THE PARABOLIC MAP $f_{1/e}(z) = \frac{1}{e}e^z$.

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ABSTRACT. We consider the exponential maps $f_\lambda : \mathbb{C} \to \mathbb{C}$ defined by the formula $f_\lambda(z) = \lambda e^z$, $\lambda \in (0, 1/e]$. Let $J_r(f_\lambda)$ be the subset of the Julia set consisting of points that do not escape to infinity under forward iterates of f. Our main result is that the function $\lambda \mapsto h_\lambda := \mathrm{HD}(J_r(f_\lambda))$, $\lambda \in (0, 1/e]$, is continuous at the point $1/e$. As a preparation for this result we deal with the map $f_{1/e}$ itself. We prove that the $h_{1/e}$-dimensional Hausdorff measure of $J_r(f_{1/e})$ is positive and finite on each horizontal strip, and that the $h_{1/e}$-dimensional packing measure of $J_r(f_\lambda)$ is locally infinite at each point of $J_r(f_\lambda)$. Our main technical devices are formed by the, associated with f_λ, maps F_λ defined on some strip P of height 2π and also associated with them conformal measures.

1. Introduction

We consider the exponential maps $f_\lambda : \mathbb{C} \to \mathbb{C}$ defined by the formula $f_\lambda(z) = \lambda e^z$, $\lambda \in (0, 1/e]$. C. McMullen has proved in [McM] that $\mathrm{HD}(J(f))$, the Hausdorff dimension of the Julia set of f is equal to 2. In fact McMullen has shown more, that the set of points escaping to infinity under f has Hausdorff dimension equal to 2. In [UZ1] and [UZ2] we extensively explored $J_r(f_\lambda)$, the complement of this latter set in $J(f)$, for hyperbolic parameters λ. Hyperbolic means here that f_λ has an attracting periodic orbit. This set has turned out to carry the interesting component of the dynamics and geometry of the maps f_λ. It was shown in [UZ2] that the function $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$ is real-analytic in some neighbourhood of every hyperbolic parameter λ. In this paper we make the first step beyond hyperbolicity and we consider the parabolic parameter $1/e$. It is called parabolic since $f_{1/e}(1) = 1$ and $f'_{1/e}(1) = 1$. Notice that $(0, 1/e)$ is a subset of hyperbolic parameters and therefore the function $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$, $\lambda \in (0, 1/e)$, is real-analytic. The natural question arises about the asymptotic behavior of this Hausdorff dimension function when $\lambda \nearrow 1/e$. Using our main technical devices formed by the, associated with f_λ, maps F_λ defined on some strip P of height 2π and also associated with them conformal measures, we answer this question by proving the following.

Theorem 1.1. *The function $\lambda \mapsto h_\lambda = \mathrm{HD}(J_r(f_\lambda))$, $\lambda \in (0, 1/e]$ is continuous.*

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A similar problem was also positively resolved in [BZ] for the quadratic family $\{z \mapsto z^2 + c\}_{c\in[0,1/4]}$. The general idea of using conformal measures is in our paper the same as in [BZ]. Our proof is however computationally simpler and clearer, mainly due to the change of variables which sends the repelling fixed points of all the maps f_λ to the one point 0, and is actually of local character, so it in fact can be applied in a much more general setting.

As a preparation for the proof of Theorem 1.1 we develop in Section 3 the theory of the map $f_{1/e}$ itself. It is partially modeled on the papers [ADU], [DU1], [DU2], [DU4] and [UZ1]. We provide sketches of proofs and indicate how to complete them using the arguments from the above mentioned paper. In particular we provide an actually complete description of the structure of conformal measures of the map $f_{1/e}$, we prove that the $h_{1/e}$-dimensional Hausdorff measure of $J_r(f_{1/e})$ is positive and finite on each horizontal strip, and that the $h_{1/e}$-dimensional packing measure of $J_r(f_\lambda)$ is locally infinite at each point of $J_r(f_\lambda)$. We also indicate that there exists a Borel probability $F_{1/e}$-invariant ergodic measure equivalent to the appropriate conformal measure. The reader familiar with parabolic maps or interested only in the results proven about them, may skip reading Section 3 or read only the statements included there and focus on Section 4, the actual proof of Theorem 1.1.

2. **Short Preliminaries.**

Throughout the entire paper we assume that $\lambda \in (0, 1/e]$. Then $f_\lambda|_{\mathbb{R}}$ has a unique (positive, repelling in case when $\lambda < 1/e$ and parabolic equal to 1 if $\lambda = 1/e$) fixed point which we denote by q_λ. Let

$$P = \{z \in \mathbb{C} \colon -\pi < \mathrm{Im}(z) \le \pi\}.$$

Let

$$\pi_0 : \mathbb{C} \to P$$

be the projection determined by the condition that $\pi_0(z) = w$ if and only if $w \in P$ and $e^z = e^w$. We define the map $F = F_\lambda : P \to P$ we intend to work with by the formula

$$F(z) = \pi_0(f(z)) \tag{2.1}$$

Recall that a Borel probability measure m supported on P is called t conformal (with $t > 0$) if for any Borel set $A \subset P$ on which F_λ is injective, we have

$$m(F_\lambda(A)) = \int_A |F_\lambda'|^t dm.$$

Throughout the entire paper we use the notation

$$f := f_{1/e}, \;\; F := F_{1/e} \;\; J_r(F_\lambda) = J_r(f_\lambda) \cap P, \;\; J_r = J_r(F_{\frac{1}{e}}) \;\text{ and } P_+ = \{z \in P : \mathrm{Re} z \ge 1\}.$$

3. **Fractal and Dynamical Properties of the Maps f and F.**

Our first goal in this section is to prove the existence of a conformal measure and to examine in detail its properties. In order to do it we begin with the definition and analysis of the sets K_M, $m > 0$. Indeed, for every $M > 0$, let

$$W_M = \{z \in J(F) : \mathrm{Re}(z) \le M \text{ and } |z - 1| \ge 1/M\}$$

and let

$$K_M=\bigcap_{k\ge 0}F^{-k}(W_M). \tag{3.1}$$

Since $F:J(F)\to J(F)$ is continuous, K_M is a forward F-invariant compact subset of $J(F)$. Notice also that if $z\in P$, $j\ge 0$, $F^j(z)\in P_+$ and $|F^j(z)-1|\ge 1/M$, then there exists a unique holomorphic inverse branch $F_z^{-j}:B(F^j(z),1/M)\to Q$ of F^j, sending $F^j(z)$ to z. Since for every $u>1$, $\inf\{|F'(z)|:\mathrm{Re}z\ge u\}>1$, and since $\inf\{\mathrm{Re}(K_M)\}>1$ for every $M>0$, we get the following.

