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# Rigidity of Conformal Iterated Function Systems

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**Abstract.** The paper extends the rigidity of mixing expanding repellers theorem by D. Sullivan announced at 1986 IMC [Su], see also the unpublished manuscript by the second author [Pr]. We show that for a regular conformal, satisfying "Open Set Condition", iterated function system of countably many holomorphic contractions of an open connected subset of a complex plane, the Radon-Nikodym derivative $d\mu/dm$ has a real-analytic extension on an open neighbourhood of the limit set of this system, where $m$ is the conformal measure and $\mu$ is the unique probability invariant measure equivalent with $m$. Next, following [Su] and [Pr], we introduce the concept of non-linearity for iterated function systems of countably many holomorphic contractions. Several necessary and sufficient conditions for non-linearity are established. We prove the following rigidity result: If $h$, the topological conjugacy between two non-linear systems $F$ and $G$, transports the conformal measure $m_F$ to the equivalence class of the conformal measure $m_G$, then $h$ has a conformal extension on an open neighbourhood of the limit set of the system $F$. Finally we prove that the hyperbolic system associated as in [MU2] to a given parabolic system of countably many holomorphic contractions is non-linear what allows to extend our rigidity result to the case of parabolic systems.

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**§1. Introduction, Preliminaries.** In [MU1] we have provided the framework to study infinite conformal iterated function systems. We shall recall first this notion and some of its basic properties. Let $I$ be a countable index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from a compact metric space $X$ into $X$ for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \le s\rho(x, y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system $S$ is uniformly contractive. Any such collection $S$ of contractions is called an iterated function system. We are particularly interested in the properties of the limit set defined by such a system. We can define this set as the image of the coding space under a coding map as follows. Let $I^n$ denote the space of words of length $n$, $I^\infty$ the space of infinite sequences of symbols in $I$, $I^* = \bigcup_{n\ge1} I^n$ and for $\omega \in I^n$, $n \ge 1$, let $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \ge 1$ does not exceed the length of $\omega$, we denote by $\omega|_n$ the word $\omega_1\omega_2 \dots \omega_n$. Since given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n \ge 1$, converge to zero and since they form a decreasing family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\omega)$, defines the coding map $\pi : I^\infty \to X$. The main object of our interest will be the limit set

$$J = \pi(I^\infty) = \bigcup_{\omega\in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|n}(X),$$

Observe that $J$ satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if $I$ is finite, then $J$ is compact and this property fails for infinite systems.

An iterated function system $S = \{\phi_i : X \to X : i \in I\}$ is said to satisfy the *Open Set Condition* if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\phi_i(U) \subset U$ for every $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \ne j$. (We do not exclude $\mathrm{cl}\phi_i(U) \cap \mathrm{cl}\phi_j(U) \ne \emptyset$.)

An iterated function system $S$ satisfying the Open Set Condition is said to be *conformal* if $X \subset I\!R^d$ for some $d \ge 1$ and the following conditions are satisfied.

(1a) $U = \mathrm{Int}_{I\!R^d}(X)$.

(1b) There exists an open connected set $V$ such that $X \subset V \subset I\!R^d$ such that all maps $\phi_i$, $i \in I$, extend to $C^1$ orientation preserving conformal diffeomorphisms of $V$ into $V$. (Note that for $d = 1$ this just means that all the maps $\phi_i$, $i \subset I$, are $C^1$ increasing diffeomorphisms, for $d \ge 2$ the words *orientation preserving conformal* mean holomorphic, and for $d > 2$ the maps $\phi_i$, $i \subset I$ are orientation preserving Möbius transformations. The proof of the last statement can be found in [BP] for example, where it is called Liouville's theorem)

(1c) There exist $\gamma, l > 0$ such that for every $x \in \partial X \subset I\!R^d$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex $x$, central angle of Lebesgue measure $\gamma$, and altitude $l$.

(1d) Bounded Distortion Property(BDP). There exists $K \geq 1$ such that

$$|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\phi'_\omega(x)|$ means the norm of the derivative.

In fact throughout the whole paper we will need one more condition which (comp. [MU1]) can be considered as a strengthening of (BDP).

(1e) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\big||\phi'_i(y)| - |\phi'_i(x)|\big| \leq L||\phi'_i|||y - x|^\alpha.$$

for every $i \in I$ and every pair of points $x, y \in V$.

**Remark 1.1.** Note that for $d = 2$, decreasing $V$ if necessary, conditions (1e) and (1d) are satisfied due to Koebe's distortion theorem.

Let us now collect some geometric consequences of (BDP). We have for all words $\omega \in I^*$ and all convex subsets $C$ of $V$

$$\text{(BDP1)} \qquad \operatorname{diam}(\phi_\omega(C)) \leq ||\phi'_\omega||\operatorname{diam}(C)$$

and, for an appropriate $V$,

$$\text{(BDP2)} \qquad \operatorname{diam}(\phi_\omega(V)) \leq D||\phi'_\omega||,$$

where the norm $||\cdot||$ is the supremum norm taken over $V$ and $D \geq 1$ is a constant depending only on $V$. Moreover,

$$\text{(BDP3)} \qquad \operatorname{diam}(\phi_\omega(X)) \geq D^{-1}||\phi'_\omega||$$

and

$$\text{(BDP4)} \qquad \phi_\omega(B(x, r)) \supset B(\phi_\omega(x), K^{-1}||\phi'_\omega||r),$$

for every $x \in X$, every $0 < r \leq \operatorname{dist}(X, \partial V)$, and every word $\omega \in I^*$.

Frequently, refering to (BDP) we will mean either (BDP) itself or one of the properties (BDP1)-(BDP4). Notice that for simplicity and clarity of our exposition we assumed the open set $U$ appearing in the open set condition to be $\operatorname{Int}(X)$.

As was demonstrated in [MU1], conformal iterated function systems naturally break into two main classes, irregular and regular. This dichotomy can be determined from either the existence of a zero of a natural pressure function or, equivalently, the existence of a conformal measure. The topological pressure function, P is defined as follows. For every integer $n \geq 1$ define

$$\psi_n(t) = \sum_{\omega \in I^n} ||\phi'_\omega||^t.$$

and

$$\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \psi_n(t).$$

For a conformal system $S$, we sometimes set $\psi_S = \psi_1 = \psi$. The finiteness parameter, $\theta_S$, of the system $S$ is defined by $\inf\{t : \psi(t) < \infty\} = \theta_S$. In [MU1], it was shown that the topological pressure function $\mathrm{P}(t)$ is non-increasing on $[0, \infty)$, strictly decreasing, continuous and convex on $[\theta, \infty)$ and $\mathrm{P}(d) \leq 0$. Of course, $\mathrm{P}(0) = \infty$ if and only if $I$ is infinite. In [MU1] (see Theorem 3.15) we have proved the following characterization of the Hausdorff dimension of the limit set $J$, which will be denoted by $\mathrm{HD}(J) = h_S$.

**Theorem 1.2.** $\mathrm{HD}(J) = \sup\{\mathrm{HD}(J_F) : F \subset I \text{ is finite}\} = \inf\{t : \mathrm{P}(t) \leq 0\}$. If $\mathrm{P}(t) = 0$, then $t = \mathrm{HD}(J)$.

We call the system $S$ *regular* if there is $t$ such that $P(t) = 0$. It follows from [MU1] that $t$ is unique. Also, the system is regular if and only if there is a $t$-conformal measure. Recall that a Borel probability measure $m$ is said to be $t$-conformal provided $m(J) = 1$ and for every Borel set $A \subset X$ and every $i \in I$

$$m(\phi_i(A)) = \int_A |\phi'_i|^t \, dm$$

and

$$m(\phi_i(X) \cap \phi_j(X)) = 0,$$

for every pair $i, j \in I$, $i \neq j$. From now on we assume that the system $S$ is regular and we denote by $\delta$ the Hausdorff dimension of its limit set. We now define the associated Perron-Frobenius operator acting on $C(X)$ as follows

$$\mathcal{L}(f)(x) = \sum_{i \in I} |\phi'_i(x)|^\delta f(\phi_i(x)).$$

Notice that the norm of $\mathcal{L}$ is equal to $||\mathcal{L}(1\!\!1)|| \leq \psi(\delta)$ and the $n$th iterate of $\mathcal{L}$ is given by the formula

$$\mathcal{L}^n(f)(x) = \sum_{|\omega|=n} |\phi'_\omega(x)|^\delta f(\phi_\omega(x)).$$

Theorem 1.3 below explains what we really need this operator for. The conformal measure $m$ is a fixed point of the operator conjugate to $\mathcal{L}$. We recall also (see [MU1, Theorem 3.8]) that there exists an invariant measure $\mu$ in the sense that for every measurable set $A$,

$$\mu(\bigcup_{i \in I} \phi_i(A)) = \mu(A)$$

equivalent to $m$ and the Radon-Nikodym derivative $d\mu/dm$ is bounded away from zero and infinity. In Sections 4 and 2 we will need better knowledge about this derivative and in particular we will need to know how it is computed. The approriate information is contained in the following (see [MU3]).

**Theorem 1.3.** The Radon-Nikodym derivative $d\mu/dm$ has a version which continuously extends to a function $\rho : X \to (0, \infty)$ and which is a unique fixed point of the Perron-Frobenius operator $\mathcal{L}$ whose integral with respect to the conformal measure $m$ is equal to 1. Moreover the iterates $\mathcal{L}^n(1\!\!1)$ converge uniformly on $X$ to $\rho$.

We call two iterated function systems $F = \{f_i : X \to X, i \in I\}$ and $G = \{g_i : Y \to Y, i \in I\}$ topologically conjugate if and only if there exists a homeomorphism $h : J_F \to J_G$ such that

$$h \circ f_i = g_i \circ h$$

for all $i \in I$. Then by induction we easily get that $h \circ f_\omega = g_\omega \circ h$ for every finite word $\omega$. The Section 2 of the paper [HU] contains the proof of the following.

**Theorem 1.4.** Suppose that $F = \{f_i : X \to X, i \in I)$ and $G = \{g_i : Y \to Y, i \in I\}$ are two topologically conjugate conformal iterated function systems. Then the following 4 conditions are equivalent.

(1) $\exists C \geq 1 \ \forall \omega \in I^*$

$$C^{-1} \leq \frac{\mathrm{diam}(g_\omega(Y))}{\mathrm{diam}(f_\omega(X))} \leq C.$$

(2) $|g'_\omega(y_\omega)| = |f'_\omega(x_\omega)|$ for all $\omega \in I^*$, where $x_\omega$ and $y_\omega$ are the only fixed points of $f_\omega : X \to X$ and $g_\omega : Y \to Y$ respectively.

(3) $\exists E \geq 1 \ \forall \omega \in I^*$

$$E^{-1} \leq \frac{||g'_\omega||}{||f'_\omega||} \leq E.$$

(4) For every finite subset $T$ of $I$, $\mathrm{HD}(J_{G,T}) = \mathrm{HD}(J_{F,T})$ and the conformal measures $m_{G,T}$ and $m_{F,T} \circ h^{-1}$ are equivalent.

Suppose additionally that both systems $F$ and $G$ are regular. Then the following condition is also equivalent to the four conditions above.

(5) $\mathrm{HD}(J_G) = \mathrm{HD}(J_F)$ and the conformal measures $m_G$ and $m_F \circ h^{-1}$ are equivalent.

