DIOPHANTINE APPROXIMATION AND SELF-CONFORMAL MEASURES

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Abstract. It is proved that the Hausdorff measure on the limit set of a finite conformal iterated function system is strongly extremal, meaning that almost all points with respect to this measure are not multiplicatively very well approximable. This proves Conjecture 10.6 from [2]. The strong extremality of all (S, P)-invariant measures is established, where S is a finite conformal iterated function system and P is a probability vector. Both above results are consequences of the much more general Theorem 1.5 concerning Gibbs states of Hölder families of functions.

1. Introduction, Preliminaries

A point $\mathbf{x} \in I\!\!R^n$ is very well multiplicatively approximable if there is $\delta > 0$, infinitely many points $\mathbf{p} = (p_1, p_2, ..., p_n) \in Z^n$, and integers $q \geq 1$ such that

$$\prod_{i=1}^{n} |qx_i - p_i| \leq q^{-(1+\delta)},$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)$. A point $\mathbf{x} = (x_1, x_2, ..., x_n) \in I\!\!R^n$ is called very well approximable if there is $\delta > 0$, infinitely many points $\mathbf{p} = (p_1, p_2, ..., p_n) \in \boldsymbol{Z}^n$, and integers $q \geq 1$ such that

$$||q\mathbf{x} - \mathbf{p}|| \leq q^{-\left(\frac{1}{n}+\delta\right)}.$$

Obviously every very well approximable point is very well multiplicatively approximable. Much effort has been devoted (Khintchine, Grosher, and others) to study these well approximable points and the complement of these points from the point of view of the Lebesgue measure on $I\!\!R^n$. It is a classical result that the set of very well multiplicatively approximable points (and so the set of very well approximable points) has Lebesgue measure zero but Hausdorff dimension equal to n. Thus the natural question arises about other measures. To be more precise, a Borel measure μ on $I\!\!R^n$ is is called extremal (strongly extremal) if μ-almost every point in $I\!\!R^n$ is not very well (multiplicatively very well) approximable. It was proven in [3] that the Riemann measure on any non-degenerate submanifold of $I\!\!R^n$ is strongly extremal, solving therefore positively a conjecture of Sprindzuk ([9]). In [2] the concept of a friendly measure has been introduced. In the class of friendly measures the authors of [2] distinguished the measures called in [8] absolutely friendly. This is the basic notion for us in this paper and it definition reads as follows. A Borel measure μ on $I\!\!R^n$ is called absolutely friendly if it

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satisfies the doubling (Federer) property and with some constants $C, R, \alpha > 0$

$$\mu\Big(B(x,r) \cap B(H,\varepsilon)\Big) \le C\left(\frac{\varepsilon}{r}\right)^{\alpha} \mu(B(x,r)) \tag{1.1}$$

for all $r \in (0,R)$, all $x \in supp(\mu)$, all $\varepsilon > 0$, and all affine hyperspaces H of codimension 1. In [2] every friendly measure was shown to be strongly extremal; comp. also [8] for related results. In [4] more diophantine consequences of absolutely friendly measures are derived. Exhibiting a large class of examples, the authors in [2] proved that the Hausdorff measure on the limit set of an irreducible iterated function system consisting of finitely many similitudes, is absolutely friendly, and consequently, strongly extremal. Utilizing our techniques worked out in the process of developing the theory of conformal iterated function systems, we prove Corollary 1.6, which is just Conjecture 10.6 from [2]. This result generalizes the just mentioned result of Kleinbock, Lindenstrauss and Weiss about strong extremality of Hausdorff measures on limit sets to the non-linear case. Corollary 1.6 is obtained as an immediate consequence of the much more general Theorem 1.5 concerning Gibbs states of Hölder families of functions. As its another immediate consequence, Corollary 1.7 is obtained establishing strong extremality of all (S,P)-invariant measures, where S is a conformal iterated function system and P is a probability vector.

Passing to preliminaries, let I be a finite index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from a compact metric X endowed with a metric d into X for which there exists $0 < s < 1$ such that $d(\phi_i(x), \phi_i(y)) \le sd(x,y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We are particularly interested in the properties of the limit set defined by such a system. We can define this set as the image of the coding space under a coding map as follows. Let $I^* = \bigcup_{n\ge1} I^n$, the space of finite words, and for $\omega \in I^n$, $n \ge 1$, let $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$. Let $I^\infty = I^{\mathbb{N}}$ be the space of all infinite sequences of elements of I. If $\omega \in I^* \cup I^\infty$ and $n \ge 1$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2\ldots\omega_n$. Since given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n \ge 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton and therefore, its unique element $\pi(\omega)$ defines the coding map $\pi : I^\infty \to X$. The main object in the theory of iterated function systems is the limit set defined as follows.

$$J = \pi(I^\infty) = \bigcup_{\omega\in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|n}(X) = \bigcap_{n\ge1} \bigcup_{|\omega|=n} \phi_\omega(X)$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that J is compact.

An iterated function system $S = \{\phi_i : X \to X : i \in I\}$ is said to satisfy the open set condition if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\phi_i(U) \subset U$ for every $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$.
An iterated function system S satisfying the open set condition is said to be conformal if $X \subset I\!\!R^d$ for some $d \geq 2$ and the following conditions are satisfied.

(1a) $U = \mathrm{Int}_{I\!\!R^d}(X)$.

(1b) There exists an open connected set $X \subset V \subset I\!\!R^d$ such that all maps ϕ_i, $i \in I$, extend to $C^{1+\varepsilon}$ conformal contracting diffeomorphisms $\overline{\phi_i}$ of V into V (throughout this entire paper we assume that if $d = 2$, then all the maps $\overline{\phi_i} : V \to V$, $i \in I$, are holomorphic).

Due to the result proved in [7], we may assume that $J \cap U \neq \emptyset$, the property known in the literature as the strong open set condition. It is by now a straightforward observation that (1b) implies the following.

(1c) Bounded Distortion Property(BDP). There exists $K \geq 1$ such that

$$|\overline{\phi}'_\omega(y)| \leq K|\overline{\phi}'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\overline{\phi}'_\omega(x)|$ means the norm of the derivative.

