}}(t_0,R)$. Therefore, the following function is well defined:

$$\Lambda^*\times D_{\mathbb{C}}(t_0,R)\ni(\lambda,t)\mapsto\nu_{\lambda,t}:=\frac{S\circ 1\!\!1_G(L_{\lambda,t})}{S\circ 1\!\!1_{G_1}(L_{\lambda,t})(1\!\!1)}\in\mathcal{H}_\alpha^*(J_{\lambda_0}).$$

Hence, the function $(\lambda,t)\mapsto\nu_{\lambda,t}\in\mathcal{H}_\alpha^*(J_{\lambda_0})$ is analytic, as a composition of analytic and linear functions. Moreover, we have, for all $g\in\mathcal{H}_\alpha(J_{\lambda_0})$, that

$$\begin{aligned}L^*_{\lambda,t}\nu_{\lambda,t}(g)=\nu_{\lambda,t}(L_{\lambda,t}(g))&=\frac{S\circ 1\!\!1_G(L_{\lambda,t})(L_{\lambda,t}g)}{S\circ 1\!\!1_G(L_{\lambda,t})(1\!\!1)}=\frac{S\circ L_{\lambda,t}(1\!\!1_G(L_{\lambda,t}(g)))}{S\circ 1\!\!1_G(L_{\lambda,t})(1\!\!1)}\\ &=\frac{S(\gamma(\lambda,t)1\!\!1_G(L_{\lambda,t})(g))}{S\circ 1\!\!1_G(L_{\lambda,t})(1\!\!1)}=\frac{\gamma(\lambda,t)S\circ 1\!\!1_G(L_{\lambda,t})(g)}{S\circ 1\!\!1_G(L_{\lambda,t})(1\!\!1)}\\ &=\gamma(\lambda,t)\nu_{\lambda,t}(g).\end{aligned}\tag{4.4}$$

In the formula above, we used the fact that the operators $1\!\!1_G(L_{\lambda,t})$ and $L_{\lambda,t}$ commute and that $1\!\!1_G(L_{\lambda,t})$ is the Riesz projection onto the one–dimensional eigenspace corresponding to the eigenvalue $\gamma(\lambda,t)$. Recall that for all $\lambda\in\Lambda$ and $t\in\mathbb{R}$ the operator $L_{\lambda,t}$ is real and positive. For each such (λ,t) put

$$\tilde\rho_{\lambda,t}=\rho_{\lambda,t}\circ\tau_\lambda.$$

At the moment, as an ingredient of the proof of Proposition 4.13, we will need and we will prove the following.

Lemma 4.14. *For every function $g\in\mathcal{H}_\alpha(J_{\lambda_0})$ and for every $(\lambda,t)\in(\Lambda^*\times\mathbb{D}_{\mathbb{C}}(R))\cap(\Lambda\times\mathbb{R})$, we have*

$$1\!\!1_G(L_{\lambda,t})(g\circ\tau_\lambda)=m_{\lambda,t}(g)\tilde\rho_{\lambda,t}.$$

Proof. A straightforward calculation shows that

$$L_{\lambda,t}(g\circ\tau_\lambda)(z)=\mathcal{L}_{\lambda,t}(g)(\tau_\lambda(z)),$$

or, denoting by $R:\mathcal{H}_1(J_\lambda)\to\mathcal{H}_\alpha(J_{\lambda_0})$ the linear operator $R(g)=g\circ\tau_\lambda$,

$$L_{\lambda,t}\circ R=R\circ\mathcal{L}_{\lambda,t}.$$

Despite the fact that usually R is not surjective, so not a conjugacy, this equality is good enough to proceed further. Because of it and the integral formula for the projection $1\!\!1_G(L_{\lambda,t})$), a straightforward calculation yields

$$1\!\!1_G(L_{\lambda,t})(g\circ\tau_\lambda)(z)=1\!\!1_G(\mathcal{L}_{\lambda,t})(g)(\tau_\lambda(z)),\tag{4.5}$$

where in here the first projection is in the space $\mathcal{H}_\alpha(J_{\lambda_0})$ while the second one is in the space $\mathcal{H}_1(J_\lambda)$. Since the operator $\mathcal{L}_{\lambda,t}$ can be treated as an operator

$$\mathcal{L}_{\lambda,t}:\mathcal{H}_1(J_\lambda)\to\mathcal{H}_1(J_\lambda),$$

for every $g\in\mathcal{H}_1(J_\lambda)$, the iterates $\gamma(\lambda,t)^{-n}\mathcal{L}^n_{\lambda,t}(g)$ converge to the function $m_{\lambda,t}(g)\cdot\rho_{\lambda,t}$. This implies that the projection $1\!\!1_G(\mathcal{L}_{\lambda,t})$ (in $\mathcal{H}_1(J_\lambda)$) onto the 1–dimensional eigenspace corresponding to the eigenvalue $\gamma(\lambda,t)$ must be given by the formula

$$g\mapsto m_{\lambda,t}(g)\cdot\rho_{\lambda,t}.$$

Consequently, using formula (4.5), we see that

$$1\!\!1_G(L_{\lambda,t})(g\circ\tau_\lambda)=m_{\lambda,t}(g)\cdot\tilde\rho_{\lambda,t}$$

□

Continuing the proof of Proposition 4.13, by making use of Lemma 4.14, we can write:

$$\begin{aligned}\nu_{\lambda,t}(g\circ\tau_\lambda)&=\frac{S\circ 1\!\!1_G(L_{\lambda,t})(g\circ\tau_\lambda)}{S\circ 1\!\!1_G(L_{\lambda,t})(1\!\!1)}=\frac{S(m_{\lambda,t}(g)\tilde\rho_{\lambda,t})}{S(m_{\lambda,t}(1\!\!1)\tilde\rho_{\lambda,t})}\\&=\frac{m_{\lambda,t}(g)S(\tilde\rho_{\lambda,t})}{m_{\lambda,t}(1\!\!1)S(\tilde\rho_{\lambda,t})}=m_{\lambda,t}(g).\end{aligned}\tag{4.6}$$

This ends the proof of Proposition 4.13. □

Remark 4.15. *In particular, it follows from Proposition* 4.13 *that for* $(\lambda,t)\in(\Lambda^*\times\mathbb{D}_\mathbb{C}(t_0,R))\cap(\Lambda\times\mathbb{R})$ *the functional* $\nu_{\lambda,t}$ *which is a priori an element of* $(\mathcal{H}^*_\alpha(J_{\lambda_0}))$ *is, actually, a restriction of the measure* $m_{\lambda,t}\circ\tau_\lambda^{-1}$ *to* $\mathcal{H}_\alpha(J_{\lambda_0})$.