Lemma 3.1. *For all $M>0$*

$$\inf\{|F'(z)|:z\in K_M\}>1.$$

Given $t\ge 0$ a Borel probability measure on P is said to be t-conformal for $F:J(F)\to J(F)$ if and only if $m(J(F))=1$ and

$$m(F(A))=\int_A|F'|^t dm \tag{3.2}$$

for every Borel set $A\subset J(F)$ such that $F_{|A}$ is one-to-one. First, following [DU1], for every $M>0$ large enough, we shall build a probability Borel measure m_M, with the topological support contained in K_M, and which will be "almost conformal" for some $t_M\ge 0$, meaning that

$$m_M(F(A))\ge\int_A|F'|^{t_M}dm_M \tag{3.3}$$

for every Borel set $A\subset Q$ such that $F|_A$ is 1-to-1, and (3.2) holds if we assume in addition that $A\cap\{z\in Q:\mathrm{Re}z\ge M \text{ or } |z-1|\le 1/M\}=\emptyset$. In what follows, throughout Lemma 3.2 we follow closely the appropriate reasoning from [UZ3]. In the sequel, we will need to refer to some details of the construction of the measure m_M, so we briefly describe it now. For every $M>0$ large enough choose a finite set $E^M\subset K_M$ such that the $B(E^M,1/2M)\supset K_M$ and that E^M contains the forward orbit of a periodic point ξ of F. Notice that, since K_M is F-forward invariant, the whole forward orbit of ξ is contained in K_M. The existence of such a periodic point follows from the density of periodic points in $J(F)$. Consider the function

$$c_M(t)=\limsup_{n\to\infty}\frac{1}{n}\log\sum_{x\in E_M}\sum_{w\in\left(F|_{K_M}\right)^{-1}(x)}|(F^n)'|^{-t}(w).$$

The function $t\mapsto c_M(t)$, $t\in I\!R$, has three important properties. First, notice that it follows from Hölder's inequality that it is convex in $I\!R$, so it is continuous. Next, it follows easily from Lemma 3.1 that this function is strictly decreasing and $\lim_{t\to+\infty}c_M(t)=-\infty$. Finally, each set $\left(F|_{K_M}\right)^{-1}(E^M)$ is not empty as it contains a point from the forward orbit of ξ. In particular $c_M(0)\ge 0$. All these properties imply that there exists a unique value $t=t_M$ with

$c_M(t_M) = 0$. Following the general construction described in [DU1] (see also [PU], Chapter 10), with the sets $E_n = \left(F|_{K_M}\right)^{-n}(E^M)$ we obtain a measure m_M, for which $m_M(K_M) = 1$ and which is "almost conformal" with the exponent t_M. We continue on with the following lemma; the idea of its proof comes from [UZ3].

Lemma 3.2. *For every M large enough there exists $p > 0$ such that* $\mathrm{HD}(K_M) \le t_{M+p}$.

Proof. It immediately follows from Lemma 3.1 that

$$L = \inf\{|(F^n)'(w)| : w \in K_M, n \ge 1\} > 1 \quad \text{and} \quad \lim_{n\to\infty} |(F^n)'(z)| = \infty \tag{3.4}$$

for all $z \in K_M$. Fix $p > 0$ so large that $KL^{-1} < p(M(M+p))^{-1}$ and consider the set J_{M+p}. Following the construction of almost conformal measures described above, we choose a finite collection of points $E^{M+p} \subset K_{M+p}$ such that the balls $B(x, (2(M+p))^{-1})$, $x \in E^{M+p}$, cover the set K_{M+p}. Let $y \in K_M \subset K_{M+p}$. Given $n \ge 0$ there exists $x \in E^{M+p}$ such that $F^n(y) \in B(x, (2(M+p))^{-1})$. By our definition of the set K_{M+p}, all holomorphic branches of F^{-i}, $i \ge 0$, are well-defined on $B(x, 1/(M+p))$. Fix $0 \le i \le n$ and let F_y^{-i} be the holomorphic branch of F^{-i} sending $F^n(y)$ to $F^{n-i}(y)$. Then, by Koebe's distortion theorem, for all $z \in B(x, (2(M+p))^{-1})$, we get

$$\frac{|(F_y^{-i})'(z)|}{|(F_y^{-i})'(F^n(y))|} \le K.$$

So, since $F^{(n-i)}(y) \in K_M$, using (3.4), we obtain $|(F_y^{-i})'(z)| \le K|(F_y^{-i})'(F^n(y))| \le KL^{-1}$. Thus, $|F_y^{-i}(x) - F^{n-i}(y)| \le KL^{-1} < p(M(M+p))^{-1}$, and consequently, using the fact that $F^{n-i}(y) \in W_M$, we see that $F_y^{-i}(x) \in W_{M+p}$ for all $0 \le i \le n$. This implies that $F_y^{-n}(x) \in K_{M+p}$, i.e. $F_y^{-n}(x) \in \left(F|_{K_{M+p}}\right)^{-n}(x)$. Let $\mathcal{F}_n(x)$ be the collection of all holomorphic inverse branches F_ν^{-n} of F^n defined on $B(x, (M+p)^{-1})$, such that $F_\nu^{-n}(x) \in K_{M+p}$. It follows from the above considerations that

$$K_M \subset \bigcup_{x\in E^{M+p}} \bigcup_{\nu\in\mathcal{F}_n(x)} F_\nu^{-n}(B(x, (2(M+p))^{-1})). \tag{3.5}$$

