DIOPHANTINE APPROXIMATION AND THE GEOMETRY OF LIMIT SETS IN GROMOV HYPERBOLIC METRIC SPACES

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ABSTRACT. In this paper, we provide a complete theory of Diophantine approximation in the limit set of a group acting on a Gromov hyperbolic metric space. This summarizes and completes what has until now been an ad hoc collection of results by many authors. In addition to providing much greater generality than any prior work, our results also give new insight into the nature of the connection between Diophantine approximation and the geometry of the limit set within which it takes place. Two results are also contained here which are purely geometric: a generalization of a theorem of C. J. Bishop and P. W. Jones to Gromov hyperbolic metric spaces, and a proof that the uniformly radial limit set of a group acting on a proper geodesic Gromov hyperbolic metric space has zero Patterson-Sullivan measure unless the group is quasiconvex-cocompact. The latter is an application of a Diophantine theorem.

CONTENTS

1. INTRODUCTION

Four of the greatest classical theorems of Diophantine approximation are Dirichlet's theorem on the approximability of every point with respect to the function $1/q^2$ (supplemented by Liouville's result regarding the optimality of this function), Jarník's theorem stating that the set of badly approximable numbers has full dimension in $\mathbb{R}$ (which we will henceforth call the Full Dimension Theorem), Khinchin's theorem on metric Diophantine approximation, and the theorem of Jarník and Besicovitch regarding the dimension of certain sets of well approximable numbers. As observed S. J. Patterson [57], these theorems can be put in the context of Fuchsian groups by noting that the set of rational numbers is simply the orbit of ∞ under the Fuchsian group $SL_2(\mathbb{Z})$ acting by Möbius transformations on the upper half-plane $\mathbb{H}^2$. In this context, they admit natural generalizations by considering a different group, a different orbit, or even a different space. Beginning with Patterson, many authors [5, 22, 39, 57, 67, 68, 71, 78, 79, 80] have considered generalizations of these theorems to the case of a nonelementary geometrically finite group acting on standard hyperbolic space $X = \mathbb{H}^{d+1}$ for some $d \in \mathbb{N}$, considering a parabolic fixed point if G has at least one cusp,

and a hyperbolic fixed point otherwise. Moreover, many theorems in the literature concerning the asymptotic behavior of geodesics in geometrically finite manifolds [2, 3, 18, 29, 41, 51, 52, 69, 73] can be recast in terms of Diophantine approximation. To date, the most general setup is that of S. D. Hersonsky and F. Paulin, who generalized Dirichlet and Khinchin's theorems to the case of pinched Hadamard manifolds [36, 37] and proved an analogue of Khinchin's theorem for locally finite uniform trees [38]; nevertheless, they still assume that the group is geometrically finite, and they only approximate by parabolic orbits.[1]

In this paper we provide a far-reaching generalization of these results to the situation of a strongly discrete group G of general type acting by isometries on an arbitrary Gromov hyperbolic metric space X (so that $X = \mathbb{H}^{d+1}$ is a special case), and we consider the approximation of a point η in the radial limit set of G by an arbitrary point ξ in the Gromov boundary of X. Thus we are generalizing

- the type of group being considered - we do not assume that G is geometrically finite,
- the space being acted on - we do not even assume that X is proper or geodesic, and
- the point being approximated with - we do not assume that ξ is a parabolic or hyperbolic fixed point of G.

Let us also emphasize that since we deal with any point of the limit set rather than a parabolic or hyperbolic fixed point, we have results which are new even in the case of the Fuchsian group $\mathrm{SL}_2(\mathbb{Z})$; see in particular Examples 1.26 and 1.43. These examples together with Theorems 3.1 and 5.10 show that the approximation theory in this case is the same as the usual approximation theory, i.e. the case $\xi = \infty$.

In addition to generalizing known results, the specific form of our theorems sheds new light on the interplay between the geometry of a group acting on a hyperbolic metric space and the Diophantine properties of its limit set. In particular, our formulation of the Jarník-Besicovitch theorem (Theorem 7.1) provides a clear relation between the density of the limit set around a particular point (measured as the rate at which the Hausdorff content of the radial limit set shrinks as smaller and smaller neighborhoods of the point are considered) and the degree to which other points can be approximated by it. Seeing how this reduces to the known result of R. M. Hill and S. L. Velani [39] is instructive; cf. Theorem 7.33 and its proof. The same is true for our formulation of Khinchin's theorem (Theorem 8.1), although to a lesser extent since we are only able to provide a complete Khinchin in special cases (which are nevertheless more general than previous results). Although our formulation of the Full Dimension Theorem (Theorem 5.10) is more or less the same as previous results (we have incorporated the countable intersection property of winning sets into the statement of the theorem), the geometrically interesting part here is the proof. It depends on constructing Ahlfors regular subsets of the uniformly radial limit set with dimension arbitrarily close to the Poincaré exponent δ (Theorem 5.9); in this way, we also generalize the seminal theorem of C. J. Bishop and P. W. Jones [7] which states that the radial and uniformly radial limit sets $\Lambda_{\mathrm{r}}(G)$ and $\Lambda_{\mathrm{ur}}(G)$ each have dimension δ. Theorem 5.9 will appear in a stronger form in [21], whose authors include the second- and third-named authors of this paper.[2]

Finally, as an application of our generalization of Khinchin's theorem we show that for any group acting on a proper geodesic hyperbolic metric space, the uniformly radial limit set has zero Patterson-Sullivan measure if and only if the group is not quasiconvex-cocompact (Proposition 8.18). In the case where $X = \mathbb{H}^{d+1}$ and G is geometrically finite, this follows from the well-known ergodicity of the geodesic flow (Theorem 1' of [74]).

Convention 1. In the introduction, propositions which are proven later in the paper will be numbered according to the section they are proven in. Propositions numbered as 1.# are either straightforward, proven in the introduction, or quoted from the literature.

Convention 2. The symbols $\lesssim$, $\gtrsim$, and $\asymp$ will denote asymptotics; a subscript of $+$ indicates that the asymptotic is additive, and a subscript of $\times$ indicates that it is multiplicative. For example, $A \lesssim_{\times, K} B$ means that there exists a constant $C > 0$ (the *implied constant*), depending only on K, such that $A \leq CB$.

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[1] Although it does not fall into our framework, let us mention in passing Hersonsky and Paulin's paper [35], which estimates the dimension of the set of geodesic rays in a pinched Hadamard manifold X which return exponentially close to a given point of X infinitely often.

[2] Let us make it clear from the start that although we cite the preprint [21] frequently for additional background, the results which we quote from [21] are not used in any of our proofs.

$A \lesssim_{+,\times} B$ means that there exist constants $C_1, C_2 > 0$ so that $A \leq C_1 B + C_2$. In general, dependence of the implied constant(s) on universal objects such as the metric space X, the group G, and the distinguished points $o \in X$ and $\xi \in \partial X$ will be omitted from the notation.

Convention 3. $x_n \underset{n}{\to} x$ means $x_n \to x$ as $n \to +\infty$. $x_n \underset{n,+}{\longrightarrow} x$ means

$$x \asymp_+ \limsup_{n \to \infty} x_n \asymp_+ \liminf_{n \to \infty} x_n,$$

and similarly for $x_n \underset{n,\times}{\longrightarrow} x$.

Convention 4. Let G be a geometrically finite subgroup of $\mathrm{Isom}(\mathbb{H}^{d+1})$ for some $d \in \mathbb{N}$. In the literature, authors have often considered "a parabolic fixed point of G if such a point exists, and a hyperbolic fixed point otherwise". To avoid repeating this long phrase we shall call such a point a *conventional* point.

1.1. **Preliminaries.**

1.1.1. *Visual metrics.* Let (X, d) be a hyperbolic metric space (see Definition 2.2). In the theorems below, we will consider its *Gromov boundary* ∂X (see Definition 2.5), together with a *visual metric* $D = D_{b,o}$ on ∂X satisfying

$$(2.15) \qquad D_{b,o}(\xi, \eta) \asymp_\times b^{-\langle \xi | \eta \rangle_o},$$

where $o \in X$ and $b > 1$, and where $\langle \cdot | \cdot \rangle$ denotes the Gromov product (see Definition 2.1). For sufficiently small $b > 1$, such a metric exists for all $o \in X$ (Proposition 2.15). We remark that when $X = \mathbb{B}^{d+1}$, $b = e$, and $o = \mathbf{0}$, then we have $\partial X = S^d$ and

$$(1.1) \qquad D_{e,\mathbf{0}}(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\|\mathbf{y} - \mathbf{x}\| \quad \forall \mathbf{x}, \mathbf{y} \in S^d.$$

(See Proposition 2.14 below.)

1.1.2. *Strongly discrete groups.* Let (X, d) be a metric space, and let $G \leq \mathrm{Isom}(X)$.[3] There are several notions of what it means for G to be discrete, which are inequivalent in general but agree in the case $X = \mathbb{H}^{d+1}$; see [21] for details. For the purposes of this paper, the most useful notion is *strong discreteness*.

Definition 1.1. A group G is *strongly discrete* if for every $r > 0$, the set

$$\{g \in G : g(o) \in B(o, r)\}$$

is finite, where $o \in X$ is a distinguished point.[4]

Remark 1.2. Strongly discrete groups are known in the literature as *metrically proper*. However, we prefer the term "strongly discrete", among other reasons, because it emphasizes that the notion is a generalization of discreteness.

1.1.3. *Groups of general type.*

Definition 1.3. An isometry g of a hyperbolic metric space (X, d) is *loxodromic* if it has two fixed points g_+ and g_- in ∂X, which are attracting and repelling in the sense that for every neighborhood U of g_-, the iterates g^n converge to g_+ uniformly on $\partial X \setminus U$, and if g^{-1} has the same property.

Definition 1.4. Let (X, d) be a hyperbolic metric space. A group $G \leq \mathrm{Isom}(X)$ is *of general type*[5] if it contains two hyperbolic isometries g_1 and g_2, such that the fixed points of g_1 are disjoint from the fixed points of g_2.

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[3]Here and from now on $\mathrm{Isom}(X)$ denotes the group of isometries of a metric space (X, d). $G \leq \mathrm{Isom}(X)$ means that G is a subgroup of $\mathrm{Isom}(X)$.

[4]Here and from now on $B(x, r)$ denotes the closed ball centered at x of radius r.

[5]This terminology was introduced in [15] to describe the cases of a classification of Gromov (§3.1 of [32]).

As in the case of $X = \mathbb{H}^{d+1}$, we say that a group $G \leq \mathrm{Isom}(X)$ is *nonelementary* if its limit set (see Definition 2.37) contains at least three points.[6] Since the fixed points of a loxodromic isometry are clearly in the limit set, we have:

Observation 1.5. Every group of general type is nonelementary.

The converse, however, is not true; there exists groups which are nonelementary and yet are not of general type. Such a group has a unique fixed point on the boundary of X, but it is not the only point in the limit set. A good example is the subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ generated by the isometries

$$g_1(z) := 2z, \qquad g_2(z) := z + 1.$$

A group which is nonelementary but not of general type is called a *focal* group.[7]

Proposition 1.6 ([21])**.** *Every nonelementary strongly discrete group is of general type, or in other words, no strongly discrete group is focal.*

Remark 1.7. Due to Proposition 1.6, in our standing assumptions below we could replace the hypothesis that G is of general type with the hypothesis that G is nonelementary. We have elected not to do so since we would prefer this paper to be as self-contained as possible.

Observation 1.8. If G is a group of general type then G has no global fixed points; i.e. for every $\xi \in \partial X$ there exists $h \in G$ such that $h(\xi) \neq \xi$.

1.1.4. *Standing assumptions.* In the statements below, we will have the following standing assumptions:

    (I) (X, d) is a hyperbolic metric space
   (II) $o \in X$ is a distinguished point
  (III) $b > 1$ is a parameter small enough to guarantee the existence of a visual metric $D = D_{b,o}$
  (IV) $G \leq \mathrm{Isom}(X)$ is a strongly discrete group of general type.

If $X = \mathbb{H}^{d+1} = \{(x_1, \ldots, x_{d+1}) : x_{d+1} > 0\}$ for some $d \in \mathbb{N}$, then we will moreover assume that $d = \frac{\|\mathrm{d}\mathbf{x}\|}{x_{d+1}}$, $o = \mathbf{e}_{d+1}$, and $b = e$. Similarly, if $X = \mathbb{B}^{d+1}$, we will assume that $d = \frac{\|\mathrm{d}\mathbf{x}\|}{1 - \|\mathbf{x}\|^2}$, $o = \mathbf{0}$, and $b = e$.

1.2. **The Bishop-Jones theorem and its generalization.** Before stating our main results about Diophantine approximation, we present the Bishop-Jones theorem, a theorem which is crucial for understanding the geometry of the limit set of a Kleinian group, and its generalization to hyperbolic metric spaces.

Theorem 1.9 (C. J. Bishop and P. W. Jones, [7])**.** *Fix $d \in \mathbb{N}$, and let G be a nonelementary discrete subgroup of $\mathrm{Isom}(\mathbb{H}^{d+1})$. Let δ be the Poincaré exponent of G, and let Λ_r and Λ_ur be the radial and uniformly radial limit sets of G, respectively. Then*

$$(1.2) \qquad\qquad \mathrm{HD}(\Lambda_\mathrm{r}) = \mathrm{HD}(\Lambda_\mathrm{ur}) = \delta.\text{[8]}$$

The following theorem is a generalization of the Bishop-Jones theroem:

Theorem 5.9. *Let (X, d, o, b, G) be as in §1.1.4. Let δ be the Poincaré exponent of G relative to b (see (2.32)), and let Λ_r and Λ_ur be the radial and uniformly radial limit sets of G, respectively (see Definition 2.37). Then (1.2) holds; moreover, for every $0 < s < \delta$ there exists an Ahlfors s-regular (see Definition 4.1) set $\mathcal{J}_s \subseteq \Lambda_\mathrm{ur}$.*

The "moreover" clause is new even in the case which Bishop and Jones considered, demonstrating that the limit set Λ_ur can be approximated by subsets which are particularly well distributed from a geometric point of view. It does not follow from their theorem since it is possible for a set to have large Hausdorff dimension without having any closed Ahlfors regular subsets of positive dimension (much less

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[6]This definition is *not* equivalent to the also common definition of a nonelementary group as one which does not preserve any point or geodesic in $\overline{X}$; rather, a group satisfies this second definition if and only if it is of general type.

[7]See footnote 5.

[8]Here and from now on $\mathrm{HD}(S)$ denotes the Hausdorff dimension of the set S.

full dimension); in fact it follows from the work of D. Y. Kleinbock and B. Weiss [43] that the set of well approximable numbers forms such a set.[9]

Remark 1.10. In Theorem 5.9 and in Theorem 5.10 below, the case $\delta = +\infty$ is possible; see [21] for an example.

Remark 1.11. The equation $\mathrm{HD}(\Lambda_\mathrm{r}) = \delta$ was proven in [60] if X is a word-hyperbolic group, a Hadamard manifold,[10] or a locally finite uniform tree.[11]

Remark 1.12. A weaker form of the "moreover" clause was cited in [50] as a private communication from the authors of this paper (Lemma 5.2 of [50]).

As an application of Theorem 5.9, we deduce the following result:

Theorem 9.3. *Fix $d \in \mathbb{N}$, let $X = \mathbb{H}^{d+1}$, and let $G \leq \mathrm{Isom}(X)$ be a discrete group acting irreducibly on X. Then*
$$\mathrm{HD}(\mathrm{BA}_d \cap \Lambda_\mathrm{ur}) = \delta = \mathrm{HD}(\Lambda_\mathrm{r}),$$
where BA_d denotes the set of badly approximable vectors in $\mathbb{R}^d$.

For the definitions and proof see Section 9.

1.3. **Dirichlet's theorem, generalizations, and optimality.** The most fundamental result in Diophantine approximation is the following:

Theorem 1.13 (Dirichlet 1842)**.** *For every irrational $x \in \mathbb{R}$,*
$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2} \text{ for infinitely many } p/q \in \mathbb{Q}.$$

Dirichlet's theorem can be viewed (up to a multiplicative constant) as the special case of the following theorem which occurs when

(1.3) $$X = \mathbb{H}^2, \qquad G = \mathrm{SL}_2(\mathbb{Z}), \qquad \xi = \infty :$$

Theorem 3.1 (Generalization of Dirichlet's Theorem)**.** *Let (X, d, o, b, G) be as in §1.1.4, and let $\xi \in \partial X$ be a distinguished point. Then for every $\sigma > 0$, there exists a number $C = C_\sigma > 0$ such that for every $\eta \in \Lambda_{\mathrm{r},\sigma}(G)$,*

(3.1) $$D(g(\xi), \eta) \leq Cb^{-d(o,g(o))} \text{ for infinitely many } g \in G.$$

Here $\Lambda_{\mathrm{r},\sigma}(G)$ denotes the σ-radial limit set of G (see Definition 2.37).
Furthermore, there exists a sequence of isometries $(g_n)_1^\infty \subseteq G$ such that
$$g_n(\xi) \underset{n}{\to} \eta \text{ and } g_n(o) \underset{n}{\to} \eta,$$
and (3.1) holds with $g = g_n$ for all $n \in \mathbb{N}$.

Remark 1.14. In some sense, it is not surprising that a point in the radial limit set should be able to be approximated well, since by definition a radial limit point already has a sequence of something resembling "good approximations", although they are in the interior rather than in the boundary. (Indeed, radial limit points are sometimes called *points of approximation*.) The proof of Theorem 3.1 is little more than simply taking advantage of this fact. Nevertheless, Theorem 3.1 is important because of its place in our general framework; in particular, its optimality (Theorem 5.10 below) is not obvious.

The fact that Theorem 3.1 reduces to Dirichlet's theorem in the special case (1.3) can be deduced from the following observations:

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[9]It could be objected that this set is not closed and so should not constitute a counterexample; however, since it has full measure, it has closed subsets of arbitrarily large measure (which in particular still have dimension 1).

[10]A *Hadamard manifold* is a complete simply connected Riemannian manifold with uniformly negative curvature. A Hadamard manifold is *pinched* if its curvature is also bounded from below.

[11]A *uniform* tree is one which admits a discrete cocompact action.

Observation 1.15. The orbit of ∞ under $\mathrm{SL}_2(\mathbb{Z})$ is precisely the extended rationals $\widehat{\mathbb{Q}}$. For each $K > 0$ and for each $p/q \in \mathbb{Q}$ in reduced form with $|p/q| \leq K$ we have

$$(1.4) \qquad q^2 \asymp_{\times, K} \min_{\substack{g \in G \\ g(\infty) = p/q}} e^{d(o, g(o))}.$$

Proof. It is well-known and easy to check that the Ford circles $\big(B_{\mathrm{e}}((p/q, 1/(2q^2)), 1/(2q^2))\big)_{p/q \in \mathbb{Q}}$ together with the set $H := \{(x_1, x_2) : x_2 > 1\}$ constitute a disjoint $G = \mathrm{SL}_2(\mathbb{Z})$-invariant collection of horoballs. (Here a subscript of e denotes the Euclidean metric on $\mathbb{R}^2$.) Now note that

$$d_{\mathrm{H}}(\partial H, G_\infty(o)) < +\infty,$$

where d_{H} denotes the Hausdorff metric induced by the hyperbolic metric on $\mathbb{H}^2$, and G_∞ is the stabilizer of ∞ in G. Thus for all $g \in G$

$$d_{\mathrm{H}}(\partial g(H), \{\widetilde{g} \in G : \widetilde{g}(\infty) = g(\infty)\}) \asymp_+ 0;$$

letting $p/q = g(\infty)$,

$$\min_{\substack{\widetilde{g} \in G \\ \widetilde{g}(\infty) = p/q}} d(o, \widetilde{g}(o)) = \quad d(o, \{\widetilde{g} \in G : \widetilde{g}(\infty) = g(\infty)\})$$

$$\asymp_+ d(o, \partial g(H))$$

$$= \quad d(o, B_{\mathrm{e}}((p/q, 1/(2q^2)), 1/(2q^2)))$$

$$\asymp_+ \log(q^2) + \log(1 + (p/q)^2) \asymp_{+, K} \log(q^2).$$

Exponentiating finishes the proof. □

Observation 1.16. In the special case (1.3), the visual metric D is equal to the spherical metric $D_{\mathrm{s}} = |\mathrm{d}x|/(1 + |x|^2)$, and is therefore bi-Lipschitz equivalent to the Euclidean metric $D_{\mathrm{e}} = |\mathrm{d}x|$ on compact subsets of $\mathbb{R}$.

Proof. Apply the Cayley transform to (1.1) above. □

Observation 1.17. The radial limit set of $\mathrm{SL}_2(\mathbb{Z})$ is precisely the set of irrational numbers; in fact $\mathbb{R} \setminus \mathbb{Q} = \Lambda_{\mathrm{r}, \sigma}$ for all $\sigma > 0$ sufficiently large.

Proof. The proof of (GF1 $\Rightarrow$ GF2) in [10] shows that for any geometrically finite group G, there exists $\sigma > 0$ such that $\Lambda = \Lambda_{\mathrm{r}, \sigma} \dot\cup \Lambda_{\mathrm{bp}}$, where Λ_{bp} is the set of (bounded) parabolic points of G. (Here $\dot\cup$ denotes a disjoint union.) In particular, when $G = \mathrm{SL}_2(\mathbb{Z})$, we have $\Lambda = \widehat{\mathbb{R}}$ and $\Lambda_{\mathrm{bp}} = \widehat{\mathbb{Q}}$, so $\mathbb{R} \setminus \mathbb{Q} = \Lambda_{\mathrm{r}, \sigma}$. □

Note that in deducing Theorem 1.13 from Theorem 3.1, the numerator of (1.13) must be replaced by a constant.[12] This sacrifice is necessary since the constant 1 depends on information about $\mathrm{SL}_2(\mathbb{Z})$ which is not available in the general case.

History of Theorem 3.1. Theorem 3.1 was previously proven

- in the case where $X = \mathbb{H}^2$, G is finitely generated, and with no assumption on ξ by S. J. Patterson in 1976 (Theorem 3.2 of [57]).
- in the case where $X = \mathbb{H}^{d+1}$, G is geometrically finite with exactly one cusp,[13] and ξ is a parabolic fixed point by B. O. Stratmann and S. L. Velani in 1995 (Theorem 1 of [71]).

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[12]This constant does not depend on x by Observation 1.17.

[13]If G has more than one cusp, then Stratmann and Velani (and later Hersonsky and Paulin) essentially prove that for every $\eta \in \Lambda_{\mathrm{r}}(G)$, there exists $\xi \in P$ such that (3.1) holds, where P is a maximal set of inequivalent parabolic points.

- in the case where X is a pinched Hadamard manifold, G is geometrically finite with exactly one cusp, and ξ is a parabolic fixed point by S. D. Hersonsky and F. Paulin in 2002 (Theorem 1.1 of [36]).[14]
- in the case where X is a locally finite tree, G is geometrically finite, and ξ is a parabolic fixed point by F. Paulin (Théorème 1.2 of [61])

1.3.1. *Optimality.* A natural question is whether the preceding theorems, Dirichlet's theorem and its generalization, can be improved. In each case we do not consider improvement of the constant factor to be significant; the question is whether it is possible for the functions on the right hand side of (1.13) and (3.1) to be multiplied by a factor tending to zero, and for the theorems to remain true. More formally, we will say that *Theorem 3.1 is optimal* for (X, d, o, b, G) if there does not exist a function $\phi \to 0$ such that for all $\eta \in \Lambda_{\mathrm{r}}(G)$, there exists a constant $C = C_\eta > 0$ such that

$$D(g(\xi), \eta) \le Cb^{-d(o,g(o))}\phi(b^{d(o,g(o))}) \text{ for infinitely many } g \in G.$$

Definition 1.18. Let (X, d, o, b, G) be as in §1.1.4, and let $\xi \in \partial X$ be a distinguished point. A point $\eta \in \Lambda_{\mathrm{r}}(G)$ is called *badly approximable with respect to* ξ if there exists $\varepsilon > 0$ such that

$$D(g(\xi), \eta) \ge \varepsilon b^{-d(o,g(o))}$$

for all $g \in G$. We denote the set of points that are badly approximable with respect to ξ by BA_ξ, and those that are well approximable by $\mathrm{WA}_\xi = \Lambda_{\mathrm{r}} \setminus \mathrm{BA}_\xi$. In the special case (1.3), the set $\mathrm{BA} := \mathrm{BA}_\infty$ is the classical set of badly approximable numbers.