Notation. The functionals $\nu_{\lambda,t}\in\mathcal{H}_\alpha(J_{\lambda_0})$, $(\lambda,t)\in\Lambda^*\times\mathbb{D}_\mathbb{C}$, where, we recall, Λ^* is a neighbourhood of λ_0) depend on "basic" point λ_0. Since we shall use a collection of "basic points" (λ_0,t_0), to underline this dependence, we will write $\nu_{\lambda,t}^{\lambda_0,t_0}$. Similarly, we will write $\tau_{\lambda}^{\lambda_0} : J_{\lambda_0} \to J_{\lambda}$, and, whenever defined, $\tau_{\lambda_2}^{\lambda_1} := \tau_{\lambda_1}^{\lambda_0} \circ \left(\tau_{\lambda_1}^{\lambda_0}\right)^{-1} : J_{\lambda_1} \to J_{\lambda_2}$.

As a straightforward consequence of holomorphic dependence of ν on λ and t, we get the following.

Proposition 4.16. *Let Λ' be a bounded open set of parameters with $\Lambda' \subset \overline{\Lambda'} \subset \Lambda$, let $\mathcal{T}$ be a bounded interval in $\mathbb{R}$. Then for every $\alpha \in (0,1)$ there exist two reals $C > 0$, $\eta > 0$, and two finite sets $\Lambda_0 \subset \Lambda'$ and $\mathcal{T}_0 \subset \mathcal{T}$, such that the collection*

$$\{B(\lambda_0, \eta/2) \times B(t_0, \eta/2) : (\lambda_0, t_0) \in \Lambda_0 \times \mathcal{T}_0\}$$

forms a cover of $\Lambda' \times \mathcal{T}$ and for every $(\lambda_0, t_0) \in \Lambda_0 \times \mathcal{T}_0$, and $(\lambda_1, t_1), (\lambda_2, t_2) \in B(\lambda_0, \eta) \times B(t_0, \eta)$, we have

$$|||\nu_{\lambda_1,t_1}^{\lambda_0,t_0} - \nu_{\lambda_2,t_2}^{\lambda_0,t_0}|||_{\alpha} \le C\left(||\lambda_1 - \lambda_2|| + |t_1 - t_2|\right), \tag{4.7}$$

where, we recall, the norm $|||\cdot|||_{\alpha}$ is calculated in $\mathcal{H}_{\alpha}^{*}(J_{\lambda_0})$.

Denote by $\mathcal{F}$ the family of all compact subsets of $\mathbb{R}^k$. We shall prove the following.

Lemma 4.17. *Let G be an arbitrary set in $\mathcal{F}$. Then for every $\delta \in (0,1)$ there exists a Lipschitz continuous function $\psi_{\delta,G} : \mathbb{R}^k \to \mathbb{R}$ such that*

(a) *with characteristic functions $\mathbb{1}_G$ and $\mathbb{1}_{B(G,\delta)}$, it holds*

$$\mathbb{1}_G \le \psi_{\delta,G} \le \mathbb{1}_{B(G,\delta)},$$

and

(b)

$$||\psi_{\delta,G}||_1 \le \delta^{-1},$$

where $||\psi_{\delta,G}||_1$ denotes the Lipschitz norm of the function $\psi_{\delta,G}$.

Proof. Define the function $\psi_{\delta} : [0,\infty) \to [0,1]$ be the following formula

$$\psi(t) = \begin{cases} 1 - \frac{t}{\delta} & \text{if } \ t \in [0,\delta] \\ 0 & \text{if } \ t \in [\delta, +\infty). \end{cases}$$

Of course ψ is well-defined and contiuous. In fact, it is straightforward to observe that ψ is Lipschitz continuous with Lipschitz constant bounded above by δ^{-1}. Now, let G be an arbitrary set in $\mathcal{F}$. Define $\psi_{\delta,G} : \mathbb{R}^k \to \mathbb{R}$ by putting

$$\psi_{\delta,G}(z) := \psi(\operatorname{dist}(z, G)).$$

Thus, as a composition of two Lipschitz continuous functions, one with Lipschitz constant bounded above by 1 and the other with Lipschitz constant bounded above by δ^{-1}, the function $\psi_{\delta,G}$ is Lipschitz continuous with Lipschitz constant bounded above by δ^{-1}. Hence

$$||\psi_{\delta,G}||_1 \le \max\{1, \delta^{-1}\} = \delta^{-1},$$

and the proof is complete. $\square$

Let Λ' be a bounded set of parameters, with $\overline{\Lambda'} \subset \Lambda$. Let $\mathcal{T} = [0, k]$ where, we recall, $k \ge 1$ is the dimension of the ambient space $\mathbb{R}^k$. Fix an arbitrary $\alpha \in (0, 1)$. Applying Proposition 4.16 we obtain two finite sets Λ_0 and $\mathcal{T}_0$, so that the statement of Proposition 4.16 holds. Let $C, \eta > 0$, be the constants provided by this proposition. By increasing C, if necessary, we can assume that for all $\lambda_0 \in \Lambda_0$, all $\lambda_1, \lambda_2 \in B(\lambda_0, \eta)$ and all $y \in J_{\lambda_1}$, we have

$$||\tau_{\lambda_2}^{\lambda_1}(y) - y|| \le C||\lambda_1 - \lambda_2||. \tag{4.8}$$

Since $\mathrm{h}_\lambda = \mathrm{HD}(J_\lambda)$ depends in a real–analytic manner on λ (see [PU], [RU], comp. [Ru2] and [UZd] and the references therein), increasing $C > 0$, if necessary, again, we get

$$\frac{1}{C}||\lambda_1 - \lambda_2|| \le |h_{\lambda_1} - h_{\lambda_2}| \le C||\lambda_1 - \lambda_2|| \tag{4.9}$$

and

$$||\tau_\lambda^{\lambda_0}(x) - \tau_\lambda^{\lambda_0}(y)|| \le C||x - y||^\alpha \tag{4.10}$$

for all $\lambda, \lambda_0 \in \Lambda'$ with $||\lambda - \lambda_0|| \le \eta$. We in addition require that $C \ge 2$. Recall from Proposition 4.16 that the collection $\{B(\lambda_0, \eta/2) \times B(t_0, \eta/2) : (\lambda_0, t_0) \in \Lambda_0 \times \mathcal{T}_0\}$ is an open cover of the compact set $\overline{\Lambda'} \times \mathcal{T}$. Let $\gamma > 0$ be its Lebesgue number. Put

$$\varepsilon := \min\{\gamma, \eta/C\}.$$

Consider an arbitrary pair of parameters $\lambda_1, \lambda_2 \in \Lambda'$ with $||\lambda_1 - \lambda_2|| < \varepsilon$. We then see that there exists $(\lambda_0, t_0) \in \Lambda_0 \times \mathcal{T}_0$ such that for all $i = 1, 2$ we have

$$||\lambda_i - \lambda_0|| < \eta \ \text{ and } \ |h_{\lambda_i} - h_{\lambda_0}| < \eta.$$

Given an arbitrary set $G \in \mathcal{F}$, consider the function $\psi_{\delta,G} : \mathbb{R}^k \to [0, 1]$, and then, two additional functions

$$\psi_{\delta,G} \circ \tau_{\lambda_i}^{\lambda_0} : J_{\lambda_0} \to [0, 1],$$