One may also deal with the case when $d = 1$ and to take (1c) as an extra assumption but we restrict ourselves in this paper exclusively to the case when $d \geq 2$. Let us now collect some geometric consequences of (BDP). We have for all words $\omega \in I^*$ and all convex subsets C of V that

$$\mathrm{diam}(\phi_\omega(C)) \leq ||\overline{\phi}'_\omega||\mathrm{diam}(C) \tag{1.2}$$

and

$$\mathrm{diam}(\phi_\omega(V)) \leq D||\overline{\phi}'_\omega||, \tag{1.3}$$

where $D \geq 1$ is a universal constant, $||\overline{\phi}'_\omega|| = \sup\{|\overline{\phi}'_\omega(x)| : x \in V\}$ and $||\phi'_\omega|| = \sup\{|\overline{\phi}'_\omega(x)| : x \in X\}$. In addition,

$$\mathrm{diam}(\phi_\omega(J)) \geq D^{-1}||\overline{\phi}'_\omega|| \tag{1.4}$$

and

$$\overline{\phi}_\omega(V) \supset \phi_\omega(B(x,r)) \supset B(\phi_\omega(x), K^{-1}||\overline{\phi}'_\omega||r), \tag{1.5}$$

for every $x \in X$, every $0 < r \leq \mathrm{dist}(X, \partial V)$, and every word $\omega \in I^*$.

Let $\sigma : I^\infty \cup I^* \to I^\infty \cup I^*$ be the shift map, i.e. cutting off the first coordinate. Passing to Hölder families of functions, introduced in [10], [1] and thoroughly explored in [6], fix $\beta > 0$ and let $F = \{f^{(i)} : X \to I\!\!R : i \in I\}$ be a family of continuous functions. For each $n \geq 1$ put

$$V_n(F) = \sup_{\omega \in I^n} \sup_{x,y \in X} \{|f^{(\omega_1)}(\phi_{\sigma(\omega)}(x)) - f^{(\omega_1)}(\phi_{\sigma(\omega)}(y))|\} \mathrm{e}^{\beta(n-1)},$$

and assume that

$$V_\beta(F) = \sup_{n\geq 1}\{V_n(F)\} < \infty.$$

The collection F is then called a Hölder family of functions (of order β). Throughout this paper the family F is always assumed to be Hölder of some order $\beta > 0$. Note that in [10], [1] and [6] we have primarily dealt with infinite countable index sets I and we needed the concept of summable Hölder families of functions. If the index set is finite, all the Hölder families are summable in the sens of [10], [1] and [6]. We have made the conventions that the empty word $\emptyset$ is the only word of length 0 and $\phi_\emptyset = \mathrm{Id}_X$. Following the classical thermodynamic formalism, we defined the topological pressure of F by setting

$$\mathrm{P}(F) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{|\omega|=n}\exp\left(\sup_X\sum_{j=1}^n f^{\omega_j}\circ\phi_{\sigma^j\omega}\right).$$

Notice that the limit indeed exists since the logarithm of the partition function

$$Z_n(F) = \sum_{|\omega|=n}\exp(\sup(S_\omega(F)))$$

is subadditive, where

$$S_\omega(F) = \sum_{j=1}^n f^{(\omega_j)}\circ\phi_{\sigma^j\omega}.$$

Moreover

$$\mathrm{P}(F) = \inf_{n\geq 1}\left\{\frac{1}{n}\log Z_n(F)\right\}.$$

Now, a Borel probability measure m_F is said to be F-conformal provided it is supported on J, for every Borel set $A \subset X$

$$m_F(\phi_\omega(A)) = \int_A \exp\Big(S_\omega(F) - \mathrm{P}(F)|\omega|\Big)\,dm_F, \quad \forall\omega\in I^* \tag{1.6}$$

and

$$m(\phi_\omega(X)\cap\phi_\tau(X)) = 0 \tag{1.7}$$

for all incomparable $\omega, \tau \in I^*$. A Borel probability measure μ supported on the limit set J is said to be S-invariant if and only if $\mu(\phi_\omega(X)\cap\phi_\tau(X)) = 0$ for all incomparable words $\omega, \tau \in I^*$ and for every set $A \subset X$

$$\sum_{i\in I}\mu(\phi_i(A)) = \mu(A).$$

In ([10], [1] comp. [6]) we have proved the following.

Theorem 1.1. *If F is a Hölder family of functions, then there exists exactly one F-conformal measure m_F and exactly one S-invariant measure μ_F absolutely continuous with respect to m_F. In addition, the Radon-Nikodym derivative $d\mu_F/dm_F$ is uniformly bounded away from zero and infinity.*

The measure μ_F is called the Gibbs state of the Hölder family F.

Remark 1.2. *It is well known that the both measures m_F and μ_F satisfy the doubling property saying that there exists a constant $E \geq 1$ such that for every $x \in I\!R^d$ and every radius $r > 0$, we have that $m_F(B(x, 2r)) \leq E m_F(B(x, r))$ and the same inequality holds with m_F replaced by μ_F.*

We end the list of facts concerning Hölder families of functions by stating the following technical but frequently used result.

Lemma 1.3. *Suppose that F is a Hölder family of functions. Then there exists a constant $Q \geq 1$ such that if $x, y \in \phi_\tau(X)$ for some $\tau \in I^*$, then for all $\omega \in I^*$*

$$|S_\omega(F)(x) - S_\omega(F)(y)| \leq Q e^{-\beta|\tau|}$$

Put

$$T = e^Q. \tag{1.8}$$

Definition 1.4. *If $d \geq 3$, then the iterated function system S is said to be irreducible provided that its limit set is not contained in a geometric sphere nor in an affine hyperplane of dimension $\leq d-1$. If $d = 2$, then the iterated function system S is said to be irreducible provided that its limit set is not contained in a union of the boundary of X and finitely many real-analytic curves of finite length.*

The main, most general result of this section is the following.