In addition, in view of Lemma 3.1, $\mathrm{diam}(F_\nu^{-n}(B(x, (2(M+p))^{-1})) \to 0$ uniformly as $n \to \infty$, and for every $t \ge 0$

$$\sum_{x\in E^{M+p}} \sum_{\nu\in\mathcal{F}_n(x)} \left(\mathrm{diam}(F_\nu^{-n}(B(x, \delta_{M+p}))\right)^t \preceq \sum_{x\in E^{M+p}} \sum_{w\in\left(F|_{K_{M+p}}\right)^{-n}(x)} \frac{1}{|(F^n)'(w)|^t} \tag{3.6}$$

Fix now an arbitrary $t > t_{M+p}$. Then $c_{M+p}(t) < 0$ and

$$\sum_{x\in E^{M+p}} \sum_{w\in\left(F|_{K_{M+p}}\right)^{-n}(x)} \frac{1}{|(F^n)'(w)|^t} \le \exp\left(\frac{1}{2}c_{M+p}(t)n\right)$$

for all n large enough. Combining this, (3.5) and (3.6), we conclude that $\mathrm{H}_t(K_M) = 0$ for all $t > t_{M+p}$ and, consequently, $\mathrm{HD}(K_M) \le t_{M+p}$. ■

The proof of the next, much easier, fact is a simplification of the argument provided in [DU1] and chapter 10 of [PU] and is written with all details as the proof of Lemma 3.1 in [UZ3].

Lemma 3.3. *It holds* $\mathrm{HD}(K_M) \ge t_M$.

The main auxiliary result of this section is the following.

Lemma 3.4. *There exist* $h \in (1, \mathrm{HD}(J_r(f))]$ *and an atomless* h*-conformal measure for* F *supported on* $J(F)$.

Proof (sketch). In view of the proof of Theorem 2.1 in [UZ1], $\mathrm{HD}(K_M) \ge s$ for some $s > 1$ and all $M > 0$ large enough. Fix one such M and put $s = t_M$. Choose p ascribed to this M according to Lemma 3.2. Then, by this lemma, $t_{M+k} \ge \mathrm{HD}(K_M) = s > 1$ for all $k \ge p$ and $t_M \le HD(K_M) \le 2$ by Lemma 3.3. Next, we check that the sequence of measures $\{m_k\}_{k=1}^\infty$ is tight. The proof goes through exactly as the proof of Proposition 3.3 in [UZ1]. Now, choose a sequence $\{m_{n_k}\}_{k=1}^\infty$ such that the limit $h = \lim_{n\to\infty} t_{n_k}$ exists. By the above, $h \in [s, \mathrm{HD}(J_r(f))]$. It is now rather straightforward to verify that any weak limit measure m of the sequence $\{m_{n_k}\}_{k=1}^\infty$ is h-conformal (the argument is a simplification of the argument provided in [DU1] and chapter 10 of [PU]). Since $|F'(z)| > 1$ for all $z \in J(F) \setminus \{1\}$, all the atoms of m must be contained in $\bigcup_{n\ge 0} F^{-n}(1)$, and in order to demonstrate that m is atomless, it suffices to show that $m(1) = 0$. This in turn can be done in the same way as the proof of Theorem 8.7 in [ADU], see also the proof of Theorem 1.1 in the next section. ■

The proof of the next lemma is the same as the proof of Proposition 3.6 in [UZ1].

Lemma 3.5. *If* $t > 1$ *and* m_t *is an arbitrary* t*-conformal probability measure for* F*, then* $m_t(J_r) = 1$.

The proof of the following proposition follows immediately from the left-hand side of the formula (4.1) established in the proof of Theorem 4.4 in [UZ1].

Proposition 3.6. *If* $t > 1$ *and* m_t *is an arbitrary* t*-conformal measure for* F*, then* $\mathrm{H}^t|_{J_r(F)} \ll m_t$*. Moreover the Radon-Nikodym derivative* $\frac{dH^t}{dm_t}$ *is bounded from above.*

As an immediate consequence of this proposition we get the following.

Corollary 3.7. *If* $t > 1$ *and a* t*-conformal measure exists, then* $\mathrm{HD}(J_r(F)) \le t$.

Combining this corollary and Lemma 3.4, we immediately get the following.

Corollary 3.8. *We have*

$$\mathrm{HD}(J_r(F)) = \mathrm{HD}(J_r(f)) = h,$$

where h *is the value described in Lemma 3.4.*

We can provide now a fairly complete description of the structure of the set of conformal measures for $F : J(F) \to J(F)$.

Theorem 3.9. *The following hold.*

(1) h *is the unique* t *for which an atomless conformal measure exists.*
(2) *There exists a unique* h*-conformal measure* m *for* $F : J(F) \to J(F)$*. The measure* m *is atomless.*
(3) *The* h*-conformal measure* m *is ergodic and conservative.*
(4) *If* ν *is a* t*-conformal measure for* F *and* $t > 1$, $t \ne h$*, then* $t > h$ *and* $\nu\Big(\bigcup_{n\ge 0} F^{-n}(1)\Big) = 1$.

Proof. Since (see Lemma 3.4) an h-conformal atomless measure exists, we can show in exactly the same way as in the proof of Theorem 4.4 in [UZ1] that if $t > h$ then $\nu\Big(\bigcup_{n\ge 0} F^{-n}(1)\Big) = 1$ for any t-conformal measure ν; as a matter of fact the argument from the proof of Theorem 4.4 in [UZ1] can be repeated for any point $z \in J_r(F)\setminus\bigcup_{n\ge 0} F^{-n}(1)$. Since, by Corollary 3.7 and (3.8), if a t-conformal measure exists and $t > 1$, then $t \ge h$, the items (4) and (1) are therefore proven. We can also show in exactly the same way as in the proof of Theorem 4.4 in [UZ1] that any two h-conformal measures restricted to $J_r(F) \setminus \bigcup_{n\ge 0} F^{-n}(1)$ are equivalent (comp. the second part of the first sentence of this proof). Suppose now that ν is an h-conformal measure with $\nu\Big(\bigcup_{n\ge 0} F^{-n}(1)\Big) > 0$. Since the set $\bigcup_{n\ge 0} F^{-n}(1)$ is completely invariant, it is straightforward to see that the measure ν_1 defined as ν restricted to $\bigcup_{n\ge 0} F^{-n}(1)$ and normalized is h-conformal. This however is a contradiction since, by Lemma 3.4 there exists an atomless h-conformal measure m. Thus, $m(J_r(F)\setminus(\bigcup_{n\ge 0} F^{-n}(1)\Big)) = 1$ and, obviously, m and ν_1 restricted to $J_r(F)\setminus\bigcup_{n\ge 0} F^{-n}(1)$ are not equivalent. Thus, any h-conformal measure ν is equivalent to m, and items (2) and (3) can be proven as in the proof of Theorem 4.4 in [UZ1]. ■