Trivially (using strong discreteness), the existence of badly approximable points in the radial limit set of G implies that Theorem 3.1 is optimal.[15] In fact, we have the following:

Theorem 1.19 (Full Dimension Theorem, Jarník 1928). $\mathrm{HD}(\mathrm{BA}) = 1$, *so Theorem 1.13 is optimal.*

Theorem 5.10 (Generalization of the Full Dimension Theorem). *Let* (X, d, o, b, G) *be as in §1.1.4, and let* $(\xi_k)_1^\ell$ *be a countable (finite or infinite) sequence in* ∂X. *Then*

$$\mathrm{HD}\left(\bigcap_{k=1}^\ell \mathrm{BA}_{\xi_k} \cap \Lambda_{\mathrm{ur}}\right) = \delta = \mathrm{HD}(\Lambda_{\mathrm{r}}).$$

In particular $\mathrm{BA}_\xi \ne \varnothing$ *for every* $\xi \in \partial X$, *so Theorem 3.1 is optimal.*

Note that the Full Dimension Theorem is simply the special case (1.3) of Theorem 5.10.

We shall prove Theorem 5.10 by proving that the sets BA_{ξ_k} are absolute winning in the sense of C. T. McMullen [51] on the sets $\mathcal{J}_s$ described in Theorem 5.9 above. Our theorem then can be deduced from the facts that the countable intersection of absolute winning sets is absolute winning, and that an absolute winning subset of an Ahlfors s-regular set has dimension s ((i), (ii), and (iii) of Proposition 4.3 below).

Remark 1.20. We use the invariance of McMullen's game under quasisymmetric maps to prove Theorem 5.10; cf. Lemma 5.11. For this reason, it is necessary in our proof to use McMullen's game rather than Schmidt's original game which is not invariant under quasisymmetric maps (see [51]).

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[14]It should be noted that the metric which Hersonsky and Paulin use, the Hamenstädt metric, corresponds to the visual metric coming from ξ rather than the visual metric coming from a point in X; however, these metrics are locally bi-Lipschitz equivalent on $\partial X \setminus \{\xi\}$. Moreover, their "depth" function is asymptotic to the expression $\min_{g(\xi)=\eta} d(o, g(o))$, by an argument similar to the proof of Observation 1.15 above. Because our setup is only equivalent asymptotically to theirs, it makes no sense for us to be interested in their computation of the exact value of the constant in Dirichlet's theorem, which of course we do not claim to generalize.

[15]In some contexts, a Dirichlet theorem can be optimal even when badly approximable points do not exist; see [30] for such an example.

1.3.2. *Geometrically finite groups and approximation by parabolic points.* In the following, we shall assume that $X = \mathbb{H}^{d+1}$ for some $d \in \mathbb{N}$, and that $G \leq \mathrm{Isom}(X)$ is a nonelementary geometrically finite group. We will say that a point $\eta \in \partial X$ is *badly approximable by parabolic points* if $\eta \in \bigcap_{k=1}^{\ell} \mathrm{BA}_{\xi_k}$ where $\{\xi_1, \ldots, \xi_\ell\}$ is a complete set of inequivalent parabolic fixed points. The set of points which are badly approximable by parabolic points admits a number of geometric interpretations:

Proposition 1.21. *For a point $\eta \in \partial X$, the following are equivalent:*
- (A) *η is badly approximable by parabolic points.*
- (B) *$[o, \eta]^{16}$ avoids some disjoint G-invariant collection of horoballs centered at the parabolic points of G.*
- (C) *The image of $[o, \eta]$ in the quotient manifold X/G is bounded.*
- (D) *η is a uniformly radial limit point of G.*

The proof of this proposition is not difficult and will be omitted. Certain equivalences have been noted in the literature (e.g. the the equivalence of (B) and (C) was noted in [69]).

Note that according to Theorem 5.10, the set of points which are badly approximable by parabolic points has full dimension in the radial limit set.

History of Theorem 5.10. Theorem 5.10 was previously proven
- in the case where $X = \mathbb{H}^2$, G is a lattice, and $\{\xi_1, \ldots, \xi_\ell\}$ is a finite set of conventional points by Patterson in 1976 (the Theorem of §10 of [57]).
- in the case where $X = \mathbb{H}^{d+1}$, G is convex-cocompact, and ξ is a hyperbolic fixed point by B. O. Stratmann in 1994 (Theorem B of [67]).
- in the case where $X = \mathbb{H}^{d+1}$, G is a lattice, and $\{\xi_1, \ldots, \xi_\ell\}$ is a finite set of parabolic points by B. O. Stratmann in 1994 (Theorem C of [67]).

Moreover, using Proposition 1.21, one can easily deduce some special cases of Theorem 5.10 from known results:
- The case where $X = \mathbb{H}^{d+1}$, G is a lattice, and $\{\xi_1, \ldots, \xi_\ell\}$ is a finite set of parabolic points can be deduced from Corollary 5.2 of [18].
- The case where $X = \mathbb{H}^{d+1}$, G is geometrically finite, and $\{\xi_1, \ldots, \xi_\ell\}$ is a finite set of parabolic points can be deduced from either [29], [69], or [7].

Also note that some further results on winning properties of the set BA_ξ can be found in [2], [3], [18], [51], and [50].

Remark 1.22. It is likely that the techniques used in [50] can be used to prove Theorem 5.10 in the case where X is a proper geodesic hyperbolic metric space, G is any strongly discrete group of general type, and $\{\xi_1, \ldots\}$ is a set of bounded parabolic points (see Subsection 2.7). However, note that such a proof would depend on our Theorem 5.9; cf. Remark 1.12.

1.4. **The Jarník-Besicovitch theorem and its generalization.** For any irrational $x \in \mathbb{R}$, we can define the *exponent of irrationality*

$$\omega(x) := \limsup_{\substack{p/q \in \mathbb{Q} \\ q \to \infty}} \frac{-\log|x - p/q|}{\log(q)}.$$

Then Dirichlet's theorem (Theorem 1.13) implies that $\omega(x) \geq 2$ for every irrational $x \in \mathbb{R}$. For each $c > 0$ let

$$W_c := \{x \in \mathbb{R} \setminus \mathbb{Q} : \omega(x) \geq 2 + c\}.$$

Furthermore, let

$$\mathrm{VWA} := \{x \in \mathbb{R} \setminus \mathbb{Q} : \omega(x) > 2\} = \bigcup_{c>0} W_c$$

$$\mathrm{Liouville} := \{x \in \mathbb{R} \setminus \mathbb{Q} : \omega(x) = +\infty\} = \bigcap_{c>0} W_c$$

---

^{16}Here and from now on $[x, y]$ denotes the geodesic segment connecting x and y.

be the set of *very well approximable* numbers and *Liouville* numbers, respectively.

Theorem 1.23 (Jarník 1929 and Besicovitch 1934)**.** *For every $c > 0$, we have*

$$\mathrm{HD}(W_c) = \frac{2}{2 + c}.$$

In particular $\mathrm{HD}(\mathrm{VWA}) = 1$ *and* $\mathrm{HD}(\mathrm{Liouville}) = 0$.

Let (X, d, o, b, G) be as in §1.1.4. For any point $\xi \in \partial X$, we can analogously define

$$\omega_\xi(\eta) := \limsup_{\substack{g \in G \\ d(o, g(o)) \to \infty}} \frac{-\log_b D(g(\xi), \eta))}{d(o, g(o))}$$

$$W_{c,\xi} := \{\eta \in \Lambda_\mathrm{r} : \omega_\xi(\eta) \geq 1 + c\}$$

$$\mathrm{VWA}_\xi := \{\eta \in \Lambda_\mathrm{r} : \omega_\xi(\eta) > 1\} = \bigcup_{c > 0} W_{c,\xi}$$

$$\mathrm{Liouville}_\xi := \{\eta \in \Lambda_\mathrm{r} : \omega_\xi(\eta) = +\infty\} = \bigcap_{c > 0} W_{c,\xi}.$$

Note that in the special case (1.3), we have $W_{c,\infty} = W_{2c}$, $\mathrm{VWA}_\infty = \mathrm{VWA}$, and $\mathrm{Liouville}_\infty = \mathrm{Liouville}$.

To formulate our generalization of Theorem 1.23, we need some preparation. Let (X, d, o, b, G) be as in §1.1.4, and fix a distinguished point $\xi \in \partial X$. We recall that for $s \geq 0$ and $S \subseteq \partial X$, the *Hausdorff content* of S in dimension s is defined as

$$(1.5) \qquad \mathcal{H}_\infty^s(S) := \inf \left\{ \sum_{A \in \mathcal{C}} \mathrm{Diam}^s(A) : \mathcal{C} \text{ is a countable cover of } S \right\}.$$

Let δ be the Poincaré exponent of G, and fix $s \in (0, \delta)$. For each $r, \sigma > 0$ we define

$$(1.6) \qquad H_{\xi,\sigma,s}(r) := \mathcal{H}_\infty^s(B(\xi, r) \cap \Lambda_{\mathrm{r},\sigma}(G))$$

$$(1.7) \qquad P_\xi(s) := \lim_{\sigma \to \infty} \liminf_{r \to 0} \frac{\log_b(H_{\xi,\sigma,s}(r))}{\log_b(r)}.$$

Theorem 7.1 (Generalization of the Jarník-Besicovitch Theorem[17])**.** *Let (X, d, o, b, G) be as in §1.1.4, and let $\xi \in \partial X$ be a distinguished point. Let δ be the Poincaré exponent of G, and suppose that $\delta < +\infty$. Then for each $c > 0$,*

$$(7.1) \qquad \begin{aligned} \mathrm{HD}(W_{c,\xi}) &= \sup \left\{ s \in (0, \delta) : P_\xi(s) \leq \frac{\delta - s}{c} \right\} \\ &= \inf \left\{ s \in (0, \delta) : P_\xi(s) \geq \frac{\delta - s}{c} \right\} \leq \frac{\delta}{c + 1}. \end{aligned}$$

In particular

$$(7.2) \qquad \mathrm{HD}(\mathrm{VWA}_\xi) = \sup\{s \in (0, \delta) : P_\xi(s) < +\infty\} = \inf \{s \in (0, \delta) : P_\xi(s) = +\infty\}$$

and $\mathrm{HD}(\mathrm{Liouville}_\xi) = 0$. *In these formulas we let* $\sup \varnothing = 0$ *and* $\inf \varnothing = \delta$.

Remark 1.24. For the remainder of this subsection, we assume that $\delta < +\infty$.

Corollary 1.25. $\mathrm{HD}(\mathrm{VWA}_\xi) = \delta$ *if and only if* $P_\xi(s) < +\infty$ *for all* $s \in (0, \delta)$.

Corollary 7.6. *The function* $c \mapsto \mathrm{HD}(W_{c,\xi})$ *is continuous on* $(0, +\infty)$.

It is obvious from (7.1) that $\mathrm{HD}(W_{c,\xi})$ depends on the function P_ξ, which in turn depends on the geometry of the limit set of G near the point ξ. We will therefore consider special cases for which the function P_ξ can be computed explicitly.

---

[17]The fact that Theorem 7.1 is truly a generalization of Theorem 1.23 is demonstrated in Remark 1.30 below.

1.4.1. *Special cases.* The simplest case for which the function P_ξ can be computed explicitly is the case where ξ is a radial limit point. Indeed, we have the following:

Example 7.30. If $\xi \in \Lambda_r$ then $P_\xi(s) = s$. Thus in this case (7.1) reduces to

$$(7.37) \qquad \qquad \mathrm{HD}(W_{c,\xi}) = \frac{\delta}{c+1},$$

i.e. the inequality of (7.1) is an equality.

This example demonstrates that the expression $\mathrm{HD}(W_{c,\xi})$ is maximized when ξ is a radial limit point. The intuitive reason for this is that there are a lot of "good approximations" to ξ, so by a sort of duality principle there are many points which are approximated very well by ξ.

We remark that if G is of divergence type, then a point in Λ_r represents a "typical" point; see Theorem 2.53 below. Thus, generically speaking, the Hausdorff dimension of the set $W_{c,\xi}$ depends only on c and δ, and depends neither on the point ξ nor the group G.

Example 1.26. Let $X = \mathbb{H}^2$ and let $G = \mathrm{SL}_2(\mathbb{Z})$. Example 7.30 shows that for every $\xi \in \partial X = \mathbb{R} \setminus \mathbb{Q}$,

$$\mathrm{HD}(W_{c,\xi}) = \frac{1}{c+1}.$$

This does not follow from any known result.

Next we consider the case where ξ is a bounded parabolic point (see Subsection 2.7):

Theorem 7.33. *Let (X, d, o, b, G) be as in §1.1.4, and let ξ be a bounded parabolic point of G. Let δ and δ_ξ be the Poincaré exponents of G and G_ξ, respectively. Then for all $s \in (0, \delta)$*

$$(7.40) \qquad \qquad P_\xi(s) = \max(s, 2s - 2\delta_\xi) = \begin{cases} s & s \geq 2\delta_\xi \\ 2s - 2\delta_\xi & s \leq 2\delta_\xi \end{cases}.$$

Thus for each $c > 0$

$$(7.41) \qquad \qquad \mathrm{HD}(W_{c,\xi}) = \begin{cases} \delta/(c+1) & c+1 \geq \delta/(2\delta_\xi) \\ \dfrac{\delta + 2\delta_\xi c}{2c+1} & c+1 \leq \delta/(2\delta_\xi) \end{cases}.$$

Remark 1.27. It is surprising that the formula (7.40) depends only on the Poincaré exponent δ_ξ. One would expect it to depend only on the orbital counting function of G_ξ (see (7.42)), but this function can be quite wild (see Appendix A), even sometimes failing to satisfy a doubling condition.

Remark 1.28. If $X = \mathbb{H}^{d+1}$, then $\delta_\xi = k_\xi/2$, where k_ξ is the rank of ξ. Thus (7.41) reduces to

$$(1.8) \qquad \qquad \mathrm{HD}(W_{c,\xi}) = \begin{cases} \delta/(c+1) & c+1 \geq \delta/k_\xi \\ \dfrac{\delta + k_\xi c}{2c+1} & c+1 \leq \delta/k_\xi \end{cases}$$

in this case.

Remark 1.29. If $\delta_\xi \geq \delta/2$, then $c+1 \geq \delta/(2\delta_\xi)$ for all $c > 0$, so (7.41) reduces to (7.37). In particular, this happens if

(1) X is a real, complex, or quaternionic hyperbolic space, and G is a lattice,
(2) X is a locally finite uniform tree which contains no vertex of degree ≤ 2, and G is a lattice, or if
(3) $X = \mathbb{H}^2$.

Proof. (1) is (i) of Proposition B.7; (2) is the fifth part of Proposition 3.1 of [13]. To demonstrate (3), note that $\delta_\xi = k_\xi/2 = 1/2$ since 1 is the only possible rank of a parabolic point in a Fuchsian group. On the other hand, it is well known that $\delta \leq 1$ for a Fuchsian group (e.g. it follows from the Bishop-Jones theorem). $\qquad \square$

Remark 1.30. Theorem 1.23 follows from Remark 1.29 applied to the special case (1.3).

Remark 1.31. If G is geometrically finite, then Theorem 7.33 and Example 7.30 together cover all possible cases for ξ.

Moving beyond the geometrically finite case, let us return to the idea of a duality principle between approximations to a point and the size of the set of points which are well approximable by that point. Another common notion of approximability is the notion of a *horospherical* limit point, i.e. a point ξ for which every horoball centered at ξ (see Definition 2.33) is penetrated by the orbit of o.

Example 7.31. If ξ is a horospherical limit point of G then $P_\xi(s) \leq 2s$. Thus in this case we have

$$\frac{\delta}{2c+1} \leq \mathrm{HD}(W_{c,\xi}) \leq \frac{\delta}{c+1}.$$

More generally, let $\langle \cdot, \cdot \rangle$ denote the Gromov product (see Definition 2.1). If we let

$$(7.38) \qquad \beta = \beta(\xi) := \limsup_{\substack{g \in G \\ d(o,g(o)) \to \infty}} \frac{\langle g(o) | \xi \rangle_o}{d(o, g(o))} \,,$$

then $P_\xi(s) \leq \beta^{-1} s$ and so

$$\frac{\delta}{\beta^{-1}c + 1} \leq \mathrm{HD}(W_{c,\xi}) \leq \frac{\delta}{c+1}.$$

In particular, if $\beta > 0$ then $\mathrm{HD}(\mathrm{VWA}_\xi) = \delta$.

Remark 1.32. If $X = \mathbb{B}^{d+1}$, then $\langle \mathbf{x} | \mathbf{y} \rangle_{\mathbf{0}} \asymp_+ -\log(\|\mathbf{y} - \mathbf{x}\|)$ whenever at least one of $\mathbf{x}, \mathbf{y}$ is in S^d [21]. Thus we may rewrite (7.38) as

$$\beta(\xi) = \limsup_{\substack{g \in G \\ \|g(\mathbf{0})\| \to 1}} \frac{\log \|\xi - g(\mathbf{0})\|}{\log(1 - \|g(\mathbf{0})\|)}.$$

From this equation, it can be seen that $\beta(\xi)$ is the supremum of $\alpha \in [0,1]$ for which $\xi \in \mathcal{L}(\alpha)$, where $\mathcal{L}(\alpha)$ is defined as in [45].

In particular, notice that if there is a sequence $g_n(\mathbf{0}) \underset{n}{\to} \xi$ such that $1/\|\xi - g_n(\mathbf{0})\|$ is bounded polynomially in terms of $1/(1 - \|g_n(\mathbf{0})\|)$, then $\mathrm{HD}(\mathrm{VWA}_\xi) = \delta$. On the other hand,

Proposition 7.32. *There exists a (discrete) nonelementary Fuchsian group G and a point $\xi \in \Lambda$ such that* $\mathrm{HD}(\mathrm{VWA}_\xi) = 0$.

History of Theorem 7.1. Although Theorem 7.1 has not been stated in the literature before (we are the first to define the function P_ξ), the equations (7.37) and (1.8) have both appeared in the literature. (7.37) was proven[18]

- in the case where $X = \mathbb{H}^{d+1}$, G is a lattice, and ξ is a parabolic fixed point by M. V. Melián and D. Pestana in 1993 (Theorems 2 and 3 of [52]).
- in the case where $X = \mathbb{H}^2$, G is a lattice, and ξ is a conventional point by S. L. Velani in 1993 (Theorem 2 of [78]).
- in the case where $X = \mathbb{H}^{d+1}$, G is a lattice, and ξ is a conventional point by S. L. Velani in 1994 (Theorem 2 of [79]).
- in the case where $X = \mathbb{H}^{d+1}$, G is convex-cocompact, and ξ is a hyperbolic fixed point by M. M. Dodson, M. V. Melián, D. Pestana, and S. L. Velani in 1995 (Theorem 3 of [22]).
- in the case where $X = \mathbb{H}^{d+1}$, G is geometrically finite and satisfies $\delta \leq k_{\max}$,[19] and ξ is a parabolic fixed point by B. O. Stratmann in 1995 (Theorem B of [68]).

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[18]Recall that (7.37) holds whenever ξ is either a radial limit point (e.g. a hyperbolic fixed point) or a (bounded) parabolic point satisfying $\delta_\xi \geq \delta/2$.

[19]Since $k_\xi = 2\delta_\xi$, the condition $\delta \leq k_{\max}$ is equivalent to the condition that $\delta_\xi \geq \delta/2$ for every parabolic point ξ, i.e. the necessary and sufficient condition for (7.41) to reduce to (7.37) (Remark 1.29). When $\delta > k_{\max}$, Stratmann proves an upper and a lower bound for $\mathrm{HD}(W_{c,\xi})$, neither of which are equal to the correct value (7.41).

Additionally, the $\leq$ direction of (7.37) was proven[20]
- in the case where $X = \mathbb{H}^2$, G is geometrically finite, and ξ is a conventional point by S. L. Velani in 1993 (Theorem 1 of [78]).
- in the case where $X = \mathbb{H}^{d+1}$, G is geometrically finite, and ξ is a conventional point by S. L. Velani in 1994 (Theorem 1 of [79]).

Finally, (1.8) has been proven in the literature only once before,
- in the case where $X = \mathbb{H}^{d+1}$, G is geometrically finite, and ξ is a parabolic fixed point by R. M. Hill and S. L. Velani in 1998 (Theorem 1 of [39]).[21]

Remark 1.33. Theorem 7.33 is more general than Theorem 1 of [39] not only because it applies to spaces $X \neq \mathbb{H}^{d+1}$, but also because it applies to geometrically infinite groups G acting on $\mathbb{H}^{d+1}$; ξ must be a bounded parabolic point, but a group can have one bounded parabolic point without being geometrically finite.

1.5. **Khinchin's theorem and its generalization.** We now consider metric Diophantine approximation, or the study of the Diophantine properties of a point chosen at random with respect to a given measure. In this case, we would like to consider a function $\Psi : \mathbb{N} \to (0, +\infty)$ and consider the points which are well or badly approximable with respect to Ψ. The classical definition and theorem are as follows:

Definition 1.34. An irrational point $x \in \mathbb{R}$ is Ψ-*approximable* if there exist infinitely many $p/q \in \mathbb{Q}$ such that
$$\left| x - \frac{p}{q} \right| \leq \Psi(q).$$

Theorem 1.35 (Khinchin 1924). *Let* $\Psi : \mathbb{N} \to (0, +\infty)$ *be any function.*

(i) *If the series*

(1.9)
$$\sum_{q=1}^{\infty} q\Psi(q)$$

converges, then almost no point is Ψ-approximable.

(ii) *Suppose that the function* $q \mapsto q^2\Psi(q)$ *is nonincreasing.*[22] *If the series* (1.9) *diverges, then almost every point is Ψ-approximable.*

In both cases the implicit measure is Lebesgue.

Again, we would like to consider a generalization of this theorem to the setting of §1.1.4. We will replace Lebesgue measure in the above theorem by an ergodic δ-quasiconformal measure (see Definition 2.51), where δ is the Poincaré exponent of G. Sufficient conditions for the existence and uniqueness of such a measure are given in Subsection 2.8. Thus, we will assume that we have such a measure, and that we want to determine the Diophantine behavior of almost every point with respect to this measure.

We will now define our generalization of the notion of a Ψ-approximable point:

Definition 1.36. Let (X, d, o, b, G) be as in §1.1.4, and fix a distinguished point $\xi \in \partial X$. Let $\Phi : [0, +\infty) \to (0, +\infty)$ be a function such that the function $t \mapsto t\Phi(t)$ is nonincreasing.[23] We say that a point $\eta \in \Lambda(G)$ is Φ, ξ-*well approximable* if for every $K > 0$ there exists $g \in G$ such that

(1.10)
$$D(g(\xi), \eta) \leq \Phi(Kb^{d(o,g(o))}).$$

We denote the set of all Φ, ξ-well approximable points by $\mathrm{WA}_{\Phi,\xi}$, and its complement by $\mathrm{BA}_{\Phi,\xi}$.

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[20]In the cases below,

[21]To verify that (7.41) is equal to the expression given in [39], note that $k_\xi = 2\delta_\xi$ for a parabolic point $\xi \in \partial\mathbb{H}^{d+1}$.

[22]There are a number of results which allow this restriction to be weakened; see [14] for a detailed account.

