Existence of Invariant Measures for Transcendental Subexpanding Functions

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Abstract

We consider the problem of the existence of absolutely continuous invariant measures for transcendental meromorphic functions. We prove sufficient conditions for a subexpanding meromorphic function f to have a σ-finite absolutely continuous invariant measure μ and we find a class of functions satisfying these assumptions.

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1 Introduction

The orbits of points under iteration by a meromorphic function fall into three categories: they may be infinite, they may become periodic and hence consist of a finite number of distinct points or they may terminate at a pole of the function. Points in the last category are called *prepoles.* For transcendental meromorphic functions with more than one pole, it follows from Picard's theorem that there are infinitely many prepoles.

The *Fatou set* $F(f)$ of a meromorphic function $f:\mathbb{C}\to\overline{\mathbb{C}}$ is defined in exactly the same manner as for rational functions; $F(f)$ is the set of points $z\in\mathbb{C}$ such that all the iterates are defined and form a normal family on a neighborhood of z. The *Julia set* $J(f)$ is the complement of $F(f)$ in $\overline{\mathbb{C}}$. Thus, $F(f)$ is open, $J(f)$ is closed, $F(f)$ is completely invariant while $f^{-1}(J(f))\subset J(f)$ and $f(J(f)\setminus\{\infty\})\subset J(f)$. For description of the dynamics of meromorphic functions see e.g. [3]. We would however like to note that it easily follows from Montel's criterion of normality that if $f:\mathbb{C}\to\overline{\mathbb{C}}$ is either entire or has exactly one pole w and $w\notin f(\mathbb{C})$ (such functions f will be called subentire and for them $f:\mathbb{C}\setminus\{w\}\to\mathbb{C}\setminus\{w\}$ is well-defined), then there exists a set $E\subset\mathbb{C}$ consisting of at most one element and such that for every $z\in J(f)\setminus\{\infty\}$ if f is entire and for every $z\in J(f)\setminus\{w,\infty\}$ if f is subentire, every $r>0$ and every $q\geq 1$

$$\bigcup_{n\geq 1} f^{qn}(B(z,r))\supset\mathbb{C}\setminus E.$$

In the sequel E will be called the set of omitted values of f. It can be also defined for meromorphic functions which are not subentire. If f is meromorphic but not subentire nor entire, then (see [3])

$$J(f)=\overline{\bigcup_{n\geq 0} f^{-n}(\infty)}.$$

The *singular set* $S(f)\subset\mathbb{C}$ of a meromorphic function f consists of those values at which f is not a regular covering. These are either critical values (algebraic singularities) or asymptotic values (transcendental singularities). The *postsingular set* $P(f)$ is the union of the forward orbits of all singular values, i.e.

$$P(f)=\bigcup_{n=0}^{\infty} f^n(S(f)).$$

If a singular value is a prepole (belongs to $\bigcup_{n\geq 0} f^{-n}(\infty)$), we take the images in this union only until the image is equal to ∞ and then we stop. It follows from Iversen's (see [8]) theorem that $E \subset S(f)$ and, and consequently, $E \subset \overline{P(f)}$. By l_2 we denote the Lebesgue measure on the plane and by m the measure induced by the spherical metric on $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Note that both measures l_2 and m are equivalent in the sense that they have the same sets of measure zero. Let

$$I_\infty(f) = \{z : f^n(z) \to \infty\}.$$

Given $z \in \mathbb{C}$ let $\omega(z)$ be the ω-limit set of z, i.e. the set of all accumulation points in $\overline{\mathbb{C}}$ of the sequence $\{f^n(z)\}_{n=1}^\infty$.

M. Lyubich has proved in [11] that there is no σ-finite measure absolutely continuous with respect to the Lebesgue measure l_2 and invariant under the action of the map $z \mapsto e^z$. Aiming to give a positive contribution in the opposite direction we shall prove as our main result the following.

Theorem 1. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function satisfying the following two conditions:*

(a) $J(f) = \overline{\mathbb{C}}$

(b) $l_2(\{z : \omega(z) \subset \overline{P(f)} \cup \{\infty\}\}) = 0$

then there exists a σ-finite ergodic conservative f-invariant measure μ equivalent with the Lebesgue measure l_2.