The following two theorems are included for completeness since they establish fundamental properties of Hausdorff and packing measures on $J_r(F)$ and provide a complete geometric interpretation of the h-conformal measure m. The proofs are straightforward modifications of corresponding proofs in [DU2] and [UZ1]. Thus, we only indicate how to use the arguments given there. We would like however to emphasize that neither of these two theorems below is needed for the proof of Theorem 1.1, the main result of this paper.

Theorem 3.10. *We have* $0 < \mathrm{H}^h(J_r(F)) < \infty$.

Proof. The proof that $\mathrm{H}^h(J_r(F)) < \infty$ repeats word by word the proof of the appropriate part of the proof of Theorem 4.5 in [UZ1]. The proof that $\mathrm{H}^h(J_r(F)) > 0$ is more involved and it combines the ideas of the proof of Theorem 4.10 in [UZ1] and the proof of Proposition 5.3 in [DU2]. Put

$$G = \overline{B(1,1)} \cap \{z : \mathrm{Re} z \geq 1\}.$$

Then there exists a unique holomorphic inverse branch $f_1^{-1} : G \to \mathbb{C}$ of f such that $f_1^{-1}(1) = 1$. Notice that $f_1^{-1}(G) \subset G$ and $f_1^{-n} : G \to G$ converges uniformly to the constant function $z \mapsto 1$, $z \in G$. Take $\theta \in (0, \pi)$ (the reader is invited to think about θ as the number appearing in the proof of Theorem 4.10 in [UZ1]) so small that

$$B(z, \theta) \cap \{f^n(0) : n \geq 0\} = \emptyset$$

for all $z \notin G$ with $\mathrm{Re} z \geq 1$. Take $\gamma \in (0, \theta/32)$ to be so small that if $f^{n-1}(z) \in P_+ \setminus G$, then the holomorphic inverse branch of f^n, sending $f^n(z)$ to z, is well-defined on $B(f^n(z), 2\gamma)$. The reader is invited to think about γ as the number appearing in Sections 4 and 5 of [DU2]. Lemmas 4.7 and 4.8 are in [DU2] are of local character and continue to be true for our map f with ω replaced by 1. By our choice of γ, the proof of Proposition 4.9 in [DU2] goes through in our case essentially word by word (one must for instance replace $||T'||$ by $||f'||_G := \sup\{|f'(w)| : w \in G\}$. Replace now $n = n(z, r)$ by $n + 1$ and follow Section 5 of [DU2] to obtain Proposition 5.3, where the assumption $(z, r) \in \Re(\omega)$ means that $f^{n(z,r)+1} \in G$. Since $h > 1$, it follows from (5.5) in [DU2] and the right-hand side inequality appearing in Proposition 5.3 of [DU2] that

$$m(B(z, r)) \preceq r^h. \tag{3.7}$$

So, suppose that that $f^{n(z,r)+1}(z) \notin G$. Since $r|(f^n)'(z)| \leq \gamma K^{-1}||f'||_G^{-1} < \gamma < \theta/32$ and $r|(f^n)'(z)| \geq \gamma K^{-1}||f'||_G^{-1}$, the proof of Theorem 4.10 in [UZ1] goes essentially word by word to give

$$m(B(z, r)) \leq K^h \left(\frac{K||f'||_G}{\gamma}\right)^{h-1} r^h.$$

Combining this with (3.7) finishes the proof. ■

The proof of Proposition 4.9 in [UZ1] goes through word by word to give the following.

Proposition 3.11. *We have* $\mathrm{P}^h(J_r(f)) = \infty$. *In fact* $\mathrm{P}^h(G) = \infty$ *for every open non-empty subset of* $J_r(f)$.

We end this section with the following two results which can be correspondingly proven in the same way as the appropriate part of Theorem 4.5 in [UZ1] and Theorem 5.2 in [UZ1].

Theorem 3.12. *There exists a unique probability F-invariant measure μ absolutely continuous with respect to h-conformal measure m. In addition, μ is equivalent to m and ergodic.*

4. Proof of Theorem 1.1

It follows from Theorem 4.4 in [UZ1] that for every $\lambda \in (0, 1/e)$ there exists a unique h_λ-conformal conformal measure for $F_\lambda : J(F_\lambda) \to J(F_\lambda)$ supported on $J_r(F_\lambda)$, where $h_\lambda = \text{HD}(J_r(F_\lambda)) = \text{HD}(J_r(f_\lambda))$. We need the following.

Lemma 4.1. *If $\lambda_n \nearrow 1/e$ as $n \to \infty$, then the sequence of measures $\{m_{\lambda_n}\}_{n=1}^\infty$ is tight.*

Proof. Writing $\text{P}_{\lambda,R}(t)$ for $\text{P}_R(t)$ in the proof of Theorem 2.1 in [UZ1], we can find $\delta > 0$ so small that the estimate of $\text{P}_{\lambda,R}(t)$ provided in this proof holds for some $R > 1$ and all $\lambda \in (1-\delta, 1+\delta)$. It is then easy to see that there exists some $s > 1$ such that $\text{P}_{\lambda,R}(s) > 0$ for all $\lambda \in (1-\delta, 1+\delta)$. Along with Corollary 3.8 this implies that

$$h_\lambda = \text{HD}(J_r(F_\lambda)) \geq \text{HD}(\pi(J_{R,\lambda})) \geq s > 1. \tag{4.1}$$

Now, with obvious modifications, the proof goes in the same way as the proof of Proposition 3.3 in [UZ1]. ■