Since [HU] deals only with real-analytic 1-dimensional systems, for completeness we provide the proof in Appendix 1.

Our main goal in this paper is to prove the rigidity theorem, $(1)-(5) \Rightarrow$ the conjugacy has a conformal extension. For finite systems arising from inverse branches of a holomorphic expanding map on a mixing repeller a sufficient condition for this implication is that the systems are non-linear, [Su, Pr]. Here we shall prove this rigidity for infinite systems. An example in which this is applicable, complex continued fractions, was considered in [MU1].

As a by-product we see that the non-linearity implies the rigidity: $(1)-(5) \Rightarrow$ the conjugacy is Lipschitz continuous. For infinite systems without the non-linearity assumption this is false, see Appendix 1. A positive result on this rigidity was obtained in [HU]. Instead of the non-linearity a so-called *bounded geometry* property was assumed and the preservation of the "scaling" of "gaps" under the conjugacy. For completeness we provide a precise statement of this theorem in Appendix 1.

We postpone the formulation of our main rigidity theorem to Section 4 where all ingredients needed to state it and to prove it will be ready. In Section 2 generalizing the approach from [PU] we prove the main technical result, the real analyticity of the Radon-Nikodym derivative $d\mu/dm$ of invariant measure $\mu$ with respect to conformal measure $m$. In Section 3 we deal with various equivalent conditions of non-linearity, in Section 4 we prove our main result, Theorem 4.1, and in Section 5 we extend the results of Section 4 to the case of parabolic iterated function systems. The Appendix 1 contains the proof of Theorem 1.4 taken from [HU] and counterexamples concerning Lipschitz continuity of the conjugacy. Appendix 2 is devoted to the proof of the continuity of the Radon-Nikodym derivative of the invariant measure with respect to the conformal measure in the parabolic case.

**§2. The Radon-Nikodym derivative $\rho$ is real-analytic.** From now on throughout the whole paper we assume that $d=2$ and $\{\phi_i\}$ is an Open Set Condition conformal regular iterated function system.

We call the system $S=\{\phi_i\}_{i\in I}$ 1-dimensional if there exists a set $D:\overline{J}\subset D\subset V$ composed of finitely many real-analytic curves with pairwise disjoint closures such that $\phi_i(D)\subset D$ for all $i\in I$.

**Lemma 2.1.** If a non-empty open subset of $\overline{J}$ is contained in a 1-dimensional real-analytic curve, then the system $S$ is 1-dimensional.

**Proof.** Since $\overline{J}$ is compact it suffices to show that each point in $\overline{J}$ has a neighbourhood contained in a real-analytic curve. The assumptions of the lemma state that there exists a point $x\in\overline{J}$, an open ball $B(x)$ centered at $x$ and $M$, a real-analytic curve, open-ended, containing $\overline{J}\cap B(x)$. Fix now an arbitrary point $z\in\overline{J}$. Since $x\in\overline{J}$ there exists $\omega\in I^*$ such that $\phi_\omega(z)\in\overline{J}\cap B(x)$, moreover $\phi_\omega(V)\subset B(x)$. Then the set $\phi_\omega(V)\cap M$ contains $\phi_\omega(V)\cap\overline{J}$, an open neighbourhood of $\phi_\omega(z)$ in $\overline{J}$ and consists of countably many real-analytic curves. Let $\Gamma$ be one of them, the connected component of $\phi_\omega(V)\cap M$ containing $\phi_\omega(z)$. It contains an open neighbourhood of $\phi_\omega(z)$ in $\overline{J}$. Then $\phi^{-1}(\Gamma)$ contains an open neighbourhood of $z$ in $\overline{J}$. ■

Our main goal in this section is to prove the following.

**Theorem 2.2.** The Radon-Nikodym derivative $\rho$ has a real-analytic extension on an open connected neighbourhood $U$ of $X$ in $V$.

**Proof.** In view of the result obtained when proving the implication $(g) \Rightarrow (a)$ of Theorem 3.1 of [HU], we may assume that our system is not 1-dimensional. First define the sequence of functions $b_n : V \to (0, \infty)$ by setting

$$b_n(z) = \sum_{|\omega|=n} |\phi'_\omega(z)|^\delta, \tag{2.1}$$

where, let us recall, $\delta = \mathrm{HD}(J)$ is the Hausdorff dimension of the limit set. In view of (2.15) in [MU1] $|b_n(z)| = b_n(z) \le K^\delta$ for all $z \in X$ and all $n \ge 1$. Hence, applying the Koebe distortion theorem we conclude that there exists $T > 0$ such that for each point $w \in X$ there exists a radius $r = r(w) > 0$ such that $B(w, 2r) \subset V$ and for all $z \in B(w, 2r)$ and all $n \ge 1$

$$|b_n(z)| = b_n(z) \le T. \tag{2.2}$$

Identify now $\mathbb{C}$, where our contractions $\phi_i$, $i \in I$, act, to $\mathbb{R}^2$ with coordinates $x, y$, the real and complex part of $z$. Embed this into $\mathbb{C}^2$ with $x, y$ complex. Denote the above $\mathbb{C} = \mathbb{R}^2$ by $\mathbb{C}_0$. We may assume that $v = 0$ in $\mathbb{C}_0$. Given $\rho \in I^*$ define the function $\rho_\omega : B_{\mathbb{C}_0}(0, 2r) \to \mathbb{C}$ by setting

$$\rho_\omega(z) = \frac{\phi'_\omega(z)}{\phi'_\omega(0)}.$$

Since $B_{\mathbb{C}_0}(0, 2r) \subset \mathbb{C}_0$ is simply connected and $\rho_\omega$ nowhere vanishes, all the branches of the $\log \rho_\omega$ are well defined on $B_{\mathbb{C}_0}(0, 2r)$. Choose this branch that maps 0 to 0 and denote it also by $\log \rho_\omega$. By Koebe's Distortion Theorem $|\rho_\omega|$ and $|\arg\rho_\omega|$ are bounded on $B(0, r)$ by universal constants $K_1, K_2$ respectively. Hence $|\log \rho_\omega| \le K = \log K_1 + K_2$. We write

$$\log \rho_\omega = \sum_{m=0}^{\infty} a_m z^m$$

and note that by Cauchy's inequalities

$$|a_m| \le K/r^m. \tag{2.3}$$

We can write for $z = x + iy$ in $\mathbb{C}_0$

$$\mathrm{Re} \log \rho_\omega = \mathrm{Re} \sum_{m=0}^{\infty} a_m (x + iy)^m = \sum_{p,q=0}^{\infty} \mathrm{Re}\Big(a_{p+q} \binom{p+q}{q} i^q\Big) x^p y^q := \sum c_{p,q} x^p y^q.$$

In view of (2.3) we can estimate $|c_{p,q}| \le |a_{p+q}| 2^{p+q} \le K r^{-(p+q)} 2^{p+q}$. Hence $\mathrm{Re} \log \rho_\omega$ extends, by the same power series expansion $\sum c_{p,q} x^p y^q$, to a complex-valued function on the polydisk $\mathbb{D}_{\mathbb{C}^2}(0, r/2)$ and

$$|\mathrm{Re} \log \rho_\omega| \le 4K \quad \text{on } \mathbb{D}_{\mathbb{C}^2}(0, r/4). \tag{2.4}$$

Now each function $b_n$, $n \geq 1$, extends to the function

$$B_n(z) = \sum_{|\omega|=n} |\phi_\omega'(0)|^\delta \mathrm{e}^{\delta \mathrm{Re} \log \rho_\omega(z)}. \tag{2.5}$$

whose domain, similarly as the domains of the functions $\mathrm{Re}\log\rho_\omega$, contains the polydisk $\mathbb{D}_{\mathbb{C}^2}(0, r/2)$. Finally, using (2.2) and (2.4) we get for all $n \geq 0$ and all $z \in \mathbb{D}_{\mathbb{C}^2}(0, r/4)$

$$\begin{aligned} |B_n(z)| &\leq \sum_{|\omega|=n} |\phi_\omega'(0)|^\delta \mathrm{e}^{\mathrm{Re}(\delta \mathrm{Re}\log\rho_\omega(z))} \leq \sum_{|\omega|=n} |\phi_\omega'(0)|^\delta \mathrm{e}^{\delta|\mathrm{Re}\log\rho_\omega(z)|} \\ &\leq \mathrm{e}^{K\delta} \sum_{|\omega|=n} |\phi_\omega'(0)|^\delta. \leq \mathrm{e}^{K\delta} T \end{aligned}$$

Now by Cauchy's integral formula (in $\mathbb{D}_{\mathbb{C}^2}(0, r/4)$) for the second derivatives we prove that the family $B_n$ is equicontinuous on, say, $\mathbb{D}_{\mathbb{C}^2}(0, r/5)$. Hence we can choose a uniformly convergent subsequence and the limit function $G$ is complex analytic and extends $\rho$ on $J \cap B(0, r/5)$, in the manner described in Theorem 1.3. Thus we have proved that $\rho$ extends to a complex analytic function in a neighbourhood of every point $v \in J$ in $\mathbb{C}^2$, i.e. real-analytic in $\mathbb{C}_0$. These extensions coincide on the intersections of the neighbourhoods, otherwise $J$ is real-analytic and we are in the [HU] case, referred to at the beginning of the proof. ■

For every $\omega \in I^*$ denote by $D_{\phi_\omega} = \frac{d\mu\circ\phi_\omega}{d\mu}$ the Jacobian of the map $\phi_\omega : J \to J$ with respect to the measure $\mu$. As an immediate consequence of Theorem 2.1, the following computation

$$\frac{d\mu \circ \phi_\omega}{d\mu} = \frac{d\mu \circ \phi_\omega}{dm \circ \phi_\omega} \cdot \frac{dm \circ \phi_\omega}{dm} \cdot \frac{dm}{d\mu} = \left(\frac{d\mu}{dm} \circ \phi_\omega\right) \cdot |\phi_\omega'|^\delta \cdot \frac{dm}{d\mu}$$

and the observation that $|\phi_\omega'|^\delta$ is real-analytic on $V$, we get the following.

**Corollary 2.3.** For every $i \in I$ the Jacobian $D_{\phi_i}$ has a real-analytic extension $\tilde{D}_{\phi_i}$ on the neighbourhood $U$ of $X$ produced in Theorem 2.1.

## §3. Non-linearity.

The main goal of this section is to prove the following.

**Theorem 3.1.** Suppose that the system $S = \{\phi_i\}_{i\in I}$ is regular and denote the corresponding conformal measure by $m$. Then the following conditions are equivalent.

(a) For each $i \in I$ the extended Jacobian $\tilde{D}_{\phi_i} : U \to \mathbb{R}$ is constant, where $U$ is the neighbourhood of $X$ produced in Corollary 2.3.

(b) There exist a continuous function $u : X \to I\!R$ and constants $c_i \in I\!R$, $i \in I$, such that

$$\log |\phi_i'| = u - u \circ \phi_i + c_i$$

for all $i \in I$.

(c) There exist a continuous function $u : \overline{J} \to I\!R$ and constants $c_i \in I\!R$, $i \in I$, such that

$$\log |\phi_i'| = u - u \circ \phi_i + c_i$$

for all $i \in I$.