Theorem 1.5. *If $S = \{\phi_i\}_{i \in I}$ is a conformal irreducible iterated function system in $I\!R^d$, $d \geq 2$, and $\{f^{(i)}\}_{i \in I}$ is a Hölder family of functions, then the corresponding Gibbs measure μ_F and, equivalently, m_F is absolutely friendly, and consequently, strongly extremal.*

Let $\mathrm{HD}(A)$ denote the Hausdorff dimension of the set A. It is well known (see for example [5], comp. [6]) that the $\mathrm{HD}(J)$-dimensional Hausdorff measure restricted to J is finite and positive; it is remarkable that this measure is a multiple of the measure $m_{h\mathrm{Log}}$, where $h\mathrm{Log} = \{h \log |\phi_i'|\}_{i \in I}$ is a Hölder family of functions. Therefore, as an immediate consequence of Theorem 1.5, we get the following result stated in [2] as Conjecture 10.6.

Corollary 1.6. *If $S = \{\phi_i\}_{i \in I}$ is a conformal irreducible iterated function system in $I\!R^d$, $d \geq 2$, then the $\mathrm{HD}(J)$-dimensional Hausdorff measure restricted to J is absolutely friendly, and consequently, strongly extremal.*

This corollary extends Theorem 2.2 in [2], where all the generators ϕ_i, $i \in I$, were assumed to be similarity self-maps of $I\!R^d$. Given a probability vector $P = (p_i)_{i\in I}$ with all positive coordinates p_i, $i \in I$, a probability measure μ on J is said to be invariant with respect to the pair (S, P) if and only if

$$\mu = \sum_{i\in I} p_i \mu \circ (\phi_i|_J)^{-1}$$

It is easy to see any (S, P)-invariant measure coincides with the Gibbs state μ_F of the H$\acute{}$older family $F = \{\log(p_i)\}_{i\in I}$. In particular there exists a unique (S, P)-invariant measure and it will be denoted by μ_P. As an immediate consequence of Theorem 1.5, we therefore get the following.

Corollary 1.7. *If $S = \{\phi_i\}_{i\in I}$ is a conformal irreducible iterated function system in $I\!R^d$, $d \geq 2$ and P is a probability vector, then the (S, P)-invariant measure is absolutely friendly, and consequently, strongly extremal.*

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2. Proofs.

If $d \geq 3$ we define $\mathcal{F}$ to be the family of all compact subsets of all intersections $H \cap X$, where H is either an arbitrary geometric sphere or an affine hyperplane of codimension 1. If $d = 2$, by $\mathcal{F}_0$ we denote the family of all intersections $X \cap L$, where L ranges over all affine straight lines in $\mathbb{C}$. We then define

$$\mathcal{F}_\infty = \{\phi_\omega^{-1}(\Gamma) : \Gamma \in \mathcal{F}_0 \text{ and } \omega \in I^*\} \quad \text{and} \quad \mathcal{F} = \overline{\mathcal{F}_\infty},$$

where the closure is taken with respect to the Hausdorff topology (metric) in the space of all compact subsets of X. We will need the following properties of $\mathcal{F}$.

Lemma 2.1. *The family $\mathcal{F}$ is compact. If $\Gamma \in \mathcal{F}$ and $\omega \in I^*$, then there exists $H \in \mathcal{F}$ such that*

$$\phi_\omega^{-1}(\Gamma) \subset H.$$

Proof. The compactness of $\mathcal{F}$ if $d \geq 3$ is clear. If $d = 2$ this is true since $\mathcal{F}$ is closed. Passing to the proof of the second part of this lemma, suppose first that $d \geq 3$. Then there exists Q, a geometric sphere or an affine hyperplane of codimension 1 such that $\Gamma \subset Q$. Thus $\phi_\omega^{-1}(\Gamma) \subset \hat{\phi}_\omega^{-1}(Q) \cap X$, where $\hat{\phi}_\omega^{-1} : \overline{I\!R}^d \to \overline{I\!R}^d$ is the unique conformal extension of $\overline{\phi} : V \to V$ from V to $\mathbb{C}$. So, we are done in this case. Thus, we may assume that $d = 2$. Then obviously

$\phi_\omega^{-1}(\Gamma) \in \mathcal{F}_\infty$ if $\Gamma \in \mathcal{F}_\infty$. So, suppose that $\Gamma = \lim_{n\to\infty} \Gamma_n$, where all $\Gamma_n \in \mathcal{F}_\infty$. Fix an integer $k \geq 1$ and take $n_k \geq 1$ so large that

$$d_H(\Gamma, \Gamma_{n_k}) < K^{-1}||\phi_\omega'||k^{-1}. \tag{2.1}$$

Consider an arbitrary $x \in \phi_\omega^{-1}(\Gamma)$. This means that $x \in X$ and $\phi_\omega(x) \in \Gamma$. Hence, by (2.1), there exists $y \in \Gamma_{n_k}$ such that $|y - \phi_\omega(x)| < K^{-1}||\overline{\phi}_\omega'||k^{-1}$. If $k \geq 1$ is large enough, then it follows from (1.5) that $[y, \phi_\omega(x)] \subset \phi_\omega(V)$. Hence $|\overline{\phi}_\omega^{-1}(y) - x| \leq K||\overline{\phi}_\omega'||^{-1}|y - \phi_\omega(x)| < 1/k$. Thus $x \in B\Big(\overline{\phi}_\omega^{-1}(\Gamma_{n_k}), 1/k\Big)$. But $x \in X$ and $\overline{\phi}_\omega^{-1}(V \setminus X) \subset V \setminus X$, and therefore, $x \in B\Big(\phi_\omega^{-1}(\Gamma_{n_k}), 1/k\Big)$. Consequently

$$\phi_\omega^{-1}(\Gamma) \subset B\Big(\phi_\omega^{-1}(\Gamma_{n_k}), 1/k\Big). \tag{2.2}$$

But each set $\phi_\omega^{-1}(\Gamma_{n_k})$ belongs to Γ_∞, and therefore (compactness of $\mathcal{F}$ has been already proved) passing yet to another subsequence, we may assume that the sequence $\{\phi_\omega^{-1}(\Gamma_{n_k})\}_{k=1}^\infty$ converges to an element $H \in \mathcal{F}$. It then follows form (2.2) that $\phi_\omega^{-1}(\Gamma) \subset H$, and we are done. ■