Lemma 4.2. *Fix a sequence $\{\lambda_k\}_{k=1}^\infty$ such that $\lambda_k \nearrow 1/e$ as $n \to \infty$. Assume that $m_{\lambda_k} \to m$ weakly and that $\lim_{n\to\infty} h_{\lambda_k} = t$ for some $t \geq 0$. Then m is a t-conformal measure for $F : P \to P$ supported on $P_+ \cap J(F)$.*

Proof. Put $m_k = m_{\lambda_k}$ and $h_k = h_{\lambda_k}$. Since each measure m_k is supported on the set $\{z \in P : \text{Re}z \geq q_{\lambda_k}\}$ (see [UZ1]) and since $\lim_{k\to\infty} q_{\lambda_k} = q = 1$, we see that $m(P_+) = 1$. Fix $w \in P \setminus J(F)$. Since the complement of $J(f)$ is the basin B of attraction of q, and $B = \bigcup n = 0^\infty f^{-n}(\text{Re}z < q)$, there exists n such that $F^n(w) = \pi(f^n(w)) \in \{\text{Re}z < q\}$. This implies (by continuity) that there exists $r = r(w))$ such that $F_\lambda^n(B(w,r)) \subset (\text{Re}z < q_\lambda)$ and, consequently, $B(w,r) \subset P \setminus J(F_\lambda)$ for all λ sufficiently close to $1/e$, say $\lambda \in (\xi, 1/e)$. Hence $m_\lambda(B(w,r)) = 0$ for all $\lambda \in (\xi, 1/e)$. Consequently, for every $w \in P \setminus J(F)$ there exists $r = r(w)$ such that $m(B(x, \frac{r}{2}) = 0$. Thus $m(P \setminus J(F)) = 0$. Thus the very last claim of this lemma is prove. Since $F(P \setminus J(F)) = P \setminus J(F)$, this implies that also $m(F(P \setminus J(F)) = 0$. Notice that for all $\lambda \in (0, \frac{1}{e}]$ $J(F_\lambda) \subset \bigcup_{l\in\mathbb{Z}} \{z : |\text{Im}(f_\lambda(z)) - 2l\pi| < \frac{\pi}{2}$. (i.e. $J(F_\lambda)$ is contained in a union of disjoint strips $L_l(\lambda) = f_\lambda^{-1}(|\{\text{Im}z - 2l\pi\} < \frac{\pi}{2} \subset \tilde{L}_k(\lambda) = f_\lambda^{-1}(|\{\text{Im}z - 2l\pi\} < \frac{3\pi}{4}$. Each F_λ is continuous and one-to-one on $\tilde{L}_l(\lambda)$. We shall check now the conformality of m.

First, notice that it is enough to assume in the formula (3.2) that the set A is bounded. So, let A be a bounded Borel set on which F is one-to-one. Then $A \cap J(F) = \bigcup_k L_k \cap A$ and it is clear that it is enough to prove conformality of the measure for each set $L_k \cap A$, i.e. one can assume that A is bounded and contained in some strip L_l. First, we assume additionally that A is a Jordan domain $A \subset L_l$ with a smooth boundary such that $m(\partial A) = m(\partial F(A)) = 0$ and we shall check the conformality formula

$$m(F(A)) = \int_A |F'|^t dm.$$

By continuity, for λ_k close to $\frac{1}{e}$, A is contained in $\tilde{L}_l(\lambda)$, so F_{λ_k} is continuous and one-to-one on A. Moreover, $F_{\lambda_k}|_A$ converges to $F|_A$ uniformly as $\lambda_k \to \frac{1}{e}$.
We shall prove first that

$$\lim_{n\to\infty} \int_A |F'_{\lambda_k}|^{h_k} dm_k = \int_A |F'|^t dm. \tag{4.2}$$

Indeed,

$$\left|\int_A |F'_{\lambda_k}|^{h_k} dm_k - \int_A |F'|^t dm\right| \le \left|\int_A |F'_{\lambda_k}|^{h_k} dm_k - \int_A |F'|^t dm_k\right| + \left|\int_A |F'|^t dm_k - \int_A |F'|^t dm\right|.$$

The second summand converges to zero since $m_{\lambda_k} \to m$ weakly and $m(\partial A) = 0$. The first summand tends to zero since $|F'_{\lambda_k}|^{h_k}$ converges to $|F'|^t$ uniformly on A. Using (4.2) and h_k-conformality for F_{λ_k}, we get that

$$m_k\Big(F_{\lambda_k}(A)\Big) = \int_A |F'_{\lambda_k}|^{h_k} dm_k \to \int_A |F'|^t dm$$

when $n \to \infty$. Therefore, in order to prove t-conformality of m, it is enough to check that

$$\lim_{n\to\infty} m_k\Big(F_{\lambda_k}(A)\Big) = m(F(A)).$$

Indeed,

$$|m_k\Big(F_{\lambda_k}(A)\Big) - m(F(A))| \le |m_k\Big(F_{\lambda_k}(A)\Big) - m_k\Big(F(A)\Big)| + |m_k\Big(F(A)\Big) - m(F(A))|.$$

The second summand converges to 0 since $m(\partial F(A)) = 0$ and $m_k \to m$ weakly. The first summand can be estimated from above by $m_k\Big(F_{\lambda_k}(A) \bigtriangleup F(A)\Big)$. Let us show that

$$\lim_{n\to\infty} m_k\Big(F_{\lambda_k}(A) \bigtriangleup F(A)\Big) = 0.$$

Indeed, $F_{\lambda_k}(A)$ is a Jordan domain enclosed by some smooth curve γ_k, $F(A)$ is also a Jordan domain enclosed by some smooth curve γ. Therefore, since F_{λ_k} converges to F uniformly on A, the symmetric difference $F_{\lambda_k}(A) \bigtriangleup F(A)$ is contained in some "collar" $Y_k = \{z : \mathrm{dist}(z, \gamma) < \delta_k\}$, where $\delta_k \to 0$ as $n \to \infty$. Now, it is enough to observe that

$$\lim_{n\to\infty} m_k(Y_k) = 0.$$

Indeed, suppose on the contrary that $m_{k_i}(Y_{k_i}) \ge \varepsilon$ for some $\varepsilon > 0$ and infinitely many k_i's. Fix one such k_i. Since $Y_{k_i} \supset Y_{k_{i+j}}$ for all j large enough, we get $m_{k_{i+j}}(Y_{k_i}) \ge \varepsilon$ which in turn, letting $j \to \infty$ gives that $m(\overline{Y_{k_i}}) \ge \varepsilon$ for all $i \ge 1$. Since $\gamma = \bigcap_{i=1}^\infty \overline{Y_{k_i}}$, this would imply that