(d1) The conformal structure on $\overline{J}$ admits a euclidean isometries refinement so that all maps $\phi_i$, $i \in I$, become affine conformal, more precisely there exists an atlas $\{\psi_t : U_t \to \mathbb{C}\}$ with open disks $U_t$, consisting of conformal injections such that $\bigcup_t U_t \supset \overline{J}$, all $U_t \cap U_s$ and $U_t \cap \phi_i(U_s)$ are connected and the compositions $\psi_t \circ \psi_s^{-1}$ and $\psi_t \circ \phi_i \circ \psi_s^{-1}$, respectively on $\psi_s(U_t \cap U_s)$ and $\psi_s \circ \phi_i^{-1}(U_t \cap \phi_i(U_s))$, are conformal affine with $|(\psi_t \circ \psi_s^{-1})'| \equiv 1$.

(d2) As (d1) but no assumptions on $|(\psi_t \circ \psi_s^{-1})'|$ (i.e. the atlas is only conformal affine).

(eh) There exist a cover $\{B_\lambda\}_{\lambda \in \Lambda}$ of $\overline{J}$ consisting of open disks and a family of harmonic functions $\gamma_\lambda : B_\lambda \to I\!R$, $\lambda \in \Lambda$, such that for all $\lambda, \lambda' \in \Lambda$ and all $i \in I$

$$\gamma_\lambda - \gamma_{\lambda'} = const \tag{3.1}$$

on $B_\lambda \cap B_{\lambda'}$ and

$$\arg_\lambda \phi_i' - \gamma_\lambda + \gamma_{\lambda'} \circ \phi_i = const \tag{3.2}$$

on $\phi_i^{-1}(B'_\lambda \cap \phi_i(B_\lambda))$, where $\arg_\lambda \phi_i' : B_\lambda \to I\!R$ is a continuous branch of argument of $\phi_i'$ defined on the simply connected set $B_\lambda$. All the sets $B_\lambda \cap B_{\lambda'}$ and $\phi_i^{-1}(B'_\lambda \cap \phi_i(B_\lambda))$ are connected.

(er) As (eh) but harmonic replaced by real-analytic.

(ec) As (eh) but harmonic replaced by continuous.

(f) $\nabla \tilde{D}\phi_i(z) = 0$ for all $z \in \overline{J}$ and all $i \in I$ if $S$ is 1-dimensional. If $S$ is not 1-dimensional then

$$\det(\nabla \tilde{D}\phi_i \circ \phi_\omega(z), \nabla \tilde{D}\phi_i(z)) = 0$$

for all $z \in \overline{J}$ and all $i \in I, \omega \in I^*$ .

**Proof.** We shall prove the following implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d1) \Rightarrow (d2) \Rightarrow (a)$, $(d2) \Rightarrow (eh) \Rightarrow (er) \Rightarrow (ec) \Rightarrow (d2)$, $(a) \Rightarrow (f)$ and $(f) \Rightarrow (er)$.

• $(a) \Rightarrow (b)$. Since for every $i \in I$, $\tilde{D}_{\phi_i} = (\rho \circ \phi_i) \cdot |\phi_i'|^\delta \cdot \rho^{-1}$, we have

$$\log(|\tilde{D}_{\phi_i}|) = \log(|\rho| \circ \phi_i) + \delta \log |\phi_i'| - \log |\rho|.$$

Thus to finish the proof of the implication $(a) \Rightarrow (b)$ it suffices to set $c_i = \frac{1}{\delta} \log(\tilde{D}_{\phi_i})$ and $u = \frac{1}{\delta} \log |\rho|$.

• The implication $(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (d1)$. Fix an element $v \in \overline{J}$ and an element $\tau \in I^\infty$. Given $n \geq 1$ and a word $\omega \in I^n$ we denote by $\overline{\omega}$ the flipped word $\omega_n\omega_{n-1}\dots\omega_1$. Our first aim is to show that the series

$$\sum_{n\geq 1}\left(\log|\phi'_{\tau_n}(\phi_{\overline{\tau|_{n-1}}}(z))| - \log|\phi'_{\tau_n}(\phi_{\overline{\tau|_{n-1}}}(v))|\right) \tag{3.3}$$

converges absolutely uniformly on $V$, where for $n = 1$ we set $\phi_{\overline{\tau|_{n-1}}} = \mathrm{Id}_V$. Indeed, it follows from (1d) and (1e), compare (4.2) of [HU], that

$$\begin{aligned}\left|\log|\phi'_{\tau_n}(\phi_{\overline{\tau|_{n-1}}}(z))| - \log|\phi'_{\tau_n}(\phi_{\overline{\tau|_{n-1}}}(v))|\right| &\leq KL\left|\phi_{\overline{\tau|_{n-1}}}(z) - \phi_{\overline{\tau|_{n-1}}}(v)\right|^\alpha \\ &\leq KLs^{(n-1)\alpha}|z-v|^\alpha \\ &\leq KL\mathrm{diam}^\alpha(V)s^{(n-1)\alpha}.\end{aligned} \tag{3.4}$$

Since

$$\sum_{n\geq 1} KL\mathrm{diam}^\alpha(V)s^{(n-1)\alpha} \leq \frac{KL\mathrm{diam}^\alpha(V)}{1-s^\alpha} < \infty,$$

the proof of the absolute uniform convergence of the series defined by (3.3) is complete. We now can define the function $u_v : V \to I\!\!R$ by setting

$$u_v(z) = u(v) + \sum_{n\geq 1}\left(\log|\phi'_{\tau_n}(\phi_{\overline{\tau|_{n-1}}}(z))| - \log|\phi'_{\tau_n}(\phi_{\overline{\tau|_{n-1}}}(v))|\right). \tag{3.5}$$

The function $u_v : V \to I\!\!R$ as the sum of an absolutely convergent series of harmonic functions, is harmonic. Iterating the formula appearing in Theorem 3.1(c), we obtain for every $n \geq 1$ and every $z \in \overline{J}$

$$u(z) - u(v) = \sum_{k=1}^{n}\left(\log|\phi'_{\tau_k}(\phi_{\overline{\tau|_{k-1}}}(z))| - \log|\phi'_{\tau_k}(\phi_{\overline{\tau|_{k-1}}}(v))|\right) + u(\phi_{\overline{\tau|_n}}(z)) - u(\phi_{\overline{\tau|_n}}(v))$$

Since, by (BDP), $|\phi_{\overline{\tau|_n}}(z) - \phi_{\overline{\tau|_n}}(v)| \leq s^n$ and since the function $u : \overline{J} \to I\!\!R$ as continuous on a compact set is uniformly continuous, it follows from the last display that $u_v(z) = u(z)$ for all $z \in \overline{J}$, i.e. $u_v$ is a harmonic extension of $u$ on $V$. From now on we will drop the subscript $v$ writing simply $u : V \to I\!\!R$. Since all the functions $\log|\phi'_i|$ and $u - u\circ\phi_i + c_i$, $i \in I$, are harmonic on $V$, each set

$$Z_i = \{z \in V : \log|\phi'_i(z)| = u(z) - u\circ\phi_i(z) + c_i\},$$

$i \in I$, is either equal to $V$ or is a real-analytic set.

Suppose first that $Z_i = V$ for all $i \in I$. For every $w \in \overline{J}$ consider a ball $B(w) \subset V$ centered at $w$. Let $l_w : B(w) \to I\!\!R$ be a harmonic conjugate function to the harmonic function $u : B(w) \to I\!\!R$ so that $u + il_w : B(w) \to C\!\!\!\!I$ is holomorphic. Write $G_w = \exp(u + il_w)$

and denote by $\psi_w : B(w) \to \mathbb{C}$ a primitive function of $G_w$. Since $\psi_w'(w) = G_w(w) \neq 0$, there exists a disk $U_w \subset B(w)$ centered at $w$ and such that $\psi_w|_{U_w}$ is injective. Using Koebe's distortion theorem for arguments (see [Hi]) we may assume that in addition all the sets $U_w$ to be so small that all the images $\phi_i(U_w)$, $i \in I$, $w \in \overline{J}$, are convex. We claim that the family $\{\psi_w : U_w \to \mathbb{C}\}_{w\in\overline{J}}$ forms an atlas demanded in (d1). Indeed, fix $w, v \in \overline{J}$ and consider an arbitrary point $z \in U_w \cap U_v$. Then

$$(\psi_w \circ \psi_v^{-1})'(\psi_v(z)) = \psi_w'(z) \cdot (\psi_v'(z))^{-1} = G_w(z) \cdot G_v^{-1}(z) = \exp\bigl(i(l_w(z) - l_v(z))\bigr)$$

and therefore $(\psi_w \circ \psi_v^{-1})'$ is constant with absolute value 1 on $\psi_v(U_v \cap U_w)$, since $h_w$ and $h_v$ differ by an additive constant on the connected set $U_w \cap U_v$ as harmonic conjugates to the same harmonic function $u$.

To discuss $\left(\psi_v \circ \phi_i \circ \psi_w^{-1}\right)'$ fix again arbitrary $w, v \in \overline{J}$ and for every $i \subset I$ consider the intersection $U_v \cap \phi_i^(U_w)$. As the intersection of two convex sets, this set is convex, and consequently connected. Take now an arbitrary point $z \in \phi_i^{-1}(U_v \cap \phi_i^(U_w))$. Since $Z_i = V$, we therefore have

$$\begin{aligned}
|\bigl(\psi_v \circ \phi_i \circ \psi_w^{-1}\bigr)'(\psi_w(z))| &= |\psi_{\phi_i(w)}'(\phi_i(z)) \cdot \phi_i'(z) \cdot (\psi_w'(z))^{-1}| = |G_v(\phi_i(z)) \cdot \phi_i'(z) \cdot G_w^{-1}(z) \\
&= |\exp\bigl(u(\phi_i(z) + il_v(\phi_i(z)) - u(z) - il_w(z)| \cdot |\phi_i'(z)|\bigr) \\
&= \exp(u(\phi_i(z) - u(z))|\phi_i'(z)| \\
&= \mathrm{e}^{c_i}
\end{aligned}$$

Hence the function $\left(\psi_v \circ \phi_i \circ \psi_w^{-1}\right)'$ as holomorphic and having constant absolute value, is constant on the connected set $\psi_w \circ \phi_i^{-1}(U_v \cap \phi_i^(U_w))$.

Suppose in turn that $Z_i \neq V$ for some $i \in I$. Since the equation (c) of Theorem 3.1 is satisfied on compact $\overline{J}$, then $\overline{J} \subset Z_i$. Since $\overline{J}$ is infinite its non-empty open part is contained in a real analytic curve, so the system is 1-dimensional. Hence by Lemma 2.1 there are finitely many real-analytic pairwise disjoint curves whose union $M$ contains $\overline{J}$. Since $\phi_i(\overline{J} \subset \overline{J}$ for all $i \in I$, decreasing $M$ if necessary, we may assume that $\phi_i(M) \subset M$ for all $i \in I$.
Change coordinates holomorphically on a neighbourhood of $M$ so that $M \subset \mathbb{R}$. (This uses the consequence of our assumptions that there is no closed curve among the components of $M$, with relaxed assumptions allowing the existence of such a curve we would change it to the unit circle and then use charts being branches of $z \mapsto \log iz$.)
Since the function $u : M \to \mathbb{R}$ is real-analytic, it uniquely extends to a complex-analytic function $\tilde{u}$ on an open neighbourhood of $M$ in $V$. Now we proceed similarly as in the previous case; we define $\psi_w$, $w \in \overline{J}$, to be a primitive of $\mathrm{e}^{\tilde{u}}$ on a sufficiently small neighbourhood of $w \in V$ and we check that $(\psi - w \circ \psi_v^{-1})' = 1$ on $\psi_v(U_v \cap U_w)$. Now note that $\tilde{u} - \tilde{u} \circ \phi_i + c_i = \widetilde{\log|\phi_i'|}$, where the latter expression is a holomorphic extension of $\log|\phi_i'|$, which extends the equality (c). Note that $\widetilde{\log|\phi_i'|} = \log \pm\phi_i'$, where $\pm$ depends as $\phi_i'$ is positive or negative. We use the fact it is real! The equality extends the equality on $\overline{J}$ because the functions on both sides are holomorphic. We conclude with

$$|\bigl(\psi_{\phi_i(w)} \circ \phi_i \circ \psi_w^{-1}\bigr)'(\psi_w(z))| = \mathrm{e}^{c_i}$$

for all $z \in \phi_i^{-1}(U_v \cap \phi_i^(U_w))$. , hence $\left(\psi_{\phi_i(w)} \circ \phi_i \circ \psi_w^{-1}\right)'$ is constant on the connected set $\psi_w \circ \phi_i^{-1}(U_v \cap \phi_i^(U_w))$. The proof of the implication $(c) \Rightarrow (d1)$ is complete.