Lemma 2.2. *If $d = 2$, then there exists an integer $N \geq 1$ such that each element of $\mathcal{F}$ is contained in a union of at most N real-analytic curves of finite length.*

Proof. Let $\rho = \mathrm{dist}(X, \partial V)$. Then for every $\omega \in I^*$, $\mathrm{dist}(\phi_\omega(X), \partial\overline{\phi}_\omega(V)) \geq K^{-1}\rho||\phi_\omega'||$. Consider an arbitrary affine straight line $L \subset \mathbb{C}$. Fix any two components C_1 and C_2 of $L \cap X$ such that the open segment Δ lying between their endpoints is disjoint from X, and that $|\Delta| \geq (2K)^{-1}\rho||\phi_\omega'||$. Since $\mathrm{diam}(\phi_\omega(X)) \leq D||\phi_\omega'||$, we conclude that the number of such segments Δ is bounded above by $2KD\rho^{-1}$. Since, if $|\Delta| < (2K)^{-1}\rho||\phi_\omega'||$, then $\Delta \subset \overline{\phi}_\omega(V)$, we therefore deduce that $L \cap \phi_\omega(X)$ can be covered by $l = [2KD\rho^{-1}] + 1$ mutually disjoint intervals $\Delta_1, \ldots, \Delta_l$, all of them contained in $\overline{\phi}_\omega(V)$. Thus, the lemma is proved for all members of the family $\mathcal{F}_\infty$. So, suppose that $\Gamma \in \mathcal{F}$. This means that there exist a sequence $\{L_n\}_{n=1}^\infty$ of straight lines in $\mathbb{C}$ and a sequence $\{\omega^{(n)}\}_{n=1}^\infty \subset I^*$ such that

$$\Gamma = \lim_{n\to\infty} \phi_{\omega^{(n)}}^{-1}(L_n \cap X), \tag{2.3}$$

where the limit is taken with respect to the Hausdorff metric d_H on compact subsets of X. Let $\Delta_1^n, \ldots, \Delta_l^n \subset X$ be the segments associated to the element L_n, $n \geq 1$. One can cover each segment $\Delta_j \cap X$, $j = 1, \ldots, l$, by at most

$$N_j = [2|\Delta_j^n|/(2K)^{-1}\rho||\phi_{\omega^{(n)}}'||] = [4K\rho^{-1}|\Delta_j^n|||\phi_{\omega^{(n)}}'||^{-1}]$$

balls (in $\mathbb{C}$) $B\Big(x_u^n, (2K)^{-1}\rho||\phi_{\omega^{(n)}}'||\Big)$, $u = 1, 2, \ldots, N_j$, all contained in $\overline{\phi}_{\omega^{(n)}}(V)$ with centers $x_u^n \in \phi_{\omega^{(n)}}(X)$. Since $\sum_{j=1}^l |\Delta_j^n| \leq \mathrm{diam}(\overline{\phi}_{\omega^{(n)}}(V)$, we see that $\sum_{j=1}^l N_j \leq 4KD\rho^{-1}$. So, repeating some balls if necessary, we have covered the set $L_n \cap \phi_{\omega^{(n)}}(X)$ by $N = [4KD\rho^{-1}]$ balls $B\Big(x_u^n, (2K)^{-1}\rho||\phi_{\omega^{(n)}}'||\Big)$, $u = 1, 2, \ldots, N_j$, all contained in $\overline{\phi}_{\omega^{(n)}}(V)$ with centers $x_u^n \in$

$\phi_{\omega(n)}(X)$. For every $n \geq 1$ and $u \in \{1, 2, \ldots, N\}$ consider now the map $F_{n,u} : \mathbb{C} \to \mathbb{C}$ given by the formula

$$F_{n,u}(z) = ||\phi'_{\omega(n)}||^{-1}|z - x_u^n|.$$

Then

$$F_{n,u}^{-1}(z) = x_u^n + ||\phi'_{\omega(n)}||z.$$

We have

$$F_{n,u}^{-1}\Big(B(0, K^{-1}\rho)\Big) \subset B\Big(x_u^n, K^{-1}\rho||\phi'_{\omega(n)}||\Big) \subset \overline{\phi}_{\omega(n)}(V),$$

and therefore the map

$$\overline{\phi}_{\omega(n)}^{-1} \circ F_{n,u}^{-1} : B(0, K^{-1}\rho) \to V$$

is well defined. Let us look now at the sets

$$F_{n,u}\Big(L_n \cap \overline{B}\Big(x_u^n, K^{-1}\rho||\phi'_{\omega(n)}||\Big)\Big).$$

These are closed segments in $\overline{B}(0, K^{-1}\rho)$. We can therefore find an unbounded increasing sequence $\{n_k\}_{k=1}^{\infty}$ and, for every $u \in \{1, 2, \ldots, N\}$, a segment $I_u \subset \overline{B}(0, K^{-1}\rho)$ such that

$$\lim_{k\to\infty} F_{n_k,u}\Big(L_{n_k} \cap \overline{B}\Big(x_u^{n_k}, K^{-1}\rho||\phi'_{\omega(n_k)}||\Big)\Big) = I_u, \tag{2.4}$$

where the limit is understood in the sense of Hausdorff metric. Since the family

$$\Big\{\overline{\phi}_{\omega(n)}^{-1} \circ F_{n,u}^{-1} : B(0, K^{-1}\rho) \to V\Big\}_{n=1}^{\infty}$$

is normal in the sense of Montel, passing to yet another subsequence and dropping the subscript k, we may assume that there exists a holomorphic function $G_u : B(0, K^{-1}\rho) \to V$ such that

$$G_u = \lim_{k\to\infty} \overline{\phi}_{\omega(n)}^{-1} \circ F_{n,u}^{-1}$$

uniformly on compact subsets of $B(0, K^{-1}\rho)$. Fix now $\xi \in \Gamma$ and then $\varepsilon > 0$. There then exists $q \geq 1$ so large that