$i = 1, 2$. Taking $\eta > 0$ small enough we may require all α-Hölder norms of $\tau_\lambda^{\lambda_0}$, with $\lambda_0 \in \Lambda_0$ and $\lambda \in B(\lambda_0, \eta)$, to be bounded above by the constant $C > 0$ dealt with above. Then

$$||\psi_{\delta,G} \circ \tau_{\lambda_i}^{\lambda_0}||_\alpha \le C||\psi_{\delta,G}||_1 \le C\delta^{-1}. \tag{4.11}$$

hence

$$m_{\lambda_1}(G) = m_{\lambda_1,h_{\lambda_1}}(G) \le m_{\lambda_1,h_{\lambda_1}}(\psi_{\delta,G}) = \nu_{\lambda_1,h_{\lambda_1}}(\psi_{\delta,G} \circ \tau_{\lambda_1}^{\lambda_0}) \tag{4.12}$$

and

$$m_{\lambda_2}(B(G,\delta)) = m_{\lambda_2,h_{\lambda_2}}(B(G,\delta)) \geq m_{\lambda_2}(\psi_{\delta,G}) = \nu_{\lambda_2,h_{\lambda_2}}(\psi_{\delta,G}\circ\tau_{\lambda_2}^{\lambda_0}). \tag{4.13}$$

Hence,

(4.14)

$$\begin{aligned}
m_{\lambda_1}(G) &\leq \nu_{\lambda_2,h_{\lambda_2}}(\psi_{\delta,G}\circ\tau_{\lambda_2}^{\lambda_0}) + \Big(\nu_{\lambda_1,h_{\lambda_1}}(\psi_{\delta,G}\circ\tau_{\lambda_1}^{\lambda_0}) - \nu_{\lambda_2,h_{\lambda_2}}(\psi_{\delta,G}\circ\tau_{\lambda_2}^{\lambda_0})\Big) \\
&\leq m_{\lambda_2}(B(G,\delta)) + (\nu_{\lambda_1,h_{\lambda_1}} - \nu_{\lambda_2,h_{\lambda_2}})(\psi_{\delta,G}\circ\tau_{\lambda_1}^{\lambda_0}) + \\
&\qquad + \nu_{\lambda_2,h_{\lambda_2}}(\psi_{\delta,G}\circ\tau_{\lambda_1}^{\lambda_0} - \psi_{\delta,G}\circ\tau_{\lambda_2}^{\lambda_0}) \\
&\leq m_{\lambda_2}(B(G,\delta)) + C(|\lambda_2-\lambda_1| + |h_{\lambda_2}-h_{\lambda_1}|)\cdot||\psi_{\delta,G}\circ\tau_{\lambda_1}^{\lambda_0}||_\alpha + \\
&\qquad + ||\psi_{\delta,G}\circ\tau_{\lambda_2}^{\lambda_0} - \psi_{\delta,G}\circ\tau_{\lambda_2}^{\lambda_0}||_\infty \\
&\leq m_{\lambda_2}(B(G,\delta)) + C^2\delta^{-1}(|\lambda_2-\lambda_1| + |h_{\lambda_2}-h_{\lambda_1}|) + \\
&\qquad + ||\psi_{\delta,G}\circ\tau_{\lambda_2}^{\lambda_0} - \psi_{\delta,G}\circ\tau_{\lambda_2}^{\lambda_0}||_\infty .
\end{aligned}$$

In the last line of this estimate we used the observation from Remark 4.15- that ν_{λ_2,h_2} is, as a matter of fact, a probability measure. The supremum norm which appears in the same last line can be easily bounded as follows:

$$\begin{aligned}
||\psi_{\delta,G}\circ\tau_{\lambda_2}^{\lambda_0} &- \psi_{\delta,G}\circ\tau_{\lambda_2}^{\lambda_0}||_\infty = \\
&= \sup_{x\in J_{\lambda_0}}\{|\psi_{\delta,G}\circ\tau_{\lambda_1}^{\lambda_0}(x) - \psi_{\delta,G}\circ\tau_{\lambda_2}^{\lambda_0}(x)|\} \\
&= \sup_{y\in J_{\lambda_1}}\{|\psi_{\delta,G}(y) - \psi_{\delta,G}\circ\tau_{\lambda_2}^{\lambda_1}(y)|\} \\
&\leq ||\psi_{\delta,G}||_1\cdot\sup_{y\in J_{\lambda_1}}\{||y-\tau_{\lambda_2}^{\lambda_1}(y)||\} \\
&\leq C\delta^{-1}||\lambda_1-\lambda_2||.
\end{aligned} \tag{4.15}$$

Using also (4.9) we can summarize the above calculations in the following.

Corollary 4.18. *Let $\Lambda'\subset\Lambda$ be a bounded set of parameters with $\overline{\Lambda'}\subset\Lambda$. There exist positive constants C' and ε such that if $\lambda_1,\lambda_2\in\Lambda'$ with $||\lambda_1-\lambda_2||<\varepsilon$ then, for every set $G\in\mathcal{F}$ and for every $\delta>0$ the following holds:*

$$m_{\lambda_1}(G) \leq m_{\lambda_2}(B(G,\delta)) + C'||\lambda_1-\lambda_2||\delta^{-1}. \tag{4.16}$$

Remark 4.19. *In fact, increasing the constant $C>0$ if necessary, we can get rid of the hypothesis $||\lambda_1-\lambda_2||<\varepsilon$ in the above corollary.*

5. Similarity IFSs satisfying the Strong Separation Condition; the case of arbitrary $k\geq 1$; Preparatory Results

Consider an admissible family of finite similarity IFSs in $\mathbb{R}^k$, $k\geq 1$. The following proposition was proved in [Ol].

Proposition 5.1. *Let $\lambda_0 \in \Lambda$. If U is a neighbourhood of λ_0 such that $\overline{U} \subset \Lambda$, then there exist positive constants A and R such that for all $\lambda \in U$,*

$$\mathrm{H}_{h_\lambda}(J_\lambda) = \inf\left\{\frac{\mathrm{diam}^{h_\lambda}(F)}{m_\lambda(F)} : F \in \mathcal{F}(A,R)\right\}, \tag{5.1}$$

where

$$\mathcal{F}(A,R) = \big\{F \subset \overline{B}(0,R) : F \text{ is closed and } \mathrm{diam}(F) \geq A\big\}.$$

Corollary 5.2. *Let Λ' be a bounded set of parameters such that $\overline{\Lambda'} \subset \Lambda$. Then there exist positive constant A and R such that for all $\lambda \in \Lambda'$,*

$$\mathrm{H}_{h_\lambda}(J_\lambda) = \inf\left\{\frac{\mathrm{diam}^{h_\lambda}(F)}{m_\lambda(F)} : F \in \mathcal{F}(A,R)\right\}. \tag{5.2}$$

We shall prove the following.