[23]Although this is not the weakest assumption which we could have used here, it has the advantage of avoiding technicalities about what regularity hypotheses on Φ are necessary for which statements; this hypothesis suffices for every statement we will consider. We have added the hypothesis to the definition itself because if a weaker hypothesis is used, then the equation (1.10) may need to be modified.

Note that our definition differs from the classical one in that factor of K has been added.[24] This is in keeping with the philosophy of hyperbolic metric spaces, since most quantities are considered to be defined only "up to a constant". Furthermore, our definition is independent of the chosen fixed point o. The relation between our definition and the classical one is as follows:

Observation 1.37. Let $\Phi : [0, +\infty) \to (0, +\infty)$ be a function such that the function $t \mapsto t\Phi(t)$ is nonincreasing. In the case (1.3), a point $x \in \mathbb{R} = \Lambda(G) \setminus \{\infty\}$ is Φ, ∞-well approximable if and only if for every $K > 0$ it is Ψ-approximable where $\Psi(q) = \Phi(Kq^2)$.

Of course, this observation follows from Observations 1.15 - 1.17.

Theorem 8.1 (Generalization of Khinchin's Theorem[25]). *Let (X, d, o, b, G) be as in §1.1.4, and fix a distinguished point $\xi \in \partial X$. Let δ be the Poincaré exponent of G, and suppose that $\mu \in \mathcal{M}(\Lambda)$ is an ergodic δ-quasiconformal probability measure satisfying $\mu(\xi) = 0$. Let*

$$\Delta_{\mu,\xi}(r) = \mu(B(\xi, r)).$$

Let $\Phi : [0, +\infty) \to (0, +\infty)$ be a function such that the function $t \mapsto t\Phi(t)$ is nonincreasing. Then:

(i) *If the series*

(8.1)
$$\sum_{g \in G} b^{-\delta d(o, g(o))} \Delta_{\mu,\xi}\left(b^{d(o,g(o))} \Phi(Kb^{d(o,g(o))})\right)$$

diverges for all $K > 0$, then $\mu(\mathrm{WA}_{\Phi,\xi}) = 1$.

(iiA) *If the series*

(8.2)
$$\sum_{g \in G} (g'(\xi))^\delta \Delta_{\mu,\xi}\left(\Phi(Kb^{d(o,g(o))})/g'(\xi)\right)$$

converges for some $K > 0$, then $\mu(\mathrm{WA}_{\Phi,\xi}) = 0$.

(iiB) *If ξ is a bounded parabolic point of G (see Definition 2.48), and if the series (8.1) converges for some $K > 0$, then $\mu(\mathrm{WA}_{\Phi,\xi}) = 0$.*

Remark 1.38. The fact that $\mu(\mathrm{WA}_{\Phi,\xi})$ is either 0 or 1 is an easy consequence of the assumption that μ is ergodic, since $\mathrm{WA}_{\Phi,\xi}$ is a G-invariant set.

Remark 1.39. (iiB) is not a special case of (iiA) as demonstrated by Proposition 8.15 below. In particular, the proof of (iiB) depends crucially on the fact that if ξ is a parabolic point, then the map $g \mapsto g(\xi)$ is highly non-injective, which reduces the number of potential approximations to consider. In general the map $g \mapsto g(\xi)$ is usually injective, so this reduction cannot be used. Thus, it seems unlikely that the converse of (i) of Theorem 8.1 should hold in general.

1.5.1. *Special cases.* When considering special cases of Theorem 8.1, we will consider the following three questions:

1. Under what hypotheses are the series (8.1) and (8.2) asymptotic? That is, when do parts (i) and (ii) of Theorem 8.1 give a value for $\mu(\mathrm{WA}_{\Phi,\xi})$ for every possible function Φ?
2. Under what hypotheses is it possible to simplify (asymptotically) the series (8.1)?
3. Under what hypotheses can we prove that $\mu(\mathrm{BA}_\xi) = 0$ and/or $\mu(\mathrm{VWA}_\xi) = 0$?

To answer these questions we consider the following asymptotic formulas:

(8.24)
$$\Delta_{\mu,\xi}(r) \asymp_\times r^\delta \quad \forall r > 0$$

(8.25)
$$f_G(t) := \#\{g \in G : d(o, g(o)) \le t\} \asymp_\times b^{\delta t} \quad \forall t > 0.$$

These asymptotics will not be satisfied for every group G and every point ξ, but we will see below that there are many reasonable hypotheses which ensure that they hold. When the asymptotics are satisfied, we have the following:

---

[24]The reason that we have called points satisfying our definition Φ, ξ-*well* approximable rather than Φ, ξ-approximable is to indicate this distinction.

[25]The fact that Theorem 8.1 is truly a generalization of Theorem 1.35 is proven in Example 1.42 below.

Proposition 8.11 (Reduction of (8.1) and (8.2) assuming power law asymptotics)**.** *If* (8.25) *holds then for each* $K > 0$

$$(8.1) \asymp_{+,\times} \int_{t=0}^{\infty} \Delta_{\mu,\xi}(K^{-1}e^t\Phi(e^t))\mathrm{d}t. \tag{8.26}$$

If (8.24) *holds then for each* $K > 0$

$$(8.1) \asymp_{\times} (8.2) \asymp_{\times} \sum_{g \in G} \Phi^{\delta}(Kb^{d(o,g(o))}). \tag{8.27}$$

If both (8.25) *and* (8.24) *hold then for each* $K > 0$

$$(8.1) \asymp_{\times} (8.2) \asymp_{+,\times} \int_{t=0}^{\infty} e^{\delta t}\Phi^{\delta}(e^t)\mathrm{d}t, \tag{8.28}$$

and so by Theorem 8.1

$$\mu(\mathrm{WA}_{\Phi,\xi}) = 0 \iff \int_{t=0}^{\infty} e^{\delta t}\Phi^{\delta}(e^t)\mathrm{d}t < +\infty. \tag{8.29}$$

Note in particular that the last series depends only on the Poincaré exponent δ, and on no other geometric information.

Corollary 8.12 (Measures of BA_ξ and VWA_ξ assuming power law asymptotics)**.** *Assume* $\xi \in \Lambda$. *If* (8.24) *holds, then* $\mu(\mathrm{VWA}_\xi) = 0$, *and* $\mu(\mathrm{BA}_\xi) = 0$ *if and only if* G *is of divergence type. If* (8.25) *holds, then* $\mu(\mathrm{BA}_\xi) = 0$. *If both hold, then* $\mu(\mathrm{BA}_\xi) = \mu(\mathrm{VWA}_\xi) = 0$.

Remark 1.40. The hypothesis $\xi \in \Lambda$ is necessary in Corollary 8.12. Indeed, if $\xi \in \partial X \setminus \Lambda$, then $\mathrm{BA}_\xi = \Lambda_{\mathrm{r}}$ (see Proposition 3.2).

1.5.2. *Verifying the hypotheses* (8.24) *and* (8.25)*.* Let us now consider (more or less) concrete examples of pairs (G, ξ) for which (8.24) and/or (8.25) are satisfied. As for the Jarník-Besicovitch theorem, the simplest case is when ξ is a uniformly radial point:

Example 8.13. If ξ is a uniformly radial limit point, then (8.24) holds.

Let us call a group G *perfect* if (8.24) and (8.25) (and thus (8.29)) are satisfied for all $\xi \in \Lambda_G$.

Example 8.14 (Quasiconvex-cocompact group)**.** Let X be a proper and geodesic hyperbolic metric space, and let $G \leq \mathrm{Isom}(X)$. G is called *quasiconvex-cocompact* if the set $G(o)$ is *quasiconvex*, i.e. there exists $\rho > 0$ such that for all $x, y \in G(o)$,

$$[x, y] \subseteq B(G(o), \rho). \tag{8.31}$$

Any quasiconvex-cocompact group is perfect.

Example 1.41 (Lattice on a real, complex, or quaternionic hyperbolic space)**.** Let X be a real, complex, or quaternionic hyperbolic space, and let $G \leq \mathrm{Isom}(X)$ be a lattice. Then G is perfect.

Proof. (8.24) follows from (ii) of Proposition B.7; (8.25) is a special case of Example 1.45 below.   □

Example 1.42 (The classical Khinchin theorem)**.** Since $\mathrm{SL}_2(\mathbb{Z})$ is a lattice, Example 1.41 implies that $\mathrm{SL}_2(\mathbb{Z})$ is perfect. Moreover, using the substitution $t = \log(q^2)$, we have

$$\int_{t=0}^{\infty} e^{\delta t}\Phi^{\delta}(e^t)\mathrm{d}t = \int_{t=0}^{\infty} e^t\Phi(e^t)\mathrm{d}t = 2\int_{q=1}^{\infty} q^2\Phi(q^2)\frac{\mathrm{d}q}{q} \asymp_{+,\times} \sum_{q=1}^{\infty} q\Phi(q^2). \tag{1.11}$$

Thus if we let $\Psi(q) = \Phi(q^2)$, then the series (8.1), (8.2), and (1.9) are all $+, \times$-asymptotic. Thus, by applying Observation 1.37, we are able to deduce the difficult direction (ii) of Theorem 1.35 from Theorem 8.1 .[26]

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[26](i) of Theorem 1.35 is not deduced from Theorem 8.1 for two trivial reasons: first of all, the hypotheses of (i) Theorem 1.35 are weaker; second of all, Φ, ∞-well-approximability implies Ψ-approximability but not vice-versa.

Example 1.43. Let $X = \mathbb{H}^2$ and let $G = \mathrm{SL}_2(\mathbb{Z})$. Theorem 8.1 shows that for every $\xi \in \partial X = \widehat{\mathbb{R}}$ and for every $\Phi : [0, +\infty) \to (0, +\infty)$ such that $t \mapsto t\Phi(t)$ is nonincreasing, $\mathrm{WA}_{\Phi, \xi}$ is either a Lebesgue null set or of full Lebesgue measure according to whether the series (1.9) converges or diverges. This result is new except when ξ is a rational or a quadratic irrational.

Our last example of a perfect group is a little more exotic:

Example 1.44 (Lattice on a locally finite uniform tree)**.** Let X be a locally finite uniform tree which contains no vertex of degree ≤ 2. Then if $G \leq \mathrm{Isom}(X)$ is a lattice, then G is perfect.

Proof. The estimate (8.25) is Theorem 2.1(2) of [38]. To demonstrate (8.24), let H be a discrete cocompact subgroup of $\mathrm{Isom}(X)$. By Example 8.14, the Patterson-Sullivan measure of H satisfies (8.24), so it remains to show that the Patterson-Sullivan measure of H is also δ-conformal with respect to G. But by Theorem 6.3 of [34], the Patterson-Sullivan measure of H is a constant multiple of the δ-dimensional Hausdorff measure on ∂X, and it is well known (e.g. p. 174 of [72]) that this is δ-conformal for any group. $\qquad \square$

Note that for each $n \geq 3$, the unique tree such that each vertex has degree n is a locally finite uniform tree.

In our next example, (8.25) is satisfied, but (8.24) is usually not satisfied. Of course, (8.24) is still satisfied if ξ is a uniformly radial point, so in this case we have (8.29). Moreover, by Corollary 8.12 we have $\mu(\mathrm{BA}_\xi) = 0$ for all $\xi \in \Lambda_G$.

Example 1.45 (Geometrically finite group on a pinched Hadamard manifold)**.** Let X be a pinched Hadamard manifold, and let $G \leq \mathrm{Isom}(X)$ be geometrically finite (meaning that $\Lambda = \Lambda_\mathrm{r} \cup \Lambda_\mathrm{bp}$). Suppose that one of the following conditions holds:

  (A) $X = \mathbb{H}^{d+1}$ for some $d \in \mathbb{N}$.
  (B) X is a real, complex, or quaternionic hyperbolic space.
  (C) For each $\xi \in \Lambda_\mathrm{bp}$, the orbital counting function f_ξ (see (7.42)) of G_ξ satisfies a *power law* i.e.

$$(1.12) \qquad\qquad f_\xi(R) \asymp_\times R^{2\delta_\xi},$$

   where δ_ξ is the Poincaré exponent of the parabolic subgroup G_ξ.
  (D) For each $\xi \in \Lambda_\mathrm{bp}$, G_ξ is of divergence type.
  (E) For each $\xi \in \Lambda_\mathrm{bp}$, $\delta_\xi < \delta$.

Then (8.25) is satisfied. In particular, if (C) holds, then for each $\xi \in \Lambda_\mathrm{bp}$ we have

$$(1.13) \qquad \mu(\mathrm{WA}_{\Phi, \xi}) = 0 \;\Leftrightarrow\; \int_{t=0}^{\infty} (e^t \Phi(e^t))^{2(\delta - \delta_\xi)} \mathrm{d}t < +\infty.$$

for every function $\Phi : [0, +\infty) \to (0, +\infty)$ such that $t \mapsto t\Phi(t)$ is nonincreasing.

Proof. The implication (E) $\Rightarrow$ (8.25) was proven by S. D. Hersonsky and F. Paulin (Lemma 3.6 of [37]) using a result of T. Roblin (Corollaire 2 of Theorémè 4.1.1 of [63]). We claim that (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (D) $\Rightarrow$ (E). Indeed, it is obvious that (A) $\Rightarrow$ (B) and that (C) $\Rightarrow$ (D). The fact that (B) $\Rightarrow$ (C) is Lemma 3.5 of [54], and the fact that (D) $\Rightarrow$ (E) is Proposition 2 of [17].

To demonstrate (1.13), note that since (8.25) holds, by Proposition 8.11 we have

$$(8.1) \asymp_{+, \times} \int_{t=0}^{\infty} \Delta_{\mu, \xi}(e^t \Phi(e^t)) \mathrm{d}t$$

$$\asymp_\times \int_{t=0}^{\infty} (e^t \Phi(e^t))^{2(\delta - \delta_\xi)} \mathrm{d}t. \qquad \text{(by Théorème 3.2 of [64])}$$

We remark that in the last step, we used the hypothesis (C) to apply B. Schapira's Global Measure Formula for pinched Hadamard manifolds.

Now since ξ is assumed to be a bounded parabolic point, Theorem 8.1 (parts (i) and (iiB)) tells us that $\mu(\mathrm{WA}_{\Phi, \xi}) = 0$ or 1 according to whether the series (8.1) converges or diverges, respectively. $\qquad \square$

Remark 1.46. The implication (A) $\Rightarrow$ (8.25) was proven by B. O. Stratmann and S. L. Velani (Corollary 4.2 of [71]). The idea of their proof, combined with the more general global measure formulas found in [54] and [64], can be used to prove that (B) $\Rightarrow$ (8.25) and that (C) $\Rightarrow$ (8.25), respectively.

Remark 1.47. If $\delta_\xi = \delta/2$, then (1.13) reduces to (8.29) (cf. Remark 1.29). In particular this happens if X is a real, complex, or quaternionic hyperbolic space and G is a lattice.

Proof. See (i) of Proposition B.7. $\square$

1.5.3. *A remark on* $\mu(\mathrm{VWA}_\xi)$ *in the above example.* In Example 1.45, since (8.25) is satisfied but not (8.24), Corollary 8.12 says that $\mu(\mathrm{BA}_\xi) = 0$ but gives no information on $\mu(\mathrm{VWA}_\xi)$ (unless ξ is a uniformly radial point).

Proposition 1.48. *Let* $X = \mathbb{H}^{d+1}$ *for some* $d \in \mathbb{N}$, *and let* $G \le \mathrm{Isom}(X)$ *be geometrically finite. Then* $\mu(\mathrm{VWA}_\xi) = 0$.

Proof. For each $c > 0$, by Theorem 7.1 we have $\mathrm{HD}(W_{c,\xi}) \le \frac{\delta}{c+1} < \delta$. But by Proposition 4.10 of [71], any set of full μ-measure has Hausdorff dimension at least δ. Thus $\mu(W_{c,\xi}) = 0$ for all $c > 0$, and so $\mu(\mathrm{VWA}_\xi) = 0$. $\square$

Remark 1.49. It appears to be difficult to prove that $\mu(\mathrm{VWA}_\xi) = 0$ directly from Theorem 8.1 (assuming that (8.24) is not satisfied and that $\xi \notin \Lambda_{\mathrm{bp}}$), since then the series (8.2) needs to be analyzed directly, as it is not clear whether it is asymptotic to (8.1).

1.5.4. *An example where* (8.1) $\not\asymp$ (8.2). In Example 1.45, we used (iiB) of Theorem 8.1 to prove (1.13). A natural question is whether we could have used (iiA) instead, as it is more general. However, we could not, as the following proposition shows:

Proposition 8.15. *Let* $X = \mathbb{H}^{d+1}$ *for some* $d \in \mathbb{N}$, *and let* $G \le \mathrm{Isom}(X)$ *be geometrically finite. If* ξ *is a bounded parabolic point whose rank is strictly greater than* $4\delta/3$, *then there exists a function* Φ *such that* $t \mapsto t\Phi(t)$ *is nonincreasing and such that for every* $K > 0$, (8.1) *converges but* (8.2) *diverges.*

If G is a nonelementary geometrically finite group and if ξ is a parabolic point of rank k_ξ, then it is well-known [4] that $\delta > k_\xi/2$; in fact, this bound is optimal, as can be seen for example from Theorem 1 of [59]. In particular, there exist examples of nonelementary geometrically finite groups with parabolic points ξ satisfying $\delta < 3k_\xi/4$ or equivalently $k_\xi > 4\delta/3$, so Proposition 8.15 is not vacuous.

1.5.5. *The measure of* Λ_{ur}. We end this subsection by stating the following result, which is an application of Theorem 8.1:

Proposition 8.18. *Let* (X, d, o, b, G) *be as in* §1.1.4, *with* X *proper and geodesic. Let* μ *be a* δ_G-*quasiconformal measure on* Λ. *Then* $\mu(\Lambda_{\mathrm{ur}}) = 0$ *if and only if* G *is not quasiconvex-cocompact.*

History of Theorem 8.1. Although Theorem 8.1 has not been stated in the literature previously, the equivalences (8.29) and (1.13) have both appeared in the literature. (8.29) was proven

- in the case where $X = \mathbb{H}^2$, G is a lattice, and ξ is a conventional point by S. J. Patterson in 1976 (the Theorem of §9 of [57]).
- in the case where $X = \mathbb{H}^{d+1}$, G is a lattice, ξ is a parabolic point, and $\Phi(e^t) = e^{-t}(t \vee 1)^{-(1+\varepsilon)/\delta}$ for some $\varepsilon \ge 0$ (the "logarithm law") by D. P. Sullivan in 1982 (Theorem 6 of [73]).
- in the case where $X = \mathbb{H}^{d+1}$, G is convex-cocompact, and ξ is a hyperbolic point by B. O. Stratmann in 1994 (Theorem A of [67]).
- in the case where X is a locally finite uniform tree which contains no vertex of degree ≤ 2, G is a lattice, and ξ is a parabolic point by S. D. Hersonsky and F. Paulin in 2007 (Theorem 1.1 of [38]).

(1.13) was proven

- in the case where $X = \mathbb{H}^{d+1}$, G is geometrically finite, and ξ is a parabolic point by B. O. Stratmann and S. L. Velani in 1995 (Theorem 4 of [71]).

- in the case where X is a pinched Hadamard manifold, G is geometrically finite, and ξ is a parabolic point satisfying (1.12) by S. D. Hersonsky and F. Paulin in 2004 (Theorem 3 of [37]).

Note that in the above lists, we ignore all differences between regularity hypotheses on the function Φ.

2. Gromov hyperbolic metric spaces

In this section we review the theory of hyperbolic metric spaces and discrete groups of isometries acting on them. There is a vast literature on this subject; see for example [11, 40, 55, 77]. We would like to single out [8], [11], and [77] as exceptions that work in not necessarily proper settings and make reference to infinite-dimensional hyperbolic space.

Definition 2.1. Let (X, d) be a metric space. For any three points $x, y, z \in X$, the *Gromov product* of x and y with respect to z is defined by

$$\langle x|y\rangle_z := \frac{1}{2}[d(x,z) + d(y,z) - d(x,y)].$$

Intuitively, the Gromov product measures the "defect in the triangle inequality".

Definition 2.2. A metric space (X, d) is called *hyperbolic* (or *Gromov hyperbolic*) if for every four points $x, y, z, w \in X$ we have

$$(2.1) \qquad \langle x|z\rangle_w \gtrsim_+ \min(\langle x|y\rangle_w, \langle y|z\rangle_w).$$

We will refer to this inequality as *Gromov's inequality*.

Hyperbolic metric spaces should be thought of as "those which have uniformly negative curvature". Note that many authors require X to be a geodesic metric space in order to be hyperbolic; we do not. If X is a geodesic metric space, then the condition of hyperbolicity can be reformulated in several different ways, including the *thin triangles condition*; see §III.H.1 of [11] for details.

For each $z \in X$, let $\mathcal{B}_z$ denote the *Busemann function*

$$(2.2) \qquad \mathcal{B}_z(x, y) := d(z, x) - d(z, y).$$

Proposition 2.3. *The Gromov product satisfies the following identities and inequalities:*

(a) $$\langle x|y\rangle_z = \langle y|x\rangle_z$$

(b) $$d(y, z) = \langle y|x\rangle_z + \langle z|x\rangle_y$$

(c) $$0 \leq \langle x|y\rangle_z \leq \min(d(x,z), d(y,z))$$

(d) $$\langle x|y\rangle_z \leq \langle x|y\rangle_w + d(z, w)$$

(e) $$\langle x|y\rangle_z \leq \langle x|z\rangle_w + d(y, z)$$

(f) $$\langle x|y\rangle_z = \langle x|y\rangle_w + \frac{1}{2}[\mathcal{B}_x(z, w) + \mathcal{B}_y(z, w)]$$

(g) $$\langle x|y\rangle_z = \frac{1}{2}[d(x, z) + \mathcal{B}_y(z, x)]$$

(h) $$\mathcal{B}_x(y, z) = \langle z|x\rangle_y - \langle y|x\rangle_z$$

(i) $$\langle x|y\rangle_z = \langle x|y\rangle_w + d(z, w) - \langle x|z\rangle_w - \langle y|z\rangle_w$$

(j) $$\langle x|y\rangle_z = \langle x|y\rangle_w + \langle x|w\rangle_z - \langle y|z\rangle_w$$

The proof is a straightforward computation. We remark that (a)-(e) may be found in Lemma 2.8 of [77].

2.1. **Examples of hyperbolic metric spaces.** The class of hyperbolic metric spaces is rich and well-studied. For example, any subspace of a hyperbolic metric space is again a hyperbolic metric space, and any geodesic metric space quasi-isometric to a geodesic hyperbolic metric space is again a hyperbolic metric space (Theorem III.H.1.9 of [11]). Moreover, for every $\kappa < 0$, every CAT(κ) space is hyperbolic (Proposition III.H.1.2 of [11]). We list below some well-known celebrated examples with no desire for completeness. Items with $*$ are CAT(κ) spaces for some $\kappa < 0$, and items with $\dagger$ are proper metric spaces, i.e. those for which closed balls are compact.

- Standard hyperbolic space $\mathbb{H}^{d+1}$ or $\mathbb{B}^{d+1}$ (*,†).
- Infinite-dimensional hyperbolic space, i.e. the open unit ball $\mathbb{B}^\infty$ in a separable Hilbert space with the Poincaré metric (*). This space and groups of isometries acting on it is investigated at length in [21].
- Trees (*,† if locally finite)
- $\mathbb{R}$-trees (*, sometimes †).
- Word-hyperbolic groups together with their Cayley metrics (†). We remark that this includes almost every finitely presented group [56].
- All Riemannian manifolds with negative sectional curvature uniformly bounded away from zero (*,†).
- Complex and quaternionic hyperbolic spaces (*,†). These will be discussed in some detail in Appendix B.
- The curve graph of an oriented surface with finitely many punctures; see [47] (†).