Remark 1. As an intermediate step in the proof of the implication $(c) \Rightarrow (d1)$ we proved (bh) (compare later (eh)), namely the property (b) with $u$ harmonic on a neighbourhood of $\overline{J}$, here $V$, in case of the system $S$ not 1-dimensional ( $Z_i = V$ for all $i$). For $S$ 1-dimensional we also can prove (bh) but indirectly, via (d1). Indeed assuming (d1) and $M$ in $\mathbb{R}$ we set the harmonic extension $u = \log|\psi_v'|$ independent of $v$.

- The implication $(d1) \Rightarrow (d2)$ is obvious.
- $(d2) \Rightarrow (a)$. Let $\{\psi_\lambda : U_\lambda \to \mathbb{C}\}_{\lambda\in\Lambda}$ be a finite conformal affine atlas for the system $S$. Fix $\beta \in \Lambda$, take a number $n_0 \geq 1$ so large that $\mathrm{diam}(V)s^{n_0}$ is less than a Lebesgue number of the cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of $\overline{J}$, consider any number $n \geq n_0$ and for every $\omega \in I^n$ choose one element $\lambda(\omega) \in \Lambda$ such that $\phi_\omega(V) \subset U_{\lambda(\omega)}$. Next, given $n \geq n_0$ and $\omega \in I^n$ consider the map

$$(\psi_{\lambda(\omega)} \circ \phi_\omega \circ \psi_\beta^{-1})' \circ \psi_\beta$$

defined on $U_\beta$. Since our atlas is affine, this function is constant on every sufficiently small neighbourhood of every point in $\overline{J} \cap U_\beta$ and therefore, as real analytic, it is constant on $U_\beta$. Denote its value there by $c_{\beta,\omega}$. Since for every $z \in U_\beta$

$$\sum_{|\omega|=n} c_{\beta,\omega}^\delta |\psi_\beta'(z)|^\delta \cdot |\psi_{\lambda(\omega)}'(\phi_\omega(z))|^{-\delta} = \sum_{|\omega|=n} |\phi_\omega'(z)|^\delta = \mathcal{L}^n(1\!\!1), \tag{3.6}$$

since by Theorem 1.3

$$\lim_{n\to\infty} \mathcal{L}^n(1\!\!1)(z) = \rho(z) \tag{3.7}$$

and since the product $|\psi_\beta'(z)|^\delta \cdot |\psi_{\lambda(\omega)}'(\phi_\omega(z))|^{-\delta}$ is uniformly bounded away from zero and infinity, we conclude that there exists a constant $M \geq 1$ such that for all $z \in U_\beta$ and all $n \geq 1$

$$M^{-1} \leq \sum_{|\omega|=n} c_{\beta,\omega}^\delta \leq M. \tag{3.8}$$

Fix now an $\varepsilon > 0$ and $n_1 \geq n_0$ so large that for all $n \geq n_1$ and all $\omega \in I^n$

$$\sup\{|\psi_{\lambda(\omega)}' \circ \phi_\omega|^{-\delta}\} - \inf\{|\psi_{\lambda(\omega)}' \circ \phi_\omega|^{-\delta}\} < \varepsilon/M.$$

Then, using (3.6), we conclude that for all $n \geq n_1$ and all $z_1, z_2 \in U_\beta$

$$\left|\sum_{|\omega|=n} \left(c_{\beta,\omega}^\delta |\psi_{\lambda(\omega)}'(\phi_\omega(z_2))|^{-\delta} - c_{\beta,\omega}^\delta |\psi_{\lambda(\omega)}'(\phi_\omega(z_1))|^{-\delta}\right)\right| \leq \varepsilon$$

and therefore

$$\lim_{n\to\infty}\left|\sum_{|\omega|=n}\left(c_{\beta,\omega}^{\delta}\psi'_{\lambda(\omega)}(\phi_\omega(z_2))|^{-\delta}-c_{\beta,\omega}^{\delta}\psi'_{\lambda(\omega)}(\phi_\omega(z_1))|^{-\delta}\right)\right|=0.$$

Combining this, (3.6) and (3.7) we conclude that there exists a constant $c_\beta\ge 0$ such that for all $z\in U_\beta$

$$\lim_{n\to\infty}\sum_{|\omega|=n}c_{\beta,\omega}^{\delta}|\psi'_{\lambda(\omega)}(\phi_\omega(z))|^{-\delta}=c_\beta.$$

Combining in turn this, (3.6) and (3.7) we conclude that for all $z\in U_\beta$

$$\rho(z)=c_\beta|\psi'_\beta(z)|^{\delta}. \tag{3.9}$$

Fix now $i\in I$, $w\in U_\beta\cap\overline{J}$, and choose $\lambda\in\Lambda$ such that $\phi_i(w)\in U_\lambda$ and a connected neighbourhood $V_w\subset U_\beta$ of $w$ such that $\phi_i(V_w)\subset U_\lambda$. Then for every $z\in V_w$

$$\begin{aligned}\tilde D_{\phi_i}(z)&=\rho\circ\phi_i(z)|\phi_i'(z)|^{\delta}\rho(z)^{-1}=c_\lambda|\psi'_\lambda(\phi_i(z))|^{\delta}\cdot|\phi_i'(z)|^{\delta}\cdot c_\beta^{-1}|\psi'_\beta(z)|^{-\delta}\\&=c_\lambda c_\beta^{-1}\left(|\psi'_\lambda(\phi_i(z))|\cdot|\phi_i'(z)|\cdot|\psi'_\beta(z)|^{-1}\right)^{\delta}\end{aligned}$$

and therefore, since our system $S$ is affine, $\tilde D_{\phi_i}$ is constant on $V_w$. Since, by Theorem 2.2, $\tilde D_{\phi_i}$ is real-analytic on $U$, we thus conclude that $\tilde D_{\phi_i}$ is constant on $U$. The proof of the implication $(d2)\Rightarrow(a)$ is finished.

• $(d2)\Rightarrow(eh)$. We can assume the sets $U_t$ appearing in condition $(d2)$ are open balls. Since $\overline{J}$ is compact, we may choose from the family $\{U_t\}$ a finite subcover $\{B_\lambda\}_{\lambda\in\Lambda}$ of $\overline{J}$. Define then for every $\lambda\in\Lambda$ the map $\gamma_\lambda:B_\lambda\to I\!\!R$ to be a continuous branch of $\arg\psi'_\lambda$ and additionally for every $i\in I$, $\arg_\lambda\phi_i':B_\lambda\to I\!\!R$ to be a continuous branch of argument of $\phi_i'$. These branches exist since $B_\lambda$ is simply connected and $\psi'_\lambda$ and $\phi_i'$ nowhere vanish. Of course all the maps $\gamma_\lambda$, $\lambda\in\Lambda$, are harmonic. Consider now two indices $\lambda,\lambda'\in\Lambda$ such that $B_\lambda\cap B_{\lambda'}\ne\emptyset$. Since our atlas is affine, $\psi_\lambda(z)=\psi_\lambda\circ\psi_{\lambda'}^{-1}(\psi_{\lambda'}(z))=a(\psi_{\lambda'}(z))+b$ for all $z\in B_\lambda\cap B_{\lambda'}$ and some $a,b\in C\!\!\!\!I$. We conclude that $\gamma_\lambda-\gamma_{\lambda'}$ is on $B_\lambda\cap B_{\lambda'}$ equal to $\arg(a)$ up to an integer multiple of $2\pi$. This means that (3.1) is satisfied. Since all the contractions $\{\phi_i\}_{i\in I}$ are affine in the atlas $\psi_\lambda:B_\lambda\to C\!\!\!\!I$, we conclude that given $\lambda,\lambda'\in\Lambda$, $i\in I$ there exist constants $d,c\in C\!\!\!\!I$ such that for every $z\in\phi_i^{-1}(B_{\lambda'}\cap\phi_i(B_\lambda))$

$$\psi_{\lambda'}\circ\phi_i(z)=\psi_{\lambda'}\circ\phi_i\circ\psi_\lambda^{-1}(\psi_\lambda(z))=d\psi_\lambda(z)+c.$$

We conclude that $\arg_\lambda\phi_i'-\gamma_\lambda+\gamma_{\lambda'}\circ\phi_i$ is equal to $\arg(d)$ up to an integer multiple of $2\pi$ on the connected set $\phi_i^{-1}(B_{\lambda'}\cap\phi_i(B_\lambda))$. This means that (3.2) is satisfied. Thus the proof of the implication $(d2)\Rightarrow(eh)$ is complete.

• The implications $(eh)\Rightarrow(er)\Rightarrow(ec)$ are obvious.

• $(ec) \Rightarrow (d2)$. The general idea is here the same as in the proof of the implication $(c) \Rightarrow (d1)$. Surprisingly, we do not get directly $(c) \Rightarrow (d1)$. For this we need to go via $(d2) \Rightarrow (a) \Rightarrow (d1)$.

