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# FRACTAL MEASURES FOR PARABOLIC IFS

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ABSTRACT. Let $h$ be the Hausdorff dimension of the limit set of a conformal parabolic iterated function system in dimension $d \geq 2$. In case the system of maps is finite, we provide necessary and sufficient conditions for the $h$-dimensional Hausdorff measure to be positive and finite and also, assuming the strong open set condition holds, characterize when the $h$-dimensional packing measure of the limit set is positive and finite. We also prove that the upper ball (box)-counting dimension and the Hausdorff dimension of this limit set coincide. As a byproduct we include a compact analysis of the behaviour of parabolic conformal diffeomorphisms in dimension 2 and separately in any dimension greater than or equal to 3.

## 1. Introduction and preliminaries

Our setting is the following. Let $X$ be a compact subset of a Euclidean space $I\!\!R^d$ with nonempty interior such that the boundary of $X$ has no isolated points. We consider a countable family of conformal maps $\phi_i : X \to X$, $i \in I$, where $I$ has at least two elements satisfying the following conditions.

**(1):** (Open Set Condition) $\phi_i(\text{Int}(X)) \cap \phi_j(\text{Int}(X)) = \emptyset$ for all $i \neq j$.

**(2):** $|\phi_i'(x)| < 1$ everywhere except for finitely many pairs $(i, x_i)$, $i \in I$, for which $x_i$ is the unique fixed point of $\phi_i$ and $|\phi_i'(x_i)| = 1$. Such pairs and indices $i$ will be called *parabolic* and the set of parabolic indices will be denoted by $\Omega$. All other indices will be called *hyperbolic*.

**(3):** (extension) There exist an open connected neighbourhood $V$ of $X$ and $s < 1$ such that $\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ if $\omega_n$ is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then $\phi_\omega$ extends conformally to $V$, maps $V$ into itself and $||\phi_\omega'|| \leq s$.

**(4):** If $i$ is a parabolic index, then $\bigcap_{n\geq 0} \phi_{i^n}(X) = \{x_i\}$ (Thus, the diameters of the sets $\phi_{i^n}(X)$ converge to 0.)

**(5):** (Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset I\!\!R^d$, there exists an open cone $\text{Con}(x, u_x, \alpha, l) \subset \text{Int}(X)$ with vertex $x$ and a central angle of Lebesgue measure $\alpha$, where $\text{Con}(x, u_x, \alpha, l) = \{y : 0 < (y - x, u_x) \leq \cos\alpha ||y - x|| \leq l\}$ and $||u_x|| = 1$.

**(6):** $\exists s < 1$ $\forall n \geq 1$ $\forall \omega \in I^n$ if $\omega_n$ is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then $||\phi_\omega'|| \leq s$.

**(7):** (Bounded Distortion Property) $\exists K \geq 1$ $\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ $\forall x, y \in V$ if $\omega_n$ is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then

$$\frac{||\phi_\omega'(y)||}{||\phi_\omega'(x)||} \leq K.$$

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**(8):** There are constants $L \geq 1$ and $\alpha > 0$ such that

$$\Big| ||\phi_i'(y)|| - ||\phi_i'(x)|| \Big| \leq L||\phi_i'||(||y-x||)^\alpha,$$

for every $i \in I$ and every pair of points $x, y \in V$.

We call such a system of maps $S = \{\phi_i : i \in I\}$ a conformal iterated function system abbreviated as conformal IFS. If $\Omega = \emptyset$, we call the system $S$ *hyperbolic*; if $\Omega \neq \emptyset$, we call it *parabolic*. Throughout this entire paper we assume that the system $S$ is parabolic.

We would like to emphasize that if $d \geq 2$, then the conditions (7) and (8) are a consequence of condition (3) alone. Indeed, in case $d = 2$, these follow from Koebe's distortion theorem (in its version stated in [Pr]) and the observation that the complex conjugation in $\mathbb{C}$ is an isometry. In case $d \geq 3$, both conditions have been proved in [U2]. Because of an extreme importance of these properties and for the sake of completeness, we include their proof taken from [U2] in the end of Section 2. Finally, since the appropriate results in case $d = 1$ have been proven in [U3], we assume throughout the entire paper that $d \geq 2$.

By $I^*$ we denote the set of all finite words with alphabet $I$ and by $I^\infty$ all infinite sequences with terms in $I$. It follows from (3) that for every hyperbolic word $\omega$, $\phi_\omega(V) \subset V$. For each $\omega \in I^* \cup I^\infty$, we define the length of $\omega$ by the uniquely determined relation $\omega \in I^{|\omega|}$. If $\omega \in I^* \cup I^\infty$ and $n \leq |\omega|$, then by $\omega|_n$ we denote the word $\omega_1\omega_2\ldots\omega_n$. In [MU4], we proved that $\lim_{n\to\infty}\sup_{|\omega|=n}\{\mathrm{diam}(\phi_\omega(X))\} = 0$. So, the map $\pi : I^\infty \to X$, $\pi(\omega) = \bigcap_{n\geq 0}\phi_{\omega|_n}(X)$, is uniformly continuous. Its range

$$J = J_S = \pi(I^\infty),$$

the main object of our interest in this paper, is called the *limit set* of the system $S$. For every integer $q \geq 1$, we denote

$$S^q = \{\phi_\omega : \omega \in I^q\}.$$

Of course, $J_{S^q} = J_S$ and sometimes in the sequel it will be more convenient to consider an appropriate family of iterates $S^q$ of $S$ rather than $S$ itself. The two basic tools we use to study limit sets of parabolic IFS are conformal measures and a hyperbolic system $S^*$ associated with $S$. The system $S^*$ is given by

$$S^* = \{\phi_{i^nj} : n \geq 1,\ i \in \Omega,\ i \neq j\} \cup \{\phi_k :\ k \in I \setminus \Omega\}.$$

Thus, $I_*$, the countable set of indices or letters for the system $S^*$ is

$$I_* = \{i^nj : n \geq 1,\ i \in \Omega,\ i \neq j\} \cup \{k :\ k \in I \setminus \Omega\}.$$

This system was described and analyzed in [MU4]. It immediately follows from our assumptions (comp. Theorem 5.2 in [MU4]) that the following is true.

**Theorem 1.1.** *The system $S^*$ is a hyperbolic conformal iterated function system.*

The limit set generated by the system $S^*$ is denoted by $J^*$. The following result (see Lemma 5.3 in [MU4]) allows us to reduce our geometric considerations to the limit set $S^*$ and we are able to apply the theory developed for infinite hyperbolic IFS.

**Lemma 1.2.** *The limit sets $J$ and $J^*$ of the systems $S$ and $S^*$ respectively differ only by a countable set: $J^* \subset J$ and $J \setminus J^*$ is countable.*

Let

$$S^*(\infty) = \bigcap_{F \in \mathcal{F}in} \overline{\bigcup_{a \in I_* \setminus F} \phi_a(X)},$$

where $\mathcal{F}in$ denotes the family all finite subsets of $I_*$. In [MU1], $S^*(\infty)$ is denoted by $X(\infty)$. The following proposition is an immediate consequence of the condition (4).

**Proposition 1.3.** *If the alphabet $I$ is finite, then*

$$S^*(\infty) = \{x_i : i \in \Omega\}, \text{ the set of parabolic fixed points.}$$

Following [MU1], given $t \geq 0$, a Borel probability measure $m$ is $t$-conformal for the system $S^*$ provided $m(J_{S^*}) = 1$ and for every Borel set $A \subset X$ and all $i, j \in I_*$ with $i \neq j$,

$$m(\phi_i(A)) = \int_A |\phi_i'|^t dm \tag{1.1}$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0. \tag{1.2}$$

For the system $S^*$, we define the functions

$$\psi(t) = \sum_{a \in I_*} ||\phi_a'||^t \text{ and } \psi_n(t) = \sum_{a \in I_*^n} ||\phi_a'||^t,$$

and $P^*$, the *topological pressure function* for the system $S^*$,

$$P^*(t) = \lim_{n \mapsto \infty} \frac{\log \psi_n(t)}{n}.$$

Finally, the *finiteness parameter* for the system $S^*$ is given by

$$\theta(S^*) = \inf\{t : \psi(t) < \infty\} = \inf\{t : P^*(t) < \infty\}.$$

The system $S^*$ is said to be *hereditarily regular* provided $\psi(\theta(S^*)) = \infty$ and *regular* provided there is some $t$ such that $P^*(t) = 0$. Of course, hreditarily regular systems are regular. Let

$$h = h_S = \dim_{\mathrm{H}}(J_S) = \dim_{\mathrm{H}}(J_{S^*})$$

be the Hausdorff dimension of the limit set $J_S$. It has been proven in [MU1] that $h = \inf\{t : P^*(t) \leq 0\}$ and if a hyperbolic IFS is regular, then an $h$-conformal measure exists and is unique. In Section 4 we shall prove the following

**Theorem 1.4.** *If $S$ is a finite parabolic IFS, then the system $S^*$ is hereditarily regular and, consequently, an $h$-conformal measure for $S^*$ exists.*

From now on, unless otherwise stated, we will assume that the alphabet $I$ is finite and $m$ will denote the $h$-conformal measure produced in Theorem 1.4.

Let $\mathcal{H}^t$ denote the $t$-dimensional Hausdorff measure and $\mathcal{P}^t$, the $t$-dimensional packing measure. We recall that the system $S$ satisfies the *strong open set condition* if $J_S \cap \mathrm{Int}X \neq \emptyset$.

Noting that in terminology of [MU1] each hereditarily regular IFS is regular, and combining Theorem 1.4, Corollary 4.7 in [MU4] and Corollary 5.10 in [MU4], we get the following.

**Theorem 1.5.** *If a finite parabolic IFS $S$ satisfies the strong open set condition, then $\mathcal{H}^t(J) < \infty$ and $\mathcal{P}^h(J) > 0$.*

Next we state the main theorem of our paper. It contains a complete description of the $h$-dimensional Hausdorff and packing measures of the limit set of a finite parabolic IFS.