2.2. **The boundary of a hyperbolic metric space.** In this subsection, we define the Gromov boundary of a hyperbolic metric space (X,d). If $X = \mathbb{H}^{d+1}$ or $\mathbb{B}^{d+1}$ for some $d \in \mathbb{N}$, then the boundary of X is exactly what you would expect, i.e. it is isomorphic to the topological boundary of X relative to $\widehat{\mathbb{R}^{d+1}}$.

In the following, we fix a distinguished point $o \in X$. However, it will be clear that the definitions given are independent of which point is chosen.

Definition 2.4. A sequence $(x_n)_1^\infty$ in X is called a *Gromov sequence* if

$$\langle x_n | x_m \rangle_o \xrightarrow[n,m]{} +\infty.$$

Two Gromov sequences $(x_n)_1^\infty$ and $(y_n)_1^\infty$ are called *equivalent* if

$$\langle x_n | y_n \rangle_o \xrightarrow[n]{} +\infty.$$

It is readily verified using Gromov's inequality that equivalence of Gromov sequences is indeed an equivalence relation.

Definition 2.5. The *Gromov boundary* of X is the set of Gromov sequences modulo equivalence. It is denoted ∂X. The *Gromov closure* of X is the set $\overline{X} := X \dot\cup \partial X$.

Remark 2.6. Let g be an isometry of X. Then g takes Gromov sequences to Gromov sequences, and equivalent Gromov sequences to equivalent Gromov sequences. Taking the quotient yields a map $g_{\partial X} : \partial X \to \partial X$. Let $g_{\overline{X}} : \overline{X} \to \overline{X}$ be the piecewise combination of g with $g_{\partial X}$. Abusing notation, we will in the sequel denote the maps g, $g_{\partial X}$, and $g_{\overline{X}}$ by the same symbol g.

2.2.1. *Extending the Gromov product and Busemann function to the boundary.* For all points $\xi, \eta \in \partial X$ and $y, z \in X$, the appropriate Gromov products and Busemann function are defined as follows:

$$(2.3) \qquad \langle \xi | \eta \rangle_z := \inf\{ \liminf_{n,m \to \infty} \langle x_n | y_m \rangle_z : (x_n)_1^\infty \in \xi, (y_m)_1^\infty \in \eta \}$$

$$(2.4) \qquad \langle \xi | y \rangle_z := \langle y | \xi \rangle_z := \inf\{ \liminf_{n \to \infty} \langle x_n | y \rangle_z : (x_n)_1^\infty \in \xi \}$$

$$(2.5) \qquad \mathcal{B}_\xi(y,z) := \langle z | \xi \rangle_y - \langle y | \xi \rangle_z,$$

keeping in mind that $\xi, \eta \in \partial X$ are collections of Gromov sequences.

The main properties of the Gromov product on the boundary come from the following lemma:

Lemma 2.7 (Lemma 5.11 of [77]). *Fix $\xi, \eta \in \partial X$ and $y, z \in X$. For all $(x_n)_1^\infty \in \xi$ and $(y_m)_1^\infty \in \eta$ we have*

$$(2.6) \qquad \langle x_n | y_m \rangle_z \xrightarrow[n,m,+]{} \langle \xi | \eta \rangle_z$$

$$(2.7) \qquad \langle x_n | y \rangle_z \xrightarrow[n,+]{} \langle \xi | y \rangle_z$$

$$(2.8) \qquad \mathcal{B}_{x_n}(y,z) \xrightarrow[n,+]{} \mathcal{B}_\xi(y,z).$$

((2.8) is not found in [77] but it follows immediately from (2.7).)

A simple but useful consequence of Lemma 2.7 is the following:

Corollary 2.8. *The formulas of Proposition 2.3 together with Gromov's inequality hold for points on the boundary as well, if the equations and inequalities there are replaced by additive asymptotics.*

2.2.2. *A topology on $\overline{X}$.* One can endow $\overline{X}$ with a topological structure. We will call a set $S \subseteq \overline{X}$ open if $S \cap X$ is open and if for each $\xi \in S \cap \partial X$ there exists $t > 0$ such that $N_t(\xi) \subseteq S$, where

$$(2.9) \qquad N_t(\xi) := \{y \in \overline{X} : \langle y | \xi \rangle_o > t\}.$$

We remark that with the topology defined above, $\overline{X}$ is metrizable; see Exercise III.H.3.18(4) of [11] if X is proper and geodesic, and [21] for the general case.

Observation 2.9. A sequence $(x_n)_1^\infty$ in $\overline{X}$ converges to a point $\xi \in \partial X$ if and only if

$$(2.10) \qquad \langle x_n | \xi \rangle_o \xrightarrow[n]{} +\infty.$$

Observation 2.10. A sequence $(x_n)_1^\infty$ in X converges to a point $\xi \in \partial X$ if and only if $(x_n)_1^\infty$ is a Gromov sequence and $(x_n)_1^\infty \in \xi$.

Lemma 2.11 (Near-continuity of the Gromov product and Busemann function)**.** *Suppose that $(x_n)_1^\infty$ and $(y_n)_1^\infty$ are sequences in $\overline{X}$ which converge to points $x_n \xrightarrow[n]{} x \in \overline{X}$ and $y_n \xrightarrow[n]{} y \in \overline{X}$. Suppose that $(z_n)_1^\infty$ and $(w_n)_1^\infty$ are sequences in X which converge to points $z_n \xrightarrow[n]{} z \in X$ and $w_n \xrightarrow[n]{} w \in X$. Then*

$$(2.11) \qquad \langle x_n | y_n \rangle_{z_n} \xrightarrow[n,+]{} \langle x | y \rangle_z$$

$$(2.12) \qquad \mathcal{B}_{x_n}(z_n, w_n) \xrightarrow[n,+]{} \mathcal{B}_x(z, w),$$

Proof. In the proof of (2.11), there are three cases:

Case 1: $x, y \in X$. In this case, (2.11) follows directly from (d) and (e) of Proposition 2.3.

Case 2: $x, y \in \partial X$. In this case, for each $n \in \mathbb{N}$, choose $\widehat{x}_n \in X$ such that either

    (1) $\widehat{x}_n = x_n$ (if $x_n \in X$), or

    (2) $\langle \widehat{x}_n | x_n \rangle_z \geq n$ (if $x_n \in \partial X$).

    Choose $\widehat{y}_n$ similarly. Clearly, $\widehat{x}_n \xrightarrow[n]{} x$ and $\widehat{y}_n \xrightarrow[n]{} y$. By Observation 2.10, $(\widehat{x}_n)_1^\infty \in x$ and $(\widehat{y}_n)_1^\infty \in y$.

    Thus by Lemma 2.7,

$$(2.13) \qquad \langle \widehat{x}_n | \widehat{y}_n \rangle_z \xrightarrow[n,+]{} \langle x | y \rangle_z.$$

    Now by Gromov's inequality and (e) of Proposition 2.3, either

$$(1) \qquad \langle \widehat{x}_n | \widehat{y}_n \rangle_z \asymp_+ \langle x_n | y_n \rangle_{z_n} \text{ or}$$

$$(2) \qquad \langle \widehat{x}_n | \widehat{y}_n \rangle_z \gtrsim_+ n,$$

    with which asymptotic is true depending on n. But for n sufficiently large, (2.13) ensures that the (2) fails, so (1) holds.

Case 3: $x \in X$, $y \in \partial X$, or vice-versa. In this case, a straightforward combination of the above arguments demonstrates (2.11).

Finally, note that (2.12) is an immediate consequence of (2.11), (2.5), and (h) of Proposition 2.3. $\qquad \square$

Remark 2.12. If g is an isometry of X, then the map $g_{\overline{X}} : \overline{X} \to \overline{X}$ defined in Remark 2.6 is a homeomorphism of $\overline{X}$.

2.3. **Different (meta)metrics.**

Definition 2.13. Recall that a *metric* on a set Z is a map $D : Z \times Z \to [0, +\infty)$ which satisfies:

(I) Reflexivity: $D(x, x) = 0$.
(II) Reverse reflexivity: $D(x, y) = 0 \Rightarrow x = y$.
(III) Symmetry: $D(x, y) = D(y, x)$.
(IV) Triangle inequality: $D(x, z) \leq D(x, y) + D(y, z)$.

Now we can define a *metametric* on Z to be a map $D : Z \times Z \to [0, +\infty)$ which satisfies (II), (III), and (IV), but not necessarily (I). This concept is not to be confused with the more common notion of a *pseudometric*, which satisfies (I), (III), and (IV), but not necessarily (II). The term "metametric" was introduced by J. Väisälä in [77].

If D is a metametric, we define its *domain of reflexivity* as the set $Z_{\text{refl}} := \{x \in Z : D(x, x) = 0\}$. Obviously, D restricted to its domain of reflexivity is a metric.

Now let (X, d) be a hyperbolic metric space. We will consider various metametrics on $\overline{X}$.

2.3.1. *The visual metametric based at a point $z \in X$.*

Proposition 2.14. *If $X = \mathbb{B}^{d+1}$, then for all $\mathbf{x}, \mathbf{y} \in \partial X$ we have*

$$(2.14) \qquad\qquad \frac{1}{2}\|\mathbf{y} - \mathbf{x}\| = e^{-\langle \mathbf{x}|\mathbf{y}\rangle_{\mathbf{o}}}.$$

Proof. See §III.H.3.19 of [11] for $\asymp_\times$ and [21] for equality. □

The following proposition generalizes (2.14) to the setting of hyperbolic metric spaces:

Proposition 2.15 (Propositions 5.16 and 5.31 of [77]). *For each $b > 1$ sufficiently small and for each $z \in X$, there exists a complete metametric $D_{b,z}$ on ∂X satisfying*

$$(2.15) \qquad\qquad D_{b,z}(y_1, y_2) \asymp_\times b^{-\langle y_1|y_2\rangle_z}$$

for all $y_1, y_2 \in \overline{X}$. The implied constant may depend on b but not on z.

We will refer to $D_{b,z}$ as the "visual (meta)metric from the point z with respect to the parameter b".

The metric $D_{b,z} \upharpoonright \partial X$ has been referred to in the literature as the *Bourdon metric*. We remark that this metric is compatible with the topology defined in §2.2.2 restricted to ∂X.

Remark 2.16. For CAT(-1) spaces, Proposition 2.15 holds for any $1 < b \leq e$; moreover, the asymptotic in (2.15) may be replaced by an equality; see [9] if X is proper and [21] for the general case.

2.3.2. *Standing assumptions for Section 2.* For the remainder of Section 2, we will have the following standing assumptions:

(I) (X, d) is a hyperbolic metric space
(II) $o \in X$ is a distinguished point
(III) $b > 1$ is a parameter small enough to guarantee for every $z \in X$ the existence of a visual metametric $D = D_{b,z}$ via Proposition 2.15 above.

2.3.3. *The visual metametric based at a point $\xi \in \partial X$.* Our next metametric is supposed to generalize the Euclidean metric on the boundary of the upper half-plane model $\mathbb{H}^{d+1}$. This metric should be thought of as "seen from the point ∞".

Notation 2.17. If X is a hyperbolic metric space and $\xi \in \partial X$, then let $\mathcal{E}_\xi := \overline{X} \setminus \{\xi\}$.

We will motivate the visual metametric based at a point $\xi \in \partial X$ by considering a sequence $(z_n)_1^\infty$ in X converging to ξ, and taking the limits of their visual metametrics.

In fact, $D_{b,z_n}(y_1, y_2) \xrightarrow[n]{} 0$ for every $y_1, y_2 \in \mathcal{E}_\xi$. Some normalization is needed.

Lemma 2.18. *Let (X, d, o, b) be as in §2.3.2, and suppose $z_n \underset{n}{\to} \xi \in \partial X$. Then for each $y_1, y_2 \in \mathcal{E}_\xi$,*

$$b^{d(o,z_n)} D_{b,z_n}(y_1, y_2) \xrightarrow[n,\times]{} b^{-[\langle y_1 | y_2 \rangle_o - \sum_{i=1}^2 \langle y_i | \xi \rangle_o]}.$$

Proof.

$$\begin{aligned}
b^{d(o,z_n)} D_{b,z_n}(y_1, y_2) &\asymp_\times b^{-[\langle y_1 | y_2 \rangle_{z_n} - d(o,z_n)]} \\
&\asymp_\times b^{-[\langle y_1 | y_2 \rangle_o - \sum_{i=1}^2 \langle y_i | z_n \rangle_o]} &&\text{(by (i) of Proposition 2.3)} \\
&\xrightarrow[n,\times]{} b^{-[\langle y_1 | y_2 \rangle_o - \sum_{i=1}^2 \langle y_i | \xi \rangle_o]}. &&\text{(by Lemma 2.11)}
\end{aligned}$$

$\square$

Corollary 2.19. *Let (X, d, o, b) be as in §2.3.2. Then for each $\xi \in \partial X$, there exists a metametric $D_{b,\xi,o}$ on $\mathcal{E}_\xi$ satisfying*

$$\tag{2.16} D_{b,\xi,o}(y_1, y_2) \asymp_\times b^{-[\langle y_1 | y_2 \rangle_o - \sum_{i=1}^2 \langle y_i | \xi \rangle_o]} .$$

The metric $D_{b,\xi,o} \upharpoonright \mathcal{E}_\xi \cap \partial X$ has been referred to in the literature as the *Hamenstädt metric*.

Proof of Corollary 2.19. Let

$$D_{b,\xi,o}(y_1, y_2) = \limsup_{z \to \xi} b^{d(o,z)} D_z(y_1, y_2).$$

Since the class of metametrics is closed under suprema and limits, it follows that $D_{b,\xi,o}$ is a metametric. $\square$

From Lemma 2.18 and Corollary 2.19 it immediately follows that

$$\tag{2.17} b^{d(o,z_n)} D_{b,z_n}(y_1, y_2) \xrightarrow[n,\times]{} D_{b,\xi,o}(y_1, y_2)$$

whenever $(z_n)_1^\infty \in \xi$.

Lemma 2.20. *Let (X, d, o, b) be as in §2.3.2. For all $x \in X$ and $\xi \in \partial X$, we have*

$$\tag{2.18} D_{\xi,o}(o, x) \asymp_\times b^{\langle x | \xi \rangle_o} \asymp_\times \frac{1}{D_o(x, \xi)}.$$

Proof. By (2.16),

$$D_{\xi,o}(o, x) \asymp_\times b^{-[\langle o | x \rangle_o - \langle o | \xi \rangle_o - \langle x | \xi \rangle_o]} = b^{\langle x | \xi \rangle_o}.$$

$\square$

2.3.4. *Comparison of different metametrics.*

Observation 2.21. *Let (X, d, o, b) be as in §2.3.2, and fix $y_1, y_2 \in \overline{X}$.*

(i) *For all $z_1, z_2 \in X$, we have*

$$\tag{2.19} \frac{D_{z_1}(y_1, y_2)}{D_{z_2}(y_1, y_2)} \asymp_\times b^{-\frac{1}{2}[\mathcal{B}_{y_1}(z_1, z_2) + \mathcal{B}_{y_2}(z_1, z_2)]}.$$

(ii) *For all $z \in X$ and $\xi \in \partial X$, we have*

$$\tag{2.20} \frac{D_{\xi,z}(y_1, y_2)}{D_z(y_1, y_2)} \asymp_\times b^{-[\langle y_1 | \xi \rangle_z + \langle y_2 | \xi \rangle_z]}.$$

(iii) *For all $z_1, z_2 \in X$ and $\xi \in \partial X$, we have*

$$\tag{2.21} \frac{D_{\xi,z_1}(y_1, y_2)}{D_{\xi,z_2}(y_1, y_2)} \asymp_\times b^{\mathcal{B}_\xi(z_1, z_2)}.$$

Proof. (i) follows from the (f) of Proposition 2.3, (ii) follows from (2.16), and (iii) follows from (2.17). $\square$

Corollary 2.22. *Let (X, d, o, b) be as in §2.3.2, and fix $y_1, y_2 \in \overline{X}$. Then for all $g \in \mathrm{Isom}(X)$,*

$$\tag{2.22} \frac{D(g(y_1), g(y_2))}{D(y_1, y_2)} \asymp_\times b^{(1/2)[\mathcal{B}_{y_1}(o, g^{-1}(o)) + \mathcal{B}_{y_2}(o, g^{-1}(o))]}.$$

Proof. Note that $D \circ g = D_{g^{-1}(o)}$, and apply (2.19). □

Notation 2.23. From now on we will write
$$g'(y) = b^{\mathcal{B}_y(o, g^{-1}(o))},$$
so that (2.22) may be rewritten as the *geometric mean value theorem*

(2.23)
$$\frac{D(g(y_1), g(y_2))}{D(y_1, y_2)} \asymp_\times (g'(y_1) g'(y_2))^{1/2}.$$

To justify this notation somewhat, note that if $y_1 \in \overline{X}$ is not an isolated point then
$$g'(y_1) \asymp_\times \lim_{y_2 \to y_1} \frac{D(g(y_1), g(y_2))}{D(y_1, y_2)}$$
by (2.23) together with Lemma 2.11.

We end this subsection with a proposition which shows the relation between the derivative of an isometry $g \in \mathrm{Isom}(X)$ at a point $\xi \in \mathrm{Fix}(g)$ and the action on the metametric space $(\mathcal{E}_\xi, D_{\xi,o})$:

Proposition 2.24. *Fix $g \in \mathrm{Isom}(X)$ and $\xi \in \mathrm{Fix}(g)$. Then for all $y_1, y_2 \in \mathcal{E}_\xi$,*
$$D_{\xi,o}(g(y_1), g(y_2)) \asymp_\times g'(\xi)^{-1} D_{\xi,o}(y_1, y_2).$$

Proof.
$$\begin{aligned}
D_{\xi,o}(g(y_1), g(y_2)) =\ & D_{\xi, g^{-1}(o)}(y_1, y_2) \\
\asymp_\times\ & b^{-\mathcal{B}_\xi(o, g^{-1}(o))} D_{\xi,o}(y_1, y_2) && \text{(by (2.21))} \\
\asymp_\times\ & g'(\xi)^{-1} D_{\xi,o}(y_1, y_2).
\end{aligned}$$
□

2.4. **The geometry of shadows.** Recall that if $X = \mathbb{H}^{d+1}$ or $\mathbb{B}^{d+1}$, then for each $z \in X$ the projection map $\pi_z : X \setminus \{z\} \to \partial X$ is defined to be the unique map so that for all $x \in X \setminus \{z\}$, x is on the geodesic ray joining z and $\pi_z(x)$. For $x \in X$ and $\sigma > 0$, it is useful to consider the set $\pi_z(B(x, \sigma))$, which is called the "shadow" of the ball $B(x, \sigma)$ with respect to the point z. This definition does not make sense in our setting, since a hyperbolic metric space does not necessarily have geodesics. Thus we replace it with the following definition which uses only the Gromov product:

Definition 2.25. Let (X, d) be a hyperbolic metric space. For each $\sigma > 0$ and $x, z \in X$, let
$$\mathrm{Shad}_z(x, \sigma) = \{\eta \in \partial X : \langle z | \eta \rangle_x \leq \sigma\}.$$
We say that $\mathrm{Shad}_z(x, \sigma)$ is the *shadow cast by x from the light source z, with parameter σ*. For shorthand we will write $\mathrm{Shad}(x, \sigma) = \mathrm{Shad}_o(x, \sigma)$.

Although we will not need it in our proofs, let us remark that the sets $\mathrm{Shad}_z(x, \sigma)$ are closed [21].

Proposition 2.26 ([21]). *Let $X = \mathbb{H}^{d+1}$ or $\mathbb{B}^{d+1}$. For every $\sigma > 0$, there exists $\tau = \tau_\sigma > 0$ such that for any $x, z \in X$ we have*
$$\pi_z(B(x, \sigma)) \subseteq \mathrm{Shad}_z(x, \sigma) \subseteq \pi_z(B(x, \tau)).$$

Let us establish up front some geometric properties of shadows.

Observation 2.27. *If $\eta \in \mathrm{Shad}_z(x, \sigma)$, then*
$$\langle x | \eta \rangle_z \asymp_{+, \sigma} d(z, x).$$

Proof.
$$\langle x | \eta \rangle_z \leq d(z, x) \asymp_+ \langle x | \eta \rangle_z + \langle z | \eta \rangle_x \leq \langle x | \eta \rangle_z + \sigma.$$
□

Lemma 2.28 (Intersecting Shadows Lemma)**.** *Let* (X, d) *be a hyperbolic metric space. For each* $\sigma > 0$*, there exists* $\tau = \tau_\sigma > 0$ *such that for all* $x, y, z \in X$ *satisfying* $d(z, y) \geq d(z, x)$ *and* $\mathrm{Shad}_z(x, \sigma) \cap \mathrm{Shad}_z(y, \sigma) \neq \varnothing$*, we have*

$$\tag{2.24} \mathrm{Shad}_z(y, \sigma) \subseteq \mathrm{Shad}_z(x, \tau)$$

and

$$\tag{2.25} d(x, y) \asymp_{+,\sigma} \mathcal{B}_z(y, x).$$

Proof. Fix $\eta \in \mathrm{Shad}_z(x, \sigma) \cap \mathrm{Shad}_z(y, \sigma)$, so that by Observation 2.27

$$\langle x|\eta\rangle_z \asymp_{+,\sigma} d(z, x) \text{ and } \langle y|\eta\rangle_z \asymp_{+,\sigma} d(z, y) \geq d(z, x).$$

Gromov's inequality along with (c) of Proposition 2.3 then gives

$$\tag{2.26} \langle x|y\rangle_z \asymp_{+,\sigma} d(z, x).$$

Rearranging yields (2.25). In order to show (2.24), fix $\xi \in \mathrm{Shad}_z(y, \sigma)$, so that $\langle y|\xi\rangle_z \asymp_{+,\sigma} d(z, y) \geq d(z, x)$. Gromov's inequality and (2.26) then give

$$\langle x|\xi\rangle_z \asymp_{+,\sigma} d(z, x),$$

i.e. $\xi \in \mathrm{Shad}_z(x, \tau)$ for some $\tau > 0$ sufficiently large (depending on σ). $\qquad\square$

Lemma 2.29 (Bounded Distortion Lemma)**.** *Let* (X, d, o, b) *be as in §2.3.2, and fix* $\sigma > 0$*. Then for every* $g \in \mathrm{Isom}(X)$ *and for every* $\xi \in \mathrm{Shad}_{g^{-1}(o)}(o, \sigma)$ *we have*

$$\tag{2.27} g'(\xi) \asymp_{\times,\sigma} b^{-d(o, g(o))}.$$

Moreover, for every $\xi_1, \xi_2 \in \mathrm{Shad}_{g^{-1}(o)}(o, \sigma)$*, we have*

$$\tag{2.28} \frac{D(g(\xi_1), g(\xi_2))}{D(\xi_1, \xi_2)} \asymp_{\times,\sigma} b^{-d(o, g(o))}.$$

Proof. By (g) of Proposition 2.3,

$$g'(\xi) = b^{\mathcal{B}_\xi(o, g^{-1}(o))} \asymp_\times b^{2\langle g^{-1}(o)|\xi\rangle_o - d(o, g(o))} \asymp_{\times,\sigma} b^{-d(o, g(o))}.$$

Now (2.28) follows from (2.27) and the geometric mean value theorem (2.23). $\qquad\square$

Lemma 2.30 (Big Shadows Lemma)**.** *Let* (X, d, o, b) *be as in §2.3.2. For every* $\varepsilon > 0$*, for every* $\sigma > 0$ *sufficiently large, and for every* $z \in X$*, we have*

$$\tag{2.29} \mathrm{Diam}(\partial X \setminus \mathrm{Shad}_z(o, \sigma)) \leq \varepsilon.$$

Proof. If $\xi, \eta \in \partial X \setminus \mathrm{Shad}_z(o, \sigma)$, then $\langle z|\xi\rangle_o > \sigma$ and $\langle z|\eta\rangle_o > \sigma$. Thus by Gromov's inequality we have

$$\langle \xi|\eta\rangle_o \gtrsim_+ \sigma.$$

Exponentiating gives $D(\xi, \eta) \lesssim_\times b^{-\sigma}$. Thus

$$\mathrm{Diam}(\partial X \setminus \mathrm{Shad}_z(o, \sigma)) \lesssim_\times b^{-\sigma} \underset{\sigma}{\to} 0,$$

and the convergence is uniform in z. $\qquad\square$

Lemma 2.31 (Diameter of Shadows Lemma)**.** *Let* (X, d, o, b) *be as in §2.3.2, and let* $G \leq \mathrm{Isom}(X)$*. Then for all* $\sigma > 0$ *sufficiently large, we have for all* $x \in G(o)$ *and for all* $z \in X$

$$\mathrm{Diam}_z(\mathrm{Shad}_z(x, \sigma)) \lesssim_{\times,\sigma} b^{-d(z, x)},$$

with $\asymp_{\times,\sigma}$ *if* $\#(\partial X) \geq 3$*.*

Proof. For any $\xi, \eta \in \mathrm{Shad}_z(x, \sigma)$, we have

$$D_{b,z}(\xi, \eta) \asymp_\times b^{-\langle \xi | \eta \rangle_z} \lesssim_\times \quad b^{-\min(\langle x | \xi \rangle_z, \langle x, \eta \rangle_z)}$$
$$\lesssim_{\times, \sigma} b^{-d(z,x)}$$

which proves the $\lesssim$ direction.

