ZERO TEMPERATURE LIMITS OF GIBBS-EQUILIBRIUM STATES FOR COUNTABLE ALPHABET SUBSHIFTS OF FINITE TYPE

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Abstract. Let Σ_A be a finitely primitive subshift of finite type on a countable alphabet. For appropriate functions $f : \Sigma_A \to \mathbb{R}$, the family of Gibbs-equilibrium states $(\mu_{tf})_{t\geq 1}$ for the functions tf is shown to be tight. Any weak*-accumulation point as $t \to \infty$ is shown to be a maximizing measure for f.

1. Introduction

Let Σ_A be a subshift of finite type on a countably infinite alphabet, and suppose that the function $f : \Sigma_A \to \mathbb{R}$ has summable variations. Further assumptions on f ensure it has a unique Gibbs-equilibrium state μ_f (see §2 for more details). The purpose of this article is to analyse the behaviour, as $t \to \infty$, of the Gibbs-equilibrium states μ_{tf} of tf. It will be shown that the family $(\mu_{tf})_{t\geq 1}$ is tight, thereby ensuring that it has a weak* accumulation point. Any such accumulation point is shown to be a maximizing measure for the function f (i.e. its f-integral dominates the integral of f with respect to other shift-invariant probability measures). This extends the analogous results of [CG, CLT, J] which were proved in the setting of finite alphabet subshifts of finite type.

The thermodynamic interpretation (cf. [Ru]) of the parameter t is as an inverse temperature of a system, while the measure μ_{tf} describes the equilibrium of the system at temperature $1/t$ (i.e. the one which minimizes the "free energy"). The $t \to \infty$ limit is therefore a zero temperature limit, and an accumulation point of the μ_{tf} can be interpreted as a *ground state*.

If f has a unique maximizing measure then our result asserts that μ_{tf} will converge to that measure. A more intriguing sitution arises when

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there are several maximizing measures, in which case we only know that μ_{tf} will accumulate on some non-empty subset of such measures. However, in all known examples it has been observed that the family μ_{tf} *does converge*, so a natural conjecture is that this is always the case; if this conjecture is true then the limit of the μ_{tf} may be regarded as the most "physically relevant" maximizing measure. This problem is open even for finite alphabet subshifts of finite type, though Brémont [Br] has shown that if f depends on only finitely many coordinates then the μ_{tf} do converge (cf. [C, J, PS] for related results).

2. Preliminaries

Let $\Sigma = \mathbb{N}^{\mathbb{N}}$ denote the full shift on the countable alphabet $\mathbb{N} = \{1, 2, \ldots\}$, equipped with the product topology.

Given an adjacency matrix $A : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$, let Σ_A denote the associated subshift of finite type

$$\Sigma_A = \{x \in \Sigma : A(x_n, x_{n+1}) = 1 \text{ for all } n \geq 1\}\,.$$

We shall suppose that A is *finitely primitive*, i.e. there exists an integer $N \geq 0$, and a finite sub-alphabet $\mathbb{M} \subset \mathbb{N}$, such that for all $x \in \Sigma_A$ and all $i \in \mathbb{N}_A$ there exists $w \in \mathbb{M}^N$ with $iwx \in \Sigma_A$. This implies that the shift map $T : \Sigma_A \to \Sigma_A$, defined by $(Tx)_n = x_{n+1}$, is topologically mixing.

For $n \in \mathbb{N}$, define $\Pi_n : \Sigma_A \to \mathbb{N}^n$ by $\Pi_n(x) = (x_1, \ldots, x_n)$, and $\pi_n : \Sigma_A \to \mathbb{N}$ by $\pi_n(x) = x_n$. If $w \in \mathbb{N}^n$ then the corresponding *cylinder set* in Σ_A is defined by $[w] = \{x \in \Sigma_A : \Pi_n(x) = w\}$. The subshift of finite type Σ_A is compact if and only if $\mathbb{N}_A := \{i \in \mathbb{N} : [i] \neq \emptyset\}$ is finite.

We shall assume that $f : \Sigma_A \to \mathbb{R}$ has *summable variations*, i.e. that

$$V(f) := \sum_{n=1}^{\infty} \mathrm{var}_n(f) < \infty\,, \tag{1}$$

where

$$\mathrm{var}_n(f) = \sup_{\Pi_n(x) = \Pi_n(y)} |f(x) - f(y)|\,.$$

In particular this implies that f is uniformly continuous (though not necessarily bounded, since $\mathrm{var}_0(f) = \sup_{x,y \in \Sigma_A} |f(x) - f(y)|$ is not included in the above sum).