Let $4\delta > 0$ be a Lebesgue number of the cover $\{B_\lambda\}_{\lambda\in\Lambda}$ of $\overline{J}$. By compactness of $\overline{J}$ there exists a finite set $T$ and points $v_t \in \overline{J}$, $t \in T$, such that the family $\{B(v_t, \delta)\}_{t\in T}$ is a cover of $\overline{J}$. Since $4\delta$ is a Lebesgue number of the cover $\{B_\lambda\}_{\lambda\in\Lambda}$, for every $t \in T$ there exists at least one element $\lambda(t) \in \Lambda$ such that $B(v_t, 2\delta) \subset B_{\lambda(t)}$. Fix now $t_0 \in T$, $\tau \in I^\infty$, that is similarly as in the implication $(c) \Rightarrow (d1)$. Then for each integer $n \geq 1$ choose $t_n \in T$ such that $\phi_{\overline{\tau|_n}}(v_{t_0}) \in B(v_{t_n}, \delta)$. Since $\phi_{\overline{\tau|_n}}$ on $B(v_{t_0}, \delta)$ shrinks distances by factor at least $s < 1$ for $n \geq 1$, we get $\phi_{\overline{\tau|_n}}(B(v_{t_0}, \delta)) \subset B(v_{t_n}, (1+s)\delta)$. Now, for every $i \in I$ and every $\lambda \in \Lambda$ let $\arg_\lambda \phi_i' : B_\lambda \to I\!\!R$ be a continuous branch of argument of $\phi_i'$. It follows from Koebe's theorem for argument (see [Hi]) that for arguments $\arg_\lambda \phi_i'$ an analogous inequality as (1e) for $\log|\phi_i'|$ is satisfied. Namely, with $L$ sufficiently large and $\alpha > 0$ sufficiently small

$$|\arg_\lambda \phi_i'(y) - \arg_\lambda \phi_i'(x)| \leq L|y-x|^\alpha$$

for all $\lambda \in \Lambda$, all $i \in I$ and all $x, y \in B_\lambda$. Hence for all $z \in B(v_{t_0}, \delta)$

$$\begin{aligned}
\sum_{n\geq 1} |\arg_{\lambda(t_{n-1})}\phi'_{\tau_n}(\phi_{\overline{\tau|_{n-1}}}(z)) - \arg_{\lambda(t_{n-1})}\phi'_{\tau_n}(\phi_{\overline{\tau|_{n-1}}}(v_{t_0}))| \\
\leq \sum_{n\geq 1} L s^{\alpha(n-1)}|z - v_{t_0}|^\alpha \\
\leq L \mathrm{diam}^\alpha(V)\frac{1}{1-s^\alpha} < \infty
\end{aligned} \tag{3.10}$$

Iterating formula (3.2) we obtain for every $n \geq 1$ and every $z \in B(v_{t_0}, \delta)$

$$\begin{aligned}
&\gamma_{\lambda(t_0)}(z) - \gamma_{\lambda_{t_0)}}(v_{t_0}) = \\
&= \sum_{k=1}^{n} \arg_{\lambda(t_{k-1})}(\phi'_{\tau_k}(\phi_{\overline{\tau|_{k-1}}}(z)) - \arg_{\lambda(t_{k-1})}\phi'_{\tau_k}(\phi_{\overline{\tau|_{k-1}}}(v_{t_0})) \\
&+ \gamma_{\lambda(t_n)}(\phi_{\overline{\tau|_n}}(z)) - \gamma_{\lambda(t_n)}(\phi_{\overline{\tau|_n}}(v_{t_0})).
\end{aligned}$$

Since for all $t \in T$, $B(v_t, (1+s)\delta) \subset B(v_t, 2\delta) \subset B_{\lambda(t)}$, all the functions $\gamma_{\lambda(t)}|_{B(v_t,(1+s)\delta)}$ are uniformly continuous. Therefore, since the set $T$ is finite, since $\phi_{\overline{\tau|_n}}(z), \phi_{\overline{\tau|_n}}(v_{t_0}) \in B(v_{t_n}, (1+s)\delta)$ and since $|\phi_{\overline{\tau|_n}}(z) - \phi_{\overline{\tau|_n}}(v_{t_0})| \leq \delta s^n$, applying (3.10) we conclude that for all $z \in B(v_{t_0}, \delta)$

$$\gamma_{\lambda(t_0)}(z) = \gamma_{\lambda(t_0)}(v_{t_0}) + \sum_{k=1}^{\infty} \arg_{\lambda(t_k)}(\phi'_{\tau_k}(\phi_{\overline{\tau|_{k-1}}}(z)) - \arg_{\lambda(t_k)}\phi'_{\tau_k}(\phi_{\overline{\tau|_{k-1}}}(v_{t_0})).$$

Thus the function $\gamma_{\lambda(t_0)}|_{B(v_{t_0},\delta)}$ as the sum of an absolutely uniformly convergent series of harmonic functions is harmonic. So, all the functions $\gamma_{\lambda(t)} : B(v_t, \delta) \to I\!\!R$. $t \in T$, are harmonic.

Remark 2. In case $S$ is not 1-dimensional the equation (ec) assumed only on $\overline{J}$ (analogously to (c)) would be sufficient for $\gamma_\lambda$ extended by the formula above to satisfy (ec) on $V$, in particular (eh) would be proved.

However, if $S$ is 1-dimensional the existence of $\gamma_\lambda$ satisfying (ec) on $\overline{J}$ is always true. Just take for $\gamma$ an argument of the direction tangent to $M$ the union of a finite family of real-analytic curves containing $\overline{J}$.

Now, for every $t \in T$ by $l_t : B(v_t, \delta) \to \mathbb{R}$ denote the harmonic conjugate to $\gamma_{\lambda(t)}$. Thus the function $G_t = \exp(l_t + i\gamma_{\lambda(t)}) : B(v_t, \delta) \to \mathbb{C}$ is holomorphic and denote by $\psi_t : B(v_t, \delta) \to \mathbb{C}$ a primitive of $G_t$. Fix $w \in \overline{J}$ and choose $t \in T$ such that $w \in B(v_t, \delta)$. Since $\psi_t'(w) = \exp(l_t(w) + i\gamma_{\lambda(t)}(w)) \neq 0$, there exists a disk $U_w \subset B(v_t, \delta)$ such that $\psi_t|_{U_w}$ is injective. Applying, as before Koebe's distortion theorem for arguments (see [Hi]) we may assume the disks $U_w$ to be so small that all the sets $\phi_i(U_w)$ are convex. We claim that the family $\{\psi_w : U_w \to \mathbb{C}\}_{w\in\overline{J}}$ forms an affine atlas for the iterated function system $S$. Indeed, fix $w, v \in \overline{J}$ and consider $t, t' \in T$ such that $U_w \subset B(v_t, \delta) \subset B_{\lambda(t)}$ and $U_v \subset B(v_{t'}, \delta) \subset B_{\lambda(t')}$. Then for every $z \in U_w \cap U_v$ we get

$$
\begin{aligned}
(\psi_w \circ \psi_v^{-1})'(\psi_v(z)) &= \psi_w'(z)(\psi_v'(z))^{-1} = G_{\lambda(t)}(z)G^{-1}_{\lambda(t')}(z) \\
&= \exp\big(l_t(z) + i\gamma_{\lambda(t)}(z) - l_{t'}(z) - i\gamma_{\lambda(t')}(z)\big) \\
&= \exp\big(i(\gamma_{\lambda(t)}(z) - \gamma_{\lambda(t')}(z)\big)\exp\big(l_t(z) - l_{t'}(z)\big).
\end{aligned}
$$

Since by (3.1) $\gamma_{\lambda(t)} - \gamma_{\lambda(t')}$ is constant on $z \in U_w \cap U_v \subset U_{\lambda(t)} \cap U_{\lambda(t')}$ and since $l_t$ and $l_{t'}$ differ on $U_{\lambda(t)} \cap U_{\lambda(t')}$ by an additive constant as harmonic conjugates to harmonic functions $\gamma_{\lambda(t)}$ and $\gamma_{\lambda(t')}$ respectively, we conclude that $(\psi_w \circ \psi_v^{-1})'$ is constant on $\psi_v(U_w \cap U_v)$.

Now fix $w, v \in \overline{J}$, $i \in I$, and write $C = \phi_i^{-1}(\phi_i(U_w) \cap U_v))$. Since $\phi_i(U_w) \cap U_v))$ is a convex set and therefore connected, its continuous image $C$ is also connected. Then there are $t, t' \in T$ such that $U_w \subset B(v_t, \delta) \subset B_{\lambda(t)}$, $U_v \subset B(v_{t'}, \delta) \subset B_{\lambda(t')}$ and $C$ is contained in a connected component of $B_{\lambda(t)} \cap \phi_i^{-1}(B_{\lambda'(t)})$. Using the chain rule we then get for all $z \in C$

$$
\begin{aligned}
(\psi_v \circ \phi_i \circ \psi_w^{-1})'(\psi_v(z)) &= \psi_v'(\phi_i(z))\phi_i'(z)(\psi_w'(z))^{-1} = G_{t'}(\phi_i(z))\phi_i'(z)G_t^{-1}(z) \\
&= \exp\big(i(\gamma_{\lambda(t')}(\phi_i(z))) + l_{t'}(\phi_i(z)) + \log|\phi_i'(z)| + i\arg_{\lambda(t)}\phi_i'(z) - i\gamma_{\lambda(t)}(z) - l_t(z)\big) \\
&= \exp\big(l_{t'}(\phi_i(z)) + \log|\phi_i'(z)| - l_t(z)\big)\exp\big(i(\arg_{\lambda(t)}\phi_i'(z) - \gamma_{\lambda(t)}(z) + \gamma_{\lambda(t')}(\phi_i(z)).
\end{aligned}
$$

Hence, using (3.2) we conclude that the derivative $(\psi_v \circ \phi_i \circ \psi_w^{-1})'$ has a constant argument on $\psi_v(C)$ and consequently $(\psi_v \circ \phi_i \circ \psi_w^{-1})'$ is constant on $\psi_v(C)$. The proof of the implication $(ec) \Rightarrow (d2)$ is complete.

- The implication $(a) \Rightarrow (f)$ is obvious.
- $(f) \Rightarrow (er)$. Suppose first that the system $S$ is 1-dimensional. Then the condition $\nabla\tilde{D}_{\phi_i} \equiv 0$ on $\overline{J}$ is similar (formally weaker) to $\tilde{D}_{\phi_i}$ constant in $(a)$. We prove $(er)$ similarly, via $(c) \Rightarrow (d1) \Rightarrow (eh)$.

Assume now that $S$ is not 1-dimensional. Suppose that $\nabla\tilde{D}_{\phi_i}=0$ on $\overline{J}$ for all $i\in I$. Since $S$ is not 1-dimensional, it implies that $\nabla D_{\phi_i}=0$ on $U$ for all $i\in I$. Thus $\tilde{D}_{\phi_i}=0$ is constant on $U$ for all $i\in I$, since $U$ is connected. So, the item (a) is proved in this case and therefore, in viev of what we have already proved, also (er2).

So, we may assume that there exists $j\in I$ and $w\in\overline{J}$ such that $\nabla D_{\phi_j}(w)\neq 0$. By continuity of the function $\nabla\tilde{D}_{\phi_j}$ there thus exists a neighbourhood $W\subset V$ of $w\in \mathbb{C}$ on which $\nabla\tilde{D}_{\phi_j}$ nowhere vanishes. Let us consider on $W$ the line field $l$ orthogonal to $\nabla\tilde{D}_{\phi_j}$. By the definition of the limit set $J$ for every $z\in\overline{J}$ there exists $\tau\in I^*$ such that $\phi_\tau(z)\in\overline{J}\cap W$. Then define

$$l(z)=(\phi_\tau^{-1})'_{\phi_\tau(z)}(l(\phi_\tau(z))), \tag{3.11}$$

where, changing temporarily notation, $(\phi_\tau^{-1})_{\phi_\tau(z)})'$ denotes the derivative of the map $\phi_\tau^{-1}$ evaluated at the point $\phi_\tau(z)$ and the display above expresses its action on a line element. We want to show first that in this manner we define a line field on $\overline{J}$. So, we need to show that if $\phi_\tau(z),\phi_\eta(z)\in\overline{J}\cap W$, then

$$(\phi_\tau^{-1})'_{\phi_\tau(z)}(l(\phi_\tau(z)))=(\phi_\eta^{-1})'_{\phi_\eta(z)}(l(\phi_\eta(z))). \tag{3.12}$$

Suppose on the contrary that (3.12) fails with some $z,\tau,\eta$ as required above. Then there exists a point $x\in W\cap\overline{J}$ and $\gamma\in I^*$ (in fact for every $x\in W$ there exists $\gamma$) such that $\phi_\gamma(x)$ is so close to $z$ that

$$(\phi_\tau^{-1})'_{\phi_\tau(\phi_\gamma(x))}(l(\phi_\tau(\phi_\gamma(x))))\neq(\phi_\eta^{-1})'_{\phi_\eta(\phi_\gamma(x))}(l(\phi_\eta(\phi_\gamma(x)))).$$

Hence

$$(\phi_{\tau\gamma}^{-1})'_{\phi_{\tau\gamma}(x)}l(\phi_{\tau\gamma}(x))\neq(\phi_{\eta\gamma}^{-1})'_{\phi_{\eta\gamma}(x)}l(\phi_{\eta\gamma}(x)).$$

So, either

$$(\phi_{\tau\gamma}^{-1})'_{\phi_{\tau\gamma}(x)}l(\phi_{\tau\gamma}(x))\neq l(x)$$

or

$$(\phi_{\eta\gamma}^{-1})'_{\phi_{\eta\gamma}(x)}l(\phi_{\eta\gamma}(x))\neq l(x)$$

Suppose for example the first incompatibility of $l$'s holds. Then

$$\det(\nabla\tilde{D}_{\phi_j}\circ\phi_{\tau\gamma}(x),\nabla\tilde{D}_{\phi_j}(x))\neq 0$$

contrary to our assumption. Thus the line field $l$ is well-defined on $\overline{J}$ and it immediately follows from the method this field is constructed that it is invariant with respect to all the contractions $\phi_i$, $i\in I$.