Now let us prove the $\gtrsim$ direction, assuming $\#(\partial X) \geq 3$. Fix $\xi_1, \xi_2, \xi_3 \in \partial X$ distinct, let $\varepsilon = \min_{i \neq j} D(\xi_i, \xi_j)/2$, and fix $\sigma > 0$ large enough so that (2.29) holds for every $z \in X$.

Now fix $x = g(o) \in G(o)$ and $z \in X$; by (2.29) we have

$$\mathrm{Diam}(\partial X \setminus \mathrm{Shad}_{g^{-1}(z)}(o, \sigma)) \leq \varepsilon,$$

and thus

$$\#\{i = 1, 2, 3 : \xi_i \in \mathrm{Shad}_{g^{-1}(z)}(o, \sigma)\} \geq 2.$$

Without loss of generality suppose that $\xi_1, \xi_2 \in \mathrm{Shad}_{g^{-1}(z)}(o, \sigma)$. By applying g, we have $g(\xi_1), g(\xi_2) \in \mathrm{Shad}_z(x, \sigma)$. Then

$$\mathrm{Diam}_z(\mathrm{Shad}_z(x, \sigma)) \geq \quad D_{b,z}(g(\xi_1), g(\xi_2))$$
$$\asymp_\times b^{-\langle g(\xi_1) | g(\xi_2) \rangle_z} = b^{-\langle \xi_1 | \xi_2 \rangle_{g^{-1}(z)}}$$
$$\gtrsim_\times b^{-\langle \xi_1 | \xi_2 \rangle_o} b^{-d(o, g^{-1}(z))} \asymp_{\times, \xi_1, \xi_2} b^{-d(z,x)}.$$

$\square$

Remark 2.32. If G is nonelementary then $\#(\partial X) \geq \#(\Lambda_G) \geq 3$ (see Definition 2.37 below). Thus in our applications, we will always have $\asymp_{\times, \sigma}$ in Lemma 2.31.

We end this subsection with the definition of a horoball:

Definition 2.33. A *horoball* centered at a point $\xi \in \partial X$ is a sublevelset of the Busemann function of ξ, i.e. a set of the form

$$H = \{x : \mathcal{B}_\xi(x, o) < t\}.$$

Such sets are called horoballs because in the case $X = \mathbb{H}^{d+1}$, they are balls tangent to the boundary.

2.5. **Modes of convergence to the boundary; limit sets.** Let (X, d) be a hyperbolic metric space, and fix a distinguished point $o \in X$. We recall (Observation 2.9) that a sequence $(x_n)_1^\infty$ in X converges to a point $\xi \in \partial X$ if and only if

$$\langle x_n | \xi \rangle_o \underset{n}{\to} +\infty.$$

In this subsection we define more restricted modes of convergence. To get an intuition let us consider the case $X = \mathbb{H}^2$.

Proposition 2.34 ([21])**.** *Let $(x_n)_1^\infty$ be a sequence in $\mathbb{H}^{d+1}$ converging to a point $\xi \in \mathcal{E}$. Then the following are equivalent:*

(A) *The sequence $(x_n)_1^\infty$ lies within a bounded distance of the geodesic ray $[o, \xi]$.*
(B) *There exists $\sigma > 0$ such that for all $n \in \mathbb{N}$,*

$$\langle o | \xi \rangle_{x_n} \leq \sigma,$$

or equivalently

(2.30) $$\xi \in \mathrm{Shad}(x_n, \sigma).$$

Condition (A) motivates calling this kind of convergence *radial*; we shall use this terminology henceforth. However, condition (B) is best suited to a general hyperbolic metric space.

Definition 2.35. Let X be a hyperbolic metric space, and let $(x_n)_1^\infty$ be a sequence in X converging to a point $\xi \in \partial X$. We will say that $(x_n)_1^\infty$ converges to ξ

- *σ-radially* if (2.30) holds for all $n \in \mathbb{N}$,
- *radially* if it converges σ-radially for some $\sigma > 0$,

- σ-*uniformly radially* if it converges σ-radially, if

(2.31) $$d(x_n, x_{n+1}) \leq \sigma \ \ \forall n \in \mathbb{N},$$

and if $x_1 = o$,
- *uniformly radially* if it converges σ-uniformly radially for some $\sigma > 0$, and
- *horospherically* if

$$\mathcal{B}_\xi(o, x_n) \underset{n}{\to} +\infty,$$

or equivalently, if $\{x_n : n \in \mathbb{N}\}$ intersects every horoball centered at ξ.

Observation 2.36. The concepts of convergence, radial convergence, uniformly radial convergence, and horospherical convergence are independent of the basepoint o, whereas the concept of σ-radial convergence depends on the basepoint. (However, see Lemma 2.39 below.)

Let $G \leq \text{Isom}(X)$. We define the limit set of G, a subset of ∂X which encodes geometric information about G. We also define a few important subsets of the limit set.

Definition 2.37. Let

$$\Lambda(G) := \{\eta \in \partial X : g_n(o) \to \eta \text{ for some sequence } (g_n)_1^\infty \text{ in } G\}$$
$$\Lambda_{\text{r}}(G) := \{\eta \in \partial X : g_n(o) \to \eta \text{ radially for some sequence } (g_n)_1^\infty \text{ in } G\}$$
$$\Lambda_{\text{ur}}(G) := \{\eta \in \partial X : g_n(o) \to \eta \text{ uniformly radially for some sequence } (g_n)_1^\infty \text{ in } G\}$$
$$\Lambda_{\text{r},\sigma}(G) := \{\eta \in \partial X : g_n(o) \to \eta \ \sigma\text{-radially for some sequence } (g_n)_1^\infty \text{ in } G\}$$
$$\Lambda_{\text{ur},\sigma}(G) := \{\eta \in \partial X : g_n(o) \to \eta \ \sigma\text{-uniformly radially for some sequence } (g_n)_1^\infty \text{ in } G\}$$
$$\Lambda_{\text{h}}(G) := \{\eta \in \partial X : g_n(o) \to \eta \text{ horospherically for some sequence } (g_n)_1^\infty \text{ in } G\}.$$

These sets are respectively called the *limit set*, *radial limit set*, *uniformly radial limit set*, *σ-radial limit set*, *σ-uniformly radial limit set*, and *horospherical limit set* of the group G.

Note that

$$\Lambda_{\text{r}} = \bigcup_{\sigma > 0} \Lambda_{\text{r},\sigma}$$
$$\Lambda_{\text{ur}} = \bigcup_{\sigma > 0} \Lambda_{\text{ur},\sigma}$$
$$\Lambda_{\text{ur}} \subseteq \Lambda_{\text{r}} \subseteq \Lambda_{\text{h}} \subseteq \Lambda.$$

Observation 2.38. The sets Λ, Λ_{r}, Λ_{ur}, and Λ_{h} are invariant under the action of G, and are independent of the basepoint o. The set Λ is closed.

The first assertion follows from Observation 2.36 and the second follows directly from the definition of Λ as the intersection of ∂X with the set of accumulation points of the set $G(o)$.

Lemma 2.39 (Near-invariance of the sets $\Lambda_{\text{r},\sigma}$). *For every $\sigma > 0$, there exists $\tau > 0$ such that for every $g \in G$, we have*

$$g(\Lambda_{\text{r},\sigma}) \subseteq \Lambda_{\text{r},\tau}.$$

Proof. Fix $\xi \in \Lambda_{\text{r},\sigma}$. There exists a sequence $(h_n)_1^\infty$ so that $h_n(o) \underset{n}{\to} \xi$ σ-radially, i.e.

$$\langle o|\xi\rangle_{h_n(o)} \leq \sigma \ \ \forall n \in \mathbb{N}$$

and $h_n(o) \underset{n}{\to} \xi$. Now

$$\langle o|g^{-1}(o)\rangle_{h_n(o)} \geq d(o, h_n(o)) - d(o, g^{-1}(o)) \underset{n}{\to} +\infty.$$

Thus, for n sufficiently large, Gromov's inequality gives

$$\langle g^{-1}(o)|\xi\rangle_{h_n(o)} \lesssim_+ \sigma \asymp_{+,\sigma} 0,$$

i.e.

$$\langle o|g(\xi)\rangle_{g \circ h_n(o)} \asymp_{+,\sigma} 0.$$

So $g \circ h_n(o) \underset{n}{\to} g(\xi)$ τ-radially, where τ is the implied constant of this asymptotic. Thus, $g(\xi) \in \Lambda_{\mathrm{r},\tau}$.

$\square$

2.6. **Poincaré exponent.** Let (X, d, o, b) be as in §2.3.2, and let $G \leq \mathrm{Isom}(X)$. For each $s \geq 0$, the series

$$\Sigma_s(G) := \sum_{g \in G} b^{-sd(o,g(o))}$$

is called the *Poincaré series* of the group G in dimension s (or "evaluated at s") relative to b. The number

(2.32) $$\delta_G := \inf\{s \geq 0 : \Sigma_s(G) < +\infty\}$$

is called the *Poincaré exponent* of the group G relative to b. Here, we let $\inf \varnothing = +\infty$.

Definition 2.40. A group of isometries G with $\delta_G < +\infty$ is said to be of *convergence type* if $\Sigma_{\delta_G}(G) < +\infty$. Otherwise, it is said to be of *divergence type*. In the case where $\delta_G = +\infty$, we say that the group is neither of convergence type nor of divergence type.

2.7. **Parabolic points.** Let (X, d, o, b) be as in §2.3.2, and let $G \leq \mathrm{Isom}(X)$.

Definition 2.41. Fix $\xi \in \partial X$, and let G_ξ be the stabilizer of ξ relative to G. We say that ξ is a *parabolic fixed point* of G if $G_\xi(o)$ is unbounded and if

(2.33) $$g'(\xi) \asymp_\times 1 \quad \forall g \in G_\xi.$$

Note that this definition together with Proposition 2.24 yields the following observation:

Observation 2.42. Let ξ be a parabolic fixed point of G. Then the action of G_ξ on $(\mathcal{E}_\xi, D_{\xi,o})$ is uniformly Lipschitz, i.e.

(2.34) $$D_{\xi,o}(g(y_1), g(y_2)) \asymp_\times D_{\xi,o}(y_1, y_2) \quad \forall y_1, y_2 \in \mathcal{E}_\xi \ \forall g \in G_\xi.$$

Notation 2.43. The implied constant of (2.34) will be denoted C_0.

Observation 2.44. Let ξ be a point satisfying (2.33). Then for all $g \in G_\xi$,

(2.35) $$D_{\xi,o}(o, g(o)) \asymp_\times b^{(1/2)d(o,g(o))}.$$

Proof. This is a direct consequence of Lemma 2.20, (g) of Proposition 2.3, and (2.33). $\square$

As a corollary we have the following:

Observation 2.45. Let ξ be a point satisfying (2.33). Then for any sequence $(g_n)_1^\infty$ in G_ξ,

(2.36) $$d(o, g_n(o)) \underset{n}{\to} +\infty \quad \Leftrightarrow \quad g_n(o) \underset{n}{\to} \xi.$$

In particular, if ξ is parabolic then $\xi \in \Lambda_G$.

Proof.

$$g_n(o) \underset{n}{\to} \xi \quad \Leftrightarrow \quad D_{\xi,o}(o, g_n(o)) \underset{n}{\to} +\infty \quad \Leftrightarrow \quad d(o, g_n(o)) \underset{n}{\to} +\infty.$$

If ξ is parabolic, then $G_\xi(o)$ is unbounded, so there exists a sequence $(g_n)_1^\infty$ in G_ξ satisfying the left hand side of (2.36). It also satisfies the right hand side of (2.36), which implies $\xi \in \Lambda_G$. $\square$

2.7.1. ξ-*bounded sets.* To define the notion of a *bounded* parabolic point, we first introduce the following definition:

Definition 2.46. A set $S \subseteq \overline{X} \setminus \{\xi\}$ is ξ-*bounded* if $\xi \notin \overline{S}$.

The motivation for this definition is that if $X = \mathbb{H}^{d+1}$ and $\xi = \infty$, then ξ-bounded sets are exactly those which are bounded in the Euclidean metric.

Observation 2.47. Fix $S \subseteq \overline{X} \setminus \{\xi\}$. The following are equivalent:

    (A) S is ξ-bounded.
    (B) $\langle x|\xi \rangle_o \asymp_+ 0$ for all $x \in X$.
    (C) $D_{\xi,o}(o,x) \lesssim_\times 1$ for all $x \in X$.
    (D) S has bounded diameter in the $D_{\xi,o}$ metametric.

Condition (D) motivates the terminology "ξ-bounded".

Proof of Observation 2.47. (A) $\Leftrightarrow$ (B) follows from the definition of the topology on $\overline{X}$, (B) $\Leftrightarrow$ (C) follows from Lemma 2.20, and (C) $\Leftrightarrow$ (D) is obvious. $\square$

2.7.2. *Bounded parabolic points.*

Definition 2.48. A parabolic point $\xi \in \Lambda_G$ is a *bounded* parabolic point if there exists a ξ-bounded set $S \subseteq \overline{X} \setminus \{\xi\}$ such that
$$G(o) \subseteq G_\xi(S).$$
We denote the set of bounded parabolic points by Λ_{bp}.

Proposition 2.49. *Let ξ be a parabolic point of a nonelementary group $G \leq \mathrm{Isom}(X)$. Then the following are equivalent:*

    (A) *ξ is a bounded parabolic point.*
    (B) (i) *There exists a ξ-bounded set S such that $\Lambda_G \setminus \{\xi\} \subseteq G_\xi(S)$, and*
        (ii) *There exists a horoball H centered at ξ which is disjoint from $G(o)$, i.e. ξ is not a horospherical limit point.*

Proof of (A) $\Rightarrow$ (B). Let S be as in Definition 2.48, and let $\widetilde{S} = S^{(1)\mathrm{e}}$. Then $\Lambda_G \setminus \{\xi\} \subseteq G_\xi(\widetilde{S})$, demonstrating (B)(i). By (g) of Proposition 2.3, for $x \in S$ we have
$$0 \asymp_+ \langle x|\xi \rangle_o \gtrsim_+ \frac{1}{2} \mathcal{B}_\xi(o,x),$$
and since ξ is parabolic, we have $\mathcal{B}_\xi(o, g(x)) \lesssim_+ 0$ for all $g \in G_\xi$. Thus $G_\xi(S)$ is disjoint from some horoball H centered at ξ, and so $G(o) \subseteq G_\xi(S)$ is also disjoint from H, demonstrating (B)(ii). $\square$

Proof of (B) $\Rightarrow$ (A). Fix $x \in G(o)$, and we will show that $D_\mathrm{e}(x, \Lambda_G \setminus \{\xi\}) \lesssim_\times 1$. Since G is nonelementary, we may fix $\eta_1, \eta_2 \in \Lambda_G$ distinct. Letting $x = g(o)$, we have
$$\langle g(\eta_1)|g(\eta_2) \rangle_x \asymp_+ 0,$$
and so by Gromov's inequality, there exists $i = 1, 2$ such that
$$\langle g(\eta_i)|\xi \rangle_x \asymp_+ 0;$$
we have
$$
\begin{aligned}
D_\mathrm{e}(x, g(\eta_i)) &\asymp_\times b^{\mathcal{B}_\xi(o,x)} D_{\xi,x}(x, g(\eta_i)) && \text{(by (2.21))} \\
&\lesssim_\times D_{\xi,x}(x, g(\eta_i)) && \text{(since } x \notin H\text{)} \\
&\asymp_\times b^{\langle g(\eta_i)|\xi \rangle_x} && \text{(by Lemma 2.20)} \\
&\asymp_\times 1.
\end{aligned}
$$

Letting r be the implied constant of this asymptotic yields $x \in (\Lambda_G \setminus \{\xi\})^{(r)\mathrm{e}}$. Thus if S is a ξ-bounded set such that $\Lambda_G \setminus \{\xi\} \subseteq G_\xi(S)$, then $G(o) \subseteq G_\xi(S^{(C_0 r)\mathrm{e}})$. $\square$

Remark 2.50. If $X = \mathbb{H}^{d+1}$, then (B)(i) of Proposition 2.49 implies (B)(ii) of Proposition 2.49. (However, it is not clear whether the implication holds in general.) Thus our definition of a bounded parabolic point agrees with the usual one in the literature, which is condition (B)(i).

Proof. Suppose that ξ satisfies (B)(i) of Proposition 2.49. By Lemma 4.1 of [10], for every $\varepsilon > 0$ there exists a horoball H_ε centered at ξ such that

$$H_\varepsilon \cap \mathcal{C}_G \subseteq T_\varepsilon(G) := \{x \in \mathbb{H}^{d+1} : \langle g \in G : d(x, g(x)) \leq \varepsilon \rangle \text{ is infinite}\}.$$

Here $\mathcal{C}_G$ denotes the convex hull of the limit set of G.

Without loss of generality suppose $o \in \mathcal{C}_G$. Choose $\varepsilon > 0$ small enough so that $o \notin T_\varepsilon(G)$; such a ε exists since G is discrete. Since $T_\varepsilon(G)$ is invariant under the action of G, we have $G(o) \cap T_\varepsilon(G) = \varnothing$. On the other hand, $G(o) \subseteq \mathcal{C}_G$, so $G(o) \cap H_\varepsilon = \varnothing$. $\qquad\square$

2.8. Quasiconformal measures.

Definition 2.51. Let (X, d, o, b, G) be as in §1.1.4. For each $s \geq 0$, a measure $\mu \in \mathcal{M}(\partial X)$ is called *s-quasiconformal* if

$$(2.37) \qquad\qquad \mu(g(A)) \asymp_\times \int_A [g'(\xi)]^s \mathrm{d}\mu(\xi)$$

for all $g \in G$ and for all $A \subseteq \partial X$. It is *ergodic* if for every G-invariant set $A \subseteq \partial X$, either $\mu(A) = 0$ or $\mu(\partial X \setminus A) = 0$.

In this subsection we discuss sufficient conditions for the existence and uniqueness of an ergodic δ-quasiconformal measure. The well-known construction of Patterson and Sullivan ([58] and [72]) can be generalized to show the following:

Theorem 2.52. *Let (X, d, o, b, G) be as in §1.1.4, and assume that X is proper and geodesic, and that $\delta = \delta_G < +\infty$. Then there exists an ergodic δ-quasiconformal measure supported on Λ.*

Proof. Everything except "ergodic" was proven in [16]; see also [21]. Let μ be a δ-quasiconformal measure. Let $\varrho : G \times \partial X \to \mathbb{R}$ satisfy (1.1)-(1.3) of [31]. Then by Theorem 1.4 of [31], μ can be written as a convex combination of ergodic measures which are "ϱ-admissible" (in the terminology of [31]). But by (1.1) of [31], we have $e^{\varrho(g,\xi)} \asymp_\times g'(\xi)^\delta$ for μ-almost every $\xi \in \partial X$, so ϱ-admissibility implies δ-quasiconformality for all measures supported on the complement of a certain μ-nullset. But this means that almost every measure in the convex combination is an ergodic δ-quasiconformal measure. $\qquad\square$

It is much more difficult to prove the existence of quasiconformal measures in the case where X is not proper. It turns out that the condition of divergence type, which in the case $X = \mathbb{H}^{d+1}$ is already known to imply uniqueness of the δ-conformal measure, is the right condition to guarantee existence in a non-proper setting:

Theorem 2.53 ([21])**.** *Let (X, d, o, b, G) be as in §1.1.4. Suppose that G is of divergence type. Then there exists a δ-quasiconformal measure μ for G, where δ is the Poincaré exponent of G. It is unique up to a multiplicative constant in the sense that if μ_1, μ_2 are two such measures then $\mu_1 \asymp_\times \mu_2$, meaning that μ_1 and μ_2 are in the same measure class and that the Radon-Nikodym derivative $\mathrm{d}\mu_1 / \mathrm{d}\mu_2$ is bounded from above and below. In addition, μ is ergodic and gives full measure to the radial limit set of G.*

3. Basic facts about Diophantine approximation

In this section we prove Theorem 3.1 and Proposition 3.2.

Theorem 3.1 (Generalization of Dirichlet's Theorem)**.** *Let (X, d, o, b, G) be as in §1.1.4, and let $\xi \in \partial X$ be a distinguished point. Then for every $\sigma > 0$, there exists a number $C = C_\sigma > 0$ such that for every $\eta \in \Lambda_{\mathrm{r},\sigma}(G)$,*

$$(3.1) \qquad\qquad D(g(\xi), \eta) \leq Cb^{-d(o, g(o))} \text{ for infinitely many } g \in G.$$

Here $\Lambda_{\mathrm{r},\sigma}(G)$ denotes the σ-radial limit set of G.

Furthermore, there exists a sequence of isometries $(g_n)_1^\infty \subseteq G$ such that

$$g_n(\xi) \underset{n}{\to} \eta \text{ and } g_n(o) \underset{n}{\to} \eta,$$

and (3.1) holds with $g = g_n$ for all $n \in \mathbb{N}$.

Proof. Since the group G is of general type, by Observation 1.8 there exists $h \in G$ such that $h(\xi) \neq \xi$. Let $\xi_1 = \xi$ and let $\xi_2 = h(\xi)$. Since η is a σ-radial limit point, there exists a sequence $(\widetilde{g}_n)_1^\infty$ in G such that $\widetilde{g}_n(o)$ converges to η σ-radially. We have

$$0 \asymp_{+,\xi} \langle \xi_1 | \xi_2 \rangle_o \gtrsim_+ \min\left(\langle \widetilde{g}_n^{-1}(o) | \xi_1 \rangle_o, \langle \widetilde{g}_n^{-1}(o) | \xi_2 \rangle_o\right).$$

Let i_n be 1 or 2, depending on which term in the minimum is smaller. We therefore have

$$\langle \widetilde{g}_n^{-1}(o) | \xi_{i_n} \rangle_o \asymp_{+,\xi} 0,$$

or equivalently

$$\langle \widetilde{g}_n(o) | \widetilde{g}_n(\xi_{i_n}) \rangle_o \asymp_{+,\xi} d(o, \widetilde{g}_n(o)).$$

On the other hand, since $\widetilde{g}_n(o)$ converges to η σ-radially we have

$$\langle \widetilde{g}_n(o) | \eta \rangle_o \asymp_{+,\sigma} d(o, \widetilde{g}_n(o)).$$

Applying Gromov's inequality yields

$$\langle \widetilde{g}_n(\xi_{i_n}) | \eta \rangle_o \gtrsim_{+,\xi,\sigma} d(o, \widetilde{g}_n(o))$$

and thus

$$D(\widetilde{g}_n(\xi_{i_n}), \eta) \lesssim_{\times,\xi,\sigma} b^{-d(o, \widetilde{g}_n(o))}.$$

If $i_n = 1$ for infinitely many n, then we are done by setting $g_n := \widetilde{g}_n$. Otherwise,

$$D(\widetilde{g}_n \circ h(\xi), \eta) \lesssim_{\times,\xi,\sigma} b^{-d(o, \widetilde{g}_n(o))} \asymp_{\times,\xi} b^{-d(o, \widetilde{g}_n \circ h(o))}$$

and we are done by setting $g_n := \widetilde{g}_n \circ h$. $\qquad\square$

If $\xi \in \partial X \setminus \Lambda$, then Theorem 3.1 together with the following proposition show that the theory of Diophantine approximation by the orbit of ξ is trivial.