$m(\partial F(A)) = m(\gamma) \geq \varepsilon > 0$ which contradicts our assumption that $m(\partial F(A)) = 0$. Thus, we have proved the equality $m(F(A)) = \int_A |F'|^t dm$ for every A with a smooth boundary, $A \subset L_l$, $m(\partial A) = m(\partial F(A)) = 0$. By the standard limit procedure, the same holds for any Jordan domain A with a smooth boundary and for any (open or closed) rectangle P contained in L_l. Now, we have to compare two Borel measures on L_l: $m_1(A) = \int_A |F'|^t$ and $m_2(A) = m(F(A))$ (the latter is well-defined since $F_{|}L_l$ is a homeomorphism). Since these two measures coincide for any rectangle $A \subset L_l$, it is standard to conclude that $m_1 = m_2$. ■

Lemma 4.3. *With the same assumptions as in Lemma 4.2, the limit measure m has no atoms.*

Proof. We change the variables. Put $\tilde{z} = z - q_\lambda$. In these coordinates f_λ takes on the form

$$\tilde{f}_\lambda(z) = \lambda e^{\tilde{z}+q_\lambda} - q_\lambda = q_\lambda \lambda e^{\tilde{z}} - q_\lambda = q_\lambda(e^{\tilde{z}} - 1).$$

So,

$$\tilde{f}_\lambda(0) = 0 \ \text{ and } \ \tilde{f}'_\lambda(0) = q_\lambda.$$

In particular we gained that 0 is a common fixed point of all maps $\tilde{f}_\lambda$. Developing the Taylor series expansion of $\tilde{f}_\lambda$ about 0, we conclude the existence of a constant $B > 0$ such that for every $\lambda \in (0, 1/e)$ sufficiently close to $1/e$ and all $z \in B(0, 1/2)$, we have that

$$\left| \tilde{f}_\lambda(z) - \left(q_\lambda z + \frac{q_\lambda}{2} z^2 \right) \right| \leq B|z|^3. \tag{4.3}$$

We change now the variables again sending the fixed point 0 to ∞ via the conjugacy $z \mapsto 1/z$. In these coordinates $\tilde{f}_\lambda$ takes on the form

$$\begin{aligned} g_\lambda(w) &= \frac{1}{\tilde{f}_\lambda\left(\frac{1}{w}\right)} = \frac{1}{q_\lambda\left(\frac{1}{w} + \frac{1}{2w^2} + O\left(\frac{1}{|w|^3}\right)\right)} \\ &= \frac{w}{q_\lambda\left(1 + \frac{1}{2w} + O\left(\frac{1}{|w|^2}\right)\right)} = \frac{w}{q_\lambda}\left(1 - \frac{1}{2w} + O\left(\frac{1}{|w|^2}\right)\right) \end{aligned}$$

for all w with $|w| \geq 2$ and the constant involved in $O(1/|w|^2)$ independent of λ. There thus exists $Q > 0$ (independent of λ) such that for all $w \in [Q, \infty)$, one can write

$$g(w) := g_\lambda(w) < \hat{g}(w) := q_\lambda^{-1}\left(w - \frac{1}{4}\right) < w.$$

Suppose now that $\hat{x}, x \in [Q, \infty)$ and $\hat{g}(\hat{x}) \leq g(x)$. Since $g(x) < \hat{g}(x)$ and since the function $\hat{g}$ is increasing, we conclude that $\hat{x} < x$. So, if $w_n > w_{n-1} > w_{n-2} > \ldots > w_0 = \xi$ is a sequence of consecutive preimages of ξ under g (i.e. $g(w_j) = w_{j-1}$) and if $\hat{w}_n > \hat{w}_{n-1} > \hat{w}_{n-2} > \ldots > \hat{w}_0 = \xi$ is a corresponding sequence of preimages of ξ under $\hat{g}$, then $w_n > \hat{w}_n$. The value of $\hat{w}_n$ can be easily calculated. Indeed,

$$\hat{w}_n = \xi q_\lambda^n + \frac{1}{4}\frac{q_\lambda^n - 1}{q_\lambda - 1} = \xi q_\lambda^n + \frac{1}{4}\sum_{j=0}^{n-1} q_\lambda^n \geq \frac{n}{4}.$$

Hence

$$w_n > \hat{w}_n > \frac{n}{4}. \tag{4.4}$$

Let $\text{Log}: \{z \in \mathbb{C}: \text{Re}z > 0\} \to \mathbb{C}$ be the branch of logarithm sending 1 to 0. Then

$$\tilde{f}_{\lambda,0}^{-1}(w) = \text{Log}(q_\lambda^{-1}w + 1), \quad w \in \{z \in \mathbb{C}: \text{Re}z > -1\},$$

is the unique holomorphic inverse branch of $\tilde{f}_\lambda$ sending 0 to 0. Fix now $w \in \{z \in \mathbb{C}: \text{Re}z \geq 0\}$. Then

$$\text{Re}\Big(\tilde{f}_{\lambda,0}^{-1}(w)\Big) = \log|q_\lambda^{-1}w + 1| \geq \log\Big(\text{Re}(q_\lambda^{-1}w) + 1\Big) \geq \log 1 = 0.$$

Hence

$$\tilde{f}_{\lambda,0}^{-1}(\{z \in \mathbb{C}: \text{Re}z \geq 0\}) \subset \{z \in \mathbb{C}: \text{Re}z \geq 0\}$$

and we can therefore speak about iterates

$$\tilde{f}_{\lambda,0}^{-n}(\{z \in \mathbb{C}: \text{Re}z \geq 0\}) \subset \{z \in \mathbb{C}: \text{Re}z \geq 0\}.$$