Notice that formula (3.11) defines an invariant line field on $V$. We can use any $\tau\in I^*$ such that $\phi_\tau(V)\subset W$. The resulting $l$ does not depend on $\tau$ because for any other such $\eta$ (3.12) holds for $z\in\overline{J}$, so it holds on entire $V$. Otherwise the system would be 1-dimensional because $l$ is real-analytic so the equation holds on a real-analytic set.

The argument $\arg l$ is of course defined up to integer multiplicity of $\pi$.

Using again Koebe's distortion theorem for arguments (see [Hi]), one can find $\{B_\lambda\}$, a finite cover of $\overline{J}$ by disks contained in $V$, small enough that all the images $\phi_i(B_\lambda)$, $i \in I$, are convex. Then all the intersections $B_\lambda \cap B_{\lambda'}$ and $B_\lambda \cap \phi_i^{(}B_{\lambda'})$ are connected.

Define $\gamma_\lambda$ as an arbitrary branch of $\arg l$ on $B_\lambda$. Then (3.1) and (3.2) follow from the invariance of $l$ by $S$, with constants $c(\lambda, \lambda'$ and $c(\lambda, \lambda', i)$ being multiplicities of $\pi$. Thus (er) is proved.

Remark 3. This is even stronger than (er) where the constants are any real numbers. Indeed the existence of an analytic invariant line field is a strictly stronger condition then others in Theorem 3.1. See [Pr] for an example.

■

**Definition 3.2** We call the iterated function system $S$ linear if one (or equivalently all) conditions of Theorem 3.1 is satisfied. Otherwise we call this system non-linear.

## §4. Rigidity.

We begin this section with the following.

**Proposition 4.1.** Suppose that $F = \{f_i : X \to X\}_{i \in I}$ and $G = \{g_i : Y \to Y\}_{i \in I}$ are two non-linear topologically conjugate systems. Suppose also that the measures $m_G$ and $m_F \circ h^{-1}$ are equivalent. If one of these systems is 1-dimensional, then so is the other one.

**Proof.** Suppose on the contrary that $G$ is not 1-dimensional. Then it follows from Theorem 3.3 that there exist $y \in J_G$, $j \in I$, $\omega \in I^*$ and a neighbourhood $W_2 \subset \mathbb{C}$ of $y$ such that the map

$$\mathcal{G} = (\tilde{D}_{g_j} \circ \gamma_\omega, \tilde{D}_{g_j})$$

is invertible on $W_2$. Since the measures $m_G$ and $m_F \circ h^{-1}$ are equivalent, after an appropriate normalization $\mu_F = \mu_G \circ h$ meaning that $D_h = \frac{d\mu_G \circ h}{d\mu_F} = 1$. Since $h \circ f_\tau = g_\tau \circ h$ for all $\tau \in I^*$ and since $D_h = 1$,

$$\mathcal{G} \circ h = \mathcal{F}$$

on $J$, where $\mathcal{F} = (\tilde{D}_{f_j} \circ \gamma_\omega, \tilde{D}_{f_j})$. Write $x = h^{-1}(y)$. Then $h = \mathcal{G}^{-1} \circ \mathcal{F}$ on $W_1 \cap J_F$ for some open neighbourhood $W_1$ of $x$ in $\mathbb{C}$ such that $\mathcal{F}(W_1) \subset \mathcal{G}(W_2)$. Since $\mathcal{F}, \mathcal{G}^{-1}$ are real-analytic, the image $\mathcal{G}^{-1} \circ \mathcal{F}(W_1 \cap M_F)$ for an adequate $W_1$ small enough is a real-analytic curve and $\mathcal{G}^{-1} \circ \mathcal{F}(W_1 \cap M_F) \cap J_G$ contains an open neighbourhood of $y$ in $J_G$. Now using Lemma 2.1 we conclude that $G$ is 1-dimensional. ■

The main result of this paper is contained in the following.

**Theorem 4.2.** If two Open Set Condition conformal regular iterated function systems $\{f_i : X \to X : i \in I\}$ and $\{g_i : Y \to Y : i \in I\}$ are non-linear and conjugate by a homeomorphism $h : J_F \to J_G$, then the following conditions are equivalent.

(a) The conjugacy between the systems $\{f_i : X \to X : i \in I\}$ and $\{g_i : Y \to Y : i \in I\}$ extends in a conformal fashion to an open neighbourhood of $X$.

(b) The conjugacy between the systems $\{f_i : X \to X : i \in I\}$ and $\{g_i : Y \to Y : i \in I\}$ extends in a real-analytic fashion to an open neighbourhood of $X$.

(c) The conjugacy between the systems $\{f_i : X \to X : i \in I\}$ and $\{g_i : Y \to Y : i \in I\}$ is bi-Lipschitz continuous.

(d) $|g'_\omega(y_\omega)| = |f'_\omega(x_\omega)|$ for all $\omega \in I^*$, where $x_\omega$ and $y_\omega$ are the only fixed points of $f_\omega : X \to X$ and $g_\omega : Y \to Y$ respectively.

(e) $\exists S \geq 1 \; \forall \omega \in I^*$
$$S^{-1} \leq \frac{\mathrm{diam}(g_\omega(Y))}{\mathrm{diam}(f_\omega(X))} \leq S.$$

(f) $\exists E \geq 1 \; \forall \omega \in I^*$
$$E^{-1} \leq \frac{||g'_\omega||}{||f'_\omega||} \leq E.$$

(g) $\mathrm{HD}(J_G) = \mathrm{HD}(J_F)$ and the measures $m_G$ and $m_F \circ h^{-1}$ are equivalent.

(h) The measures $m_G$ and $m_F \circ h^{-1}$ are equivalent.

**Proof.** The implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are obvious. That $(c) \Rightarrow (d)$ results from the fact that (c) implies condition (1) of Theorem 1.4 which in view of this theorem is equivalent with condition (2) of Theorem 1.4 which finally is the same as condition (d) of Theorem 4.2. The implications $(d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g)$ have been proved in Theorem 1.4. The implication $(g) \Rightarrow (h)$ is again obvious. We are left to prove that $(h) \Rightarrow (a)$. We shall first prove that $(h) \Rightarrow (b)$. So, suppose that $(h)$ holds. Then, after an appropriate normalization $\mu_F = \mu_G \circ h$ meaning that $D_h = \frac{d\mu_G \circ h}{d\mu_F} = 1$. If $F$ is 1-dimensional, then by Proposition 4.1, so is $G$ and the implication $(h) \Rightarrow (b)$ follows from Theorem 3.1 of [HU]. Hence, we may assume that neither system $F$ or $G$ is 1-dimensional. Therefore, since $G$ is non-linear, there exist $y \in J_G$, $j \in I$, $\omega \in I^*$ and a neighbourhood $W_2 \subset \mathbb{C}$ of $y$ such that the map
$$\mathcal{G} = (\tilde{D}_{g_j} \circ g_\omega, \tilde{D}_{g_j})$$
is invertible on $W_2$. Since $h \circ f_\tau = g_\tau \circ h$ for all $\tau \in I^*$ and since $D_h = 1$,
$$\mathcal{G} \circ h = \mathcal{F}$$
on $W_1 \cap J_f$, where $\mathcal{F} = (\tilde{D}_{f_j} \circ g_\omega, \tilde{D}_{f_j})$ and $W_1$ is a neighbourhood of $x = h^{-1}(y) \subset \mathbb{C}$. Since $\mathcal{G}$ is invertible on $W_2$, $\mathcal{G}(y) = \mathcal{F}(x)$ and $\mathcal{F}$ is continuous, we may assume that $\mathcal{F}(W_1) \subset \mathcal{G}(W_2)$. Hence $\mathcal{G}^{-1} \circ \mathcal{F}$ is well-defined on $W_1$ and $\mathcal{G}^{-1} \circ \mathcal{F}|_{W_1 \cap J_F} = h$. Consider now $\omega \in I^*$ such that $f_\omega(J_F) \subset W_1$. Since
$$\mathcal{G}^{-1} \circ \mathcal{F}(f_\omega(J_F)) = h \circ f_\omega(J_F) = g_\omega \circ h(J_F) = g_\omega(J_G) \subset g_\omega(V_G),$$
since $g_\omega(V_G)$ is open, since $f_\omega$ and $\mathcal{G}^{-1} \circ \mathcal{F}$ are continuous, there exists an open neighbourhood $V'_F \subset V_F$ of $X$ such that $f_\omega(V'_F) \subset W_1$ and $\mathcal{G}^{-1} \circ \mathcal{F}(f_\omega(V'_F)) \subset g_\omega(V_G)$. Hence, the map
$$g_\omega^{-1} \circ (\mathcal{G}^{-1} \circ \mathcal{F}) \circ f_\omega : V'_F \to \mathbb{C}$$
is well-defined, by Corollary 2.3 is real-analytic, and $g_\omega^{-1} \circ (\mathcal{G}^{-1} \circ \mathcal{F}) \circ f_\omega|_{J_F} = h$. Thus, the property (b) is proved. The last step of the proof of Theorem 4.2, that is the implication $(b) \Rightarrow (a)$ can be carried out similarly as the proof of Lemma 7.2.7 in [Pr]. ■

§**5. Rigidity of parabolic systems.** We first recall from [MU2] the concept of conformal parabolic iterated function systems. Let $X$ be a compact connected subset of a Euclidean space $I\!R^d$. Suppose that we have countably many conformal maps $\phi_i : X \to X$, $i \in I$, where $I$ has at least two elements and the following conditions are satisfied.

(5a) (Open Set Condition) $\phi_i(\mathrm{Int}(X)) \cap \phi_j(\mathrm{Int}(X)) = \emptyset$ for all $i \neq j$.