Proposition 3.2. *Let (X, d, o, b, G) be as in §1.1.4. For every $\xi \in \partial X \setminus \Lambda$, we have $\mathrm{BA}_\xi = \Lambda_\mathrm{r}$.*

Proof. We have $D(\xi, \Lambda) > 0$ and so

$$\langle \xi | \eta \rangle_o \asymp_{+,\xi} 0$$

for all $\eta \in \Lambda$. Now fix $\eta \in \Lambda_\mathrm{r}(G)$; for all $g \in G$, we have

$$\langle g(\xi) | \eta \rangle_{g(o)} = \langle \xi | g^{-1}(\eta) \rangle_o \asymp_{+,\xi} 0.$$

Thus by (d) of Proposition 2.3 we have

$$\begin{aligned}\langle g(\xi) | \eta \rangle_o &\lesssim_+ \quad t \langle g(\xi) | \eta \rangle_{g(o)} + d(o, g(o)) \\ &\asymp_{+,\xi} d(o, g(o)).\end{aligned}$$

So, $D(g(\xi), \eta) \gtrsim_{\times,\xi} b^{-d(o,g(o))}$, i.e. η is badly approximable with respect to ξ. We are done. $\qquad\square$

4. Schmidt's game and McMullen's absolute game

Throughout this section (Z, D) denotes a metric space. In our applications, Z will not be a hyperbolic metric space; rather, it will be the boundary of a hyperbolic metric space (or some subset thereof), and D will be a visual metric as defined in Proposition 2.15.

We recall that for $0 < c < 1$, the metric space (Z, D) is said to be *c-uniformly perfect* if for every $r \in (0, 1)$ and for every $z \in Z$, $B(z, r) \setminus B(z, cr) \neq \varnothing$; it is *uniformly perfect* if it is c-uniformly perfect for some $0 < c < 1$. Thus every uniformly perfect space is perfect but not necessarily vice versa.

Definition 4.1. For $s > 0$, a measure $\mu \in \mathcal{M}(Z)$ is called *Ahlfors s-regular* if

$$\mu(B(z, r)) \asymp_\times r^s$$

for all $z \in \mathrm{Supp}(\mu)$[27] and for all $r \in (0, 1)$. The set Z itself is called Ahlfors s-regular if it is equal to the $\mathrm{Supp}(\mu)$ for some Ahlfors s-regular measure $\mu \in \mathcal{M}(Z)$.

We note that it is well-known that if Z is an Ahlfors s-regular metric space then its Hausdorff, packing, and box-counting dimensions are all equal to s (see e.g. [81]). Moreover, every Ahlfors regular space is uniformly perfect, since if Z is Ahlfors s-regular then

$$B(z, r) \setminus B(z, cr) = \varnothing \Rightarrow B(z, r) = B(z, cr) \Rightarrow r^s \asymp_\times (cr)^s \Rightarrow c \asymp_\times 1;$$

taking the contrapositive yields that for $0 < c < 1$ sufficiently small, Z is c-uniformly perfect.

We now define Schmidt's game (introduced by W. M. Schmidt in [65]) for the metric space (Z, D). When defining the game in this level of generality, we must specify that when choosing a ball in (Z, D), a player is really choosing a pair $(z, r) \in Z \times (0, +\infty)$ which determines to a ball $B(z, r) \subseteq Z$. Unlike in the case of $Z = \mathbb{R}^d$, it is possible that the same ball may be written in two different ways, i.e. $B(z, r) = B(\tilde{z}, \tilde{r})$ for some $(z, r) \neq (\tilde{z}, \tilde{r})$. In this case, it is the player's job to disambiguate.

Now fix $0 < \alpha, \beta < 1$, and suppose that two players, Bob and Alice, take turns choosing balls $B_k = B(z_k, r_k)$ and $A_k = B(\tilde{z}_k, \tilde{r}_k)$, satisfying

$$B_1 \supseteq A_1 \supseteq B_2 \supseteq \ldots$$

and

$$\tilde{r}_k = \alpha r_k, \quad r_{k+1} = \beta \tilde{r}_k.$$

A set $S \subseteq Z$ is said to be *(α, β)-winning* if Alice has a strategy guaranteeing that the unique point in the set $\bigcap_{k=1}^\infty B_k = \bigcap_{k=1}^\infty A_k$ belongs to S, regardless of the way Bob chooses to play. It is said to be *α-winning* if it is (α, β)-winning for all $\beta > 0$, and *winning* if it is α-winning for some α.

In [51], C. T. McMullen introduced the following modification:[28] Let Z be a c-uniformly perfect metric space. Fix $0 < \beta \leq c/5$,[29] and suppose that Bob and Alice take turns choosing balls $B_k = B(z_k, r_k)$ and $A_k = B(\tilde{z}_k, \tilde{r}_k)$ so that

(4.1) $$B_1 \supseteq B_1 \setminus A_1 \supseteq B_2 \supseteq B_2 \setminus A_2 \supseteq B_3 \supseteq \cdots$$

and

(4.2) $$\tilde{r}_k \leq \beta r_k, \quad r_{k+1} \geq \beta \tilde{r}_k.$$

A set $S \subseteq Z$ is said to be *β-absolute winning* if Alice has a strategy which leads to

$$\bigcap_{k=1}^\infty B_k \cap S \neq \varnothing$$

regardless of how Bob chooses to play; S is said to be *absolute winning* if S is β-absolute winning for all $0 < \beta \leq c/5$. Note a significant difference between Schmidt's original game and the modified game: now Alice has rather limited control over the situation, since she can block fewer of Bob's possible moves at the

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[27]Here and from now on $\mathrm{Supp}(\mu)$ denotes the topological support of a measure μ.

[28]Strictly speaking, McMullen only defined his game for $Z = \mathbb{R}^d$.

[29]The constant $c/5$ is chosen to ensure that Bob will always have a legal move; see Lemma 4.2 below.

next step. Also, in the new version the r_ks do not have to tend to zero; therefore $\bigcap_1^\infty B_k$ does not have to be a single point (however the outcome with $r_k \not\to_k 0$ is clearly winning for Alice as long as S is dense).

The following lemma guarantees that Bob always has a legal move in the β-absolute winning game:

Lemma 4.2. *Let Z be a c-uniformly perfect metric space and fix $0 < \beta \le c/5$. Then for any two sets $B_1 = B(z_1, r_1)$ and $A_1 = B(\widetilde{z}_1, \widetilde{r}_1) \subseteq B_1$ with $\widetilde{r}_1 \le \beta r_1$, there exists a set $B_2 = B(z_2, r_2) \subseteq B_1$ with $r_2 = \beta r$ which is disjoint form A_1.*

Proof. Let $r = r_1$ and $w_1 = z_1$. Since Z is c-uniformly perfect, there exists a point $w_2 \in B(z_1, (1-\beta)r) \setminus B(z_1, c(1-\beta)r)$. Then the balls $U_i = B(w_i, \beta r)$ are contained in B_1; moreover,

$$D(U_1, U_2) \ge D(w_1, w_2) - 2\beta r \ge c(1-\beta)r - 2\beta r > (c - 3\beta)r.$$

On the other hand,

$$\text{Diam}(A_1) \le 2\widetilde{r}_1 \le 2\beta r \le (c - 3\beta)r,$$

which shows that A_1 can intersect at most one of the sets U_1 and U_2. Letting B_2 be one that it does not intersect finishes the proof. $\square$

The following proposition summarizes some important properties of winning and absolute winning subsets of a c-uniformly perfect Ahlfors s-regular metric space (Z, D):

Proposition 4.3.
   (i) *Winning sets are dense and have Hausdorff dimension s.*
   (ii) *Absolute winning implies α-winning for any $0 < \alpha \le c/5$.*
   (iii) *The countable intersection of α-winning (resp., absolute winning) sets is again α-winning (resp., absolute winning).*
   (iv) *The image of an α-winning set under a bi-Lipschitz map f is $K^{-2}\alpha$-winning, where K is the bi-Lipschitz constant of f.*
   (v) *The image of an absolute winning set under a quasisymmetric map (and in particular any bi-Lipschitz map) is absolute winning.*

Proofs.
   (i) Proposition 5.1 of [44]
   (ii) Fix $\beta > 0$, and let $\gamma = \min(\alpha, \beta)$. We can convert a winning strategy for Alice in the γ-absolute winning game into a winning strategy in the (α, β)-game by using Lemma 4.2 with β replaced by α; the idea is that if Alice would have made the move A_1 in the γ-absolute winning game, then she should make the move B_2 in the (α, β)-game. Details are left to the reader.
   (iii) Theorem 2 of [65] (the argument can be easily modified to account for absolute winning)
   (iv) Proposition 5.3 of [20]
   (v) Theorem 2.2 of [51][30]

$\square$

Note that winning sets arise naturally in many settings in dynamics and Diophantine approximation [18, 19, 20, 24, 27, 28, 42, 44, 53, 65, 66, 75, 76]. Several examples of absolute winning sets were exhibited by McMullen in [51], most notably the set of badly approximable numbers in $\mathbb{R}$. However absolute winning does not occur as frequently as winning. In particular, the set of badly approximable vectors in $\mathbb{R}^d$ for $d > 1$ is winning but not absolute winning (see [12]).

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[30]Full disclosure: McMullen's proof appears to the casual observer to be using crucially the fact that $Z = \mathbb{R}^d$. In fact, his proof has the following form:

(4.3)   Quasisymmetric $\Rightarrow$ Conditions 1-4 on p.5-6 $\Rightarrow$ Preserves absolute winning sets.

The first arrow depends on the assumption $Z = \mathbb{R}^d$; the second arrow does not. Moreover, maps satisfying McMullen's conditions 1-4 are what are usually referred to as quasisymmetric maps by those studying spaces beyond $\mathbb{R}^d$ (and by us below in Lemma 5.11). (Maps satisfying McMullen's definition of quasisymmetric are termed "weakly quasisymmetric".) Thus the second arrow of (4.3), which holds for any Z, is accurately represented by our statement (v) above.

For the purposes of this paper, we will introduce a slight modification of McMullen's absolute winning game. The only difference will be that in (4.2) we will require equality, i.e.

$$(4.4) \qquad \widetilde{r}_k = \beta r_k, \quad r_{k+1} = \beta \widetilde{r}_k.$$

For each $0 < \beta \leq c/5$, a set for which Alice can win this modified game will be called *β-modified absolute winning*.

Proposition 4.4. *Fix $0 < \beta \leq c/5$. Then every β-absolute winning set is β-modified absolute winning, and every $\beta^2/4$-modified absolute winning set is β-absolute winning.*

Proof. First of all, let us notice that it makes no difference whether or not Alice is required to have equality in her choice of radii; choosing the largest possible radius always puts the greatest restriction on Bob's moves, and is therefore most advantageous for Alice. On the other hand, requiring Bob to choose the minimal possible radius is an advantage for Alice, since it restricts Bob's choice of moves. This demonstrates the first statement, that every β-absolute winning set is β-modified absolute winning.

Let $S \subseteq Z$ be a $\beta^2/4$-modified absolute winning set. By Theorem 7 of [65], there exists a *positional* strategy for Alice to win the $\beta^2/4$-modified absolute winning game with target set S, i.e. a map $\mathcal{F}$ which inputs Bob's ball and outputs the next move Alice should make, without any information about previous moves. Based on this map $\mathcal{F}$, we will give a positional strategy $\mathcal{G}$ for Alice to win the β-absolute winning game. Given any ball $B(z,r)$, let $n = n_r$ be the unique integer so that $(\beta/2)^{2n+1} \leq r < (\beta/2)^{2n-1}$. Write $\mathcal{F}(B(z,(\beta/2)^{2n})) = B(w,(\beta/2)^{2n+2})$ (the radius $(\beta/2)^{2n+2}$ is forced by the definition of the modified absolute winning game). Let $\mathcal{G}(B(z,r)) = B(w, 2(\beta/2)^{2n+2})$; this move is legal in the β-absolute winning game since $2(\beta/2)^{2n+2} \leq \beta r$.

We now claim that with the positional strategy $\mathcal{G}$, Alice is assured to win the β-absolute winning game. Let $\left(B_k = B(z_k, r_k)\right)_1^\infty$ and $\left(A_k = B(\widetilde{z}_k, \widetilde{r}_k)\right)_1^\infty$ be a sequence of moves for Bob and Alice which are legal in the β-absolute winning game, and such that $A_k = \mathcal{G}(B_k)$ for all $k \in \mathbb{N}$. Let us assume that $r_k \underset{k}{\to} 0$, since otherwise Alice wins automatically since S is dense. Let $n_1 \in \mathbb{N}$ be such that $(\beta/2)^{2n_1} < r_1$. For each $n \geq n_1$, let k_n be such that $(\beta/2)^{2n+1} \leq r_{k_n} < \frac{1}{2}(\beta/2)^{2n}$; such a k_n exists by (4.2), although it may not be unique. Note that $n_{r_{k_n}} = n$. Let $\widetilde{B}_n = B(z_{k_n}, (\beta/2)^{2n})$, and let $\widetilde{A}_n = \mathcal{F}(\widetilde{B}_n) = B(w_n, (\beta/2)^{2n+2})$.

Claim 4.5. *The sequences $(\widetilde{B}_n)_{n_1}^\infty$ and $(\widetilde{A}_n)_{n_1}^\infty$ form a sequence of legal moves in the β-modified absolute winning game.*

Proof. Alice's moves are legal because she chose them according to her positional strategy. Bob's first move $\widetilde{B}_{n_1}$ is clearly legal, so fix $n \geq n_1$ and let us consider the legality of the move $\widetilde{B}_{n+1}$. It suffices to show

$$(4.5) \qquad \widetilde{B}_{n+1} \subseteq \widetilde{B}_n \setminus \widetilde{A}_n.$$

Let $k = k_n$; then since the sequences $(B_k)_1^\infty$ and $(A_k)_1^\infty$ are legal for the β-absolute winning game, we have (4.1) and (4.2); in particular,

$$z_{k_{n+1}} \in B_{k_{n+1}} \subseteq B_{k_n+1} \subseteq B_k \setminus A_k,$$

i.e.

$$D(z_{k_{n+1}}, z_k) \leq r_k \text{ and } D(z_{k_{n+1}}, \widetilde{z}_k) > \widetilde{r}_k.$$

Now by the definition of A_k,

$$\widetilde{z}_k = w_n \text{ and } \widetilde{r}_k = 2(\beta/2)^{2n+2},$$

and by the definition of $k = k_n$,

$$r_k \leq \frac{1}{2}(\beta/2)^{2n}.$$

Thus

$$D(z_{k_{n+1}}, z_k) \leq \frac{1}{2}(\beta/2)^{2n} \text{ and } D(z_{k_{n+1}}, w_n) > 2(\beta/2)^{2n+2},$$

from which it follows that

$$B(z_{k_{n+1}}, (\beta/2)^{2n+2}) \subseteq B(z_k, (\beta/2)^{2n}) \setminus B(w_n, (\beta/2)^{2n+2}),$$

which is just (4.5) expanded.   ◁

Now clearly, the intersection of the sequence $(B_k)_1^\infty$ is the same as the intersection of the sequence $(\widetilde{B}_n)_{n_1}^\infty$, which is in S by Claim 4.5. Thus Alice wins the β-absolute winning game with her positional strategy $\mathcal{G}$. $\square$

Now since β-absolute winning implies $\widetilde{\beta}$-absolute winning for every $\beta \leq \widetilde{\beta} \leq c/5$, we have the following:

Corollary 4.6. *If $S \subseteq Z$ is β_m-modified absolute winning for a sequence $\beta_m \xrightarrow[m]{} 0$, then S is absolute winning.*

5. Partition structures

In this section we introduce the notion of a partition structure, an important technical tool for proving Theorems 5.9, 5.10, and 9.3. Moreover, we introduce stronger formulations of the former two theorems, from which they follow as corollaries. Subsection 5.1 can also be found with some modifications in [21].

Throughout this section, (Z, D) denotes a metric space.

5.1. **Partition structures.**

Notation 5.1. Let

$$\mathbb{N}^* = \bigcup_{n=0}^{\infty} \mathbb{N}^n.$$

If $\omega \in \mathbb{N}^* \cup \mathbb{N}^{\mathbb{N}}$, then we denote by $|\omega|$ the unique element of $\mathbb{N} \cup \infty$ such that $\omega \in \mathbb{N}^{|\omega|}$ and call $|\omega|$ the *length* of ω. For each $N \in \mathbb{N}$, we denote the initial segment of ω of length N by

$$\omega_1^N := (\omega_n)_1^N \in \mathbb{N}^N.$$

For two words $\omega, \tau \in \mathbb{N}^{\mathbb{N}}$, let $\omega \wedge \tau$ denote their longest common initial segment, and let

$$\rho_2(\omega, \tau) = 2^{-|\omega \wedge \tau|}.$$

Then $(\mathbb{N}^{\mathbb{N}}, \rho_2)$ is a metric space.

Definition 5.2. A *tree* on $\mathbb{N}$ is a set $T \subseteq \mathbb{N}^*$ which is closed under initial segments.

Notation 5.3. If T is a tree on $\mathbb{N}$, then we denote its set of infinite branches by

$$T(\infty) := \{\omega \in \mathbb{N}^{\mathbb{N}} : \omega_1^n \in T \ \forall n \in \mathbb{N}\}.$$

On the other hand, for $n \in \mathbb{N}$ we let

$$T(n) := T \cap \mathbb{N}^n.$$

For each $\omega \in T$, we denote the set of its children by

$$T(\omega) := \{a \in \mathbb{N} : \omega a \in T\}.$$

The set of infinite branches in $T(\infty)$ of which ω is an initial segment will be denoted $[\omega]_T$ or simply $[\omega]$.

Definition 5.4. A *partition structure* on Z consists of a tree $T \subseteq \mathbb{N}^*$ together with a collection of closed subsets $(\mathcal{P}_\omega)_{\omega \in T}$ of Z, each having positive diameter and enjoying the following properties:

(I) If $\omega \in T$ is an initial segment of $\tau \in T$ then $\mathcal{P}_\tau \subseteq \mathcal{P}_\omega$. If neither ω nor τ is an initial segment of the other then $\mathcal{P}_\omega \cap \mathcal{P}_\tau = \varnothing$.

(II) For each $\omega \in T$ let

$$D_\omega = \text{Diam}(\mathcal{P}_\omega).$$

There exist $\kappa > 0$ and $0 < \lambda < 1$ such that for all $\omega \in T$ and for all $a \in T(\omega)$, we have

(5.1) $$D(\mathcal{P}_{\omega a}, Z \setminus \mathcal{P}_\omega) \geq \kappa D_\omega$$

and

(5.2) $$\kappa D_\omega \leq D_{\omega a} \leq \lambda D_\omega.$$

Fix $s > 0$. The partition structure $(\mathcal{P}_\omega)_{\omega \in T}$ is called s-*thick* if for all $\omega \in T$,

$$(5.3) \qquad \sum_{a \in T(\omega)} D_{\omega a}^s \geq D_\omega^s.$$

Definition 5.5. If $(\mathcal{P}_\omega)_{\omega \in T}$ is a partition structure, a *substructure* of $(\mathcal{P}_\omega)_{\omega \in T}$ is a partition structure of the form $(\mathcal{P}_\omega)_{\omega \in \widetilde{T}}$, where $\widetilde{T} \subseteq T$ is a subtree.

Observation 5.6. Let $(\mathcal{P}_\omega)_{\omega \in T}$ be a partition structure on a complete metric space (Z, D). For each $\omega \in T(\infty)$, the set

$$\bigcap_{n=1}^{\infty} \mathcal{P}_{\omega_1^n}$$

is a singleton. If we define $\pi(\omega)$ to be the unique member of this set, then the map $\pi : T(\infty) \to Z$ is continuous (in fact quasisymmetric; see Lemma 5.11 below).

Definition 5.7. The set $\pi(T(\infty))$ is called the *limit set* of the partition structure.

We remark that a large class of examples of partition structures comes from the theory of conformal iterated function systems [48] (or in fact even graph directed Markov systems [49]) satisfying the strong separation condition (also known as the disconnected open set condition [62]; see also [26], where the limit sets of iterated function systems satisfying the strong separation condition are called dust-like). Indeed, the notion of a partition structure was intended primarily to generalize these examples. The difference is that in a partition structure, the sets $(\mathcal{P}_\omega)_\omega$ do not necessarily have to be defined by dynamical means. We also note that if $Z = \mathbb{R}^d$ for some $d \in \mathbb{N}$, and if $(\mathcal{P}_\omega)_{\omega \in T}$ is a partition structure on Z, then the tree T has bounded degree, meaning that there exists $N < +\infty$ such that $\#(T(\omega)) \leq N$ for every $\omega \in T$.

Theorem 5.12 (Proven below)**.** *Fix $s > 0$. Then any s-thick partition structure $(\mathcal{P}_\omega)_{\omega \in T}$ on a complete metric space (Z, D) has a substructure $(\mathcal{P}_\omega)_{\omega \in \widetilde{T}}$ whose limit set is Ahlfors s-regular. Furthermore the tree $\widetilde{T}$ can be chosen so that for each $\omega \in \widetilde{T}$, we have that $\widetilde{T}(\omega)$ is an initial segment of $T(\omega)$, i.e. $\widetilde{T}(\omega) = T(\omega) \cap \{1, \ldots, N_\omega\}$ for some $N_\omega \in \mathbb{N}$.*

Lemma 5.13 (Proven below)**.** *Let (X, d, o, b, G) be as in §1.1.4. Then for every $0 < s < \delta_G$ and for all $\sigma > 0$ sufficiently large, there exist a tree T on $\mathbb{N}$ and an embedding $T \ni \omega \mapsto x_\omega \in G(o)$ such that if*

$$\mathcal{P}_\omega := \mathrm{Shad}(x_\omega, \sigma),$$

then $(\mathcal{P}_\omega)_{\omega \in T}$ is an s-thick partition structure on $(\partial X, D)$, whose limit set is a subset of Λ_{ur}.

Combining these results in the obvious way yields the following corollary:

Corollary 5.8. *Let (X, d, o, b, G) be as in §1.1.4. Then for every $0 < s < \delta_G$ and for all $\sigma > 0$ sufficiently large, there exist a tree T on $\mathbb{N}$ and an embedding $T \ni \omega \mapsto x_\omega \in G(o)$ such that if*

$$\mathcal{P}_\omega := \mathrm{Shad}(x_\omega, \sigma),$$

then $(\mathcal{P}_\omega)_{\omega \in T}$ is a partition structure on $(\partial X, D)$, whose limit set $\mathcal{J}_s \subseteq \Lambda_{\mathrm{ur}}(G)$ is Ahlfors s-regular.

Using Corollary 5.8, we prove the following generalization of the Bishop-Jones theorem:

Theorem 5.9. *Let (X, d, o, b, G) be as in §1.1.4. Then*

$$(1.2) \qquad \mathrm{HD}(\Lambda_{\mathrm{r}}) = \mathrm{HD}(\Lambda_{\mathrm{ur}}) = \delta;$$

moreover, for every $0 < s < \delta$ there exists an Ahlfors s-regular set $\mathcal{J}_s \subseteq \Lambda_{\mathrm{ur}}$.