Recall that ergodicity means that if G is a Borel set satisfying $f^{-1}(G) = G$, then either $\mu(G) = 0$ or $\mu(G^c) = 0$ and conservativity means that for every set G with positive measure, the measure of those z for which $f^n(z) \in G$ only for finitely many n's is equal to zero. Of course condition (b) implies that $l_2(I_\infty(f)) = 0$. Note that due to Lyubich's result from [11] the condition (b) of Theorem 1 fails for the function $z \mapsto e^z$ and due to Bock's result from [6] it fails for the map $z \mapsto \tan\left(\frac{\pi i z}{2}\right)$. The simplest examples of functions satisfying the assumptions of Theorem 1 are given by the formula $f(z) = 2\pi i e^z$ and $g(z) = \pi i \tan(z)$. For the proof that these functions actually satisfy (b) it is important to know that $l_2(I_\infty(f)) = 0$ and $l_2(I_\infty(g)) = 0$ (see [9] and [6] respectively). Here we present a larger class of functions f with $l_2(I_\infty(f)) = 0$.

Theorem 2. *If*

$$f(z) = \frac{Ae^{z^p} + Be^{-z^p}}{Ce^{z^p} + De^{-z^p}}$$

$p \in \mathbb{N}$, $AD - BC \neq 0$, *then* $l_2(I_\infty(f)) = 0$.

An example of a function satisfying the assumptions of Theorem 1 and holomorphically conjugate to a function from the class involved in Theorem 2 with $p = 2$ is given by the formula

$$f(z) = \sqrt{\pi} i \tan(z^2) + \sqrt{\pi}.$$

Indeed, this easily follows from the fact that the asymptotic values 0 and $2\sqrt{\pi}$ as well as the critical point 0 are mapped by f on the repelling fixed point $\sqrt{\pi}$ and the property that $l_2(I_\infty(f)) = 0$ following from Theorem 2.

We will frequently use the following two versions of Koebe's distortion theorem.

Theorem A. (Koebe's Distortion Theorem, I) *There exists a function $k : [0,1) \to [1,\infty)$ such that for all $z \in \mathbb{C}$, all $r > 0$, all $t \in [0,1)$ and any univalent analytic function $H : B(z,r) \to \mathbb{C}$, we have*

$$\sup\{|H'(x)| : x \in B(z,tr)\} \le k(t) \inf\{|H'(x)| : x \in B(z,tr)\}.$$

Theorem B. (Koebe's Distortion Theorem, II) *Given a number $s > 0$ there exists a function $k_s : [0,1) \to [1,\infty)$ such that for any $z \in \overline{\mathbb{C}}$, $r > 0$, $t \in [0,1)$ and any univalent analytic function $H : B(z,r) \to \overline{\mathbb{C}}$ such that the complement $\overline{\mathbb{C}} \setminus H(B(z,r))$ contains a ball of radius s we have*

$$\sup\{|H'(x)|_\rho : x \in B(z,tr)\} \le k_s(t) \inf\{|H'(x)|_\rho : x \in B(z,tr)\},$$

where $|H'(x)|_\rho$ means that the derivative is taken with respect to the spherical metric on $\overline{\mathbb{C}}$.

We put $K = \max\{k(1/2), k_s(1/2)\}$.

2 Proof of Theorem 1

We start with the description of our setting. Let X be a compact metric space, m be a Borel measure such that $m(X) = 1$. Suppose $T : X \to X$ is a measurable map and m is a quasi-invariant measure, i.e. $m \circ T^{-1} << m$. In the proof of Theorem 1 we apply the following result of M. Martens (see [12]).

Theorem 2.1. *Let (X, m, T) be as above. Suppose we have a partition $\mathcal{A} = \{A_i : i \in \mathbb{N} \cup \{0\}\}$ of X such that A_i are Borel sets of positive measure, $m(X \setminus \bigcup_{n=0}^\infty A_i) = 0$ and they satisfy the following conditions:*

1. T is ergodic and conservative with respect to the measure m.

2. $\forall i,\ j \geq 0\ \exists k \geq 0$ such that up to measure zero $T^k(A_i) \supset A_j$

3. $\forall i \geq 0\ \exists K_i \geq 1$, for all Borel sets $A, B \subset A_i$ and for all integers $n \geq 0$

$$\frac{m(T^{-n}(A))}{m(T^{-n}(B))} \leq K_i \frac{m(A)}{m(B)}.$$

Then there exists σ-finite ergodic conservative measure μ equivalent with m and such that

$$\mu \circ T^{-1} = \mu$$

and for every Borel set A

$$\mu(A) = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} m(T^{-k}(A))}{\sum_{k=0}^{n} m(T^{-k}(A_0))}.$$

(Note that due to conservativity of f, $\sum_{k=0}^{\infty} m(T^{-k}(A_0)) = \infty$.)

