REAL ANALYTICITY OF HAUSDORFF DIMENSION OF FINER JULIA SETS OF EXPONENTIAL FAMILY

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ABSTRACT. We deal with all the mappings $f_\lambda(z) = \lambda e^z$ that have an attracting periodic orbit. We consider the set $J_r(f_\lambda)$ consisting of those points of the Julia set of f_λ that do not escape to infinity under positive iterates of f_λ. Our ultimate result is that the function $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$ is real analytic. In order to prove it we develop the thermodynamic formalism of potentials of the form $-t\log|F'_\lambda|$, where F_λ is the natural map associated with f_λ closely related to the corresponding map introduced in [UZd]. It includes appropriately defined topological pressure, Perron-Frobenius operators, geometric and invariant generalized conformal measures (Gibbs states). We show that our Perron-Frobenius operators are quasicompact, that they embed into a family of operators depending holomorphically on an appropriate parameter and we obtain several other properties of these operators. We also study in detail the properties of quasiconformal conjugacies between the maps f_λ. As a byproduct of our main course of reasoning we prove stochastic properties of the dynamical system generated by F_λ and the invariant Gibbs states μ_t such as the Central Limit Theorem and the exponential decay of correlations.

1. Introduction

In this section we continue our investigations of the subsets $J_r(f_\lambda)$ of the Julia sets of maps $f_\lambda(z) = \lambda e^z$ that have an attracting periodic orbit. We call the family of corresponding parameters λ by Hyp. Our main result is that the function $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$, $\lambda \in \mathrm{Hyp}$, is real analytic. In Section 2 we prove that the maps from the family Hyp are uniformly expanding on their Julia sets and we define the maps F_λ mapping an appropriate infinite cylinder into itself. In Section 3 we define appropriate in this context topological pressure of the potentials $-t\log|F'_\lambda|$, $t \geq 0$, Perron-Frobenius operators with some more general potentials and generalized conformal (Gibbs) measures. Using the existence of these measures we prove three basic properties of the Perron-Frobenius operators in Lemmas 3.4, 3.5 and 3.6. We end this section with the proof of the uniqueness and ergodicity of conformal measures. In Section 4 we show that our Perron-Frobenius operators satisfy the assumptions of the Ionescu-Tulcea and Marinescu theorem. In particular the Perron-Frobenius operator acting on the space of Hölder continuous functions (with a fixed exponent) is quasicompact and the full description of its spectral properties is provided in Theorem 4.4. This permits us to prove in Section 5 the existence and uniqueness of invariant measures equivalent to conformal measures. We

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also obtain here finer stochastic properties of the dynamical system generated by the map F_λ and these invariant measures such as the Central Limit Theorem and the exponential decay of correlations. The short 6th section is devoted to the presentation of a proof of an appropriated Bowen's formula (Hausdorff dimension of the set $J_r(f_\lambda)$ is equal to the unique zero of the pressure function) which also follows from the results proven in [UZd]. Section 7 , closely related to the last section of the paper [UZi], provides a sufficient condition for our Perron-Frobenius to depend holomorphically on the appropriate parameter. Section 8 establishes uniform Hölder continuity of quasiconformal conjugacies in connected components of Hyp and some other interesting itself properties. In Section 9, perhaps most technical part of our paper, we first prove continuity of the topological pressure with respect to the parameter $\lambda \in \mathrm{Hyp}$, then we check that the conditions presented in Section 7 are satisfied, and we relatively easy conclude the proof of our main result by applying the perturbation theory for linear operators and the implicit function theorem for the topological pressure.

2. Hyperbolic maps - Preliminaries

We denote by Hyp the set of all parameters λ such that f_λ has a periodic attracting orbit. Fix $\lambda \in \mathrm{Hyp}$. We shall check that f_λ is expanding on its Julia set.

Proposition 2.1. *For every $\lambda \in \mathrm{Hyp}$ then there exist $c > 0$ and $\gamma > 1$ such that for $z \in J(f_\lambda)$*

$$|(f_\lambda^n)'(z)| > c\gamma^n.$$

Proof. Let $\lambda \in \mathrm{Hyp}$ and denote f_λ by f. Using Proposition 6.1 in [McM], we conclude that for every $z \in J(f)$, $\lim_{n\to\infty} |(f^n)'(z)| = \infty$. For every $m \geq 1$ let

$$A_m(\lambda) = \{z \in J(f) : |(f_\lambda^m)'(z)| > 3\}.$$

Since each set A_m is open and since there exists $M > 0$ such that $\{z : \mathrm{Re} z > M\} \subset A_1(\lambda)$, the open sets $A_2(\lambda), A_3(\lambda), A_4(\lambda), \dots$ cover

$$Y(\lambda) = \{z \in J(f) : 0 \leq \mathrm{Im}(z) \leq 2\pi\} \setminus \{z : \mathrm{Re} z > M\},$$

a compact subset of $J(f_\lambda)$. Thus, we can chose from the cover $\{A_m(\lambda)\}_{m=1}^\infty$ a finite subcover of $Y(\lambda)$ say, $A_1(\lambda), A_2(\lambda), \dots, A_k(\lambda)$ for some $k_\lambda \geq 1$. Since all the sets A_m are invariant under the shift map $z \to z + 2\pi i$, this shows that $J(f) \subset A_1(\lambda) \cup A_2(\lambda) \cup \dots \cup A_k(\lambda)$. Let

$$I = \inf_{z\in J(f)} \{|f'(z)|\} = \inf_{z\in J(f)} \{|z|\}$$

The number I is positive since a neighbourhood of 0 is attracted by the attracting periodic orbit. Take $z \in J(f)$ and $n \geq 1$. Then, using the above finite cover $A_1(\lambda), A_2(\lambda), \dots, A_k(\lambda)$ one can divide the trajectory of z of length n (i.e. the sequence $z, f(z), \dots, f^n(z)$) into pieces of length $\leq k_\lambda$ such that the derivative of the composition along each (except for the last one) piece of the trajectory is larger than 2. This gives us the following estimate

$$|(f^n)'(z)| \geq 3^{[\frac{n}{k}]} I^k$$

which is the required property. ■

Since the map f_λ is periodic, we consider it rather on the cylinder than on $\mathbb{C}$. So, let P be the quotient space (the cylinder),

$$P = \mathbb{C}/\sim,$$

where $z_1 \sim z_2$ if and only if $z_1 - z_2 = 2k\pi i$ for some $k \in \mathbb{Z}$. Let $\pi : \mathbb{C} \to P$ be the natural projection. Since the map $\pi \circ f_\lambda : \mathbb{C} \to P$ is constant on equivalence classes of relation $\sim$, it canonically induces a conformal map

$$F_\lambda : P \to P.$$

The map $F_\lambda : P \to P$ will be the main object of our technical considerations. The Julia set of F is defined to be

$$J(F_\lambda) = \pi(J(f_\lambda)). \tag{2.1}$$

and

$$F_\lambda(J(F_\lambda)) = J(F_\lambda) = F^{-1}(J(F_\lambda)).$$

The cylinder P is canonically endowed with a Euclidean metric which without confusion will be denoted by the same symbol $|w - z|$ for all $z, w \in P$.

3. Pressure, Perron-Frobenius Operators and Generalized Conformal Measures

We fix now $\lambda \in \mathrm{Hyp}$ until Section 8, we put

$$f = f_\lambda \text{ and } F = F_\lambda$$

and we start to deal with topological pressure. For every $t \geq 0$ and every $z \in J(F)$ define the lower and upper topological pressure respectively by

$$\underline{\mathrm{P}}_z(t) = \liminf_{n\to\infty} \frac{1}{n} \log \sum_{x \in F^{-n}(z)} |(F^n)'(x)|^{-t} \text{ and } \overline{\mathrm{P}}_z(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in F^{-n}(z)} |(F^n)'(x)|^{-t}.$$

Since any two points in $J(F)$ belong to an open simply connected set disjoint from $\pi\Big(\{f^n(0) : n \geq 0\}\Big)$, the forward trajectory of $\pi(0)$ under F, it follows from Koebe's distortion theorem that $\underline{\mathrm{P}}_z(t)$ and $\overline{\mathrm{P}}_z(t)$ are independent of z and we denote their respective values by $\underline{\mathrm{P}}(t)$ and $\overline{\mathrm{P}}(t)$. Let

$$\mathrm{P}_z(1,t) := \sum_{x \in F^{-1}(z)} |F'(x)|^{-t} = \sum_{k=-\infty}^{+\infty} |\tilde{z} + 2\pi i k|^{-t}, \tag{3.1}$$

where $\tilde{z}$ is an arbitrary point in $\pi^{-1}(z)$. Notice that for every $t > 1$ the series $\mathrm{P}_z(1,t)$ converges and $||\mathrm{P}(1,t)||_\infty = \sup_{z\in J(F)}\{\mathrm{P}_z(1,t)\} < \infty$. Observe now that for every $n \ge 1$ and every $z \in J(F)$

$$\begin{aligned}\sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-t} &= \sum_{w\in F^{-(n-1)}(z)} \sum_{x\in F^{-1}(w)} |(F^{n-1})'(w)|^{-t}|F'(x)|^{-t} \\ &= \sum_{w\in F^{-(n-1)}(z)} |(F^{n-1})'(w)|^{-t} \sum_{x\in F^{-1}(w)} |F'(x)|^{-t} \\ &\le ||\mathrm{P}(1,t)||_\infty \sum_{w\in F^{-(n-1)}(z)} |(F^{n-1})'(w)|^{-t}.\end{aligned}$$

Therefore, we obtain by induction that $\sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-t} \le ||\mathrm{P}(1,t)||_\infty^n$ and consequently

$$\overline{\mathrm{P}}(t) = \overline{\mathrm{P}}_z(t) \le \log ||\mathrm{P}(1,t)||_\infty \tag{3.2}$$

for all $t > 1$. It follows from Hölder's inequality that both functions $\overline{\mathrm{P}}(t)$ and $\underline{\mathrm{P}}(t)$ are convex. Since in addition (see Proposition 2.1), $F : J(F) \to J(F)$ is expanding, we get the following.

Proposition 3.1. *Both functions* $t \mapsto \overline{\mathrm{P}}(t), \underline{\mathrm{P}}(t)$, $t \in (1,\infty)$, *are continuous, strictly decreasing and* $\lim_{t\to+\infty} \overline{\mathrm{P}}(t) = -\infty$.

In this section we establish the existence of conformal measures for the map $F : J(F) \to J(F)$. The details are presented in the appendix for the sake of completeness and the convenience of the reader. A Borel measure m_t is called (t,α_t)-conformal (with $t > 1$ and $\alpha_t \ge 0$) if for any Borel set $A \subset P$ on which F is injective, we have

$$m_t(F(A)) = \int_A \alpha_t |F'|^t dm_t$$

Since the proof of the existence of conformal measures was technically and conceptually considerably easier to carry out working with a strip S rather then the cylinder P and since the strip $S \subset \mathbb{C}$ was needed exclusively for the proof of the existence of conformal measures, we placed it in the appendix. As an immediate consequence of Theorem 10.3 from this appendix, we get the following.

Theorem 3.2. *There exist* $\alpha_t \ge 0$ *and a* (t,α_t)*-conformal measure* m_t *for the map* $F : J(F) \to J(F)$ *determined by the condition that* $m_t(\pi(A)) = m(A)$ *for every Borel set* A *such that* $\pi|_A$ *is 1-to-1, where* m *is the measure produced in Theorem 10.3. In addition* $m_t(J(F)) = 1$.

Let $C_b = C_b(J(F))$ be the Banach space of all bounded continuous complex-valued functions on $J(F)$. Notice that for each $z \in J(F)$ and each $\tilde{z} \in \pi^{-1}(z)$, the series

$$\sum_{k=-\infty}^{+\infty} |\tilde{z} + 2\pi ik|^{-t}$$

is independent of the choice of $\tilde{z}$ and it converges if and only if $\mathrm{Re}(t) > 1$. This enables us to define for these t's the Perron-Frobenius operator $\mathcal{L} = \mathcal{L}_t : C_b \to C_b$ by the formula

$$\mathcal{L}_t g(z) = \sum_{x \in F^{-1}(z)} |F'(x)|^{-t} g(x) = \sum_{k=-\infty}^{+\infty} |\tilde{z} + 2\pi ik|^{-t} g(z_k), \tag{3.3}$$

where $\tilde{z}$ is an arbitrary point from $\pi^{-1}(z)$ and z_k is the only point of the the singleton $\pi\Big(f^{-1}(\tilde{z} + 2\pi ik)\Big)$. It immediately follows from (3.3) that

$$\mathcal{L}_t g(z) \le ||\mathcal{L}_t 1\!\!1||_\infty ||g||_\infty \quad \text{and} \quad \lim_{\mathrm{Re} z \to +\infty} \mathcal{L}_t 1\!\!1(z) = 0. \tag{3.4}$$

Notice also that $\mathcal{L}_t : C_b \to C_b$ is a bounded operator and its norm is equal to $||\mathrm{P}(1,t)||_\infty$. Assume from now on throughout this section that $t \in (1, \infty)$ and consider the dual operator $\mathcal{L}_t^* : C_b^* \to C_b^*$ given by the formula $\mathcal{L}_t^* \mu(g) = \mu(\mathcal{L}_t g)$. A straighforward calculation (see Proposition 2.2 in [DU1] for example, where the finiteness of the partition can be replaced by its countability) shows the following.

Proposition 3.3. *For every $t > 1$, $\mathcal{L}_t^* m_t = \alpha_t m_t$.*

Let

$$\delta = \frac{1}{2} \min\left\{\frac{1}{2}, \mathrm{dist}\Big(J(F), \pi\Big(\{f^n(0) : n \ge 0\}\Big)\Big)\right\} \tag{3.5}$$

Observe that for every $v \in J(F)$ and every $n \ge 1$ there exists a unique holomorphic inverse branch $F_v^{-n} : B(F^n(v), 2\delta) \to P$ of F^{-n} sending $F^n(v)$ to v. In particular $F_v^{-n}(J(F) \cap B(F^n(v), 2\delta)) \subset J(F)$. Fix now $t > 1$ and define

$$\hat{\mathcal{L}}_t = \alpha_t^{-1} \mathcal{L}.$$

For every $x \in I\!\!R$ let

$$P_x = \{z \in J(F) : \mathrm{Re}(z) \le x\}.$$

Fix any two points w and z in P_x. There then exists the shortest smooth arc $\gamma_{w,z}$ joining w and z in $P \setminus B\Big(\{f^n(0) : n \ge 0\}, 2\delta\Big)$. The supremum of (Euclidean) lengths of arcs $\gamma_{w,z}$ taken over all pairs $w, z \in P_x$ is finite and consequently there exists a number $l_x \ge 1$ such that each such arc $\gamma_{w,z}$ can be covered by a chain of at most balls l_x balls of radius δ centered at points of $\gamma_{w,z}$. We may assume in addition that $U_{w,z}$, the union of these balls is a simply connected set. It then follows from Koebe's distortion theorem that there exists $K_x \ge 1$ such that if $F_*^{-n} : U_{w,z} \to \Gamma$ is a holomorphic branch of F^{-n}, then

$$\frac{|(F_*^{-n})'(w)|}{|(F_*^{-n})'(z)|} \le K_x$$

and consequently

$$K_x^{-t} \le \frac{\hat{\mathcal{L}}_t(1\!\!1)(w)}{\hat{\mathcal{L}}_t(1\!\!1)(z)} \le K_x^t. \tag{3.6}$$

We shall prove the following.

Lemma 3.4. $Q = \sup_n\{||\hat{\mathcal{L}}_t^n(1\!\!1)||_\infty\} < \infty$.

Proof. Fix $x \ge 0$ so large that for every $w \in P_x^c$

$$\alpha_t^{-1} \sum_{k=-\infty}^{+\infty} |\tilde{w} + 2\pi ik|^{-t} \le 1, \tag{3.7}$$

where $\tilde{w}$ is an arbitrary element of $\pi^{-1}(w)$. We shall prove by induction that for every $n \ge 0$,

$$||\hat{\mathcal{L}}_t^n(1\!\!1)||_\infty \le \frac{K_x^t}{m_t(P_x)}.$$

And indeed, for $n = 0$ this estimate is immediate. So, suppose that it holds for some $n \ge 0$ and let $\hat{z}_{n+1} \in P$ be such a point that $\hat{\mathcal{L}}_t^{n+1}(1\!\!1)(\hat{z}_{n+1}) = ||\hat{\mathcal{L}}_t^{n+1}(1\!\!1)||_\infty$ (such a point exists due to (3.4)). If $\hat{z}_{n+1} \in P_x$, then using (3.6) we obtain

$$1 = \int \hat{\mathcal{L}}_t^{n+1}(1\!\!1)dm \ge \int_{P_x} \hat{\mathcal{L}}_t^{n+1}(1\!\!1)dm \ge K_x^{-t}||\hat{\mathcal{L}}_t^{n+1}(1\!\!1)||_\infty m(P_x)$$

and consequently $||\hat{\mathcal{L}}_t^{n+1}(1\!\!1)||_\infty \le K_x^t\Big(m(P_x)\Big)^{-1}$. If $\hat{z}_{n+1} \notin P_x$, then it follows from (3.7) and the inductive assumption that

$$\begin{aligned}
||\hat{\mathcal{L}}_t^{n+1}(1\!\!1)||_\infty &= \hat{\mathcal{L}}_t^{n+1}(1\!\!1)(\hat{z}_{n+1}) = \sum_{k=-\infty}^{+\infty} \hat{\mathcal{L}}_t^n(1\!\!1)\Big((\hat{z}_{n+1})_k\Big)\alpha_t^{-1}|\tilde{\hat{z}}_{n+1} + 2\pi ik|^{-t} \\
&\le \sum_{k=-\infty}^{+\infty} \alpha_t^{-1}||\hat{\mathcal{L}}_t^n(1\!\!1)||_\infty|\tilde{\hat{z}}_{n+1} + 2\pi ik|^{-t} \le K_x^t\Big(m(P_x)\Big)^{-1}\alpha_t^{-1} \sum_{k=-\infty}^{+\infty} |\tilde{\hat{z}}_{n+1} + 2\pi ik|^{-t} \\
&\le K_x^t\Big(m(P_x)\Big)^{-1}.
\end{aligned}$$

We are done. ■

Lemma 3.5. *There exists $x_0 \ge 0$ such that for every $x \ge x_0$*

$$\inf_{n\ge 0} \sup_{z\in P_x}\{\hat{\mathcal{L}}_t^n(1\!\!1)(z)\} \ge \frac{1}{4}.$$

Proof. Let Q come from Lemma 3.4. Let x_0 be so large that $m(P_{x_0}^c) \le 1/(4Q)$. Suppose for the contrary that $\mathcal{L}_0^n(1\!\!1)(z) < 1/4$ for some $n \ge 0$ and all $z \in P_{x_0}$. Then

$$1 = \int \hat{\mathcal{L}}_t^n(1\!\!1)dm = \int_{P_{x_0}} \hat{\mathcal{L}}_t^n(1\!\!1)dm + \int_{P_{x_0}^c} \hat{\mathcal{L}}_t^n(1\!\!1)dm \le \frac{1}{4}m(P_{x_0}) + Qm(P_{x_0}^c) \le \frac{1}{4} + Q\frac{1}{4Q} = \frac{1}{2}.$$

This contradiction finishes the proof. ■

As an immediate consequence of this lemma and (3.6) we get the following.