**Theorem 1.6.** *Let $S$ be a finite parabolic IFS satisfying the strong open set condition. Then*

- **(a):** *If $h < 1$, then $0 < \mathcal{P}^h(J) < \infty$ and $\mathcal{H}^h(J) = 0$.*
- **(b):** *If $h = 1$, then $0 < \mathcal{H}^h(J) \le \mathcal{P}^h(J) < \infty$.*
- **(c):** *If $h > 1$, then $0 < \mathcal{H}^h(J) < \infty$ and $\mathcal{P}^h(J) = \infty$.*

This sort of theorem has appeared in several contexts, for Kleinian groups in [Su], in the context of parabolic rational functions in [DU], for rational functions with no recurrent critical points in the Julia set (abbreviated as NCP maps), in [U1] and for parabolic Cantor sets (which comprise 1-dimensional parabolic IFS) in [U3]. The idea behind the proofs here are different from those cited. It replies on developing, extending, simplifying and clarifying the approach which originated in [MU3], and employing the necessary and sufficient conditions for the Hausdorff and packing measures to be positive and finite, provided in [MU1] and [MU2]. We note that the inducing procedures proposed in [UZ] indicate that in the case of parabolic rational functions and perhaps even in the case of NCP maps, one can demonstrate appropriate versions of our main theorem as a corollary of Theorem 1.6. We shall also prove in the Section 4 the following.

**Theorem 1.7.** *If $S$ is a finite parabolic IFS, then*

$$\overline{\dim_{\mathrm{B}}(J)} = \dim_{\mathrm{H}}(J),$$

*where $\overline{\dim_{\mathrm{B}}(J)}$ denotes the upper ball-counting dimension, also called the box-counting dimension, Minkowski dimension or capacity.*

One more note for the reader. The dynamical properties of the parabolic IFS proven in Sections 2 and 3 and needed for the proofs of Theorem 1.6 and Theorem 1.7 are provided in the beginning of Section 4 in a unified fashion. Therefore, the reader only interested in Theorem 1.6 and Theorem 1.7 may actually read Section 4 independently of Section 2 and Section 3.

Section 2 mainly concerns the dynamical properties of a single parabolic conformal diffeomorphism in $\mathbb{R}^d$, $d \ge 3$ and can be viewed as an introduction to the technically more complicated Section 3 which deals with dynamical properties of a single simple parabolic holomorphic map in $\mathbb{R}^2$. Both sections provide a compact systematic description of the quantitative behaviour of parabolic maps needed for the proofs in Section 4. The qualitative behaviour of a single parabolic holomorphic map considerd in Section 3 is known as Fatou's flower theorem (see [Al] for additional historical information). Some quantitative results can be also found in these papers. At the end of Sections 2 and 3 some facts about parabolic iterated function systems are proven.

We end this section with two terminologies. Given two sets $A, B \subset I\!R^d$, we denote

$$\text{dist}(A,B) = \inf\{||a-b|| : (a,b) \in A \times B\} \text{ and } \text{Dist}(A,B) = \sup\{||a-b|| : (a,b) \in A \times B\}.$$

## 2. The case $d \geq 3$

As we mentioned in the introduction, it is known (see [BP] and [Ha]) that in every dimension $d \geq 3$ each $C^1$ conformal homeomorphism $A$ defined on an open connected subset of $I\!R^d$ extends to the entire space $I\!R^d$ and takes on the form

$$A = \eta D \circ i_{a,r} + b, \tag{2.1}$$

where $0 < \eta \in R$ is a positive scalar, $D$ is a linear isometry of $I\!R^d$, $i_{a,r}$ is either the inversion with respect to some sphere centered at a point $a$ and with radius $r$, or the identity map, and $b \in I\!R^d$. If $i_{a,r}$ is an inversion, then for every $z \in I\!R^d$

$$||A'(z)|| = \frac{\eta r^2}{||z-a||^2}.$$

**Definition 2.1.** *We say a conformal map $A : I\!R^d \to I\!R^d$ is parabolic provided it has a fixed point $\omega \in I\!R^d$ such that $||A'(\omega)|| = 1$ and there is a point $\xi \in I\!R^d \setminus \{\omega\}$ and $\lim_{n\to\infty} A^n\xi = \omega$.*

If $A$ is a conformal map and fixes $\omega$, then setting

$$\tilde{A} = i_{\omega,1}^{-1} \circ A \circ i_{\omega,1} = i_{\omega,1} \circ A \circ i_{\omega,1},$$

we have $\tilde{A}$ is conformal and $\tilde{A}(\infty) = \infty$. Therefore,

$$\tilde{A} = \lambda D + c,$$

where $\lambda > 0$, $D$ is an orthonormal matrix, and $c \in I\!R^d$. From now on, without loss of generality, we will assume that $\omega = 0$, *i.e.*, $\omega$ is the origin and we will write $i$ for $i_{0,1}$.

**Lemma 2.2.** *If $A : I\!R^d \to I\!R^d$ is a parabolic conformal map and if $\lambda$ is the scalar involved in the formula for $\tilde{A}$, then $\lambda = 1$.*

*Proof.* If $\lambda < 1$, then $\tilde{A} : I\!R^d \to I\!R^d$ is a strict contraction and due to Banach's contraction principle, it has a fixed point $b \in I\!R^d$ such that $\lim_{n\to\infty} \tilde{A}^n(z) = b$ for every $z \in I\!R^d$. However, this is a contradiction, since $\lim_{n\to\infty} \tilde{A}^n(i(\xi)) = \infty$. Thus, $\lambda \geq 1$. Assume $\lambda > 1$. Then for every $z \in I\!R^d \setminus \{0\}$,

$$\begin{aligned} ||A'(z)|| &= ||i'(\tilde{A}(i(z))\tilde{A}'(i(z))i'(z)|| = \lambda||\tilde{A}(i(z))||^{-2}||z||^{-2} = \lambda||z||^{-2}||\lambda D(||z||^{-2}(z)) + c||^{-2} \\ &= \lambda||(\lambda||z||^{-1}D(z) + c||z||)||^{-2} = \lambda^{-1}[||D(z/||z||) + (||z||/\lambda)c||]^{-2}. \end{aligned}$$

Since $\lim_{z\to 0} ||z|| = 0$ and since $||D(z)|| = ||z||$, we deduce that $||A'(0)|| = \lim_{z\to 0} ||A'(z)|| = \lambda^{-1} < 1$. This contradiction shows that $\leq 1$, and consequently $\lambda = 1$. The proof is complete. ■

Next, we want to estimate the rate at which $\tilde{A}^n(z)$ goes to $+\infty$.

**Lemma 2.3.** *If $A : I\!R^d \to I\!R^d$ is a parabolic conformal map, then there exists a non-zero vector $b \in I\!R^d$ and a positive constant $\kappa$ such that for every $z \in I\!R^d$ and every positive integer $n$*

$$||\tilde{A}^n z - nb|| \le ||z|| + \kappa.$$

*Proof.* By a straightforward induction, we get

$$\tilde{A}^n z = D^n z + \sum_{j=0}^{n-1} D^j(c).$$

Write $c = b + a$, where $b$ is a fixed point (*a priori* perhaps 0) of $D$ and $a$ belongs to $W$, the orthogonal complement of the vector space of the fixed points of $D$. Since $\lim_{n\to\infty} \tilde{A}^n(i(\xi)) = \infty$, $W$ is not the trivial subspace of $I\!R^d$. In addition, $D(W) = W$ and $D - \mathrm{Id} : W \to W$ is invertible. Since

$$(D - \mathrm{Id})\left(\sum_{j=0}^{n-1} D^j(a)\right) = D^n a - a$$

and since $||D^n a - a|| \le 2||a||$, we therefore conclude that for every $n \ge 1$

$$\left\|\sum_{j=0}^{n-1} D^j(a)\right\| \le 2||a|| \cdot \left\|(D - \mathrm{Id})|_W^{-1}\right\|.$$

Hence,

$$||\tilde{A}^n z - nb|| = \left\|D^n z + \sum_{j=0}^{n-1} D^j(a)\right\| \le ||z|| + 2\left\|(D - \mathrm{Id})|_W^{-1}\right\| \cdot ||a||.$$

Again, since $\lim_{n\to\infty} \tilde{A}^n(i(\xi)) = \infty$, we finally conclude that $b \ne 0$ and the proof is complete. ■

As an immediate consequence of this lemma we get the following.

**Corollary 2.4.** *Let $A : I\!R^d \to I\!R^d$ be a parabolic conformal map. For every compactum $F \subset I\!R^d$, there exists a constant $B_F \ge 1$ and integer $M_F \in I\!N$ such that for every $n \ge M_F$ and every $z \in F$*

$$B_F^{-1} n \le ||\tilde{A}^n z|| \le B_F n.$$

**Lemma 2.5.** *Let $A : I\!R^d \to I\!R^d$ be a parabolic conformal map. For every compactum $L \subset I\!R^d \setminus \{0\}$, there exist a constant $C_{L,1} \ge 1$ and integer $N_L \in I\!N$ such that for every $n \ge N_L$ and every $z \in L$*

$$C_{L,1}^{-1} n^{-2} \le ||(A^n)'(z)|| \le C_{L,1} n^{-2} \text{ and } \mathrm{diam}(A^n(L)) \le C_{L,1} n^{-2}$$

*Proof.* By the Chain Rule, we find for every $z \in I\!R^d \setminus \{0\}$

$$||(A^n)'(z)|| = ||i'(\tilde{A}^n(i(z)))|| \cdot ||(\tilde{A}^n)'(i(z))|| \cdot ||i'(z)|| = ||\tilde{A}^n(i(z))||^{-2} ||z||^{-2}.$$

For every $z \in L$, $\mathrm{Dist}^{-2}(0,L) \le ||z||^{-2} \le \mathrm{dist}^{-2}(0,L)$, and in view of Corollary 2.4, if $n \ge M_{i(L)}$, then $B_{i(L)}^{-1} n \le ||\tilde{A}^n z|| \le B_{i(L)} n$. Consequently, if $z \in L$ and $n \ge M_{i(L)}$, we have

$$\Big(B_{i(L)}\mathrm{Dist}(0,L)\Big)^{-2} n^{-2} \le ||(A^n)'(z)|| \le B_{i(L)}^2 \mathrm{dist}^{-2}(0,L) n^{-2}$$

and the proof is complete. ■

**Lemma 2.6.** *Let $A : I\!\!R^d \to I\!\!R^d$ be a parabolic conformal map. For every compactum $L \subset I\!\!R^d \setminus \{0\}$, there exists a constant $C_{L,2} \ge 1$ such that for all integers $k, n$ with $n \ge k \ge 1$,*

$$\mathrm{Dist}(A^k(L), A^n(L)) \le C_{L,2}\left|k^{-1} - (n+1)^{-1}\right|$$

*and*

$$\mathrm{Dist}(A^n(L), 0) \le C_{L,2} n^{-1}.$$

*Proof.* Let us start with the second inequality. If $n \ge M_{i(L)}$ and $z \in L$, then, by Corollary 2.4, we get $||A^n z|| = ||\tilde{A}^n(i(z))||^{-1} \le B_{i(L)} n^{-1}$ and the second inequality follows provided $C_{L,2}$ is sufficiently large.