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function such that

$$l_2(\{z : \omega(z) \subset \overline{P(f)} \cup \{\infty\}\}) = 0.$$

Obviously this assumption implies that

$$l_2(\overline{P(f)}) = 0. \tag{1}$$

First we construct the partition $\mathcal{A}$, next we check that it satisfies the assumptions of Theorem 2.1. We define a new metric on the plane $\mathbb{C}$ by putting

$$d_{min}(x, y) := \min\{1, |x - y|)\}$$

and we consider the family of balls

$$\left\{ B\left(z, \frac{1}{2} d_{min}(z, \overline{P(f)})\right)\right\}_{z\in\mathbb{C}\setminus\overline{P(f)}}.$$

This family obviously covers $\mathbb{C} \setminus \overline{P(f)}$. Since $\mathbb{C} \setminus \overline{P(f)}$ is an open set, it is a Lindelöf space, and therefore we can choose a countable subcover of $\mathbb{C}\setminus\overline{P(f)}$, which we denote by

$$\left\{ B\left(z_i, \frac{1}{2} d_{min}(z_i, \overline{P(f)})\right)\right\}_{i=1}^{\infty}.$$

We inductively define a partition $\mathcal{A} = \{A_i\}_{i=0}^{\infty}$ of $\mathbb{C} \setminus \overline{P(f)}$ as follows. Let

$$A_0 = \left\{ B\left(z_0, \frac{1}{2} d_{min}(z_0, \overline{P(f)})\right) \right\}.$$

Assume that we have defined the set $A_1, \ldots, A_n$ such that

$$A_j \subset \left\{ B\left(z_j, \frac{1}{2} d_{min}(z_j, \overline{P(f)})\right) \right\}$$

and

$$\mathrm{Int} A_j \neq \emptyset.$$

Then A_{n+1} we define as

$$A_{n+1} = \left\{ B\left(z_{n+1}, \frac{1}{2} d_{min}(z_{n+1}, \overline{P(f)})\right) \right\} \setminus \bigcup_{j=1}^{n} A_j.$$

The set A_{n+1} is disjoint with the sets $A_1, \ldots, A_n$ and

$$A_{n+1} \subset B\left(z_{n+1}, \frac{1}{2} d_{min}(z_{n+1}, \overline{P(f)})\right) \setminus \bigcup_{j=1}^{n} B\left(z_j, \frac{1}{2} d_{min}(z_j, \overline{P(f)})\right).$$

Thus either $A_{n+1} = \emptyset$ or $\mathrm{Int} A_{n+1} \neq \emptyset$ and we remove all the empty sets.

Remark 1. *Since $\mathcal{A}$ is the partition of $\mathbb{C} \setminus \overline{P(f)}$, we have $S(f) \cap A_j = \emptyset$ for each $j \in \mathbb{N} \cup \{0\}$.*

Lemma 2.2. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function. If $z \in J(f) \setminus \{\infty\}$, $r > 0$ and $K \subset \mathbb{C}$ is a compact set disjoint from the exceptional set E, then there exists $n \geq 1$ such that $f^n(B(z,r)) \supset K$.*

