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Hausdorff measures versus equilibrium states of conformal infinite iterated function systems

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Abstract. In this paper, similarly as in [HU] and [HMU], we introduce and develope the ergodic theory of Hölder systems of functions. In the context of conformal infinite iterated function systems we prove the volume lemma, the Billingsley type result, for the Hausdorff dimension of the projection onto the limit set of a shift invariant measure. Our central result is to demonstrate in this context the appearence for equilibrium states the "singularity-absolute contnuity" dichotomy observed in [PUZ,I] and [PUZ,II] (comp. also [DU1] and [DU2]) in the setting of rational functions of the Riemann sphere.

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§**1. Introduction, Preliminaries.** The main goal of this paper is to develope the ergodic theory of Hölder systems of functions (comp. also [HU] and [HMU]) and to demonstrate in the context of conformal infinite iterated function systems the existence of phenomena observed in [PUZ,I] and [PUZ,II] (comp. also [DU1] and [DU2]) in the setting of rational functions of the Riemann sphere. Let us recall now that in [MU1] we have provided the framework to study conformal infinite iterated function systems. In order to recall this notion and its basic properties let X be a nonempty compact subset of a Euclidean space $I\!R^d$. Let I be a countable index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from X into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \le s\rho(x,y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called a hyperbolic iterated function system. We are particularly interested in the properties of the limit set defined by such a system. We can define this set as the image of the coding space under a coding map as follows. Let $I^* = \bigcup_{n\ge1} I^n$, the space of finite words, and for $\omega \in I^n$, $n \ge 1$, let $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \ge 1$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2\ldots\omega_n$. Since given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n \ge 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\omega)$, defines the coding map $\pi : I^\infty \to X$. The main object of our interest will be the limit set

$$J = \pi(I^\infty) = \bigcup_{\omega\in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|n}(X),$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if I is finite, then J is compact. However, our main interest concerns systems S which are infinite.

An iterated function system $S = \{\phi_i : X \to X : i \in I\}$, is said to satisfy the Open Set Condition (abbreviated (OSC)) if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\phi_i(U) \subset U$ for every $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \ne j$.

A hyperbolic iterated function system S satisfying (OSC), is said to be conformal (c.i.f.s.) if the following conditions are satisfied.

(a) X is a compact connected subset of a Euclidean space $I\!R^d$ and $U = \text{Int}_{I\!R^d}(X)$.
(b) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset I\!R^d$ there exists an open cone $Con(x, u_x, \alpha, l) \subset \text{Int}(X)$ with vertex x, direction vector u_x, central angle of Lebesgue measure α, and altitude l.
(c) There exists an open connected set $X \subset V \subset I\!R^d$ such that all maps ϕ_i, $i \in I$, extend to $C^{1+\varepsilon}$ diffeomorphisms on $\phi_i : V \to V$ and are conformal on V.

(d) Bounded Distortion Property(BDP). There exists $K \geq 1$ such that

$$|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\phi'_\omega(x)|$ means the norm of the derivative.

We provide below without proofs all the geometrical consequences of the bounded distortion property (d) derived in [MU1]. We have for all words $\omega \in I^*$ and all convex subsets C of V

$$\text{(BDP1)} \qquad \mathrm{diam}(\phi_\omega(C)) \leq ||\phi'_\omega||\mathrm{diam}(C)$$

and

$$\text{(BDP2)} \qquad \mathrm{diam}(\phi_\omega(V)) \leq D||\phi'_\omega||,$$

where the norm $||\cdot||$ is the supremum norm taken over V and $D \geq 1$ is a universal constant. Moreover,

$$\text{(BDP3)} \qquad \mathrm{diam}(\phi_\omega(X)) \geq D^{-1}||\phi'_\omega||$$

and

$$\text{(BDP4)} \qquad \phi_\omega(B(x,r)) \supset B(\phi_\omega(x), K^{-1}||\phi'_\omega||r),$$

for every $x \in X$, every $0 < r \leq \mathrm{dist}(X, \partial V)$, and every word $\omega \in I^*$. Also, there exists $0 < \beta \leq \alpha$ such that for all $x \in X$ and for all words $\omega \in I^*$

$$\text{(BDP5)} \quad \phi_\omega(\mathrm{Int}(X)) \supset \mathrm{Con}\big(\phi_\omega(x), \beta, D^{-1}||\phi'_\omega||\big) \supset \mathrm{Con}\big(\phi_\omega(x), \beta, D^{-2}\mathrm{diam}(\phi_\omega(V))\big),$$

where $\mathrm{Con}\big(\phi_\omega(x), \beta, D^{-1}||\phi'_\omega||\big)$ and $\mathrm{Con}\big(\phi_\omega(x), \beta, D^{-2}\mathrm{diam}(\phi_\omega(V))\big)$ denote some cones with vertices at $\phi_\omega(x)$, angles β, and altitudes $D^{-1}||\phi'_\omega||$ and $D^{-2}\mathrm{diam}(\phi_\omega(V))$ respectively. Frequently, refering to (BDP) we will mean either (BDP) itself or one of the properties (BDP1)-(BDP5). We will need also the following result stated in [MU1] as Lemma 2.6, where S^{d-1} is the $d-1$-dimensional unit sphere containd in $I\!R^d$ and λ_{d-1} is the $d-1$-dimensional Lebesgue measure in $I\!R^d$.

Lemma 1.1. If S is a conformal iterated function system, then for every $x \in X$ and every integer $n \geq 1$, we have $\#\pi_n^{-1}(x) \leq \lambda_{d-1}(S^{d-1})/\beta$. In particular, S is uniformly pointwise finite. More precisely $\sup_{x\in X} \#\{i \in I : x \in \phi_i(X)\} \leq \lambda_{d-1}(S^{d-1})/\beta < \infty$.

Our paper is organized as follows, In the second section which is introductory but interesting itself, similarly as in [HU] and [HMU], we introduce and develope the ergodic theory of Hölder systems of functions proving basic theorems concerning topological pressure, Perron-Frobenius operator, conformal and invariant measures and equilibrium states. The third section, similarly as in [HMU] is devoted to prove the volume lemma in our setting, the result proved many times in various contexts beginning from the classical work of Billingsley. The 4th section dealing with the Ionescu-Tulcea and Marinescu theorem and its spectral consequences improves on some considerations from [DU1] and [DU2]. Section 5 is modelled mainly on Section 3 of [DU1] and is devoted to derive stochastic consequences of the Ionescu-Tulcea and Marinescu theorem. The last section, Section 6, containing central conclusions combines and developes the approach from [DU1], [PUZ,I], and [PUZ,II] and adjusts it to the context of conformal iterated function systems.

§**2. Thermodynamic formalism for iterated function systems.** As in the previous section let I be a countable alphabet, let $I^* = \bigcup_{n\geq 1} I^n$ be the space of all finite words and let $\Sigma = I^\infty$ be the infinitely dimensional shift space equipped with the product topology. Denote by $\sigma : \Sigma \to \Sigma$ be the shift transformation (cutting out the first coordinate), $\sigma(\{x_n\}_{n=1}^\infty) = (\{x_n\}_{n=2}^\infty)$. We also consider $\sigma : I^* \to I^* \cup \{\emptyset\}$ defined similarly, $\sigma(\omega_1\omega_2,\ldots,\omega_n) = (\omega_2,\ldots,\omega_n)$, $\sigma(\omega_1) = \emptyset$. Fix $\beta > 0$. In this section $S = \{\phi_i : X \to X : i \in I\}$ is a hyperbolic iterated function system and $\phi = \{\phi^{(i)} : X \to I\!R : i \in I\}$ is a family of continuous functions such that if

$$V_n(\phi) = \sup_{\omega\in I^n}\sup_{x\in X}\{|\phi^{(\omega_1)}(\phi_{\sigma(\omega)}(x)) - \phi^{(\omega_1)}(\phi_{\sigma(\omega)}(y))|\}\mathrm{e}^{\beta n},$$

then

$$V_\beta(\phi) = \sup_{n\geq 1}\{V_n(\phi)\} < \infty \tag{2.1}$$

and $\mathcal{L}_\phi(1\!\!1) \in C(X)$, where $1\!\!1$ is the function identically equal to 1 and

$$\mathcal{L}_\phi(\psi)(x) = \sum_{i\in I}\mathrm{e}^{\phi^{(i)}(x)}\psi(\phi_i(x)),\ \psi \in C(X),$$

is the associated Perron-Frobenius operator. The family ϕ is then called a Hölder system of functions of order β. Notice that then $\mathcal{L}_\phi$ acts on $C(X)$ as a continuous operator and $||\mathcal{L}_\phi||_0 \leq \mathcal{L}_\phi(1\!\!1)$. Let $\mathcal{L}_\phi^* : C(X)^* \to C(X)^*$ be the dual operator and following [Bo] define the following map on the space of probability measures on X:

$$\nu \mapsto \frac{\mathcal{L}_\phi^*(\nu)}{\mathcal{L}_\phi^*(\nu)(1\!\!1)}$$

This map is continuous and therefore in view of the Shauder-Tichonov theorem it has a fixed point, say m_ϕ. Thus

$$\mathcal{L}_\phi^*(m_\phi) = \lambda m_\phi, \tag{2.2}$$

where $\lambda = \mathcal{L}_\phi^*(m_\phi)(1\!\!1)$. Following the classical thermodynamic formalism we define the topological pressure of ϕ by setting

$$\begin{aligned}
\mathrm{P}(\phi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \left\| \exp\left(\sum_{j=1}^{n} \phi^{\omega_j} \circ \phi_{\sigma^j \omega} \right) \right\|_0 \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\left(\sup_X \sum_{j=1}^{n} \phi^{\omega_j} \circ \phi_{\sigma^j \omega} \right),
\end{aligned}$$

where $\|\cdot\|_0$ denotes the supremum norm. Notice also that the limit above exists indeed since the partition function

$$Z_n(\phi) = \log \sum_{|\omega|=n} \left\| \exp\left(\sum_{j=1}^{n} \phi^{\omega_j} \circ \phi_{\sigma^j \omega} \right) \right\|_0$$

is subadditive. Given $n \geq 1$ and $\omega \in I^n$ denote $\sum_{j=1}^n \phi^{(\omega_j)} \circ \phi_{\sigma^j\omega} : X \to I\!R$ by $S_\omega(\phi) : X \to I\!R$. Let us then prove the following.

Lemma 2.1. If $x, y \in \phi_\tau(X)$ for some $\tau \in I^*$, then for all $\omega \in I^*$

$$|S_\omega(\phi)(x) - S_\omega(\phi)(y)| \leq \frac{V(\phi)}{1 - \mathrm{e}^{-\beta}} \mathrm{e}^{-\beta|\tau|}$$

Proof. Let $n = |\omega|$. Write $x = \phi_\tau(u)$, $y = \phi_\tau(w)$, where $u, w \in X$. By (2.1) we get

$$\begin{aligned}
\left| \sum_{j=1}^{n} \phi^{(\omega_j)}(\phi_{\sigma^j\omega}(x)) - \sum_{j=1}^{n} \phi^{(\omega_j)}(\phi_{\sigma^j\omega}(y)) \right| &= \left| \sum_{j=1}^{n} \phi^{(\omega\tau)_j} \circ \phi_{\sigma^j\omega\tau}(u) - \sum_{j=1}^{n} \phi^{(\omega\tau)_j} \circ \phi_{\sigma^j\omega\tau}(w) \right| \\
&\leq \sum_{j=1}^{n} \left| \phi^{(\omega\tau)_j} \circ \phi_{\sigma^j\omega\tau}(u) - \phi^{(\omega\tau)_j} \circ \phi_{\sigma^j\omega\tau}(w) \right| \\
&\leq \sum_{j=1}^{n} V(\phi) \mathrm{e}^{-\beta(n+|\tau|-j)} \\
&\leq \frac{V(\phi)}{1 - \mathrm{e}^{-\beta}} \mathrm{e}^{-\beta|\tau|}
\end{aligned}$$

The proof is finished. ■

Remark. We allow in Lemma 2.1 τ to be the empty word $\emptyset$. Then $\phi_\emptyset = \mathrm{Id}_X$ and $|\emptyset| = 0$.

Set

$$Q = \exp\left(\frac{V(\phi)}{1 - \mathrm{e}^{-\beta}} \right).$$

We shall prove the following.

Lemma 2.2. The eigenvalue λ (see 2.2) of the dual Perron-Frobenius operator is equal to $e^{P(\phi)}$.

Proof. Iterating (2.2) we get

$$\begin{aligned}\lambda^n = \lambda^n m_\phi(1\!\!1) = \mathcal{L}_\phi^{*n}(1\!\!1) = \int_X \mathcal{L}_\phi^n(1\!\!1)dm_\phi \\ = \int_X \sum_{|\omega|=n} \exp(S_\omega(\phi)(x)) \le \sum_{|\omega|=n} ||\exp(S_\omega(\phi))||_0.\end{aligned}$$

So,

$$\log\lambda \le \lim_{n\to\infty}\frac{1}{n}\log\sum_{|\omega|=n}||\exp(S_\omega(\phi))||_0 = P(\phi).$$

Fix now $\omega \in I^n$ and take a point x_ω where the function $S_\omega(\phi)$ takes on its maximum. In view of Lemma 2.1, for every $x \in X$ we have

$$\sum_{|\omega|=n}\exp(S_\omega(\phi)(x)) \ge Q^{-1}\sum_{|\omega|=n}\exp(S_\omega(\phi)(x_\omega)) = Q^{-1}\sum_{|\omega|=n}||\exp(S_\omega(\phi))||_0.$$

Hence, iterating (2.2) as before,

$$\lambda^n = \int_X \sum_{|\omega|=n}\exp(S_\omega(\phi))dm_\phi \ge Q^{-1}\sum_{|\omega|=n}||\exp(S_\omega(\phi))||_0.$$

So, $\log\lambda \ge \lim_{n\to\infty}\frac{1}{n}\log\sum_{|\omega|=n}||\exp(S_\omega(\phi))||_0 = P(\phi)$. The proof is finished. ■

Let $\mathcal{L}_0$ denote the normalized Perron-Frobenius operator, i.e. $\mathcal{L}_0 = e^{-P(\phi)}\mathcal{L}_\phi$. We shall prove the following.

Proposition 2.3. $m_\phi(J) = 1$.

Proof. Since by (2.2) and Lemma 2.2

$$\mathcal{L}_0^*(m_\phi) = m_\phi \tag{2.3}$$

and consequently $\mathcal{L}_0^{*n}(m_\phi) = m_\phi$ for all $n \ge 0$, we have

$$\int_X \sum_{|\omega|=n}\exp\big(S_\omega(\phi) - P(\phi)n\big)\cdot(f\circ\phi_\omega)dm_\phi = \int_X f dm_\phi \tag{2.4}$$

for all $n \ge 0$ and all continuous functions $f : X \to I\!\!R$. Since this equality extends to all bounded measurable functions f, we get

(2.5) $$m_\phi(A) = \sum_{\tau\in I^n}\int \exp\big(S_\tau(\phi)-P(\phi)n\big)\cdot 1\!\!1_{\phi_\omega(A)}\circ\phi_\tau dm_\phi \ge \int_A \exp\big(S_\omega(\phi)-P(\phi)n\big)dm_\phi$$

for all $n \geq 0$, all $\omega \in I^n$, and all Borel sets $A \subset X$, where $1\!\!1_E$ is the characteristic function of the set E. Now, for each $n \geq 1$ set $X_n = \bigcup_{|\omega|=n} \phi_\omega(X)$. Then $1\!\!1_{X_n} \circ \phi_\omega = 1\!\!1$ for all $\omega \in I^n$. Thus apllying (2.4) to the function $f = 1\!\!1_{X_n}$ and later to the function $f = 1\!\!1$, we obtain

$$
\begin{aligned}
m_\phi(X_n) &= \int_X \sum_{|\omega|=n} \exp\bigl(S_\omega(\phi) - \mathrm{P}(\phi)n\bigr) \cdot (1\!\!1_{X_n} \circ \phi_\omega) dm_\phi \\
&= \int_X \sum_{|\omega|=n} \exp\bigl(S_\omega(\phi) - \mathrm{P}(\phi)n\bigr) dm_\phi = \int 1\!\!1 dm_\phi = 1.
\end{aligned}
$$

Hence $m_\phi(J) = m_\phi\bigl(\bigcap_{n\geq 1} X_n\bigr) = 1$. The proof is complete. ■

Theorem 2.4. For all $n \geq 1$

$$Q^{-1} \leq \mathcal{L}_0^n(1\!\!1) \leq Q.$$

Proof. Given $n \geq 1$ by (2.4) there exits $x_n \in X$ such that $\mathcal{L}_0^n(1\!\!1)(x_n) \leq 1$. It then follows from Lemma 2.1 that for every $x \in X$, $\mathcal{L}_0^n(1\!\!1) \leq Q$. Similarly by (2.4) there exists $y_n \in X$ such that $\mathcal{L}_0^n(1\!\!1) \geq 1$. It then follows from Lemma 2.1 that for every $x \in X$, $\mathcal{L}_0^n(1\!\!1) \geq Q^{-1}$. The proof is finihed. ■

If $\omega \in I^*$ set $[\omega] = \{\tau \in I^\infty : \tau|_{|\omega|} = \omega\}$. We shall prove the following.