Towards obtaining the first inequality, for every set $M \subset I\!\!R^d$, let $\mathrm{conv}(M)$ denote the convex hull of $M$. Obviously, $\mathrm{conv}(M) \subset B(M, \mathrm{diam}(M))$ and $\mathrm{diam}(\mathrm{conv}(M)) = \mathrm{diam}(M)$. By using Lemma 2.3, we have for every $u \in L$ and $n \in I\!\!N$,

$$\begin{aligned} ||\tilde{A}^{n+1}(i(u)) - \tilde{A}^n(i(u))|| &\le ||\tilde{A}^{n+1}(i(u)) - (n+1)b - (\tilde{A}^n(i(u)) - nb) + b|| \\ &\le 2(||i(u)|| + \kappa) + ||b|| \le 2(\mathrm{Dist}(0, i(L)) + \kappa) + ||b|| := M. \end{aligned}$$

Next, choose a positive integer $N_0$ such that $\mathrm{Dist}(0, \mathrm{conv}(\cup_{t \ge N_0}\tilde{A}^t(i(L)))) = H > 0$ and $N_0||b|| > \mathrm{Dist}(0, i(L)) + \kappa + ||b|| := M$. We claim there is a positive constant $C$ such that if $u, v \in L, k \ge N_0$ and $j \ge 0$, then

$$||A^{k+j+1}(v) - A^{k+j}(u)|| \le C\frac{1}{(k+j+1)^2}.$$

In order to see this, note that

$$\begin{aligned} &||A^{k+j+1}(v) - A^{k+j}(u)|| \le \\ &\quad \le ||i(\tilde{A}^{k+j+1}(i(v))) - i(\tilde{A}^{k+j+1}(i(u)))|| + ||i(\tilde{A}^{k+j+1}(i(u))) - i(\tilde{A}^{k+j}(i(u)))|| \\ &\quad \le \sup\{||i'(w)|| : w \in [\tilde{A}^{k+j+1}(i(v)), \tilde{A}^{k+j+1}(i(u))]\}||\tilde{A}^{k+j+1}i((v)) - \tilde{A}^{k+j+1}i((u))|| \\ &\qquad + \sup\{||i'(w)|| : w \in [\tilde{A}^{k+j}(i(u)), \tilde{A}^{k+j+1}(i(u))]\}||\tilde{A}^{k+j+1}i((u)) - \tilde{A}^{k+j}i((u))|| \\ &\quad \le \mathrm{diam}(i(L)) \sup\{||w||^{-2} : w \in [\tilde{A}^{k+j+1}(i(v)), \tilde{A}^{k+j+1}(i(u))]\} \\ &\qquad + M \sup\{||w||^{-2} : w \in [\tilde{A}^{k+j}(i(u)), \tilde{A}^{k+j+1}(i(u))]\} \end{aligned}$$

Now, if $w \in [\tilde{A}^{k+j+1}(i(v)), \tilde{A}^{k+j+1}(i(u))]$, then by Lemma 2.3, $||w-(k+j+1)b|| \le \mathrm{Dist}(0, i(L)) + \kappa$ and $||w|| \ge (k+j+1)[||b|| - (\mathrm{Dist}(0, i(L)) + \kappa)/N_0]$. Also, since $||\tilde{A}^{k+j}(i(u)) - (k+j+1)b|| \le ||i(u)|| + \kappa + ||b||$, if $w \in [\tilde{A}^{k+j}(i(u)), \tilde{A}^{k+j+1}(i(u))]$, then $||w - (k+j+1)b|| \le \mathrm{Dist}(0, i(L)) + \kappa + ||b||$ and $||w|| \ge (k+j+1)[||b|| - (\mathrm{Dist}(0, i(L)) + \kappa + ||b||)/N_0] \ge (k+j+1)[||b|| - M/N_0]$.

Combining these inequalities establishes our claim.

Therefore, if $N_0 \le k \le n$ we have

$$\begin{aligned}\mathrm{Dist}(A^k(L), A^n(L)) &\le \sum_{j=0}^{n-k-1} \mathrm{Dist}\Big(A^{k+j+1}(L), A^{k+j}(L)\Big) \\ &\le \sum_{j=0}^{n-k} C(k+j)^{-2} \le C_{L,2}(k^{-1} - (n+1)^{-1})\end{aligned}$$

for some constant $C_{L,2} \ge 1$. Clearly, increasing $C_{L,2}$ appropriately, we see that the last inequality is also true for all $1 \le k \le n$. The proof of the first part of our lemma is thus complete. ■

**Lemma 2.7.** *For every compactum $L \subset I\!\!R^d \setminus \{0\}$ there exist a constant $C_{L,3} \ge 1$ and an integer $q \ge 0$ such that for all $k \ge 1$ and all $n \ge k + q$*

$$\mathrm{dist}(A^k(L), A^n(L)) \ge C_{L,3}\Big(k^{-1} - n^{-1}\Big)$$

*and*

$$\mathrm{dist}(A^n(L), 0) \ge C_{L,3} n^{-1}.$$

*Proof.* First, notice that it follows from Lemma 2.3 that if $w, z \in i(L)$ and $k, n \in N$, then

$$(n-k)||b|| - 2(\mathrm{Dist}(0, i(L)) + \kappa) \le ||\tilde{A}^n(w) - \tilde{A}^k(z)||.$$

Therefore, there is a positive integer $q_0$ such that if $n - k \ge q_0$, then $||\tilde{A}^n(w) - \tilde{A}^k(z)|| \ge (1/2)||b||(n-k)$. Let $N_0$ be as in the proof of Lemma 2.6 and $M_{i(L)}$ be as in Corollary 2.4. Let $k, n \ge N_1 = \max\{N_0, M_{i(L)}\}$. Consider two arbitrary points $z, w \in i(L)$ and parametrize the line segment $\gamma$ joining $\tilde{A}^k(z)$ and $\tilde{A}^n(w)$ as

$$\gamma(t) = \tilde{A}^k(z) + t(\tilde{A}^n(w) - \tilde{A}^k(z)),\ \ t \in [0,1].$$

The curve $i(\gamma)$ is a subarc of either a cirlce or a line and let $l(i(\gamma))$ be its length. We have

$$\begin{aligned} l(i(\gamma)) &= \int_0^1 ||(i \circ \gamma)'(t)||dt = \int_0^1 ||i'(\gamma(t))||||\gamma'(t)||dt = ||\tilde{A}^n(w) - \tilde{A}^k(z)|| \int_0^1 ||\gamma(t)||^{-2} dt \\ &= ||\tilde{A}^n(w) - \tilde{A}^k(z)|| \int_0^1 ||\tilde{A}^k(z) + t(\tilde{A}^n(w) - \tilde{A}^k(z)||^{-2} dt \\ &\ge ||\tilde{A}^n(w) - \tilde{A}^k(z)|| \int_0^1 \Big(||\tilde{A}^k(z)|| + t||\tilde{A}^n(w) - \tilde{A}^k(z)||\Big)^{-2} dt \\ &= ||\tilde{A}^n(w) - \tilde{A}^k(z)|| \cdot ||\tilde{A}^n(w) - \tilde{A}^k(z)||^{-1} \int_{||\tilde{A}^k(z)||}^{||\tilde{A}^k(z)||+||\tilde{A}^n(w)-\tilde{A}^k(z)||} u^{-2} du \qquad (2.2) \\ &= ||\tilde{A}^k(z)||^{-1} - \Big(||\tilde{A}^k(z)|| + ||\tilde{A}^n(w) - \tilde{A}^k(z)||\Big) \\ &= \frac{||\tilde{A}^n(w) - \tilde{A}^k(z)||}{||\tilde{A}^k(z)|| \cdot \Big(||\tilde{A}^k(z)|| + ||\tilde{A}^n(w) - \tilde{A}^k(z)||\Big)} \end{aligned}$$

We have $||\tilde{A}^k(z)|| + ||\tilde{A}^n(w) - \tilde{A}^k(z)|| \leq B_{i(L)}k + C_{i(L),1}(1/k - 1/(n+1))$. So, there is a constant $U$ such that $||\tilde{A}^k(z)|| + ||\tilde{A}^n(w) - \tilde{A}^k(z)|| \leq Un$. In view of Corollary 2.4, there is a constant $Q_0$ such that

$$l(i(\gamma)) \geq Q_0 \frac{||\tilde{A}^n(w) - \tilde{A}^k(z)||}{kn}.$$

Thus, there is a constant $Q$ such that if $k \geq N_1$ and $n \geq k + q_0$, then

$$l(i(\gamma)) \geq Q(k^{-1} - n^{-1}). \tag{2.3}$$

If $i(\gamma)$ is a line segment, then

$$||A^n(i(w)) - A^k(i(z))|| = l(i(\gamma)) \geq Q(k^{-1} - n^{-1}). \tag{2.4}$$

If, however, $i(\gamma)$ is an arc of a circle, then consider the ray

$$g(t) = \tilde{A}^k(z) + t(\tilde{A}^n(w) - \tilde{A}^k(z)),\ t \in (-\infty, 0].$$

Proceeding exactly as in the formula (2.2) and using the estimate $||g(t)|| \leq ||\tilde{A}^k(z)|| - t||\tilde{A}^n(w) - \tilde{A}^k(z)||$, we get

$$l(i(g)) \geq \int_{||\tilde{A}^k(z)||}^{\infty} u^{-2} du = ||\tilde{A}^k(z)||^{-1}.$$

And applying Corollary 2.4 we get $l(i(g)) \geq B_{i(L)}^{-1}k^{-1} \geq B_{i(L)}(k^{-1} - n^{-1})$. Therefore, invoking (2.3), we deduce that both arcs joining the points $A^k(i(z))$ and $A^n(i(w))$ on the circle $i(\{\tilde{A}^k(z) + t(\tilde{A}^n(w) - \tilde{A}^k(z)) : t \in I\!R \cup \{\infty\}\})$ have the length $\geq \min\{B_{i(L)}, Q\}(k^{-1} - n^{-1})$. Thus, taking also in account (2.4), we see there is a constant $P_0$ such that if $k, n \geq N_1$ and $n - k \geq q_0$, then

$$\text{dist}(A^k(L), A^n(L)) \geq P_0(k^{-1} - n^{-1}).$$

Since 0 is not an element of $\cup_{j=1}^{N_1} A^j(L)$, and since it follows from Lemma 2.3 that $A^k(L) \to 0$ as $k \to \infty$, there is a constant $C_{L,3}$ such that the first part of the conclusion of the lemma holds. Applying the proven part of the lemma, we conclude that

$$\text{dist}(A^n(L), 0) = \lim_{k\to\infty} \text{dist}(A^n(L), A^k(L)) \geq \lim_{k\to\infty} C_{L,3}(n^{-1} - k^{-1}) = C_{L,3}n^{-1}.$$

The proof is complete. ■

We end this section we by proving the following two results concerning general parabolic IFS in dimension $d \geq 3$. The first is a straightforward consequence of Lemma 2.3.