Proof: Suppose first that f is either entire or subentire. Since due to Baker's and Bhattacharyya's theorem (see [1] and [4], comp. [3]), the set of repelling periodic points is dense in the Julia set, we see that there exists a periodic point $x \in B(z,r)$, say of period $q \geq 1$. Since x is repelling there exists $s > 0$ so small that $B(x,s) \subset B(z,r)$ and $f^q(B(x,s)) \supset B(x,s)$. Since $\bigcup_{j\geq 1} f^{qj}(B(x,s)) \supset \mathbb{C} \setminus E$, since K is a compact subset of $\mathbb{C} \setminus E$ and since $\{f^{qj}(B(x,s))\}_{j=1}^{\infty}$ is an increasing family of open sets, there thus exists $k \geq 1$ such that $f^{qk}(B(x,s)) \supset K$. So, we are done in this case. Assume in turn that f is not entire nor subentire. Then $\overline{\bigcup_{n\geq 1} f^{-n}(\infty)} = \mathbb{C}$ and fix a point

$$w \in B(z,r) \cap \bigcup_{n \geq 1} f^{-n}(\infty).$$

So, $w \in f^{-n}(\infty)$ for some $n \geq 1$ and there exists $t > 0$ so small that $B(w,t) \cap \bigcup_{j=0}^{n-1} f^{-j}(\infty) = \emptyset$. Hence $f^n(B(w,t))$ is well-defined and it forms an open neighbourhood of $\infty \in \overline{\mathbb{C}}$. Since ∞ is an essential singularity of f, by Picard's theorem the set $f(f^n(B(w,t)) \setminus \{\infty\})$ contains the whole $\overline{\mathbb{C}} \setminus E$. The proof is complete. ■

As an immediate consequence of this lemma we get the following.

Corollary 2.3. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function such that $J(f) = \overline{\mathbb{C}}$ and $l_2(\{z : \omega(z) \subset \overline{P(f)} \cup \{\infty\}\}) = 0$. If $\mathcal{A}$ is a partition defined above, then $l_2(\mathbb{C} \setminus \bigcup_{n=0}^{\infty} A_i) = 0$ and $\mathcal{A}$ satisfies the second assumption of Theorem 2.1 i.e.*

$$\forall i, j \geq 0 \ \ \exists k \geq 0 \ \ \textit{such that} \quad \textit{up} \quad \textit{to} \quad \textit{measure} \quad \textit{zero} \quad f^k(A_i) \supset A_j.$$

Lemma 2.4. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function such that $J(f) = \overline{\mathbb{C}}$. If $l_2(\{z : \omega(z) \subset \overline{P(f)} \cup \{\infty\}\}) = 0$, then f is ergodic and conservative with respect to the measure m.*

Proof: Let $\overline{P(f)}_- = \{z \in \mathbb{C} : \omega(z) \subset \overline{P(f)} \cup \{\infty\}\}$. We shall prove first that every forward invariant ($f(F) \subset F$) subset F of $J(f)$ is either of measure 0 or 1. Indeed, suppose on the contrary that $0 < m(F) < 1$. Since $m(\overline{P(f)_-}) = 0$, it suffices to show that

$$m(F \setminus \overline{P(f)}_-) = 0.$$

Denote by Z the set of all points $z \in F \setminus \overline{P(f)}_-$ such that

$$\lim_{r \to 0} \frac{m(B(z,r) \cap (F \setminus \overline{P(f)}_-))}{m(B(z,r))} = 1. \tag{2}$$

In view of the Lebesgue density theorem (see for example Theorem 2.9.11 in [7]), $m(Z) = m(F)$. Since $m(F) > 0$ we find at least one point $z \in Z$. Since $z \in J(f) \setminus \overline{P(f)}_-$, there exists $x \in \mathbb{C} \setminus \overline{P(f)}$ and an increasing sequence $\{n_k\}_{k=1}^{\infty}$ such that

$$x = \lim_{k \to \infty} f^{n_k}(z) \ \text{ and } |f^{n_k}(z) - x| < \delta/2$$

for every $k \geq 1$, where $\delta = \text{dist}(x, \overline{P(f)}) > 0$. Suppose that $m(B(x,\delta) \setminus F) = 0$. Obviously $m(f(Y)) = 0$ for all Borel sets Y such that $m(Y) = 0$. Hence,

$$\begin{aligned} 0 = m\big(f^n(B(x,\delta) \setminus F)\big) &\geq m\big(f^n(B(x,\delta)) \setminus f^n(F)\big) \\ &\geq m\big(f^n(B(x,\delta)) \setminus F\big) \geq m\big(f^n(B(x,\delta))\big) - m(F) \end{aligned} \tag{3}$$