Lemma 2.5. There exixts a unique Borel probability measure $\tilde{m}_\phi$ on I^∞ such that $\tilde{m}_\phi([\omega]) = \int \exp\bigl(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\bigr) dm_\phi$ for all $\omega \in I^*$.

Proof. In view of (2.4), $\int \exp\bigl(S_\omega(\phi) - \mathrm{P}(\phi)n\bigr) dm_\phi = 1$ for all $n \geq 1$ and therefore one can define a Borel probability measure m_n on C_n, the algebra generated by the cylinder sets of the form $[\omega]$, $\omega \in I^n$, putting $m_n([\omega]) = \int \exp\bigl(S_\omega(\phi) - \mathrm{P}(\phi)n\bigr) dm_\phi$. Hence, applying (2.4) again we ge for all $\omega \in I^n$.

$$
\begin{aligned}
m_{n+1}([\omega]) &= \sum_{i\in I} m_{n+1}([\omega i]) = \sum_{i\in I} \int \exp\bigl(S_{\omega i}(\phi) - \mathrm{P}(\phi)(n+1)\bigr) dm_\phi \\
&= \int \sum_{i\in I} \exp\left(\sum_{j=1}^{n} \phi^{(\omega_j)} \circ \phi_{\sigma^j(\omega i)} - \mathrm{P}(\phi)n + \phi^{(i)} - \mathrm{P}(\phi)\right) dm_\phi \\
&= \int \sum_{i\in I} \exp\bigl(S_\omega \circ \phi_i - \mathrm{P}(\phi)n\bigr) \exp\bigl(\phi^{(i)} - \mathrm{P}(\phi)\bigr) dm_\phi \\
&= \int \mathcal{L}_0\bigl(\exp(S_\omega(\phi) - \mathrm{P}(\phi)n)\bigr) dm_\phi = \int \exp\bigl(S_\omega(\phi) - \mathrm{P}(\phi)n\bigr) dm_\phi = m_n([\omega])
\end{aligned}
$$

and therefore in view of Kolmogorov's extension theorem there exists a unique probability measure $\tilde{m}_\phi$ on I^∞ such that $\tilde{m}_\phi([\omega]) = \tilde{m}_{|\omega|}([\omega])$ for all $\omega \in I^*$. The proof is complete. ■

As an immediate consequence of this lemma and Lemma 2. 1we see that if R is a collection of incomparable words such that $\bigcup_{\omega\in R}[\omega] = I^\infty$, then we have
(2.5)

$$1 \le \sum_{\omega\in R} ||\exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)||_0 \le Q \text{ and } Q^{-1} \le \sum_{\omega\in R} \inf_X \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big) \le 1.$$

Lemma 2.6. The measures m_ϕ and $\tilde m_\phi \circ \pi^{-1}$ are equal.

Proof. Let $A \subset J$ be an arbitrary closed subset of J and for every $n \ge 1$ let $A_n = \{\omega \in I^n : \phi_\omega(X) \cap A \ne \emptyset\}$. In view of (2.4) applied to the characteristic function $1\!\!1_A$ we have for all $n \ge 1$

$$\begin{aligned} m_\phi(A) &= \sum_{\omega\in I^n} \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)(1\!\!1_A \circ \phi_\omega)\, dm_\phi \\ &= \sum_{\omega\in A_n} \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)(1\!\!1_A \circ \phi_\omega)\, dm_\phi \\ &\le \sum_{\omega\in A_n} \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)\, dm_\phi = \sum_{\omega\in A_n} \tilde m_\phi([\omega]) = \tilde m_\phi\big(\bigcup_{\omega\in A_n}[\omega]\big) \end{aligned}$$

Since the family of sets $\{\bigcup_{\omega\in A_n}[\omega] : n \ge 1\}$ is descending and $\bigcap_{n\ge1}\bigcup_{\omega\in A_n}[\omega] = \pi^{-1}(A)$ we therefore get $m_\phi(A) \le \lim_{n\to\infty} \tilde m_\phi\big(\bigcup_{\omega\in A_n}[\omega]\big) = \tilde m_\phi(\pi^{-1}(A))$. Since both measures m_ϕ and $\tilde m_\phi \circ \pi^{-1}$ are regular (as J is a metric separable space), this inequality extends to the family of all Borel subsets of J. Since both measures are probabilistic we get $m_\phi = \tilde m_\phi \circ \pi^{-1}$. The proof is finished. ■

Let us recall that in the beginning of this section by $\sigma : I^\infty \to I^\infty$ we have denoted the left shift map (cutting out the first coordinate) on I^∞. We also recall that a measure preserving endomorphism is said to be totally ergodic if and only if all its (positive) iterates are ergodic. Now we shall prove that the shift map $\sigma : I^\infty \to I^\infty$ has a unique invariant (totally ergodic) probability measure equivalent with $\tilde m_\phi$.

Theorem 2.7. There exists a unique σ-invariant probability measure $\tilde\mu_\phi$ absolutely continuous with respect to $\tilde m_\phi$. Moreover $\tilde\mu_\phi$ is equivalent with $\tilde m_\phi$, $Q^{-1} \le d\tilde\mu_\phi/d\tilde m_\phi \le Q$ and the dynamical system $\sigma : I^\infty \to I^\infty$ is totally ergodic with respect to the mesure $\tilde\mu_\phi$.

Proof. First notice that, using Lemma 2.5 and Lemma 2.1, for each $\omega \in I^*$ and each $n \ge 0$ we have

$$\begin{aligned} \tilde m_\phi(\sigma^{-n}([\omega])) &= \sum_{\tau\in I^n} \tilde m_\phi([\tau\omega]) = \sum_{\tau\in I^n} \int |\exp\big(S_{\tau\omega}(\phi) - \mathrm{P}(\phi)|\tau\omega|\big)\, dm_\phi \\ &\ge \sum_{\tau\in I^n} Q^{-1}||\exp\big(S_\tau(\phi) - \mathrm{P}(\phi)|\tau|\big)||_0 \int \exp\big(S_\omega(\phi - \mathrm{P}(\phi)|\omega|\big)\, dm_\phi \\ &= Q^{-1} \int \exp\big(S_\omega(\phi - \mathrm{P}(\phi)|\omega|\big)\, dm_\phi \sum_{\tau\in I^n} ||\exp\big(S_\tau(\phi - \mathrm{P}(\phi)|\tau|\big)||_0 \\ &\ge Q^{-1}\tilde m_\phi([\omega])\tilde m_\phi(I^\infty) = Q^{-1}\tilde m_\phi([\omega]) \end{aligned}$$

and

$$
\begin{aligned}
\tilde{m}_\phi(\sigma^{-n}([\omega])) &= \sum_{\tau\in I^n}\tilde{m}_\phi([\tau\omega]) = \sum_{\tau\in I^n}\int \exp\bigl(S_{\tau\omega}(\phi-\mathrm{P}(\phi)|\tau\omega|\bigr)\,dm_\phi \\
&\le \sum_{\tau\in I^n}||\exp\bigl(S_\tau(\phi-\mathrm{P}(\phi)|\tau|\bigr)||_0\int \exp\bigl(S_\omega(\phi)-\mathrm{P}(\phi)|\omega|\bigr)\,dm_\phi \\
&= \int \exp\bigl(S_\omega(\phi)-\mathrm{P}(\phi)|\omega|\bigr)\,dm_\phi \sum_{\tau\in I^n}||\exp\bigl(S_\tau(\phi)-\mathrm{P}(\phi)|\tau|\bigr)||_0 \\
&\le Q\tilde{m}_\phi([\omega]).
\end{aligned}
$$

Let now L be a Banach limit defined on the Banach space of all bounded sequences of real numbers. We define $\mu([\omega]) = L\bigl((\tilde{m}_\phi(\sigma^{-n}([\omega])))_{n\ge 0}\bigr)$. Hence $Q^{-1}\tilde{m}_\phi([\omega]) \le \mu([\omega]) \le Q\tilde{m}_\phi([\omega])$ and therefore it is not difficult to check that the formula $\mu(A) = L\bigl((\tilde{m}_\phi(\sigma^{-n}(A)))_{n\ge 0}\bigr)$ defines a finite non-zero finitely additive measure on Borel sets of I^∞ satisfying $Q^{-1}\tilde{m}_\phi(A)\le\mu(A)\le Q\tilde{m}_\phi(A)$. Using now a Calderon's theorem (see Theorem 3.13 of [Fr]) and its proof, one constructs a Borel probability (σ-additive) measure $\tilde{\mu}_\phi$ on I^∞ satisfying the formula

$$
Q^{-1}\tilde{m}_\phi(A)\le\tilde{\mu}_\phi(A)\le Q\tilde{m}_\phi(A)
$$

for all Borel sets $A\subset I^\infty$ with perhaps a larger constant Q. Thus, to complete the proof of our theorem we only need to show the total ergodicity of $\tilde{\mu}_\phi$ or equivalently of $\tilde{m}_\phi$. Toward this end take a Borel set $A\in I^\infty$ with $\tilde{m}_\phi(A)>0$. Using Lemma 2.5 and Lemma 2.1 it is straightforward to check that for every $\omega\in I^*$, $\tilde{m}_\phi(\omega A)\ge Q^{-1}||\exp\bigl(S_\omega(\phi)-\mathrm{P}(\phi)|\omega|\bigr)||_0\tilde{m}_\phi(A)>0$, where $\omega A=\{\omega\rho:\rho\in A\}$. Hence, since the nested family of sets $\{[\tau]:\tau\in I^*\}$ generates the Borel σ-algebra on I^∞, for every $n\ge 0$ and every $\omega\in I^n$ we can find a subfamily Z of I^* consisting of mutually incomparable words and such that $A\subset\bigcup\{[\tau]:\tau\in Z\}$ and $\sum_{\tau\in Z}\tilde{m}_\phi([\omega\tau])\le 2\tilde{m}_\phi(\omega A)$. Then

$$
\begin{aligned}
\tilde{m}_\phi\bigl(\sigma^{-n}(A)\cap[\omega]\bigr) = \tilde{m}_\phi(\omega A) &\ge \frac{1}{2}\sum_{\tau\in Z}\tilde{m}_\phi([\omega\tau]) = \frac{1}{2}\sum_{\tau\in Z}\int|\exp\bigl(S_{\omega\tau}(\phi-\mathrm{P}(\phi)|\omega\tau|\bigr)\,dm_\phi \\
&\ge \frac{1}{2}Q^{-1}|\exp\bigl(S_\omega(\phi-\mathrm{P}(\phi)|\omega|\bigr)||_0\sum_{\tau\in Z}\int|\exp\bigl(S_\tau(\phi-\mathrm{P}(\phi)|\tau|\bigr)\,dm_\phi \\
&\ge \frac{1}{2}Q^{-1}\int\exp\bigl(S_\omega(\phi-\mathrm{P}(\phi)|\omega|\bigr)\,dm\sum_{\tau\in Z}\tilde{m}_\phi([\tau]) \\
&\ge \frac{1}{2}Q^{-1}\tilde{m}_\phi([\omega])\tilde{m}_\phi\bigl((\bigcup\{[\tau]:\tau\in Z\}\bigr)\ge\frac{1}{2}Q^{-1}\tilde{m}_\phi(A)\tilde{m}_\phi([\omega]).
\end{aligned}
$$

Therefore $\tilde{m}_\phi\bigl(\sigma^{-n}(I^\infty\backslash A)\cap[\omega]\bigr) = \tilde{m}_\phi\bigl([\omega]\backslash(\sigma^{-n}(A)\cap[\omega])\bigr) = \tilde{m}_\phi([\omega]) - \tilde{m}_\phi\bigl(\sigma^{-n}(A)\cap[\omega] \le \bigl(1-(2Q)^{-1}\tilde{m}_\phi(A)\bigr)\tilde{m}_\phi([\omega])$. Hence for every Borel set $B\subset I^\infty$ with $\tilde{m}_\phi(B)<1$, for every $n\ge 0$, and for every $\omega\in I^n$ we get

$$
\tilde{m}_\phi(\sigma^{-n}(B)\cap[\omega]\bigr)\le\bigl(1-(2Q)^{-1}(1-\tilde{m}_\phi(B))\bigr)\tilde{m}_\phi([\omega]). \tag{2.6}
$$

In order to conclude the proof of the complete ergodicity of σ fix $r \geq 1$, and suppose that $\sigma^{-r}(B) = B$ with $0 < \tilde{m}_\phi(B) < 1$. Put $\gamma = 1 - (2Q)^{-1}(1 - \tilde{m}_\phi(B))$. Note that $0 < \gamma < 1$. In view of (2.6), for every $\omega \in (I^r)^*$ we get $\tilde{m}_\phi(B \cap [\omega]) = \tilde{m}_\phi\big(\sigma^{-|\omega|}(B) \cap [\omega]\big) \leq \gamma \tilde{m}_\phi([\omega])$. Take now $\eta > 1$ so small that $\gamma\eta < 1$ and choose a subfamily R of $(I^r)^*$ consisting of mutually incomparable words and such that $B \subset \bigcup\{[\omega] : \omega \in R\}$ and $\tilde{m}_\phi\big(\bigcup\{[\omega] : \omega \in R\}\big) \leq \eta \tilde{m}_\phi(B)$. Then $\tilde{m}_\phi(B) \leq \sum_{\omega \in R} \tilde{m}_\phi(B \cap [\omega]) \leq \sum_{\omega \in R} \gamma \tilde{m}_\phi([\omega]) = \gamma \tilde{m}_\phi\big(\bigcup\{[\omega] : \omega \in R\}\big) \leq \gamma\eta \tilde{m}_\phi(B) < \tilde{m}_\phi(B)$. This contradiction finishes the proof. ■

Theorem 2.8. *m_ϕ is the only probability measure m satisfying $\mathcal{L}_0(m) = m$.*

Proof. Since m_ϕ satisfies this equality we are only left to prove its uniqueness. So, let m_1 be another such a measure and let $\tilde{m}_1$ be the probability measure produced in Lemma 2.5 applied to the measure m_1. Then for every $\omega \in I^*$ we have $Q^{-1} \leq \tilde{m}_1([\omega])/\tilde{m}_\phi([\omega]) \leq Q$, whence $\tilde{m}_1$ and $\tilde{m}_\phi$ are equivalent and the Radon-Nikodym derivative $\rho = \frac{d\tilde{m}_1}{d\tilde{m}_\phi}$ satisfies $Q^{-1} \leq \rho \leq Q$. We also have $\tilde{m}_\phi([\sigma(\omega)]) = \int \exp\big(S_{\sigma(\omega)}(\phi) - \mathrm{P}(\phi)|\sigma(\omega)|\big)\, dm_\phi$ and therefore

$$
\begin{aligned}
\tilde{m}_\phi([\omega]) &= \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)\, dm_\phi \\
&= \int \exp\big(\phi^{(\omega_1)}(\phi_{\sigma(\omega)}(x)) - \mathrm{P}(\phi)\big) \exp\big(S_{\sigma(\omega)}(\phi)(x) - \mathrm{P}(\phi)|\sigma(\omega)|\big)\, dm_\phi(x).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\inf\{\exp\big(\phi^{(\omega_1)}(x) - \mathrm{P}(\phi)\big) : x \in \phi_{\sigma(\omega)}(X)\}\tilde{m}_\phi([\sigma(\omega)]) &\leq \tilde{m}_\phi([\omega]) \\
&\leq \sup\{\exp\big(\phi^{(\omega_1)}(x) - \mathrm{P}(\phi)\big) : x \in \phi_{\sigma(\omega)}(X)\}\tilde{m}_\phi([\sigma(\omega)]).
\end{aligned}
$$

Since $\phi^{(\omega_1)}$ is a continuous function on X we thus obtain that for every $\omega \in I^\infty$

$$
\lim_{n\to\infty} \frac{\tilde{m}_\phi\big([\omega|_n]\big)}{\tilde{m}_\phi\big([\sigma(\omega)|_{n-1}]\big)} = \exp\big(\phi^{(\omega_1)}(\pi(\sigma(\omega))) - \mathrm{P}(\phi)\big) \tag{2.7}
$$

and the same formula is true with $\tilde{m}_\phi$ replaced by $\tilde{m}_1$. In view of Theorem 2.7 there exists a set of points $\omega \in I^\infty$ with $\tilde{m}_\phi$ measure 1 for which the Radon-Nikodym derivatives $\rho(\omega)$ and $\rho(\sigma(\omega))$ both are defined. Let $\omega \in I^\infty$ be such a point. Then using (2.7) and its version for $\tilde{m}_1$ we obtain

$$
\begin{aligned}
\rho(\omega) = \lim_{n\to\infty} \left(\frac{\tilde{m}_1\big([\omega|_n]\big)}{\tilde{m}_\phi\big([\omega|_n]\big)}\right) &= \lim_{n\to\infty} \left(\frac{\tilde{m}_1\big([\omega|_n]\big)}{\tilde{m}_1\big([\sigma(\omega)|_{n-1}]\big)} \cdot \frac{\tilde{m}_1\big([\sigma(\omega)|_{n-1}]\big)}{\tilde{m}_\phi\big([\sigma(\omega)|_{n-1}]\big)} \cdot \frac{\tilde{m}_\phi\big([\sigma(\omega)|_{n-1}]\big)}{\tilde{m}_\phi\big([\omega|_n]\big)}\right) \\
&= \exp\big(\phi^{(\omega_1)}(\pi(\sigma(\omega))) - \mathrm{P}(\phi)\big)\rho(\sigma(\omega)) \exp\big(\phi^{(\omega_1)}(\pi(\sigma(\omega))) - \mathrm{P}(\phi) = \rho(\sigma(\omega))
\end{aligned}
$$

But since, in view of Theorem 2.7, σ is ergodic with respect to $\tilde{m}_\phi$, we conclude that ρ is $\tilde{m}_\phi$-almost everywhere constant. Since moreover $\tilde{m}_1$ and $\tilde{m}_\phi$ are both probabilistic and equivalent, $\tilde{m}_1 = \tilde{m}_\phi$. So, applying Lemma 2.6 finishes the proof. ■

We call a Borel probability measure m on J, ϕ-conformal if

$$m(\phi_\omega(X)) = \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)\,dm,\ \omega \in I^* \tag{2.8}$$

and

$$m(\phi_\omega(X) \cap \phi_\tau(X)) = 0 \tag{2.9}$$

for all incomparable words $\omega, \tau \in I^*$. Notice that in fact it suffices to require (2.8) and (2.9) to hold for words of length 1. Now we shall prove the existence (and uniqueness) of ϕ-conformal measures. In fact we shall show that every measure fulfilling slightly weaker requirements than being a fixed point of the dual operator $\mathcal{L}_0^*$, is conformal.