(5b) $|\phi_i'(x)| < 1$ everywhere except for finitely many pairs $(i, x_i)$, $i \in I$, for which $x_i$ is the unique fixed point of $\phi_i$ and $|\phi_i'(x_i)| = 1$. Such pairs and indices $i$ will be called parabolic and the set of parabolic indices will be denoted by $\Omega$. All other indices will be called hyperbolic.

(5c) $\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ if $\omega_n$ is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then $\phi_\omega$ extends conformally to an open connected set $V \subset I\!R^d$ and maps $V$ into itself.

(5d) If $i$ is a parabolic index, then $\bigcap_{n \geq 0} \phi_{i^n}(X) = \{x_i\}$ and the diameters of the sets $\phi_{i^n}(X)$ converge to 0.

(5e) (Bounded Distortion Property) $\exists K \geq 1$ $\forall n \geq 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ $\forall x, y \in V$ if $\omega_n$ is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then

$$\frac{|\phi_\omega'(y)|}{|\phi_\omega'(x)|} \leq K.$$

(5f) $\exists s < 1$ $\forall n \geq 1$ $\forall \omega \in I^n$ if $\omega_n$ is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then $||\phi_\omega'|| \leq s$.

(5g) (Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset I\!R^d$ there exists an open cone $\mathrm{Con}(x, \alpha, l) \subset \mathrm{Int}(X)$ with vertex $x$, central angle of Lebesgue measure $\alpha$, and altitude $l$.

(5h) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\big||\phi_i'(y)| - |\phi_i'(x)|\big| \leq L||\phi_i'|||y - x|^\alpha,$$

for every $i \in I$ and every pair of points $x, y \in V$.

We call such a system of maps $S = \{\phi_i : i \in I\}$ a subparabolic iterated function system. Let us note that conditions (5a),(5c),(5e)-(5g) are modeled on similar conditions which were used to examine hyperbolic conformal systems in Section 1. Condition (5h) also held for many of the systems studied in [MU1] but was not a general requirement. We need this condition in the sequel. If $\Omega \neq \emptyset$ we call the system $\{\phi_i : i \in I\}$ parabolic. As declared in (5b) the elements of the set $I \setminus \Omega$ are called hyperbolic. We extend this name to all the words appearing in (5e) and (5f). Fix a finite set $\tilde{\Omega} \supset \Omega$. For every $i \in \tilde{\Omega}$ denote

$$X_i = \bigcup_{j \in I \setminus \{i\}} \phi_j(X).$$

In this paper we also need the following technical condition whose meaning will be explained by Theorem 5.2 below. For all $i \in \tilde{\Omega}$

$$\sum_{n \geq 0} ||\phi_{i^n}'||_{X_i}^\alpha < \infty. \tag{5.i}$$

Since the set $\tilde{\Omega}$ is finite, the number

$$T = \max_{i \in \tilde{\Omega}} \left\{ \sum_{n \geq 0} ||\phi'_{i^n}||^{\alpha}_{X_i} \right\} \tag{5.1}$$

is finite. We would also like to recall that in [MU2] the main construction was to associate to a parabolic system $S$ an infinite but hyperbolic conformal iterated function system. Generalizing it a little bit, i.e. working with $\tilde{\Omega}$ instead of $\Omega$, this construction goes as follows. The system $S^*_{\tilde{\Omega}}$ is generated by $I_*$, the set of maps of the form $\phi_{i^n j}$, where $n \geq 1$, $i \in \tilde{\Omega}$, $i \neq j$, and the maps $\phi_k$, where $k \in I \setminus \tilde{\Omega}$. Note that $J_{S^*_{\tilde{\Omega}}} = J_S \setminus \{\phi_\omega(x_i) : i \in \tilde{\Omega}, \omega \in I^*\}$.

It immediately follows from our assumptions that the following is true (comp. Teorem 5.2 from [MU2]).

**Theorem 5.1.** If the system $S$ satisfies all the conditions (5a)-(5h), then the system $S^*_{\tilde{\Omega}}$ satisfies the conditions (1a)-(1d).

As a complement to this theorem we shall prove the following.

**Theorem 5.1'.** If the system $S$ satisfies all the conditions (5a)-(5i), then the system $S^*_{\tilde{\Omega}}$ satisfies the conditions (1a)-(1e).

**Proof.** In view of Theorem 5.1 we only need to prove condition (1e). So, fix $i \in \tilde{\Omega}$ and $j \in I \setminus \{i\}$. Consider arbitrary $n \geq 1$ and $x, y \in X$. Write $t = \min\{|\phi'_i(x)| : i \in \tilde{\Omega}, x \in X\} > 0$. We then have, assuming for example $|\phi'_{i^n j}(y)| < |\phi'_{i^n j}(x)|$,

$$\begin{aligned}
\Big||\phi'_{i^n j}(y)| - |\phi'_{i^n j}(x)|\Big| &= |\phi'_{i^n j}(x)| \left|1 - \frac{|\phi'_{i^n j}(y)|}{|\phi'_{i^n j}(x)|}\right| \leq ||\phi'_{i^n j}|| \left|\log \frac{|\phi'_{i^n j}(y)|}{|\phi'_{i^n j}(x)|}\right| \\
&\leq \Big|\log|\phi'_j(y)| - \log|\phi'_j(x)|\Big| + \sum_{k=0}^{n-1} \Big|\log|\phi'_i(\phi_{i^k j}(y)| - \log|\phi'_i(\phi_{i^k j}(x)|\Big| \\
&\leq \left( \frac{K}{||\phi'_j||} ||\phi'_j(y)| - |\phi'_j(x)|| + \sum_{k=0}^{n-1} \frac{1}{t} \, ||\phi'_i(\phi_{i^k j}(y)| - |\phi'_i(\phi_{i^k j}(x)|| \right) \\
&\leq \left( KL|y-x|^\alpha + \frac{1}{t} \sum_{k=0}^{n-1} L|\phi_{i^k j}(y) - \phi_{i^k j}(x)|^\alpha \right) \\
&\leq \left( KL|y-x|^\alpha + \frac{L}{t} \sum_{k=0}^{n-1} ||\phi'_{i^k}||^\alpha_{X_i} |\phi_j(y) - \phi_j(x)|^\alpha \right) \\
&\leq \left( KL + \frac{L}{t} \sum_{k=0}^{\infty} ||\phi'_{i^k}||^\alpha_{X_i} \right) |y-x|^\alpha \\
&\leq L\left(K + \frac{T}{t}\right) |y-x|^\alpha.
\end{aligned}$$

The proof is complete. ■

From now on we assume that the system $S$ satisfies all the conditions (5a)-(5i). We shall prove the following.

**Proposition 5.3.** If the system $S$ is regular and parabolic ($\Omega \neq \emptyset$), then the associated hyperbolic system $S^* = S^*_\Omega$ is non-linear.

**Proof.** We keep for the hyperbolic system $S^*$ the same notation and terminology as for the hyperbolic system $S$ in the sections 1-4. Theorem 5.7 from [MU2] says that the system $S^*$ is regular and the $\delta$-conformal measure for $S^*$ is also conformal for $S$. This permits us to extend for every $k \in I$ (even for parabolic $k$) the Jacobian

$$D_{\phi_k}(z) = \frac{\rho(\phi_k(z))}{\rho(z)}|\phi_k'(z)|^\delta.$$

In view of Theorem 2.1 all these functions $D_{\phi_k}$ have a real-analytic extensions on $U$. Suppose now on the contrary that the system $S^*$ is linear. Fix $i \in \Omega$ and $j \in I \setminus \{i\}$. There then exist two numbers $D_{ij}$ and $D_{i^2j}$ such that $D_{\phi_{ij}}(z) = D_{ij}$ and $D_{\phi_{i^2j}}(z) = D_{i^2j}$ for all $z \in U$. Now, for every $z \in X$

$$\begin{aligned}
D_{\phi_i}(\phi_{ij}(z)) &= \frac{\rho(\phi_{i^2j}(z))}{\rho(\phi_{ij}(z)}|\phi_i'\phi_{ij}(z))|\delta \\
&= \frac{\rho(\phi_{i^2j}(z))}{\rho(z)}|\phi_{i^2j}'(z)|^\delta \frac{\rho(z)}{\rho(\phi_{ij}(z)} \frac{1}{|\phi_{ij}'(z)|^\delta} \\
&= \frac{D_{\phi_{i^2j}}(z)}{D_{\phi_{ij}}(z)} = \frac{D_{i^2j}}{D_{ij}}.
\end{aligned}$$

Since $D_{\phi_i}$ is real-analytic on $U$ and since $\phi_{ij}(X) \supset \phi_{ij}(\mathrm{Int}(X))$, an open subset of $U$, we therefore conclude that $D_{\phi_i}(z) = D_{i^2j}/D_{ij} := D_i$ for every $z \in U$. Hence for every $z \in X$

$$\frac{\rho(\phi_{i^n}(z))}{\rho(z)}|\phi_{i^n}'(z)|^\delta = D_i^n \tag{5.2}$$

Applying this equality with $n = 1$ and $z = x_i$ we obtain

$$D_i = \frac{\rho(x_i)}{\rho(x_i)}|\phi_i'(x_i)| = |\phi_i'(x_i)| = 1.$$

Thus, it follows from (5.2) and (5.d) that for every $z \in X$

$$\lim_{n\to\infty} |\phi_{i^n}'(z)| = \left(\frac{\rho(z)}{\rho(\phi_{i^n}(z))}\right)^{1/\delta} = \rho(z)^{1/\delta} \tag{5.3}$$

Now, on one hand, in view of Theorem 3.8 in [MU1], $\rho(z) > 0$ for all $z \in J$ and, on the other hand, it follows from (5.3) and (5.i) that $\rho(z) = 0$ for all $z \in X$. This contradiction finishes the proof. ■

As an immediate consequence of this proposition we get the following.

**Corollary 5.4.** If the system $S$ is regular and parabolic ($\Omega \neq \emptyset$), then for every finite set $\tilde{\Omega} \supset \Omega$, the associated hyperbolic system $S^* = S^*_{\tilde{\Omega}}$ is non-linear.

The main result of this section is the following.

**Theorem 5.5.** If both topologically conjugate systems $F = \{f_i : X \to X, i \in I\}$ and $G = \{g_i : Y \to Y, i \in I\}$ are regular and at least one of them is parabolic, then the conditions listed in Theorem 4.2 are mutually equivalent where in the items $(d), (e), (f)$ the words $\omega$ are required to be hyperbolic.

**Proof.** Without loosing generality we may assume that the system $G$ is parabolic. Let $\tilde{\Omega} = \Omega_G \cup \Omega_F$ and let $F^*$ and $G^*$ be the corresponding hyperbolic systems. Let $J_F \to J_G$ be the topological conjugacy between the systems $F$ and $G$. The chain of implications $(a) \Rightarrow \ldots \Rightarrow (h)$ can be proved in exactly the same way as in the proof of Theorem 4.2. Notice that although $(h)$ establishes also a topological conjugacy between the systems $F^*$ and $G^*$, we could not invoke this fact to give a proof of implications $(a) \Rightarrow \ldots \Rightarrow (h)$ since not all hyperbolic words of $F$ (or $G$), for ex. the words of the form $iji$, $i \in \Omega_F$, $j \in I \setminus \Omega_F$, can be represented as concatenations of words from $F^*$ (or $G^*$).