Lemma 3.6. *For every $x \ge x_0$ we have*

$$\inf_{n\ge 0} \inf_{z\in P_x} \{\hat{\mathcal{L}}_t^n(1\!\!1)(z)\} \ge \frac{1}{4}\Big(\max\{K_x, K_{x_0}\}\Big)^{-t}.$$

We shall prove the following.

Proposition 3.7. *For every $t > 1$ we have $\underline{\mathrm{P}}(t) = \overline{\mathrm{P}}(t) = \log \alpha_t$.*

Proof. It follows from Lemma 3.4 that $\mathcal{L}_t^n(1\!\!1)(z) \le Q\alpha_t^n$ for every $z \in P$. Hence

$$\overline{\mathrm{P}}(t) = \overline{\mathrm{P}}_z(t) = \limsup_{n\to\infty} \frac{1}{n}\mathcal{L}^n(1\!\!1)(z) \le \log \alpha_t.$$

In view of Lemma 3.6, $\mathcal{L}_t^n(1\!\!1)(x_0) \ge \frac{1}{4}K_{x_0}^{-t}\alpha_t^n$ and therefore

$$\underline{\mathrm{P}}(t) = \underline{\mathrm{P}}_z(t) = \liminf_{n\to\infty} \frac{1}{n}\mathcal{L}_t^n(1\!\!1)(z) \ge \log \alpha_t.$$

We are done. ■

Denote the common value of $\underline{\mathrm{P}}(t)$ and $\overline{\mathrm{P}}(t)$ by $\mathrm{P}(t)$.

Let $\{z_n\}_{n=1}^\infty$ be a sequence of points in the cylinder P. We say that $\lim_{n\to\infty} z_n = \infty$ if and only if $\lim_{n\to\infty} \mathrm{Re}(z_n) = +\infty$.

Let

$$I_\infty(F) = \{z \in J(F) : \lim_{n\to\infty} F^n(z) = \infty\},$$

i.e. $I_\infty(F)$ is the set of points escaping to infinity under forward iterates of F. Analogously define

$$I_\infty(f) = \{z \in J(f) : \lim_{n\to\infty} f^n(z) = \infty\}.$$

Denote

$$J_r(F) = J(F) \setminus I_\infty(F) \quad \text{and} \quad J_r(f) = J(f) \setminus I_\infty(f)$$

and notice that

$$I_\infty(f) = \pi^{-1}(I_\infty(F)).$$

Put

$$Y_M = \{z \in P : \mathrm{Re} z > M\}.$$

Let m_t be the $(t, e^{\mathrm{P}(t)})$-conformal measure constructed in Theorem 3.2 (due to Proposition 3.7 $\alpha_t = e^{\mathrm{P}(t)}$). We shall prove the following.

Proposition 3.8. *There exists $M > 0$ such that for m_t-a.e. x*

$$\liminf_{n\to\infty} \mathrm{Re}(F^n(x)) \le M.$$

In particular, $m_t(I_\infty(F)) = 0$ or equivalently $m_t(J_r(F)) = 1$.

Proof. Fix $M > 0$. Let $B \subset Y_M$ be an arbitrary Borel set. We shall estimate from above the measure $m_t(B \cap F^{-1}(B))$. We have

$$m_t(B \cap F^{-1}(B)) \le m_t(F^{-1}(B)) = \sum_{k\in\mathbf{Z}} m_t(\{x : f(x) \in B + 2k\pi i\}).$$

If $f(x) \in B + 2k\pi i$, then

$$|F'(x)| = |f'(x)| = |f(x)| > (M^2 + k^2)^{\frac{1}{2}}.$$

Thus

$$m_t\Big(\{x : F(x) \in B\}\Big) \le 2\sum_{k=0}^{\infty} m_t(B) \cdot \frac{e^{-\mathrm{P}(t)}}{(M^2 + k^2)^{\frac{t}{2}}} \le const\, m_t(B) M^{1-t}.$$

Therefore, in particular, one gets

$$m_t(B \cap F^{-1}(B)) < \frac{C}{M^{t-1}} m_t(B) \tag{3.8}$$

for every Borel set $B \subset Y_M$ and for some constant C independent of M and B. Since $B \cap F^{-1}(B) \subset Y_M$, one can now use the estimate (3.8) to get inductively

$$m_t\Big(B \cap F^{-1}(B) \cap \cdots \cap F^{-n}(B)\Big) \le (CM^{1-t})^n m_t(B)$$

This implies that for all M large enough

$$m_t(\bigcap_{n=0}^{\infty} F^{-n}(Y_M)) = 0$$

and consequently

$$m_t(\bigcup_{k=0}^{\infty} F^{-k}(\bigcap_{n=0}^{\infty} F^{-n}(Y_M))) = 0.$$

The proof is finished. ■

Let us show now that the estimates used in Proposition 3.8 and Proposition 10.2 lead to the following.

Corollary 3.9.

$$m_t(Y_M) \le Ce^{(1-t)M}$$

for some constant C and all $M \ge 0$ large enough.

Proof. It follows from the proof of Proposition 3.8 that

$$m_t\Big(\{x \in Y_M : F(x) \in Y_M\}\Big) < m_t(Y_M)CM^{1-t}$$

and from the proof of Proposition 10.2 (formula (10.5) with m_N replaced by m_t that

$$m_t\Big(\{x \in Y_M : \mathrm{Re}F(x) \le M\}\Big) \le Ce^{(1-t)M}.$$

These two sets cover the whole set Y_M. The first inequality says that (for all M sufficiently large) the first set covers less than, say, one half of the measure of Y_M. Thus,

$$m_t(Y_M) \le 2m\Big(\{x \in Y_M : \mathrm{Re}F(x) \le M\}\Big) \le 2Ce^{(1-t)M}$$

and the proof is complete. ■

Theorem 3.10. *The $(t, e^{\mathrm{P}(t)})$-conformal measure $m = m_t$ is a unique (t, β)-conformal measure for F with $t > 1$. In addition it is ergodic with respect to each iterate of F.*

Proof. Fix $j \ge 1$. Suppose that ν is a (t, β^j)-conformal measure for F^j with some $t > 1$ and $\beta > 0$. The same proof as in the case of the measure m shows that $\nu(I_\infty(F)) = 0$. Let $J_{r,N}(F)$ be the subset of $J_r(F)$ defined as follows: $z \in J_{r,N}(F)$ if the trajectory of z under F^j has an accumulation point in $\{z \in J(F) : \mathrm{Re}z < N\}$. Obviously, $\bigcup_N J_{r,N}(F) = J_r(F)$ and by Proposition 3.8 there exists $M' > 0$ such that $\nu(J_{r,M}(F)) = m(J_{r,M'}(F)) = 1$. Fix $z \in J_{r,N}(F)$ and let recall that $\delta \le \mathrm{dist}\Big(\{J(F), \pi\Big(\{f^n(0)\}_{n\ge 0}\Big)\Big)/2$. Then there exist $y \in J(F)$ such that $\mathrm{Re}(y) < N$ and an increasing sequence $\{n_k\}_{k=1}^\infty$ such that $y = \lim_{k\to\infty} F^{jn_k}(z)$. Considering for k large enough the sets $F_z^{-jn_k}(B(y,\delta))$ and $F_z^{-jn_k}(B(y,\delta/K))$, where $F_z^{-jn_k}$ is the holomorphic inverse branch of F^{n_k} defined on $B(y, 2\delta)$ and sending $F^{jn_k}(z)$ to z, using conformality of measures m and ν along with Koebe's distortion theorem we easily deduce that

$$B_N(\nu)^{-1}\beta^{-jn_k}|(F^{jn_k})'(z)|^{-t} \le \nu\Big(B_N(z, c|(F^{jn_k})'(z)|^{-1})\Big) \le B_N(\nu)\beta^{-jn_k}|(F^{jn_k})'(z)|^{-t} \tag{3.9}$$

for all $k \ge 1$ large enough, where $c \asymp 1$, $K \ge 1$ is the constant appearing in the Koebe's distortion theorem and ascribed to the scale $1/2$, $B_N(\nu)$ is some constant depending on ν and N. Let M be fixed as above. Fix now E, an arbitrary bounded Borel set contained in $J_r(F)$ and let $E' = E \cap J_{r,M'}(F)$. Since m is regular, for every $x \in E'$ there exists a radius $r(x) > 0$ of the form from (3.9) (and the corresponding number $n(x) = n_k(x)$ for an appropriate k) such that

$$m\big(\bigcup_{x\in E'} B(x, r(x)) \setminus E'\big) \le \varepsilon. \tag{3.10}$$

Now by the Besicovič theorem (see [G]) we can choose a countable subcover $\{B(x_i, r(x_i))\}_{i=1}^{\infty}$ with $r(x_i) \le \varepsilon$ and $jn(x_i) \ge \varepsilon^{-1}$, from the cover $\{B(x, r(x))\}_{x\in E'}$ of E', of multiplicity bounded by some constant $C \ge 1$, independent of the cover. Therefore, assuming $e^{\mathrm{P}(t)} < \beta$ and using (3.9) along with (3.10), we obtain

$$
\begin{aligned}
\nu(E) = \nu(E') &\le \sum_{i=1}^{\infty} \nu(B(x_i, r(x_i)))\beta^{-jn(x_i)} \le B_M(\nu) \sum_{i=1}^{\infty} r(x_i)^t \beta^{-jn(x_i)} \\
&\le B_M(\nu)B_M(m) \sum_{i=1}^{\infty} m(B(x_i, r(x_i)))\beta^{-jn(x_i)} e^{\mathrm{P}(t)jn(x_i)} \\
&\le B_M(\nu)B_M(m)Cm(\bigcup_{i=1}^{\infty} B(x_i, r(x_i)))\left(e^{\mathrm{P}(t)}\beta^{-1}\right)^{jn(x_i)} \\
&\le B_M(\nu)B_M(m)Cm(\bigcup_{i=1}^{\infty} B(x_i, r(x_i)))\left(e^{\mathrm{P}(t)}\beta^{-1}\right)^{\varepsilon^{-1}} \\
&\le CB_M(\nu)B_M(m)\left(e^{\mathrm{P}(t)}\beta^{-1}\right)^{\varepsilon^{-1}}(\epsilon + m(E')) \\
&= CB_M(\nu)B_M(m)\left(e^{\mathrm{P}(t)}\beta^{-1}\right)^{\varepsilon^{-1}}(\epsilon + m(E)).
\end{aligned}
\tag{3.11}
$$

Hence letting $\varepsilon \searrow 0$ we obtain $\nu(E) = 0$ and consequently $\nu(J(F)) = 0$ which is a contradiction. We obtain a similar contradiction assuming that $\beta < e^{\mathrm{P}(t)}$ and replacing in (3.11) the roles of m and ν. Thus $\beta = e^{\mathrm{P}(t)}$ and letting $\epsilon \searrow 0$ again, we obtain from (3.11) that $\nu(E) \le CB_M(\nu)B_M(m)m(E)$. Exchanging m and ν, we obtain $m(E) \le CB_M(\nu)B_M(m)\nu(E)$. These two conclusions along with the already mentioned fact that $m(J_r(F)) = \nu(J_r(F)) = 1$, imply that the measures m and ν are equivalent with Radon-Nikodym derivatives bounded away from zero and infinity.

Let us now prove that any $(t, e^{\mathrm{P}(t)})$-conformal measure ν is ergodic with respect to F^j. Indeed, suppose to the contrary that $F^{-j}(G) = G$ for some Borel set $G \subset J(F)$ with $0 < m(G) < 1$. But then the two conditional measures ν_G and $\nu_{J(F)\setminus G}$

$$
\nu_G(B) = \frac{\nu(B \cap G)}{\nu(G)}, \quad \nu_{J(F)\setminus G}(B) = \frac{\nu(B \cap J(F) \setminus G)}{\nu(J(F) \setminus G)}
$$

would be $(t, e^{j\mathrm{P}(t)})$-conformal for F^j and mutually singular. This contradiction finishes the proof. ■

4. Old and New Perron-Frobenius Operators and Their Finer Properties

As in the previous section we fix the map $f = f_\lambda$ and $F = F_\lambda$ and we assume that $f : \mathbb{C} \to \mathbb{C}$ has an attracting periodic orbit, i.e. that $\lambda \in$ Hyp. Recall that $C_b = C_b(J(F))$ is the space of all bounded continuous complex valued functions defined on $J(F)$. Fix $\alpha \in (0, 1]$. Given $g \in C_b$ let

$$
v_\alpha = \inf\{L \ge 0 : |g(y) - g(x)| \le L|y - x|^\alpha \text{ for all } x, y \in J(F) \text{ with } |y - x| \le \delta\},
$$

be the α-variation of the function g, where $\delta > 0$ was defined in formula (3.5) and let

$$||g||_\alpha = v_\alpha(g) + ||g||_\infty.$$

Clearly the space

$$\mathrm{H}_\alpha = \mathrm{H}_\alpha(J(F)) = \{g \in J(F) : ||g||_\alpha < \infty\}$$

endowed with the norm $||\cdot||_\alpha$ is a Banach space densly contained in C_b with respect to the $||\cdot||_\infty$ norm.

Recall that for every $n \geq 1$ and every $v \in J(F)$,

$$F_v^{-n} : B(F^n(v), 2\delta) \to P$$

was defined to be the holomorphic inverse branches of F^n defined on $B(F^n(v), 2\delta)$ and sending $F^n(v)$ to v. It follows from Proposition 2.1 and Koebe's distortion theorem that there exist constants $L > 0$ and $0 < \beta < 1$ such that for every $n \geq 0$, every $v \in J(F)$ and every $z \in B(F^n(v), \delta)$, we have

$$|(F_v^{-n})'(z)| \leq L\beta^n \tag{4.1}$$

We say that a continuous function $\phi : J(F) \to \mathbb{C}$ is dynamically Hölder with an exponent $\alpha > 0$ if there exists $c_\phi > 0$ such that

$$|\phi_n(F_v^{-n}(y)) - \phi_n(F_v^{-n}(x))| \leq c_\phi |\phi_n(F_v^{-n}(x))||y - x|^\alpha \tag{4.2}$$

for all $n \geq 1$, all $x, y \in J(F)$ with $|x - y| \leq \delta$ and all $v \in F^{-n}(x)$, where

$$\phi_n(z) = \phi(z)\phi(F(z))\ldots\phi(F^{n-1}(z)).$$

We say that a continuous function $\phi : J(F) \to \mathbb{C}$ is summable if

$$\sup_{z \in J(F)} \left\{ \sum_{v \in F^{-1}(z)} ||\phi \circ F_v^{-1}||_\infty \right\} < \infty.$$

If the continuous function ϕ is summable then the formula

$$\mathcal{L}_\phi g(z) = \sum_{x \in F^{-1}(z)} \phi(x)g(x)$$

defines a bounded operator $\mathcal{L}_\phi : C_b \to C_b$ called the Perron–Frobenius operator associated with the potential ϕ. We shall prove the following.

Lemma 4.1. *If $\phi : J(F) \to \mathbb{C}$ is a summable dynamically Hölder potential with an exponent $\alpha > 0$ then $\mathcal{L}_\phi(\mathrm{H}_\alpha) \subset \mathrm{H}_\alpha$. If, in addition, $\phi(J(F)) \subset [0, \infty)$ and $\sup_{n \geq 1}\{||\mathcal{L}_\phi^n(1\!\!1)||_\infty\} < \infty$, then there exists a constant $c_1 > 0$ such that*

$$||\mathcal{L}_\phi^n g||_\alpha \leq \frac{1}{2}||g||_\alpha + c_1||g||_\infty$$

for all $n \geq 1$ large enough and every $g \in \mathrm{H}_\alpha$.