First, let us note that Lemma 2.6 shows that a conformal parabolic map in $I\!R^d, d \geq 3$ has a unique fixed point.

**Proposition 2.8.** *If $\{\phi_i : X \to X\}_{i \in I}$ is an at least 3-dimensional parabolic conformal IFS (I is allowed to be infinite), then $x_i$, the only fixed point of a parabolic map $\phi_i$, belongs to $\partial X$.*

*Proof.* In view of Lemma 2.3, for every $R > 0$ large enough and every $n \geq 1$, the set $\tilde{\phi}_i(\{z : ||z|| > R\})$ is not contained in $\{z : ||z|| > R\}$. Consequently, for every neighbourhood $U$ of $x_i$, the set $\phi_i^n(U)$ does not converge to $x_i$. Since however $\lim_{n\to\infty} \phi_i^n(X) = x_i$, the point $x_i$ cannot belong to Int$X$. The proof is complete. ■

In [U2] we have demonstrated that in the case $d \geq 3$ the Bounded Distortion Property (1d) and the property (1e) are satisfied automatically. Because of the importance of these properties for our geometric considerations in Section 4 and for the sake of completeness, we present below their proof taken from [U2].

**Theorem 2.9.** *If $\{\phi_i\}_{i\in I}$ is a collection of maps satisfying condition (3), then the conditions (7) and (8) are also satisfied, perhaps with a smaller set $V$ and a sufficiently high iterate $S^q$ of $S$. The property (8) takes on the following stronger form*

$$\Big| ||\phi_\omega'(y) - ||\phi_\omega'(x)|| \Big| \leq K||\phi_\omega'||||y - x|| \tag{2.5}$$

*for all hyperbolic words $\omega \in I^*$, all $x, y \in V$ and some sufficiently large $K$.*

*Proof.* Let $U$ be an open neighbourhood of $X$ such that $\text{dist}(U, \partial V) > 0$ . Fix a hyperbolic word $\omega \in I^*$. In view of (2.1) there exist $\lambda_\omega > 0$, a linear isometry $A_\omega$, an inversion (or the identity map) $i_\omega = i_{a_\omega, r_\omega}$ and a vector $b_\omega \in I\!R^d$ such that $\phi_\omega = \lambda_\omega A_\omega \circ i_\omega + b_\omega$. In case when $i_\omega$ is the identity map the statement of our theorem is obvious. So, we may assume that $i_\omega$ is an inversion. Then for every $z \in I\!R^d$

$$||\phi_\omega'(z)|| = \frac{\lambda_\omega r_\omega^2}{||z - a_\omega||^2}.$$

Hence, for all $x, y \in I\!R^d$

$$\frac{||\phi_\omega'(y)||}{||\phi_\omega'(x)||} = \frac{||x - a_\omega||^2}{||y - a_\omega||^2}. \tag{2.6}$$

Since $\phi_\omega(V) \subset V$, $a_\omega \notin V$. Therefore, for all $x, y \in U$

$$\frac{||x - a_\omega||}{||y - a_\omega||} \leq \frac{||x - y|| + ||y - a_\omega||}{||y - a_\omega||} = 1 + \frac{||x - y||}{||y - a_\omega||} \leq 1 + \frac{\text{diam}(U)}{\text{dist}(U, \partial V)} \tag{2.7}$$

Thus,

$$\frac{||\phi_\omega'(y)||}{||\phi_\omega'(x)||} \leq \left(1 + \frac{\text{diam}(U)}{\text{dist}(U, \partial V)}\right)^2.$$

and condition (7) holds. In order to prove the second part we may assume without loosing generality that $||\phi'_\omega(x)|| \le ||\phi'_\omega(y)||$. Using (2.6) and (2.7), we then get

$$\begin{aligned}
\Big|||\phi'_\omega(y)|| - ||\phi'_\omega(x)||\Big| &\le ||\phi'_\omega|| \left(\frac{||\phi'_\omega(y)||}{||\phi'_\omega(x)||} - 1\right) = ||\phi'_\omega|| \left(\frac{||x-a_\omega||^2}{||y-a_\omega||^2} - 1\right) \\
&= ||\phi'_\omega|| \left(\frac{||x-a_\omega||}{||y-a_\omega||} - 1\right)\left(\frac{||x-a_\omega||}{||y-a_\omega||} + 1\right) \\
&\le ||\phi'_\omega|| \left(2 + \frac{\mathrm{diam}(U)}{\mathrm{dist}(U,\partial V)}\right) \frac{||x-y||}{||y-a_\omega||} \\
&\le ||\phi'_\omega|| \left(2 + \frac{\mathrm{diam}(U)}{\mathrm{dist}(U,\partial V)}\right) \frac{||x-y||}{\min\{||y-a_\omega||, ||x-a_\omega||\}} \\
&\le \left(2 + \frac{\mathrm{diam}(U)}{\mathrm{dist}(U,\partial V)}\right) \frac{1}{\mathrm{dist}(U,\partial V)} ||\phi'_\omega|| ||y-x||.
\end{aligned}$$

Now cover $X$ by finitely many balls with a positive distance to $\partial V$. Join them by smooth compact arcs contained in $V$ to obtain a connected set $M$ whose closure is contained in $V$. Form the new set $U$, an open connected neighbourhood of $X$ with a positive distance to the boundary of $V$, by adding to $M$ sufficiently small open neighbourhoods of these compact arcs. We may require these neighbourhoods to be topological closed balls (in $\mathbb{R}^d$) with smooth boundaries. Finally, the boundary of $U$ itself can be taken to be smooth and combining (3) and (4) along with proven distortion property we can easily deduce that $\phi_\omega(U) \subset U$ if only $|\omega|$ is large enough. The proof is complete. ■

## 3. **The plane case,** $d = 2$

We call a holomorphic map $\phi$, defined around a point $\omega \in \mathbb{C}$, simple parabolic if $\phi(\omega) = \omega$, $\phi'(\omega) = 1$ and $\phi$ is not the identity map. Then on a sufficiently small neighbourhood of $\omega$, the map $\phi$ has the following Taylor series expansion:

$$\phi(z) = z + a(z-\omega)^{p+1} + b(z-\omega)^{p+2} + \dots$$

with some integer $p \ge 1$ and $a \in \mathbb{C} \setminus \{0\}$. Being in the circle of ideas related to Fatou's flower theorem (see [Al] for extended historical information), we now want to analyze qualitatively and especially quantitatively the behaviour of $\phi$ in a sufficiently small neighbourhood of the parabolic point $\omega$. Let us recall that the rays coming out from $\omega$ and forming the set

$$\{z : (a(z-\omega)^p < 0\}$$

are called attracting directions and the rays forming the set

$$\{z : (a(z-\omega)^p > 0\}$$

are called repelling directions. Fix an attractive direction, say $A = \omega + \sqrt[p]{-a^{-1}(0,\infty)}$, where $\sqrt[p]{\cdot}$ is a holomorphic branch of the $p$th radical defined on $\mathbb{C} \setminus a^{-1}(0,\infty)$. In order to simplify our analysis let us change the system of coordinates with the help of the affine map $\rho(z) = \sqrt[p]{-a^{-1}} + \omega$. We then get

$$\phi_0(z) = \rho^{-1} \circ \phi \circ \rho(z) = z - z^{p+1} + b\sqrt[p]{-a^{-1}}z^{p+2} + \dots$$

and $\rho^{-1}(A) = (0, \infty)$ is an attractive direction for $\phi_0$. We want to analyze the behaviour of $\phi_0$ on an appropriate neighbourhoods of $(0, \epsilon)$, for $\epsilon > 0$ sufficiently small. In order to do it, similarly as in the previous section, we conjugate $\phi_0$ on $\mathbb{C} \setminus (-\infty, 0]$ to a map defined "near" infinity. Precisely, we consider $\sqrt[p]{\cdot}$, the holomorphic branch of the $p$th radical defined on $\mathbb{C} \setminus (-\infty, 0]$ and leaving the point 1 fixed. Then we define the map

$$H(z) = \frac{1}{\sqrt[p]{z}}$$

and consider the conjugate map

$$\tilde{\phi} = H^{-1} \circ \phi_0 \circ H.$$

Straighforward calculations show that

$$\tilde{\phi}(z) = z + 1 + O(|z|^{-\frac{1}{p}}) \tag{3.1}$$

and

$$\tilde{\phi}'(z) = 1 + O(|z|^{-\frac{p+1}{p}}). \tag{3.2}$$

Given now a point $x \in (0, \infty)$ and $\alpha \in (0, \pi)$, let

$$S(x, \alpha) = \{z : -\alpha < \arg(z - x) < \alpha\}.$$

The formula (3.1) shows that for every $\alpha \in (0, \pi)$ there exists $x(\alpha) \in (0, \infty)$ such that for every $x \geq x(\alpha)$

$$\overline{\tilde{\phi}(S(x, \alpha))} \subset S\left(x + \frac{1}{2}, \alpha\right), \tag{3.3}$$

$$|z| \geq B^p \tag{3.4}$$

and

$$\mathrm{Re}(\tilde{\phi}(z)) \geq \mathrm{Re}(z) + \frac{1}{2} \tag{3.5}$$

for all $z \in (S(x, \alpha)$, where $B$ is the constant responsible for $O(|z|^{-\frac{1}{p}})$ in (3.1). The following lemma immediately follows from (3.4), (3.1) and (3.5) by a straightforward induction.

**Lemma 3.1.** *For every compactum $F \subset S(x(\alpha), \alpha)$ there exists a constant $C_F \geq 1$ such that for every $z \in F$ and every $n \geq 1$*

$$C_F^{-1} n \leq |\tilde{\phi}^n(z)| \leq C_F n.$$

Using a straightforward induction, one gets from (3.1) and Lemma 3.1 that

$$\tilde{\phi}^n(z) = z + n + O\Big(\max\{n^{1-\frac{1}{p}}, \log n\}\Big) \tag{3.6}$$

and

$$\tilde{\phi}^n(z) = \tilde{\phi}^k(z) + (n-k) + O\Big(|n^{1-\frac{1}{p}} - k^{1-\frac{1}{p}}|\Big), \tag{3.7}$$

where the constant involved in $''O''$ depends only on $F$ and $\phi_0$. Using Lemma 3.1 and (3.2) we shall prove the following.