An iterated function system $\{\phi_i : i \in I\}$ by definition does not have ovrlaps if $\phi_i(X) \cap \phi_j(X) = \emptyset$ for all $i, j \in I$, $i \neq j$.

Lemma 2.9. Suppose that the iterated function system $\{\phi_i : i \in I\}$ does not have overlaps or it is conformal. A Borel probability measure ν on X is ϕ-conformal if and only if $\nu(\phi_\omega(A)) \geq \int_A \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)\big)\,d\nu$ for all $\omega \in I^*$ and for all Borel subsets A of X.

Proof. That conformal measures satisfy the requirements appearing in this lemma follows immediately from their definition. In order to prove the harder part first we shall show that condition (2.9) is satisfied, then that $\nu(J) = 1$, and finally that (2.8) holds. If the system does not have overlaps then (2.9) is immediate. So, suppose that it is conformal and suppose on the contrary that $\nu(\phi_\rho(X) \cap \phi_\tau(X)) > 0$ for some $q \geq 1$ and two distinct words $\rho, \tau \in I^q$. Let $E = \phi_\rho(X) \cap \phi_\tau(X)$ and for every $n \geq 1$ let $E_n = \bigcup_{\omega \in I^n} \phi_\omega(E)$. Since each element of E_n admits at least two different codes of length $n + q$ which agree on the initial segment of length n, it follows from Lemma 1.1 that $\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n = \emptyset$. On the other hand, by (2.5) and Lemma 1.1, we get $\nu(E_n) \geq Q^{-1}\beta\lambda_{d-1}S^{d-1}\nu(E)$, thus $\nu\big(\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n\big) \geq Q^{-1}\beta\lambda_{d-1}S^{d-1}\nu(E) > 0$. This contradiction shows that

$$\nu(\phi_\rho(X) \cap \phi_\tau(X)) = 0 \tag{2.10}$$

for all incomparable words $\rho, \tau \in I^*$. From now on the proof runs simultaneously for confromaI systems and those without overlaps. In order to show that $\nu(J) = 1$ suppose to the contrary that $\nu(X \setminus J) > 0$. In view of (2.10) for all $\omega \in I^*$ we have $\nu(\phi_\omega(X \setminus J) \cap J) = \nu\big(\bigcup_{\tau \in I^{|\omega|}} \phi_\omega(X \setminus J) \cap \phi_\tau(J)\big) \leq \sum_{\tau \in I^{|\omega|}} \nu(\phi_\omega(X \setminus J) \cap \phi_\tau(J)) = 0$. Hence setting $E_n = \bigcup_{\omega \in I^n} \phi_\omega(X \setminus J)$ we get $\nu\big(J \cap \bigcup_{n \geq 1} E_n\big) = 0$. On the other hand, as above, $\nu(E_n) \geq Q^{-1}\nu(X \setminus J)$ (because of (2.10) we could skip the factor $\beta\lambda_{d-1}^{-1}(S^{d-1})$ here) and therefore $\nu\big(\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n\big) \geq K^{-\delta}\nu(X \setminus J) > 0$. Moreover

$$\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n \subset \bigcap_{k=1}^\infty \Big(\bigcup_{n=k}^\infty \bigcup_{\omega \in I^n} \phi_\omega(X)\Big) = \bigcap_{k=1}^\infty \bigcup_{\omega \in I^k} \phi_\omega(X) = J.$$

Combining the formulae occuring at the ends of the last three sentences we fall into a contradiction which proves that $\nu(J) = 1$.

Now we need and we are in position to prove, that the measure m_ϕ is ϕ-conformal. Indeed, m_ϕ satisfies all conditions placed in the right-hand side of Lemma 2.9. Moreover, using (2.10), (2.3), and Lemma 2.5, given an integer $n \ge 1$, we can write $1 = m_\phi(X) = m_\phi\big(\bigcup_{\omega\in I^n} \phi_\omega(X)\big) = \sum_{\omega\in I^n} m_\phi(\phi_\omega(X)) \ge \sum_{\omega\in I^n} \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)\, dm_\phi = 1$. Therefore $m_\phi(\phi_\omega(X)) = \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)\, dm_\phi$ for all $\omega \in I^n$. Define now two finite measures m_1 and m_2 on X in the following way: $m_1(A) = \int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)\, dm_\phi$ and $m_2(A) = m_\phi(\phi_\omega(A))$. Since we know that $m_1(X) = m_2(X)$ and $m_1(A) \le m_2(A)$ for all Borel sets A, we conclude that $m_1 = m_2$. Hence, conformality of m_ϕ is proved.

Let us now return to the measure ν. We shall show that m_ϕ is absolutely continuous with respect ν. Indeed, it follows from conformality of m_ϕ and (BDP) that $Q^{-1}||\exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)||_0 \le m_\phi(\phi_\omega(X)) \le ||\exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)||_0$ for all $\omega \in I^*$. Since, by the assumptions, $\nu(\phi_\omega(X)) \ge Q^{-1}||\exp\big(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\big)||_0$, we therefore obtain $m_\phi(\phi_\omega(X)) \le Q\nu(\phi_\omega(X))$. So, using (2.10), we conclude that m_ϕ is absolutely continuous with respect to ν and $\rho = dm_\phi/d\nu \le Q$ ν-a.e.. Repeating essentially the argument from the proof of Theorem 2.8 to show that ρ is almost everywhere constant, we proceed as follows. In view of Lemma 2.6 and Theorem 2.7 there exists a set of points $\omega \in I^\infty$ with $\tilde m_\phi$ measure 1 for which the Radon-Nikodym derivatives $\rho\circ\pi(\omega)$ and $\rho\circ\pi(\sigma(\omega))$ both are defined. Let $\omega \in I^\infty$ be such a point. Then

$$
\begin{aligned}
\rho\circ\pi(\omega) &= \lim_{n\to\infty}\left(\frac{m_\phi(\phi_{\omega|_n}(X))}{\nu(\phi_{\omega|_n}(X))}\right)\\
&= \lim_{n\to\infty}\left(\frac{m_\phi(\phi_{\omega|_n}(X))}{m_\phi(\phi_{\sigma(\omega)|_{n-1}}(X))}\cdot\frac{m_\phi(\phi_{\sigma(\omega)|_{n-1}}(X))}{\nu(\phi_{\sigma(\omega)|_{n-1}}(X))}\cdot\frac{\nu(\phi_{\sigma(\omega)|_{n-1}}(X))}{\nu(\phi_{\omega|_n}(X))}\right)\\
&\le \lim_{n\to\infty}\left(\frac{\int_{\phi_{\sigma(\omega)|_{n-1}}(X)}\exp(\phi^{(\omega_1)} - \mathrm{P}(\phi))\,dm_\phi}{m_\phi(\phi_{\sigma(\omega)|_{n-1}}(X))}\right)\cdot\rho(\pi(\sigma(\omega)))\cdot\\
&\cdot\lim_{n\to\infty}\frac{\nu\big(\phi_{\sigma(\omega)|_{n-1}}(X)\big)}{\int_{\phi_{\sigma(\omega)|_{n-1}}(X)}\exp(\phi^{(\omega_1)}(x) - \mathrm{P}(\phi))\,d\nu(x)}\\
&= \exp(\phi^{(\omega_1)}(\pi(\sigma(\omega))) - \mathrm{P}(\phi))\rho(\pi(\sigma(\omega)))\big(\exp(\phi^{(\omega_1)}(\pi(\sigma(\omega))) - \mathrm{P}(\phi))\big)^{-1}\\
&= \rho(\pi(\sigma(\omega)))
\end{aligned}
$$

So, by the Birkhoff ergodic theorem, $\rho\circ\pi(\omega)$ is m_ϕ-a.e. constant and so is the Radon-Nikodym derivative $\rho : J \to [0,\infty)$. Keep the same symbol ρ for this value. Since both measures m and ν are probabilistic, $\rho \ge 1$. In the proof of the previous theorem we were done at this point concluding that $\rho = 1$ since $\tilde m_1$ and $\tilde m_\phi$ were equivalent. Here an additional argument is needed. And indeed, if $\rho > 1$ m_ϕ-almost everywhere, define the set $Z = \{x \in J : \rho(x) = 0\}$. Then $\nu(Z) = 1 - 1/\rho > 0$. We claim that

$$
\nu((J\setminus Z)\cap\phi_\omega(Z)) = 0 \tag{2.11}
$$

for all $\omega \in I^*$. Indeed, if $\nu((J\setminus Z)\cap\phi_\omega(Z)) > 0$ for some $\omega\in I^*$, then $m_\phi(\phi_\omega(Z)) \ge m_\phi((J\setminus Z)\cap\phi_\omega(Z)) = \rho\nu((J\setminus Z)\cap\phi_\omega(Z)) > 0$ which by conformality of m_ϕ implies that $m_\phi(Z) > 0$. This contradiction finishes the proof of (2.11). But now it follows from (2.11) that the probability measure $\nu|_Z/\nu(Z)$ satisfies the assumptions of the right-hand side of Lemma 2.9, hence from what has been proved we conclude that m_ϕ is absolutely continuous with respect to $\nu|_Z/\nu(Z)$. This however contradicts the definition of the set Z and finishes the proof. ■

Corollary 2.10. m_ϕ is the only probability measure satisfying $\mathcal{L}_0^*(m_\phi) = m_\phi$ and m_ϕ is ϕ-conformal. Also m_ϕ-almost every point $x \in J$ has a unique representation in the form $x = \pi(\omega)$, $\omega \in I^\infty$, that is $\pi^{-1}(x)$ is a singleton. In particular in view of Theorem 2.7 and Lemma 2.6 the measure $\mu_\phi = \tilde{\mu}_\phi \circ \pi^{-1}$ is equivalent with m_ϕ with bounded Radon-Nikodym derivatives.

In the and of this section we further investigate the σ-invariant measure $\tilde{\mu}_\phi$ introduced in Theorem 2.7. Let $\rho = \frac{d\tilde{\mu}_\phi}{d\tilde{m}_\phi}$. We begin with the following technical result.

Lemma 2.11. The following 3 conditions are equivalent

(a) $\int -\phi d\tilde{\mu}_\phi < \infty$.

(b) $\sum_{i\in I} \inf(-\phi|_{[i]}) \exp(\inf \phi|_{[i]}) < \infty$.

(c) $\mathrm{H}_{\tilde{\mu}_\phi}(\alpha) < \infty$, where $\alpha = \{[i] : i \in I\}$ is the partition of I^∞ into initial cylinders of length 1.

Proof. Suppose that $\int -\phi d\tilde{\mu}_\phi < \infty$. It means that $\sum_{i\in I} \int_{[i]} -\phi d\tilde{\mu}_\phi < \infty$ and consequently, using Theorem 2.7 and Lemma 2.5, we get

$$\begin{aligned}
\infty &> \sum_{i\in I} \inf(-\phi|_{[i]}) \int_{[i]} d\tilde{\mu}_\phi = \sum_{i\in I} \inf(-\phi|_{[i]}) \int_{[i]} \rho d\tilde{m}_\phi \\
&\geq Q^{-1} \sum_{i\in I} \inf(-\phi|_{[i]}) \tilde{m}_\phi([i]) = Q^{-1} \sum_{i\in I} \inf(-\phi|_{[i]}) \int_X \exp(\phi^{(i)}(x) - \mathrm{P}(\phi)) dm_\phi(x) \\
&= Q^{-1} \mathrm{e}^{-\mathrm{P}(\phi)} \sum_{i\in I} \inf(-\phi|_{[i]}) \int_X \exp(\phi^{(i)}(x)) dm_\phi(x)
\end{aligned}$$

Thus

$$\begin{aligned}
\infty &> \sum_{i\in I} \inf(-\phi|_{[i]}) \int_X \exp(\phi^{(i)}(x)) dm_\phi(x) \geq \sum_{i\in I} \inf(-\phi|_{[i]}) \exp(\inf_X(\phi^{(i)}) \\
&= \sum_{i\in I} \inf(-\phi|_{[i]}) \exp(\inf \phi|_{[i]}).
\end{aligned}$$

Now suppose that $\sum_{i\in I} \inf(-\phi|_{[i]}) \exp(\inf \phi|_{[i]}) < \infty$. We shall show that $\mathrm{H}_{\tilde{\mu}_\phi}(\alpha) < \infty$. So, using Theorem 2.7 again,

$$\mathrm{H}_{\tilde{\mu}_\phi}(\alpha) = \sum_{i\in I} -\tilde{\mu}_\phi([i]) \log \tilde{\mu}_\phi([i]) \leq \sum_{i\in I} -Q^{-1} \tilde{m}_\phi([i]) \big(\log \tilde{m}_\phi([i]) - \log Q\big).$$

But $\sum_{i\in I} -Q^{-1}\tilde{m}_\phi([i])(-\log Q) = Q^{-1}\log Q$, so it suffices to show that

$$\sum_{i\in I} -\tilde{m}_\phi([i])\log \tilde{m}_\phi([i]) < \infty.$$

But

$$\sum_{i\in I} -\tilde{m}_\phi([i])\log \tilde{m}_\phi([i]) = \sum_{i\in I} -\tilde{m}_\phi([i])\log\left(\int_X \exp\big(\phi^{(i)} - \mathrm{P}(\phi)\big)\right) dm_\phi$$
$$\leq \sum_{i\in I} -\tilde{m}_\phi([i])(\inf_X(\phi^{(i)}) - \mathrm{P}(\phi)).$$

But $\sum_{i\in I}\tilde{m}_\phi([i])\mathrm{P}(\phi) = \mathrm{P}(\phi)$, so it suffices to show that $\sum_{i\in I\!N} -\tilde{m}_\phi([i])\inf_X(\phi^{(i)}) < \infty$. And indeed,

$$\sum_{i\in I} -\tilde{m}_\phi([i])\inf_X(\phi^{(i)}) = \sum_{i\in I}\tilde{m}_\phi([i])\sup_X(-\phi^{(i)}) \leq \sum_{i\in I}\tilde{m}_\phi([i])\big(\inf_X(-\phi^{(i)}) + \log Q\big).$$

Since $\sum_{i\in I}\tilde{m}_\phi([i])\log Q = \log Q$, it is enough to show that $\sum_{i\in I}\tilde{m}_\phi([i])\inf_X(-\phi^{(i)}) < \infty$. And indeed,

$$\sum_{i\in I}\tilde{m}_\phi([i])\inf_X(-\phi^{(i)}) = \sum_{i\in I}\int \exp\big(\phi^{(i)} - \mathrm{P}(\phi)\big)dm_\phi \inf_X(-\phi^{(i)})$$