Proof. The "moreover" clause follows directly from Corollary 5.8; applying the mass distribution principle yields $\mathrm{HD}(\Lambda_{\mathrm{ur}}) \geq \delta$. The proof of the remaining inequality $\mathrm{HD}(\Lambda_{\mathrm{r}}) \leq \delta$ is a straightforward adaptation of the argument in the standard case (see [70]). $\square$

5.2. **Diophantine approximation and partition structures.** Recall that for $\xi \in \partial X$, BA_ξ denotes the set of points in Λ_r which are badly approximable with respect to ξ (Definition 1.18). We have the following:

Theorem 6.1. *Let (X, d, o, b, G) be as in §1.1.4. Fix $0 < s < \delta_G$ and let $\mathcal{J}_s$ be defined as in Corollary 5.8. Then for each $\xi \in \partial X$, the set $\mathrm{BA}_\xi \cap \mathcal{J}_s$ is absolute winning on $\mathcal{J}_s$.*

From Theorem 6.1 we deduce the following corollary, which was stated in the introduction:

Theorem 5.10 (Generalization of the Full Dimension Theorem). *Let (X, d, o, b, G) be as in §1.1.4, and let $(\xi_k)_1^\ell$ be a countable (finite or infinite) sequence in ∂X. Then*

$$\mathrm{HD}\left(\bigcap_{k=1}^{\ell} \mathrm{BA}_{\xi_k} \cap \Lambda_\mathrm{ur} \right) = \delta = \mathrm{HD}(\Lambda_\mathrm{r}).$$

In particular $\mathrm{BA}_\xi \neq \varnothing$ for every $\xi \in \partial X$, so Theorem 3.1 cannot be improved by more than a multiplicative constant.

Proof of Theorem 5.10 assuming Theorem 6.1. For each $0 < s < \delta_G$, applying Theorem 6.1 (and thus implicitly Corollary 5.8) and parts (i), (ii), and (iii) of Proposition 4.3 gives that the set $\bigcap_1^\ell \mathrm{BA}_{\xi_k} \cap \mathcal{J}_s$ has full Hausdorff dimension in $\mathcal{J}_s$, i.e. its dimension is s. Letting s tend to δ_G yields

$$\mathrm{HD}\left(\bigcap_{k=1}^{\ell} \mathrm{BA}_{\xi_k} \cap \Lambda_\mathrm{ur} \right) \geq \delta.$$

On the other hand, we clearly have $\mathrm{HD}\left(\bigcap_1^\ell \mathrm{BA}_{\xi_k} \cap \Lambda_\mathrm{ur} \right) \leq \mathrm{HD}(\Lambda_\mathrm{r})$, and the equation $\delta = \mathrm{HD}(\Lambda_\mathrm{r})$ is given by Theorem 5.9. This completes the proof. $\qquad \square$

5.3. **Basic facts about partition structures.** In this subsection we prove Theorem 5.12, and the following lemma which will be used in the proof of Theorem 6.1:

Lemma 5.11. *Let $(\mathcal{P}_\omega)_{\omega \in T}$ be a partition structure on a complete metric space (Z, D). Then the map $\pi : T(\infty) \to \pi(T(\infty))$ is quasisymmetric.*

Proof. Recall that a surjective map $\Phi : (Z_1, D_1) \to (Z_2, D_2)$ is said to be *quasisymmetric* if there exists an increasing homeomorphism $\eta : (0, +\infty) \to (0, +\infty)$ such that for every $z_1, z_2, z_3 \in Z_1$, we have

$$(5.4) \qquad \frac{D_2(\Phi(z_1), \Phi(z_2))}{D_2(\Phi(z_1), \Phi(z_3))} \leq \eta\left(\frac{D_1(z_1, z_2)}{D_1(z_1, z_3)} \right).$$

In our case we have $(Z_1, D_1) = (T(\infty), \rho_2)$, $(Z_2, D_2) = (\pi(T(\infty)), D)$, and $\Phi = \pi$.

Fix $\omega, \tau^{(1)}, \tau^{(2)} \in T(\infty)$, and let $m_i = |\omega \wedge \tau^{(i)}|$, $i = 1, 2$. Suppose first that $m_1 \leq m_2$. By (5.1) and the first inequality of (5.2), we have

$$D(\pi(\omega), (\tau^{(2)})) \geq \kappa D_{\omega_1^{m_2}} \geq \kappa^{m_2 - m_1 + 1} D_{\omega_1^{m_1}}$$
$$\geq \kappa^{m_2 - m_1 + 1} D(\pi(\omega), \pi(\tau^{(1)}))$$
$$\frac{D(\pi(\omega), \pi(\tau^{(1)}))}{D(\pi(\omega), \pi(\tau^{(2)}))} \leq \kappa^{-1} \left(\frac{\rho_2(\omega, \tau^{(1)})}{\rho_2(\omega, \tau^{(2)})} \right)^{-\log_2(\kappa)}.$$

On the other hand, suppose that $m_1 \geq m_2$. By (5.1) and the second inequality of (5.2), we have

$$D(\pi(\omega), (\tau^{(2)})) \geq \kappa D_{\omega_1^{m_2}} \geq \kappa \lambda^{m_2 - m_1} D_{\omega_1^{m_1}}$$
$$\geq \kappa \lambda^{m_2 - m_1} D(\pi(\omega), \pi(\tau^{(1)}))$$
$$\frac{D(\pi(\omega), \pi(\tau^{(1)}))}{D(\pi(\omega), \pi(\tau^{(2)}))} \leq \kappa^{-1} \left(\frac{\rho_2(\omega, \tau^{(1)})}{\rho_2(\omega, \tau^{(2)})} \right)^{-\log_2(\lambda)}.$$

Thus by letting

$$\eta(t) = \kappa^{-1} \max \left(t^{-\log_2(\kappa)}, t^{-\log_2(\lambda)} \right)$$

we have proven (5.4).   □

Theorem 5.12. *Fix $s > 0$. Then any s-thick partition structure $(\mathcal{P}_\omega)_{\omega \in T}$ on a complete metric space (Z, D) has a substructure $(\mathcal{P}_\omega)_{\omega \in \widetilde{T}}$ whose limit set is Ahlfors s-regular. Furthermore the tree $\widetilde{T}$ can be chosen so that for each $\omega \in \widetilde{T}$, we have that $\widetilde{T}(\omega)$ is an initial segment of $T(\omega)$, i.e. $\widetilde{T}(\omega) = T(\omega) \cap \{1, \ldots, N_\omega\}$ for some $N_\omega \in \mathbb{N}$.*

Proof. This proof will also appear in [21].

We will recursively define a sequence of maps

$$\mu_n : T(n) \to [0, 1]$$

with the following consistency property:

$$(5.5) \qquad \mu_n(\omega) = \sum_{a \in T(\omega)} \mu_{n+1}(\omega a).$$

The Kolmogorov consistency theorem will then guarantee the existence of a measure $\widetilde{\mu} \in \mathcal{M}(T(\infty))$ satisfying

$$(5.6) \qquad \widetilde{\mu}([\omega]) = \mu_n(\omega)$$

for each $\omega \in T(n)$.

Let $c = 1 - \lambda^s > 0$, where λ is as in (5.2). For each $n \in \mathbb{N}$, we will demand of our function μ_n the following property: for all $\omega \in T(n)$, if $\mu_n(\omega) > 0$, then

$$(5.7) \qquad cD_\omega^s \leq \mu_n(\omega) < D_\omega^s.$$

We now begin our recursion. For the case $n = 0$, let $\mu_0(\varnothing) := cD_\varnothing^s$; (5.7) is clearly satisfied.

For the inductive step, fix $n \in \mathbb{N}$ and suppose that μ_n has been constructed satisfying (5.7). Fix $\omega \in T(n)$, and suppose that $\mu_n(\omega) > 0$. Formulas (5.3) and (5.7) imply that

$$\sum_{a \in T(\omega)} D_{\omega a}^s > \mu_n(\omega).$$

Let $N_\omega \in T(\omega)$ be the smallest integer such that

$$(5.8) \qquad \sum_{a \leq N_\omega} D_{\omega a}^s > \mu_n(\omega).^{31}$$

Then the minimality of N_ω says precisely that

$$\sum_{a \leq N_\omega - 1} D_{\omega a}^s \leq \mu_n(\omega).$$

Using the above, (5.8), and (5.2), we have

$$(5.9) \qquad \mu_n(\omega) < \sum_{a \leq N_\omega} D_{\omega a}^s \leq \mu_n(\omega) + D_{\omega N_\omega}^s \leq \mu_n(\omega) + \lambda^s D_\omega^s.$$

For each $a \in T(\omega)$ with $a > N_\omega$, let $\mu_{n+1}(\omega a) = 0$, and for each $a \leq N_\omega$, let

$$\mu_{n+1}(\omega a) = \frac{D_{\omega a}^s \mu_n(\omega)}{\sum_{b \leq N_\omega} D_{\omega b}^s}.$$

---

[31]Obviously, this and similar sums are restricted to $T(\omega)$.

Obviously, μ_{n+1} defined in this way satisfies (5.5). Let us prove that (5.7) holds (of course, with $n = n+1$). The second inequality follows directly from the definition of μ_{n+1} and from (5.8). Using (5.9), (5.7) (with $n = n$), and the equation $c = 1 - \lambda^s$, we deduce the first inequality as follows:

$$\mu_{n+1}(\omega a) \geq \frac{D^s_{\omega a}\mu_n(\omega)}{\mu_n(\omega) + \lambda^s D^s_\omega} = D^s_{\omega a}\left[1 - \frac{\lambda^s D^s_\omega}{\mu_n(\omega) + \lambda^s D^s_\omega}\right]$$
$$\geq D^s_{\omega a}\left[1 - \frac{\lambda^s}{c + \lambda^s}\right]$$
$$= cD^s_{\omega a}.$$

The proof of (5.7) (with $n = n + 1$) is complete. This completes the recursive step.

Let
$$\widetilde{T} = \bigcup_{n=1}^\infty \{\omega \in T(n) : \mu_n(\omega) > 0\}.$$

Clearly, the limit set of the partition structure $(\mathcal{P}_\omega)_{\omega \in \widetilde{T}}$ is exactly the topological support of $\mu := \pi[\widetilde{\mu}]$, where $\widetilde{\mu}$ is defined by (5.6). Furthermore, for each $\omega \in \widetilde{T}$, we have $\widetilde{T}(\omega) = T(\omega) \cap \{1, \ldots, N_\omega\}$. Thus, to complete the proof of Theorem 5.12 it suffices to show that the measure μ is Ahlfors s-regular.

To this end, fix $z = \pi(\omega) \in \text{Supp}(\mu)$ and $0 < r \leq \kappa D_\varnothing$, where κ is as in (5.1) and (5.2). For convenience of notation let
$$\mathcal{P}_n := \mathcal{P}_{\omega_1^n}, \quad D_n := \text{Diam}(\mathcal{P}_n),$$

and let $n \in \mathbb{N}$ be the largest integer such that $r < \kappa D_n$. We have

(5.10) $$\kappa^2 D_n \leq \kappa D_{n+1} \leq r < \kappa D_n.$$

(The first inequality comes from (5.2), whereas the latter two come from the definition of r.)

We now claim that
$$B(z, r) \subseteq \mathcal{P}_n.$$

Indeed, by contradiction suppose that $w \in B(z, r) \setminus \mathcal{P}_n$. By (5.1) we have
$$D(z, w) \geq D(z, Z \setminus \mathcal{P}_n) \geq \kappa D_n > r$$

which contradicts the fact that $w \in B(z, r)$.

Let $k \in \mathbb{N}$ be large enough so that $\lambda^k \leq \kappa^2$. It follows from (5.10) and repeated applications of the second inequality of (5.2) that
$$D_{n+k} \leq \lambda^k D_n \leq \kappa^2 D_n \leq r,$$

and thus
$$\mathcal{P}_{n+k} \subseteq B(z, r) \subseteq \mathcal{P}_n.$$

Thus, invoking (5.7), we get

(5.11) $$(1 - \lambda^s)D^s_{n+k} \leq \mu(\mathcal{P}_{n+k}) \leq \mu(B(z, r)) \leq \mu(\mathcal{P}_n) \leq D^s_n.$$

On the other hand, it follows from (5.10) and repeated applications of the first inequality of (5.2) that

(5.12) $$D_{n+k} \geq \kappa^k D_n \geq \kappa^{k-1} r.$$

Combining (5.10), (5.11), and (5.12) yields
$$(1 - \lambda^s)\kappa^{s(k-1)} r^s \leq \mu(B(z, r)) \leq \kappa^{-2s} r^s,$$

i.e. μ is Ahlfors s-regular. This completes the proof of Theorem 5.12. $\square$

5.4. **Proof of Lemma 5.13 (A partition structure on ∂X).** In this subsection we prove Lemma 5.13. The entire subsection is repeated with some modification from [21].

Lemma 5.13. *Let (X, d, o, b, G) be as in §1.1.4. Then for every $0 < s < \delta_G$ and for all $\sigma > 0$ sufficiently large, there exist a tree T on $\mathbb{N}$ and an embedding $T \ni \omega \mapsto x_\omega \in G(o)$ such that if*

$$\mathcal{P}_\omega := \mathrm{Shad}(x_\omega, \sigma),$$

then $(\mathcal{P}_\omega)_{\omega \in T}$ is an s-thick partition structure on $(\partial X, D)$, whose limit set is a subset of Λ_{ur}.

We begin by stating our key lemma. This lemma will also be used in the proof of Theorem 7.1.

Lemma 5.14 (Construction of children)**.** *Let (X, d, o, b, G) be as in §1.1.4. Then for every $0 < s < \delta_G$, for every $0 < \lambda < 1$, for all $\sigma > 0$ sufficiently large, and for every $w \in G(o)$, there exists a finite subset $T(w) \subseteq G(o)$ (the children of w) such that if we let*

$$\mathcal{P}_x := \mathrm{Shad}(x, \sigma)$$
$$D_x := \mathrm{Diam}(\mathcal{P}_x)$$

then the following hold:

(i) *The family $(\mathcal{P}_x)_{x \in T(w)}$ consists of pairwise disjoint shadows contained in $\mathcal{P}_w$.*
(ii) *There exists $\kappa > 0$ independent of w such that for all $x \in T(w)$,*

$$D(\mathcal{P}_x, \partial X \setminus \mathcal{P}_w) \geq \kappa D_w$$
$$\kappa D_w \leq D_x \leq \lambda D_w.$$

(iii)

$$\sum_{x \in T(w)} D_x^s \geq D_w^s.$$

It is not too hard to deduce Lemma 5.13 from Lemma 5.14. We do it now:

Proof of Lemma 5.13 assuming Lemma 5.14. Let $\lambda = 1/2$, and let $\sigma > 0$ be large enough so that Lemma 5.14 holds. Let $(x_n)_1^\infty$ be an enumeration of $G(o)$. Let

$$T = \bigcup_{n=1}^\infty \{\omega \in \mathbb{N}^n : x_{\omega_{j+1}} \in T(x_{\omega_j}) \ \forall j = 1, \ldots, n-1\},$$

and for each $\omega \in T$ let

$$x_\omega = x_{\omega_{|\omega|}}$$
$$\mathcal{P}_\omega = \mathcal{P}_{x_\omega}.$$

Then the conclusion of Lemma 5.14 precisely implies that $(\mathcal{P}_\omega)_{\omega \in T}$ is an s-thick partition structure on $(\partial X, D)$.

To complete the proof, we must show that the limit set of the partition structure $(\mathcal{P}_\omega)_{\omega \in T}$ is contained in $\Lambda_{\mathrm{ur}}(G)$. Indeed, fix $\omega \in T(\infty)$. Then for each $n \in \mathbb{N}$, $\pi(\omega) \in \mathcal{P}_{\omega_1^n} = \mathrm{Shad}(x_{\omega_1^n}, \sigma)$ and $d(o, x_{\omega_1^n}) \to \infty$. So, the sequence $(x_{\omega_1^n})_1^\infty$ converges radially to $\pi(\omega)$. On the other hand,

$$d(x_{\omega_1^n}, x_{\omega_1^{n+1}}) \asymp_{+,\sigma} \mathcal{B}_o(x_{\omega_1^{n+1}}, x_{\omega_1^n}) \qquad\qquad \text{(by (2.25))}$$

$$\asymp_{+,\sigma} -\log_b \left(\frac{D_{\omega_1^{n+1}}}{D_{\omega_1^n}} \right) \qquad \text{(by the Diameter of Shadows Lemma)}$$

$$\leq \ -\log_b(\kappa) \asymp_{+,\kappa} 0. \qquad\qquad \text{(by (5.2))}$$

Thus the sequence $(x_{\omega_1^n})_1^\infty$ converges to $\pi(\omega)$ uniformly radially. $\qquad\square$

Proof of Lemma 5.14. Choose an arbitrary $t \in (s, \delta_G)$. A point $\eta \in \Lambda$ will be called t-*divergent* if for every neighborhood B of η the restricted Poincaré series

$$\Sigma_t(G \restriction B) := \sum_{\substack{g \in G \\ g(o) \in B}} b^{-td(o,g(o))} \tag{5.13}$$

diverges.

The following lemma was proven in [7] for the case $X = \mathbb{H}^{d+1}$, using the fact that the space $\overline{\mathbb{H}^{d+1}}$ is compact. A new proof is needed for the general case, since in general $\overline{X}$ will not be compact.

Lemma 5.15. *There exists at least one t-divergent point.*

Proof. Since G is of general type, there exists a loxodromic isometry $g \in G$. Let g_+ and g_- be the attracting and repelling fixed points of g, respectively. Let B_+ and B_- be disjoint neighborhoods of g_+ and g_-, respectively. Since the series $\Sigma_t(G)$ diverges, it follows that the series (5.13) diverges for either $B = \partial X \setminus B_-$ or $B = \partial X \setminus B_+$. Without loss of generality, let us assume that it diverges for $B = \partial X \setminus B_-$. Then $\Sigma_t(G \restriction g^n(\partial X \setminus B_-))$ also diverges for every $n \in \mathbb{N}$. By the definition of a loxodromic isometry, g^n tends to g_+ uniformly on $\partial X \setminus B_-$, so for any neighborhood U of g_+, we eventually have $g^n(\partial X \setminus B_-) \subseteq U$. But then $\Sigma_t(G \restriction U)$ diverges, which proves that g_+ is a t-divergent point. ◁

Remark 5.16. It is not hard to see that the set of t-divergent points is invariant under the action of the group. It follows from this and from Observation 1.8 that there are at least two t-divergent points.

Sublemma 5.17. *Let η be a t-divergent point, and let B_η be a neighborhood of η. Then for all $\sigma > 0$ sufficiently large, there exists a set $S_\eta \subseteq G(o) \cap B_\eta$ such that for all $z \in X \setminus B_\eta$,*

(i) *If*

$$\mathcal{P}_{z,x} := \text{Shad}_z(x, \sigma),$$

then the family $(\mathcal{P}_{z,x})_{x \in S_\eta}$ consists of pairwise disjoint shadows contained in $\mathcal{P}_{z,o} \cap B_\eta$.

(ii) *There exists $\kappa > 0$ independent of z such that for all $x \in S_\eta$,*

$$D_{b,z}(\mathcal{P}_{z,x}, \partial X \setminus \mathcal{P}_{z,o}) \geq \kappa \, \text{Diam}_z(\mathcal{P}_{z,o}) \tag{5.14}$$

$$\kappa \, \text{Diam}_z(\mathcal{P}_{z,o}) \leq \text{Diam}_z(\mathcal{P}_{z,x}) \leq \lambda \, \text{Diam}_z(\mathcal{P}_{z,o}). \tag{5.15}$$

(iii)

$$\sum_{x \in S_\eta} \text{Diam}_z^s(\mathcal{P}_{z,x}) \geq \text{Diam}_z^s(\mathcal{P}_{z,o}).$$

Sublemma 5.17 will be proven below; for now, let us complete the proof of Lemma 5.14 assuming Sublemma 5.17.

Let η_1 and η_2 be two distinct t-divergent points. Let B_1 and B_2 be disjoint neighborhoods of η_1 and η_2, respectively, and let $S_1 \subseteq G(o) \cap B_1$ and $S_2 \subseteq G(o) \cap B_2$ be the sets guaranteed by Sublemma 5.17. Now suppose that $w = g_w(o) \in G(o)$. Let $z = g_w^{-1}(o)$. Then either $z \notin B_1$ or $z \notin B_2$; say $z \notin B_i$. Let $T(w) = g_w(S_i)$; then (i)-(iii) of Sublemma 5.17 exactly guarantee (i)-(iii) of Lemma 5.14. □

Proof of Sublemma 5.17. Choose $\rho > 0$ large enough so that

$$\{x \in \overline{X} : \langle x | \eta \rangle_o \geq \rho\} \subseteq B_\eta.$$

Then fix $\sigma > 0$ large to be announced below, depending only on ρ.

For all $x \in X$ we have

$$0 \asymp_{+,\rho} \langle z | \eta \rangle_o \gtrsim_+ \min(\langle x | \eta \rangle_o, \langle x | z \rangle_o).$$

Fix $\widetilde{\rho} \geq \rho$ large to be announced below, depending only on ρ and σ. Let

$$\widetilde{B}_\eta = \{x \in X : \langle x | \eta \rangle_o \geq \widetilde{\rho}\}.$$

It follows that for all $x \in \widetilde{B}_\eta$, we have

$$\langle x | z \rangle_o \asymp_{+,\rho} 0, \tag{5.16}$$

assuming $\widetilde{\rho}$ is chosen large enough. We emphasize that the implied constants of these asymptotics are independent of w.

For each $n \in \mathbb{N}$ let

$$A_n := B(o, n+1) \setminus B(o, n)$$

be the nth annulus centered at o. We shall need the following variant of the Intersecting Shadows Lemma:

Claim 5.18. *There exists $\tau > 0$ depending on ρ and σ such that for all $n \in \mathbb{N}$ and for all $x, y \in A_n \cap \widetilde{B}_\eta$, if*

$$\mathcal{P}_{z,x} \cap \mathcal{P}_{z,y} \neq \varnothing,$$

then

$$d(x, y) < \tau.$$

Proof. Without loss of generality suppose $d(z, y) \geq d(z, x)$. Then by the Intersecting Shadows Lemma we have

$$d(x, y) \asymp_{+,\sigma} \mathcal{B}_z(y, x) = \mathcal{B}_o(y, x) + 2\langle x|z \rangle_o - 2\langle y|z \rangle_o.$$

Now $\mathcal{B}_o(y, x) \leq 1$ since $x, y \in A_n$. On the other hand, since $x, y \in \widetilde{B}_\eta$, we have

$$\langle x|z \rangle_o \asymp_{+,\rho} \langle y|z \rangle_o \asymp_{+,\rho} 0.$$

Combining gives

$$d(x, y) \asymp_{+,\rho,\sigma} 0,$$

i.e. there exists τ depending only on ρ and σ such that $d(x, y) < \tau$.   ◁

Let

$$M = \#\{g \in G : g(o) \in B(o, \tau)\};$$

M is finite since G is strongly discrete. Then fix $\widetilde{M} > 0$ large to be announced below, depending on ρ and M (and thus implicitly on σ). Since η is t-divergent, we have

$$
\begin{aligned}
+\infty = \Sigma_t(G \upharpoonright \widetilde{B}_\eta) &= \sum_{n=1}^{\infty} \Sigma_t(G \upharpoonright \widetilde{B}_\eta \cap A_n) \\
&= \sum_{n=1}^{\infty} \sum_{\substack{g \in G \\ g(o) \in \widetilde{B}_\eta \cap A_n}} b^{-d(o, g(o))} \\
&\asymp_{\times} \sum_{n=1}^{\infty} b^{-(t-s)n} b^{-sn} \#\{g \in G : g(o) \in \widetilde{B}_\eta \cap A_n\}.
\end{aligned}
$$

It follows that there exist arbitrarily large numbers $n_\eta \in \mathbb{N}$ such that

$$(5.17) \qquad\qquad b^{-sn_\eta} \#\{g \in G : g(o) \in \widetilde{B}_\eta \cap A_{n_\eta}\} \geq \widetilde{M}.$$

Fix such an n_η, also to be announced below, depending on λ, ρ, $\widetilde{\rho}$, and $\widetilde{M}$ (and thus implicitly on M and σ). Let S_η be a maximal τ-separated subset of $G(o) \cap \widetilde{B}_\eta \cap A_{n_\eta}$.[32] We have

$$\#(S_\eta) \geq \frac{\#\{g \in G : g(o) \in \widetilde{B}_\eta \cap A_{n_\eta}\}}{M}$$

and so

$$(5.18) \qquad\qquad \sum_{x \in S_\eta} b^{-sd(o,x)} \asymp_{\times} b^{-sn_\eta} \#(S_\eta) \geq \frac{\widetilde{M}}{M} \asymp_{\times, M} \widetilde{M}.$$

---

[32]A set S is *τ-separated* if

$$x, y \in S \text{ distinct} \Rightarrow d(x, y) \geq \tau.$$

The existence of a maximal τ-separated set follows from Zorn's lemma.