**Lemma 3.2.** *For every compactum $F \subset S(x(\alpha), \alpha)$ there exists a constant $D_F \geq 1$ such that for every $z \in F$ and every $n \geq 1$*

$$D_F^{-1} \leq |(\tilde{\phi}^n)'(z)| \leq D_F.$$

*Proof.* For every $z \in S(x(\alpha), \alpha)$ let $g(z) = \tilde{\phi}'(z) - 1$. By the Chain Rule, we have for every $z \in S(x(\alpha), \alpha)$ and every $n \geq 1$

$$(\tilde{\phi}^n)'(z) = \prod_{j=0}^{n-1} \phi'(\tilde{\phi}^j(z)) = \tilde{\phi}'(z) \prod_{j=1}^{n-1} \Big(1 + g(\tilde{\phi}^j(z))\Big).$$

Using (3.2) and and the right-hand side of of Lemma 3.1, we get for every $z \in F$ and every $j \geq 1$ that

$$|g(\tilde{\phi}^j(z))| = O\Big(|\tilde{\phi}^j(z)|^{-\frac{p+1}{p}}\Big) \leq C_F^{-\frac{p+1}{p}} O\Big(j^{-\frac{p+1}{p}}\Big).$$

Since the series $\sum_{j=1}^{\infty} j^{-\frac{p+1}{p}}$ converges, the proof is complete. ■

For every $x \in (0, \infty)$ and $\alpha \in (0, \pi)$ let

$$S_0(x, \alpha) = H(S(x, \alpha))$$

and

$$S_\phi^A(x, \alpha) = \rho \circ H(S(x, \alpha)) = \rho(S_0(x, \alpha)).$$

The regions $S_0(x, \alpha)$ and $S_\phi^A(x, \alpha)$ look like flower petals containning symmetrically a part of the ray $(0, \infty)$ and the ray $A = \omega + \sqrt[p]{-a^{-1}(0, \infty)}$ respectively and form with these rays two "angles" of measures $\alpha/\pi$ at the point 0 and $\omega$ respectively. We recall from the previous section that $\mathrm{conv}(M)$ denotes the convex hull of the set $M$. Combining Lemma 3.1 and Lemma 3.2 we deduce the following.

**Lemma 3.3.** *For every $\alpha \in (0, \pi/2)$ and for every compactum $F \subset S(x(\alpha), \alpha)$ there exists a constant $C_F \geq 1$ such that for every $n \geq 1$*

$$C_F^{-1} n \leq \mathrm{dist}(0, \mathrm{conv}(\tilde{\phi}^n(F))) \leq \mathrm{Dist}(0, \mathrm{conv}(\tilde{\phi}^n(F))) \leq C_F n.$$

Let us now use the properties of the map $\tilde{\phi}$ and establish useful facts about the map $\phi$.

**Lemma 3.4.** *For every compactum $L \subset S_\phi^A(x,\alpha)$ there exists a constant $C_L \ge 1$ such that for every $z \in L$ and every $n \ge 1$*

$$C_L^{-1} n^{-\frac{p+1}{p}} \le |(\phi^n)'(z)|, \ \operatorname{diam}(\phi_n(L)) \le C_L n^{-\frac{p+1}{p}}.$$

*Proof.* It of course suffices to prove this lemma for $\phi$ replaced by $\phi_0$. Since $H^{-1}(L)$ is a compact subset of $S(x(\alpha),\alpha)$ and since $H'(z) = -\frac{1}{p} z^{-\frac{p+1}{p}}$, using the Chain Rule along with Lemma 3.1, Lemma 3.2, and (3.4), we deduce that for every $z \in L$ and every $n \ge 1$

$$\begin{aligned}
|(\phi_0^n)'(z)| &= |(H \circ \tilde\phi^n \circ H^{-1})'(z)| = |H'(\tilde\phi^n(H^{-1}(z))| \cdot |(\tilde\phi^n)'(H^{-1}(z)| \cdot |(H^{-1})'(z)| \\
&= \frac{1}{p} |\tilde\phi^n(H^{-1}(z))|^{-\frac{p+1}{p}} |(\tilde\phi^n)'(H^{-1}(z))| p |z|^{-(p+1)} \\
&\le D_{H^{-1}(L)}^{\frac{p+1}{p}} C_{H^{-1}(L)} (\operatorname{dist}(0, H^{-1}(L)))^{-(p+1)} n^{-\frac{p+1}{p}}
\end{aligned}$$

and

$$|(\phi_0^n)'(z)| \le D_{H^{-1}(L)}^{-\frac{p+1}{p}} C_{H^{-1}(L)}^{-1} \operatorname{Dist}(0, H^{-1}(L))^{-(p+1)} n^{-\frac{p+1}{p}}.$$

The proof is complete. ■

**Lemma 3.5.** *For every compactum $L \subset S_\phi^A(x,\alpha)$ there exists a constant $C_{L,1} \ge 1$ such that for all $k, n \ge 1$*

$$\operatorname{Dist}(\phi^k(L), \phi^n(L)) \le C_{L,1} \left| \min(k,n)^{-\frac{1}{p}} - (\max(k,n)+1)^{-\frac{1}{p}} \right|$$

*and*

$$\operatorname{Dist}(\phi^n(L), \omega) \le C_{L,1} n^{-\frac{1}{p}}.$$

*Proof.* It suffices again to prove this lemma for $\phi$ replaced by $\phi_0$. Let us prove the first inequality. Without loss of generality we may assume that $n \ge k$. Since $H^{-1}(L)$ and $\operatorname{conv}(H^{-1}(L))$ are compact subsets of $S(x(\alpha),\alpha)$, using (3.1), Lemma 3.3, Lemma 3.1, and

Lemma 3.2, we can estimate for every $j \geq 0$ and all $z, \xi \in L$ as follows

$$
\begin{aligned}
|\phi_0^{k+j+1}(\xi) - \phi_0^{k+j}(z)| &\leq |\phi_0^{k+j+1}(\xi) - |\phi_0^{k+j+1}(z)| + |\phi_0^{k+j+1}(z) - \phi_0^{k+j}(z)| \leq \\
&\leq \sup\{|H'(w)| : w \in \mathrm{conv}\Big(\tilde{\phi}^{k+j+1}(H^{-1}(L))\Big)\}\mathrm{diam}\Big(\mathrm{conv}\Big(\tilde{\phi}^{k+j+1}(H^{-1}(L))\Big)\Big) + \\
&+ \Big(1 + B|\tilde{\phi}^{k+j}(H^{-1}(z))|^{-\frac{1}{p}}\Big) \sup\{|H'(w)| : w \in [\tilde{\phi}^{k+j}(H^{-1}(z)), (\tilde{\phi}^{k+j+1}(H^{-1}(z))]\} \\
&\leq \frac{1}{p}\sup\{|w|^{-\frac{p+1}{p}} : w \in \mathrm{conv}\Big(\tilde{\phi}^{k+j+1}(H^{-1}(L))\Big)\}\mathrm{diam}(\tilde{\phi}^{k+j+1}(H^{-1}(L)) + \\
&+ \frac{2}{p}\sup\{|w|^{-\frac{p+1}{p}} : w \in [\tilde{\phi}^{k+j}(H^{-1}(z)), (\tilde{\phi}^{k+j+1}(H^{-1}(z))]\} \\
&\leq \frac{1}{p} D_{H^{-1}(L)} C_{H^{-1}(L)} \mathrm{diam}(H^{-1}(L))(k+j+1)^{-\frac{p+1}{p}} + \\
&+ \frac{2}{p}\Big(|\tilde{\phi}^{k+j+1}(H^{-1}(z))| - |\tilde{\phi}^{k+j}(H^{-1}(z))|\Big)^{-\frac{p+1}{p}} \\
&\leq \frac{1}{p} D_{H^{-1}(L)} C_{H^{-1}(L)} \mathrm{diam}(H^{-1}(L))(k+j+1)^{-\frac{p+1}{p}} + \\
&+ \frac{2}{p}\left(C_{H^{-1}(L)}(k+j+1) - B\Big(|\tilde{\phi}^{k+j}(H^{-1}(z))|^{-\frac{1}{p}} + 1\Big)\right)^{-\frac{p+1}{p}} \\
&\leq \frac{1}{p} D_{H^{-1}(L)} C_{H^{-1}(L)} \mathrm{diam}(H^{-1}(L))(k+j+1)^{-\frac{p+1}{p}} + \\
&+ \frac{2}{p}\left(C_{H^{-1}(L)}(k+j+1) - B\Big(C_{H^{-1}(L)}^{\frac{1}{p}}(k+j)^{-\frac{1}{p}} + 1\Big)\right) \\
&\leq \frac{1}{p} D_{H^{-1}(L)} C_{H^{-1}(L)} \mathrm{diam}(H^{-1}(L))(k+j+1)^{-\frac{p+1}{p}} + \frac{4}{p} C_{H^{-1}(L)}^{\frac{p+1}{p}}(k+j+1)^{-\frac{p+1}{p}} \\
&= \frac{1}{p}\Big(D_{H^{-1}(L)} C_{H^{-1}(L)} \mathrm{diam}(H^{-1}(L)) + 4C_{H^{-1}(L)}^{\frac{p+1}{p}}\Big)(k+j+1)^{-\frac{p+1}{p}}
\end{aligned}
$$

where the last inequality has been written assuming that $k \geq 1$ is large enough, say $k \geq q$ and $B$ is the constant coming from (3.1). Denote the constant appearing in the last row of the above formula by $C_L'$. Using also Lemma 3.4 we then get

$$
\begin{aligned}
\mathrm{Dist}(\phi_0^k(L), \phi_0^n(L)) &\leq \sum_{j=0}^{n-k-1} \mathrm{Dist}(\phi_0^{k+j}(L), \phi_0^{k+j+1}(L)) + \sum_{j=0}^{n-k} \mathrm{diam}(\phi_0^{k+j}(L)) \\
&\leq \sum_{j=0}^{n-k} C_L'(k+j)^{-\frac{p+1}{p}} = C_{L,1}(k^{-\frac{1}{p}} - (n+1)^{-\frac{1}{p}})
\end{aligned}
$$

for some constant $C_{L,1} \geq 1$. Clearly, increasing the constant $C_{L,1}$ appropriately, we see that the last inequality is also true for all $1 \leq k \leq q$. The proof of the first part of Lemma 3.6 is thus complete. The second part is a straightforward consequence of the first one. Indeed, it follows from (3.3) that $\phi^k(L)$ converges to $\omega$ if $k \to \infty$. Hence, applying the first part of the lemma, we get

$$\mathrm{Dist}(\phi^n(L),\omega)=\lim_{k\to\infty}\mathrm{Dist}(\phi^n(L),\phi^k(L))\le\lim_{k\to\infty}C_{L,1}(n^{-\frac{1}{p}}-(k+1)^{-\frac{1}{p}})=C_{L,1}n^{-\frac{1}{p}}.$$