Proof. Fix $n \geq 1$, $g \in \mathrm{H}_\alpha$ and $x, y \in J(F)$ with $|y-x| \leq \delta$. Put $V_n = F^{-1}(x)$. Then we have

$$
\begin{aligned}
&|\mathcal{L}_\phi^n g(y) - \mathcal{L}_0^n g(x)| = \\
&= \left|\sum_{v\in V_n} \phi_n(F_v^{-n}(y))g(F_v^{-n}(y)) - \sum_{v\in V_n} \phi_n(F_v^{-n}(x))g(F_v^{-n}(x))\right| \\
&= \Big|\sum_{v\in V_n} \phi_n(F_v^{-n}(x))(g(F_v^{-n}(y)) - g(F_v^{-n}(x))) + \sum_{v\in V_n} g(F_v^{-n}(y))(\phi_n(F_v^{-n}(y)) - \phi_n(F_v^{-n}(x)))\Big| \\
&\leq \sum_{v\in V_n} \left|g(F_v^{-n}(y))\right| \left|\phi_n((F_v^{-n})(y)) - \phi_n((F_v^{-n})(x))\right| + \\
&+ \sum_{v\in V_n} |\phi_n((F_v^{-n})(x))||g(F_v^{-n}(y) - g(F_v^{-n}(x)))| \\
&\leq \sum_{v\in V_n} ||g||_\infty \sum_{v\in V_n} |\phi_n(F_v^{-n}(x))| \cdot |x-y|^\alpha + \sum_{v\in V_n} |\phi_n(F_v^{-n})(x)| v_\alpha(g)|F_v^{-n}(y) - F_v^{-n}(x)|^\alpha \\
&\leq c_\phi ||g||_\infty \mathcal{L}_{|\phi|}^n(1\!\!1)(x)|y-x|^\alpha + v_\alpha(g)(L\beta^n)^\alpha |y-x|^\alpha \sum_{v\in V_n} |\phi_n(F_v^{-n}(x))| \\
&\leq ||\mathcal{L}_{|\phi|}^n(1\!\!1)||(c_\phi ||g||_\infty + L^\alpha \beta^{\alpha n} v_\alpha(g)))|y-x|^\alpha.
\end{aligned}
$$

This shows that

$$
v_\alpha(\mathcal{L}_\phi^n g) \leq \mathcal{L}_{|\phi|}^n(1\!\!1)(c_\phi ||g||_\infty + L^\alpha \beta^{\alpha n}||g||_\alpha) < \infty \tag{4.3}
$$

and, in particular, $\mathcal{L}_\phi^n(g) \in \mathrm{H}_\alpha$. The inclusion $\mathcal{L}_\phi(H_\alpha) \subset H_\alpha$ is proved. Suppose now that $\phi(J(F)) \subset [0,\infty)$ and $Q_\phi = \sup_{n\geq 1}\{||\mathcal{L}_\phi^n(1\!\!1)||_\infty\}$ is finite. It then follows from (4.3) that

$$
||\mathcal{L}_\phi^n g||_\alpha \leq Q_\phi L^\alpha \beta^{\alpha n}||g||_\alpha + c_\phi Q_\phi ||g||_\infty + ||\mathcal{L}_\phi^n g||_\infty \leq Q_\phi L^\alpha \beta^{\alpha n}||g||_\alpha + Q_\phi(c_\phi + 1)||g||_\infty.
$$

The proof is thus finished by taking $n \geq 1$ so large that $Q_\phi L^\alpha \beta^{\alpha n} \leq \frac{1}{2}$. ■

We say that a summable dynamically Hölder potential $\phi : J(F) \to (0, \infty)$ satisfies condition (*) if

$$
Q_\phi = \sup_{n\geq 1}\{||\mathcal{L}_\phi^n(1\!\!1)||_\infty\} < \infty
$$

and we say that ϕ is rapidly decreasing if

$$
\lim_{\mathrm{Re}z\to\infty} \mathcal{L}_\phi(1\!\!1)(z) = 0.
$$

In order to apply the theorem of Ionescu-Tulcea and Marinescu we also need the following.

Lemma 4.2. *Suppose that $\phi : J(F) \to (0,\infty)$ is a rapidly decreasing summable dynamically Hölder potential satisfying condition (*). If B is a bounded subset of H_α (with the $||\cdot||_\alpha$ norm), then $\mathcal{L}_\phi(B)$ is a pre-compact subset of C_b (with the $||\cdot||_\infty$ norm).*

Proof. Fix an arbitrary sequence $\{g_n\}_{n=1}^\infty \subset B$. Since, by Lemma 4.1, the family $\mathcal{L}_\phi(B)$ is equicontinuous and, since the operator $\mathcal{L}_\phi$ is bounded, this family is bounded, it follows from Ascoli's theorem that we can choose from $\{\mathcal{L}_\phi(g_n)\}_{n=1}^\infty$ an infinite subsequence $\{\mathcal{L}_\phi(g_{n_j})\}_{j=1}^\infty$ converging uniformly on compact subsets of $J(F)$ to a function $\psi \in C_b$. Fix now $\varepsilon > 0$. Since B is a bounded subset of C_b, it follows from (3.4) that there exists $T > 0$ such that $|\mathcal{L}_\phi g(z)| \le \varepsilon/2$ for all $g \in B$ and all $z \in J(F)$ with $\mathrm{Re} z \ge T$. Hence

$$|\psi(z)| \le \varepsilon/2 \tag{4.4}$$

for all $z \in J(F)$ with $\mathrm{Re} z \ge T$. Thus $|\mathcal{L}_\phi(g_{n_j})(z) - \psi(z)| \le \varepsilon$ for all $j \ge 1$ and all $z \in J(F)$ with $\mathrm{Re} z \ge T$. In addition, there exists $p \ge 1$ such that $|\mathcal{L}_\phi(g_{n_j})(z) - \psi(z)| \le \varepsilon$ for every $j \ge p$ and every $z \in P_T$. Therefore $|\mathcal{L}_\phi(g_{n_j})(z) - \psi(z)| \le \varepsilon$ for all $j \ge p$ and all $z \in J(F)$. This means that $||\mathcal{L}_\phi(g_{n_j}) - \psi||_\infty \le \varepsilon$ for all $j \ge p$. Letting $\varepsilon \searrow 0$ we conclude from this and from (4.4) that $\mathcal{L}_\phi(g_{n_j})$ converges uniformly on $J(F)$ to $\psi \in C_b$. We are done. ■

Combining now Lemma 4.1 and Lemma 4.2, we see that the assumptions of Theorem 1.5 in [IM] are satisfied with Banach spaces $\mathrm{H}_\alpha \subset C_b$ and the bounded operator $\mathcal{L}_\phi : C_b \to C_b$ preserves H_α. It gives us the following.

Theorem 4.3. *If the assumptions of Lemma 4.2 are satisfied then there exist finite numbers $\gamma_1, \dots, \gamma_p \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$, finitely many bounded finitely dimensional operators $Q_1, \dots, Q_p : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ and an operator $S : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ such that*

$$\mathcal{L}_\phi^n = \sum_{i=1}^p \gamma_i^n Q_i + S^n$$

for all $n \ge 1$,

$$Q_i^2 = Q_i,\ Q_i \circ Q_j = 0, (i \ne j),\ Q_i \circ S = S \circ Q_i = 0$$

and

$$||S^n||_\alpha \le C\xi^n$$

for some constant $C > 0$, some constant $\xi \in (0,1)$ and all $n \ge 1$. In particular all numbers $\gamma_1, \dots, \gamma_p$ are isolated eigenvalues of the operator $\mathcal{L}_\phi : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ and this operator is quasi-compact.

Since for all $t \in \mathbb{C}$ with $\mathrm{Re} t \ge 0$, all $n \ge 1$, all $x, y \in J(F)$ with $|y - x| \le \delta$, all $v \in F^{-1}(x)$ and some constant $M_t > 0$,

$$\Big||(F_v^{-n})'(y)|^t - |(F_v^{-n})'(x)|^t\Big| \le M_t |(F_v^{-n})'(x)|^{\mathrm{Re} t} |y - x|,$$

it follows from (3.1), (3.4) and Lemma 3.4 that if t is real and $\mathrm{Re} t > 1$, then $\phi_t(z) = e^{-\mathrm{P}(t)}|F'(z)|^t$ is a rapidly decreasing summable dynamically Hölder potential satisfying condition (*) which means that all the assumptions of Theorem 4.3 are satisfied. Note that $\mathcal{L}_{\phi_t} = \hat{\mathcal{L}}_t$. Using heavily Theorem 4.3 we shall prove the following

Theorem 4.4. *If $t > 1$ then we have the following.*

(a) *The number 1 is a simple isolated eigenvalue of the operator $\hat{\mathcal{L}}_t : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$.*
(b) *The eigenspace of the eigenvalue 1 is generated by nowhere vanishing function $\psi_t \in \mathrm{H}_\alpha$ such that $\int \psi_t dm_t = 1$ and $\lim_{\mathrm{Re} z \to +\infty} \psi_t(z) = 0$.*
(c) *The number 1 is the only eigenvalue of modulus 1.*
(d) *With $S : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ as in Theorem 4.3, we have*

$$\hat{\mathcal{L}}_t = Q_1 + S,$$

where $Q_1 : \mathrm{H}_\alpha \to \mathbb{C}\psi_t$ is a projector on the eigenspace $\mathbb{C}\psi_t$, $Q_1 \circ S = S \circ Q_1 = 0$ and

$$||S^n||_\alpha \le C\xi^n$$

for some constant $C > 0$, some constant $\xi \in (0, 1)$ and all $n \ge 1$.

Proof. Let us show that 1 is an eigenvalue of $\hat{\mathcal{L}}_t$ and let us identify the eigenfunction claimed in part (b). And indeed, in view of Lemma 4.1, $||\hat{\mathcal{L}}_t^n(1\!\!1)||_\alpha \le C_1$ for some constant $C_1 > 0$ and all $n \ge 0$. Thus,

$$\left\| \frac{1}{n}\sum_{j=1}^{n} \hat{\mathcal{L}}_t^j(1\!\!1) \right\|_\alpha = \left\| \hat{\mathcal{L}}_t \left(\frac{1}{n}\sum_{j=0}^{n-1} \hat{\mathcal{L}}_t^j(1\!\!1) \right) \right\|_\alpha \le C_1$$

for every $n \ge 1$. Therefore, it follows from Lemma 4.2 that there exists a strictly increasing sequence of positive integers $\{n_k\}_{k\ge1}$ such that the sequence $\left\{ \frac{1}{n_k}\sum_{j=1}^{n_k} \hat{\mathcal{L}}_t^j(1\!\!1) \right\}_{k\ge1}$ converges in the Banach space C_b to a function $\psi_t : J(F) \to \mathbb{R}$. Obviously, $||\psi_t||_\alpha \le C_1$ and, in particular $\psi_t \in \mathrm{H}_\alpha$. Since m is a fixed point of the operator conjugate to $\hat{\mathcal{L}}_t$, $\int \hat{\mathcal{L}}_t^j(1\!\!1)d\,m_t = 1$ for every $j \ge 0$. Consequently, $\int \frac{1}{n}\sum_{j=0}^{n-1} \hat{\mathcal{L}}_t^j(1\!\!1)d\,m_t = 1$ for every $n \ge 1$. So, applying Lebesgue's dominated convegence theorem along with Lemma 3.4, we conclude that $\int \psi_t d\,m_t = 1$. It immediately follows from Lemma 3.6 that $\psi_t > 0$ throughout $J(F)$. Since $\psi_t = \hat{\mathcal{L}}_t\psi_t$, it follows from (3.4) that $\lim_{\mathrm{Re} z \to +\infty} \psi_t(z) = 0$. Thus, in order to complete the proof of the items (a), (b), (c) (that 1 is an isolated eigenvalue of $\hat{\mathcal{L}}_t : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ follows from Theorem 4.3) it suffices to show that if $\beta \in S^1$ is an eigenvalue of $\hat{\mathcal{L}}_t : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ and ρ is its eigenfunction, then $\beta = 1$ and $\rho \in \mathbb{C}\psi_t$. But this can be done in exactly the same way as in the proof of Theorem 35(ii) in [DU2]. The item (d) is now an immediate consequence of Theorem 4.3 and items (a), (b) and (c). ■

5. Invariant Measures

The following theorem immediately follows from Theorem 4.4, Proposition 3.8 and Theorem 3.10.

Theorem 5.1. *If $t > 1$, then the measure $\mu = \mu_t = \psi_t m_t$ is F-invariant, ergodic with respect to each iterate of F and equivalent with the measure m_t. In particular $\mu(J_r(F)) = 1$.*

Due to Theorem 4.4 the F-invariant measure μ has much finer stochastic properties than ergodicity of all iterates of F. Here these follow.

Theorem 5.2. *The dynamical system (F, μ_t) is metrically exact i.e., its Rokhlin natural extension is a K-system.*

The proof of this fact is the same as the proof of Corollary 37 in [DU2]. The next two theorems are standard consequences of Theorem 4.4 (see [DPU] and [PU] for example). Let g_1 and g_2 be real square-μ integrable functions on $J_r(F)$. For every positive integer n the n-th correlation of the pair g_1, g_2, is the number

$$C_n(g_1, g_2) := \int g_1 \cdot (g_2 \circ F^n)\, d\mu - \int g_1\, d\mu \int g_2\, d\mu.$$

provided the above integrals exist. Notice that due to the F-invariance of μ we can also write

$$C_n(g_1, g_2) = \int (g_1 - Eg_1)\Big((g_2 - Eg_2) \circ F^n\Big)\, d\mu,$$

where we write $Eg = \int g\, d\mu$. We have the following.

Theorem 5.3. *There exists $C \geq 1$ and $\rho < 1$ such that for all $g_1 \in \mathrm{H}_\alpha(P), g_2 \in L^1(\mu_t)$*

$$C_n(g_1, g_2) \leq C\rho^n \|g_1 - Eg_1\|_\alpha \|g_2 - Eg_2\|_{L^1}.$$

Let $g : J_r(F) \to R$ be a square-integrable function. The limit

$$\sigma^2(g) = \lim_{n\to\infty} \frac{1}{n} \int \Big(\sum_{j=0}^{n-1} g \circ F^j - nEg\Big)^2 d\mu_t$$

is called asymptotic variance or dispersion, provided it exists.

Theorem 5.4. *If $g \in \mathrm{H}_\alpha(\mathrm{P})$, $\alpha \in (0, 1)$, then $\sigma^2(g)$ exists and, if $\sigma^2(g) > 0$, then the sequence of random variables $\{g \circ F^n\}_{n=0}^\infty$ with respect to the probability measure μ_t satisfies the Central Limit Theorem, i.e.*

$$\mu\left(\left\{x \in J_r(F) : \frac{\sum_{j=0}^{n-1} g \circ F^j - nEg}{\sqrt{n}} < r\right\}\right) \to \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{r} e^{-t^2/2\sigma^2}\, dt.$$

6. Bowen's Formula

For every $n \geq 1$ let

$$A_n = \{z \in J(F) : n-1 \leq \text{Re}z \leq n\}$$

We shall prove the following

Lemma 6.1. *If $t > 1$ then, $\int \log |F'| d\mu_t < +\infty$.*

Proof. Since $\psi_t : J(F) \to (0, +\infty)$ is bounded (even more $\lim_{\text{Re}z \to +\infty} \psi_t(z) = 0$), applying Corollary 3.9, we obtain

$$\begin{aligned}\int \log |F'| d\mu_t \preceq \int \log |F'| dm_t &= \sum_{n=1}^{\infty} \int_{A_n} \log |F'| dm_t \\ &\leq \sum_{n=1}^{\infty} m_t(\Big(Y_{n-1}\Big) \log\Big(|\lambda| \exp(n)\Big) \\ &\leq \sum_{n=1}^{\infty} C \exp\Big((1-t)(n-1)\Big)\Big(\log|\lambda| + n\Big) < +\infty.\end{aligned}$$

It is well-known (see [PU] for example) that $h_M = \text{HD}(J_M)$ (see the appendix or [UZd] for the definition Of J_M)) is the unique zero of the pressure function $t \mapsto \text{P}_M(t)$. Since $\text{HD}(J_r(F)) = \lim_{n\to\infty} \text{HD}(J_M)$ (see [UZd]) and since $\text{HD}(J_r(F)) > 1$ (also see [UZd]), there exists M so large that $h_M = \text{HD}(J_M) > 1$. Then $\text{P}(h_M) \geq \text{P}_M(h_M) > 0$. Since, in addition $\text{P}(h_M) < \infty$, we conclude from Proposition 3.1 and Proposition 3.7 that there exists a unique $h > 1$ such that $\text{P}(h) = 0$. So, the measure m_h is actually h-conformal in a sense of [UZd]. It therefore follows from Theorem 4.5 of this paper that $h = \text{HD}(J_r(F))$. Below we provide a direct independent proof of this result.

Theorem 6.2. $\text{HD}(J_r(F)) = h$*, the unique zero of the pressure function $t \mapsto \text{P}(t)$.*

Proof. Given $k \geq 1$ let

$$X_k = \{z \in J_r(F) : \liminf_{n\to\infty} \text{Re}(F^n(z)) < k\}.$$

Choose an arbitrary point $z \in J(F)$. Fix $t > h$. Take $n \geq 1$ so large that

$$\frac{1}{j} \log \sum_{x \in F^{-j}(z)} |(F^j)'(x)|^{-t} \leq \frac{1}{2}\text{P}(t) \quad \text{(notice that P}(t) < 0).$$

for all $j \geq n$. Cover P_k by finitely many open balls $B(z_1, \delta), B(z_2, \delta), \dots, B(z_l, \delta)$. Since

$$X_k \subset \bigcup_{j=n}^{\infty} \bigcup_{i=1}^{l} \bigcup_{x \in F^{-j}(z_i)} F_x^{-j}(P_k),$$

we conclude that

$$\mathrm{H}^t(X_k) \leq \lim_{n\to\infty} \sum_{j=n}^{\infty}\sum_{i=1}^{l} \sum_{x\in F^{-j}(z_i)} K^t|(F^j)'(x)|^{-t}(2\delta)^t \leq l(2\delta K)^t \lim_{n\to\infty}\sum_{j=n}^{\infty} \exp(\frac{1}{2}\mathrm{P}(t)j) = 0$$

since $\mathrm{P}(t)/2 < 0$. Hence $\mathrm{HD}(X_k) \leq t$. Since $J_r(F) = \bigcup_{k=0}^{\infty} X_k$, this implies that $\mathrm{HD}(J_r(F)) \leq t$. Letting now $t \searrow h$, we conclude that $\mathrm{HD}(J_r(F)) \leq h$.