Fix now $z_0 \in (0, 1/Q)$ and for every $n \geq 0$ put $z_n = \tilde{f}_{\lambda,0}^{-1}(z_0) \in (0, \infty)$. Since the maps $\tilde{f}_\lambda$ and g_λ are conjugate via the map $z \mapsto 1/z$, it follows from (4.4) that

$$z_n \leq \frac{4}{n}. \tag{4.5}$$

Let

$$S = \left\{z \in \mathbb{C}: 1 \leq |z| \leq \frac{3\pi}{4}\right\} \cap \{z \in \mathbb{C}: \text{Re}z \geq 0\}.$$

We shall prove now that for every $\lambda \in (0, 1/e)$ sufficiently close to $1/e$ and for every $r \in (0, 1)$, we have

$$B(0, r) \cap J(\tilde{f}_\lambda) \subset \bigcup_{n=n_*(r)}^{\infty} \tilde{f}_{\lambda,0}^{-n}(S), \tag{4.6}$$

where $n_*(r) = \left[\frac{-\log(r)}{\log(2e)}\right]$ and $[x]$ is the integer part of x. In particular

$$\lim_{r \to 0} n_*(r) = \infty. \tag{4.7}$$

Indeed, notice first that if $\text{Re}z \geq 0$, then

$$\begin{aligned} |\tilde{f}_{\lambda,0}^{-1}(z)| = |\tilde{f}_{\lambda,0}^{-1}(z) - \tilde{f}_{\lambda,0}^{-1}(0)| &\leq |z| \sup\{|(\tilde{f}_{\lambda,0}^{-1})'(w)| : w \in [0, z]\} \\ &\leq |z| \sup\left\{\frac{1}{q_\lambda|1 + q_\lambda^{-1}w|} : w \in [0, z]\right\} \leq |z|q_\lambda^{-1}. \end{aligned}$$

Since $J(\tilde{f}_\lambda) = J(f_\lambda) - q_\lambda \subset \{z \in \mathbb{C}: \text{Re}z \geq 0\}$, since $\tilde{f}_\lambda(J(\tilde{f}_\lambda)) = J(\tilde{f}_\lambda) = \tilde{f}_\lambda^{-1}(J(\tilde{f}_\lambda))$ and since $\tilde{f}_{\lambda,0}^{-1}(f_\lambda(z)) = z$ for all $z \in \{z \in \mathbb{C}: \text{Re}z \geq 0\} \cap B(0, 1)$, we therefore conclude that $|\tilde{f}_\lambda(z)| \geq q_\lambda|z|$ for all $z \in J(\tilde{f}_\lambda) \cap B(0, 1)$. Since for every $\lambda \in (0, 1/e)$, $q_\lambda > 1$, we thus see that for every $z \in B(0,1) \cap J(\tilde{f}_\lambda) \setminus \{0\}$ there exists a least $n \geq 1$ such that $\tilde{f}_\lambda^n(z) \notin B(0,1)$. It then follows from the Mean Value Inequality that

$$|\tilde{f}_\lambda^n(z)| = |\tilde{f}_\lambda(\tilde{f}_\lambda^{n-1}(z)) - \tilde{f}_\lambda(0)| \leq q_\lambda e|\tilde{f}_\lambda^{n-1}(z) - 0| \leq q_\lambda e|\tilde{f}_\lambda^{n-1}(z)| \leq q_\lambda e \leq \frac{3}{4}\pi,$$

where the last inequality was written assuming that $\lambda \in (0, 1/e)$ is sufficiently close to $1/e$. Hence, the formula (4.6) has been established with an appropriate $n = n_*(z,r)$ and we need to prove the required lower bound on $n_*(z,r)$ independent of z. So, suppose that $z \in B(0,r)$, $r \in (0,1)$, and let $n \geq 1$ be the least integer such that $\tilde{f}_\lambda^n(z) \notin B(0,1)$. Then

$$1 \leq |\tilde{f}_\lambda^n(z)| = |\tilde{f}_\lambda^n(z) - \tilde{f}_\lambda^n(0)| \leq |z| \sup\{|(\tilde{f}_\lambda^n)'(w)| : w \in [0,z]\} \leq |z|(q_\lambda e)^n \leq r(2e)^n,$$

where the last inequality was written assuming that $\lambda \in (0, 1/e)$ is sufficiently close to $1/e$. Hence, $n \geq \frac{-\log(r)}{\log(2e)}$ and the proof of (4.6) is complete.

Fix now $r \in (0,1)$ and $\lambda \in (0,1/e)$ so close to $1/e$ as required in formula (4.6). Let $\tilde{m}_\lambda$ be the image of m_λ under the translation $z \mapsto z - q_\lambda$. Notice that

$$\tilde{m}_\lambda\Big(\tilde{f}_{\lambda,0}^{-n}(S)\Big) = \int_S |(\tilde{f}_{\lambda,0}^{-n})'|^{h_\lambda} d\tilde{m}_\lambda$$

and that $\tilde{m}_\lambda$ is atomless. Therefore, using (4.6), we get

$$\begin{aligned} \tilde{m}_\lambda(B(0,r)) &\leq \sum_{n \geq n_*(r)} \tilde{m}_\lambda\Big(\tilde{f}_{\lambda,0}^{-n}(S)\Big) = \sum_{n \geq n_*(r)} \int_S |(\tilde{f}_{\lambda,0}^{-n})'|^{h_\lambda} d\tilde{m}_\lambda \\ &\leq \tilde{m}_\lambda(S) \sum_{n \geq n_*(r)} \sup\Big\{|(\tilde{f}_{\lambda,0}^{-n})'(z)| : z \in S\Big\} \leq \sum_{n \geq n_*(r)} \sup\Big\{|(\tilde{f}_{\lambda,0}^{-n})'(z)| : z \in S\Big\} \end{aligned} \tag{4.8}$$