The proof is complete. ■

**Lemma 3.6.** *For every compactum $L\subset S_\phi^A(x,\alpha)$ there exist a constant $C_{L,2}\le 1$ and an integer $q\ge 0$ such that for all $k\ge 1$ and $n\ge k+q$,*

$$\mathrm{dist}(\phi^k(L),\phi^n(L))\ge C_{L,2}|n^{-\frac{1}{p}}-k^{-\frac{1}{p}}|$$

*and*

$$\mathrm{dist}(\phi^n(L),\omega)\ge C_{L,2}n^{-\frac{1}{p}}.$$

*Proof.* It suffices of course to prove this lemma with $\phi$ replaced by $\phi_0$. Consider two arbitrary points $z,\xi\in H^{-1}(L)$ and the line segment $\gamma$ joining $\tilde\phi^k(z)$ and $\tilde\phi^n(\xi)$. Parametrize it as

$$\gamma(t)=\tilde\phi^k(z)+t(\tilde\phi^n(\xi)-\tilde\phi^k(z)),\ t\in[0,1].$$

Let $l(H(\gamma))$ be the length of the curve (a subarc of either a circle or a line) $H(\gamma)$. We have

$$\begin{aligned}
l(H(\gamma))&=\int_0^1|(H\circ\gamma)'(t)|dt=\int_0^1|H'(\gamma(t))||\gamma'(t)|dt\\
&=|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\int_0^1|H'(\gamma(t))|dt=\frac{1}{p}|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\int_0^1|\gamma(t)|^{-\frac{p+1}{p}}dt\\
&=\frac{1}{p}|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\int_0^1\Big(\tilde\phi^k(z)+t(\tilde\phi^n(\xi)-\tilde\phi^k(z))\Big)^{-\frac{p+1}{p}}dt\\
&\ge\frac{1}{p}|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\int_0^1\Big(|\tilde\phi^k(z)|+t|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\Big)^{-\frac{p+1}{p}}dt\\
&=\frac{1}{p}\int_{|\tilde\phi^k(z)|}^{|\tilde\phi^k(z)|+|\tilde\phi^n(\xi)-\tilde\phi^k(z)|}u^{-\frac{p+1}{p}}du=\Big(|\tilde\phi^k(z)|^{-\frac{1}{p}}-\Big(|\tilde\phi^k(z)|+|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\Big)^{-\frac{1}{p}}\Big) \qquad (3.8)\\
&=\frac{\Big(|\tilde\phi^k(z)|+|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\Big)^{\frac{1}{p}}-|\tilde\phi^k(z)|^{\frac{1}{p}}}{|\tilde\phi^k(z)|^{\frac{1}{p}}\Big(|\tilde\phi^k(z)|+|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\Big)^{\frac{1}{p}}}\\
&\ge C_{H^{-1}(L)}^{-\frac{1}{p}}(3C_{H^{-1}(L)})^{-\frac{1}{p}}\frac{\Big(|\tilde\phi^k(z)|+|\tilde\phi^n(\xi)-\tilde\phi^k(z)|\Big)^{\frac{1}{p}}-|\tilde\phi^k(z)|^{\frac{1}{p}}}{k^{\frac{1}{p}}n^{\frac{1}{p}}},
\end{aligned}$$

where the last inequality has been written due to Lemma 3.1. By the Mean Value Theorem there exists $\eta \in [|\tilde\phi^k(z)|, |\tilde\phi^k(z)| + |\tilde\phi^n(\xi) - \tilde\phi^k(z)|]$ such that

$$\begin{aligned}\Big(|\tilde\phi^k(z)| + |\tilde\phi^n(\xi) - \tilde\phi^k(z)|\Big)^{\frac{1}{p}} - |\tilde\phi^k(z)|^{\frac{1}{p}} = \\ = \frac{1}{p}|\tilde\phi^n(\xi) - \tilde\phi^k(z)|\eta^{\frac{1-p}{p}} \geq \frac{1}{p}|\tilde\phi^n(\xi) - \tilde\phi^k(z)|\Big(|\tilde\phi^k(z)| + |\tilde\phi^n(\xi) - \tilde\phi^k(z)|\Big)^{\frac{1-p}{p}} \\ \geq \frac{1}{p}(3C_{H^{-1}(L)})^{\frac{1-p}{p}}|\tilde\phi^n(\xi) - \tilde\phi^k(z)|\end{aligned} \tag{3.9}$$

Now, in view of (3.6), $\tilde\phi^n(\xi) - \tilde\phi^k(z) = \xi - z + O(\max\{n^{1-\frac{1}{p}}, \log n\})$. Hence

$$|\tilde\phi^n(\xi) - \tilde\phi^k(z)| \geq \operatorname{diam}(H^{-1}(L)) + (n-k) - O\Big(\max\{n^{1-\frac{1}{p}}, \log n\}\Big) \geq \frac{1}{2}(n-k)$$

if only $n-k$ is large enough, say $n-k \geq q$. Using this, (3.8) and (3.9), if $n \geq k+q$, then

$$l(H(\gamma)) \geq \frac{1}{2p}(3C_{H^{-1}(L)})^{\frac{1-p}{p}}\frac{(n-k)n^{1-\frac{1}{p}}}{k^{\frac{1}{p}}n^{\frac{1}{p}}} \tag{3.10}$$

Since $t \leq t^{\frac{1}{p}}$ for $t \in [0,1]$, we get $1-t \geq 1-t^{\frac{1}{p}}$ for these $t$, and consequently $1-\frac{k}{n} \geq 1-\left(\frac{k}{n}\right)^{\frac{1}{p}}$ or $\frac{n-k}{n} \geq 1-\left(\frac{k}{n}\right)^{\frac{1}{p}}$. Multiplying this last inequality by $n^{\frac{1}{p}}$, we get $(n-k)n^{\frac{1-p}{p}} \geq n^{\frac{1}{p}} - k^{\frac{1}{p}}$. Combining this and (3.10) yields

$$l(H(\gamma)) \geq \frac{1}{2p}(3C_{H^{-1}(L)})^{\frac{1-p}{p}}\Big(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}\Big). \tag{3.11}$$

If $H(\gamma)$ is a segment of the line, then

$$|\phi_0^k(H(z)) - \phi_0^n(H(\xi))| = l(H(\gamma)) \geq \frac{1}{2p}(3C_{H^{-1}(L)})^{\frac{1-p}{p}}\Big(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}\Big). \tag{3.12}$$

If however $H(\gamma)$ is an arc of a circle, then consider the curve

$$g(t) = \tilde\phi^k(z) + t(\tilde\phi^n(\xi) - \tilde\phi^k(z)),\ t \in (-\infty, 0].$$

Proceeding exactly as in the formula (3.8) with the estimate $|g(t)| \leq |\tilde\phi^k(z)| - t|(\tilde\phi^n(\xi) - \tilde\phi^k(z)|$, we get

$$l(H(\gamma)) \geq \frac{1}{p}\int_{|\tilde\phi^k(z)|}^{\infty} u^{-\frac{p+1}{p}}du = |\tilde\phi^k(z)|^{-\frac{1}{p}}.$$

Applying now Lemma 3.1 this gives

$$l(H(\gamma)) \geq (C_{H^{-1}(L)})^{-\frac{1}{p}}k^{-\frac{1}{p}} \geq (C_{H^{-1}(L)})^{-\frac{1}{p}}\Big(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}\Big).$$

Therefore, invoking (3.11), we deduce that both arcs joining the points $\phi_0^k(H(z))$ and $\phi_0^n(H(z))$ on the circle $H(\{\tilde\phi^k(z) + t(\tilde\phi^n(\xi) - \tilde\phi^k(z)) : t \in I\!R \cup \{\infty\})$ have the length $\geq C(\Big(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}\Big)$, where $C = \min\left\{\frac{1}{2p}(3C_{H^{-1}(L)})^{\frac{1-p}{p}}, C_{H^{-1}(L)})^{-\frac{1}{p}}\right\}$. Hence $|\phi_0^k(H(z)) - \phi_0^n(H(\xi))| \geq \frac{C}{\pi}\Big(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}\Big)$. This and (3.12) imply that

$$\mathrm{dist}\Big(\phi_0^k(H(z)), \phi_0^n(H(\xi))\Big) \geq \frac{C}{\pi}\Big(k^{-\frac{1}{p}} - n^{-\frac{1}{p}}\Big)$$

and the proof of the first part of our lemma is complete. Since it follows from (3.3) that $\phi^k(L)$ converges to $\omega$ if $k \to \infty$, applying the proven part of the lemma, we conclude that

$$\mathrm{dist}(\phi^n(L), \omega) = \lim_{k\to\infty} \mathrm{dist}(\phi^n(L), \phi^k(L)) \geq \lim_{k\to\infty} C_{L,2}\Big(n^{-\frac{1}{p}} - k^{-\frac{1}{p}}\Big) = C_{L,2} n^{-\frac{1}{p}}.$$

The proof is complete. ■

**Remark 3.7.** *We would like to remark that all statements proven in this section about the map $\phi$ continue to be true if we replace the assumption $L \subset S_\phi^A(x(\alpha), \alpha)$ by the assumption $\phi^j(L) \subset S_\phi^A(x(\alpha), \alpha)$ for some $j \geq 0$.*

**Lemma 3.8.** *If $L \subset \mathbb{C} \setminus \omega$ is a compactum and $\lim_{n\to\infty} \phi^n(L) = \omega$, then there exists an attracting direction $A$ such that for every $\alpha \in (0, \pi)$, $\phi^n(L) \subset S_\phi^A(x(\alpha), \alpha)$ for every $n \geq 0$ large enough.*

*Proof.* First notice that due to (3.3), if $\phi^k(L) \subset S_\phi^A(x(\alpha), \alpha)$, then $\phi^n(L) \subset S_\phi^A(x(\alpha), \alpha)$ for all $n \geq k$. Suppose now that the statement converse than that claimed in our lemma is true. Since the set of attracting directions is finite, there thus exist $\beta \in (0, \pi)$ and such that for every $n \geq k$

$$\phi^n(L) \cap \bigcup_{i=1}^p S_\phi^{A_i^+}(x(\beta), \beta) = \emptyset, \tag{3.13}$$

where $\{A_1^+, A_2^+, \ldots, A_p^+\}$ is the set of all attracting directions for $\phi$ at $\omega$. Taking now $\gamma \in (\pi - \beta, \pi)$ we see that the union

$$\bigcup_{i=1}^p S_\phi^{A_i^+}(x(\beta), \beta) \cup \bigcup_{i=1}^p S_{\phi^{-1}}^{A_i^-}(x(\gamma), \gamma)$$