In order to prove the opposite inequality fix $\varepsilon > 0$. Since $\mu_h(J_r(F)) = 1$ and since μ_h is ergodic F-invariant, it follows from Birkhoff's ergodic theorem and Jegorov's theorem that there exist a Borel set $Y \subset J_r(F)$ and the integer $k \geq 1$ such that $\mu(Y) \geq \frac{1}{2}$ and for every $x \in Y$ and every $n \geq k$

$$\left|\frac{1}{n}\log|(F^n)'(x)| - \chi\right| < \epsilon, \tag{6.1}$$

where $\chi = \int \log|F'|d\mu_h$ is finite due to Lemma 6.1. Put $\nu = m_{h|Y}$. Given $x \in Y$ and $0 < r < \delta$ let $n \geq 0$ be the largest integer such that

$$B(x,r) \subset F_x^{-n}(B(F^n(x),\delta)). \tag{6.2}$$

Then $B(x,r)$ is not contained in $F_x^{-(n+1)}(B(F^{n+1}(x),\delta))$ and applying Koebe's distortion theorem we get

$$r \geq K^{-1}\delta|(F^{n+1})'(x)|^{-1}. \tag{6.3}$$

Taking $r > 0$ sufficiently small, we may assume that $n \geq k$. Combining now (6.2) along with $\mathrm{P}(h) = 0$, Koebe's Distortion Theorem and (6.3), we obtain

$$\begin{aligned} m_h(B(x,r)) &\leq m_h(F_x^{-n}(B(F^n(x),\delta))) \asymp |(F_x^{-n})'(F^n(x))|^h m_h(B(F^n(x),\delta)) \\ &\preceq |(F_x^{-n})'(F^n(x))|^h \preceq r^h \frac{|(F^{n+1})'(x)|^h}{|(F^n)'(x)|^h} \end{aligned}$$

Employing now (6.1), we thus get

$$m_h(B(x,r)) \leq r^h e^{(\chi+\epsilon)(n+1)} e^{-(\chi-\varepsilon)n} \asymp r^h e^{2\varepsilon n} \tag{6.4}$$

Now, it follows from (6.2), Koebe's distortion theorem and (6.1) that $r \leq K\delta|(F^n)'(x)|^{-1} \leq K\delta e^{-(\chi-\varepsilon)n}$. Thus $e^{(\chi-\varepsilon)n} \preceq r^{-1}$ and consequently $e^{2\varepsilon n} \preceq r^{-\frac{2\varepsilon}{\chi-\varepsilon}}$. This and (6.4) imply that $\nu(B(x,r)) \leq m_h(x,r)) \preceq r^{h-\frac{2\epsilon}{\chi-\epsilon}}$. Consequently $\mathrm{HD}(J_r(F)) \geq \mathrm{HD}(\nu) \geq h - \frac{2\epsilon}{\chi-\epsilon}$ and letting $\epsilon \to 0$ we finally obtain $\mathrm{HD}(J_r(F)) \geq h$. We are done. ■

7. Analyticity of Perron-Frobenius operators

We start with the following.

Lemma 7.1. *Supppose that $\{\phi_\sigma : J(F) \to \mathbb{C}\}_{\sigma\in G}$ is a family of continuous summable potentials , where G is an open connected subset of $\mathbb{C}$. If for every $z \in J(F)$ the function $\sigma \mapsto \phi_\sigma(z), \sigma \in G$, is holomorphic and the map $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(\mathrm{H}_\alpha)$ is continuous on G, then the map $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(\mathrm{H}_\alpha)$ is holomorphic on G.*

Proof. Let $\gamma \subset G$ be a simple closed curve. Fix $g \in H_\alpha$ and $z \in J(F)$. Let $W \subset G$ be a bounded open set such that $\gamma \subset W \subset \bar{W} \subset G$. Since for each $x \in F^{-1}(z)$ the function $\sigma \mapsto g(x)\phi_\sigma(x)$ is holomorphic on G and since for each $\sigma \in W$

$$\left|\sum_{x\in F^{-1}(z)} g(x)\phi_\sigma(x)\right| \le ||\mathcal{L}_{\phi_\sigma} g||_\infty \le ||\mathcal{L}_{\phi_\sigma} g||_\alpha \le ||g||_\alpha \sup\{||\mathcal{L}_{\phi_\theta}||_\alpha : \theta \in \bar{W}\} < \infty$$

by compactness of $\bar{W}$ and continuity of the mapping $\sigma \mapsto \mathcal{L}_{\phi_\sigma}$, we conclude that the function

$$\sigma \mapsto \mathcal{L}_{\phi_\sigma} g(z) = \sum_{x\in F^{-1}(z)} \phi(x)g(x) \in \mathbb{C},\ \sigma \in W,$$

is holomorphic. Hence, by Cauchy's theorem $\int_\gamma \mathcal{L}_{\phi_\sigma} g(z)d\sigma = 0$. Since the function $\sigma \mapsto \mathcal{L}_{\phi_\sigma} g \in H_\alpha$ is continuous, the integral $\int_\gamma \mathcal{L}_{\phi_\sigma} g d\sigma$ exists and for every $z \in J(F)$, we have $\int_\gamma \mathcal{L}_{\phi_\sigma} g d\sigma(z) = \int_\gamma \mathcal{L}_{\phi_\sigma} g(z)d\sigma = 0$. Hence, $\int_\gamma \mathcal{L}_{\phi_\sigma} g d\sigma = 0$. Now, since the function $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(H_\alpha)$ is continuous, the integral $\int_\gamma \mathcal{L}_{\phi_\sigma} d\sigma$ exists and for every $g \in H_\alpha$, $\int_\gamma \mathcal{L}_{\phi_\sigma} d\sigma(g) = \int_\gamma \mathcal{L}_{\phi_\sigma} g d\sigma = 0$. Thus, $\int_\gamma \mathcal{L}_{\phi_\sigma} d\sigma = 0$ and in view of Morera's theorem, the function $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(H_\alpha)$ is holomorphic in G. The proof is complete. ■

In order to prove the main result of this section we need the following auxiliary definitions and few elementary lemmas. Given $w \in J(F)$ we define $\mathrm{H}_{\alpha,w}$ to be the set of all bounded functions $g : B(w,\delta) \to \mathbb{C}$ such that there exists a constant $C \ge 0$ such that if $x, y \in B(w,\delta)$ and $|y - x| \le \delta$, then $|g(y) - g(x)| \le C|y - x|^\alpha$. The α-variation $v_\alpha(g)$ is defined to be the least C with this property. $\mathrm{H}_{\alpha,w}$ endowed with the norm $||g||_\alpha = v_\alpha(g) + ||g||_\infty$ is a Banach space.

Lemma 7.2. *If $v \in J(F)$ and $\phi \in \mathrm{H}_a$ then the operator $A_{v,\phi} : \mathrm{H}_\alpha \to \mathrm{H}_{\alpha,F(v)}$ given by the formula*

$$A_{v,\phi} g(z) = \phi(F_v^{-1}(z))g(F_v^{-1}(z)),\ z \in B(F(v),\delta)$$

is continuous, and

$$||A_{v,\phi}||_\alpha \le (2 + (L\beta)^\alpha)||\phi \circ F_v^{-1}||_\alpha$$

.

Proof. For every $g \in \mathrm{H}_\alpha$ and $z \in B(F(v),\delta)$ we have

$$|A_{v,\phi} g(z)| = |\phi(F_v^{-1}(z))| \cdot |g(F_v^{-1}(z))| \le ||\phi \circ F_v^{-1}||_\alpha \cdot ||g||_\alpha \tag{7.1}$$

If, in addition, $w \in B(F(v),\delta)$ and $|w-z| \le \delta$, then similarly as in the proof of Lemma 4.1, we get

$$\begin{aligned}|A_{v,\phi}g(w) - A_{v,\phi}g(z)| \le \\ &|g(F_v^{-1}(w))||\phi\circ F_v^{-1}(w) - \phi\circ F_v^{-n}(z)| + |\phi\circ F_v^{-n}(z)||g(F_v^{-1}(w)) - g(F_v^{-1}(z))| \\ &\le ||g||_\infty ||\phi\circ F_v^{-1}||_\alpha |w-z|^\alpha + ||\phi\circ F_v^{-1}||_\infty L^\alpha\beta^\alpha|w-z|^\alpha \\ &\le ||g||_\alpha(1+(L\beta)^\alpha)||\phi\circ F_v^{-1}||_\alpha|w-z|^\alpha\end{aligned}$$

Hence, $v_\alpha(A_{v,\phi}g) \le (1+(L\beta)^\alpha)||\phi\circ F_v^{-1}||_\alpha||g||_\alpha$ and combining this with (7.1), we obtain $||A_{v,\phi}g||_\alpha \le (2+(L\beta)^\alpha)||\phi\circ F_v^{-1}||_\alpha||g||_\alpha$. Consequently, $A_{v,\phi}(\mathrm{H}_\alpha) \subset \mathrm{H}_{\alpha,F(v)}$, the operator $A_{v,\phi} : \mathrm{H}_\alpha \to \mathrm{H}_{\alpha,F(v)}$ is continuous, and $||A_{v,\phi}||_\alpha \le (2+(L\beta)^\alpha)||\phi\circ F_v^{-1}||_\alpha$. The proof is complete. ■

Lemma 7.3. *If $\phi : J(F) \to \mathbb{C}$ is dynamically Hölder then for every $v \in J(F)$,*

$$||\phi\circ F_v^{-1}||_\alpha \le (c_\phi+1)||\phi\circ F_v^{-1}||_\infty.$$

Proof. It follows from (4.2) that for all $x, y \in B(F(v),\delta)$ with $|x-y| \le \delta$ we have

$$|\phi\circ F_v^{-1}(y) - \phi\circ F_v^{-1}(x)| \le c_\phi|\phi(F_v^{-1}(x))|\cdot|y-x|^\alpha \le c_\phi||\phi\circ F_v^{-1}||_\infty|y-x|^\alpha$$

and, therefore, $v_\alpha(\phi\circ F_v^{-1}) \le c_\phi||\phi\circ F_v^{-1}||_\infty$. Thus, $||\phi\circ F_v^{-1}||_\alpha \le (c_\phi+1)\cdot||\phi\circ F_v^{-1}||_\infty$. We are done. ■

A straightforward calculation proves the following.

Lemma 7.4. *If $\phi \in \mathrm{H}_\alpha$, then for every $n \ge 1$ and every $v \in J(F)$*

$$||\phi\circ F_v^{-n}||_\alpha \le (1+L^\alpha\beta^{\alpha n})||\phi||_\alpha,$$

where $g \mapsto g\circ F_v^{-n} : B(F^n(v),\delta) \to \mathbb{C}$ is an operator from H_α to $\mathrm{H}_{\alpha,F^n(v)}$.

Lemma 7.5. *If $\rho : X \to \mathrm{H}_\alpha$ is a continuous mapping defined on a metric space X , then for every $v \in J(F)$ the function $x \mapsto A_{v,\rho(x)} \in L(\mathrm{H}_\alpha, \mathrm{H}_{\alpha,F^n(v)})$, $x \in X$, is continuous.*

Proof. Fix $x_0 \in X$, $\varepsilon > 0$ and take $\theta > 0$ so small that for every $x \in B(x_0,\theta)$ and every $v \in J(F)$, $||\rho(x)-\rho(x_0)||_\alpha \le (2+(L\beta)^\alpha)^{-2}\varepsilon$. Then applying Lemma 4.1 and Lemma 7.4 we see that for every $x \in B(x_0,\theta)$ and every $v \in V_1$, we have

$$\begin{aligned}||A_{v,\rho(x)} - A_{v,\rho(x_0)}||_\alpha &= ||A_{v,\rho(x)-\rho(x_0)}||_\alpha \le (2+(L\beta)^\alpha)||\cdot||(\rho(x)-\rho(x_0))\circ F_v^{-1}||_\alpha \\ &\le (2+(L\beta)^\alpha)(1+(L\beta)^\alpha)||\rho(x)-\rho(x_0)||_\alpha \le \varepsilon\end{aligned}$$

The proof is complete. ■

Denote the class of Hölder continuous summable functions on $J(F)$ by H_α^s. We are now in a position to prove the main result of this section.

Theorem 7.6. *Suppose that G is an open connected subset of the complex plane $\mathbb{C}$ and that $\phi_\sigma : J(F) \to \mathbb{C}$, $\sigma \in G$, is a family of mappings such that the following assumptions are satisfied.*

(a) *For every $\sigma \in G$, ϕ_σ is in H_α^s.*
(b) *For every $\sigma \in G$ the function ϕ_σ is dynamically Hölder.*
(c) *The function $\sigma \mapsto \phi_\sigma \in \mathrm{H}_\alpha$ ($\sigma \in G$) is continuous.*
(d) *The family $\{c_{\phi_\sigma}\}_{\sigma \in G}$ is bounded.*
(e) *The function $\sigma \mapsto \phi_\sigma(z) \in \mathbb{C}$, $\sigma \in G$, is holomorphic for every $z \in J(F)$.*
(f) $\forall(\sigma_2 \in G)\, \exists(r > 0)\, \exists(\sigma_1 \in G)$

$$\sup\left\{\left|\frac{\phi_\sigma}{\phi_{\sigma_1}}\right| : \sigma \in \overline{B(\sigma_2, r)}\right\} < \infty.$$

Then the function $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(\mathrm{H}_\alpha)$, $\sigma \in G$, is holomorphic.

Proof. In view of Lemma 7.1 it suffices to demonstrate that the function $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(\mathrm{H}_\alpha)$, $\sigma \in G$, is continuous. First notice that in view of Lemma 7.2, Lema 7.3 and the assumption (d), we have for every $v \in J(F)$ and every $\sigma \in \overline{B(\sigma_2, r)}$ that

$$||A_{v,\phi_\sigma}||_\alpha \le (2 + (L\beta)^\alpha)||\phi_\sigma \circ F_v^{-1}||_\alpha \le M||\phi_\sigma \circ F_v^{-1}||_\infty,$$

where $M = (2+(L\beta)^\alpha)\sup\{c_{\phi_\sigma}, \sigma \in G\} < \infty$. We can continue the above estimate as follows.

$$\begin{aligned}||A_{v,\phi_\sigma}||_\alpha \le M||\phi_\sigma \circ F_v^{-1}||_\infty &= M\left|\left|\phi_{\sigma_1} \circ F_v^{-1} \cdot \frac{\phi_\sigma \circ F_v^{-1}}{\phi_{\sigma_1} \circ F_v^{-1}}\right|\right|_\infty \\ &\le M||\phi_{\sigma_1} \circ F_v^{-1}||_\infty \left|\left|\frac{\phi_\sigma \circ F_v^{-1}}{\phi_{\sigma_1} \circ F_v^{-1}}\right|\right|_\infty \\ &\le M\left|\left|\frac{\phi_\sigma}{\phi_{\sigma_1}}\right|\right|_\infty ||\phi_{\sigma_1} \circ F_v^{-1}||_\infty \le MT||\phi_{\sigma_1} \circ F_v^{-1}||_\infty,\end{aligned} \tag{7.2}$$

where T is the supremum taken from the assumption (f). For every $z \in J(F)$ define the operator $\mathcal{L}_{\phi_\sigma,z} : \mathrm{H}_\alpha \mapsto \mathrm{H}_{\alpha,z}$ by the formula

$$\mathcal{L}_{\phi_\sigma,z} = \sum_{v \in F^{-1}(z)} \phi \circ F - v^{-1} \cdot g \circ F_v^{-1} = \sum_{v \in F^{-1}(z)} A_{v,\phi_\sigma}. \tag{7.3}$$

Notice that

$$\mathcal{L}_{\phi_\sigma,z}(g) = \mathcal{L}_{\phi_s g}(g)|_{B(z,\delta)} \tag{7.4}$$

for every $g \in \mathrm{H}_\alpha$. Fix now $\varepsilon > 0$ and two elements $\sigma, \tau \in B(\sigma_2, r)$. Then there exist $g_\varepsilon \in B_{\mathrm{H}_\alpha}(0,1)$ and two points $x, y \in J(F)$ such that

$$\begin{aligned}
||\mathcal{L}_{\phi_\sigma} - \mathcal{L}_{\phi_\tau}||_\alpha &= \sup\{||\mathcal{L}_{\phi_\sigma}(g) - \mathcal{L}_{\phi_\tau}(g)||_\alpha : g \in B_{\mathrm{H}_\alpha}(0,1)\} \\
&\le ||\mathcal{L}_{\phi_\sigma}(g_\varepsilon) - \mathcal{L}_{\phi_\tau}(g_\varepsilon)||_\alpha + \frac{\varepsilon}{5} \\
&= v_\alpha\Big(\mathcal{L}_{\phi_\sigma}(g_\varepsilon) - \mathcal{L}_{\phi_\tau}(g_\varepsilon)\Big) + \frac{\varepsilon}{5}||\mathcal{L}_{\phi_\sigma}(g_\varepsilon) - \mathcal{L}_{\phi_\tau}(g_\varepsilon)||_\infty \\
&\le v_\alpha\Big(\mathcal{L}_{\phi_\sigma,x}(g_\varepsilon) - \mathcal{L}_{\phi_\tau,x}(g_\varepsilon)\Big) + \frac{\varepsilon}{5} + ||\mathcal{L}_{\phi_\sigma,y}(g_\varepsilon) - \mathcal{L}_{\phi_\tau,y}(g_\varepsilon)||_\infty + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} \\
&\le ||\mathcal{L}_{\phi_\sigma,x}(g_\varepsilon) - \mathcal{L}_{\phi_\tau,x}(g_\varepsilon)||_\alpha + ||\mathcal{L}_{\phi_\sigma,y}(g_\varepsilon) - \mathcal{L}_{\phi_\tau,y}(g_\varepsilon)||_\alpha + \frac{3\varepsilon}{5} \qquad (7.5) \\
&\le 2||\mathcal{L}_{\phi_\sigma,w}(g_\varepsilon) - \mathcal{L}_{\phi_\tau,w}(g_\varepsilon)||_\alpha + \frac{3\varepsilon}{5} \\
&\le 2||\mathcal{L}_{\phi_\sigma,w} - \mathcal{L}_{\phi_\tau,w}||_\alpha + \frac{3\varepsilon}{5},
\end{aligned}$$

where w is either x or y depending upon which number $||\mathcal{L}_{\phi_\sigma,x}(g_\varepsilon) - \mathcal{L}_{\phi_\tau,x}(g_\varepsilon)||_\alpha$ or $||\mathcal{L}_{\phi_\sigma,y}(g_\varepsilon) - \mathcal{L}_{\phi_\tau,y}(g_\varepsilon)||_\alpha$ is larger. Since ϕ_{σ_1} is a summable function (see (a)), there exists a finite set $V \subset F^{-1}(w)$ such that

$$\sum_{v \in F^{-1}(w)\setminus V} ||\phi_{\sigma_1} \circ F_v^{-1}||_\infty \le \frac{\varepsilon}{10MT}. \qquad (7.6)$$

Since, by (c), the function $\xi \mapsto \phi_\xi \in \mathrm{H}_\alpha$ is continuous, it follows from Lemma 7.5 that the function $\xi \mapsto A_{v,\phi_\xi} \in L(\mathrm{H}_\alpha, \mathrm{H}_{\alpha,F(v)})$ is continuous. Consequently there exists $\eta \in (0,r)$ such that

$$||A_{v,\phi_\sigma} - A_{v,\phi_\tau}||_\alpha \le \frac{\varepsilon}{10\#V} \qquad (7.7)$$

for all $\sigma, \tau \in B(\sigma_2, \eta)$ and all $v \in V$. Combining now (7.5), (7.4), (7.3), (7.7) and (7.6), we get

$$||\mathcal{L}_{\phi_\sigma} - \mathcal{L}_{\phi_\tau}||_\alpha \le \frac{3\varepsilon}{5} + 2\sum_{v\in V} ||A_{v,\phi_\sigma} - A_{v,\phi_\tau}||_\alpha + 2\sum_{v \in F^{-1}(w)\setminus V}\Big(||A_{v,\phi_\sigma}||_\alpha + ||A_{v,\phi_\tau}||_\alpha\Big) \le \frac{3\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon.$$

We are done. ■

Due to Hartogs' theorem, as an immediate consequence of Theorem 7.6 we obtain the following.