Lemma 5.3. *Let $\Lambda' \subset \overline{\Lambda'} \subset \Lambda$ be a bounded set of parameters. Then for every $\lambda \in \Lambda'$ there exists a set $F_\lambda \in \mathcal{F}(A,R)$ (with A, R given by the previous corollary) such that*

$$\mathrm{H}_\lambda(J_\lambda) = \frac{\mathrm{diam}^{h_\lambda}(F_\lambda)}{m_\lambda(F_\lambda)}. \tag{5.3}$$

Proof. By Corollary 5.2 there exists a sequence $F_n \in \mathcal{F}(A,R)$, $n \geq 1$, such that

$$\lim_{n\to\infty} \frac{\mathrm{diam}^{h_\lambda}(F_n)}{m_\lambda(F_n)} = \mathrm{H}_{h_\lambda}(J_\lambda).$$

Since, by Blaschke Selection Theorem the space $\mathcal{F}(A,R)$, endowed with Hausdorff metric, is compact, passing to a subsequence, we may assume that

$$F_n \longrightarrow F_\lambda \in \mathcal{F}(A,R),$$

where the converges is with respect to the Hausdorff metric on $\mathcal{F}(A,R)$. Then, obviously,

$$\lim_{n\to\infty} \mathrm{diam}^{h_\lambda}(F_n) = \mathrm{diam}^{h_\lambda}(F_\lambda)$$

and

$$m_\lambda(F_\lambda) \geq \limsup_{n\to\infty} m_\lambda(F_n).$$

Thus, $\frac{\mathrm{diam}^{h_\lambda}(F_\lambda)}{m_\lambda(F_\lambda)} \leq \mathrm{H}_{h_\lambda}(J_\lambda)$. Since the opposite inequality immediately follows from Corollary 5.2, the proof is thus complete. $\square$

Now, fix some bounded set of parameters Λ' with $\overline{\Lambda'} \subset \Lambda$, and then the constants $C', \varepsilon > 0$ according to Corollary 4.18. For each $\lambda \in \Lambda'$ let $F_\lambda \in$

$\mathcal{F}(A,R)$ be the set produced in Lemma 5.3. Fix $\lambda_1, \lambda_2 \in \Lambda'$ with $||\lambda_1 - \lambda_2|| < \varepsilon$. In order to simplify notation we write

$$h_1 := h_{\lambda_1},\ h_2 := h_{\lambda_2},\quad m_1 := m_{\lambda_1},\ m_2 := m_{\lambda_2},$$

and

$$F_1 := F_{\lambda_1},\ F_2 := F_{\lambda_2},\quad J_1 := J_{\lambda_1},\ J_2 := J_{\lambda_2}.$$

Fix some $\delta > 0$. Then, since $\mathrm{H}_{\mathrm{h}_{\lambda_1}}(J_1) < +\infty$ and $\mathrm{diam}(F_1) > 0$, it follows from Lemma 5.3 that $m_1(B(F_1,\delta) \geq m_1(F_1) > 0$. Corollary 5.2 then yields

$$\begin{aligned}\mathrm{H}_{h_2}(J_2) &= \frac{\mathrm{diam}^{h_2}(F_2)}{m_2(F_2)} = \frac{\mathrm{diam}^{h_2}(F_2)}{\mathrm{diam}^{h_1}(F_2)} \cdot \frac{\mathrm{diam}^{h_1}(F_2)}{m_1(\overline{B}(F_2,\delta))} \cdot \frac{m_1(\overline{B}(F_2,\delta))}{m_2(F_2)} \\ &\geq \mathrm{diam}^{h_2-h_1}(F_2) \cdot \frac{\mathrm{diam}^{h_1}(F_2)}{\mathrm{diam}^{h_1}(\overline{B}(F_2,\delta))} \cdot \mathrm{H}_{h_1}(J_1) \cdot \frac{m_1(\overline{B}(F_2,\delta))}{m_2(F_2)}\end{aligned} \tag{5.4}$$

Writing $\delta = \mathrm{diam}(F_2)\gamma$, we get:

$$\frac{\mathrm{diam}^{h_1}(\overline{B}(F_2,\delta))}{\mathrm{diam}^{h_1}(F_2)} \leq \frac{\mathrm{diam}^{h_1}(F_2)(1+2\gamma)^h}{\mathrm{diam}^{h_1}(F_2)}. = (1+2\gamma)^{h_1}.$$

Thus,

$$\frac{\mathrm{diam}^{h_1}(F_2)}{\mathrm{diam}^{h_1}(\overline{B}(F_2,\delta))} \geq \frac{1}{(1+2\gamma)^{h_1}} \geq 1 - 2h_1\gamma \geq 1 - \frac{2h_1}{A}\delta.$$

Therefore, the estimate (5.4) implies that

$$\mathrm{H}_{h_2}(J_2) \geq \mathrm{H}_{h_1}(J_1) \cdot \mathrm{diam}^{h_2-h_1}(F_2) \cdot \left(1 - \frac{2h_1}{A}\delta\right) \cdot \frac{m_1(\overline{B}(F_2,\delta))}{m_2(F_2)}. \tag{5.5}$$

Thus, in order to proceed with continuity issues, we need a good lower estimate of the ratio $m_1(\overline{B}(F_2,\delta))/m_2(F_2)$. This is done in the next section.

6. Similarity IFSs satisfying the Strong Separation Condition; the case of arbitrary $k \geq 1$; Holder Continuity of Hausdorff Measure

We keep the setting, notation, and hypotheses the same as in Section 5. In order to estimate the ratio $m_1(\overline{B}(F_2,\delta))/m_2(F_2)$ we shall prove the following.

Lemma 6.1. *There exists a positive constant c such that $m_\lambda(F_\lambda) \geq c$ for all $\lambda \in \Lambda'$.*

Proof. We already know, see [Ol] or [SUZ], that the function $\lambda \mapsto \mathrm{H}_{h_\lambda}(J_\lambda)$ is continuous, thus bounded in the set $\overline{\Lambda'}$. Denote the corresponding supremum by B. Then, with the use of Lemma 5.3, we get for every $\lambda \in \Lambda'$ that

$$B \geq \mathrm{H}_{h_\lambda}(J_\lambda) = \frac{\mathrm{diam}(F_\lambda)}{m_\lambda(F_\lambda)} \geq \frac{A}{m_\lambda(F_\lambda)}.$$

Hence $m_\lambda(F_\lambda) \geq A/B$, and the proof is complete. □

Using Corollary 4.18 and Remark 4.19, with $G := F_2$, we get

$$m_2(F_2) \leq m_1(B(F_2, \delta)) + C'||\lambda_1 - \lambda_2||\delta^{-1}.$$

Therefore, taking also into account Lemma 6.1, we get

$$\frac{m_1(B(F_2,\delta))}{m_2(F_2)} \geq 1 - C'\delta^{-1}||\lambda_2 - \lambda_1|| \cdot \frac{1}{m_2(F_2)} \geq 1 - C''\delta^{-1}||\lambda_2 - \lambda_1||, \tag{6.1}$$

where $C'' = C'/c$. Along with (5.5) this gives

$$\mathrm{H}_{h_2}(J_2) \geq \mathrm{H}_{h_1}(J_1) \cdot \mathrm{diam}^{h_2 - h_1}(F_2) \cdot \left(1 - \frac{2h_1}{A}\delta\right) \cdot (1 - C''\delta^{-1}||\lambda_2 - \lambda_1||). \tag{6.2}$$