We also assume that

$$\sum_{i \in \mathbb{N}} \exp(\sup f|_{[i]}) < \infty\,, \tag{2}$$

so in particular f is bounded above, and is unbounded below if and only if $I\!N_A$ is infinite. The summability condition (2) allows much of the thermodynamic formalism (cf. [Bo, Ru]) for finite alphabet subshifts of finite type to be generalised to the non-compact setting[1]. In particular it is equivalent [MU, Prop. 2.1.9] to the finiteness of the topological pressure

$$P(f) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n y = y} \exp\left(\sup_{x\in[\Pi_n(y)]} \sum_{i=0}^{n-1} f(T^i x) \right),$$

and implies the variational characterisation [MU, Thm. 2.1.8]

$$P(f) = \sup\{h(m) + \int f\, dm : m \in \mathcal{M}, \int f\, dm > -\infty\}, \qquad (3)$$

where $\mathcal{M}$ denotes the set of T-invariant Borel probability measures on Σ_A, and $h(m)$ the metric entropy of $m \in \mathcal{M}$.

Moreover, there is (see [MU, Theorems 2.2.4 and 2.3.3]) a unique measure $\mu_f \in \mathcal{M}$ for which there exist constants $C_2 > C_1 > 0$ such that

$$C_1 \le \frac{\mu_f[\Pi_n(x)]}{\exp(\sum_{i=0}^{n-1} f(T^i x) - nP(f))} \le C_2 \qquad (4)$$

for all $x \in \Sigma_A$, $n \ge 1$. In fact [MU, Thm. 2.2.7] we may choose

$$C_2 = \exp(4V(f)).$$

The measure μ_f is called the *Gibbs state* for f.

Suppose furthermore that[2]

$$\sum_{i\in I\!N} \inf(-f|_{[i]}) \exp(\inf f|_{[i]}) < \infty, \qquad (5)$$

so that in particular $\int f\, d\mu_f > -\infty$.

In this case (see [MU, Lem. 2.2.8, Thm. 2.2.9]) μ_f is an *equilibrium state* for f, in the sense that

$$P(f) = h(\mu_f) + \int f\, d\mu_f. \qquad (6)$$

Indeed it is the *unique* equilibrium state for f: if $m \in \mathcal{M} \setminus \{\mu_f\}$ is any other invariant measure with $\int f\, dm > -\infty$, then $h(m) + \int f\, dm <$

[1] Our reference to this generalised thermodynamic formalism is [MU] (though see also [Sa1, Sa2]), in which f is assumed to be locally Hölder (i.e. $\mathrm{var}_n(f) \to 0$ exponentially fast). The proofs in [MU] can, however, be easily adapted to the more general case where f has summable variations.

[2] Note that the lefthand side of (5) is well-defined: (2), together with the fact that $\mathrm{var}_1(f) < \infty$, implies that $\inf(-f|_{[i]}) \to \infty$, so that $\inf(-f|_{[i]})$ is positive except for finitely many i.

$P(f)$. Since μ_f is both the unique Gibbs state and the unique equilibrium state for f, we shall henceforth refer to it as the *Gibbs-equilibrium state* for f.

Since f satisfies (1), (2) and (5), so does the function tf for any $t \geq 1$. It follows that each such tf also has a unique Gibbs-equilibrium state μ_{tf}.

A *maximizing measure* for f is a measure $\mu \in \mathcal{M}$ such that $\int f\,d\mu \geq \int f\,dm$ for all $m \in \mathcal{M}$. Our assumptions on f ensure (see [JMU]) that this definition of a maximizing measure is equivalent to requiring that

$$\int f\,d\mu = \sup_{x \in \Sigma_A} \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)\,.$$

They also ensure (see [JMU]) the *existence* of a maximizing measure. The set of maximizing measures for f, which in general is not a singleton, will be denoted $\mathcal{M}_{\max}(f)$. The general properties of a maximizing measure are rather different from those of a Gibbs-equilibrium state. For example the support of a Gibbs-equilibrium state is always the full space Σ_A, whereas a maximizing measure has full support only in the trivial situation where f is cohomologous to a constant (i.e. $f = c + \varphi \circ T - \varphi$ for some $c \in I\!R$ and some bounded continuous φ). This latter fact is because if f is as above then there exists a bounded continuous φ such that the set of maxima of $f + \varphi - \varphi \circ T$ contains the support of a T-invariant measure (see [JMU]).