for all $n \geq 0$. Since by Lemma 2.2, $\sup_{n\geq 1}\{m\big(f^n(B(x,\delta))\big)\} = 1$, this implies that $0 \geq 1 - m(F)$ which is a contradiction. Consequently $m(B(x,\delta)\setminus F) > 0$. Hence for every $j \geq 1$ large enough, $m\big(B(f^{n_j}(z), 2\delta)\setminus F\big) \geq m\big(B(x,\delta)\setminus F\big) > 0$. Therefore, as $f^{-1}(J(f)\setminus F) \subset J(f)\setminus F$, the standard application of Koebe's Distortion Theorem II (Theorem B) shows that

$$\limsup_{r\to 0} \frac{m(B(z,r)\setminus F)}{m(B(z,r))} > 0$$

which contradicts (2). Thus either $m(F) = 0$ or $m(F) = 1$. In particular ergodicity is proven and conservativity is now straightforward. One needs to prove that for every Borel set $B \subset J(f)$ with $m(B) > 0$ one has $m(G) = 0$, where

$$G = \{x \in J(f) : \sum_{n\geq 0} \chi_B(f^n(x)) < +\infty\}.$$

Indeed, suppose that $m(G) > 0$ and for all $n \geq 0$ let

$$G_n = \{x \in J(f) : \sum_{k\geq n} \chi_B(f^n(x)) = 0\} = \{x \in J(f) : f^k(x) \notin B \text{ for all } k \geq n\}.$$

Since $G = \bigcup_{n\geq 0} G_n$, there exists $k \geq 0$ such that $m(G_k) > 0$. Since all the sets G_n are forward invariant we conclude that $m(G_k) = 1$. But on the other hand all the sets $f^{-n}(B)$, $n \geq k$, are of positive measure and are disjoint from G_k. This contradiction finishes the proof. ■

Remark 2. *Notice that the same result under slightly weaker assumptions was proved by H. Bock in [5] and [6]. We presented our independent proof for the sake of completeness.*

Lemma 2.5. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a transcendental meromorphic function such that $l_2(\{z : \omega(z) \subset \overline{P(f)} \cup \{\infty\}\}) = 0$. If $\mathcal{A} = \{A_i\}_{i=0}^{\infty}$ is a partition defined above, then for every $i \geq 0$ there exists $K_i \geq 1$ such that for each $n \geq 0$ and all Borel sets $A, B \subset A_i \in \mathcal{A}$ with $m(B) > 0$, we have*

$$\frac{m(f^{-n}(A))}{m(f^{-n}(B))} \leq K_i \frac{m(A)}{m(B)}.$$

Proof: Fix $i \geq 0$. Note that all holomorphic inverse branches of f^n, $n \geq 1$, are well-defined on $(B(z_i, (1/2)d_{min}(z_i, \overline{P(f)}))$. Denote the set they form by $\mathcal{F}_i$. It is well-known (see [2] where the proof is provided in the setting of rational functions) that $\mathcal{F}_i$ is a normal family. Since $J(f) = \overline{\mathbb{C}}$, all the limit functions of $\mathcal{F}_i$ are constant. Therefore there exists $r_i > 0$ such that if $f_\nu^{-n} \in \mathcal{F}_i$, then $f_\nu^{-n}(B(z_i, (3/4)d_{min}(z_i, \overline{P(f)}))$ is disjoint from a ball of radius

r_i (with respect to the spherical metric). It therefore follows from Koebe's Distortion Theorem, II (Theorem B) that there exists $\tilde{K}_i \geq 1$ such that

$$\frac{|(f_\nu^{-n})'(y)|_\rho}{|(f_\nu^{-n})'(x)|_\rho} \leq \tilde{K}_i$$

for all $f_\nu^{-n} \in \mathcal{F}_i$ and all $x, y \in B(z_i, (1/2)d_{min}(z_i, \overline{P(f)}))$, where the subscript ρ indicates that the derivative is taken with respect to the spherical metric. Hence for all Borel sets $A, B \subset A_i$ we get

$$\frac{m(f_\nu^{-n}(A))}{m(f_\nu^{-n}(B))} = \frac{\int_A |(f_\nu^{-n})'|_\rho^2 dm}{\int_A |(f_\nu^{-n})'|_\rho^2 dm} \leq \frac{\sup_{A_i}\{|(f_\nu^{-n})'|_\rho\}^2 m(A)}{\inf_{A_i}\{|(f_\nu^{-n})'|_\rho\}^2 m(B)} \leq \tilde{K}_i^2 \frac{m(A)}{m(B)}.$$