Corollary 7.7. *Suppose that G is an open connected subset of $\mathbb{C}^n$, $n \ge 1$, and that $\phi_\sigma : J(F) \to \mathbb{C}$, $\sigma \in G$, is a family of mappings such that the following assumptions are satisfied.*

(a) *For every $\sigma \in G$, ϕ_σ is in H_α^s.*
(b) *For every $\sigma \in G$ the function ϕ_σ is dynamically Hölder.*
(c) *The function $\sigma \mapsto \phi_\sigma \in \mathrm{H}_\alpha$ ($\sigma \in G$) is continuous.*

(d) *The family* $\{c_{\phi_\sigma}\}_{\sigma\in G}$ *is bounded.*
(e) *The function* $\sigma \mapsto \phi_\sigma(z) \in \mathbb{C}$, $\sigma \in G$, *is holomorphic for every* $z \in J(F)$.
(f) $\forall(\sigma_2 \in G)\,\exists(r > 0)\,\exists(\sigma_1 \in G)$

$$\sup\left\{\left|\frac{\phi_\sigma}{\phi_{\sigma_1}}\right| : \sigma \in \overline{B(\sigma_2, r)}\right\} < \infty.$$

Then the function $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(\mathrm{H}_\alpha)$, $\sigma \in G$, *is holomorphic.*

8. Quasiconformal conjugacies in the family $\{\lambda e^z\}$

Fix $\lambda_0 \in \mathrm{Hyp}$. It is known that $f_{\lambda_0} : \mathbb{C} \to \mathbb{C}$ is structurally stable, i.e. if λ is sufficiently close to λ_0, then there exists a conjugating homeomorphism $h_\lambda : \mathbb{C} \to \mathbb{C}$, $h_\lambda \circ f_{\lambda_0} = f_\lambda \circ h_\lambda$. Moreover, h_λ can be chosen to be quasiconformal, for every $x \in \mathbb{C}$, the function $\lambda \mapsto h_\lambda(x)$ is holomorphic and the quasiconformal constant converges to 1 when λ approaches λ_0. In the following proposition we use the construction of h_λ given in [EL] (which, in turn, follows [MSS]).

Proposition 8.1. *If* $\lambda_0 \in \mathrm{Hyp}$, *then* h_λ *can be chosen so that* $\sup_{z\in\mathbb{C}}\left\{\left|\frac{dh_\lambda(z)}{d\lambda}\right|\right\}$ *is bounded in some neighbourhood of* λ_0.

Proof. We follow now the way in which h_λ is constructed. Let $\alpha_0, \dots, \alpha_{p-1}$ be the attracting periodic orbit of period p for f_{λ_0}. There exist an open set Ω_λ containing the attracting periodic orbit $\alpha_0, \dots, \alpha_{p-1}$ and the sets Ω_λ^k, $k = 0, 1, \dots, p-1$, such that

$$\Omega_\lambda = \bigcup_{0\le k\le p-1} \Omega_\lambda^k,$$

$$f_\lambda(\Omega_\lambda^k) = \Omega_\lambda^{k+1}, k = 0, \dots, p-2,$$

and

$$f_\lambda(\Omega_f^{p-1}) \subset \Omega_\lambda^0.$$

For every λ sufficiently close to λ_0 there also exists an analytic map (linearization) $z \mapsto H_\lambda(z)$ mapping the set Ω_λ onto the unit disk $\mathbb{D}$ such that H_λ maps each set Ω_λ^k conformally onto $\mathbb{D}$, $H_\lambda \circ f(z) = H_\lambda(z)$ for all $z \in \Omega_\lambda^k$ and all $k = 0, 1, \dots, p-2$, and $H_\lambda \circ f_\lambda^p = \Lambda(\lambda)H_\lambda$, where $\Lambda(\lambda) = (f_{\lambda_0}^p)'(\alpha_0)$ is the multiplier of the periodic orbit $\alpha_0, \dots, \alpha_{p-1}$. Since there is only one singular value 0, this last (conjugacy) equation can be extended to a domain containing 0, i.e. we may assume that $0 \in \Omega_\lambda$. Moreover, H_λ can be chosen so that $H_\lambda(0) = \mathrm{const.}$ not intersect Next, one constructs a conjugacy $h_\lambda^{loc} : \Omega_{\lambda_0} \to \Omega_\lambda$ such that h_λ^{loc} is quasiconformal for every λ sufficiently close to λ_0, $h_\lambda^{loc} \circ f_{\lambda_0} = f_\lambda \circ h_\lambda^{loc}$, $h_{\lambda_0} = id$, and $h_\lambda^{loc}(0) = 0$ Notice, that in the construction of h_λ we have to modify $H_\lambda^{-1} \circ H_{\lambda_0}$, on the annulus $\Omega_{\lambda_0} \setminus f^p(\Omega_{\lambda_0}) \subset \Omega_{\lambda_0}$ which forms a fundamental domain for the map $f_{\lambda_0}^p|_{\Omega_{\lambda_0}}$. In addition, it can be arranged so that $h_\lambda^{loc} = H_\lambda^{-1} \circ H_{\lambda_0}$ in some sufficiently small neighbourhood of 0 and, in particular, it is holomorphic with respect to z in this neighbourhood of 0. Now, the global quasiconformal conjugacy h_λ is given by pulling h_λ^{loc} back using the dynamics of both f_λ and f_{λ_0}. More precisely, for every ξ in the Fatou set which is the basin of attraction to the periodic orbit $\alpha_0, \dots, \alpha_{p-1}$, the value $h_\lambda(\xi)$ is determined by the equation $f_\lambda^k \circ h_\lambda(\xi) = h_\lambda \circ f_{\lambda_0}^k(\xi)$, where $k \geq 0$ is the minimal integer n such that $f_{\lambda_0}^k(\xi) \in \Omega_\lambda$. So, if $z = f_{\lambda_0}^k(\xi)$ is in the fundamental domain $\Omega_{\lambda_0} \setminus f^p(\Omega_{\lambda_0})$, then $w = h_\lambda(\xi)$ satisfies the equation

$$f_\lambda^k(w) = h_\lambda^{loc}(z).$$

Using the Implicit Function Theorem we get

$$\frac{dw}{d\lambda} = \frac{\frac{\partial}{\partial\lambda} h_\lambda^{loc}(z) - \frac{\partial f_\lambda^k(w)}{\partial\lambda}}{\frac{\partial f_\lambda^k(w)}{\partial w}}.$$

We will need formulas for both partial derivatives $\frac{\partial f_\lambda^k(w)}{\partial\lambda}$ and $\frac{\partial f_\lambda^k(w)}{\partial w}$. The second one is simply the derivative of the k-th iterate of f_λ, i.e. it is equal to $w_1 w_1 \dots w_k$ where $w_i = f_\lambda^i(w)$. For the first derivative we have the formula

$$\lambda \frac{\partial}{\partial\lambda} f_\lambda^k(w) = \sum_{i=1}^k f_\lambda^i(w)(f_\lambda^{k-i})'(f_\lambda^i(w)) = \sum_{i=1}^k w_i w_{i+1} \dots w_k. \tag{8.1}$$

So, we have to estimate

$$\frac{\frac{\partial}{\partial\lambda} h_\lambda^{loc}(z) - \frac{1}{\lambda}\sum_{i=1}^k w_i w_{i+1} \dots w_k}{w_1 \dots w_k}, \tag{8.2}$$

where z is in the fundamental domain F_{λ_0} and w satisfies $f_\lambda^k(w) = h_\lambda^{loc}(z)$. Notice that (8.2) takes on the form

$$\frac{\frac{\partial}{\partial\lambda} h_\lambda^{loc}(z)}{w_1 \dots w_k} - \frac{\frac{1}{\lambda}\sum_{i=1}^k w_i w_{i+1} \dots w_k}{w_1 \dots w_k} = \frac{\frac{\partial}{\partial\lambda} h_\lambda^{loc}(w_k)}{w_1 \dots w_k} - \frac{1}{\lambda}\left(1 + \frac{1}{w_1} + \frac{1}{w_1 w_2} + \dots + \frac{1}{w_1 \dots w_{k-1}}\right). \tag{8.3}$$

We know that $h_\lambda(\cdot) \to \mathrm{id}$ as $\lambda \to \lambda_0$ uniformly on compact subsets of $\mathbb{C}$. Using this observation for $Y(\lambda_0)$, the compact set introduced in the proof of Proposition 2.1, it easy to show that if $z \in J(f_\lambda)$ then $|(f_\lambda^n)'(z)| \geq c\gamma^n$ for some $c > 0$ and $\gamma > 1$ independent of λ being sufficiently close to λ_0. We will also need the following, easy to prove strengthening of the expanding property. There exists $l \in \mathbb{N}$ (independent of λ in a neighbourhood of λ_0) such that

$$w = w_0, w_1, \dots, w_l \notin \Omega_\lambda \to |(f_\lambda^l)'(w)| > 3$$

Remembering that $|w_i| > \xi$ for all $i = 1, \dots, k-1$ and some constant $\xi > 0$, it follows from the above property that the sum $(1 + \frac{1}{w_1} + \frac{1}{w_1 w_2} + \dots + \frac{1}{w_1 \dots w_{k-1}})$ is uniformly bounded. It remains to estimate from above

$$\left|\frac{\frac{\partial}{\partial\lambda} h_\lambda^{loc}(z)}{w_1 \dots w_k}\right|.$$

Again, by the strengthening of Proposition 2.1, we get $|w_1 \dots w_{k-1}| \geq |w_1 \dots w_{k-l}|\xi^{l-1} \geq$ const. So, it is enough to estimate $|\frac{\partial}{\partial\lambda} h_\lambda^{loc}(z)/w_k| = \frac{\partial}{\partial\lambda} h_\lambda^{loc}(z)/h_\lambda^{loc}(z)$, where $z \in \Omega_{\lambda_0} \setminus f^p(\Omega_{\lambda_0})$. The function $\left|\frac{\partial}{\partial\lambda} h_\lambda^{loc}(z)\right|$ is bounded in $\Omega_{\lambda_0} \setminus f^p(\Omega_{\lambda_0})$, so the only problem arises when $h_\lambda^{loc}(z)$

is close to 0. But then we use the fact that the conjugacy h_λ in the neighbourhood of 0 is simply the composition $H_\lambda^{-1} \circ H_{\lambda_0}$, so it is a holomorphic function of (z, λ). Using the fact that $H_\lambda(0) = 0$ for all λ we conclude that h_λ in the neighbourhood of the point $(0, \lambda_0)$, h_λ has the form $h_\lambda(z) = zg(z, \lambda)$, where g is an analytic function of z, λ non-vanishing in a neighbourhood of 0 for every λ sufficiently close to λ_0. Then

$$\frac{\partial}{\partial\lambda}h_\lambda(z) = z\frac{\partial}{\partial\lambda}g(z,\lambda) = h_\lambda^{loc}(z)\frac{\frac{\partial}{\partial\lambda}g(z,\lambda)}{g(z,\lambda)}$$

and therefore the ratio $\left|\frac{\partial}{\partial\lambda}h_\lambda^{loc}(z)/z\right|$ is bounded in a neighbourhood of 0. Thus $\left|\frac{\partial}{\partial\lambda}h_\lambda(z)\right|$ is bounded throughout the whole Fatou set by a number say T. Fix now $z \in J(f)$. Then there exists a sequence $\{z_n\}_{n=1}^\infty$ of points in the Fatou set of f converging to z. Since $h_\lambda(z_n) \to h_\lambda(z)$ uniformly in λ and since all the functions $\lambda \mapsto h_\lambda(\xi)$ are holomorphic for every $\xi \in \mathbb{C}$, we conclude that $\frac{\partial h_\lambda(z_n)}{\partial\lambda} \to \frac{\partial h_\lambda(z)}{\partial\lambda}$. Thus $\left|\frac{\partial h_\lambda(z)}{\partial\lambda}\right| \le T$ and we are done. ■

Given $K, \alpha > 0$ we say that a map $h : \mathbb{C} \to \mathbb{C}$ is (K, α)-Hölder continuous if $|h(x) - h(y)| \le K|x-y|^\alpha$ for all $x, y \in \mathbb{C}$ such that $|x - y| \le 1$.

Proposition 8.2. *The conjugating homeomorphism $h_\lambda : \mathbb{C} \to \mathbb{C}$ is $(K_Q, 1/Q)$-Hölder continuous where, Q is the quasiconformality constant of h_λ and $K_{(\cdot)} : [1, \infty) \to (0, \infty)$ is an increasing function.*

Proof. It is well known that every quasiconformal homeomorphism is Hölder continuous on compact sets. But here, we know in addition that $\sup_{z\in\mathbb{C}}\left\{\left|\frac{dh_\lambda(z)}{d\lambda}\right|\right\} < \infty$. In particular, this implies that in some sufficiently small neighbourhood of λ_0 we have $\sup_{z\in\mathbb{C}}\{|z - h_\lambda(z)|\} < 1$. We now follow the standard proof of Hölder continuity of quasiconformal homeomorphisms. Fix $x \in \mathbb{C}$. Consider the open disk G of radius 1 with the center at x. Then the topological disk $G' = h_\lambda(G)$ is contained in the disk of radius 2 with center at $h_\lambda(x)$. Let $R : \mathbb{D} \to G'$ be its conformal representation such that $R(0) = h_\lambda(x)$. Then the map $g = R^{-1} \circ h_\lambda : G \to \mathbb{D}$ is a quasiconformal homeomorphism between two disks of radius 1. Hence, by Mori's theorem $|g(z_1) - g(z_2)| < 16|z_1 - z_2|^{1/Q}$. In particular, for every $z \in G$, $|g(z)| = |g(z) - g(x)| \le 16|z - x|^{1/Q}$. This implies that if $|z - x| \le 1/2$, then $|g(z)| \le (1/2)^{1/Q}$. Therefore, if $|z_1 - x|, |z_2 - x| \le 1/2$, then

$$|h_\lambda(z_2) - h_\lambda(z_1)| = |R(g(z_2)) - R(g(z_2))| \le \hat{K}_Q|R'(0)||g(z_2) - g(z_1)| \le 16\hat{K}_Q|R'(0)| \cdot |z_2 - z_1|^{\frac{1}{Q}}$$

where $\hat{K}_Q$ is the Koebe distortion constant corresponding to the scale $(1/2)^{1/Q}$. Now, by Koebe's $\frac{1}{4}$-theorem the image $R(D)$ contains a ball of radius $\frac{1}{4}|R'(0)|$ centered at $h_\lambda(x)$. Hence $|R'(0)| \le 8$ and consequently $|h_\lambda(z_2) - h_\lambda(z_1)| \le 128\hat{K}_Q$. We are done. ■

We end this section with the following simple observation.

Proposition 8.3. *For every $\lambda_0 \in \mathrm{Hyp}$, every λ sufficiently close to λ_0 and every $z \in \mathbb{C}$, we have $h_\lambda(z+2\pi i) = h_\lambda(z) + 2\pi i$.*

Proof. Since h_λ conjugates f_λ and f_{λ_0}, we have

$$\lambda e^{h_\lambda(z)} = h_\lambda(\lambda_0 e^z) = h_\lambda(\lambda_0 e^{z+2\pi i}) = \lambda e^{h_\lambda(z+2\pi i)}.$$

Hence $h_\lambda(z+2\pi i) - h_\lambda(z) \in 2\pi i\mathbb{Z}$ and, since the function $z \mapsto h_\lambda(z+2\pi i) - h_\lambda(z)$ is continuous, there exists $k(\lambda) \in \mathbb{Z}$ such that $h_\lambda(z+2pi) - h_\lambda(z) = 2\pi i k(\lambda)$ for all $z \in \mathbb{C}$. Since the function $\lambda \mapsto h_\lambda(2\pi i) - h_\lambda(0) = 2\pi i k(\lambda)$ is holomorphic, we conclude that $k(\lambda)$ is a constant independent of λ. Since $h_{\lambda_0}(2\pi i) - h_{\lambda_0}(0) = 2\pi i - 0 = 2\pi i$, the proof is complete. ■

In particular, h_λ induces the map of the cylinder P onto itself and we reserve for it the same symbol h_λ.

9. Real Analyticity of the Hausdorff Dimension

In this section we prove Theorem 9.3, our main result in this paper. Let us recall that by Hyp we denoted the set of all those parameters $\lambda \in \mathbb{C}\setminus\{0\}$ for which the mapping $f_\lambda : \mathbb{C} \to \mathbb{C}$ has an attracting periodic orbit. We will need the following continuity result.