But since $\mathcal{L}_\phi(1\!\!1) \in C(X)$, $\phi^{(i)}$ are negative everywhere for all i large enough, say $i \geq k$. Then using Lemma 2.1 again we get

$$\sum_{i\geq k}\tilde{m}_\phi([i])\inf_X(-\phi^{(i)}) \leq \mathrm{e}^{-\mathrm{P}(\phi)}Q\sum_{i\geq k}\exp(\inf_X(\phi^{(i)}))\inf_X(-\phi^{(i)})$$

which is finite due to our assumption. Hence, $\sum_{i\in I\!N}\tilde{m}_\phi([i])\inf_X(-\phi^{(i)}) < \infty$. . Finally suppose that $\mathrm{H}_{\tilde{\mu}_\phi}(\alpha) < \infty$. We need to show that $\int -\phi d\tilde{\mu}_\phi < \infty$. We have

$$\infty > \mathrm{H}_{\tilde{\mu}_\phi}(\alpha) = \sum_{i\in I} -\tilde{\mu}_\phi([i])\log\big(\tilde{\mu}_\phi([i])\big) \leq \sum_{i\in I} -Q\tilde{m}_\phi([i])\big(\inf(\phi|_{[i]}) - \mathrm{P}(\phi) - 2\log Q\big).$$

Hence $\sum_{i\in I} -\tilde{m}_\phi([i])\inf(\phi|_{[i]}) < \infty$ and therefore

$$\int -\phi d\tilde{\mu}_\phi = \sum_{i\in I}\int_{[i]} -\phi d\tilde{\mu}_\phi \leq \sum_{i\in I}\sup(-\phi|_{[i]})\tilde{m}_\phi([i]) = \sum_{i\in I} -\inf(\phi|_{[i]})\tilde{m}_\phi([i]) < \infty.$$

The proof is complete. ■

Abusing a little bit notation we can define a function $\phi : I^\infty \to I\!\!R$ by setting

$$\phi(\omega) = \phi^{(\omega_1)}(\pi(\sigma(\omega))).$$

Since $\{\phi^{(i)} : i \in I\}$ is a Hölder system of functions of order β, ϕ is a Hölder continuous function of order β meaning that $V_\beta(\phi) = \sup_{n\geq 1}\{e^{\beta n} V_n(\phi)\} < \infty$, where

$$V_n(\phi) = \sup\{|\phi(\omega) - \phi(\tau)| : \omega|_n = \tau|_n\}.$$

It is easy to see that

$$\mathrm{P}(\phi) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{|\omega|=n} \exp\left(\sup_{\tau\in[\omega]} \sum_{j=0}^{n-1} \phi\circ\sigma^j(\tau)\right)\right), \tag{2.12}$$

where the topological pressure $\mathrm{P}(\phi)$ has been defined at the beginning of the section. It is also not difficult to check (see [HU] and [MU2]) that formula (2.12) gives the same value as the definition introduced by Sarig in [Sa]. Therefore, it follows from Theorem 3 of [Sa] that $\sup\{\mathrm{h}_\mu(\sigma) + \int \phi d\mu\} = \mathrm{P}(\phi)$, where the supremum is taken over all σ-invariant probability measures such that $\int -\phi d\mu < \infty$. We call a σ-invariant probability measure μ an equilibrium state of the potential ϕ if $\mathrm{h}_\mu(\sigma) + \int \phi d\mu = \mathrm{P}(\phi)$. We shall prove the following.

Theorem 2.12. If $\sum_{i\in I} \inf(-\phi|_{[i]}) \exp(\inf \phi|_{[i]}) < \infty$, then $\tilde{\mu}_\phi$ is an equilibrium state of the potential ϕ satisfying $\int -\phi d\tilde{\mu}_\phi < \infty$.

Proof. It follows from Lemma 2.11 that $\int -\phi d\tilde{\mu}_\phi < \infty$. To show that $\tilde{\mu}_\phi$ is an equilibrium state of the potential ϕ consider $\alpha = \{[i] : i \in I\}$, the partition of I^∞ into initial cylinders of length one. By Lemma 2.11, $\mathrm{H}_{\tilde{\mu}_\phi}(\alpha) < \infty$. Applying the Breiman-Shanon-McMillan theorem, the Birkhoff ergodic theorem, and using Theorem 2.7 and Lemma 2.5, we therefore get for $\tilde{\mu}_\phi$-a.e. $\omega \in \Sigma$

$$\begin{aligned}
\mathrm{h}_{\tilde{\mu}_\phi}(\sigma) \geq \mathrm{h}_{\tilde{\mu}_\phi}(\sigma, \alpha) &= \lim_{n\to\infty} \frac{-1}{n} \log(\tilde{\mu}_\phi([\omega|_n])) \\
&= \lim_{n\to\infty} \frac{-1}{n} \log\left(\int \exp\big(S_\omega(\phi) - \mathrm{P}(\phi)n\big) dm_\phi\right) \\
&= \lim_{n\to\infty} \frac{-1}{n} \log\left(\int \exp\big(\sum_{j=0}^{n-1} \phi(\sigma^j(\omega|_n\tau)) d\tilde{m}_\phi(\tau) - \mathrm{P}(\phi)n\big)\right) \\
&\geq \liminf_{n\to\infty} \frac{-1}{n} \log\left(\exp\big(\sum_{j=0}^{n-1} \phi(\sigma^j(\omega)) - \log Q - \mathrm{P}(\phi)n\big)\right) \\
&= \lim_{n\to\infty} \frac{-1}{n} \sum_{j=0}^{n-1} \phi(\sigma^j(\omega)) + \mathrm{P}(\phi) = -\int \phi d\tilde{\mu}_\phi + \mathrm{P}(\phi).
\end{aligned}$$

Hence $h_{\mu_\phi}(\sigma) + \int \phi d\tilde{\mu}_\phi \geq P(\phi)$, which in view of the variational principle (see Theorem 3 in [Sa]) implies that $\tilde{\mu}_\phi$ is an equilibrium state for the potential ϕ. The proof is finished. ■

§**3. Volume Lemma.** Recal that if ν is a finite Borel measure on X, then $HD(\nu)$, the Hausdorff dimension of ν, is the infimum of Hausdorff dimensions of sets of ful measure ν. From now on throughout the whole paper we assume that the system $\{\phi_i : i \in I\}$ is conformal. By $\alpha = \{[i] : i \in I\}$ we denote the partition of I^∞ into initial cylinders of length 1. If μ is a Borel shift-invariant ergodic probability measure on I^∞ by $h_\mu(\sigma)$ we denote its entropy with respect to the shift map σ and by $\chi_\mu(\sigma) = \int g d\mu$ its characteristic Lyapunov exponent, where

$$g(\omega) = -\log |\phi'_{\omega_1}(\pi(\sigma(\omega)))|.$$

Note that $g(\omega) \geq -\log s$ for all $\omega \in I^\infty$ and therefore $\chi_\mu(\sigma) \geq -\log s > 0$. In this section we shall prove the following.

Theorem 3.1.(Volume Lemma) Suppose that μ is a Borel shift-invariant ergodic probability measure on I^∞ such that

$$\mu \circ \pi^{-1}\big(\phi_\omega(X) \cap \phi_\tau(X)\big) = 0 \tag{3.1}$$

for all incomparable words $\omega, \tau \in I^*$. If

(a) the series $\sum_{i \in I} -\mu([i]) \log(||\phi'_i||_0)$ converges and $H_\mu(\alpha) < \infty$
or

(b) $H_\mu(\alpha) < \infty$ and $\chi_\mu(\sigma) < \infty$
then

$$HD(\mu \circ \pi^{-1}) = \frac{h_\mu(\sigma)}{\chi_\mu(\sigma)},$$

where $H_\mu(\alpha)$ is the entropy of the partition α with respect to the measure μ.

Proof. If (a) holds then the series $\sum_{i \in I} -\mu([i]) \log(||\phi'_i||_0)$ converges and, using (BDP), we conclude that the function g is integrable. So, in case (a) the assumptions of (b) are also satisfied and from now on we assume that (b) holds. Since $H_\mu(\alpha) < \infty$ and α is a generating partition, the entropy $h_\mu(\sigma) = h_\mu(\sigma, \alpha) \leq H_\mu(\alpha)$ is finite. Thus, in view of the Birkhoff ergodic theorem and the Breimann-Shannon-McMillan theorem there exists a set $I_0 \subset I^\infty$ such that $\mu(I_0) = 1$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g \circ \sigma^j(\omega) = \chi_\mu(\sigma) \quad \text{and} \quad \lim_{n\to\infty} \frac{-\log(\mu([\omega|_n]))}{n} = h_\mu(\sigma) \tag{3.2}$$

for all $\omega \in I_0$. Fix now $\omega \in I_0$ and $\eta > 0$. For $r > 0$ let $n = n(\omega, r) \geq 0$ be the least integer such that $\phi_{\omega|_n}(X) \subset B(\pi(\omega), r)$. Then

$$\log\big(\mu \circ \pi^{-1}(B(\pi(\omega), r))\big) \geq \log\big(\mu \circ \pi^{-1}(\phi_{\omega|_n}(X))\big) \geq \log(\mu([\omega|_n]) \geq -(h_\mu(\sigma) + \eta)n$$

for every $r > 0$ small enough (which implies that $n = n(\omega, r)$ is large enough) and $\mathrm{diam}\big(\phi_{\omega|_{n-1}}(X)\big) \geq r$. The last inequality along with (BDP) imply that

$$\begin{aligned}\log r &\leq \log\big(\mathrm{diam}(\phi_{\omega|_{n-1}}(X))\big) \leq \log\big(DK|\phi'_{\omega|_{n-1}}(\pi(\sigma^{n-1}(\omega)))|\big) \\ &\leq \log(DK) + \sum_{j=1}^{n-1} \log|\phi'_{\omega_j}(\pi(\sigma^j(\omega)))| \leq \log(DK) - (n-1)(\chi_\mu(\sigma) - \eta)\end{aligned}$$

for all $r > 0$ small enough. Therefore, for these r

$$\begin{aligned}\frac{\log\big(\mu\circ\pi^{-1}(B(\pi(\omega), r))\big)}{\log r} &\leq \frac{-(\mathrm{h}_\mu(\sigma) + \eta)n}{\log(DK) - (n-1)(\chi_\mu(\sigma) - \eta)} \\ &= \frac{\mathrm{h}_\mu(\sigma) + \eta}{\frac{-\log(DK)}{n} + \frac{n-1}{n}(\chi_\mu(\sigma) - \eta)}.\end{aligned}$$

Hence letting $r \to 0$, and consequently $n(\omega, r) \to \infty$, we obtain

$$\limsup_{r\to 0} \frac{\log\big(\mu\circ\pi^{-1}(B(\pi(\omega), r))\big)}{\log r} \leq \frac{\mathrm{h}_\mu(\sigma) + \eta}{\chi_\mu(\sigma) - \eta}.$$

Since η was an arbitrary positive number we finitely obtain

$$\limsup_{r\to 0} \frac{\log\big(\mu\circ\pi^{-1}(B(\pi(\omega), r))\big)}{\log r} \leq \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}$$

for all $\omega \in I_0$. Hence (see [Ma], [PU]), as $\mu\circ\pi^{-1}(\pi(I_0)) = 1$, $\mathrm{HD}(\mu\circ\pi^{-1}) \leq \mathrm{h}_\mu(\sigma)/\chi_\mu(\sigma)$. Let now $J_1 \subset J$ be an arbitrary Borel set such that $\mu\circ\pi^{-1}(J_1) > 0$. Fix $\eta > 0$. In view of (3.2) and Jegorov's theorem there exist $n_0 \geq 1$ and a Borel set $\tilde{J}_2 \subset \pi^{-1}(J_1)$ such that $\mu(\tilde{J}_2) > \mu(\pi^{-1}(J_1))/2 > 0$,

$$\mu([\omega|_n]) \leq \exp\big((-\mathrm{h}_\mu(\sigma) + \eta)n\big) \tag{3.3}$$

and $|\phi'_{\omega|_n}(\pi(\sigma^n(\omega))| \geq \exp\big((-\chi_\mu(\sigma) - \eta)n\big)$ for all $n \geq n_0$ and all $\omega \in \tilde{J}_2$. Due to the (BDP), the last inequality implies that there exists $n_1 \geq n_0$ such that

$$\mathrm{diam}\big(\phi_{\omega|_n}(X)\big) \geq (DK)^{-1}\mathrm{e}^{(-\chi_\mu(\sigma)-\eta)n} \geq \mathrm{e}^{-(\chi_\mu(\sigma)+2\eta)n} \tag{3.4}$$

for all $n \geq n_1$ and all $\omega \in \tilde{J}_2$. Given now $0 < r < \exp\big(-(\chi_\mu(\sigma) + 2\eta)n_1\big)$ and $\omega \in \tilde{J}_2$ let $n(\omega, r) \geq 0$ be the least number n such that $\mathrm{diam}\big(\phi_{\omega|_{n+1}}(X)\big) < r$. Using (3.4) we deduce that $n(\omega, r) + 1 > n_1$, hence $n(\omega, r) \geq n_1$ and $\mathrm{diam}\big(\phi_{\omega|_n}(X)\big) \geq r$. In view of Lemma 2.7 of [MU1] there exists a universal constant $L \geq 1$ such that for every $\omega \in \tilde{J}_2$ and $0 < r < \exp\big(-(\chi_\mu(\sigma) + 2\eta)n_1\big)$ there exist $k \leq L$ points $\omega^{(1)}, \ldots, \omega^{(k)} \in \tilde{J}_2$ such that

$\pi(\tilde{J}_2) \cap B(\pi(\omega), r) \subset \bigcup_{j=1}^k \phi_{\omega^{(j)}|_{n(\omega^{(j)},r)}}(X)$. Let $\tilde{\mu} = \mu|_{\tilde{J}_2}$ be the restriction of the measure μ to the set $\tilde{J}_2$. Using (3.1), (3.3) and (3.4) we get

$$
\begin{aligned}
\tilde{\mu} \circ \pi^{-1}(B(\pi(\omega), r)) &\leq \sum_{j=1}^{k} \mu \circ \pi^{-1}\big(\phi_{\omega^{(j)}|_{n(\omega^{(j)},r)}}(X)\big) = \sum_{j=1}^{k} \mu\big([\omega^{(j)}|_{n(\omega^{(j)},r)}]\big) \\
&\leq \sum_{j=1}^{k} \exp\big((-\mathrm{h}_\mu(\sigma) + \eta) n(\omega^{(j)}, r)\big) \\
&= \sum_{j=1}^{k} \Big(\exp\big(-(\chi_\mu(\sigma) + 2\eta)(n(\omega^{(j)}, r) + 1)\big)\Big)^{\frac{n(\omega^{(j)},r)}{n(\omega^{(j)},r)+1} \cdot \frac{-\mathrm{h}_\mu(\sigma)+\eta}{-(\chi_\mu(\sigma)+2\eta)}} \\
&\leq \sum_{j=1}^{k} \mathrm{diam}\big(\phi_{\omega^{(j)}|_{n(\omega^{(j)},r)+1}}(X)\big)^{\frac{n(\omega^{(j)},r)}{n(\omega^{(j)},r)+1} \cdot \frac{\mathrm{h}_\mu(\sigma)-\eta}{\chi_\mu(\sigma)+2\eta}} \leq \sum_{j=1}^{k} r^{\frac{n(\omega^{(j)},r)}{n(\omega^{(j)},r)+1} \cdot \frac{\mathrm{h}_\mu(\sigma)-\eta}{\chi_\mu(\sigma)+2\eta}} \\
&\leq L r^{\frac{\mathrm{h}_\mu(\sigma)-2\eta}{\chi_\mu(\sigma)+2\eta}},
\end{aligned}
$$

where the last inequality sign was written assuming n_1 so large that $\frac{n_1}{n_1+1} \cdot \frac{\mathrm{h}_\mu(\sigma)-\eta}{\chi_\mu(\sigma)+2\eta} \geq \frac{\mathrm{h}_\mu(\sigma)-2\eta}{\chi_\mu(\sigma)+2\eta}$. Hence (see [Ma],[PU]), $\mathrm{HD}(J_1) \geq \mathrm{HD}(\pi(\tilde{J}_2)) \geq \frac{\mathrm{h}_\mu(\sigma)-2\eta}{\chi_\mu(\sigma)+2\eta}$ and since η was an arbitrary number $\mathrm{HD}(J_1) \geq \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}$. Thus $\mathrm{HD}(\mu \circ \pi^{-1}) \geq \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}$ and the proof is complete. ■

Remark 3.2. Note that proveing $\mathrm{HD}(\mu \circ \pi^{-1}) \leq \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}$ we did not use the property $\mu([\omega]) = \mu \circ \pi^{-1}(\phi_\omega(X))$, $\omega \in I^*$, which is equivalent with (3.1).