($A_i^-$ being attracting directions for $\phi^{-1}$) forms a deleted neighbourhood of $\omega$. Along with (3.13) this implies that $\phi^n(L) \subset S_{\phi^{-1}}^{A_i^-}(x(\gamma), \gamma)$ for some $i \in \{1, 2, \ldots p\}$ and all $n \geq k$. But since, by (3.3), $\lim_{n\to\infty} \phi^{-n}(S_{\phi^{-1}}^{A_i^-}(x(\gamma), \gamma)) = \omega$, we conclude that $L = \lim_{n\to\infty} \phi^{-n}(\phi^(L)) = \omega$. This contradiction finishes the proof. ■

We end this section with a result concerning parabolic IFS in dimension $d = 2$

**Proposition 3.9.** *If $S = \{\phi_i : X \to X\}_{i\in I}$ is a parabolic IFS and $d = 2$, then the fixed point of each parabolic element $\phi_i$ belongs to the boundary of $X$. In addition, the derivative of each parabolic element evaulated at the corresponding parabolic fixed point is a root of unity.*

*Proof.* Suppose that $i \in I$ is a parabolic index and that the corresponding fixed point $x_i$ is in Int$X$. Let $C_i$ be the component of $\mathrm{Int}(X)$ containing $x_i$. So, $C_i$ is an open connected subset of $\mathbb{C}$ missing at least three points, since $X$ is a compact subset of $\mathbb{C}$. Therefore, due to the uniformization theorem, there exists a holomorphic covering map $R : D \to C_i$ sending 0 to $x_i$, where $D = \{z \in \mathbb{C}: |z| < 1\}$ is the open unit disk in $\mathbb{C}$. Since $\phi_i(x_i) = x_i$, $\phi_i(C_i) \subset C_i$. Considering, if necessary, the second iterate of $\phi_i$ we may assume that $\phi_i$ is holomorphic. Hence, all its lifts to $D$ (*i.e.*, satisfying the equality $\phi_i \circ R = R \circ \psi$) are holomorphic. Take $\psi : D \to D$, the lift fixing the point 0. Then $\psi'(0) = \phi_i'(x_i)$, whence $|\psi'(0)| = 1$. Therefore, in view of Schwarz's lemma, $\psi : D \to D$ is a rotation with the center at 0. In particular

$$\phi_i(C_i) = \phi_i \circ R(D) = R \circ \psi(D) = R(D) = C_i.$$

This contradicts condition (4) from Section 1. Finally, suppose $i$ is a parabolic index. If $\phi_i'(x_i)$ were not a root of unity, then the images of finitely many iterates of $\phi_i$ of an open cone witnessing the cone condition at $x_i$ would cover a punctured neighborhood of $X$. This contradicts the fact the the boundary of $X$ has no isolated points. ■

## 4. **Proofs of the main theorems**

In order to be apply the results of sections 2 and 3 we need the following. Recall for each parabolic index $i$, $x_i$ is the unique fixed point of the map $\phi_i$.

**Proposition 4.1.** *If $\{\phi_i : X \to X\}_{i\in I}$ is a parabolic IFS (I is allowed to be infinite), then for every parabolic index $i \in I$ and every $j \in I \setminus \{i\}$, we have $x_i \notin \phi_j(X)$.*

*Proof.* Suppose on the contrary that $x_i \in \phi_j(X)$ for some parabolic index $i \in I$ and some $j \in I \setminus \{i\}$. Then by the Cone Condition and conformality of $\phi_j$, the set $\phi_j(X)$ contains a central cone with positive measure and vertex $x_i$. On the other hand, since $\phi_i$ is conformal, $X \setminus \phi_i(X)$ contains no central cone with positive measure and vertex $x_i$. This is a contradiction since, by the Open Set Condition, $\mathrm{Int}(\phi_i(X)) \cap \mathrm{Int}(\phi_j(X)) = \emptyset$. The proof is complete. ■

Consider a parabolic IFS, $S = \{\phi_i : X \to X\}_{i\in I}$. If $S$ is 2-dimensional, then dealing with the family of second iterates $S^2 = \{\phi_{ij} : i, j \in I\}$, instead of $S$, we may assume that all the parabolic maps are holomorphic. Also, from Proposition 3.9 the derivative of each parabolic element evaluated at the corresponding parabolic fixed point, is a root of unity. Therefore, for some appropriate positive integer $q$, the derivative of each parabolic element of $S^q$ evaluated at the corresponding parabolic fixed point is equal to 1. Thus, without loss of generality, we may assume that in case $d = 2$, all the parabolic elements of $S$ are simple parabolic mappings in the sense of Section 3. Grouping now together the results of sections 2 and 3, we deduce that for any given $d \geq 2$, there exists a constant $Q \geq 1$ and an integer $q \geq 0$ such that for every parabolic index $i \in I$ there exists an integer $p_i \geq 1$ such that for every $j \in I \setminus \{i\}$ and all $n, k \geq 1$ we have

$$Q^{-1} n^{-\frac{p_i+1}{p_i}} \leq \inf_X\{||\phi_{i^n j}'(x)||\}, ||\phi_{i^n j}'||, \mathrm{diam}(\phi_{i^n j}(X)) \leq Q n^{-\frac{p_i+1}{p_i}}, \tag{4.1}$$

$$Q^{-1}n^{-\frac{1}{p_i}} \le \mathrm{dist}(x_i, \phi_{i^n j}(X)) \le \mathrm{Dist}(x_i, \phi_{i^n j}(X)) \le Qn^{-\frac{1}{p_i}}, \tag{4.2}$$

$$\mathrm{Dist}(\phi_{i^n j}(X), \phi_{i^k j}(X)) \le Q\left|\min\{k,n\}^{-\frac{1}{p_i}} - (\max\{k,n\}+1)^{-\frac{1}{p_i}}\right| \tag{4.3}$$

and, furthermore, if $|n-k| \ge q$, then

$$\mathrm{dist}(\phi_{i^n j}(X), \phi_{i^k j}(X)) \ge Q|n^{-\frac{1}{p_i}} - k^{-\frac{1}{p_i}}|. \tag{4.4}$$

We also need the following.

**Theorem 4.2.** *If $\{\phi_i : X \to X\}_{i\in I}$ is a parabolic IFS (I is allowed to be infinite), then*

$$dim_H(J_S) > \max\left\{\frac{p_i}{p_i+1} : i \text{ is parabolic}\right\},$$

*where $p_i$ is the integer indicated in (4.4).*

*Proof.* Using (4.1), if we take $t$ slightly larger than $\frac{p_i}{p_i+1}$, then $\psi(t)$ can be made as large as we like. Since $P^*(t) \ge -t\log K + \log\psi(t)$, $P^*(t) > 0$. Therefore, $h = \dim_{\mathrm{H}}(J_{S^*}) > \frac{p_i}{p_i+1}$. It therefore immediately follows from Lemma 1.2 that

$$\dim_{\mathrm{H}}(J_S) = \dim_{\mathrm{H}}(J_{S^*}) > \max\left\{\frac{p_i}{p_i+1} : i \text{ is parabolic}\right\}.$$

The proof is complete. ■

If, in addition $S$ is finite, then we conclude from (4.1) that

$$\theta_{S^*} = \max\left\{\frac{p_i}{p_i+1} : i \text{ is parabolic}\right\}$$

and $\psi(\theta_{S^*}) = \infty$. This means that the system $S^*$ is hereditarily regular and we have proved Theorem 1.4.

**Lemma 4.3.** *For every parabolic index $i \in I$, there exists an open cone $C_i \subset X$ with vertex $x_i$ and such that $x_i \in \overline{J \cap C_i}$.*

*Proof.* In case $d \ge 3$ this is an immediate consequence of Lemma 2.3. In case $d \ge 3$ this is an immediate consequence of (3.6) and Lemma 3.8. ■

In view of Theorem 1.5 in order to prove Theorem 1.6 it suffices to demonstrate the following four lemmas assuming the finite parabolic system $S$ satisfies the strong open set condition.

**Lemma 4.4.** *If $h < 1$, then $\mathcal{H}^h(J) = 0$.*

**Lemma 4.5.** *If $h \le 1$, then $\mathcal{P}^h(J) < \infty$.*

**Lemma 4.6.** *If $h > 1$, then $\mathcal{P}^h(J) = \infty$.*

**Lemma 4.7.** *If $h \ge 1$, then $\mathcal{H}^h(J) > 0$.*

**Proof of Lemma 4.4.** Let $i \in I$ be a parabolic index. Fix $j \in I \setminus \{i\}$. Since $\phi_{i^n j}(X) \subset B(x_i, r)$ if and only if $\text{Dist}(x_i, \phi_{i^n j}(X)) < r$, it follows from (4.2) that if $Qn^{-\frac{1}{p_i}} < r$, then $\phi_{i^n j}(X) \subset B(x_i, r)$. Hence using (4.1) and the conformality of $m$, we get

$$\begin{aligned} r^{-h} m(B(x_i, r)) &\ge r^{-h} \sum_{n: Qn^{-\frac{1}{p_i}} < r} m(\phi_{i^n j}(X)) \ge r^{-h} \sum_{n > (Qr^{-1})^{p_i}} Q^{-h} n^{-\frac{p_i+1}{p_i} h} \\ &\ge Q^{-h} (\text{const})\, r^{-h} (Q^{p_i} r^{-p_i})^{1 - \frac{p_i+1}{p_i} h} \ge (\text{const})\, r^{-h} r^{-p_i + (p_i+1)h} \\ &= (\text{const})\, r^{p_i (h-1)}. \end{aligned}$$

Since $h < 1$, this implies that $\lim_{r \to 0} r^{-h} m(B(x_i, r)) = \infty$. By Proposition 1.3, $x_i \in S^*(\infty)$, it therefore follows immediately from Lemma 4.9 in [MU1] that $\mathcal{H}^h(J_S) = \mathcal{H}^h(J_{S^*}) = 0$. The proof is finished. ■