$$|G_u(z) - \overline{\phi}_{\omega(n)}^{-1} \circ F_{n,u}^{-1}(z)| < \frac{\varepsilon}{3} \tag{2.5}$$

all $u \in \{1, 2, \ldots, N\}$, all $z \in \overline{B}(0, (2K)^{-1}\rho)$, and all $k \geq q$. By (2.4) we may assume $q \geq 1$ to be so large that

$$d_H\Big(F_{n,u}\Big(L_n \cap \overline{B}\Big(x_u^n, K^{-1}\rho||\phi'_{\omega(n)}||\Big)\Big), I_u\Big) < \min\Big\{(4K)^{-1}\rho, \frac{\varepsilon}{3}||G'_u|_{\overline{B}(0,((4/3)K)^{-1}\rho)}||_{\infty}^{-1}\Big\} \tag{2.6}$$

for all $n \geq q$. By (2.3) there exist $n \geq q$ and $\xi_n \in L_n \cap \phi_{\omega(n)}(X)$ such that

$$|\xi - \phi_{\omega(n)}^{-1}(\xi_n)| < \frac{\varepsilon}{3}. \tag{2.7}$$

There now exists $u_n \in \{1, 2, \ldots, N\}$ such that $\xi_n \in B\Big(x_u^n, (2K)^{-1}\rho||\phi'_{\omega^{(n)}}||\Big)$. Passing yet to another subsequence, we may assume that u_n does not depend on n; say $u_n = u$. We have $F_{n,u}(\xi_n) \in B(0, (2K)^{-1}\rho)$. Therefore, by (2.7) and (2.5), we get that

$$\begin{aligned} |\xi - G_u(F_{n,u}(\xi_n))| &\le |\xi - \phi_{\omega^{(n)}}^{-1}(\xi_n)| + |\phi_{\omega^{(n)}}^{-1}(\xi_n) - G_u(F_{n,u}(\xi_n))| \\ &= |\xi - \phi_{\omega^{(n)}}^{-1}(\xi_n)| + |\phi_{\omega^{(n)}}^{-1} \circ F_{n,u}^{-1}(F_{n,u}(\xi_n)) - G_u(F_{n,u}(\xi_n))| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2}{3}\varepsilon. \end{aligned} \tag{2.8}$$

Since $\xi_n \in L_n \cap B\Big(x_u^n, (2K)^{-1}\rho||\phi'_{\omega^{(n)}}||\Big)$, it follows from (2.6) that there exists $\theta_n \in I_u$ such that

$$|F_{n,u}(\xi_n) - \theta_n| < \min\left\{(4K)^{-1}\rho, \frac{\varepsilon}{3}||G'_u|_{\overline{B}(0,((4/3)K)^{-1}\rho)}||_\infty^{-1}\right\}.$$

Hence, $\theta_n \in B(0, ((4/3)K)^{-1}\rho)$. Thus

$$|G_u(F_{n,u}(\xi_n)) - G_u(\theta_n)| \le ||G'|_{\overline{B}(0,((4/3)K)^{-1}\rho)}||_\infty^{-1}|F_{n,u}(\xi_n) - \theta_n| < \frac{\varepsilon}{3}.$$

(If $||G'_u|_{\overline{B}(0,((4/3)K)^{-1}\rho)}||_\infty = 0$, then G_u is a constant function, and omitting the middle term, we get even better estimate: $\varepsilon/3$ replaced by 0.) So, combining this and (2.8), we conclude that $|\xi - G_u(\theta_n)| < \varepsilon$. Since $\theta_n \in I_u$, this implies that $\xi \in B(G_u(I_u), \varepsilon)$. Consequently, $\Gamma \subset B\left(\bigcup_{u=1}^N G_u(I_u), \varepsilon\right)$, and letting $\varepsilon \searrow 0$, we see that $\Gamma \subset \bigcup_{u=1}^N G_u(I_u)$. We are done. ■

Proof of Theorem 1.5. Let $\mathcal{F}$ be the family defined in the beginning of this section. We will conduct this proof without distinguishing the cases $d = 2$ and $d \ge 3$. It will consist of two preparatory lemmas and the concluding argument. Put $m = m_F$ and $\mu = \mu_F$. It is known (see [1], comp. [6]) that there exist a Borel probability measure $\tilde{m}$ on I^∞ and a unique ergodic σ-invariant measure $\tilde{\mu}$ equivalent to $\tilde{m}$ such that $m = \tilde{m} \circ \pi^{-1}$ and $\mu = \tilde{\mu} \circ \pi^{-1}$. We shall prove first the following.

Lemma 2.3. *We have that $m(H) = 0$ for every $H \in \mathcal{F}$.*

Proof. Indeed, seeking contradiction, suppose that $m(H) > 0$ for some $H \in \mathcal{F}$. Since the system S satisfies the strong open set condition with the set X, there exists $x \in J$ and $R > 0$ such that $B(x, 2R) \subset \mathrm{Int}X$. Since $x \in J$, $m(B(x, R)) > 0$. Since the measure $\tilde{m}$ is equivalent to the shift-invariant ergodic measure $\tilde{\mu}$, it follows from Birkhoff's ergodic theorem that there exists a Borel set $G \subset I^\infty$ such that $\tilde{m}(G) = 1$, and the set $\{n \ge 0 : \pi(\sigma^n(\omega)) \in B(x, R)\}$ is infinite for all $\omega \in G$. Fix one $\omega \in G$ such that $\pi(\omega) \in H$ ($\tilde{m}(G \cap \pi^{-1}(H)) > 0$) is a Lebesgue density point of the set H with respect to the measure m. For every $n \ge 0$ put

$$B_n = B(\pi(\sigma^n(\omega)), R).$$

Fix $\varepsilon > 0$ and then fix such an $n = n_\varepsilon \ge 0$ (sufficiently large) that $\pi(\sigma^n(\omega)) \in B(x, R)$ and

$$m\Big(B\Big(\pi(\omega), R||\phi'_{\omega|_n}||\Big) \setminus H\Big) \le \varepsilon m\Big(B\Big(\pi(\omega), R||\phi'_{\omega|_n}||\Big)\Big). \tag{2.9}$$