Corollary 3.3. If $\{\phi_i : i \in I\}$ is a Hölder system of functions and the series

$$\sum_{i \in I} -m_\phi(\phi_i(X)) \log(||\phi_i'||_0) \text{ and } \sum_{i \in I} -m_\phi(\phi_i(X)) \log(m_\phi(\phi_i(X)))$$

converge, then

$$\mathrm{HD}(m_\phi) = \mathrm{HD}(\mu_\phi) = \frac{\mathrm{h}_\mu(\sigma)}{\chi_\mu(\sigma)}.$$

Proof. The proof is an immediate consequence of Theorem 3.1, Lemma 2.6, and Corollary 2.10. ■

§4. Ionescu-Tulcea and Marinescu theorem.

Let

$$\mathcal{H}_0 = \{f : I^\infty \to \mathbb{C} : f \text{ is bounded and continuous}\}$$

and for every $\alpha > 0$ let

$$\mathcal{H}_\alpha = \{f \in \mathcal{H}_0 : V_\alpha(f) < \infty\},$$

where

$$V_\alpha(f) = \sup\left\{\frac{|f(\omega) - f(\tau)|}{d_\alpha(\omega, \tau)} : \omega, \tau \in I^\infty, \omega \neq \tau \text{ and } \omega_1 = \tau_1\right\}$$

and $d_\alpha(\omega, \tau) = \mathrm{e}^{-\alpha k}$, k - the maximal integer such that $\omega|_k = \tau|_k$. Notice that $\mathcal{H}_0$ is a Banach space with the supremum norm $||\cdot||_0$ and each $\mathcal{H}_\alpha$ is a Banach space with the norm $||\cdot||_\alpha$ defined by the formula $||f||_\alpha = ||f||_0 + V_a(f)$. Now we introduce the main object of this section, the normalized Perron-Frobenius operator $L_0 : \mathcal{H}_0 \to \mathcal{H}_0$ defined as follows.

$$L_0(f)(\omega) = \sum_{\tau \in \sigma^{-1}(\omega)} \exp\big(\phi(\tau) - \mathrm{P}(\phi)\big) f(\tau) = \sum_{i \in I} \exp\big(\phi(i\omega) - \mathrm{P}(\phi)\big) f(i\omega),$$

where, let us recall from Section 2, $\phi(\tau) = \phi^{(\tau_1)}(\pi(\sigma(\tau)))$.

Theorem 4.1. The normalized operator $L_0 : \mathcal{H}_0 \to \mathcal{H}_0$ preserve the space $\mathcal{H}_\beta$ and moreover there exist constants $0 < \gamma < 1$, $C > 0$, and an integer $q \geq 1$ such that for every $f \in \mathcal{H}_\beta$

$$||L_0^q(f)||_\beta \leq \gamma ||f||_\beta + C||f||_0.$$

Proof. Let $\tau, \rho \in I^\infty$, $\tau|_k = \rho|_k$ and $\tau_{k+1} \neq \rho_{k+1}$ for some $k \geq 1$. Then for every $n \geq 1$

$$\begin{aligned}
&L_0^n(f)(\rho) - L_0^n(f)(\tau) \\
&\quad= \sum_{\omega \in I^n} \exp(S_n\phi(\omega\rho) - \mathrm{P}(\phi)) f(\omega\rho) - \sum_{\omega \in I^n} \exp(S_n\phi(\omega\tau) - \mathrm{P}(\phi)) f(\omega\tau) \\
&\quad= \sum_{\omega \in I^n} \exp(S_n\phi(\omega\rho) - \mathrm{P}(\phi)) \big(f(\omega\rho) - f(\omega\tau)\big) \\
&\quad+ \sum_{\omega \in I^n} f(\omega\tau) \big(\exp(S_n\phi(\omega\rho) - \mathrm{P}(\phi)) - \exp(S_n\phi(\omega\tau) - \mathrm{P}(\phi))\big)
\end{aligned} \tag{4.1}$$

But $|f(\omega\rho) - f(\omega\tau)| \leq V_\beta(f)\mathrm{e}^{-\beta(n+k)}$ and therefore employing Theorem 2.4 and the definition of ϕ, we obtain

$$\begin{aligned}
\sum_{\omega \in I^n} \exp(S_n\phi(\omega\rho) - \mathrm{P}(\phi)n)|f(\omega\rho) - f(\omega\tau)| &\leq V_\beta(f)\mathrm{e}^{-\beta(n+k)} Q \\
&\leq \mathrm{e}^{-\beta n} Q ||f||_\beta d_\beta(\rho, \tau)
\end{aligned} \tag{4.2}$$

Now notice that there exists a constant $M \geq 1$ such that $|1 - \mathrm{e}^x| \leq M|x|$ for all x with $|x| \leq \mathrm{e}^{-\beta} \log Q$. Since by Lemma 2.1, $|S_n\phi(\omega\rho) - S_n\phi(\omega\tau)| \leq \mathrm{e}^{-\beta k} \log Q \leq \mathrm{e}^{-\beta} \log Q$ we can estimate as follows.

$$\begin{aligned}
&|\exp(S_n\phi(\omega\rho) - \mathrm{P}(\phi)n) - \exp(S_n\phi(\omega\tau) - \mathrm{P}(\phi)n)| \\
&\qquad= \exp(S_n\phi(\omega\rho) - \mathrm{P}(\phi)n)|1 - \exp(S_n\phi(\omega\tau) - S_n\phi(\omega\rho))| \\
&\qquad\leq M \exp(S_n\phi(\omega\rho) - \mathrm{P}(\phi)n)|S_n\phi(\omega\rho) - S_n\phi(\omega\tau)| \\
&\qquad\leq M \exp(S_\phi(\omega\rho) - \mathrm{P}(\phi)n) \log Q \mathrm{e}^{-\beta k} \\
&\qquad= M \log Q \exp(S_n\phi(\omega\rho) - \mathrm{P}(\phi)n) d_\beta(\rho, \tau)
\end{aligned}$$

Hence, using Theorem 2.4 again, we get

$$\sum_{\omega\in I^n} |f(\omega\tau)||\exp(S_n\phi(\omega\rho)-\mathrm{P}(\phi)n)-\exp(S_n\phi(\omega\tau)-\mathrm{P}(\phi)n)|$$
$$\le ||f||_0 M\log Q d_\beta(\rho,\tau)\sum_{\omega\in I^n}\exp(S_n\phi(\omega\rho)-\mathrm{P}(\phi)n)$$
$$\le MQ\log Q||f||_0 d_\beta(\rho,\tau)$$

Combining this inequality, (4.2) and (4.1), we finally get

$$|L_0^n(f)(\rho)-L_0^n(f)(\tau)|\le \mathrm{e}^{-\beta n}Q||f||_\beta d_\beta(\rho,\tau)+MQ\log Q||f||_0 d_\beta(\rho,\tau).$$

Taking now n so large that $\gamma=\mathrm{e}^{-\beta n}Q<1$ finishes the proof. ■

Applying now the theorem of Ionescu-Tulcea and Marinescu (see [IM], comp. [PU]) and Theorem 2.7 we shall prove the following.

Theorem 4.2. If L_0 is the normalized operator, then

(a) $\lambda=1$ is the only eigenvalue of modulus 1 for $L_0:\mathcal{H}_0\to\mathcal{H}_0$ and its eigenspace E has dimension 1. In fact, $\psi=\frac{d\tilde{\mu}_\phi}{d\tilde{m}_\phi}$ has a version in $\mathcal{H}_\beta$ and $E=\mathbb{C}\psi$

(b) $L_0=P+S$, where $P:\mathcal{H}_0\to E$ is a projector from $\mathcal{H}_0$ to E, $P\circ S=S\circ P=0$ and $\sup_{n\ge1}||S^n||_0<\infty$.

(c) S acts on $\mathcal{H}_\beta$ and there exist constants $M>0$ and $0<\gamma_1<1$ such that

$$||S^n||_\beta\le M\gamma_1^n$$

for every $n\ge1$.

Proof. Since, by the Ascoli-Arzela theorem the unit ball in $\mathcal{H}_\beta$ is compact as a subset of $\mathcal{H}_0$, Theorem 4.1 and Theorem 2.4 complete the demostration that the assumptions of the Ionescu-Tulcea and Marinescu theorem (see [IM]) are satisfied. The latter claims that $\sigma(L_0)\cap S^1$ is finite, consists, say, of elements $\lambda_1,\lambda_2,\dots,\lambda_q$, and for every $n\ge1$

$$L_0^n=\sum_{i=1}^q\lambda_i^nP_i+S^n,$$

where $P_i:\mathcal{H}_0\to E_i$, $i=1,2,\dots,q$, are projectors onto eigenspaces E_i of eigenvalues λ_i respectively, $P_i\circ S=S\circ P_i$, $P_i\circ P_j=0$ for all $i\ne j$, and S has the properties listed in part (b) and part (c) of Theorem 4.2. It now follows from the theory of positive operators on latices (see [Sch]) that each eigenvalue λ_i is a root of unity. We shall first show that

$$\mathcal{H}_\beta\cap\bigoplus_{i=1}^q E_i\ne\{0\}. \tag{4.3}$$

Indeed, suppose on the contrary that the above intersection is equal to $\{0\}$. Since $1\!\!1 \in \mathcal{H}_\beta$, then on the one hand $\int L_0(1\!\!1) d\tilde{m}_\phi = \int 1\!\!1 d\tilde{m}_\phi = 1$ for all $n \geq 1$ and on the other hand $\lim_{n\to\infty} ||L^n(1\!\!1)||_0 = \lim_{n\to\infty} ||S^n(1\!\!1)||_0 = 0$ which implies that $\lim_{n\to\infty} \int L_0(1\!\!1) d\tilde{m}_\phi = 0$. This contradiction finishes the proof of (4.3). Choose now $r \geq 1$ so that $\lambda_i^r = 1$ for all $i = 1, 2, \dots, q$. and consider an arbitrary $f \in \bigoplus_{i=1}^q E_i$. Then $L_0^r(f) = f$ and consequently $L_0^r(\mathrm{Re}(f)) = \mathrm{Re}(f)$ and $L_0^r(\mathrm{Im}(f)) = \mathrm{Im}(f)$. Writing $u = \mathrm{Re}(f)$ (resp. $u = \mathrm{Im}(f)$)we have

$$L_0^r(u) = u.$$

Let M be so large that $u + M\psi > 0$ and let

$$\tilde{\psi} = \frac{u + M\psi}{\int (u + M\psi) d\tilde{m}_\phi}.$$

Then, using Theorem 2.7 we conclude that $L_0^r(\tilde{\psi}) = \tilde{\psi}$ $\tilde{m}_\phi$-a.e., $\tilde{\psi} > 0$, and $\int \tilde{\psi} d\tilde{m}_\phi = 1$. Hence the probability measure $\tilde{\psi}\tilde{m}_\phi$ is equivalent with $\tilde{\mu}_\phi$ and σ^r-invariant. Thus, it follows from the total ergodicity of the shift map σ with respect to the measure $\tilde{\mu}_\phi$, proven in Theorem 2.7, that $\tilde{\psi}\tilde{m}_\phi = \tilde{\mu}_\phi = \psi\tilde{m}_\phi$. Hence, $\tilde{\psi} = \psi$ $\tilde{m}_\phi$-a.e. Therefore $u + M\psi = \psi \int (u + M\psi) d\tilde{m}_\phi$ m_ϕ-a.e. and consequently

$$u = \Big(\int (u + M\psi) d\tilde{m}_\phi - M\Big)\psi. \tag{4.4}$$

This completes the part (a) of Theorem 4.2 (To observe that ψ has a version in $\mathcal{H}_\beta$ take an arbitrary $f \in \mathcal{H}_\beta \cap \bigoplus_{i=1}^q E_i \backslash \{0\}$ whose existence follows from (4.3). Then at least one of the functions $\mathrm{Re}(f)$ or $\mathrm{Im}(f)$ does not vanish and since both $\mathrm{Re}(f), \mathrm{Im}(f) \in \mathcal{H}_\beta \cap \bigoplus_{i=1}^q E_i \backslash \{0\}$, the claim follows from (4.4)) and part (b) and (c) follow now immediately from the Ionescu-Tulcea and Marinescu theorem. The proof is complete. ■

§**5. Stochastic laws.** In this section we closely follow §3 of [DU1]. Let Γ be a finite or countable measurable partition of a probability space $(Y, \mathcal{F}, \nu)$ and let $S : Y \to Y$ be a measure preserving transformation. For $0 \leq a \leq b \leq \infty$, set $\Gamma_a^b = \bigvee_{a \leq l \leq b} S^{-l}\Gamma$. The measure ν is said to be absolutely regular with respect to the filtration defined by Γ, if there exists a sequence $\beta(n) \downarrow 0$ such that

$$\int_Y \sup_a \sup_{A \in \Gamma_{a+n}^\infty} |\nu(A|\Gamma_0^a) - \nu(A)| d\nu \leq \beta(n).$$

The numbers $\beta(n)$, $(n \geq 1)$, are called coefficients of absolute regularity. Let α be the partition of I^∞ into initial cylinders of length 1. Using Theorem 4.2, Theorem 2.7, and proceeding exactly as in the proof of [Ry, §3 Theorem 5] we derive the following (with the notation of previous sections).

Theorem 5.1. The measure $\tilde{\mu}_\phi$ is absolutely regular with respect to the filtration defined by the partition α. The coefficients of absolute regularity decrease to 0 at an exponential rate.

It follows from this theorem that the theory of absolutely regular processes applies ([IL], [PS]). We sketch this application briefly. We say that a measurable function $f : I^\infty \to I\!R$ belongs to the space $L^*(\sigma)$ if there exist constants $\alpha, \gamma, M > 0$ such that $\int \|f\|_0^{2+\alpha}\, d\tilde{\mu}_\phi < \infty$ and

$$\int \|f - E_{\tilde{\mu}_\phi}\big(f|(\alpha)^n\big)\|_0^{2+\alpha}\, d\tilde{\mu}_\phi \leq M n^{-2-\gamma}$$

for all $n \geq 1$, where $E_{\tilde{\mu}_\phi}\big(f|(\alpha)^{n-1}\big)$ denotes the conditional expectation of g with respect to the partition $(\alpha)^{n-1}$ and the measure $\tilde{\mu}_\phi$. $L^*(\sigma)$ is a linear space. It follows from Theorem 4.1, [IL] and [PS] that with $\tilde{\mu}_\phi(f) = \int f\, d\tilde{\mu}_\phi$ the series

$$\sigma^2 = \sigma^2(f) = \int_{I^\infty} (f - \tilde{\mu}_\phi(f))^2\, d\tilde{\mu}_\phi + 2\sum_{n=1}^{\infty} \int_{I^\infty} (f - \tilde{\mu}_\phi(f))(f \circ \sigma^n - \tilde{\mu}_\phi(f))\, d\tilde{\mu}_\phi$$

is absolutely convergent and non–negative. The reader should not be confused by two different meanings of the symbol σ: the number defined above and the shift map. If $\sigma^2 > 0$, then the process $(f \circ \sigma^n : n \geq 1)$ satisfies the central limit theorem and an a.s. invariance principle. The latter theorem means that one can redefine the process $(f \circ \sigma^n : n \geq 1)$ on some probability space on which there is defined a standard Brownian motion $(B(t) : t \geq 0)$ such that for some $\lambda > 0$

$$\sum_{0 \leq j \leq t} \big[f \circ \sigma^j - \tilde{\mu}_\phi(f)\big] - B(\sigma^2 t) = O(t^{\frac{1}{2}-\lambda}) \quad \tilde{\mu}_\phi \text{ a.e.}$$

Let $h : [1, \infty) \longrightarrow I\!R$ be a positive non–decreasing function. The function h is said to belong to the lower class if

$$\int_1^\infty \frac{h(t)}{t} \exp\big(-\frac{1}{2} h(t)^2\big)\, dt < \infty$$

and to the upper class if

$$\int_1^\infty \frac{h(t)}{t} \exp\big(-\frac{1}{2} h(t)^2\big)\, dt = \infty.$$

Well known results for Brownian motion imply (see Theorem A in [PS]) the following.

Theorem 5.2. If $f \in L^*(\sigma)$ and $\sigma^2(f) > 0$ then

$$\tilde{\mu}_\phi\Big(\{\omega \in I^\infty : \sum_{j=0}^{n-1} \big(f(\sigma^j(\omega)) - \tilde{\mu}_\phi(f)\big) > \sigma(f) h(n)\sqrt{n} \text{ for infinitely many } n \geq 1\}\Big)$$
$$= \begin{cases} 0 & \text{if } h \text{ belongs to the lower class,} \\ 1 & \text{if } h \text{ belongs to the upper class.} \end{cases}$$

Our last goal in this section is to provide a sufficient condition for the functions ϕ and $g : I^\infty \to I\!R$ to belong to the space $L^*(\sigma)$, where, we recall,

$$g(\omega) = -\log|\phi'_{\omega_1} \circ \pi \circ \sigma(\omega)|$$

Lemma 5.3. Each Hölder continuous function which has some finite moment greater than 2 belongs to $L^*(\sigma)$.