3. Proofs of results

To prove our main result, Theorem 1, we first require two preparatory lemmas. For the first of these we only use the fact that f is continuous and bounded above.

Lemma 1. *The map*

$$\mathcal{M} \longrightarrow I\!R$$

$$\mu \longmapsto \int f\,d\mu$$

is upper semi-continuous with respect to the weak topology on $\mathcal{M}$.*

Proof. Suppose that $\mu_i \to \mu$ in the weak* topology. That is, $\int g\,d\mu_i \to \int g\,d\mu$ for all bounded continuous functions g. We must prove that

$$\limsup_{i \to \infty} \int f\,d\mu_i \leq \int f\,d\mu\,. \tag{7}$$

Let $f_k \searrow f$ be a sequence of bounded continuous functions converging pointwise to f, for example $f_k = \max(f, -k)$. If $I \in I\!R$ is such that $I > \int f d\mu$ then $\int f_k \, d\mu < I$ for all sufficiently large $k \geq 1$, by the monotone convergence theorem. Choose one such k, and let $\delta > 0$ be arbitrary. Since f_k is a bounded continuous function, and $\mu_i \to \mu$ in the weak* topology, $\int f_k \, d\mu > \int f_k \, d\mu_i - \delta$ for all i sufficiently large. But $\int f_k \, d\mu_i \geq \int f \, d\mu_i$ since $f_k \geq f$, hence

$$I > \int f_k \, d\mu > \int f_k \, d\mu_i - \delta \geq \int f d\mu_i - \delta$$

for all i sufficiently large. But $\delta > 0$ and $I > \int f \, d\mu$ were arbitrary, so in fact

$$\int f \, d\mu \geq \int f \, d\mu_i \tag{8}$$

for all i sufficiently large, and (7) follows. □

For the second lemma we only use the fact that the μ_{tf} are *Gibbs* states.

Lemma 2. *The family of Gibbs-equilibrium states* $(\mu_{tf})_{t\geq 1}$ *is tight, i.e. for all* $\varepsilon > 0$ *there exists a compact set* $K \subset \Sigma_A$ *such that* $\mu_{tf}(K) > 1 - \varepsilon$ *for all* $t \geq 1$.

Proof. Given $\varepsilon > 0$, we will find an increasing sequence (n_k) in $\mathbb{N}$ such that the compact set $K = \{x \in \Sigma_A : 1 \leq x_k \leq n_k \ \forall k \in \mathbb{N}\}$ satisfies $\mu_{tf}(K) > 1 - \varepsilon$ for all $t \geq 1$. Now

$$\begin{aligned} \mu_{tf}(K) &= \mu_{tf}\left(\Sigma_A \setminus \cup_{k=1}^{\infty}\{x \in \Sigma_A : x_k > n_k\}\right) \\ &\geq 1 - \sum_{k=1}^{\infty} \mu_{tf}\left(\{x \in \Sigma_A : x_k > n_k\}\right) \\ &= 1 - \sum_{k=1}^{\infty} \sum_{i=n_k+1}^{\infty} \mu_{tf}(\pi_k^{-1}(i)) \\ &= 1 - \sum_{k=1}^{\infty} \sum_{i=n_k+1}^{\infty} \mu_{tf}[i] \, , \end{aligned}$$

so to ensure that $\mu_{tf}(K) > 1 - \varepsilon$ it suffices to choose the integers n_k such that

$$\sum_{i=n_k+1}^{\infty} \mu_{tf}[i] < \frac{\varepsilon}{2^k} \quad \text{for all } k \in \mathbb{N}, t \geq 1 \, . \tag{9}$$

We now show that such a choice is possible. First, the Gibbs property (4), with $n = 1$ and f replaced by tf, gives

$$\mu_{tf}[i] \leq e^{4tV(f)} \exp\left(\sup\{tf|_{[i]}\} - P(tf)\right) \, . \tag{10}$$

Now let $m \in \mathcal{M}$ be any measure for which $I := \int f\,dm$ is finite (e.g. we may take m to be supported on a periodic orbit). From (3) we have

$$P(tf) - tI = P\left(t(f-I)\right) \geq \int t(f-I)\,dm + h(m) \geq 0\,,$$

so together with (10) we deduce that

$$\begin{aligned} \mu_{tf}[i]_A &\leq e^{4tV(f)} \exp\left(\sup\{t(f-I)|_{[i]}\}\right) e^{-P(t(f-I))} \\ &\leq e^{4tV(f)} \exp\left(\sup\{t(f-I)|_{[i]}\}\right) \\ &= \exp\left(t\left(4V(f) - I + \sup\{f|_{[i]}\}\right)\right)\,. \end{aligned} \tag{11}$$