Since $B_n \subset X$, we get

$$\begin{aligned} m\Big(\phi_{\omega|_n}\Big(B_n \setminus \phi_{\omega|_n}^{-1}(H)\Big)\Big) &= \int_{B_n\setminus\phi_{\omega|_n}^{-1}(H)} \exp\Big(S_{\omega|_n}F - n\mathrm{P}(F)\Big)dm \\ &\geq T^{-1}\exp\Big(\sup\Big(S_{\omega|_n}F\Big) - n\mathrm{P}(F)\Big)m\Big(B_n \setminus \phi_{\omega|_n}^{-1}(H)\Big), \end{aligned}$$

where $T \geq 1$ is the number defined in (1.8). Hence, using (2.9) and applying Lemma 1.3, we get

$$\begin{aligned} m\Big(B_n \setminus \phi_{\omega|_n}^{-1}(H)\Big) &\leq T\exp\Big(\mathrm{P}(F)n - \sup\Big(S_{\omega|_n}F\Big)\Big)m\Big(\phi_{\omega|_n}\Big(B_n \setminus \phi_{\omega|_n}^{-1}(H)\Big)\Big) \\ &\leq T\exp\Big(\mathrm{P}(F)n - \sup\Big(S_{\omega|_n}F\Big)\Big)m\Big(B\Big(\pi(\omega), R||\phi_{\omega|_n}'||\Big) \setminus H\Big) \\ &\leq \varepsilon T\exp\Big(\mathrm{P}(F)n - \sup\Big(S_{\omega|_n}F\Big)m\Big(B\Big(\pi(\omega), R||\phi_{\omega|_n}'||\Big)\Big). \end{aligned} \tag{2.10}$$

Again, since $B_n \subset X$, using the bounded distortion property (BDP) and the doubling property of the measure m (see Remark 1.2), we obtain

$$\begin{aligned} \exp\Big(\sup\Big(S_{\omega|_n}F\Big) - n\mathrm{P}(F)\Big) &\geq \exp\Big(\sup\Big(S_{\omega|_n}F\Big) - n\mathrm{P}(F)\Big)m(B_n) \geq m\Big(\phi_{\omega|_n}(B_n)\Big) \\ &\geq m\Big(B\Big(\pi(\omega), K^{-1}R||\phi_{\omega|_n}'||\Big)\Big) \geq Cm\Big(B\Big(\pi(\omega), R||\phi_{\omega|_n}'||\Big)\Big) \end{aligned}$$

with some universal constant $C > 0$. Combining this and (2.10), we get that

$$m\Big(B_n \setminus \phi_{\omega|_n}^{-1}(H)\Big) \leq TC^{-1}\varepsilon. \tag{2.11}$$

Putting now $n_k = n_{1/k}$, $k \geq 1$, and passing to a subsequence if necessary, we may assume that $\pi\Big(\sigma^{n_k}(\omega)\Big)$ converges to a point $z \in J$ when $k \nearrow \infty$. In view of Lemma 2.1 that for every $k \geq 1$ there exists $M_k \in \mathcal{F}$ such that $\phi_{\omega|_{n_k}}^{-1}(H) \subset M_k$. It follows again from Lemma 2.1 that passing to a subsequence, we may assume without loss of generality that the sequence $\{M_k\}_{k=1}^\infty$ converges in the Hausdorff metric d_H to some element $M \in \mathcal{F}$. Letting $k \nearrow \infty$, it therefore follows from (2.11) that

$$m(B(z, R/2) \setminus M) = 0. \tag{2.12}$$

Suppose now that $(J \cap B(z, R/2)) \setminus M \neq \emptyset$. Since this is a nonempty open subset of J, it would be of positive measure, contrary to (2.12). Thus

$$J \cap B(z, R/2) \subset M. \tag{2.13}$$

Since $z \in J$, we have that $z \in \pi(\tau)$ for some $\tau \in I^\infty$. Then for every $n \geq 1$ sufficiently large, $\phi_{\tau|_n}(J) \subset B(z, R/2)$, and as $\phi_{\tau|_n}(J) \subset J$, we conclude from (2.13) that $\phi_{\tau|_n}(J) \subset M$, or equivalently, $J \subset \phi_{\tau|_n}^{-1}(M)$. Applying now Lemma 2.1 and Lemma 2.2, we see that this contradicts irreducibility of the system S, and finishes the proof of Lemma 2.3. ■

For every $\varepsilon > 0$ let

$$t(\varepsilon) = \sup\{m(H, \varepsilon) : H \in \mathcal{F}\}.$$

Since the family $\mathcal{F}$ is compact, applying Lemma 2.3, we easily see that

$$\lim_{\varepsilon\searrow 0} t(\varepsilon) = 0. \tag{2.14}$$

Put

$$u = \min_{i\in I}\{\inf\{|\phi_i'(z)| : z \in V\}\} \text{ and } \rho = \mathrm{dist}(X, \partial V).$$

Our next step is to prove the following.

Lemma 2.4. *There is a constant $\alpha > 0$ such that for every $\varepsilon \in (0, \min\{1, K^{-1}u\rho\})$*

$$t(\varepsilon^2) \le Tt^2(\alpha\varepsilon).$$

Proof. Fix $\varepsilon \in (0, \min\{1, K^{-1}u\rho\})$ and $H \in \mathcal{F}$. Define

$$P = \{\omega \in I^* : ||\overline{\phi}_\omega'|| \le \varepsilon \text{ and } ||\overline{\phi}_{\omega|_{|\omega|-1}}'|| > \varepsilon\}.$$

Obviously all the elements of P are mutually incomparable and each element of I^∞ has an initial block belonging to P. Put

$$\hat{P} = \{\omega \in P : \phi_\omega(J) \cap B(H, \varepsilon^2) \ne \emptyset\}.$$

Then

$$J \cap B(H, \varepsilon^2) = \bigcup_{\omega\in\hat{P}} \phi_\omega(J) \cap B(H, \varepsilon^2). \tag{2.15}$$