Proof. It suffices to show that any Hölder continuous function $\psi : \Sigma \to I\!R$ satisfies the requirement $\int \|\psi - E_{\tilde{\mu}_\phi}(\psi|(\alpha)^n)\|_0^3 \, d\tilde{\mu}_\phi \le Mn^{-2-\gamma}$ which will finish the proof. So, given $n \ge 1$ suppose that $\omega, \tau \in A$ for some $A \in \alpha^n$. In particular $\omega|_n = \tau|_n$. Hence $|\psi(\omega) - \psi(\tau)| \le V_\beta(\psi)\mathrm{e}^{-\beta n}$ which means that $\psi(\tau) - V_\beta(\psi)\mathrm{e}^{-\beta n} \le \psi(\omega) \le \psi(\tau) + V_\beta(\psi)\mathrm{e}^{-\beta n}$. Integrating these inequalities against the measure $\tilde{\mu}_\phi$ and keeping ω fixed, we obtain

$$\int_A \psi d\tilde{\mu}_\phi - V_\beta(\psi)\mathrm{e}^{-\beta n}\tilde{\mu}_\phi(A) \le \psi(\omega)\tilde{\mu}_\phi(A) \le \int_A \psi d\tilde{\mu}_\phi + V_\beta(\psi)\mathrm{e}^{-\beta n}\tilde{\mu}_\phi(A).$$

Dividing these inequalities by $\tilde{\mu}_\phi(A)$ we deduce that

$$\left|\psi(\omega) - \frac{1}{\tilde{\mu}_\phi(A)}\int_A \psi d\tilde{\mu}_\phi\right| \le V_\beta(\psi)\mathrm{e}^{-\beta n}.$$

Thus $\int \|\psi(\omega) - E_{\tilde{\mu}_\phi}(\psi|(\alpha)^n)\|_0^3 \, d\tilde{\mu}_\phi \le V_\beta(\psi)^3\mathrm{e}^{-3\beta n}$ and we are done. ■

§6. **Refined Geometry.** Let $\psi = \phi + \kappa g - \mathrm{P}(\phi)$, where $\kappa = \mathrm{HD}(\mu_\phi)$ and, as in the previous section, $g(\omega) = -\log|\phi'_{\omega_1} \circ \pi \circ \sigma(\omega)|$. Throughout the whole section we assume that $\int |\phi|^{2+\gamma} d\tilde{\mu}_\phi < \infty$ and $\int |\phi|^{2+\gamma} d\tilde{\mu}_\phi < \infty$ for some $\gamma > 0$. In view of Lemma 5.3 and since $L^*(\sigma)$ is a linear space, $\psi \in L^*(\sigma)$ and, in particular $\sigma^2 = \sigma^2(\psi)$ exists. The following lemma has been proved in [DU1] as Lemma 4.3. We provide its formulation and short proof for the sake of completeness.

Lemma 6.1. Let $\eta, \chi > 0$ and let $\rho : [(\chi + \eta)^{-1}, \infty) \to I\!R_+$ belong to the upper (lower) class. Let $\theta : [(\chi + \eta)^{-1}, \infty) \to I\!R_+$ be a function such that $\lim_{t\to\infty} \rho(t)\theta(t) = 0$. Then there exists an upper (lower) class function $\rho_+ : [1, \infty) \to I\!R_+$, ($\rho_- : [1, \infty) \to I\!R_+$) with the following properties.

(a) $\rho(t(\chi + \eta)) + \theta(t(\chi + \eta)) \le \rho_+(t)$, $(t \ge 1)$

(b) $\rho(t(\chi - \eta)) - \theta(t(\chi - \eta)) \ge \rho_-(t)$, $(t \ge 1)$.

Proof. Since $\lim_{t\to\infty} \rho(t)\theta(t) = 0$, there exists a constant M such that $(\rho(t) + \theta(t))^2 \le \rho(t)^2 + M$. Let ρ belong to the upper class. Then $t \mapsto \rho(t/(\chi + \eta))$ also belongs to the upper class. Hence we may assume that $\chi + \eta = 1$. Define

$$\rho_+(t)^2 = \inf\{u(t)^2 : u \text{ is non-decreasing and } u(t) \ge \rho(t) + \theta(t)\}.$$

Then $\rho_+(t) \ge \rho(t) + \theta(t)$ for $t \ge 1$ and ρ_+ is non-decreasing. Since $\rho_+(t)^2 \le \rho(t)^2 + M$, we also get

$$\int_1^\infty \frac{\rho_+(t)}{t}\exp\bigl(-(1/2)\rho_+^2(t)\bigr)dt \ge \exp(-M/2)\int_1^\infty \frac{\rho_+(t)}{t}\exp\bigl(-(1/2)\rho^2(t)\bigr)dt = \infty.$$

The proof in the case of a function of the lower class is similar. ■

A function $h:[1,\infty)\to I\!R_+$ is said to be slowly growing if $h(t)=o(t^\alpha)$ for all $\alpha>0$. Let $\chi=\chi_{\tilde\mu_\phi}(\sigma)=\int g d\tilde\mu_\phi$. First we shall prove the main geometrical lemma.

Lemma 6.2. (Refined Volume Lemma) Suppose that $\sigma^2=\sigma^2(\psi)>0$. If a slowly growing function h belongs to the upper class, then for μ_ϕ-a.e. $x\in J$,

$$\limsup_{r\to 0}\frac{\mu_\phi(B(x,r))}{r^\kappa\exp\bigl(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\bigr)}=\infty.$$

If, on the other hand, h belongs to the lower class, then for every $\varepsilon>0$ there exists a Borel set $J_\varepsilon\subset J$ such that $\mu_\phi(J_\varepsilon)\ge 1-\varepsilon$, and there exists a constant $k(\varepsilon)\ge 1$ such that for all $x\in J_\varepsilon$ and all $0<r\le 1/k(\varepsilon)$

$$\frac{\mu_\phi(J_\varepsilon\cap B(x,r))}{r^\kappa\exp\bigl(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\bigr)}\le\varepsilon.$$

Proof. Given $x=\pi(\omega)\in J$ and $r>0$ let $n=n(\omega,r)$ be the least integer such that $\phi_{\omega|_n}(X)\subset B(x,r)$. Then $r\le \mathrm{diam}(\phi_{\omega|_{n-1}}(X))$ and

$$m_\phi(B(x,r))\ge m_\phi(\phi_{\omega|_n}(X))=\int_X\exp\bigl(S_{\omega|_n}(\phi)-\mathrm{P}(\phi)n\bigr)(z)dm_\phi(z).$$

Hence,

$$\frac{m_\phi(B(x,r))}{r^\kappa\exp\bigl(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\bigr)}\ge \tag{6.1}$$

$$\begin{aligned}
&\ge\frac{\int_X\exp\bigl(S_{\omega|_n}(\phi)-\mathrm{P}(\phi)n\bigr)(z)dm_\phi(z)}{r^\kappa\exp\bigl(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\bigr)}\ge\frac{Q^{-1}\exp\bigl(S_{\omega|_n}(\phi)-\mathrm{P}(\phi)n\bigr)(x)}{r^\kappa\exp\bigl(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\bigr)}\\
&\ge\frac{Q^{-1}\exp\bigl(S_{\omega|_n}(\phi)-\mathrm{P}(\phi)n\bigr)(x)}{\mathrm{diam}(\phi_{\omega|_{n-1}}(X))^\kappa\exp\bigl(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\bigr)}\\
&\ge\frac{Q^{-1}\exp\bigl(\sum_{j=0}^{n-1}\phi\circ\sigma^j(\omega)-\mathrm{P}(\phi)n\bigr)}{D^\kappa\exp\bigl(-\kappa\sum_{j=0}^{n-2}g\circ\sigma^j(\omega)\bigr)\exp\bigl(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\bigr)}\\
&=\frac{\exp\Bigl(\sum_{j=0}^{n-1}(\phi\circ\sigma^j(\omega)+\kappa g\circ\sigma^j(\omega))-\mathrm{P}(\phi)n-\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}-\kappa g(\sigma^{n-1}(\omega))\Bigr)}{QD^\kappa},
\end{aligned}$$

where the second inequality is true due to Lemma 2.1. In view of the Birkhoff ergodic theorem there exists a Borel set $Y_1\subset I^\infty$ of $\tilde\mu_\phi$ measure 1 such that for every $\eta>0$, every $\omega\in Y_1$ and every n large enough, say $n\ge n_1(\omega,\eta)$

$$-\log r\le-\log\bigl(\mathrm{diam}(\phi_{\omega|_n}(X))\bigr)+\log 2\le(\chi+\eta)n. \tag{6.2}$$

In fact, in what follows we will need a better upper estimate on $-\log r$. In order to provide it notice that in view of Lemma 6.1 the function g is a member of $L^*(\sigma)$. Let τ^2 denote the assymptotic variance of g. If $\tau^2=0$, then by [IL] g is cohomologous to the constant χ by an L^1 couboundary. It turns out that the following proof, where we assume $\tau^2>0$ becomes much simpler when $\tau^2=0$. Since the function $t\mapsto 2\sqrt{t\log\log t}$ belongs to the lower class there exists $Y_2\subset Y_1$ of $\tilde\mu_\phi$ measure 1 such that for all $\omega\in Y_2$, $\tau_1>\tau$, and all n large enough, say $n\ge n_2(\omega)\ge n_1(\omega,\eta)$,

$$-\log r\le -\log\big(\mathrm{diam}(\phi_{\omega|_n}(X)\big)+\log 2\le \log D+\log 2+\sum_{j=0}^{n-1}g(\sigma^j(\omega))$$

$$\le \chi n+2\tau\sqrt{n\log\log n}+\log D+\log 2\le \chi n+2\tau_1\sqrt{n\log\log n} \tag{6.3}$$

It is a simple exercise in the measure theory to check that if $t>0$ and $\int|f|^td\mu<\infty$, then for every $a>0$, $\mu(|f|\ge a)\le a^{-t}\int|f|^td\mu$. Since by our assumptions $\int|g|^{2+4\gamma}d\tilde\mu_\phi<\infty$ for some $0<\gamma<1/2$, for all $\eta>0$ we get

$$\tilde\mu_\phi\big(\{\omega\in I^\infty:\kappa|g(\omega)|\ge\sigma(h((\chi+\eta)n)\sqrt{n})^{1-\gamma}\}\big)$$
$$\le\big(\sigma(h((\chi+\eta)n)\sqrt{n})^{1-\gamma}\big)^{-2-4\gamma}\int|g|^{2+4\gamma}d\tilde\mu_\phi.$$

Since for every $n\ge1$, $h((\chi+\eta)n)\ge h(\chi+\eta)>0$, $\frac12(1-\gamma)(-2-4\gamma)<-1$ and the measure $\tilde\mu_\phi$ is σ-invariant, we get

$$\sum_{n=1}^\infty\tilde\mu_\phi\left(\{\omega\in I^\infty:\kappa|g(\sigma^{n-1}(\omega))|\ge\sigma(h((\chi+\eta)n)\sqrt{n})^{1-\gamma}\}\right)$$
$$\le\sum_{n=1}^\infty\big(\sigma(h((\chi+\eta)n)\sqrt{n})^{1-\gamma}\big)^{-2-4\gamma}\int|g|^{2+4\gamma}d\tilde\mu_\phi<\infty$$

Therefore, in view of the Borel Canteli lemma there exists a set $Y_3\subset Y_2$ of $\tilde\mu_\phi$ measure 1 such that for all $\omega\in Y_3$ there exists $n_3(\omega)\ge n_2(\omega)$ such that for all $n\ge n_3(\omega)$

$$\kappa|g(\sigma^{n-1}(\omega))|\le\sigma(h((\chi+\eta)n)\sqrt{n})^{1-\gamma}. \tag{6.4}$$

Combining this, (6.3), (6.2) and (6.1) we get for all $a<0$

$$\frac{m_\phi(B(x,r))\mathrm{e}^{\sigma a n^{1/4}}}{r^\kappa\exp\big(\sigma\mu_\chi^{-1/2}h(-\log r)\sqrt{-\log r}\big)}\ge \tag{6.5}$$

$$\ge\frac{1}{QD^\kappa}\exp\Bigg(\sum_{j=0}^{n-1}\psi\circ\sigma^j(\omega)-\sigma\chi^{-1/2}h((\chi+\eta)n)\sqrt{\chi n+2\tau_1\sqrt{n\log\log n}}-$$
$$-\sigma(h((\chi+\eta)n)\sqrt{n})^{1-\gamma}\Bigg)\mathrm{e}^{\sigma a n^{1/4}}$$

$$=\frac{1}{QD^\kappa}\exp\Bigg(\sum_{j=0}^{n-1}\psi\circ\sigma^j(\omega)-\sigma\sqrt{n}\Bigg(h((\chi+\eta)n)\sqrt{1+\frac{2\tau_1}{\chi}\sqrt{\frac{\log\log n}{n}}}-an^{-1/4}+$$
$$+h((\chi+\eta)n)\sqrt{n})^{1-\gamma}n^{-\gamma/2}\Bigg)\Bigg)$$

Now, consider the function

$$\theta(t) = h(t)\left(\sqrt{1+\frac{2\tau_1}{\chi}\sqrt{\frac{\log\log(t(\chi+\eta)^{-1}}{t(\chi+\eta)^{-1}}}}-1\right) - a(t(\chi+\eta)^{-1})^{-1/4} + \frac{h(t)^{1-\gamma}}{(t(\chi+\eta)^{-1})^{\gamma/2}}.$$

So, $\theta(t) > 0$ and since $h(t)$ is slowly growing, $\lim_{t\to\infty} h(t)\theta(t) = 0$. Therefore it follows from Lemma 6.1(a) that there exists $h_+(t)$ in the upper class such that $h_+(t) \geq h(t(\chi+\eta)) + \theta(t(\chi+\eta))$. Since, by Theorem 2.12 and Theorem 3.1, $\int \psi d\tilde{\mu}\phi = \int \phi d\tilde{\mu}_\phi + \frac{\mathrm{h}_{\tilde{\mu}_\phi}}{\chi}\chi - \mathrm{P}(\phi) = \int \phi d\tilde{\mu}_\phi + \mathrm{h}_{\tilde{\mu}_\phi} - \mathrm{P}(\phi) = 0$, it follows from Theorem 5.2 that for infinitely many n's

$$\begin{aligned}
0 &\leq \sum_{j=0}^{n-1}\psi\circ\sigma^j(\omega) - \sigma\sqrt{n}h_+(n) \leq \sum_{j=0}^{n-1}\psi\circ\sigma^j(\omega) - \sigma\sqrt{n}\big(h(n(\chi+\eta)) + \theta(n(\chi+\eta))\big) \\
&\leq \sum_{j=0}^{n-1}\psi\circ\sigma^j(\omega) - \sigma\sqrt{n}\Bigg(h(n(\chi+\eta))\sqrt{1+\frac{2\tau_1}{\chi}\sqrt{\frac{\log\log n}{n}}} \\
&\qquad\qquad - an^{-1/4} + h((\chi+\eta)n)^{1-\gamma}n^{-\gamma/2}\Bigg).
\end{aligned}$$

Combining this and (6.3) we see that

$$\frac{m_\phi(B(x,r))}{r^\kappa\exp\big(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\big)}) \geq (QD^\kappa)^{-1}\exp\big(-\sigma a n^{1/4}\big)$$

for $\tilde{\mu}_\phi$ almost all ω and infiniteley many n's provided they are of the form $n(\omega,r)$. But since there exists $n_3(\omega)$ such that each $n \geq n_3(\omega)$ is of the form $n(\omega,r)$, fixing $a<0$, we eventually get

$$\limsup_{r\to 0}\frac{m_\phi(B(x,r))}{r^\kappa\exp\big(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\big)} = \infty.$$

for μ_ϕ a. e. $x \in J$. Since, by Corollary 2.10, μ_ϕ and m_ϕ are equivalent with bounded Radon-Nikodym derivatives, the proof of the first part of Lemma 6.2 is complete.