Fix now $z_0 = 3\pi/4$. Since $-q_\lambda$ is the only singular point of $\tilde{f}_\lambda$ and since $\{\tilde{f}_\lambda^n(-q_\lambda)\}_{n=0}^\infty \subset \mathbb{R} \cap \{z \in \mathbb{C} : \mathrm{Re} z < 0\}$, we conclude that there exists an open connected simply connected set $V \supset S$ such that each holomorphic inverse branch of $\tilde{f}_\lambda^n$ defined on S extends holomorphically to V. It therefore follows from Koebe's Distortion Theorem that for every $n \geq 0$

$$\inf\Big\{|(\tilde{f}_{\lambda,0}^{-n})'(w)| : w \in S\Big\} \asymp \sup\Big\{|(\tilde{f}_{\lambda,0}^{-n})'(w)| : w \in S\Big\}.$$

Since $[z_1, z_0] = [\log(q_\lambda^{-1}3\pi/4 + 1), 3\pi/4] \subset [1, \pi]$ for all $\lambda \in (0,1/e)$ sufficiently close to $1/e$, we obtain the following

$$\begin{aligned} |z_0 - z_1| &\leq \sup\{|(\tilde{f}_\lambda^n)'(w)| : w \in [z_n, z_{n-1}]\}|z_n - z_{n-1}| \\ &= \Big(\inf\Big\{|(\tilde{f}_{\lambda,0}^{-n})'(w)| : w \in [z_1, z_0]\Big\}\Big)^{-1} |z_n - z_{n-1}| \\ &\asymp \Big(\sup\Big\{|(\tilde{f}_{\lambda,0}^{-n})'(w)| : w \in S\Big\}\Big)^{-1} |z_n - z_{n-1}|. \end{aligned} \tag{4.9}$$

Now, in view of (4.3)

$$\begin{aligned} |z_n - z_{n-1}| = |z_n - \tilde{f}_\lambda(z_n)| &= \left|z_n - \Big(q_\lambda z_n - \frac{q_\lambda}{2} z_n^2 + O(|z_n|^3)\Big)\right| \\ &\leq (q_\lambda - 1)|z_n| + C|z_n|^2 \end{aligned}$$

for some constant $C > 0$ independent of λ and n. Combining this along with (4.8), (4.9) and (4.5), we get

$$\begin{aligned}\tilde{m}_\lambda(B(0,r)) &\le \sum_{n\ge n_*(r)} |z_n - z_{n-1}|^{h_\lambda}|z_0 - z_1|^{-h_\lambda} \asymp \sum_{n\ge n_*(r)} |z_n - z_{n-1}|^{h_\lambda} \\ &\le \sum_{n\ge n_*(r)} \Big((q_\lambda - 1)|z_n| + C|z_n|^2\Big)^{h_\lambda} \preceq \sum_{n\ge n_*(r)} \frac{1}{n^{h_\lambda}}.\end{aligned}$$

Since, in view of (4.1), $h_\lambda \ge s > 1$ for all $\lambda \in (0, 1/e)$ sufficiently close to $1/e$, the last series can be estimated from above by

$$\sum_{n\ge n_*(r)} \frac{1}{n^s}$$

which converges to 0 when $r \searrow 0$ since $\lim_{r\to 0} n_*(r) = \infty$. Since $\lim_{k\to\infty} \tilde{m}_{\lambda_k} = \tilde{m}$, where $\tilde{m}$ is the image of the limit measure m under the translation $z \mapsto z - 1$, we therefore conclude that $\tilde{m}(0) = 0$ and consequently $m(1) = 0$. Hence by conformality of m, $m\left(\bigcup_{n\ge 0} f^{-n}(1)\right) = 0$ and, as m cannot have other atoms, we are done. ■

We are now ready to finish the proof of Theorem 1.1. Indeed, fix a sequence $\lambda_k \nearrow 1/e$. Since $1 < s \le h_{\lambda_k} \le 2$, each accumulation point t of the sequence $\{h_{\lambda_k}\}_{k=1}^\infty$ is in the interval $[s, 2]$. Our aim is to show that $t = h(= \mathrm{HD}(J_r(f_{1/e}))$. Passing to a subsequence, we may assume that $\lim_{k\to\infty} h_{\lambda_k} = t$ and, in view of Lemma 4.1, we may assume that the sequence $\{m_{\lambda_n}\}_{n=1}^\infty$ converges weakly to a Borel probability measure m. In view of Lemma 4.2, (4.1) and Lemma 4.3, m is a t-conformal measure for $F_{1/e}$ supported on $J(F_{1/e})$ with $t \ge s > 1$. Since by Lemma 4.3, the measure m is atomless, it therefore follows from Theorem 3.9(1) that $t = h$ and we are done. ■

References

[ADU] J. Aaronson, M. Denker, M. Urbański, Ergodic theory for Markov fibered systems and parabolic rational maps, Transactions of A.M.S. 337 (1993), 495-548.

[BZ] B. Bodart, M. Zinsmeister, Quelques résultats sur la dimension de Hausdorff des ensembles de Julia des polynomes quadratiques. Fund. Math. (1996), 121-137.

[DU1] M. Denker, M. Urbański, On Sullivan's conformal measures for rational maps of the Riemann sphere, Nonlinearity 4 (1991), 365-384.

[DU2] M. Denker, M. Urbański, The capacity of parabolic Julia sets, Math. Zeitsch. 211, (1992), 73-86.

[DU3] M. Denker, M. Urbański, On the existence of conformal measures, Trans. A.M.S. 328 (1991), 563-587.

[DU4] M. Denker, M. Urbański, Hausdorff and conformal measures on Julia sets with a rationally indifferent periodic point, J. London Math. Soc. 43 (1991), 107-118.

[McM] C. McMullen, Area and Hausdorff dimension of Julia sets of entire functions, Trans. A.M.S. 300 (1987), 329-342.

[PU] F. Przytycki, M. Urbański, Fractals in the plane - Ergodic theory methods, to appear in the Cambridge University Press, available on Urbański's webpage.

[UZ1] M. Urbański, A. Zdunik, The finer geometry and dynamics of exponential family, Michigan Math. J.51 (2003), 227-250.

[UZ2] M. Urbański, A. Zdunik, Real analyticity of Hausdorff dimension of finer Julia sets of exponential family, Preprint 2002, to appear Ergod. Th. & Dynam. Sys.

[UZ3] M. Urbański, A. Zdunik, Geometry and ergodic theory of non-hyperbolic exponential maps, Preprint 2003.

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