In order to conclude the argument, note that

$$\begin{aligned} m(f^{-n}(A)) &= \sum_{\mathcal{F}_i} m\big(f_\nu^{-n}(A)\big) \leq \sum_{\mathcal{F}_i} \tilde{K}_i^2 \frac{m(A)}{m(B)} m\big(f_\nu^{-n}(B)\big) \\ &= \tilde{K}_i^2 \frac{m(A)}{m(B)} \sum_{\mathcal{F}_i} m\big(f_\nu^{-n}(B)\big) = \tilde{K}_i^2 \frac{m(A)}{m(B)} m(f^{-n}(B)) \end{aligned}$$

We are done. ■

The proof of Theorem 1 follows now immediately from Corollary 2.3, Lemma 2.4, Lemma 2.5 and from Theorem 2.1.

3 Proof of Theorem 2

The idea of the proof is to obtain good estimates of the derivative of the function f around poles and to follow the scheme worked out in [10]. We can rewrite $f(z)$ in the form

$$f(z) = \frac{Ae^{2z^p} + B}{Ce^{2z^p} + D}.$$

It is easy to calculate that

$$f'(z) = \frac{2p(AD - BC)z^{p-1}e^{2z^p}}{(Ce^{2z^p} + D)^2} = \frac{2p(AD - BC)z^{p-1}e^{2z^p}(f(z))^2}{(Ae^{2z^p} + B)^2} \tag{4}$$

For $p = 1$ the function f has no critical points. Let $p > 1$. Then $f'(z) = 0$ iff $z = 0$ and $f(0) = \frac{A+B}{C+D}$ is a critical value. Note that the assumption $AD - BC \neq 0$ implies that either $C \neq 0$ or $D \neq 0$. Assume that $C = 0$

then f is a transcendental entire map with one finite asymptotic value $\frac{B}{D}$. Analogously, if $D = 0$ then f is also a transcendental entire map with one finite asymptotic value $\frac{A}{C}$. For transcendental entire function the theorem follows from Theorem 7 in [9]. So, suppose that $C \neq 0$ and $D \neq 0$. It is straightforward to verify that the function $g(z) = (Ae^z + Be^{-z})/(Ce^z + De^{-z})$ satisfies the Riccati equation $g' = a + bg + cg^2$ with $a = -2AB/(AD - BC)$, $b = -2(AD + BC)/(AD - BC)$, $c = 2CD/(AD - BC)$. Since $f(z) = g(z^p)$, we therefore conclude that

$$f'(z) = pz^{p-1}(a + bf(z) + c(f(z))^2). \tag{5}$$

Fix now $R >> 1$, a pole z_q of f with $|z_q| \geq R$. Let $A_R = \{z \in \overline{\mathbb{C}} : |z| \geq R\}$ and let V_q be the connected component of $f^{-1}(A_R)$ containing z_q. Since $c \neq 0$, it then follows from (5) that with R sufficiently large

$$|f'(z)| \geq \frac{p}{2}|c||z|^{p-1}R^2 \geq R^p \tag{6}$$

for all $z \in V_q$. Fix now $R_1 > R$, put $A_{R,R_1} = \{z \in \mathbb{C} : R < |z| < R_1\}$ and consider $\tilde{V}_{q,R_1}$, the connected component of $f^{-1}(A_{R,R_1})$ enclosing (in $\mathbb{C}$) the point z_q. It then follows from (5) that

$$|f'(z)| \leq 2p|c||z|^{p-1}R_1^2 \tag{7}$$

for all $z \in \tilde{V}_{q,R_1}$. Combining this with the first part of (6) we get that

$$\frac{\sup_{z \in \tilde{V}_{q,R_1}} |f'(z)|}{\inf_{z \in \tilde{V}_{q,R_1}} |f'(z)|} \leq L = 4C\left(\frac{R_1}{R}\right)^2, \tag{8}$$

where

$$C = \sup_q \frac{\sup\{|z| : z \in V_q\}}{\inf\{|z| : z \in V_q\}} < \infty$$

if R is large enough.