Lemma 9.1. *For every $t > 1$ the function $\lambda \mapsto \mathrm{P}_\lambda(t)$, $\lambda \in \mathrm{Hyp}$, is continuous.*

Proof. Fix $\lambda_0 \in \mathrm{Hyp}$ and then $r > 0$ so small that $B(\lambda_0, r) \subset \mathrm{Hyp}$ and for every $\lambda \in B(\lambda_0, r)$ f_λ, f_{λ_0} are quasiconformally conjugate via the homeomophism $h_\lambda : \mathbb{C} \to \mathbb{C}$ described in Section 8. Then h_λ conjugates also F_λ and F_{λ_0}. Fix now $\gamma > 1$. Let $0 < r_1 \le r$ be so small that $\inf_{\lambda\in B(\lambda_0,r_1)}\{\mathrm{dist}(0, J(f_\lambda))\} > 0$. Then there exists $0 < r_2 \le r_1$ so small that if $|z-w| \le r_2$ then

$$\gamma^{-1} < \frac{|w|}{|z|} < \gamma.$$

In view of Proposition 8.1 there exists $0 < r_3 < r_2$ so small that if $|\lambda - \lambda_0| < r_3$ then $|h_\lambda(z) - z| < r_2$ away from a small fixed neighbourhood of the attracting periodic orbit. Fix now $z \in J(F_{\lambda_0})$. Take $n \ge 1$ and $x \in F_{\lambda_0}^{-n}(z)$. Since h_λ conjugates F_λ and F_{λ_0}, $h_\lambda(F_{\lambda_0}^{-n}(z)) = F_\lambda^{-n}(z)$. Also, for every $0 \le i \le n$ and every $x \in F_{\lambda_0}^{-n}(z)$, $h_\lambda(f_{\lambda_0}^i(x)) = f^i(h_\lambda)(x))$ and, therefore $|f_{\lambda_0}^i(x) - f_\lambda^i(h_\lambda(x))| \le r_2$. Hence,

$$\frac{|(F_\lambda^n)'(h_\lambda(x))|}{(F_{\lambda_0}^n)'(x)|} = \frac{(|f_\lambda^n)'(h_\lambda(x))|}{(f_{\lambda_0}^n)'(x)|} = \frac{|\prod_{i=0}^{n-1} f_\lambda'(f_\lambda^i(h_\lambda(x)))|}{|\prod_{i=0}^{n-1} f_{\lambda_0}'(f_{\lambda_0}^i(x))|} = \prod_{i=1}^{n} \frac{|f_\lambda^i(h_\lambda(x))|}{|f_{\lambda_0}^i(x)} \in (\gamma^{-n}, \gamma^n).$$

Since $h_\lambda : F_{\lambda_0}^{-n}(z) \to F_\lambda^{-n}(h_\lambda(z))$ is a bijection, we therefore conclude that

$$\frac{\sum_{x\in F_\lambda^{-n}(h_\lambda(z))} |(F_\lambda^n)'(x)|^{-t}}{\sum_{x\in F_{\lambda_0}^{-n}(z)} |(F_{\lambda_0}^n)'(x)|^{-t}} \in (\gamma^{-tn}, \gamma^{tn})$$

and from this,

$$\frac{1}{n}\log\sum_{x\in F_\lambda^{-n}(h_\lambda(z))}|(F_\lambda^n)'(x)|^{-t}-\frac{1}{n}\sum_{x\in F_{\lambda_0}^{-n}(z)}|(F_{\lambda_0}^n)'(x)|^{-t}\in(-t\log\gamma,t\log\gamma).$$

So, $\mathrm{P}_\lambda(t)-\mathrm{P}_{\lambda_0}(t)\in(-t\log\gamma,t\log\gamma)$ for all $\lambda\in B(\lambda_0,r_3)$. The continuity of the function $\lambda\to P_\lambda(t)$ is established. ■

Fix now $\lambda_0\in\mathrm{Hyp}$ and $t_0\in(1,\infty)$. Since by Proposition 8.2, $h_\lambda:J(F_{\lambda_0})\to J(F_\lambda)$ is Hölder continuous with the Hölder exponent $\alpha(\lambda)$ depending on λ and since $\alpha(\lambda)$ converges to 1 as $\lambda\to\lambda_0$, we get that for every $r>0$ sufficiently small that

$$\alpha=\inf\{\alpha(\lambda):\lambda\in B(\lambda_0,r)\}>0. \tag{9.1}$$

For every $\lambda\in B(\lambda_0,r)$ and every $t>1$ let $\mathcal{L}^0_{\lambda,t}:\mathrm{H}_\alpha(J(F_{\lambda_0}))\to\mathrm{H}_\alpha(J(F_{\lambda_0}))$ be the operator induced by the weight function $|F'_\lambda\circ h_\lambda|^{-t}:J(F_{\lambda_0})\to\mathbb{R}$, i.e.

$$\mathcal{L}^0_{\lambda,t}g(z)=\sum_{x\in F_{\lambda_0}^{-1}(z)}|F'_\lambda(h_\lambda(x))|^{-t}g(x).$$

Our aim is to use Corollary 7.7. However, the potential $|F'_\lambda\circ h_\lambda|^{-t}$ does not depend on $(\lambda,t)\in\mathbb{C}^2$ in a holomorphic way. For this reason, we have to embed λ into $\mathbb{C}^2$ and t into $\mathbb{C}$. We embed the complex plane $\mathbb{C}$ into $\mathbb{C}^2$ by the formula $x+iy\mapsto(x,y)\in\mathbb{C}^2$. So, $\lambda\in\mathbb{C}=\mathbb{R}^2$ may be treated as an element of $\mathbb{C}^2$. Fix

$$f=f_{\lambda_0}\text{ and }F=F_{\lambda_0}.$$

The technical result of this section is provided by the following.

Proposition 9.2. *Fix $\lambda_0\in\mathrm{Hyp}$ and $t_0>1$. There then exist $R>0$ and a holomorphic function*

$$L:\mathbb{D}_{\mathbb{C}^3}((\lambda_0,t_0),R)\to L(\mathrm{H}_a(J(F(\lambda_0)))$$

(λ_0 is treated here as elements of $\mathbb{C}^2$, t_0 as an element of $\mathbb{C}$) and α comes from (9.1) with r replaced by R) such that for every $(\lambda,t)\in B(\lambda_0,R)\times B(t_0,R)\subset\mathbb{C}\times\mathbb{R}$

$$L(\lambda,t)=\mathcal{L}^0_{\lambda,t}. \tag{9.2}$$

Proof. For every $\lambda\in\mathbb{C}$ sufficiently close to λ_0, say $\lambda\in U$, let $\theta_\lambda=F'_\lambda\circ h_\lambda$ and for every $z\in J(F)$ let

$$\phi_{(\lambda,t)}(z)=|\theta_\lambda|^{-t} \tag{9.3}$$

and

$$\psi_z(\lambda)=\frac{\theta_\lambda(z)}{\theta_{\lambda_0}(z)},\ (\lambda,z)\in U\times J(F).$$

We claim that there exists $r > 0$ such that for every $z \in J(F)$ the holomorphic function $\log \psi_z : B(\lambda_0, r) \to \mathbb{C}$ is well defined and there exists a universal constant (independent of $z \in J(F)$ in particular) $M_1 > 0$ such that

$$|\log \psi_z(\lambda)| \le M_1 \tag{9.4}$$

for all $\lambda \in B(\lambda_0, r)$, where the branch $\log \psi_z(\lambda)$ is determined by the requirement that $\log \psi_z(\lambda_0) = 0$. Indeed

$$\psi_z(\lambda) = \frac{F'_\lambda(h_\lambda(z))}{F'_{\lambda_0}(z)} = \frac{f_\lambda(h_\lambda(z))}{f_{\lambda_0}(z)} = \frac{h_\lambda(f_{\lambda_0}(z))}{f_{\lambda_0}(z)} = \frac{h_\lambda(f_{\lambda_0}(z)) - f_{\lambda_0}(z)}{f_{\lambda_0}(z)} + 1$$

and, using Proposition 8.1 along with the fact that $J(f_{\lambda_0})$ lies at a positive distance from the origin, we see that there exists $r > 0$ so that for all $\lambda \in B(\lambda_0, r)$ and all $z \in J(F)$ we have $|\psi_z(\lambda) - 1| < 1/2$. So, formula (9.4) is proven. Fix now $z_1, z_2 \in P$ with $|z_2 - z_1| \le \delta$. There then exist the lifts $\tilde z_1 \in \pi^{-1}(z_1)$ and $\tilde z_2 \in \pi^{-1}(z_2)$ such that $|\tilde z_2 - \tilde z_1| = |z_2 - z_1|$. We have

$$\begin{aligned}\frac{\psi_{z_2}(\lambda)}{\psi_{z_1}(\lambda)} &= \frac{f_\lambda(h_\lambda(\tilde z_2)) \cdot f_{\lambda_0}(\tilde z_1)}{f_\lambda(h_\lambda(\tilde z_1)) \cdot f_{\lambda_0}(\tilde z_2)} = \frac{\lambda e^{h_\lambda(\tilde z_2)} \lambda_0 e^{\tilde z_1}}{\lambda e^{h_\lambda(\tilde z_1)} \lambda_0 e^{\tilde z_2}} \\ &= \exp\Big(h_\lambda(\tilde z_2) - h_\lambda(\tilde z_1) + \tilde z_1 - \tilde z_2\Big)\end{aligned}$$

Hence, applying Proposition 8.2 and (9.1), we get

$$\begin{aligned}|\log \psi_{z_2}(\lambda) - \log \psi_{z_1}(\lambda)| &= |h_\lambda(\tilde z_2) - h_\lambda(\tilde z_1) + \tilde z_1 - \tilde z_2| \le |h_\lambda(\tilde z_2) - h_\lambda(\tilde z_1)| + |\tilde z_1 - \tilde z_2| \\ &\le (C+1)|\tilde z_1 - \tilde z_2|^\alpha = (C+1)|z_1 - z_2|^\alpha \le 2C|z_1 - z_2|^\alpha.\end{aligned}$$

Hence for every $\lambda \in B(\lambda_0, r)$ the function $z \mapsto \log \psi_z(\lambda)$, $z \in J(F)$, belongs to H_α and its Hölder constant is bounded from above by $2C$.

Since the function $\log \psi_z : B(\lambda_0, r) \to \mathbb{C}$ is holomorphic, it is uniquely represented as a power series

$$\log \psi_z(\lambda) = \sum_{n=1}^{\infty} a_n(z)(\lambda - \lambda_0)^n.$$

By Cauchy's inequalities,

$$|a_n(z)| \le \frac{M_1}{r^n} \tag{9.5}$$

for all $n \ge 0$. For every $\lambda = x + iy \in B(\lambda_0, r) \subset \mathbb{C}$, we have

$$\begin{aligned}\operatorname{Re} \log \psi_z(\lambda) &= \operatorname{Re}\left(\sum_{n=1}^{\infty} a_n(z)\Big((x - \operatorname{Re}(\lambda_0)) + (y - \operatorname{Im}(\lambda_0))i\Big)^n\right) \\ &= \sum_{p,q=0}^{\infty} c_{p,q}(z)\Big(x - \operatorname{Re}(\lambda_0)\Big)^p \Big(y - \operatorname{Im}(\lambda_0)\Big)^q,\end{aligned} \tag{9.6}$$

where $c_{p,q}(z) = a_{p+q}(z)\binom{p+q}{q} i^q$. Due to (9.5)

$$|c_{p,q}(z)| \le |a_{p+q}(z)| \cdot 2^{p+q} \le M_1 2^{p+q} r^{-(p+q)} \tag{9.7}$$

Hence, $\mathrm{Re}\log\psi_z$ extends by the same power series expansion

$$\sum_{p,q=0}^{\infty} c_{p,q}(z)\Big(x-\mathrm{Re}(\lambda_0)\Big)^p\Big(y-\mathrm{Im}(\lambda_0)\Big)^q$$

to a holomorphic function on the polydisk $\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/4)$. We denote this extension by the same symbol $\mathrm{Re}\log\psi_z$ and we have

$$|\mathrm{Re}\log\psi_z(\lambda)|\le\sum_{p,q=0}^{\infty}M_1 2^{-(p+q)}=4M_1 \tag{9.8}$$

on $\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/4)$. So, for every $t\in B_{\mathbb{C}}(t_0,\rho)$, where $\rho=t_0-1$, the formula

$$\zeta_{(\lambda,t)}(z)=-\Big(t\mathrm{Re}\log\psi_z(\lambda)+t\log|\theta_{\lambda_0}(z)|\Big) \tag{9.9}$$

extends $-t\log|\theta_\lambda(z)|$ on the polydisk $\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/4)\times B_{\mathbb{C}}(t_0,\rho)$. Now, due to (9.8), for every $(\lambda,t)\in B_{\mathbb{C}^2}(\lambda_0,r/4)\times B_{\mathbb{C}}(t_0,\rho)$ and every $z\in J(F)$, we have

$$\begin{aligned}|e^{\zeta_{\lambda,t}(z)}|&=\exp\left(\mathrm{Re}(-t\mathrm{Re}\log\psi_z(\lambda))-t\log|\theta_{\lambda_0}(z)|\right)\\&=\exp(\mathrm{Re}(-t\mathrm{Re}\log\psi_z(\lambda)))|\theta_{\lambda_0}(z)|^{-\mathrm{Re}(t)}\le\exp(|t||\mathrm{Re}\log\psi_z(\lambda)|)|\theta_{\lambda_0}(z)|^{-\mathrm{Re}(t)}\\&\le e^{4M_1|t|}|\theta_{\lambda_0}(z)|^{-\mathrm{Re}t}\end{aligned} \tag{9.10}$$

Since the function $|\theta_{\lambda_0}|^{-\mathrm{Re}t}$ is summable, it therefore follows that each function

$$\phi_{(\lambda,t)}=e^{\zeta_{(\lambda,t)}},\ (\lambda,t)\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/4)\times B_{\mathbb{C}}(t_0,\rho),$$

is summable and one part of the assumption (a) of Corollary 7.7 is proven. Of course putting $L(\lambda,t)=\mathcal{L}^0_t$, $(\lambda,t)\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/4)\times B_{\mathbb{C}}(t_0,\rho)$, the condition (9.2) is satisfied. Obviously, the function $(\lambda,t)\mapsto\phi_{(\lambda,t)}(z)$, $(\lambda,t)\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/4)\times B_{\mathbb{C}}(t_0,\rho)$, is holomorphic for every $z\in J(F)$ and the assumption (a) of Corollary 7.7 is established.

We shall now show that for every $\lambda\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/4)$ the function $z\mapsto\mathrm{Re}\log\psi_z(\lambda)$, $z\in J(F)$, is in H_α. Since we have already proved that for every $\lambda\in B(\lambda_0,r)$ the function $z\mapsto\log\psi_z(\lambda)$, $z\in J(F)$, is in H_α and its Hölder constant is bounded from above by $2C$, using Cauchy's inequalities, we conclude that

$$|a_n(z)-a_n(w)|\le 2C\left(\frac{4}{r}\right)^n|z-w|^\alpha$$

for all $z,w\in J(F)$ with $|z-w|\le\delta$. Therefore,

$$|c_{p,q}(z)-c_{p,q}(w)|\le 2C\cdot 2^{p+q}\left(\frac{4}{r}\right)^{p+q}|z-w|^\alpha=2C\left(\frac{8}{r}\right)^{p+q}|z-w|^\alpha \tag{9.11}$$

Hence,

$$|\mathrm{Re}\log\psi_z(\lambda)-\mathrm{Re}\log\psi_w(\lambda)|\le 2C\sum_{p,q=0}^{\infty}\left(\frac{8}{r}\right)^{p+q}\left(\frac{r}{16}\right)^p\left(\frac{r}{16}\right)^q|z-w|^\alpha=8C|z-w|^\alpha \tag{9.12}$$

for every $\lambda\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16))$ and all $z,w\in J(F)$ with $|z-w|\le\delta$. Hence, using (9.8), we see that the function $z\mapsto\mathrm{Re}\log\psi_z(\lambda)$, $z\in J(F)$, is in H_α for every $\lambda\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/4)$. Since

$\theta_{\lambda_0}(z) = F'_{\lambda_0}(z) = \lambda_0 e^z$, we get that $\log|\theta_{\lambda_0}(z)| = \log|\lambda_0|\text{Re}(z)$. Combining this, (9.12) and (9.9) we conclude that

$$|\zeta_{(\lambda,t)}(z) - \zeta_{(\lambda,t)}(w)| \le 9C|t||z-w|^\alpha \le 9C(|t_0|+\rho)|z-w|^\alpha \tag{9.13}$$

for every $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times B_{\mathbb{C}}(t_0, \rho)$.

We shall now check the second part of the assumption (a) of Corollary 7.7 that $\phi_{(\lambda,t)} = e^{\zeta_{(\lambda,t)}} \in \text{H}_\alpha$ for all $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times B_{\mathbb{C}}(t_0, \rho)$. Indeed, first observe that due to (9.10) there exists a constant $M_2 > 0$ such that

$$|\phi(\lambda,t)(z)| = |e^{\zeta_{(\lambda,t)}(z)}| \le M_2 \tag{9.14}$$

for all $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times B_{\mathbb{C}}(t_0, \rho)$ and all $z \in J(F)$. Obviously, there exists a constant $M_3 > 0$ such that $|e^\eta - 1| < M_3|\eta|$ for all $\eta \in \mathbb{C}$ with $|\eta| \le 9C\delta^\alpha$. Applying (9.13) and (9.14), we obtain

$$\begin{aligned}|\phi_{(\lambda,t)}(z) - \phi_{(\lambda,t)}(w)| &= |e^{\zeta_{(\lambda,t)}(w)}| \cdot |e^{\zeta_{(\lambda,t)}(z) - \zeta_{(\lambda,t)}(w)} - 1| \le M_3 M_2 |\zeta_{(\lambda,t)}(z) - \zeta_{(\lambda,t)}(w)| \\ &\le 9CM_2M_3(|t_0|+\rho)|z-w|^\alpha\end{aligned}$$

for all $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times B_{\mathbb{C}}(t_0, \rho)$ and all $z, w \in J(F)$ with $|z-w| \le \delta$. In particular, $\phi_{(\lambda,t)} \in \text{H}_\alpha$ and assumption (a) of Corollary 7.7 is verified.