Fix now $\omega \in \hat{P}$ and $y \in \phi_\omega(J) \cap B(H, \varepsilon^2)$. If $x \in \phi_\omega(J)$, then it follows from the definition of P that

$$\begin{aligned}\mathrm{dist}(x, H) &\le ||x - y|| + \mathrm{dist}(y, H) \le \mathrm{diam}(\phi_\omega(J)) + \varepsilon^2 \le D||\phi_\omega'|| + \varepsilon^2 \\ &\le D\varepsilon + \varepsilon^2 = (D + \varepsilon)\varepsilon \le (D + 1)\varepsilon.\end{aligned}$$

Hence, $\phi_\omega(J) \subset B(H, (D + 1)\varepsilon)$ and consequently

$$\bigcup_{\omega\in\hat{P}} \phi_\omega(J) \subset B(H, (D + 1)\varepsilon).$$

Thus,

$$\begin{aligned} t((D+1)\varepsilon) &\ge m\Big(B(H,(D+1)\varepsilon)\Big) \ge m\left(\bigcup_{\omega\in\hat{P}} \phi_\omega(J)\right) = \sum_{\omega\in\hat{P}} m(\phi_\omega(J)) \\ &\ge \sum_{\omega\in\hat{P}} T^{-1}\exp\Big(\sup(S_\omega\phi - \mathrm{P}(\phi)|\omega|\Big) \\ &= T^{-1}\sum_{\omega\in\hat{P}}\Big(\sup(S_\omega\phi - \mathrm{P}(\phi)|\omega|\Big). \end{aligned} \tag{2.16}$$

Take now $\omega \in \hat{P}$ and consider the set $J \cap \phi_\omega^{-1}(B(H,\varepsilon^2))$. Fix $x \in J \cap \phi_\omega^{-1}(B(H,\varepsilon^2))$. This relation means that $\phi_\omega(x) \in \phi_\omega(J) \cap B(H,\varepsilon^2)$. Thus, there exists $y \in H$ such that $||y - \phi_\omega(x)|| < \varepsilon^2 < K^{-1}u\rho\varepsilon$. Since $\omega \in P$, we get using (1.5) that

$$\overline{\phi}_\omega(V) \supset B(\phi_\omega(X), K^{-1}||\overline{\phi}'_\omega||\rho) \supset B(\phi_\omega(J), K^{-1}||\overline{\phi}'_\omega||\rho) \supset B(\phi_\omega(J), K^{-1}u\rho\varepsilon).$$

Therefore $[\phi_\omega(x), y] \subset \overline{\phi}_\omega(V)$ and consequently,

$$\begin{aligned}||x - \overline{\phi}_\omega^{-1}(y)|| &= ||\overline{\phi}\phi_\omega^{-1}(\phi_\omega(x)) - \overline{\phi}_\omega^{-1}(y)|| \le ||(\overline{\phi}_\omega^{-1})'||_{\overline{\phi}_\omega(V)}||\phi_\omega(x) - y|| \\ &\le K||\overline{\phi}'_\omega||^{-1}\varepsilon^2 \le Ku^{-1}\varepsilon^{-1}\varepsilon^2 = K2u^{-1}\varepsilon.\end{aligned}$$

This shows that $x \in B(\overline{\phi}_\omega^{-1}(H), Ku^{-1}\varepsilon)$, and in consequence

$$J \cap \phi_\omega^{-1}(B(H,\varepsilon^2)) \subset B\Big(\overline{\phi}_\omega^{-1}(H), Ku^{-1}\varepsilon\Big).$$

Therefore, using (2.15), Lemma 2.1 and (2.16), we obtain

$$\begin{aligned}m(B(H,\varepsilon^2)) &= \sum_{\omega\in\hat{P}} m\Big(\phi_\omega(J) \cap B(H,\varepsilon^2)\Big) = \sum_{\omega\in\hat{P}} m\Big(\phi_\omega\Big(J \cap \phi_\omega^{-1}(B(H),\varepsilon^2))\Big)\Big) \\ &\le \sum_{\omega\in\hat{P}} \exp\Big(\sup(S_\omega\phi) - \mathrm{P}(\phi)|\omega|\Big) m\Big(J \cap \phi_\omega^{-1}(B(H,\varepsilon^2))\Big) \\ &\le \sum_{\omega\in\hat{P}} \exp\Big(\sup(S_\omega\phi) - \mathrm{P}(\phi)|\omega|\Big) m\Big(B\Big(phi_\omega^{-1}(H), Ku^{-1}\varepsilon\Big)\Big) \\ &\le t(K^2u^{-1}\varepsilon)\sum_{\omega\in\hat{P}} \exp\Big(\sup(S_\omega\phi) - \mathrm{P}(\phi)|\omega|\Big) \\ &\le Tt((D+1)\varepsilon)t(K^2u^{-1}\varepsilon).\end{aligned}$$

So, taking $\alpha = \max\{D+1, Ku^{-1}\}$ finishes the proof of our lemma. ■

We are now ready to do the last step of the proof of Theorem 1.5. It follows from (2.14) that we can choose $\gamma \in (0, \min\{1, K^{-1}u\rho\})$ so small that

$$t(\gamma) \le (eT)^{-1}. \tag{2.17}$$

Keep now $\varepsilon \in (0,\gamma)$ and consider the largest $n \ge 1$ such that $\alpha^{2-2^{1-n}}\varepsilon^{2^{-n}} < \gamma$. Applying Lemma 2.4 n times and using (2.17), we then get

$$t(\varepsilon) \le T^{2^n-1}t^{2^n}(\gamma) \le T^{-1}\left(T\frac{1}{eT}\right)^{2^n} = T^{-1}e^{-2^n}. \tag{2.18}$$