Let us now prove the second part of the lemma. For every $\omega \in I^\infty$ and every $r > 0$ let $n = n(\omega,r) \geq 0$ be the least integer n such that $\mathrm{diam}(\phi_{\omega|_{n+1}}(X)) < r$. Clearly $\lim_{r\to 0} n(\omega,r) = \infty$ and therefore there exists $r_1(\omega)$ such that for all $0 < r \leq r_1(\omega)$

$$\mathrm{diam}(\phi_{\omega|_n}(X)) \geq r. \tag{6.6}$$

Fix now $\omega \in I^\infty$ and $0 < r \leq r_1(\omega)$. By Lemma 2.5 and Lemma 2.1 we get

$$\begin{aligned}
\tilde{m}_\phi([\omega|_n]) &= \int_X \exp\big(S_{\omega|_n}(\phi) - \mathrm{P}(\phi)n\big)dm_\phi \leq Q\exp\big(S_{\omega|_n}(\phi)(\pi(\sigma^n(\omega))) - \mathrm{P}(\phi)n\big) \\
&= Q\exp\left(\sum_{j=0}^{n-1}\phi\circ\sigma^j(\omega) - \mathrm{P}(\phi)n\right).
\end{aligned}$$

Hence

$$\frac{\tilde{m}_\phi([\omega|_n])}{r^\kappa \exp\bigl(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\bigr)} \le \tag{6.7}$$

$$\begin{aligned}
&\le \frac{Q\exp\left(\sum_{j=0}^{n-1}\phi\circ\sigma^j(\omega) - \mathrm{P}(\phi)n\right)}{\mathrm{diam}^\kappa(\phi_{\omega|_{n+1}}(X))\exp\bigl(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\bigr)} \\
&\le \frac{Q\exp\left(\sum_{j=0}^{n-1}\phi\circ\sigma^j(\omega) - \mathrm{P}(\phi)n\right)}{D^{-\kappa}|\phi'_{\omega|_{n+1}}(\pi(\sigma^{n+1}(\omega)))|\exp\bigl(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\bigr)} \\
&= QD^\kappa\exp\left(\sum_{j=0}^{n-1}\phi\circ\sigma^j(\omega) + \kappa\sum_{j=0}^{n} g\circ\sigma^j(\omega) - \mathrm{P}(\phi)n - \sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\right) \\
&= QD^\kappa\exp\left(\sum_{j=0}^{n-1}\phi\circ\sigma^j(\omega) - \sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r} + \kappa g\circ\sigma^n(\omega)\right).
\end{aligned}$$

In view of the Birkhoff ergodic theorem there exists a Borel set $Y_1 \subset I^\infty$ of $\tilde{\mu}_\phi$ measure 1 such that for every $\eta > 0$, every $\omega \in Y_1$ and every n large enough, say $n \ge n_1(\omega,\eta)$

$$-\log r \ge -\log\bigl(\mathrm{diam}(\phi_{\omega|_n}(X))\bigr) \ge (\chi-\eta)n. \tag{6.8}$$

In fact, in what follows we will need a better upper estimate on $-\log r$. In order to provide it notice that in view of Lemma 6.1 the function g is a member of $L^*(\sigma)$. Let τ^2 denote the assymptotic variance of g. If $\tau^2 = 0$, then by [IL] g is cohomologous to the constant χ by an L^1 couboundary. It turns out that the following proof, where we assume $\tau^2 > 0$ becomes much simpler when $\tau^2 = 0$. Since the function $t \mapsto 2\sqrt{t\log\log t}$ belongs to the lower class there exists $Y_2 \subset Y_1$ of $\tilde{\mu}_\phi$ measure 1 such that for all $\omega \in Y_2$, $\tau_1 > \tau$, and all n large enough, say $n \ge n_2(\omega) \ge n_1(\omega,\eta)$,

$$\begin{aligned}
-\log r &\ge -\log\bigl(\mathrm{diam}(\phi_{\omega|_n}(X)\bigr) \ge -\log(DK) + \sum_{j=0}^{n-1} g(\sigma^j(\omega)) \\
&\ge \chi n - 2\tau\sqrt{n\log\log n} - \log(DK) \ge \chi n - 2\tau_1\sqrt{n\log\log n}.
\end{aligned} \tag{6.9}$$

The same argument as that leading to (6.4) shows that there exists a Borel set $Y_3 \subset Y_2$ of $\tilde{\mu}_\phi$ measure 1 such that for all $\omega \in Y_3$ there exists $n_3(\omega) \ge n_2(\omega)$ such that for all $n \ge n_3(\omega)$, $\kappa|g(\sigma^n(\omega))| \le \sigma(h((\chi-\eta)n)\sqrt{n})^{1-\gamma}$. Combining this, (6.7), (6.8) and (6.9) we get for all $a > 0$

$$\frac{\tilde{m}_\phi([\omega|_n])e^{\sigma a n^{1/4}}}{r^\kappa \exp\bigl(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\bigr)}$$

$$\leq QD^\kappa \exp\Big(\sum_{j=0}^{n-1} \phi\circ\sigma^j(\omega) - \sigma\chi^{-1/2}h((\chi-\eta)n)\sqrt{\chi n - 2\tau_1\sqrt{n\log\log n}+}$$

$$+\sigma(h((\chi-\eta)n)\sqrt{n})^{1-\gamma}\Big)\mathrm{e}^{\sigma a n^{1/4}}$$

$$= QD^\kappa \exp\Big(\sum_{j=0}^{n-1} \psi\circ\sigma^j(\omega) - \sigma\sqrt{n}\Big(h((\chi-\eta)n)\sqrt{1-\frac{2\tau_1}{\chi}\sqrt{\frac{\log\log n}{n}}}-$$

$$- a n^{-1/4} - h((\chi-\eta)n)\sqrt{n})^{1-\gamma}n^{-\gamma/2}\Big)\Big) \tag{6.10}$$

Now, consider the function

$$\theta(t) = h(t)\left(1-\sqrt{1-\frac{2\tau_1}{\chi}\sqrt{\frac{\log\log(t(\chi-\eta)^{-1}}{t(\chi-\eta)^{-1}}}}\right) + a(t(\chi-\eta)^{-1})^{-1/4} + \frac{h(t)^{1-\gamma}}{(t(\chi-\eta)^{-1})^{\gamma/2}}.$$

So, $\theta(t)>0$ and since $h(t)$ is slowly growing, $\lim_{t\to\infty} h(t)\theta(t)=0$. Therefore it follows from Lemma 6.1(b) that there exists $h_-(t)$ in the lower class such that $h_-(t)\leq h(t(\chi-\eta))-\theta(t(\chi-\eta))$. Since, by Theorem 2.12 and Theorem 3.1, $\int\psi d\tilde\mu\phi = \int\phi d\tilde\mu_\phi + \frac{\mathrm{h}_{\tilde\mu_\phi}}{\chi}\chi - \mathrm{P}(\phi) = \int\phi d\tilde\mu_\phi + \mathrm{h}_{\tilde\mu_\phi} - \mathrm{P}(\phi) = 0$, it follows from Theorem 5.2 that there exists a Borel set $Y_4\subset Y_3$ of $\tilde\mu_\phi$ measure 1 such that for all $\omega\in Y_4$ and all n large enough, say $n\geq n_4(\omega)\geq n_3(\omega)$

$$0\geq \sum_{j=0}^{n-1}\psi\circ\sigma^j(\omega) - \sigma\sqrt{n}h_-(n) \geq \sum_{j=0}^{n-1}\psi\circ\sigma^j(\omega) - \sigma\sqrt{n}\big(h(n(\chi-\eta)) - \theta(n(\chi-\eta))\big)$$

$$\geq \sum_{j=0}^{n-1}\psi\circ\sigma^j(\omega) - \sigma\sqrt{n}\Big(h(n(\chi-\eta))\sqrt{1-\frac{2\tau_1}{\chi}\sqrt{\frac{\log\log n}{n}}}-$$

$$- a n^{-1/4} - h((\chi-\eta)n)^{1-\gamma}n^{-\gamma/2}\Big).$$

Combining this and (6.10) we conclude that for every $\omega\in Y_4$ and every $n\geq n_4(\omega)$

$$\frac{\tilde m_\phi([\omega|_n]}{r^\kappa\exp\big(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\big)} \leq QD^\kappa)\mathrm{e}^{-\sigma a n^{1/4}}.$$

In other words, for every $\omega\in Y_4$ and every $r>0$ small enough, say $r\leq r(\omega)\leq r_1(\omega)$,

$$\frac{\tilde m_\phi([\omega|_{n(\omega,r)}]}{r^\kappa\exp\big(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\big)} \leq QD^\kappa)\mathrm{e}^{-\sigma a n(\omega,r)^{1/4}}. \tag{6.11}$$

Fix now $\varepsilon>0$ and take q so large that $QD^\kappa)\mathrm{e}^{-\sigma a q^{1/4}}\leq\varepsilon$. Then, since $\lim_{r\searrow 0} n(\omega,r)=\infty$, there exists $k(\omega)\geq 1$ such that for all $0<r\leq 1/k(\omega)$, (6.10) holds and $n(\omega,r)\geq q$. Since

$Y_4 = \bigcup_{k=1}^{\infty}\{\omega \in Y_4 : k(\omega) \le k\}$, there exists $k(\varepsilon)$ so large that if $\tilde{J}_\varepsilon = \{\omega \in: Y_4 : k(\omega) \le k(\varepsilon)\}$, then

$$\tilde{m}_\phi(J \setminus \tilde{J}_\varepsilon) \le \varepsilon. \tag{6.12}$$

Moreover for every $\omega \in \tilde{J}_\varepsilon$ and every $r < 1/k(\varepsilon)$ (so $r \le 1/k(\omega)$)

$$\frac{\tilde{m}_\phi([\omega_{n(\omega,r)})}{r^\kappa \exp\big(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\big)} \le QD^\kappa \mathrm{e}^{-a\sigma n(\omega,r)^{1/4}} \le QD^\kappa \mathrm{e}^{-a\sigma q^{1/4}} \le \varepsilon. \tag{6.13}$$

Let $J_\varepsilon = \pi(\tilde{J}_\varepsilon)$. It then follows from (6.12) and lemma 2.6 that

$$m_\phi(J_\varepsilon) = \tilde{m}_\phi \circ \pi^{-1}(J_\varepsilon) = \tilde{m}_\phi \circ \pi^{-1}(\pi(\tilde{J}_\varepsilon)) \ge \tilde{m}_\phi(\tilde{J}_\varepsilon) \ge 1 - \varepsilon.$$

Now, in view of Lemma 2.7 of [MU1] there exists a universal constant $L \ge 1$ such that for every $x \in J_\varepsilon$ and every $< r < 1/k(\varepsilon)$ there exist points $\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(L)} \in \tilde{J}_\varepsilon$ such that $J_\varepsilon \cap B(x,r) \subset \bigcup_{j=1}^{L} \phi_{\omega^{(j)}|_{n(\omega^{(j)},r)}}(X)$. Therefore, by (6.13)

$$\frac{m_\phi(J_\varepsilon \cap B(x,r))}{r^\kappa \exp\big(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\big)} \le \sum_{j=1}^{L} \frac{m_\phi\big(\phi_{\omega^{(j)}|_{n(\omega^{(j)},r)}}(X)\big)}{r^\kappa \exp\big(\sigma\chi^{-1/2}h(-\log r)\sqrt{-\log r}\big)}\Big) \le L\varepsilon.$$

Since the measures μ_ϕ and m_ϕ are equivalent with bounded Radon-Nikodym derivatives, looking at the last two displays we conclude that the proof of the second part of Lemma 6.2 is complete. ■

For a function $h : [1,\infty) \to (0,\infty)$ define for sufficiently small $t > 0$

$$\tilde{h}(t) = t^\kappa \exp\left(\frac{\sigma}{\sqrt{\chi}} h(-\log t)\sqrt{-\log t}\right).$$

Theorem 6.3. Suppose that $\sigma^2(\psi) > 0$ and that $h : [1,\infty) \to (0,\infty)$ is a slowly growing function.

(a) If h belongs to the upper class, then the measures μ_ϕ and $\mathcal{H}^{\tilde{h}}$ on J are singular.

(b) If h belongs to the lower class, then μ_ϕ is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^{\tilde{h}}$.

Proof. Suppose first that h belongs to the upper class. For every $n \ge 1$ and every $\varepsilon > 0$, by Lemma 6.2 there exists a Borel set $E_n \subset J$ such that $\mu_\phi(E_n) > 1 - \varepsilon 2^{-n}$ and such that for every $x \in E_n$ and some closed ball $B(x)$ centered at x and with diameter $< 1/n$, $\mu_\phi(B(x)) > n\tilde{h}(B(x))$. By Besicovic's covering theorem there exists a universal constant $C > 0$ such that from the cover $\{B(x) : x \in E_n\}$ one can choose a countable subcover $\{B(x_j) : j \ge 1\}$ of multiplicity $\le C$. Since for every $j \ge 1$, $\mathrm{diam}(B(x_j)) < 1/n$, we get

$$\mathcal{H}^{\tilde{h}}(E_n, 1/n) \le \frac{1}{n}\sum_{j=1}^{\infty} \mu_\phi(B(x_j)) \le \frac{C}{n}\mu_\phi(J) = \frac{C}{n}.$$

Setting $E_\varepsilon = \bigcap_{n\ge 1} E_n$ we then have $\mathcal{H}^{\tilde{h}}(E_\varepsilon) = 0$ and $\mu_\phi(E_\varepsilon) \ge 1 - \varepsilon$. Finally the set $E = \bigcup_{q \ge 1} E_{1/q}$ satisfies $\mathcal{H}^{\tilde{h}}(E) = 0$ and $\mu_\phi(E) = 1$. The proof of item (a) is therefore complete.

Soppose in turn that h belongs to the lower class and consider a Borel set $E \subset J$ with $\mu_\phi(E) > 0$. Take $\varepsilon = \mu_\phi(E)/2$. Then by Lemma 6.2 $\mu_\phi(J_\varepsilon \cap E) \geq \mu_\phi(E) - \varepsilon = \varepsilon$. Fix $0 < \delta \leq 1/k(\varepsilon)$ and consider $\mathcal{B} = \{B(x_i, r_i)\}$, a cover of $J_\varepsilon \cap E$ by balls centered at points of $J_\varepsilon \cap E$ and with radii $\leq \delta$. Then by Lemma 6.2

$$\sum_i \tilde{h}(r_i) \geq \frac{1}{\varepsilon} \sum_i \mu_\phi(J_\varepsilon \cap B(x_i, r_i)) \geq \frac{1}{\varepsilon} \mu_\phi(J_\varepsilon \cap E) \geq 1.$$

Hence $\mathcal{H}_\delta^{\tilde{h}}(E) \geq B > 0$, where B is a universal constant (see [Ma], comp. [PU]). Thus $\mathcal{H}^{\tilde{h}}(E) \geq B > 0$ and we are done. ■

Remark. Taking $h := 0$ it follows from Theorem 6.3(a) that the measure μ_ϕ is singular with respect to the κ-dimensional Hausdorff measure $\mathcal{H}^\kappa$ on J.

We say that two functions $f_1, f_2 : I^\infty \to I\!R$ are cohomologous in a class H if there exists a function $u \in H$ such that

$$f_2 - f_1 = u \circ \sigma - u.$$

As a complementary result to Theorem 6.3 we shall prove the following.

Theorem 6.4. If $\sigma^2(\psi) = 0$, then $\kappa = \mathrm{HD}(\mu_\phi) = \mathrm{HD}(J) := h$, the functions $-hg$ and $\phi - \mathrm{P}(\phi)$ are cohomologous in the class of Hölder continuous bounded functions, the system $\{\phi_i : i \in I\}$ is regular and m_ϕ is equivalent with the h-conformal measure on J, that is with m_{-hg}, with bounded Radon-Nikodym derivatives and the invariant measures $\tilde{\mu}_\phi$ and $\tilde{\mu}_{hg}$ are equal.

In order to prove Theorem 6.4 we need some preparations. First, let α_- be the partition of the two-sided shift space $I^{\boldsymbol{Z}}$ into elements of the form $\omega \times I^{\{1,2,\dots\}}$, where $\omega \in I^{\{\dots,-2,-1,0\}}$. Given $-\infty \leq m \leq n \leq +\infty$ let $\omega|_m^n = \omega_m \omega_{m+1} \dots \omega_n$ and let

$$[\omega]_m^n = \{\tau \in I^{\boldsymbol{Z}} : \tau_k = \omega_k \text{ for all } m \leq k \leq n\}.$$

Finally let $\overline{\mu}_\phi$ be Rokhlin's natural extentsion of the invariant measure $\tilde{\mu}_\phi$ onto the two-sided shift space $I^{\boldsymbol{Z}}$. Let us recall that $\overline{\mu}_\phi$ is defined on a cylinder $C = \pi_{n_1}^{-1}(C_1) \cap \pi_{n_2}^{-1}(C_2) \cap \dots \cap \pi_{n_k}^{-1}(C_k)$ with $n_1 \leq n_2 \leq \dots \leq n_k$, by the formula

$$\overline{\mu}_\phi(C) = \tilde{\mu}_\phi\big(\sigma^{-(n_1+1)}(C)|_1^\infty\big),$$

where for every $k \in \boldsymbol{Z}$, $\pi_k : I^{\boldsymbol{Z}} \to I$ is the projection onto the kth coordinate given by the formula $\pi_k(\omega) = \omega_k$ and for every set $B \subset I^{\boldsymbol{Z}}$, $B|_1^\infty$ is the projection of B onto $I^{\{1,2,\dots\}}$ denoted also by $\pi(B)$. Let us recall that the measure $\overline{\mu}_\phi$ is shift-invariant. We shall prove the following.