Since the map g is the composition of a Möbius transformation and the map $z \mapsto e^{2z}$ for every q large enough and since each holomorphic branch of $z^{1/p}$ sending the point z_q^p to z_q is univalent on the balls containing z_q^p, so big that applying Koebe's Distortion Theorem I (Theorem A) produces some radius γ_q such that

$$B(z_q, \gamma_q/4) \subset V_q \subset B(z_q, \gamma_q/2). \tag{9}$$

Let $\mathcal{V} = f^{-1}(A_R)$. A straightforward calculations show that

$$\lim_{R_1 \to \infty} \frac{l_2(A_{R,R_1} \setminus \mathcal{V})}{l_2(A_{R,R_1})} = 1 \tag{10}$$

and

$$\lim_{R_1\to\infty}\frac{l_2\big(\tilde V_{q,R_1}\big)}{l_2(V_q)}=1 \tag{11}$$

uniformly with respect to q. Therefore for every $R_1>0$ large enough

$$\frac{l_2(A_{R,R_1}\setminus\mathcal{V})}{l_2(A_{R,R_1})}\ge\frac{1}{2}\quad\text{and}\quad\frac{l_2\big(\tilde V_{q,R_1}\big)}{l_2(V_q)}\ge\frac{1}{2}. \tag{12}$$

We want to show that for every q we have

$$\frac{l_2(V_q\setminus f^{-1}(\mathcal{V}))}{l_2(V_q)}\ge(4L^2)^{-1}, \tag{13}$$

where L is the upper bound on distortion given by (8). And indeed, using (8) and (12), we get

$$\begin{aligned}\frac{l_2(V_q\setminus f^{-1}(\mathcal{V}))}{l_2(V_q)}&=\frac{l_2([(V_q\setminus\tilde V_{q,R_1})\setminus f^{-1}(\mathcal{V})]\cup[\tilde V_{q,R_1}\setminus f^{-1}(\mathcal{V})])}{l_2(V_q)}\\&\ge\frac{l_2(\tilde V_{q,R_1}\setminus f^{-1}(\mathcal{V}))}{l_2(V_q)}=\frac{l_2(f_q^{-1}(A_{R,R_1}\setminus\mathcal{V}))}{l_2(f_q^{-1}(A_{R,R_1}))}\cdot\frac{l_2(\tilde V_{q,R_1})}{l_2(V_q)}\\&\ge\frac{1}{2}L^{-2}\frac{l_2(A_{R,R_1}\setminus\mathcal{V})}{l_2(A_{R,R_1})}\ge(4L^2)^{-1}.\end{aligned}$$

Suppose now on the contrary that $l_2(I_\infty(f))>0$. Since

$$I_\infty(f)\subset\bigcap_{n\ge1}\bigcup_{k\ge n}\bigcap_{l\ge k}f^{-l}(A_R)$$

there in particular exists $k\ge1$ such that $l_2\Big(I_\infty(f)\cap\bigcap_{j\ge k}f^{-j}(A_R)\Big)>0$. Let ξ_0 be a density point of the Lebesgue measure of the set $I_\infty(f)\cap\bigcap_{j\ge k}f^{-j}(A_R)$. For every $n\ge0$ put

$$\xi_n=f^n(\xi_0).$$

Since $\lim_{n\to\infty}\xi_n=\infty$, for every n large enough there exists $q(n)$ such that $\xi_n\in V_{q(n)}$, $\lim_{n\to\infty}|\xi_n^p-z_{q(n)}^p|=0$ and $\lim_{n\to\infty}q(n)=\infty$. Hence, using Koebe's Distortion Theorem I (Theorem A), we deduce that for all n large enough $|\xi_n-z_{q(n)}|\le\gamma_{q(n)}/8$ (γ_q are the numbers defined in (9)) and combining this with (9), we conclude that

$$B(\xi_n,\gamma_{q(n)}/8)\subset V_{q(n)}\subset B(\xi_n,\gamma_{q(n)}) \tag{14}$$