**Proof of Lemma 4.5.** Fix a parabolic index $i \in I$, $j \in I \setminus \{i\}$, $n \ge 1$ and fix $r$, $2\text{diam}(\phi_{i^n j}(X)) < r \le 1$. Take an arbitrary point $x \in \phi_{i^n j}(X)$. It follows from (4.3) and the inequality $r > 2\text{diam}(\phi_{i^n j}(X))$ that if $k \le n$ and $\overline{Q}\Big(k^{-\frac{1}{p_i}} - n^{-\frac{1}{p_i}}\Big) < r$, where we take an appropriate constant $\overline{Q} \ge Q$, then $B(x, r) \supset \phi_{i^k j}(X)$. Hence, using (4.1),Theorem 4.2 and letting $E(x)$ denote the greatest integer in $x$, we get

$$\begin{aligned} m(B(x, r)) &\ge \sum_{k = E\left(\left(\overline{Q}^{-1} r + n^{-\frac{1}{p_i}}\right)^{-p_i}\right) + 1}^{n} m(\phi_{i^k j}(X)) \ge \sum_{k = E\left(\left(\overline{Q}^{-1} r + n^{-\frac{1}{p_i}}\right)^{-p_i}\right) + 1}^{n} \overline{Q}^{-h} k^{-\frac{p_i+1}{p_i} h} \\ &\ge (\text{const}) \left( \left( \overline{Q}^{-1} r + n^{-\frac{1}{p_i}} \right)^{-p_i \left(1 - \frac{p_i+1}{p_i} h\right)} - n^{1 - \frac{p_i+1}{p_i} h} \right) \\ &\ge (\text{const}) \left( \left( \overline{Q}^{-1} r + n^{-\frac{1}{p_i}} \right)^{(p_i+1)h - p_i} - n^{-\frac{1}{p_i}((p_i+1)h - p_i)} \right). \end{aligned} \tag{4.5}$$

It follows from the Mean Value Theorem that there exists some $\eta$ with $n^{-\frac{1}{p_i}} \le \eta \le \overline{Q}^{-1} r + n^{-\frac{1}{p_i}}$ such that

$$\begin{aligned} \left( \overline{Q}^{-1} r + n^{-\frac{1}{p_i}} \right)^{(p_i+1)h - p_i} - n^{-\frac{1}{p_i}((p_i+1)h - p_i)} &= ((p_i+1)h - p_i)(\overline{Q}^{-1} r) \eta^{(p_i+1)h - p_i - 1} \\ &= ((p_i+1)h - p_i) \overline{Q}^{-1} r \eta^{(p_i+1)(h-1)} \\ &\ge ((p_i+1)h - p_i) \overline{Q}^{-1} r \left( \overline{Q}^{-1} r + n^{-\frac{1}{p_i}} \right)^{(p_i+1)(h-1)}. \end{aligned} \tag{4.6}$$

But, by our constraints on $r$ and by (4.1), $n^{-\frac{1}{p_i}} \le Q^{\frac{1}{p_i+1}} \mathrm{diam}^{\frac{1}{p_i+1}}(\phi_{i^nj}(X)) \le (1/2)Q^{\frac{3}{p_i+1}} r^{\frac{1}{p_i+1}}$. Thus, combining this, (4.6) and (4.5), we get

$$\begin{aligned} m(B(x,r)) &\ge (\mathrm{const})\, r \left(\overline{Q}^{-1} r + n^{-\frac{1}{p_i}}\right)^{(p_i+1)(h-1)} \\ &\ge (\mathrm{const})\, r \left(r^{\frac{1}{p_i+1}}\right)^{(p_i+1)(h-1)} = (\mathrm{const})\, r^h. \end{aligned}$$

Therefore, the proof follows by applying Theorem 2.5(2) in [MU2] with $\xi = 1$, $\gamma = 1$ and $F$ consisting of hyperbolic indices. ■

**Proof of Lemma 4.6.** Fix a parabolic index $i \in I$. Since the system is finite, by applying Proposition 4.1, there is some $R > 0$ such that if $0 < r < R$, then $B(x_i, r)$ does not intersect $\phi_j(X)$, for any $j \ne i$. Fix such a radius $r$. Using (4.2) and (4.1), we derive

$$\begin{aligned} r^{-h} m(B(x_i,r)) &\le r^{-h} \sum_{j\ne i} \sum_{n: Q^{-1} n^{-\frac{1}{p_i}} < r} m(\phi_{i^nj}(X)) \le r^{-h} \sum_{j \ne i} \sum_{n > (Qr)^{-p_i}} Q^h ||\phi'_{i^nj}||^h \\ &\le Q^h r^{-h} \sum_{j\ne i} \sum_{n > (Qr)^{-p_i}} n^{-\frac{p_i+1}{p_i} h} \\ &\le (\mathrm{const}) \# I Q^h \left(\frac{p_i+1}{p_i} h - 1\right) r^{-h} (Qr)^{(-p_i)\left(1 - \frac{p_i+1}{p_i} h\right)} \\ &= (\mathrm{const})\, r^{-h + (p_i+1)h - p_i} = (\mathrm{const})\, r^{p_i(h-1)}. \end{aligned}$$

Since $h > 1$, this implies that $\lim_{r\to 0} r^{-h} m(B(x_i, r)) = 0$. Applying Lemma 4.13 in [MU1] along with Lemma 4.3 and Proposition 1.3, we conclude that $\mathcal{P}^h(J) = \infty$. ■

**Proof of Lemma 4.7.** Fix a parabolic index $i \in I$, $j \in I \setminus \{i\}$, $n \ge \max\{2q, q+1\}$ and $x \in \phi_{i^nj}(X)$. Given $1 \ge r > \mathrm{diam}(\phi_{i^nj}(X))$ and using (4.1) twice we obtain

$$\begin{aligned} \Sigma_1 := \sum_{a \ne i} \sum_{k=n-q}^{n+q} m(\phi_{i^ka}(X)) &\le \sum_{a\ne i} \sum_{k=n-q}^{n+q} ||\phi'_{i^ka}||^h \\ \le \sum_{a\ne i} \sum_{k=n-q}^{n+q} Q^h k^{-\frac{p_i+1}{p_i} h} &\le \# I Q^h 2q (n-q)^{-\frac{p_i+1}{p_i} h} = 2\# I q Q^h \left(\frac{n}{n-q}\right)^{\frac{p_i+1}{p_i} h} n^{-\frac{p_i+1}{p_i} h} \\ \le 2q Q^h \# I 2^{\frac{p_i+1}{p_i} h} Q \mathrm{diam}^h(\phi_{i^nj}(X)) &\le 2q Q^{h+1} 2^{\frac{p_i+1}{p_i} h} \# I r^h. \end{aligned} \tag{4.7}$$

Put $l = E\left(\left(n^{-\frac{1}{p_i}} - Qr\right)^{-p_i}\right) + 1$ if $Qr < n^{-\frac{1}{p_i}}$ and $l = \infty$ otherwise. Using (4.1) we get

$$\Sigma_2 := \sum_{a\neq i} \sum_{k:|n^{-\frac{1}{p_i}} - k^{-\frac{1}{p_i}}|<Qr} m(\phi_{i^k a}(X)) \le \sum_{a\neq i} \sum_{k=E\left(\left(Qr+n^{-\frac{1}{p_i}}\right)^{-p_i}\right)}^{l} Q^h k^{-\frac{p_i+1}{p_i}h}$$

$$\le \#IQ^h \sum_{k=E\left(\left(Qr+n^{-\frac{1}{p_i}}\right)^{-p_i}\right)}^{l} k^{-\frac{p_i+1}{p_i}h}.$$

Suppose first that $Qr < n^{-\frac{1}{p_i}}$. Then

$$\Sigma_2 \le \#IQ^h \left(\frac{p_i+1}{p_i}h - 1\right)\left(\left(Qr + n^{-\frac{1}{p_i}}\right)^{-p_i+(p_i+1)h} - \left(n^{-\frac{1}{p_i}} - Qr\right)^{-p_i+(p_i+1)h}\right).$$

It follows now from the Mean Value Theorem that there exists $\eta \in [n^{-\frac{1}{p_i}} - Qr, n^{-\frac{1}{p_i}} + Qr]$ such that

$$\left(Qr + n^{-\frac{1}{p_i}}\right)^{-p_i+(p_i+1)h} - \left(n^{-\frac{1}{p_i}} - Qr\right)^{-p_i+(p_i+1)h} = ((p_i+1)h - p_i)2Qr\eta^{(p_i+1)(h-1)}.$$

Since by (4.1),

$$n^{-\frac{1}{p_i}} \le Q^{\frac{1}{p_i}} \operatorname{diam}(\phi_{i^n j}(X))^{\frac{1}{p_i}} \le Q^{\frac{1}{p_i}} r^{\frac{1}{p_i}},$$

we therefore find

$$\begin{aligned}\Sigma_2 &\le (\text{const})\, r\eta^{(p_i+1)(h-1)} \le (\text{const})\, r\left(Qr + n^{-\frac{1}{p_i}}\right)^{(p_i+1)(h-1)} \\ &\le (\text{const})\, r\left(Q^{\frac{1}{p_i}} r^{\frac{1}{p_i}} + Qr\right)^{(p_i+1)(h-1)} \le (\text{const})\, r\left(r^{\frac{1}{p_i}}\right)^{(p_i+1)(h-1)} \qquad (4.8) \\ &= (\text{const})\, r^h.\end{aligned}$$

Suppose in turn that $Qr \ge n^{-\frac{1}{p_i}}$. Then

$$\begin{aligned}\Sigma_2 &\le Q^h \#I \sum_{k=E\left(\left(Qr+n^{-\frac{1}{p_i}}\right)^{-p_i}\right)}^{\infty} k^{-\frac{p_i+1}{p_i}h} \le Q^h \#I \left(\frac{p_i+1}{p_i}h - 1\right)\left(Qr + n^{-\frac{1}{p_i}}\right)^{-p_i\left(1-\frac{p_i+1}{p_i}h\right)} \\ &= Q^h \#I \left(\frac{p_i+1}{p_i}h - 1\right)\left(Qr + n^{-\frac{1}{p_i}}\right)^{(p_i+1)h-p_i} \le (\text{const})\, r^{(p_i+1)h-p_i} \qquad (4.9) \\ &= (\text{const})\, r^h r^{p_i(h-1)} \le (\text{const})\, r^h.\end{aligned}$$

Since, by (4.4), $m(B(x,r)) \le \Sigma_1 + \Sigma_2$, it follows from (4.7)-(4.9) that $m(B(x,r) \le (\text{const})\, r^h$. Finally, applying Theorem 2.4(3) in [MU2] completes the proof. ■

**Proof of Theorem 1.7.** It is a straightforward consequence of formulae (4.2), (4.3) and (4.4) that for every parabolic index $i \in I$, every $j \in I \setminus \{i\}$ and every $x \in X$, $\overline{\mathrm{BD}}\left(\{\phi_{i^n j}(x)\}_{n\geq 1}\right) = \frac{1}{\frac{1}{p_i}+1} = \frac{p_i}{p_i+1}$. Hence, it follows from Theorem 4.1 and Theorem 2.11 in [MU2] along with Theorem 3.1 in [MU1] that $\overline{\mathrm{BD}}(J) = \dim_{\mathrm{H}}(J)$. ■

## References


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