Lemma 6.5. If $\{\mu_{\alpha_-(\omega)} : \omega \in I^{\boldsymbol{Z}}\}$ is the Rokhlin canonical system of measures of the measure $\overline{\mu}_\phi$ on the partition α_-, then for $\overline{\mu}_\phi$-a.e. $\omega \in I^{\boldsymbol{Z}}$ the conditional measure $\mu_{\alpha_-(\omega)}$ considered on $I^{I\!N}$ is equivalent with $\tilde{\mu}_\phi$. Moreover, the Radon-Nikodym derivative $d\mu_{\alpha_-(\omega)}/d\tilde{\mu}_\phi$ is bounded from above and from below respectively by Q^4 and Q^{-4}.

Proof. By the martingale theorem, for $\overline{\mu}_\phi$-a.e. $\omega \in I^{\boldsymbol{Z}}$ and every Borel set $B \subset I^{\boldsymbol{Z}}$,

$$\mu_{\alpha_-(\omega)}(B) = \lim_{n\to\infty} \frac{\overline{\mu}_\phi(B \cap [\omega]_{-n}^0)}{\overline{\mu}_\phi([\omega]_{-n}^0)}.$$

It therefore suffices to show that for every $\tau \in I^*$

$$Q^{-4} \le \frac{\mu_{\alpha_-(\omega)}([\omega|_{-\infty}^0 \tau])}{\mu_\phi([\tau])} \le Q^4.$$

And indeed, in view of Lemma 2.5 and Theorem 2.7 we get

$$\begin{aligned}
\overline{\mu}_\phi([\omega]_{-n}^0) &= \tilde{\mu}_\phi(\sigma^{-(n+1)}([\omega]_{-n}^0)|_1^\infty) \le Q\tilde{m}_\phi(\sigma^{-(n+1)}([\omega]_{-n}^0)|_1^\infty) \\
&= Q\int \exp\bigl(S_{\sigma^{-(n+1)}([\omega]_{-n}^0)|_1^{n+1}}(\phi) - \mathrm{P}(\phi)(n+1)\bigr)dm_\phi \\
&= Q\int \exp\bigl(S_{\omega_{-n}\ldots\omega_0}(\phi) - \mathrm{P}(\phi)(n+1)\bigr)dm_\phi \\
&\le Q\exp\bigl(S_{\omega_{-n}\ldots\omega_0}(\phi) - \mathrm{P}(\phi)(n+1)\bigr)
\end{aligned}$$

and putting $k = |\tau|$

$$\begin{aligned}
\overline{\mu}_\phi([\tau] \cap [\omega]_{-n}^0) &= \tilde{\mu}_\phi(\sigma^{-(n+1)}([\tau] \cap [\omega]_{-n}^0)|_1^\infty)) \ge Q^{-1}\tilde{m}_\phi(\sigma^{-(n+1)}([\tau] \cap [\omega]_{-n}^0)|_1^\infty)) \\
&= Q^{-1}\int \exp\bigl(S_{\sigma^{-(n+1)}([\tau]\cap[\omega]_{-n}^0)|_1^{n+1}}(\phi) - \mathrm{P}(\phi)(n+1+k)\bigr)dm_\phi \\
&= Q^{-1}\int \exp\bigl(S_{\omega_{-n}\ldots\omega_0\tau_1\ldots\tau_k}(\phi) - \mathrm{P}(\phi)(n+1+k)\bigr)dm_\phi \\
&\ge Q^{-1}\exp\bigl(\inf(S_{\omega_{-n}\ldots\omega_0\tau_1\ldots\tau_k}(\phi)) - \mathrm{P}(\phi)(n+1+k)\bigr) \\
&\ge Q^{-1}\exp\bigl(\inf(S_{\omega_{-n}\ldots\omega_0}(\phi)) + \inf(S_{\tau_1\ldots\tau_k}(\phi)) - \mathrm{P}(\phi)(n+1+k)\bigr) \\
&\ge Q^{-3}\exp\bigl(\sup(S_{\omega_{-n}\ldots\omega_0}(\phi)) - \mathrm{P}(\phi)(n+1)\bigr)\exp\bigl(\sup(S_{\tau_1\ldots\tau_k}(\phi)) - \mathrm{P}(\phi)k\bigr).
\end{aligned}$$

Applying Lemma 2.1 we therefore obtain

$$\begin{aligned}
\frac{\overline{\mu}_\phi([\tau] \cap [\omega]_{-n}^0)}{\overline{\mu}_\phi([\omega]_{-n}^0)} &\ge Q^{-3}Q^{-1}\exp\bigl(\sup S_\tau(\phi) - \mathrm{P}(\phi)k\bigr) \\
&= Q^{-4}\exp\bigl(\sup S_\tau(\phi) - \mathrm{P}(\phi)k\bigr) \ge Q^{-4}\mu_\phi([\tau]).
\end{aligned}$$

Hence $\mu_{\alpha_-(\omega)}([\omega|_{-\infty}^0 \tau]) \ge Q^{-4}\mu_\phi([\tau])$. Similar computations show that $\mu_{\alpha_-(\omega)}([\omega|_{-\infty}^0 \tau]) \le Q^4\mu_\phi([\tau])$. The proof is complete. ■

As an immediate consequence of this lemma we get the following.

Corollary 6.6. If $\{\mu_{\alpha_-(\omega)} : \omega \in I^{\boldsymbol{Z}}\}$ is the Rokhlin canonical system of measures of the measure $\overline{\mu}_\phi$ on the partition α_-, then for $\overline{\mu}_\phi$-a.e. $\omega \in I^{\boldsymbol{Z}}$, $supp(\mu_{\alpha_-(\omega)}) = \alpha_-(\omega)$, where $\alpha_-(\omega)$ is the only atom of α_- containing ω.

Lemma 6.7. If $\eta : I^\infty \to I\!R$ is a Hölder continuous function of some order $\beta > 0$ such that $\int |\eta|^{2+\gamma} d\tilde{\mu}_\phi < \infty$, $\int \eta d\tilde{\mu}_\phi = 0$ and $\sigma^2(\eta) = 0$, then there exists a bounded Hölder continuous function u of order $\beta > 0$ such that $\eta = u - u \circ \sigma$. In particular η turns out to be bounded.

Proof. It follows from Theorem 5.1 and [IL] that there exists $u \in L_2(\tilde{\mu}_\phi)$ such that

$$\eta = u - u \circ \sigma \tag{6.14}$$

$\tilde{\mu}_\phi$ a.e. Our aim is to show that u has a Hölder continuous version of order β. We first extend η and u on the two-sided shift space $I^{\boldsymbol{Z}}$ by declaring

$$\eta(\omega) = \eta(\omega|_1^\infty) \ \text{ and } \ u(\omega) = u(\omega|_1^\infty).$$

wherever $u(\omega|_1^\infty)$ is defined. The cohomological equation (6.14) remains satisfied since

$$u(\omega) - u \circ \sigma(\omega) = u(\omega|_1^\infty) - u(\sigma(\omega)|_1^\infty) = u(\omega|_1^\infty) - u(\sigma((\omega|_1^\infty))) = \eta(\omega). \tag{6.15}$$

In view of Luzin's theorem there exists a compact set $D \subset I^{\boldsymbol{Z}}$ such that $\overline{\mu}_\phi(D) > 1/2$ and the function $u|_D$ is continuous. In view of Birkhoff's ergodic theorem there exists a Borel set $B \subset I^{\boldsymbol{Z}}$ such that $\overline{\mu}_\phi(B) = 1$, for every $x \in B$, $\sigma^n(x)$ visits D with the asymptotic frequence $> 1/2$, u is well-defined on $\bigcup_{n \in \boldsymbol{Z}} \sigma^{-n}(B)$ and (6.14) holds on $\bigcup_{n \in \boldsymbol{Z}} \sigma^{-n}(B)$. By the definition of conformal measures and by Lemma 6.5 there exists a Borel set $F \subset I^{\boldsymbol{Z}}$ such that $\overline{\mu}_\phi = 1$, for all $\omega \in F$, $\mu_{\alpha_-(\omega)}(B \cap \alpha_-(\omega)) = 1$, and $supp(\mu_{\alpha_-(\omega)}) = \alpha_-(\omega)$. In particular, for every $\omega \in F$, the set $B \cap \alpha_-(\omega)$ is dense in $\alpha_-(\omega)$. Fix one $\omega \in F$ and consider two arbitrary elements $\rho, \tau \in \alpha_-(\omega)$. Then there exists a countinuous increasing to infinity sequence $\{n_j\}$ such that $\sigma^{-n_j}(\rho), \sigma^{-n_j}(\tau) \in D$ for all $j \geq 1$. Using (6.14) we get

$$\begin{aligned} |u(\rho) - u(\tau)| &= \left| u(\sigma^{-n_j}(\rho)) - \sum_{k=1}^{n_j} \eta(\sigma^{-k}(\rho)) - \left(u(\sigma^{-n_j}(\tau)) - \sum_{k=1}^{n_j} \eta(\sigma^{-k}(\rho)) \right) \right| \\ &\leq |u(\sigma^{-n_j}(\rho)) - u(\sigma^{-n_j}(\tau))| + \sum_{k=1}^{n_j} |\eta(\sigma^{-k}(\rho)) - u(\sigma^{-n_j}(\tau))|. \end{aligned} \tag{6.16}$$

Now, since $\lim_{j\to\infty} \text{dist}(\sigma^{-n_j}(\rho), \sigma^{-n_j}(\tau)) = 0$, since both $\sigma^{-n_j}(\rho)$ and $\sigma^{-n_j}(\tau)$ belong to D and since $u|_D$ is uniformly continuous (as D is compact), we conclude that

$$\lim_{j\to\infty} |u(\sigma^{-n_j}(\rho)) - u(\sigma^{-n_j}(\tau))| = 0.$$

Since η is Hölder continuous of order β,

$$\sum_{k=1}^{n_j} |\eta(\sigma^{-k}(\rho) - u(\sigma^{-n_j}(\tau)| \leq \sum_{k=1}^{n_j} V_\beta(\eta) \mathrm{e}^{-\beta k} d_\beta(\rho, \tau) \leq \frac{V_\beta \mathrm{e}^{-\beta}}{1 - \mathrm{e}^{-\beta}} d_\beta(\rho, \tau).$$

Therefore, it follows from (6.16) that

$$|u(\rho) - u(\tau)| \le \frac{V_\beta e^{-\beta}}{1 - e^{-\beta}} d_\beta(\rho, \tau).$$

Hence, as $\alpha_-(\omega) \cap B$ is dense in $\alpha_-(\omega)$, u has a bounded Hölder continuous extension from $\alpha_-(\omega) \cap B$ on $\alpha_-(\omega) = \underline{\omega} \times I^{\mathbb{N}}$, where $\underline{\omega} = \omega|_{-\infty}^{0}$. Denote this extension by $\overline{u} : \alpha_-(\omega) \to \mathbb{R}$ and for every $\tau \in I^{\mathbb{N}}$ set

$$\overline{u}(\tau) = \overline{u}(\underline{\omega}\tau).$$

This obviously defines a bounded Hölder continuous function $\overline{u} : I^{\mathbb{N}} \to \mathbb{R}$. Define now the set B_ω to be determined by the condition $\underline{\omega} B_\omega = \alpha_-(\omega) \cap B$. The function $\overline{u} : I^{\mathbb{N}} \to \mathbb{R}$ is a version of u. Indeed, since $\mu_{\alpha_-(\omega)}(\omega B_\omega) = 1$, it follows from Lemma 6.5 that $\tilde{\mu}_\phi(B_\omega) = 1$ and additionally, for every $\tau \in B_\omega$, $\overline{u}(\tau) = \overline{u}(\underline{\omega}\tau) = u(\tau)$. Since the measure $\tilde{\mu}_\phi$ is shift-invariant, $\tilde{\mu}_\phi(B_\omega \cap \sigma^{-1}(B_\omega)) = 1$. Take now an arbitrary element $\rho \in B_\omega \cap \sigma^{-1}(B_\omega)$. Then $\sigma(\omega) \in B_\omega$ and we have $\eta(\rho) = u(\rho) - u(\sigma(\rho)) = \overline{u}(\rho) - \overline{u}(\sigma(\rho))$. But since $supp(\tilde{\mu}_\phi) = I^{\mathbb{N}}$, the set $B_\omega \cap \sigma^{-1}(B_\omega)$ is dense in $I^{\mathbb{N}}$ and therefore $\eta = \overline{u} - \overline{u} \circ \sigma$ on $I^{\mathbb{N}}$. The proof is complete. ■

Proof of Theorem 6.4. First notice that in view of Theorem 3.1, Theorem 2.12 and Lemma 2.11

$$\int \psi d\tilde{\mu}_\phi = \int \phi d\tilde{\mu}_\phi + \frac{h_{\tilde{\mu}_\phi}}{\chi_{\tilde{\mu}_\phi}} \chi_{\tilde{\mu}_\phi} - P(\phi) = \int \phi d\tilde{\mu}_\phi + h_{\tilde{\mu}_\phi} - P(\phi) = P(\phi) - P(\phi) = 0.$$

Hence the assumptions of Lemma 6.6 are satisfied with $\eta = \psi$ and therefore there exists a bounded function $u \in \mathcal{H}_\beta$ such that $\phi - P(\phi) + \kappa g = u - u \circ \sigma$, that is the functions $-\kappa g$ and $\phi - P(\phi)$ are cohomologous in the class of bounded functions of $\mathcal{H}_\beta$. As we have already pointed out in the paragraph proceeding Theorem 2.12 our definition of pressure coincides with that introduced in [Sa]. That is for every Hölder continuous function $f : I^{\mathbb{N}} \to \mathbb{R}$

$$P(f) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega \in \mathrm{Per}_i(n)} \exp \sum_{j=0}^{n-1} f \circ \sigma^j(\omega) \right)$$

for every $i \in I$, where $\mathrm{Per}_i(n) \subset I^{\mathbb{N}}$ is the set of all fixed points ω of σ^n with $\omega_1 = i$. Hence

$$\begin{aligned}
P(-\kappa g) &= P(\phi - P(\phi) + u \circ \sigma - u) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega \in \mathrm{Per}_i(n)} \exp\big(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\omega) - P(\phi) + u \circ \sigma^{j+1}(\omega) - u \circ \sigma^j(\omega)\big) \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega \in \mathrm{Per}_i(n)} \exp\big(\sum_{j=0}^{n-1} (\phi \circ \sigma^j(\omega)) + u(\sigma^n(\omega)) - u(\omega)\big) \exp(-P(\phi)n) \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega \in \mathrm{Per}_i(n)} \exp\big(\sum_{j=0}^{n-1} \phi \circ \sigma^j(\omega)\big) - P(\phi) \right) = 0.
\end{aligned}$$

Therefore, in the terminology of [MU1] the system $\{\phi_i : i \in I\}$ is regular and it follows from Theorem 3.15 of [MU1] that $\kappa = \mathrm{HD}(J) := h$. Thus, it is only left to show that m_ϕ and m_{-hg} are equivalent with bounded Radon-Nikodyn derivatives and $\mu_\phi = \mu_{-hg}$. And indeed, for every $\omega \in I^*$ we have

$$\tilde{m}_\phi([\omega]) = \int \exp\bigl(S_\omega(\phi) - \mathrm{P}(\phi)|\omega|\bigr) dm_\phi = \int \exp\Bigl(\sum_{j=0}^{n-1}(\phi \circ \sigma^j(\omega\tau) - \mathrm{P}(\phi)\Bigr) d\tilde{m}_\phi(\tau)$$

$$\asymp \exp\Bigl(\sup_{[\omega]} \sum_{j=0}^{n-1}(-\kappa g + u - u \circ \sigma) \circ \sigma^j)\Bigr) = \exp\Bigl(\sup_{[\omega]} \sum_{j=0}^{n-1}(-\kappa g \circ \sigma^j + u - u \circ \sigma^n)\Bigr)$$

$$\asymp \exp\Bigl(\sup_{[\omega]} \sum_{j=0}^{n-1}(-\kappa g \circ \sigma^j)\Bigr) \asymp \tilde{m}_{-hg}([\omega])$$

Hence $\tilde{m}_\phi$ and $\tilde{m}_{-hg}$ are equivalent with bounded Radon-Nikodym derivatives. Therefore $\tilde{\mu}_\phi$ and $\tilde{\mu}_{-hg}$ are equivalent and, as ergodic shift-invariant, they must coincide. Since, by lemma 2.6, $m_\phi = \tilde{m}_\phi \circ \pi^{-1}$ and $m_{-hg} = \tilde{m}_{-hg} \circ \pi^{-1}$, we conclude that m_ϕ and m_{-hg} are equivalent with bounded Radon-Nikodym derivatives. ■

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