The summability condition (2) implies that $\sup f|_{[i]} \to -\infty$ as $i \to \infty$, with the convention that $f|_{[i]} = -\infty$ if $[i] = \emptyset$. In particular there exists $J \in \mathbb{N}$ such that if $i \geq J$ then

$$4V(f) - I + \sup f|_{[i]} < 0\,.$$

So if $t \geq 1$ and $i \geq J$ then $t(4V(f) - I + \sup f|_{[i]}) < 4V(f) - I + \sup f|_{[i]} < 0$, and from (11) we obtain

$$\mu_{tf}[i] \leq e^{4V(f)-I} e^{\sup f|_{[i]}}\,. \tag{12}$$

The summability condition (2) means there exists $n_k \geq J$ such that

$$\sum_{i=n_k+1}^{\infty} e^{\sup f|_{[i]}} < \frac{\varepsilon}{2^k} e^{I-4V(f)}\,,$$

and combined with (12) we deduce (9), as required. $\square$

Theorem 1. *The family of Gibbs measures $(\mu_{tf})_{t\geq 1}$ has a weak* accumulation point as $t \to \infty$. Any such accumulation point μ is a maximizing measure for f, and $\int f\,d\mu = \lim_{t\to\infty} \int f\,d\mu_{tf}$.*

Proof. By Lemma 2 the family $(\mu_{tf})_{t\geq 1}$ is tight, so by Prohorov's theorem [Bi, p. 37] there exists at least one weak* accumulation point.

Now suppose μ is any such accumulation point. If $p(t) = P(tf)$ for $t \geq 1$ then $p'(t) = \int f\,d\mu_{tf}$ (cf. [MU, Prop. 2.6.13]). But (3) implies that p is convex, so that $t \mapsto p'(t) = \int f\,d\mu_{tf}$ is non-decreasing, and bounded above by $\sup f$. It follows that the limit $\lim_{t\to\infty} p'(t) = \lim_{t\to\infty} \int f\,d\mu_{tf}$ *exists* and is *finite*. Moreover, Lemma 1 gives

$$\lim_{t\to\infty} \int f\,d\mu_{tf} \leq \int f\,d\mu\,. \tag{13}$$

In particular $\int f\,d\mu > -\infty$. From (3) and (6) it follows that

$$\int tf\,d\mu_{tf} + h(\mu_{tf}) \geq \int tf\,d\mu + h(\mu)\,,$$

so
$$\int f\,d\mu_{tf} + \frac{h(\mu_{tf})}{t} \ge \int f\,d\mu + \frac{h(\mu)}{t}\,. \tag{14}$$
Now $h(\mu_{tf}) = P(tf) - t\int f\,d\mu_{tf} = p(t) - tp'(t)$, so
$$\frac{d}{dt}h(\mu_{tf}) = -tp''(t) < 0$$
for $t \ge 1$. Therefore $h(\mu_{tf})$ is a decreasing function of $t \ge 1$, and in particular is bounded, so letting $t \to \infty$ in (14) gives
$$\lim_{t\to\infty} \int f\,d\mu_{tf} \ge \int f\,d\mu\,.$$
Combining this with (13) we see that $\lim_{t\to\infty}\int f\,d\mu_{tf} \ge \int f\,d\mu$, as required.

We now show that μ is f-maximizing. If not then there exists $\nu \in \mathcal{M}$ with $\int f\,d\nu - \int f\,d\mu = \varepsilon > 0$. Now f is bounded above, so $\int f\,d\nu < \infty$. Moreover $P(f) < \infty$, so (3) and (6) imply that $h(\nu) < \infty$. We can then define the affine map $l_\nu : I\!R \to I\!R$ by $l_\nu(t) = h(\nu) + t\int f\,d\nu$. Now $t \mapsto p'(t) = \int f\,d\mu_{tf}$ is a function which increases to its limit $\int f\,d\mu$, so in particular
$$\int f d\mu \ge \int f\,d\mu_{tf} = p'(t) \quad \text{for all } t \ge 1\,,$$
and hence
$$l'_\nu(t) = \int f\,d\nu = \int f\,d\mu + \varepsilon \ge p'(t) + \varepsilon$$
for all $t \ge 1$. Therefore $l_\nu(t) > p(t)$ for all sufficiently large t. That is, $h(\nu) + \int tf\,d\nu > P(tf)$ for all sufficiently large t, contradicting (3). Therefore μ is f-maximizing. □

Note that in the case of a finite alphabet subshift of finite type Σ_A, the identity $\int f\,d\mu = \lim_{t\to\infty}\int f\,d\mu_{tf}$ in Theorem 1 follows immediately from the fact that μ is a weak* accumulation point of μ_{tf}, since the continuous function f is automatically bounded on the compact space Σ_A.