Putting $M_- = \min\{A, (2R)^{-1}\}$ we thus get, for all $\lambda_1, \lambda_2 \in \Lambda'$ that,

$$\frac{\mathrm{H}_{h_2}(J_2)}{\mathrm{H}_{h_1}(J_1)} \geq M_-^{|h_2 - h_1|}\left(1 - \frac{2h_1}{A}\delta\right)(1 - C''\delta^{-1}||\lambda_2 - \lambda_1||). \tag{6.3}$$

Now, we relate δ to $||\lambda_1 - \lambda_2||$. Fix some $\kappa \in (0,1)$ and write δ in the form $\delta = ||\lambda_1 - \lambda_2||^\kappa$. Using formula (4.9) and (6.3); the latter as it is and also with exchanged places of λ_1 and λ_2, we get

$$\left|\frac{\mathrm{H}_{h_2}(J_2)}{\mathrm{H}_{h_1}(J_1)} - 1\right| \leq D\max\{||\lambda_2 - \lambda_1||^\kappa, ||\lambda_2 - \lambda_1||^{(1-\kappa)}\} \tag{6.4}$$

with some appropriate finite constant D. Since the value $\Lambda' \ni \lambda \mapsto \mathrm{H}_{h_\lambda}(J_\lambda)$ is, by its continuity, bounded, putting $\kappa = 1/2$ in (6.4), we get, for all $\lambda_1, \lambda_2 \in \Lambda'$, that

$$\begin{aligned} \left|\mathrm{H}_{h_{\lambda_2}}(J_{\lambda_2}) - \mathrm{H}_{h_{\lambda_1}}(J_{\lambda_1})\right| &= \mathrm{H}_{h_{\lambda_1}}(J_{\lambda_1})\left|\frac{\mathrm{H}_{h_{\lambda_2}}(J_{\lambda_2})}{H_{h_{\lambda_1}}(J_{\lambda_1})} - 1\right| \\ &\leq \mathrm{H}_{h_{\lambda_1}}(J_{\lambda_1})D||\lambda_2 - \lambda_1||^{\frac{1}{2}} \\ &\leq E||\lambda_2 - \lambda_1||^{\frac{1}{2}} \end{aligned} \tag{6.5}$$

with some other finite constant E. We thus proved the following.

Theorem 6.2. *If $\Lambda' \subset \Lambda$ is an arbitrary bounded domain such that $\overline{\Lambda'} \subset \Lambda$, and Φ^λ, $\lambda \in \Lambda$, forms an admissible family of IFSs consisting of similarities and satisfying the Strong Separation Condition, then the function*

$$\Lambda' \ni \lambda \mapsto \mathrm{H}_\lambda(J_\lambda)$$

is Hölder continuous with Hölder exponent $1/2$.

7. Similarity IFSs in the real line; Hausdorff measure is piecewise real–analytic

In this Section we keep the setting, notation, and assumptions of Section 5, assuming in addition that the phase space X is contained in the real line $\mathbb{R}$. In particular, $J_\lambda \subset \mathbb{R}$. Assume without loss of generality that $X = I$, the latter being the unit interval $[0,1]$. Such systems will be referred to as linear real systems. The set F_λ coming from Lemma 5.3 can be taken to be compact and convex, i. e. a closed bounded interval. Aiming to establish real analyticity of Hausdorff measure, we start with the following simple fact.

Proposition 7.1. *Let Λ' be bounded set of parameters, with $\overline{\Lambda'} \subset \Lambda$. There exist a constant $k \in \mathbb{N}$ such that, for every $\lambda \in \Lambda'$, the set F_λ is a convex hull of some intervals $\varphi_\lambda^{\mathbf{i}}(I)$, with $|\mathbf{i}| \le k$.*

Proof. Fix some $\lambda \in \Lambda'$. Put $F := F_\lambda$, $m := m_\lambda$, $h := h_\lambda$, $J := J_\lambda$, $L := \mathrm{diam}(F)$, $M := m(F)$. Write $F = [a,b]$. Of course then $|b-a| = L$. Since F minimizes the ratio in Lemma 5.3, the endpoints a and b of F are in J.

Now consider a point $x \in J$ with $x < a$. Put $r := a - x$. Set

$$F' = [a-2r, b] = [x-r, x+r] \cup [a,b].$$

Using Lemma 4.9, we get

$$m(F') = m(F) + m([x-r,x+r]) \ge m(F) + cr^h = M + cr^h.$$

Since the set F was chosen to minimize the ratio $\mathrm{diam}^h(G)/m(G)$ amongst all the sets G with diameter larger than A, we thus have:

$$\frac{(L+2r)^h}{M+cr^h} \ge \frac{\mathrm{diam}^h(F')}{m(F')} \ge \frac{\mathrm{diam}^h(F)}{m(F)} = \frac{L^h}{M}$$

or, equivalently,

$$\left(1 + \frac{2r}{L}\right)^h \ge \left(1 + \frac{c}{M} r^h\right). \tag{7.1}$$

Since $h \le 1$, $(1+x)^h \le 1+x$ for $x > 0$, and (7.1) gives

$$\left(1 + \frac{2r}{L}\right) \ge \left(1 + \frac{c}{M} r^h\right) \tag{7.2}$$

Equivalently,

$$r \geq \left(\frac{cL}{2M}\right)^{\frac{1}{1-h}}. \tag{7.3}$$

The conclusion is that for all $r < (cL/(2M))^{\frac{1}{1-h}}$, we have that $[a-r,a)\cap J = \emptyset$. Likewise, with the same method of proof, $(b, b+r]\cap J = \emptyset$. This means that there are large gaps of the Cantor set J_λ on both sides of the interval $F = F_\lambda$. Consequently, there is $k \in \mathbb{N}$ such that for every $\lambda \in \Lambda'$ the set F_λ is a union of some sets of the form $\varphi_\lambda^{\mathbf{i}}(I)$, with $|\mathbf{i}| \leq k$. $\square$

Now we can prove the main theorem of this section.

Theorem 7.2. *The function $\Lambda \ni \lambda \mapsto \mathrm{H}_\lambda(J_\lambda)$ is piecewise real–analytic.*

Proof. This is consequence of Proposition 7.1. Indeed, let $k \in \mathbb{N}$ be taken from Proposition 7.1. Fix some cylinders $\mathbf{i}_j$, $j = 1, \dots n$, $|\mathbf{i}_j| \leq k$, and consider the respective sets $\varphi_\lambda^{\mathbf{i}_j}(I)$. Let G_λ be the convex hull of their union. Each set constructed in this way will be called a (λ, k)–set. The endpoints of the interval G_λ move in a real-analytic way with respect to λ. Likewise, the function

$$\Lambda \ni \lambda \mapsto m_\lambda(G_\lambda) = \sum_{j=1}^{n} m_\lambda\big(\varphi_\lambda^{i_j}(I)\big) = \sum_{j=1}^{n} |(\varphi_\lambda^{i_j})'|^{h_\lambda}$$

is real–analytic. This is so since obviously the function $\lambda \mapsto h_\lambda$ is real-analytic, as h_λ is a unique solution to the equation $\sum |(\varphi_\lambda^{i})'|^t = 1$. We thus conclude that the function

$$\Lambda \ni \lambda \mapsto \frac{\mathrm{diam}^{h_\lambda}(G_\lambda)}{m_\lambda(G_\lambda)}$$

is real–analytic. Denote by $\mathcal{G}_\lambda^k$ the family of all (λ, k)– sets G_λ defined as above, with $|\mathbf{i}_j| \leq k$. This is a finite family of sets. Now, Proposition 7.1 implies that for every $\lambda \in \Lambda$,

$$\mathrm{H}_{h_\lambda}(J_\lambda) = \max\left\{\frac{\mathrm{diam}^{h_\lambda}(G_\lambda)}{m_\lambda(G_\lambda)} : G_\lambda \in \mathcal{G}_\lambda^k\right\}. \tag{7.4}$$