To prove $(h) \Rightarrow (a)$, we can use the fact that $h$ establishes a topological conjugacy between the systems $F^*$ and $G^*$, apply Theorem 4.2 and Corollary 5.4. The proof is complete. ■

## Appendix 1. Conjugacies and scaling.

**Proof of Theorem 1.4.** Let us first demonstrate that conditions (2) and (3) are equivalent. Indeed, suppose that (2) is satisfied and let $K_F$ and $K_G$ denote the distortion constants of the systems $F$ and $G$ respectively. Then for all $\omega \in I^*$, $||g'_\omega|| \leq K_G|g'_\omega(y_\omega)| = K_G|f'_\omega(x_\omega)| \leq K_G||f'_\omega||$ and similarly $||f'_\omega|| \leq K_F||g'_\omega||$. So suppose that (3) holds and (2) fails, that is that there exists $\omega \in I^*$ such that $|g'_\omega(y_\omega)| \neq |f'_\omega(x_\omega)|$. Without loosing generality we may assume that $|g'_\omega(y_\omega)| < |f'_\omega(x_\omega)|$. For every $n \geq 1$ let $\omega^n$ be the concatenation of $n$ words $\omega$. Then $g_{\omega^n}(y_\omega) = g^n_\omega(y_\omega) = y_\omega$ and similarly $f_{\omega^n}(x_\omega) = x_\omega$. So, $x_{\omega^n} = x_\omega = \pi_F(\omega^\infty)$ and $y_{\omega^n} = y_\omega = \pi_G(\omega^\infty)$. Moreover $|g'_{\omega^n}(y_\omega)| = |g'_\omega(y_\omega)|^n$ and $|f'_{\omega^n}(x_\omega)| = |f'_\omega(x_\omega)|^n$. Hence

$$\lim_{n\to\infty} \frac{|g'_{\omega^n}(y_\omega)|}{|f'_{\omega^n}(x_\omega)|} = 0.$$

On the other hand, by (3) and the Bounded Distortion Property

$$\frac{|g'_{\omega^n}(y_\omega)|}{|f'_{\omega^n}(x_\omega)|} \geq \frac{K_G^{-1}||g'_{\omega^n}||}{||f'_{\omega^n}||} \geq E^{-1}K_G^{-1}$$

for all $n \geq 1$. This contradiction finishes the proof of equivalence of conditions (2) and (3). Since the equivalence of (1) and (3) is by (BDP2) and (BDP3) immediate, the proof of the

equivalence of conditions (1)-(3) is finished. We shall now prove that (3) $\Rightarrow$ (5). Indeed, it follows from (3) that $E^{-1}\psi_{G,n}(t) \le \psi_{F,n}(t) \le E\psi_{G,n}(t)$ for all $t \ge 0$ and all $n \ge 1$. Hence $P_G(t) = P_F(t)$ and therefore by Theorem 1.2, $\mathrm{HD}(J_G) = \mathrm{HD}(J_F)$. Denote this common value by $h$. Although we keep the same symbol for the homeomorphism establishing conjugacy between the systems $F$ and $G$, it will never cause misunderstandings.
Suppose now that both systems are regular (in fact assuming (3) regularity of one of these systems implies regularity of the other). Then for every $\omega \in I^*$

$$(K_F E)^{-h} \le \frac{K_F^{-h}||f'_\omega||^h}{||g'_\omega||^h} \le \frac{m_F(f_\omega(J_F))}{m_G(g_\omega(J_G))} \le \frac{||f'_\omega||^h}{K_G^{-h}||g'_\omega||^h} \le (EK_G)^h.$$

So, the measures $m_G$ and $m_F \circ h^{-1}$ are equivalent, and even more

$$(K_F E)^{-h} \le \frac{dm_F \circ h^{-1}}{dm_G} \le (EK_G)^h.$$

Let us show now that (5) $\Rightarrow$ (3). Indeed, if (5) is satisfied then the measure $\mu_F \circ h^{-1}$ is equivalent to $\mu_G$. Since additionally $\mu_F \circ h^{-1}$ and $\mu_G$ are both ergodic (see Theorem 3.8 of [MU], they are equal. Hence, using the equality $\mathrm{HD}(J_F) = \mathrm{HD}(J_G) := h$, we get

$$\begin{aligned}
||g'_\omega||^h &\asymp \int |g'_\omega|^h \, dm_G = m_G(g_\omega(J_G)) \asymp \mu_G(g_\omega(J_G)) \\
&= \mu_F \circ h^{-1}(g_\omega(J_G)) = \mu_F(f_\omega(J_F)) \asymp m_F(f_\omega(J_F)) \\
&= \int |f'_\omega|^h \, dm_F \asymp ||f'_\omega||^h
\end{aligned}$$

and raising the first and the last term of this sequence of comparabilities to the power $1/h$, we finish the proof of the implication (5) $\Rightarrow$ (3).
The equivalence of (4) and conditions (1) - (3) is now a relatively simple corollary. Indeed, to prove that (3) implies (4) fix a finite subset $T$ of $I$. By (3) $E^{-1} \le ||f'_\omega||/||g'_\omega|| \le E$ for all $\omega \in T^*$, and as every finite system is regular, the equivalence of measures $m_{G,T}$ and $m_{F,T} \circ h^{-1}$ follows from the equivalence of conditions (3) and (5) applied to the systems $\{f_i : i \in T\}$ and $\{g_i : i \in T\}$. If in turn (4) holds and $\omega \in I^*$, then $\omega \in T^*$, where $T$ is the (finite) set of letters making up the word $\omega$ and the measures $m_{G,T}$ and $m_{F,T} \circ h^{-1}$ are equivalent. Hence, by the equivalence of (2) and (5) applied to the systems $\{f_i : i \in T\}$ and $\{g_i : i \in T\}$ we conclude that $|g'_\omega(y_\omega)| = |f'_\omega(x_\omega)|$. Thus (2) follows and the proof of Theorem 1.4 is finished. ■

We now recall from [HU] the following.

**Definition.** A conformal system $S = \{\phi_i : X \to X : i \in I\}$ is said to be of *bounded geometry* if there exists $C \ge 1$ such that for all $i, j \in I$, $i \ne j$

$$\max\{\mathrm{diam}(\phi_i(X)), \mathrm{diam}(\phi_j(X))\} \le C\mathrm{dist}\big(\phi_i(X), \phi_j(X)\big).$$

**Theorem,** [HU]. If both conformal iterated function systems $F = \{f_i : X \to X : i \in I\}$ and $G = \{g_i : Y \to Y : i \in I\}$ are of bounded geometry, then the topological conjugacy $h : J_F \to J_G$ is bi-Lipschitz continuous if and only if the following two conditions are satisfied.

(a) $$Q^{-1} \le \frac{\operatorname{diam}(f_\omega(X))}{\operatorname{diam}(g_\omega(Y))} \le Q$$

for some $Q \ge 1$ and all $\omega \in I^*$.

(b) $$D^{-1} \le \frac{\operatorname{dist}\big(g_i(Y), g_j(Y)\big)}{\operatorname{dist}\big(f_i(X), f_j(X)\big)} \le D$$

for some $D \ge 1$ and all $i, j \in I$, $i \ne j$.

**Example 1.** For infinite system, even in $I\!R$, it is not true that (a) implies $h$ Lipschitz continuous. We shall construct such $F, G$, with bounded geometry.

Let $A_i = [\frac{1}{i}, \frac{1}{i} + \exp -2i]$ for $i = 2, 3, ...$ and $A_i' = [\exp -i, \exp -i + \exp -2i]$, $f_i : [0,1] \to A_i$ and $g_i : [0,1] \to A_i'$ affine, onto, preserving orientation. Let $h$ map the end points of $f_\omega([0,1])$ to the end points of $g_\omega([0,1])$ for all $\omega \in I^*$. Then $f$ extends uniquely, continuously, to the limit sets of the systems due to $\operatorname{diam}(f_\omega([0,1]), \operatorname{diam}(f_\omega([0,1]) \to 0$ if the length of $\omega$ tends to $\infty$. By the construction it is a continuous conjugacy, but it is not Lipschitz even on $\bigcup_i f_i(\{0,1\})$.

If the sets $X$ and $Y$ are both contained in the real line $I\!R$, then it can be relatively easily to prove that already conditions (a) and (b) (without boundedness of geometry) imply that the conjugacy $h$ is Lipschitz continuous.

**Appendix 2. The Radon-Nikodym derivative $\rho = d\mu/dm$ in the parabolic case.** To fix terminology, $\mu$ in this Appendix is a $\sigma$-finite $S$-invariant measure equivalent with $\delta$-conformal measure $m$. The existence and (obvious) uniqueness of $\mu$ up to a multiplicative constant have been proved in Corollary 5.11 of [MU2]. In this appendix we establish the continuity property of $\rho = d\mu/dm$ in the parabolic case. In order to complete terminology, by $\mu^*$ we will denote the unique probability measure that is $S^*$-invariant and equivalent with conformal measure $m$ and by $\rho^*$ the Radon-Nikodym derivative $\rho = d\mu^*/dm$. Our result in this appendix is the following.

**Theorem A2.1.** If a regular parabolic system $S$ satisfies all the conditions $(5a) - (5h)$ and the alphabet $I$ is finite, then the Radon-Nikodym derivative $\rho = d\mu/dm$ is continuous on the set $J \setminus \{x_i : i \in \Omega\}$

**Proof.** According to formula (5.1) from [MU2] and the definition of conformal measure we obtain
$$\frac{d\mu}{dm} = \rho^* + \sum_{k \ge 1} \sum_{i \in \Omega} (\rho^* \circ \phi_{i^k}) \cdot |\phi_{i^k}'|^\delta.$$

Given now $i \in \Omega$, $j \in I \setminus \Omega$ and $n \geq 0$ we shall prove the the series $\sum_{k\geq 1} |\phi_{i^k}'|^\delta$ converges absolutely uniformly on $\phi_{i^n j}(X)$. Indeed, fix $x \in X$. Then it follows from (5e) that putting $T_{i,j,n} = \inf\{|\phi_{i^n j}'(z)| : z \in X\} > 0$, we get

$$\sum_{k\geq 1} |\phi_{i^k}'(\phi_{i^n j}(x)))|^\delta = \sum_{k\geq 1} \frac{|\phi_{i^{k+n}j}'(x)|^\delta}{|\phi_{i^n j}'(x)|^\delta} \leq \frac{1}{T_{i,j,n}} \sum_{k\geq 1} ||\phi_{i^{k+n}j}'||^\delta$$
$$\leq \frac{K^\delta}{T_{i,j,n}^\delta} \sum_{k\geq 1} m(\phi_{i^{k+n}j}(X)) \leq \frac{K}{T_{i,j,n}}^\delta < \infty.$$

Since $\rho^*$ is bounded from above by $K^\delta$ we therefore conclude that the series

$$\Sigma(i) = \sum_{k\geq 1} (\rho^* \circ \phi_{i^k}) \cdot |\phi_{i^k}'|^\delta$$

converges absolutely uniformly on the set $\phi_{i^n j}(X)$. Employing now (5d) and using finiteness of $I$ we therefore deduce that the function $\Sigma(i)$ is continuous on the set

$$\bigcup_{j\neq i} \bigcup_{k\geq 0} \phi_{i^k j}(X) \supset J \setminus \{x_i\}.$$

Since $\Omega$ is finite we finally get that $\rho = \rho^* + \sum_{i\in\Omega} \Sigma(i)$ is continuous on the set $J \setminus \{x_i : i \in \Omega\}$. The proof is complete. ■

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