We shall now check the assumptions (b) and (d) of Corollary 7.7, i.e. that all the functions $\phi_{(\lambda,t)} = e^{\zeta_{(\lambda,t)}}$, $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times B_{\mathbb{C}}(t_0, \rho)$, are dynamically Hölder (with the exponent α) with uniformly bounded constants c_{ϕ_α}. So, fix $\lambda \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$, $n \ge 1$, $v \in F^{-n}(x)$ and $x, y \in J(F)$ with $|x-y| \le \delta$. Applying (4.1) and (9.13), we obtain

$$\begin{aligned}\Big|\sum_{i=0}^{n-1} \zeta_{(\lambda,t)}(F^i(F_v^{-n}(y))) &- \sum_{i=0}^{n-1} \zeta_{(\lambda,t)}(F^i(F_v^{-n}(x)))\Big| \\ &\le \sum_{i=0}^{n-1} \Big|\zeta_{\lambda,t)}(F^i(F_v^{-n}(y))) - \zeta_{(\lambda,t)}(F^i(F_v^{-n}(x)))\Big| \\ &\le \sum_{i=0}^{n-1} 9C(|t_0|+\rho)|F^i(F_v^{-n}(y)) - F^i(F_v^{-n}(x))|^\alpha \\ &\le 9CL^\alpha(|t_0|+\rho)\sum_{i=0}^{\infty} \beta^{(n-i)\alpha}|y-x|^\alpha \\ &\le \frac{9CL^\alpha}{1-\beta^\alpha}(|t_0|+\rho)|y-x|^\alpha\end{aligned}$$

Therefore, putting $M_4=\sup\left\{\left|\frac{e^z-1}{z}\right| : |z|\le \frac{9CL^\alpha}{1-\beta^\alpha}(|t_0|+\rho)\right\}<\infty$, we get

$$
\begin{aligned}
&|\phi_{(\lambda,t),n}(F_v^{-n}(y))-\phi_{(\lambda,t),n}(F_v^{-n}(x))|\le\\
&\qquad=|\phi_{(\lambda,t),n}(F_v^{-n}(x))|\cdot\Big|\frac{\phi_{(\lambda,t),n}(F_v^{-n}(y))}{\phi_{(\lambda,t),n}(F_v^{-n}(x))}-1\Big|\\
&\qquad=|\phi_{(\lambda,t),n}(F_v^{-n}(x))|\left|\exp\left(\sum_{i=0}^{n-1}\zeta_{(\lambda,t)}(F^i(F_v^{-n}(y)))-\sum_{i=0}^{n-1}\zeta_{(\lambda,t)}(F^i(F_v^{-n}(x)))\right)-1\right|\\
&\qquad\le\frac{9CM_4L^\alpha}{1-\beta^\alpha}(|t_0|+\rho)|\phi_{(\lambda,t),n}(F_v^{-n}(x))|\cdot|y-x|^\alpha.
\end{aligned}
$$

and the assumptions (b) and (d) of Corollary 7.7 have been verified.

We shall now check the assumption (c) of Corollary 7.7 that the function

$$(\lambda,t)\mapsto\phi_{(\lambda,t)}\in \mathrm{H}_\alpha$$

is continuous in some neighbourhood of (λ,t_0) in $\mathbb{C}^3$. Since

$$\phi_{(\lambda,t)}(z)=e^{-t\mathrm{Re}\log\psi_z(\lambda)}\cdot|\theta_{\lambda_0}(z)|^{-t},$$

it is enough to show that both maps $z\mapsto e^{-t\mathrm{Re}\log\psi_z(\lambda)}$ and $z\mapsto|\theta_{\lambda_0}(z)|^{-t}$ are in H_α and that both maps

$$(\lambda,t)\mapsto e^{-t\mathrm{Re}\log\psi_{(\cdot)}(\lambda)}\in\mathrm{H}_\alpha$$

and

$$(\lambda,t)\mapsto|\theta_{\lambda_0}(\cdot)|^{-t}\in\mathrm{H}_\alpha$$

are continuous.

First recall that the function $z\mapsto\mathrm{Re}\log\psi_z(\lambda)$, $z\in J(F)$, is in H_α and consequently the function $z\mapsto t\mathrm{Re}\log\psi_z(\lambda)$, $z\in J(F)$, is in H_α for every $t\in\mathbb{R}$. Our most direct aim now is to show first that the mapping $(\lambda,t)\mapsto t\mathrm{Re}\log\psi_{(\cdot)}(\lambda)\in\mathrm{H}_\alpha$ is continuous on a polydisk $\mathbb{D}_{\mathbb{C}^3}((\lambda_0,t_0),R)$ with sufficiently small $R>0$. This function is obviously continuous with respect to the variable t on the polydisk $\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)\times B_{\mathbb{C}}(t_0,\rho)$. It is therefore sufficient to prove the Lipschitz continuity of the functions $\lambda\mapsto -t\mathrm{Re}\log\psi_{(\cdot)}(\lambda)\in\mathrm{H}_\alpha$ with Lipschitz constants independent of t. In order to do it, fix $\lambda=(\lambda_x,\lambda_y),\lambda'=(\lambda_x',\lambda_y')\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)$ In view of (9.6) we have for all $z\in J(F)$ that

$$
\begin{aligned}
&|\mathrm{Re}\log\psi_z(\lambda')-\mathrm{Re}\log\psi_z(\lambda)|=\\
&\qquad=\sum_{p,q=0}^{\infty}c_{p,q}(z)\Big((\lambda_x'-\mathrm{Re}\lambda_0)^p(\lambda_y'-\mathrm{Im}\lambda_0)^q-(\lambda_x-\mathrm{Re}\lambda_0)^p(\lambda_y-\mathrm{Im}\lambda_0)^q\Big)
\end{aligned}
\tag{9.15}
$$

Put $a_x = \lambda'_x - \mathrm{Re}\lambda_0$, $a_y = \lambda'_y - \mathrm{Im}\lambda_0$, $b_x = \lambda_x - \mathrm{Re}\lambda_0$ and $b_y = \lambda_y - \mathrm{Im}\lambda_0$. We then have

$$
\begin{aligned}
|a_x^p a_y^q - b_x^p b_y^q| &= |a_x^p(a_y^q - b_y^q) + b_y^q(a_x^p - b_x^p)| \\
&\le |a_x^p||a_y - b_y|\sum_{i=0}^{q-1}|a_y|^i|b_y|^{q-1-i} + |b_y^q||a_x - b_x|\sum_{i=0}^{p-1}|a_x|^i|b_x|^{p-1-i} \\
&\le \left(q\left(\frac{r}{16}\right)^p\left(\frac{r}{16}\right)^{q-1} + p\left(\frac{r}{16}\right)^q\left(\frac{r}{16}\right)^{p-1}\right)||\lambda' - \lambda|| \qquad (9.16) \\
&\le \frac{16}{r}(p+q)\left(\frac{r}{16}\right)^p\left(\frac{r}{16}\right)^q||\lambda' - \lambda||
\end{aligned}
$$

Combining this, (9.15) and (9.7), we obtain

$$
|\mathrm{Re}\log\psi_z(\lambda') - \mathrm{Re}\log(\psi_z(\lambda)| \le \frac{16}{r}||\lambda' - \lambda||\sum_{p,q=0}^{\infty}(p+q)8^{-(p+q)} = \frac{16C_1}{r}||\lambda' - \lambda||,
$$

(where $C_1 = \sum_{p,q=0}^{\infty}(p+q)8^{-(p+q)}$ is finite) for all $t \in B_{\mathbb{C}}(t_0, \rho/2)$ and all $\lambda, \lambda' \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$. Fix now $z, w \in J(F)$ with $|z - w| \le \delta$. It follows from (9.11) and (9.16) that

$$
\begin{aligned}
&|t|\Big|\mathrm{Re}\log\psi_w(\lambda') - \mathrm{Re}\log\psi_w(\lambda) - (\mathrm{Re}\log\psi_z(\lambda') - \mathrm{Re}\log\psi_z(\lambda))\Big| \\
&= |t|\left|\sum_{p,q=0}^{\infty}(c_{p,q}(w) - c_{p,q}(z))\Big((\lambda'_x - \mathrm{Re}\lambda_0)^p(\lambda'_y - \mathrm{Im}\lambda_0)^q - (\lambda_x - \mathrm{Re}\lambda_0)^p(\lambda_y - \mathrm{Im}\lambda_0)^q)\Big)\right| \\
&\le (|t_0| + \rho)\frac{32C}{r}|z - w|^{\alpha}||\lambda' - \lambda||\sum_{p,q=0}^{\infty}(p+q)2^{-(p+q)} \\
&\le \frac{32CC_2}{r}(|t_0| + \rho)||\lambda' - \lambda|||z - w|^{\alpha}
\end{aligned}
$$

where $C_2 = \sum_{p,q=0}^{\infty}(p+q)2^{-(p+q)}$ is finite. Thus,

$$
v_\alpha\Big(-t\mathrm{Re}\log\psi_{(\cdot)}(\lambda) - (-t\mathrm{Re}\log\psi_{(\cdot)}(\lambda'))\Big) \le 32CC_2(|t_0| + \rho)r^{-1}||\lambda' - \lambda||.
$$

for all $\lambda, \lambda' \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$ and $t \in B_{\mathbb{C}}(t_0, \rho)$. Thus the proof of the continuity of the function $(\lambda, t) \mapsto -t\mathrm{Re}\log\psi_{(\cdot)}(\lambda)$, $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times B_{\mathbb{C}}(t_0, \rho)$, is complete. The continuity of the function $(\lambda, t) \mapsto \phi_{(\lambda,t)}(\cdot) = \exp\Big(-t\mathrm{Re}\log\psi_{(\cdot)}(\lambda)\Big)$, $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times B_{\mathbb{C}}(t_0, \rho)$ follows now immediately from (9.8) and inequality $|e^b - e^a| = |e^a||e^{b-a} - 1| \le A|b - a|$, where A depends on the upper bound of $|a|$.

We shall now show that for every $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times B_{\mathbb{C}}(t_0, \rho)$ the function $z \mapsto |\theta_{\lambda_0}(\cdot)|^{-t}$, $z \in J(F)$, is $\in \mathrm{H}_\alpha$. Indeed, this immediately follows from the formula $\log|\theta_{\lambda_0}(z)| = \log|\lambda_0| + \mathrm{Re}(z)$, along with the facts that $\mathrm{Re}(t) > 0$ and

$$
\kappa = \inf\{\mathrm{Re}(\xi) : \xi \in J(F)\} > 0. \qquad (9.17)
$$

So, it remains to show that for every $z \in J(F)$ the function $(\lambda, t) \mapsto |\theta_{\lambda_0}(z)|^{-t} = |\lambda_0|^{-t} e^{t\mathrm{Re}(z)} \in \mathrm{H}_\alpha$ is continuous. Since $|\theta_{\lambda_0}(z)|^{-t}$ does not depend on λ, we only need to check its continuity with respect to the variable t. Let $\tilde\theta_{\lambda_0} : J(f) \to I\!R$ be defined as $\tilde\theta_{\lambda_0} = \theta_{\lambda_0} \circ \pi$. Then $|\tilde\theta_{\lambda_0}(z)|^{-t} = |f'_{\lambda_0}(z)|^{-t} = |f_{\lambda_0}(z)|^{-t} = |\lambda_0|^{-t}|e^z|^{-t} = |\lambda_0|^{-t} e^{-t\mathrm{Re}(z)}$. This function is $I\!R$-differentiable with respect to $z = (x, y)$ and its gradient is equal to $-(e^{-t\log\lambda_0} t e^{-tx}, 0)$. Recall that the Julia set $J(f)$ is contained in the half-plane $\mathrm{Re}(z) \ge \kappa$. So, given t_1, t_2 close enough to t_0, we can estimate from above the norm of gradient of the function $|\tilde\theta_{\lambda_0}(\cdot)|^{-t_1} - |\tilde\theta_{\lambda_0}(\cdot)|^{-t_2}$ in the half-plane $\mathrm{Re}(z) > R$ by

$$\sup_{x \ge \kappa} \left| -(e^{-t_1 \log|\lambda_0|} t_1 e^{-t_1 x}) - (-(e^{-t_2\log|\lambda_0|} t_2 e^{-t_2 x}))\right|.$$

Since $\mathrm{Re}(t_1), \mathrm{Re}(t_2) > 0$, it is obvious that this supremum tends to 0 as $t_2 \to t_1$. This implies that $v_1\Big(|\tilde\theta_{\lambda_0}(\cdot)|^{-t_1} - |\tilde\theta_{\lambda_0}(\cdot)|^{-t_2}\Big)$ is arbitrarily small if t_2 and t_1 are close enough. Since $\lim_{z\to\infty} |\tilde\theta_{\lambda_0}(z)| = \infty$, it is easy to see that the function $t \mapsto |\tilde\theta_{\lambda_0}(\cdot)|^t \in C_b$ is continuous in the supremum norm. Consequently the function $t \mapsto |\tilde\theta_{\lambda_0}(\cdot)|^t \in \mathrm{H}_1 \subset \mathrm{H}_\alpha$ is continuous. Thus the same is also true for the function $t \mapsto |\theta_{\lambda_0}(\cdot)|^t \in \mathrm{H}_\alpha$ and the proof of the item (c) is complete.

So, it remains to check item (f), the last assumption in Corollary 7.7. In order to do it fix arbitrary $\lambda_2 \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$ and $t_2 \in B_{\mathbb{C}}(t_0, \rho)$. Take $\gamma > 0$ so small that $\mathbb{D}_{\mathbb{C}^2}(\lambda_2, \gamma) \subset \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$ and $B_{\mathbb{C}}(t_2, 2\gamma) \subset B_{\mathbb{C}}(t_0, \rho)$. Then fix arbitrary $\lambda_1 \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$ and $t_1 \in (1, \mathrm{Re}(t_2) - \gamma)$ Then for every $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_2, \gamma) \times B_{\mathbb{C}}(t_2, \gamma)$, we have

$$\begin{aligned}\left|\frac{\phi_{(\lambda,t)}(z)}{\phi_{(\lambda_1,t_1)}(z)}\right| &= \frac{e^{-t\mathrm{Re}\log\psi_z(\lambda)}}{e^{-t_1\mathrm{Re}\log\psi_z(\lambda_1)}} |\theta_{\lambda_0}(z)|^{-(t-t_1)} = e^{t_1\mathrm{Re}\log\psi_z(\lambda_1) - t\mathrm{Re}\log\psi_z(\lambda)} \cdot |\theta_{\lambda_0}(z)|^{-(t-t_1)} \\ &= e^{t_1(\mathrm{Re}\log\psi_z(\lambda_1) - \mathrm{Re}\log\psi_z(\lambda))} \cdot e^{(t_1-t)\mathrm{Re}\log\psi_z(\lambda)} \cdot |\theta_{\lambda_0}(z)|^{-(t-t_1)}\end{aligned}$$

Using (9.8) we can estimate

$$|e^{t_1(\mathrm{Re}\log\psi_z(\lambda_1) - \mathrm{Re}\log\psi_z(\lambda))}| = e^{\mathrm{Re}(t_1(\mathrm{Re}\log\psi_z(\lambda_1) - \mathrm{Re}\log\psi_z(\lambda)))} \le e^{|t_1(\mathrm{Re}\log\psi_z(\lambda_1) - \mathrm{Re}\log\psi_z(\lambda))|} \le e^{8t_1 M_1}.$$

and

$$|e^{(t_1-t)\mathrm{Re}\log\psi_z(\lambda)}| = e^{\mathrm{Re}((t_1-t)\mathrm{Re}\log\psi_z(\lambda))} \le e^{|(t_1-t)\mathrm{Re}\log\psi_z(\lambda)|} \le e^{4\rho M_1}.$$

Since $A = \inf_{z\in J(F)} |\theta_{\lambda_0}(z)|$ is finite, since $\mathrm{Re}(t_1 - t) < 0$ and since $\mathrm{Re}(t_1 - t) > -\rho$, we can write

$$|\theta_{\lambda_0}(z)|^{-(t-t_1)} = |\theta_{\lambda_0}(z)|^{\mathrm{Re}(t_1-t)} \le \min\{1, |\theta_{\lambda_0}(z)|\}^{\mathrm{Re}(t_1-t)} \le \min\{1, |\theta_{\lambda_0}(z)|\}^{-\rho} \le \min\{1, A\}^{-\rho}.$$

Therefore

$$\left|\frac{\phi_{(\lambda,t)}(z)}{\phi_{(\lambda_1,t_1)}(z)}\right| \le \exp\Big(8t_1 M_1 + 4\rho M_1\Big)\min\{1, A\}^{-\rho}$$

and the item (e) is verified. The assumptions of Corollary 7.7 are therefore checked with $G = \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times B_{\mathbb{C}}(t_0, \rho)$. ■

We are now in a position to conclude the proof of the following main result of our paper.