This function is piecewise real–analytic and the proof is complete. $\square$

Formula (7.4) suggests that, although the map $\lambda \mapsto \mathrm{H}_{h_\lambda}(J_\lambda)$ is piecewise real–analytic, one should not expect to have more, namely a real–analytic dependence throghout the whole set Λ. Below, we describe a simple example where the real analyticity does fail.

Example, where real analyticity of the Hausdorff measure fails: We start with four basic intervals contained in $I = [0,1]$:

$$I_1 = \left[0, \frac{1}{9}\right],\ I_2 = \left[\frac{2}{9}, \frac{3}{9}\right],\ I_3 = \left[\frac{6}{9}, \frac{7}{9}\right],\ I_4 = \left[\frac{8}{9}, 1\right].$$

Denote by Δ_1,Δ_2, Δ_3 the corresonding intervals gaps between those basic intervals. In particular Δ_1, Δ_3 are of length 1/9 while Δ_2 is of length 1/3.

Denote by φ^i, $i = 1,2,3,4$, the linear increasing maps, mapping the interval I onto respective intervals I_i. The limit set J_0 of this iterated function system is just the standard Middle-Third Cantor set. Denote by $h_0 = \log 2/\log 3$ its Hausdorff dimension. Let m_0 be the corresponding conformal measure on J_0. The proof of Lemma 5.3, in which the involvements of λ and Λ were irrelevant, gives that

$$\mathrm{H}_{h_0}(J_0) = \frac{\mathrm{diam}^{h_0}(F_0)}{m_0(F_0)},$$

where F_0 is some interval containing one of the gaps Δ_1, Δ_2 or Δ_3, thus of diameter at least 1/9. Obviously, the endpoints of F_0 are in J_0. Denote by L_0 the interval $[0,1/3]$ and by R_0 the interval $[2/3,1]$. This means that L_0 is the convex hull of $I_1 \cup I_2$ while R_0 is the convex hull of $I_3 \cup I_4$. Since $\mathrm{H}_{h_0}(J_0) = 1$, we have, for every interval G intersecting J_0, that

$$\frac{\mathrm{diam}^{h_0}(G)}{m_0(G)} \geq 1. \tag{7.5}$$

We need the following.

Lemma 7.3. *If a closed interval F contains the gap Δ_2 and $F \neq I$, then*

$$\frac{\mathrm{diam}^{h_0}(F)^{h_0}}{m_0(F)} > 1 = \frac{\mathrm{diam}^{h_0}(I)}{m_0(I)}.$$

Similarly, if a closed interval F either contains the gap Δ_1 or Δ_3 and $F \neq I, L, R$ then

$$\frac{\mathrm{diam}^{h_0}(F)^{h_0}}{m_0(F)} > 1 = \frac{\mathrm{diam}^{h_0}(L_0)^{h_0}}{m_0(L_0)} = \frac{\mathrm{diam}^{h_0}(R_0)}{m_0(R_0)}.$$

Proof. Assume that F contains the gap Δ_2. Denoting by r_1 and r_2 the lenghts of two intervals forming the connected components of the difference $F \setminus \Delta_2$ and using (7.5) we get $\mathrm{diam}(F) = \frac{1}{3} + r_1 + r_2$, while $m(F) \leq r_1^{h_0} + r_2^{h_0}$. Thus,

$$\frac{\mathrm{diam}^{h_0}(F)}{m_0(F)} \geq \frac{(\frac{1}{3} + r_1 + r_2)^{h_0}}{r_1^{h_0} + r_2^{h_0}}.$$

The following Lemma is based on elementary calculus. Its proof is omitted.

Lemma 7.4. *Consider the function* $[1,1/3]\times[0,1/3]\ni(r_1,r_2)\mapsto\phi(r_1,r_2)$*:*

$$\phi(r_1,r_2)=\frac{(\frac{1}{3}+r_1+r_2)^{h_0}}{r_1^{h_0}+r_2^{h_0}}.$$

Then for all (r_1,r_2) *we have* $\phi(r_1,r_2)\ge 1$*. Moreover,* $\phi(r_1,r_2)=1$ *iff* $r_1=1/3$ *and* $r_2=1/3$*.*

Using this lemma we conclude that if the maximizing interval F_0 contains the gap Δ_2, then $r_1=1/3$ and $r_2=1/3$, thus $F_0=I$. We check in the same way that, if the interval F_0 contains the gap $\Delta_1=(\frac{1}{9},\frac{2}{9})$ then either $F_0=L_0$ or $F_0=I$, and, similarly, if F_0 contains the gap $(\frac{7}{9}.\frac{8}{9})$ then either $F_0=L_0$ or $F_0=I$. $\square$

Put $r_0=\frac{1}{9}$. For r close to r_0 consider the linear iterated function system built in an analogous way with the four basic subintervals $I_1^r:=[0,r]$, $I_2^r:=I_2$, $I_3^r:=I_3$, $I_4^r:=I_4$; the only difference is thus that the first interval I_1^r is now equal $[0,r]$ rather than $[0,1/9]$. Denote by J_r the limit set of this linear iterated function system. The intervals L_r and R_r are defined analogously as L_0 and R_0. The Hausdorff dimension h_r of the set J_r is now given by the equation:

$$3\cdot(1/9)^{h_r}+r^{h_r}=1.$$

Thus,

$$r<r_0\iff h_r<h_{r_0}. \tag{7.6}$$

Let F_r be an interval for which

$$\mathrm{H}_{h_r}(J_r)=\frac{\mathrm{diam}^{h_r}(F_r)}{m_{h_r}(F_r)}.$$

As before, we can assume that F_r contains one of the gaps Δ_1^r, Δ_2^r or Δ_3^r. We have checked, for $r=r_0$, that the set F_0 is one of three sets L_0, R_0 or I. For every other set G containing one of the gaps, the ratio $\mathrm{diam}^{h_0}(G)/m_0(G)$ is strictly larger than that for L_0, R_0 or I. It follows from Proposition 7.1 that there exists $k\in\mathbb{N}$ such that for all r close to r_0 the set F_r must be an element of the finite family of sets $\mathcal{G}_k^r$. Each set $G^r\in\mathcal{G}_k^r$ is a convex hull of some cylinders of generation at most k. Since, for every $G^0\in\mathcal{G}_k^0$ we have