In the finite alphabet case some extra information is known about μ: it is of maximal entropy within the class of f-maximizing measures (see [CG, CLT, J]). In the infinite alphabet case this is an open problem:

Question 1. *If μ is a weak* accumulation point of $(\mu_{tf})_{t\ge1}$, is it the case that*
$$h(\mu) = \sup_{m\in\mathcal{M}_{max}(f)} h(m)\ ? \tag{15}$$

An approach to proving (15) is to first show that

$$h(\mu) = \lim_{t\to\infty} h(\mu_{tf}) = \inf_{t\geq 1} h(\mu_{tf})\,. \tag{16}$$

The second equality in (16) is certainly true in the inifinite alphabet case, since $t \mapsto h(\mu_{tf})$ is decreasing (as noted in the proof of Theorem 1), and bounded below. Moreover

$$\lim_{t\to\infty} h(\mu_{tf}) \geq h(\mu)\,, \tag{17}$$

since μ_{tf} is the equilibrium state for tf, while μ is f-maximizing and hence tf-maximizing for $t \geq 0$, so

$$h(\mu_{tf}) - h(\mu) \geq \int tf\, d\mu - \int tf\, d\mu_{tf} \geq 0$$

for all $t \geq 1$.

We do not know, however, if *equality* holds in (17):

Question 2. *If μ is a weak* accumulation point of $(\mu_{tf})_{t\geq 1}$, is it the case that $h(\mu) = \lim_{t\to\infty} h(\mu_{tf})$?*

As noted above, in the finite alphabet case the answer is affirmative; this is proved by combining (17) with the well known fact [Wa, Thm. 8.2] that the entropy map $\nu \mapsto h(\nu)$ is upper semi-continuous on $\mathcal{M}$.

By contrast, for infinite alphabet subshifts of finite type the entropy map is in general *not* upper semi-continuous. To see this, let Σ be the full shift on $\mathbb{N}$, and define the probability vector P_n by

$$P_n = (1 - n^{-1}\,, \underbrace{(nk_n)^{-1}, \ldots, (nk_n)^{-1}}_{k_n \text{ terms}}, 0, 0, \ldots)\,,$$

where $k_n = \lceil e^{n^2} \rceil$. Let μ_n be the Bernoulli measure corresponding to P_n (so the support of μ_n is the full shift on the symbols $\{1, \ldots, n+1\}$). Then

$$h(\mu_n) \;=\; -(1-n^{-1})\log(1-n^{-1}) + n^{-1}\log(nk_n) \;>\; n\,.$$

In particular $h(\mu_n) \to \infty$ as $n \to \infty$, whereas the weak* limit of (μ_n) is the Dirac measure concentrated on the fixed point $(1, 1, \ldots)$. This measure has zero entropy, so the entropy map is not upper semi-continuous.

Of course this absence of upper semi-continuity does not rule out an affirmative answer to Question 2. In this case Question 1 could also be answered affirmatively, by the following argument. If $h(\mu) = \sup_{m\in\mathcal{M}_{\max}(f)} h(m)$ were *not* true then we could find $m \in \mathcal{M}_{\max}(f)$

with $h(m) - h(\mu) = \varepsilon > 0$. The affirmative answer to Question 2 then gives

$$h(m) - \lim_{t\to\infty} h(\mu_{tf}) = \varepsilon\,,$$

so that

$$h(m) - h(\mu_{tf}) \geq \frac{\varepsilon}{2} \tag{18}$$

for sufficiently large $t \geq 1$.

But m is f-maximizing, so $\int f\,dm \geq \int f\,d\mu_{tf}$ for all $t \geq 1$, and therefore

$$\int tf\,dm \geq \int tf\,d\mu_{tf} \tag{19}$$

for all $t \geq 1$. Combining (18) and (19) gives

$$h(m) + \int tf\,dm > h(\mu_{tf}) + \int tf\,d\mu_{tf}$$

for $t \geq 1$ sufficiently large. But this is a contradiction, because μ_{tf} is an equilibrium state for the function tf.

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