Theorem 9.3. *The function* $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$, $\lambda \in \mathrm{Hyp}$, *is real-analytic.*

Proof. In view of we are equivalently to prove that the function $\lambda \mapsto \mathrm{HD}(J_r(F_\lambda))$, $\lambda \in Hyp$, is real-analytic. So, Fix $\lambda_0 \in \mathrm{Hyp}$ and $t_0 \in (1, \infty)$. Since by Proposition 8.2, $h_\lambda : J(F_{\lambda_0}) \to J(F_\lambda)$ is Hölder continuous with the Hölder exponent $\alpha(\lambda)$ depending on λ but converging to 1 as $\lambda \to \lambda_0$, we get that for every $r > 0$ sufficiently small

$$\alpha = \inf\{\alpha_\lambda : \lambda \in B(\lambda_0, r)\} > 0.$$

Recall now that for every $\lambda \in B(\lambda_0, r)$ and every $t > 1$, $\mathcal{L}^0_{\lambda,t} : \mathrm{H}_\alpha(J(F_{\lambda_0})) \to \mathrm{H}_\alpha(J(F_{\lambda_0}))$ is the operator induced by the weight function $|F'_\lambda \circ h_\lambda|^{-t} : J(F_{\lambda_0}) \to I\!R$, i.e.

$$\mathcal{L}^0_{\lambda,t} g(z) = \sum_{x \in F^{-1}_{\lambda_0}(z)} |F'_\lambda(h_\lambda(x))|^{-t} g(x).$$

Proposition 9.2 says that there exist $R > 0$ and a holomorphic function

$$L : \mathbb{D}_{\mathbb{C}^3}((\lambda_0, t_0), R) \to L(\mathrm{H}_\alpha(J(F\lambda_0))$$

(λ_0 is treated here as elements of $\mathbb{C}^2$ and t_0 as an element of $\mathbb{C}$) such that for every $(\lambda, t) \in B(\lambda_0, R) \times B(t_0, \rho) \subset \mathbb{C} \times I\!R$

$$L(\lambda, t) = \mathcal{L}^0_{\lambda,t}. \tag{9.18}$$

Now, in view of Theorem 4.4 and Proposition 3.7, $e^{\mathrm{P}_{\lambda_0}(t)}$ $(t \in B(t_0, R)$ is a simple isolated eigenvalue of the operator $L(\lambda_0, t) = \mathcal{L}^0_{\lambda_0,t} : \mathrm{H}_\alpha(J(F_{\lambda_0}) \to \mathrm{H}_\alpha(J(F_{\lambda_0}))$. Applying now the perturbation theory for linear operators (see [Ka]), we see that there exists $0 < R_1 \le R$ and a holomorphic function $\gamma : \mathbb{D}_{\mathbb{C}^3}((\lambda_0, t), R_1) \to \mathbb{C}$ such that $\gamma(\lambda_0, t_0) = e^{P_{\lambda_0}(t_0)}$ and for every $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^3}((\lambda_0, t), R_1)$ the number $\gamma(\lambda, t)$ is a simple isolated eigenvalue of $L(\lambda, t)$ with the remainder part of the spectrum uniformly separated from $\gamma(\lambda, t)$. In particular there exist $0 < R_2 \le R_1$ and $\kappa > 0$ such that

$$\sigma(L(\lambda, t)) \cap B(e^{\mathrm{P}_{\lambda_0}(t_0)}, \kappa) = \{\gamma(\lambda, t)\} \tag{9.19}$$

for all $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^3}((\lambda_0, t_0), R_2)$. Consider now for each $(\lambda, t) \in B(\lambda_0, t_0) \times (t_0 - R, t_0 + R$ the operator $\mathcal{L}_{\lambda,t} : \mathrm{H}_1(J(F_\lambda)) \to \mathrm{H}_1(J(F_\lambda))$ (see Lemma 4.1) given by the formula

$$\mathcal{L}_{\lambda,t} g(z) = \sum_{x \in F^{-1}(z)} |F'_\lambda(x)|^{-t} g(x).$$

It is easy to see that the map $T_\lambda : C_b(J(F_\lambda)) \to C_b(J(F_{\lambda_0}))$ defined by the formula $T_\lambda(g) = g \circ h_\lambda$ establishes a bounded linear conjugacy between $\mathcal{L}_{\lambda,t} : C_b(J(F_\lambda)) \to C_b(J(F_\lambda))$ and $\mathcal{L}^0_{\lambda,t} : C_b(J(F_{\lambda_0})) \to C_b(J(F_{\lambda_0}))$. Since the map $h_\lambda : J(F_{\lambda_0}) \to J(F_\lambda)$ is Hölder continuous with the Hölder exponent α, we obtain

$$T_\lambda(\mathrm{H}_1(J(F_\lambda))) \subset H_\alpha(J(F_{\lambda_0})).$$

Hence $e^{P_\lambda(t)}$ is an eigenvalue of the operator

$$\mathcal{L}^0_{\lambda,t} : \mathrm{H}_\alpha(J(F_{\lambda_0})) \to \mathrm{H}_\alpha(J(F_{\lambda_0}))$$

and, by Lemma 9.1, $e^{P_\lambda(t)} \in B(e^{P_\lambda(t_0)}, \kappa)$ for all $\lambda \in B(\lambda_0, R_3)$ and all $t \in (t_0 - \rho, t_0 + \rho)$ if $\rho \in (0, \min\{t_0, R_2\}$ and $R_3 \in (0, I\!R_2)$ are sufficiently small. Combining this, (9.18) and (9.19) we see that $\gamma(\lambda, t) = e^{\mathrm{P}_\lambda(t)}$ for λ, t as above. Therefore the function $(\lambda, t) \to P_\lambda(t)$, $(\lambda, t) \in B(\lambda_0, R_3) \times (t_0 - \rho, t_0 + \rho)$, is real-analytic. Since, by Theorem 6.2, $\mathrm{P}_\lambda(s_\lambda) = 0$ where $s_\lambda = \mathrm{HD}(J(F_\lambda))$, in order to conclude the proof it suffices to show that

$$\frac{\partial P_\lambda(t)}{\partial t} \neq 0$$

for all $(\lambda, t) \in B(\lambda_0, R_3) \times (t_0 - \rho, t_0 + \rho)$. So fix such λ and t. Fix $z \in J(F_\lambda)$. Since for every $u \geq 0$ and every $n \geq 1$

$$\sum_{x \in F_\lambda^{-n}(z)} |(F_\lambda^n)'(x)|^{-(t+u)} = \sum_{x \in F_\lambda^{-n}(z)} |(F_\lambda^n)'(x)|^{-u} |(F_\lambda^n)'(x)|^{-t} \leq L^{-u} \beta^{un} \sum_{x \in F_\lambda^{-n}(z)} |(F_\lambda^n)'(x)|^{-t},$$

we conclude that $\mathrm{P}_\lambda(t+u) - \mathrm{P}_\lambda(t) \leq u \log \beta$ which implies that $\frac{\partial \mathrm{P}_\lambda(t)}{\partial t}(\lambda, t) \leq \log \beta < 0$. We are done. ■

10. Appendix. Construction of conformal measures

Let $\lambda \in \mathrm{Hyp}$. In what follows we rely on the description given in [BD], we also use the notation of this paper. We assume that $z_0, z_1, \ldots, z_n = z_0$ is an attracting cycle of $f = f_\lambda$. Assume that the singular value 0 is contained in the domain A_1, the immediate basin of attraction of z_1. The Jordan domain with smooth boundary $B_{n+1} \subset A_1$ is chosen so that $0, z_1 \in B_{n+1}$, $\mathrm{cl} f^n(B_{n+1}) \subset B_{n+1}$. Then B_n is defined as $B_n = f^{-1}(B_{n+1})$. Notice that B_n contains some half-plane $\mathrm{Re} z < -M$ and $z_0 \in B_n$. The set $B_{n-1}, B_{n-2}, \ldots, B_0$ are then defined inductively as follows. B_i is a connected component of $f^{-1}(B_{i+1})$, containing z_i. Notice that for $i < n$ B_{n-i} is a simply-connected unbounded domain. The set B_0 is a complement of a union of infinitely many sets F_i. Obviously, $F_i = F_0 + 2k\pi i)$ (see Fig 3. in [BD]). (Here, $F_0 + 2k\pi i$ is the image of F_0 under the map $z \to z + 2k\pi i$.) To build an appropriate dynamics, we fix one component F_0 of the complement of B_0. Since the construction is very transparent in the case of a fixed attracting point, we describe it first. So, assume that $n = 1$. Then we start with B_2. B_1 is an unbounded domain containing some half-plane, $f^{-1}(B_1)$ is connected and $B_1 \subset B_0 = f^{-1}(B_1)$. Each component F_i of the complement of B_0 is mapped by f onto $\mathbb{C} \setminus \mathrm{cl} B_1 \supset \mathrm{cl} \bigcup F_i$. Let $S = F_0$. Then we see that

$$f(S) \supset \mathrm{cl}\left(\bigcup_k (S + 2k\pi i)\right)$$

and $f^{-1}(S) \cap S$ is a union of countably many disjoint unbounded simply-connected domains ("fingers"), each of them being mapped by f univalently onto $S + 2k\pi i$ for some $k \in \boldsymbol{Z}$. We define $\hat{F} : f^{-1}(S) \cap S \to S$ as $\hat{F} = \pi \circ f$ where π is the natural projection $\pi : \bigcup F_i \to F_0$.

In the general case of periodic attracting orbit the construction is slightly more complicated. Again, let F_0 be a component of the complement of B_0. Let

$$S = F_0 \setminus \pi^{-1}(\bigcup_{i=1}^{n-1} B_i),$$

where π is the natural projection $\pi : \bigcup F_i \to F_0$. So,

$$f(S) \supset \bigcup_k (S + 2k\pi i)$$

and, actually, modifying the set S slightly, we can require that $f(S) \supset \mathrm{cl} \bigcup_k (S + 2k\pi i)$. Again, $S \cap f^{-1}(\pi^{-1}(S))$ is a union of disjoint unbounded, simply connected domains and $\hat{F} : f^{-1}(S) \cap S \to S$ as $\hat{F} = \pi \circ f$ is defined as $\hat{F} = \pi \circ f$. Let

$$J(F) = \{z \in S : \hat{F}^n \text{ is defined for all n}\}.$$

One can easily see that

$$J(f) \cap S = J(\hat{F}).$$

In this section developing the arguments worked out in [UZd] we construct first conformal measures for the map $\hat{F} : S \cap J(f) \to S \cap J(f)$. The details are presented for the sake of completeness and the convenience of the reader. A Borel measure m is called (t, α_t)-conformal (with $t > 1$) if for any Borel set $A \subset S$ on which $\hat{F}$ is injective, we have

$$m(\hat{F}(A)) = \int_A \alpha_t |\hat{F}'|^t dm$$

In [UZd] we only considered the case when $\alpha_t = 1$. Let

$$\tilde{S}_M = \{z \in S : \mathrm{Re} z \leq M\}.$$

Consider the preimage $\hat{F}^{-1}(\tilde{S}_M)$. This set is a union of infinitely many topological disks Q_i and

$$\overline{Q_i} \cap \overline{Q_j} = \emptyset.$$

Now, we consider the finite family of disks Q_i^M, which are contained in $\tilde{S}_M$. In this way we obtain the finite iterated function system:

$$\{\phi_i : \tilde{S}_M \to Q_i^M\},$$

where ϕ_i are appropriate holomorphic branches of $\hat{F}^{-1}$. Let J_M be the limit set of this system and let m_M be its unique $(t, \exp(\mathrm{P}_M(t)))$-conformal measure, where

$$\mathrm{P}_M(t) \leq \overline{\mathrm{P}}(t) < \infty \tag{10.1}$$

is the standard topological pressure for the repeller $F|_{J_M} : J_M \to J_M$ and the potential $-t \log |\hat{F}'|$.

Remark 10.1. *We have $J_M \subset J_{M+1}$ for all M large enough. In order to see this, take Q_i^M and let Q_i^{M+1}, be the preimage of $\tilde{P}_{M+1}$ under the same holomorphic branch $\hat{F}_*^{-1}$ of $\hat{F}^{-1}$. Then, obviously, $Q_i^{M+1} \supset Q_i^M$. Since $\hat{F}(Q_i^{M+1} \setminus Q_i^M) \subset \{z \in \tilde{P}_{M+1} : M < \mathrm{Re}z \le M+1\}$ and since the derivative of $\hat{F}_*^{-1}$ on $\{z \in \tilde{P}_{M+1} : M < \mathrm{Re}z \le M+1\}$ is bounded from above by $C_1 M^{-1}$, we conclude that $\mathrm{diam}(Q_i^{M+1} \setminus Q_i^M) \le C_2 M^{-1}$ for some appropriate constants C_1 and C_2. Since $Q_i^M \subset \{\mathrm{Re}z \le M\}$, this implies that*

$$Q_i^{M+1} \subset \{\mathrm{Re}z \le M+1\}$$

for all M large enough. Hence, each $Q_i^{M+1} \supset Q_i^M$ is (see the definition) used in the construction of J_{M+1}. Thus, the corresponding limit set J_{M+1} contains J_M.

Proposition 10.2. *If $t > 1$, then the sequence of measures m_M, $M \in \mathbb{N}$ is tight, i.e. for every $\varepsilon > 0$ there exists M so large that for every N*

$$m_N(\{z \in S : \mathrm{Re}z > M\}) < \varepsilon.$$

Proof. Fix $\varepsilon > 0$, $M > 0$ and $N \ge \hat{q}$. We shall estimate separately the measure m_N of two sets, which cover $\{z \in S : \mathrm{Re}z > M\}$. First, we have

$$m_N(\{x \in J_N : \mathrm{Re}F(x) \ge M\}) = \sum_{k \in \boldsymbol{Z}} m_N(\{x \in J_N : f(x) \in [M,N] \times ([-\pi,\pi] + 2k\pi i).$$

If $x \in J_N$ and $\mathrm{Re}f(x) \in [M,N] \times [-\pi,\pi] + 2k\pi i\}$, then

$$|\hat{F}'(x)| = |f(x)| \ge \frac{1}{2}(M + \pi|k|) \ge \frac{1}{2}(M + |k|)$$

which gives

$$\begin{aligned} m_N(\{x : \mathrm{Re}\hat{F}(x) \ge M\}) &\le 2\sum_{k=0}^{\infty} m_N(\{x : M \le \mathrm{Re}x \le N\}) \cdot \frac{2^t e^{-\mathrm{P}_N(t)}}{(M+k)^h} \\ &\le 2^{t+1} e^{-\mathrm{P}_N(t)} \sum_{k=0}^{\infty} \frac{1}{(M+k)^h}, \end{aligned} \tag{10.2}$$

If $N \le M$, then

$$m_N(\{z \in S : \mathrm{Re}z > M\}) = 0. \tag{10.3}$$

Since, by Remark 10.1, the sequence $N \mapsto \mathrm{P}_N(t)$ is eventually non-decreasing, we see that$\gamma(t) = \sup_N\{-\mathrm{P}_N(t)\} < \infty$. If $N > M$, then it follows from this along with (10.2) and (10.1) that

$$m_N(\{x : \mathrm{Re}F(x) \ge M\}) \le \frac{2^{t+1} e^{-\mathrm{P}_N(t)}}{t-1} M^{1-t} \le \frac{2^{t+1}\gamma(t)}{t-1} M^{1-t}. \tag{10.4}$$

Keeping $N > M$ we now estimate the measure of the second set. Namely

$$m_N(\{x : M < \mathrm{Re}x < N \text{ and } \mathrm{Re}F(x) < M\}).$$

If $\mathrm{Re}x > M$, then $|f(x)| > \lambda e^M$ and therefore $|\mathrm{Im}f(x)| \geq \sqrt{\lambda^2 e^{2M} - M^2}$. Hence $|F'(x)| = |f'(x)| = |f(x)| \geq |\mathrm{Im}f(x)| \geq \sqrt{\lambda^2 e^{2M} - M^2}$. Thus,

$$m_N(\{x : M < \mathrm{Re}x < N \text{ and } \mathrm{Re}F(x) < M\} \leq const\,\gamma(t) \sum_{k \geq \sqrt{\lambda^2 e^{2M} - M^2}}^{\infty} k^{-t}$$

$$\leq const \frac{\gamma(t)}{t-1} e^{M(1-t)}. \tag{10.5}$$

Combining this along with (10.3) and (10.4) we obtain

$$m_N(\{x : \mathrm{Re}x > M\}) < \varepsilon$$

for all N and all M large enough. ■

Since, by Prop 10.2 the sequence m_N is tight, it follows from Prochorov's theorem that there exists a subsequence m_{N_i} weakly convergent to some limit probability measure $m = m_t$. This is the measure we are looking for. Recall that

$$J(\hat{F}) = S \cap J(f).$$

The proof of the following theorem requires only minor obvious modifications of the proof of Theorem 3.4 in [UZd].

Theorem 10.3. *The measure m is (t, α_t)-conformal, where $\alpha_t = \lim_{N\to\infty} e^{\mathrm{P}_N(t)}$ and $m(J(\hat{F})) = 1$.*

References

[BD] R. Bhattacharjee, R. Devaney, Tying hairs for structurally stable exponentials, Ergod. Th. Dyn. Sys. 20 (2000),1603-1617

[DPU] M. Denker, F. Przytycki, M. Urbański, On the transfer operator for rational functions on the Riemann sphere, Ergod. Th. and Dynam. Sys. 16 (1996), 255-266.

[DU1] M. Denker, M. Urbański, On the existence of conformal measures, Trans. A.M.S. 328 (1991), 563-587.

[DU2] M. Denker, M. Urbański, Ergodic theory of equilibrium states for rational maps, Nonlinearity 4 (1991), 103-134.

[EL] A. Eremenko, M.Lyubich, Dynamical properties of some classes of entire functions, Ann. Inst. Fourier, Grenoble, 42 (1992), 989-1020.

[IM] C. Ionescu-Tulcea, G. Marinescu, Théorie ergodique pour des classes d'operations non-complement continues, Ann. Math. 52, (1950), 140-147.

[Ka] T. Kato, Perturbation theory for linear operators, Springer (1995).

[MSS] On the dynamics of rational maps, Ann. Sci. Ec. Norm. Sup., 4e serie (16) (1983), 193-217.

[McM] C. McMullen, Area and Hausdorff dimension of Julia sets of entire functions, Trans. A.M.S. 300 (1987), 329-342.

[MU] D. Mauldin, M. Urbański, Graph Directed Markov Systems, preprint 2000, to appear Cambridge Univ. Press.

[PU] F. Przytycki, M. Urbański, Fractals in the plane - Ergodic theory methods, to appear in the Cambridge University Press, available on Urbański's webpage.

[UZd] M. Urbański, A. Zdunik, The finer geometry and dynamics of exponential family, Preprint 2001.

[UZi] M. Urbański, M. Zinsmeister, Geometry of hyperbolic Julia-Lavaurs sets, Indagationes Math. 12(2), (2001), 273-292

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