$$\frac{\mathrm{diam}^{h_0}(G^0)}{m_0(G)}>\min\left\{\frac{\mathrm{diam}^{h_0}(L_0)}{m_0(L_0)},\frac{(\mathrm{diam}^{h_0}(R_0)}{m_0(R_0)},\frac{\mathrm{diam}^{h_0}(I)}{m_0(I)}\right\},$$

the analogous inequality holds, by continuity, for all r in some interval $(r_0-\delta,r_0+\delta)$. We conclude that, for $r\in(r_0-\delta,r_0+\delta)$ the set F_r is, again, one of the sets L_r, R_r or I. Thus, we have the following formula:

$$\mathrm{H}_{h_r}(J_r) = \max\left\{\tilde{L}_r, \tilde{R}_r, \tilde{I}_r\right\}, \tag{7.7}$$

where $\tilde{L}_r$, $\tilde{R}_r$, and $\tilde{I}_r$ are the values of appropriate ratios $\mathrm{diam}^{h_r}(\cdot)/m_{h_r}(\cdot)$, calculated for for the arguments (i. e. sets) L_r, R_r, and I:

$$\tilde{L}_r = \frac{(\frac{1}{3})^{h_r}}{r^{h_r} + (\frac{1}{9})^{h_r}}, \quad \tilde{R}_r = \frac{(\frac{1}{3})^{h_r}}{(\frac{1}{9})^{h_r} + (\frac{1}{9})^{h_r}}, \quad \tilde{I}_r = 1.$$

A direct calculation, using (7.6) entails the following Lemma.

Lemma 7.5. *For $r < r_0$ we have*

$$\tilde{R}_r < \tilde{I}_r < \tilde{L}_r$$

while, for $r > r_0$ we have

$$\tilde{L}_r < \tilde{I}_r < \tilde{R}_r.$$

Hence,

$$\mathrm{H}_{h_r}(J_r) = \begin{cases} \frac{(1/3)^{h_r}}{(1/9)^{h_r}+(1/9)^{h_r}} & \text{if} \quad r < r_0 \\ \\ \frac{(1/3)^{h_r}}{(1/9)^{h_r}+r^{h_r}} & \text{if} \quad r \geq r_0 \end{cases}. \tag{7.8}$$

This function is not even differentiable at $r = r_0$, not to mention real-analytic.

8. Hölder continuity of the Hausdorff measures; self-conformal case

In this Section, we consider simultaneously admissible families either of conformal IFS in $\mathbb{R}^k$, $k \geq 3$, satisfying Strong Separation Condition, or admissible families of conformal expanding repellers in the complex plane $\mathbb{C}$. We shall prove the following main result of this section.

Theorem 8.1. *Let S_λ, $\lambda \in \Lambda$, (resp. T_λ, $\lambda \in \Lambda$), be an admissible family either of conformal IFS in $\mathbb{R}^k$, $k \geq 3$, or conformal expanding repellers in $\mathbb{C}$; we do not need and we do not specify which case holds. Let Λ' be a bounded open set such that $\overline{\Lambda'} \subset \Lambda$. Then the function*

$$\Lambda' \ni \lambda \mapsto \mathrm{H}_{h_\lambda}(J_\lambda)$$

is Hölder continuous with an exponent equal to $(3 + \sup\{h_\lambda : \lambda \in \Lambda'\})^{-1}$.

Proof. Fix some set Λ' satisfyng the assumptions and some $\kappa \in (0,1)$. Let $M = \sup\{|T'_\lambda|_{|J_\lambda} : \lambda \in \Lambda'\}$. Fix the constants $C, C', \varepsilon > 0$ with which the statement of Corollary 4.18 holds. Let $\lambda_1, \lambda_2 \in \Lambda'$ with $||\lambda_1 - \lambda_2|| < \varepsilon$. To simplify notation we shall write

$$h_1 = h_{\lambda_1},\ h_2 = h_{\lambda_2},\ m_1 = m_{\lambda_1},\ m_2 = m_{\lambda_2},\ J_1 = J_{\lambda_1},\ J_2 = J_{\lambda_2}.$$

By Propositon 3.1 there exists a set $F_2 \in \mathcal{F}$ intersecting J_2 for which $\operatorname{diam}(F_2) \le ||\lambda_2 - \lambda_1||^\kappa$ and

$$\frac{\operatorname{diam}^{h_2}(F_2)}{m_2(F_2)} \ge \mathrm{H}_{h_2}(J_2) \ge (1 - ||\lambda_2 - \lambda_1||^\kappa)\frac{\operatorname{diam}^{h_2}(F_2)}{m_2(F_2)}. \tag{8.1}$$

Let $n \ge 1$ be the least integer for which

$$\operatorname{diam}(T_2^n(F_2)) \ge ||\lambda_2 - \lambda_1||^\kappa.$$

Then

$$\operatorname{diam}(T_2^n(F_2)) \le M||\lambda_2 - \lambda_1||^\kappa.$$

with some constant M independent of λ and n. Due to distortion estimates (see Proposition 2.6 and Theorem 2.8), we have

$$\begin{aligned} &\left|\frac{\operatorname{diam}^{h_2}(T_2^n(F_2))}{m_2(T_2^n(F_2))}\Big/\frac{\operatorname{diam}^{h_2}(F_2)}{m_2(F_2)} - 1\right| = \\ &\qquad = \left|\left(\frac{\operatorname{diam}(T_2^n(F_2)}{\operatorname{diam}(F_2)}\right)^{h_2} \cdot \frac{m_2(F_2)}{m_2(T_2^n(F_2))} - 1\right| \\ &\qquad = \mathcal{O}(||\lambda_2 - \lambda_1||)^\kappa). \end{aligned} \tag{8.2}$$

So,

$$\begin{aligned} \mathrm{H}_{h_2}(J_2) &\ge (1 - ||\lambda_2 - \lambda_1||^\kappa) \cdot \frac{\operatorname{diam}^{h_2}(F_2)}{m_2(F_2)} \\ &\ge (1 - \mathcal{O}(||\lambda_2 - \lambda_1||^\kappa))(1 - \mathcal{O}(||\lambda_2 - \lambda_1||^\kappa)) \cdot \frac{\operatorname{diam}^{h_2}(T_2^n(F_2))}{m_2(T_\lambda^n(F_2))} \\ &\ge (1 - \mathcal{O}(||\lambda_2 - \lambda_1||^\kappa)) \cdot \frac{\operatorname{diam}^{h_2}(T_2^n(F_2))}{m_2(T_2^n(F_2))}. \end{aligned} \tag{8.3}$$

On the other hand, let $G \in \mathcal{F}$ be an arbitrary set intersecting J_2 with

$$||\lambda_2 - \lambda_1||^\kappa \le \operatorname{diam}(G) \le M||\lambda_2 - \lambda_1||^\kappa.$$