But $\alpha^{2-2^{1-(n+1)}}\varepsilon^{2^{-(n+1)}} \ge \gamma$, which implies that $\varepsilon^{2^{-(n+1)}} \ge \gamma\alpha^{-2}$. Equivalently $(\gamma\alpha^{-2})^{2^n} \le \varepsilon^{1/2}$ or $\exp\Big(-2^n\log(\alpha^2\gamma^{-1})\Big) \le \varepsilon^{1/2}$. This gives that $e^{-2^n} \le \varepsilon^\beta$, where $\beta = \Big(2\log(\alpha^2\gamma^{-1})\Big)^{-1}$. It therefore follows from (2.18) that $t(\varepsilon) \le T^{-1}\varepsilon^\beta$. Since m is a probability measure, replacing if necessary T^{-1} by a bigger constant, say C, we get that

$$t(\varepsilon) \le C\varepsilon^\beta \tag{2.19}$$

for all $\varepsilon > 0$. Fix now $z \in J$, $H \in \mathcal{F}$, $\varepsilon \in (0, \min\{1, K^{-1}u\rho\})$ and $r \in (0, u)$. Define

$$P_r = \{\omega \in I^* : ||\overline{\phi}'_\omega|| \le r \text{ and } ||\overline{\phi}'_{\omega|_{|\omega|-1}}|| > r\}$$

and set

$$\tilde{P} = \{\omega \in P_r : \phi_\omega(J) \cap B(z, r) \ne \emptyset\}.$$

Then

$$J \cap B(z, r) = \bigcup_{\omega \in \tilde{P}} \phi_\omega(J) \cap B(z, r). \tag{2.20}$$

Fix now $\omega \in \tilde{P}$ and $y \in \phi_\omega(J) \cap B(z, r)$. If $x \in \phi_\omega(J)$, then it follows from the definition of P that

$$||x - z|| \le ||x - y|| + ||y - z|| < \operatorname{diam}(\phi_\omega(J)) + r \le D||\phi'_\omega|| + r \le Dr + r = (D + 1)r.$$

Hence,

$$\bigcup_{\omega \in \tilde{P}} \phi_\omega(J) \subset B(z, (D + 1)r).$$

So,

$$\begin{aligned} m(B(z, (D + 1)r)) \ge m\left(\bigcup_{\omega \in \tilde{P}} \phi_\omega(J)\right) &= \sum_{\omega \in \tilde{P}} m\Big(\phi_\omega(J)\Big) \ge \sum_{\omega \in \tilde{P}} T^{-1} \exp\Big(\sup(S_\omega \phi) - \mathrm{P}(\phi)|\omega|\Big) \\ &= T^{-1} \sum_{\omega \in \tilde{P}} \exp\Big(\sup(S_\omega \phi) - \mathrm{P}(\phi)|\omega|\Big). \end{aligned} \tag{2.21}$$

Take now $\omega \in \tilde{P}$ and consider the set $J \cap \phi_\omega^{-1}(B(H, r\varepsilon))$. Fix $x \in J \cap \phi_\omega^{-1}(B(H, r\varepsilon))$. This means that $\phi_\omega(x) \in \phi_\omega(J) \cap B(H, r\varepsilon)$. Thus, there exists $y \in H$ such that $||y - \phi_\omega(x)|| < r\varepsilon$. Since $\omega \in P_r$ and since $\varepsilon \le K^{-1}u\rho$, using (1.5), we get that

$$\overline{\phi}_\omega(V) \supset B(\phi_\omega(X), K^{-1}||\overline{\phi}'_\omega||\rho) \supset B(\phi_\omega(J), K^{-1}||\overline{\phi}'_\omega||\rho) \supset B(\phi_\omega(J), K^{-1}u\rho r) \supset B(\phi_\omega(J), r\varepsilon)$$

Therefore $[\phi_\omega(x), y] \subset \overline{\phi}_\omega(V)$ and consequently,

$$||x - \overline{\phi}_\omega^{-1}(y)|| = ||\overline{\phi}_\omega^{-1}(\phi_\omega(x)) - \overline{\phi}_\omega^{-1}(y)|| \le ||(\overline{\phi}_\omega^{-1})'||_{\overline{\phi}_\omega(V)}||\phi_\omega(x) - y|| \le K||\overline{\phi}'_\omega||^{-1}r\varepsilon \le Ku^{-1}\varepsilon.$$

This shows that $x \in B\Big(\phi_\omega^{-1}(H), K^2u^{-1}\varepsilon\Big)$, and in consequence

$$J \cap \phi_\omega^{-1}(B(H, r\varepsilon)) \subset B\Big(\phi_\omega^{-1}(H), K^2u^{-1}\varepsilon\Big).$$

Therefore, using (2.20) and (2.21), we obtain

$$\begin{aligned}
m\Big(B(H,r\varepsilon)\cap B(z,r)\Big) &= \sum_{\omega\in\hat{P}} m\Big(\phi_\omega(J)\cap B(z,r)\Big) = \sum_{\omega\in\hat{P}} m\Big(\phi_\omega\Big(J\cap\overline{\phi}_\omega^{-1}(B(z,r))\Big)\Big) \\
&\le \sum_{\omega\in\hat{P}} \exp\Big(\sup(S_\omega\phi)-\mathrm{P}(\phi)|\omega|\Big)m\Big(J\cap\overline{\phi}_\omega^{-1}(B(z,r))\Big) \\
&\le \sum_{\omega\in\hat{P}} \exp\Big(\sup(S_\omega\phi)-\mathrm{P}(\phi)|\omega|\Big)m\Big(B\Big(B\Big(\overline{\phi}_\omega^{-1}(H),Ku^{-1}\varepsilon\Big)\Big) \\
&\le Tm\Big(B\Big(B\Big(\overline{\phi}_\omega^{-1}(H),Ku^{-1}\varepsilon\Big)\Big)m(B(z(D+1)r)) \\
&\le t(Ku^{-1}\varepsilon)m(B(z,(D+1)r)).
\end{aligned}$$

Using (2.19) and the doubling property of the measure m, we therefore get

$$m\Big(B(H,r\varepsilon)\cap B(z,r)\Big)\le C\varepsilon^\beta m(B(z,r)).$$

Thus the proof that the measure m is absolutely firendly (and so, by a result from [2], strongly extremal), is complete. ■

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