Since each ξ_n is also a density point of the set $\bigcap_{j\geq k} f^{-j}(A_R)$, we may if needed replace ξ_0 by an appropriate iterate ξ_n and assume that all the numbers $q(n)$, $n \geq 1$ are so large as one wishes. Since for every q the map $f : V_q \to A_R$ is univalent, there exists its inverse map $g^{(q)} : A_R \to V_q$. Since in addition for all $n \geq 1$, $V_{q(n)} \subset A_R$, therefore the composition

$$g_n = g^{q(0)} \circ g^{q(1)} \circ g^{q(2)} \ldots \circ g^{q(n)} : A_R \to A_R$$

is well-defined (and obviously univalent). Moreover $g_n(V_{q(n)}) \subset V_{q(0)}$. By Koebe's $\frac{1}{4}$-Theorem and Koebe's Distortion Theorem I (Theorem A), we get

$$B\left(\xi_0, \frac{1}{4}|g_n'(\xi_n)|\gamma_n\right) \subset g_n(B(\xi_n, \gamma_n)) \subset B(\xi_0, K|g_n'(\xi_n)|\gamma_n),$$

where we abbreviated $\gamma_{q(n)}$ to γ_n. Using Koebe's Distortion Theorem I (Theorem A) again along with (14), we can estimate as follows.

$$\begin{aligned}
&\frac{l_2\big(B(\xi_0, K|g_n'(\xi_n)|\gamma_n) \cap f^{-(n+2)}(A_R)\big)}{l_2\big(B((\xi_0, K|g_n'(\xi_n)|\gamma_n)\big)} \\
&= 1 - \frac{l_2\big(B(\xi_0, K|g_n'(\xi_n)|\gamma_n) \setminus f^{-(n+2)}(A_R)\big)}{l_2\big(B((\xi_0, K|g_n'(\xi_n)|\gamma_n)\big)} \\
&\leq 1 - \frac{l_2(g_n(B(\xi_n, \gamma_n)) \setminus f^{-(n+2)}(A_R))}{l_2\big(B((\xi_0, K|g_n'(\xi_n)|\gamma_n)\big)} = 1 - \frac{l_2(g_n(B(\xi_n, \gamma_n) \setminus f^{-1}(\mathcal{V})))}{(4K)^2 l_2\big(B(\xi_0, \frac{1}{4}|g_n'(\xi_n)|\gamma_n)\big)} \\
&\leq 1 - \frac{1}{16K^2}\frac{l_2(g_n(B(\xi_n, \gamma_n) \setminus f^{-1}(\mathcal{V})))}{l_2\big(g_n(B(\xi_n, \gamma_n))\big)} \leq 1 - \frac{1}{16K^4}\frac{l_2\big(B(\xi_n, \gamma_n) \setminus f^{-1}(\mathcal{V})\big)}{l_2\big(B(\xi_n, \gamma_n)\big)} \\
&\leq 1 - \frac{1}{2^{10}K^4}\frac{l_2\big(V_{q(n)} \setminus f^{-1}(\mathcal{V})\big)}{l_2\big(B(\xi_n, \gamma_n/8)\big)} \leq 1 - \frac{1}{2^{10}K^4}\frac{l_2\big(V_{q(n)} \setminus f^{-1}(\mathcal{V})\big)}{l_2\big(V_{q(n)}\big)} \\
&\leq 1 - \frac{1}{2^{12}K^4L^2},
\end{aligned}$$

where writing the last inequality we have used (13). Hence, for every $n \geq k$

$$\begin{aligned}
\frac{l_2\big(B(\xi_0, K|g_n'(\xi_n)|\gamma_n) \cap \bigcap_{j\geq k} f^{-j}(A_R)\big)}{l_2\big(B((\xi_0, K|g_n'(\xi_n)|\gamma_n)\big)} &\leq \frac{l_2\big(B(\xi_0, K|g_n'(\xi_n)|\gamma_n) \cap f^{-(n+2)}(A_R)\big)}{l_2\big(B((\xi_0, K|g_n'(\xi_n)|\gamma_n)\big)} \\
&\leq 1 - \frac{1}{2^{12}K^4L^2}.
\end{aligned}$$

Thus ξ_0 is not a density point of the set $\bigcap_{j\geq k} f^{-j}(A_R)$ and consequently not a density point of the set $I_\infty(f) \cap \bigcap_{j\geq k} f^{-j}(A_R)$. This contradiction finishes the proof of our theorem.

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