Fix

$$C||\lambda_2 - \lambda_1|| \le \delta \le ||\lambda_2 - \lambda_1||^\kappa.$$

Then, by (4.8), the open set $B(G,\delta)$ intersects J_1, and $\mathrm{diam}(B(G,\delta)) \le (M+2)||\lambda_2-\lambda_1||^\kappa$. Therefore, for m_1–almost every point $x \in J_1$ we have that $T_1^k(x) \in B(G,\delta)$ for infinitely many k. For each such k put

$$G_x^k := T_{1,x}^{-k}(\overline{B(G,\delta)}),$$

where $T_{1,x}^{-k}$ is the unique continuous branch of T_1^{-k} sending $T_1^k(x)$ to x. Using (8.2), by virtue of Theorem 3.2, for m_1–almost every point $x \in J_1$, we thus get

$$\mathrm{H}_{h_1}(J_1) \le \liminf_{k\to\infty} \frac{\mathrm{diam}^{h_1}(G_x^k)}{m_1(G_x^k)} \le (1+\mathcal{O}(||\lambda_2-\lambda_1||^\kappa))\frac{\mathrm{diam}^{h_1}(B(G,\delta))}{m_1(\overline{B(G,\delta)})}$$

Applying this to $G := T_2^n(F_2)$ we thus get,

$$\mathrm{H}_{h_1}(J_1) \le (1+\mathcal{O}(||\lambda_2-\lambda_1||^\kappa))\frac{\mathrm{diam}^{h_1}(B(T_2^n(F_2),\delta))}{m_1(\overline{B(T_2^n(F_2),\delta)})}. \tag{8.4}$$

Therefore, the inequalities (8.3) and (8.4) give an estimate of $\mathrm{H}_{h_2}(J_2)$ from below by the following product

$$\begin{aligned}&\mathrm{H}_{h_1}(J_1)\cdot(1-\mathcal{O}(||\lambda_2-\lambda_1||^\kappa)\cdot \mathrm{diam}^{h_2-h_1}(T_2^n(F_2))\cdot \\ &\qquad\qquad \cdot\left(\frac{\mathrm{diam}(T_2^n(F_2))}{\mathrm{diam}(B(T_2^n(F_2),\delta))}\right)^{h_1}\cdot\frac{m_1(\overline{B(T_2^n(F_2),\delta)})}{m_2(T_2^n(F_2))}.\end{aligned} \tag{8.5}$$

The ratio $\big(\mathrm{diam}(T_2^n(F_2))/\mathrm{diam}(B(T_2^n(F_2),\delta))\big)^{h_1}$ is obviously bounded below by

$$\left(1-\frac{2C''\delta}{\mathrm{diam}\big(T_2^n(F_2)\big)}\right)^{h_1}$$

with some appropriate finite constant $C''>0$. Using Corollary 4.18, we can continue as follows:

$$\begin{aligned}\mathrm{H}_{h_2}(J_2) \ge (1-&\mathcal{O}(||\lambda_2-\lambda_1||^\kappa)\cdot \mathrm{H}_{h_1}(J_1)\left(1-\frac{C'}{m_2(T_2^n(F_2))}\delta^{-1}||\lambda_2-\lambda_1||\right)\cdot \\ &\cdot(M||\lambda_2-\lambda_1||)^{-\kappa|h_2-h_1|)}\cdot\left(1-\frac{2C''\delta}{||\lambda_2-\lambda_1||^\kappa}\right)^{h_1}.\end{aligned} \tag{8.6}$$

In order to continue this estimate we need the following.

Lemma 8.2. *There exists a constant $d>0$ such that, for all $\lambda \in \Lambda'$,*

$$m_\lambda(T_\lambda^n(F_\lambda)) \ge d\,\mathrm{diam}^{h_\lambda}(T_\lambda^n(F_\lambda)).$$

Proof. Using only the bounded distortion property and the right-hand side of (8.1), we get

$$\frac{m_\lambda(T_\lambda^n(F_\lambda))}{\operatorname{diam}^{h_\lambda}(T_\lambda^n(F_\lambda))} \ge K^{-2h}\frac{m_\lambda(F_\lambda)}{\operatorname{diam}^{h_\lambda}(F_\lambda)} \ge \frac{K^{-2h}}{2\mathrm{H}_{h_\lambda}(J_\lambda)},$$

with some (distortion) constant $K > 0$. So, it is enough to find a common upper bound for $\mathrm{H}_{h_\lambda}(J_\lambda)$, $\lambda \in \Lambda'$. This however, in turn, follows from continuity of the function $\Lambda \ni \lambda \mapsto \mathrm{H}_{h_\lambda}(J_\lambda)$. $\square$

Let us also note that, due to (4.9),

$$\left|1 - ||\lambda_2 - \lambda_1||^{-\kappa|h_2 - h_1|}\right| = \mathcal{O}\left(||\lambda_2 - \lambda_1|| \log \frac{1}{||\lambda_2 - \lambda_1||}\right) = o(||\lambda_2 - \lambda_1||^{1-\gamma})$$

for every $\gamma > 0$. Therefore, fixing some small $\gamma > 0$ we can continue the estimate (8.6), with some another constant c, as follows:

(8.7)
$$\mathrm{H}_{h_2}(J_2) \ge \mathrm{H}_{h_1}(J_1)(1 - c||\lambda_2 - \lambda_1||^\kappa) \cdot \left(1 - c||\lambda_2 - \lambda_1||^{-h_2\kappa}\delta^{-1}||\lambda_2 - \lambda_1||\right) \cdot \\ \cdot (1 - c(||\lambda_2 - \lambda_1||^{1-\gamma})) \cdot (1 - c||\lambda_2 - \lambda_1||^{-\kappa}\delta).$$

Suppose now that we can choose $\delta > 0$ so that

$$||\lambda_2 - \lambda_1||^{-h_2\kappa}\delta^{-1}||\lambda_2 - \lambda_1|| \le ||\lambda_2 - \lambda_1||^\kappa \tag{8.8}$$

and

$$||\lambda_2 - \lambda_1||^{-\kappa}\delta \le ||\lambda_1 - \lambda_2||^\kappa \tag{8.9}$$

then (8.7) gives, possibly with a modified positive constant c, that

$$\mathrm{H}_{h_2}(J_2) \ge \mathrm{H}_{h_1}(J_1)(1 - c||\lambda_2 - \lambda_1||^\kappa) \tag{8.10}$$

for all $\lambda_1, \lambda_2 \in \Lambda'$. One verifies easily that the choice of δ satisfying (8.8) and (8.9) is possible if $\kappa \le \left(3 + \sup\{h_\lambda : \lambda \in \Lambda'\}\right)^{-1}$. Exchanging in (8.10) the roles of λ_1 and $\lambda_2 \in \Lambda'$, and using also the common lower bound for $H_{h_\lambda}(J_\lambda)$, in the same way as in (6.5)), we conclude that the function

$$\Lambda' \ni \lambda \mapsto \mathrm{H}_{h_\lambda}(J_\lambda)$$

is κ–Hölder continuous with $\kappa = \left(3 + \sup\{h_\lambda : \lambda \in \Lambda'\}\right)^{-1}$. The proof is complete. $\square$

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