Proof of (i). In order to see that the shadows $(\mathcal{P}_{z,x})_{x \in S_\eta}$ are pairwise disjoint, suppose that $x, y \in S_\eta$ are such that $\mathcal{P}_{z,x} \cap \mathcal{P}_{z,y} \neq \varnothing$. By Claim 5.18 we have $d(x, y) < \tau$. Since S_η is τ-separated, this implies $x = y$.

Fix $x \in S_\eta$. Using (5.16) and the fact that $x \in A_{n_\eta}$, we have

$$\langle o|z \rangle_x \asymp_+ d(o, x) - \langle x|z \rangle_o \asymp_{+,\rho} d(o, x) \asymp_+ n_\eta.$$

Thus for all $\xi \in \mathcal{P}_{z,x}$,

$$0 \asymp_{+,\sigma} \langle z|\xi \rangle_x \gtrsim_+ \min(\langle o|z \rangle_x, \langle o|\xi \rangle_x) \asymp_+ \min(n_\eta, \langle o|\xi \rangle_x);$$

taking n_η sufficiently large (depending on σ), this gives

$$\langle o|\xi \rangle_x \asymp_{+,\sigma} 0,$$

from which it follows that

$$\langle x|\xi \rangle_o \asymp_+ d(o, x) - \langle o|\xi \rangle_x \asymp_{+,\sigma} n_\eta.$$

Therefore, since $x \in \widetilde{B}_\eta$, we get

$$\langle \xi|\eta \rangle_o \gtrsim_+ \min(\langle x|\xi \rangle_o, \langle x|\eta \rangle_o) \gtrsim_{+,\sigma} \min(n_\eta, \widetilde{\rho}).$$

Thus $\xi \in B_\eta$ as long as $\widetilde{\rho}$ and n_η are large enough (depending on σ). Thus $\mathcal{P}_{z,x} \subseteq B_\eta$.

Finally, note that we do not need to prove that $\mathcal{P}_{z,x} \subseteq \mathcal{P}_{z,o}$, since it is implied by (5.14) which we prove below.

$\triangleleft$

Proof of (ii). Take any $x \in S_\eta$. Then by (5.16), we have

$$(5.19) \qquad d(x, z) - d(o, z) = d(o, x) - 2\langle x|z \rangle_o \asymp_{+,\rho} d(o, x) \asymp_+ n_\eta.$$

Combining with the Diameter of Shadows Lemma gives

$$(5.20) \qquad \frac{\mathrm{Diam}_z(\mathcal{P}_{z,x})}{\mathrm{Diam}_z(\mathcal{P}_{z,o})} \asymp_{\times,\sigma} \frac{b^{-d(z,x)}}{b^{-d(z,o)}} \asymp_{\times,\rho} b^{-n_\eta}.$$

Thus by choosing n_η sufficiently large depending on σ, λ, and ρ (and satisfying (5.17)), we guarantee that the second inequality of (5.15) holds. On the other hand, once n_η is chosen, (5.20) guarantees that if we choose κ sufficiently small, then the first inequality of (5.15) holds.

In order to prove (5.14), let $\xi \in \mathcal{P}_{z,x}$ and let $\gamma \in \partial X \setminus \mathcal{P}_{z,o}$. We have

$$\langle x|\xi \rangle_z \asymp_+ d(x, z) - \langle z|\xi \rangle_x \geq d(x, z) - \sigma$$
$$\langle o|\gamma \rangle_z \asymp_+ d(o, z) - \langle o|\xi \rangle_x \leq d(o, z) - \sigma.$$

Also, by (5.16) we have

$$\langle o|x \rangle_z \asymp_+ d(o, z) - \langle x|z \rangle_o \asymp_{+,\rho} d(o, z).$$

Applying Gromov's inequality twice and then applying (5.19) gives

$$d(o, z) - \sigma \gtrsim_+ \langle o|\gamma \rangle_z \gtrsim_+ \quad \min\left(\langle o|x \rangle_z, \langle x|\xi \rangle_z, \langle \xi|\gamma \rangle_z\right)$$
$$\gtrsim_{+,\rho} \min\left(d(o, z), d(x, z) - \sigma, \langle \xi|\gamma \rangle_z\right)$$
$$\asymp_+ \quad \min\left(d(o, z), d(o, z) + n_\eta - \sigma, \langle \xi|\gamma \rangle_z\right).$$

By choosing n_η and σ sufficiently large (depending on ρ), we can guarantee that neither of the first two expressions can represent the minimum without contradicting the inequality. Thus

$$d(o, z) - \sigma \gtrsim_{+,\rho} \langle \xi|\gamma \rangle_z;$$

exponentiating and the Diameter of Shadows Lemma give

$$D_{b,z}(\xi, \gamma) \gtrsim_{\times,\rho} b^{-(d(o,z)-\sigma)} \asymp_{\times,\sigma} b^{-d(o,z)} \asymp_{\times,\sigma} \mathrm{Diam}_z(\mathcal{P}_{z,o}).$$

Thus we may choose κ small enough, depending on ρ and σ, so that (5.14) holds.

$\triangleleft$

Proof of (iii).

$$\sum_{x \in S_\eta} \mathrm{Diam}_z^s(\mathcal{P}_{z,x}) \asymp_\times \sum_{x \in S_\eta} b^{-sd(z,x)} \qquad \text{(by the Diameter of Shadows Lemma)}$$

$$\asymp_{\times,\rho} b^{-sd(z,o)} \sum_{x \in S_\eta} b^{-d(o,x)} \qquad \text{(by (5.19))}$$

$$\gtrsim_{\times,M} \widetilde{M} b^{-sd(z,o)} \qquad \text{(by (5.18))}$$

$$\asymp_\times \widetilde{M} \mathrm{Diam}_z^s(\mathcal{P}_{z,o}). \qquad \text{(by the Diameter of Shadows Lemma)}$$

Letting $\widetilde{M}$ be larger than the implied constant yields the result.   ◁

□

6. Proof of Theorem 6.1 (Absolute winning of BA_ξ)

In this section we prove Theorem 6.1. We repeat both Theorem 6.1 and Corollary 5.8 for convenience.

Corollary 5.8 (Proven in Sections 5 - 5.4). *Let (X, d, o, b, G) be as in §1.1.4. Then for every $0 < s < \delta_G$ and for all $\sigma > 0$ sufficiently large, there exist a tree T on $\mathbb{N}$ and an embedding $T \ni \omega \mapsto x_\omega \in G(o)$ such that if*

$$\mathcal{P}_\omega := \mathrm{Shad}(x_\omega, \sigma),$$

then $(\mathcal{P}_\omega)_{\omega \in T}$ is a partition structure on $(\partial X, D)$, whose limit set $\mathcal{J}_s \subseteq \Lambda_{\mathrm{ur}}(G)$ is Ahlfors s-regular.

Theorem 6.1. *Let (X, d, o, b, G) be as in §1.1.4. Fix $0 < s < \delta_G$ and let $\mathcal{J}_s$ be defined as in Corollary 5.8. Then for each $\xi \in \partial X$, the set $\mathrm{BA}_\xi \cap \mathcal{J}_s$ is absolute winning on $\mathcal{J}_s$.*

Reductions. First, note that by (iii) of Proposition 4.3, it is enough to show that $(\mathrm{BA}_\xi \cup G(\xi)) \cap \mathcal{J}_s$ is absolute winning on $(\mathcal{J}_s, D)$,. Next, by (v) of Proposition 4.3 together with Lemma 5.11, it is enough to show that $\pi^{-1}(\mathrm{BA}_\xi \cup G(\xi))$ is absolute winning on $(E^{\mathbb{N}}, \rho_2)$. Finally, by Corollary 4.6, it is enough to show that $\pi^{-1}(\mathrm{BA}_\xi \cup G(\xi))$ is 2^{-m}-modified absolute winning for every $m \in \mathbb{N}$.

Thus Theorem 6.1 is a direct corollary of the following theorem:

Theorem 6.2. *Let (X, d, o, b, G) be as in §1.1.4. Fix $0 < s < \delta_G$, let $(\mathcal{P}_\omega)_{\omega \in T}$ be defined as in Corollary 5.8, and let $\pi : T^{\mathbb{N}} \to \Lambda_{\mathrm{ur}}$ be as in Definition 5.7. Then for each $\xi \in \partial X$ and for each $m \in \mathbb{N}$, the set $\pi^{-1}(\mathrm{BA}_\xi \cup G(\xi))$ is 2^{-m}-modified absolute winning on $(E^{\mathbb{N}}, \rho_2)$.*

The remainder of this section will be devoted to the proof of Theorem 6.2. Notice that a ball in $(E^{\mathbb{N}}, \rho_2)$ of radius 2^{-n} is simply a cylinder of length n, so multiplying the radius of a ball by 2^{-m} is the same as increasing the length of the corresponding cylinder by m.

Let $\sigma > 0$ be large enough so that both Corollary 5.8 and the Diameter of Shadows lemma hold. For each $x \in X$ let $\mathcal{P}_x = \mathrm{Shad}(x, \sigma)$, so that $\mathcal{P}_\omega = \mathcal{P}_{x_\omega}$.

Fix $c > 0$ to be announced below (depending only on m and on the partition structure). We define

$$(6.1) \qquad \phi(x) := \sum_{\substack{g \in G \\ \langle g(o)|g(\xi)\rangle_o \le d(o,x) \\ g(\xi) \in \mathcal{P}_x}} b^{-c\,\mathcal{B}_o(g(o),x)}$$

and for each $\omega \in T$, let $\phi(\omega) := \phi(x_\omega)$. Intuitively, $\phi(x)$ measures "how many points in the orbit of ξ are close to x, up to a fixed height". Since badly approximable points stay away from the orbit of ξ, to find a badly approximable point we should minimize ϕ.

Thus Alice's strategy is as follows: If Bob has just chosen his nth ball $B_n = [\omega^{(n)}]$, then Alice will choose (i.e. remove) the ball $A_n = [\tau^{(n)}]$, where $\tau^{(n)}$ is an extension of $\omega^{(n)}$ of length $m + |\omega^{(n)}|$, and has the largest value for ϕ among such extensions. We will show that this strategy forces the sequence $(\phi(\omega_1^n))_1^\infty$ to remain bounded, which in turn forces $\pi(\omega) \in \mathrm{BA}_\xi \cup G(\xi)$.

Lemma 6.3. *Fix $\omega \in T(\infty)$, and suppose that the sequence $(\phi(\omega_1^n))_1^\infty$ is bounded. Then $\omega \in \pi^{-1}(\mathrm{BA}_\xi \cup G(\xi))$.*

Proof. Fix $\varepsilon > 0$ to be announced below, and let $\eta = \pi(\omega)$. Fix $g \in G$, and suppose for a contradiction that

$$D(g(\xi), \eta) \leq \varepsilon b^{-d(o, g(o))}$$

but $\eta \neq g(\xi)$. For convenience of notation let

$$x_n = x_{\omega_1^n}, \quad \mathcal{P}_n = \mathcal{P}_{\omega_1^n}.$$

Since $\eta \neq g(\xi)$, we have $g(\xi) \notin \mathcal{P}_n$ for all n sufficiently large. Let $n \in \mathbb{N}$ be the largest integer such that $g(\xi) \in \mathcal{P}_n$. In particular, $g(\xi) \notin \mathcal{P}_{n+1}$. On the other hand, $\eta \in \mathcal{P}_{n+2}$ and so

$$\varepsilon b^{-d(o, g(o))} \geq D(g(\xi), \eta) \geq \quad D(\mathcal{P}_{n+2}, \partial X \setminus \mathcal{P}_{n+1})$$
$$\asymp_\times b^{-d(o, x_{n+1})} \asymp_\times b^{-d(o, x_n)}, \text{[33]}$$

where the last two asymptotics follow from (5.1) and (5.2), respectively, together with the Diameter of Shadows Lemma. Taking negative logarithms we have

(6.2) $$d(o, g(o)) \lesssim_+ \log_b(\varepsilon) + d(o, x_n).$$

Combining with (c) of Proposition 2.3 yields

$$\langle g(o) | g(\xi) \rangle_o \lesssim_+ \log_b(\varepsilon) + d(o, x_n).$$

If ε is chosen small enough, then this bound is a regular inequality, i.e.

$$\langle g(o) | g(\xi) \rangle_o \leq d(o, x_n).$$

Together with the fact that $g(\xi) \in \mathcal{P}_n$, this guarantees that the term $b^{-c \mathcal{B}_o(g(o), x_n)}$ will be included in the summation (6.1) (with $x = x_n$). In particular, since $(\phi(x_n))_1^\infty$ is bounded we have

$$b^{-c \mathcal{B}_o(g(o), x_n)} \lesssim_\times 1$$

and thus

$$d(o, x_n) \lesssim_+ d(o, g(o)).$$

If ε is small enough, then this is a contradiction to (6.2). $\qquad\square$

To prove that Alice's strategy forces the sequence $(\phi(x_n))_1^\infty$ to remain bounded, we first show that it begins bounded. Now clearly,

$$\phi(x) \leq b^{cd(o, x)} \sum_{\substack{g \in G \\ \langle o | \xi \rangle_{g^{-1}(o)} \leq d(o, x)}} b^{-cd(o, g(o))}$$
$$= b^{cd(o, x)} \sum_{\substack{g \in G \\ \xi \in \mathrm{Shad}(g^{-1}(o), d(o, x))}} b^{-cd(o, g(o))}.$$

Thus, the following lemma demonstrates that $\phi(x)$ is finite for every $x \in X$:

Lemma 6.4. *For all $\rho > 0$, we have*

$$\sum_{\substack{g \in G \\ \xi \in \mathrm{Shad}(g^{-1}(o), \rho)}} b^{-cd(o, g(o))} < +\infty.$$

---

[33]Here and from now on we omit the dependence on κ, λ, and σ when writing asymptotics.

Proof. Let
$$S_\rho := \{z \in X : \xi \in \mathrm{Shad}(z, \rho)\}.$$

As in the proof of Lemma 5.13 we let
$$A_n := B(o, n + 1) \setminus B(o, n).$$

Then for all $z, w \in S_\rho \cap A_n$, we have $\mathrm{Shad}(z, \rho) \cap \mathrm{Shad}(w, \rho) \neq \varnothing$, and so by the Intersecting Shadows Lemma
$$d(z, w) \asymp_{+,\rho} |\mathcal{B}_o(z, w)| \asymp_+ 0.$$

Let $\tau = \tau_\rho > 0$ be the implied constant of this asymptotic, so that

(6.3) $$\mathrm{diam}(S_\rho \cap A_n) \leq \tau$$

for all n. Let $M = \#\{g \in G : g(o) \in B(o, \tau)\}$; $M < +\infty$ since G is strongly discrete.

For each $n \in \mathbb{N}$, choose $z_n \in S_\rho \cap A_n \cap G(o)$ if possible, otherwise not. It follows from (6.3) that $S_\rho \cap G(o) \subseteq \bigcup_n B(z_n, \tau)$. We have

$$\sum_{\substack{g \in G \\ \xi \in \mathrm{Shad}(g^{-1}(o), \rho)}} b^{-cd(o, g(o))} = \sum_{\substack{g \in G \\ g(o) \in S_\rho}} b^{-cd(o, g(o))}$$

$$\leq \sum_{n=0}^{\infty} \sum_{\substack{g \in G \\ g(o) \in B(z_n, \tau)}} b^{-cd(o, g(o))}$$

$$\leq \sum_{n=0}^{\infty} M b^{-c(d(o, z_n) - \tau)} \asymp_\times M b^{c\tau} \sum_{n=0}^{\infty} b^{-cn} < +\infty,$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $\alpha > 0$ be large enough so that $d(o, x_{\omega a}) \leq d(o, x_\omega) + \alpha$ for all $\omega \in T$ and for all $a \in T(\omega)$; such an α exists by (5.1) and the Diameter of Shadows Lemma.

Lemma 6.5. *For each $\omega \in T$,*

(6.4) $$\sum_{a \in T(\omega)} \phi(x_{\omega a}) \lesssim_+ b^{c\alpha} \phi(x_\omega).$$

Proof. For convenience of notation let $x = x_\omega$. We have

$$b^{-c\alpha} \sum_{a \in T(\omega)} \phi(x_{\omega a}) \leq \sum_{a \in T(\omega)} b^{-c\,\mathcal{B}_o(x_{\omega a}, x)} \phi(x_{\omega a}) \qquad\qquad (\text{since } d(o, x_{\omega a}) \leq d(o, x_\omega) + \alpha)$$

$$= \sum_{a \in T(\omega)} \sum_{\substack{g \in G \\ \langle g(o) | g(\xi) \rangle_o \leq d(o, x_{\omega a}) \\ g(\xi) \in \mathcal{P}_{x_{\omega a}}}} b^{-c\,\mathcal{B}_o(g(o), x)} \qquad\qquad (\text{by the definition of } \phi)$$

$$\leq \sum_{\substack{g \in G \\ \langle g(o) | g(\xi) \rangle_o \leq d(o, x) + \alpha \\ g(\xi) \in \mathcal{P}_x}} b^{-c\,\mathcal{B}_o(g(o), x)} \qquad\qquad (\text{since } (\mathcal{P}_{\omega a})_{a \in T(\omega)} \text{ are disjoint})$$

$$= \phi(x) + \sum_{\substack{g \in G \\ d(o, x) < \langle g(o) | g(\xi) \rangle_o \leq d(o, x) + \alpha \\ g(\xi) \in \mathcal{P}_x}} b^{-c\,\mathcal{B}_o(g(o), x)}. \qquad\qquad (\text{by the definition of } \phi)$$

Thus, to prove Lemma 6.5 it is enough to show that the series

(6.5) $$\sum_{\substack{g \in G \\ d(o, x) < \langle g(o) | g(\xi) \rangle_o \leq d(o, x) + \alpha \\ g(\xi) \in \mathcal{P}_x}} b^{-c\,\mathcal{B}_o(g(o), x)}$$

is bounded independent of x. To this end, fix $g \in G$ which gives a term in (6.5), i.e.

$$(6.6) \qquad d(o,x) < \langle g(o)|g(\xi)\rangle_o \leq d(o,x) + \alpha$$

$$(6.7) \qquad g(\xi) \in \mathcal{P}_x = \mathrm{Shad}(x,\sigma).$$

Now by (6.7) we have

$$\langle x|g(\xi)\rangle_o \asymp_+ d(o,x);$$

combining with (6.6) and Gromov's inequality yields

$$\langle x|g(o)\rangle_o \asymp_+ d(o,x),$$

which together with (b) of Proposition 2.3 gives

$$(6.8) \qquad \langle o|g(o)\rangle_x \asymp_+ 0.$$

By (j) of Proposition 2.3,

$$\langle o|\xi\rangle_{g^{-1}(x)} = \langle g(o)|g(\xi)\rangle_x \asymp_+ \langle g(o)|g(\xi)\rangle_o + \langle o|g(o)\rangle_x - \langle x|g(\xi)\rangle_o$$
$$\asymp_+ d(o,x) + 0 - d(o,x) = 0.$$

Thus, if we choose $\rho > 0$ large enough, we see that $\xi \in \mathrm{Shad}(g^{-1}(x),\rho)$ for all g satisfying (6.6) and (6.7). On the other hand, rearranging (6.8) yields

$$\mathcal{B}_o(g(o),x) \asymp_+ d(g(o),x) = d(o,g^{-1}(x)).$$

We return to our original series; fixing $\rho > 0$ large enough we have

$$b^{-c\alpha} \sum_{a \in T(\omega)} \phi(x_{\omega a}) - \phi(x) \leq \sum_{\substack{g \in G \\ d(o,x) < \langle g(o)|g(\xi)\rangle_o \leq d(o,x)+\alpha \\ g(\xi) \in \mathcal{P}_x}} b^{-c\,\mathcal{B}_o(g(o),x)}$$

$$\lesssim_\times \sum_{\substack{g \in G \\ \xi \in \mathrm{Shad}(g^{-1}(x),\rho)}} b^{-cd(o,g^{-1}(x))}$$

$$= \sum_{\substack{g \in G \\ \xi \in \mathrm{Shad}(g^{-1}(o),\rho)}} b^{-cd(o,g^{-1}(o))},$$

since $x = x_\omega \in G(o)$ by construction. The sum is clearly independent of ω, and by Lemma 6.4 it is finite. This completes the proof. $\qquad\square$

Iterating (6.4) yields

$$(6.9) \qquad \sum_\tau \phi(\tau) \lesssim_{+,m} b^{cm\alpha}\phi(\omega),$$

where the sum is taken over all τ of which ω is an initial segment and for which $|\tau| = m + |\omega|$.

Recall that Alice's strategy is to remove the ball $A_n = [\tau^{(n)}]$, where $\tau^{(n)}$ is the extension of $\omega^{(n)}$ of length $m + |\omega^{(n)}|$ which has the largest value for ϕ. Now let $B_{n+1} = [\omega^{(n+1)}]$ be the next ball that Bob plays. By the rules of the game, we must have $|\omega^{(n+1)}| = m + |\omega^{(n)}|$ and $[\omega^{(n+1)}] \cap [\tau^{(n)}] = \varnothing$, and so $\omega^{(n+1)}$ and $\tau^{(n)}$ represent distinct terms in the sum (6.9). In particular,

$$\phi(\tau^{(n)}) + \phi(\omega^{(n+1)}) \lesssim_{+,m} b^{cm\alpha}\phi(\omega^{(n)}).$$

On the other hand, we have

$$\phi(\omega^{(n+1)}) \leq \phi(\tau^{(n)})$$

and so

$$(6.10) \qquad \phi(\omega^{(n+1)}) \lesssim_{+,m} \frac{b^{cm\alpha}}{2}\phi(\omega^{(n)}).$$

This inequality suggests that we should choose c small enough so that $b^{cm\alpha} < 2$. As claimed at the beginning of this section, such a choice depends only on m and on the partition structure $(\mathcal{P}_\omega)_{\omega \in T}$.

It follows that for all $n \in \mathbb{N}$ we have

$$\phi(\omega^{(n)}) \leq \max\left(C \sum_{i=0}^{\infty} \left(\frac{b^{cm\alpha}}{2}\right)^i, \phi(\omega^{(0)})\right) < +\infty,$$

where C is the implied constant in (6.10). Thus by Lemma 6.3, $\omega \in \pi^{-1}(\mathrm{BA}_\xi \cup G(\xi))$.

7. Proof of Theorem 7.1 (Generalization of the Jarník-Besicovitch Theorem)

Theorem 7.1 (Generalization of the Jarník-Besicovitch Theorem)**.** *Let (X, d, o, b, G) be as in §1.1.4, and let $\xi \in \partial X$ be a distinguished point. Let δ be the Poincaré exponent of G, and suppose that $\delta < +\infty$. Then for each $c > 0$,*

(7.1)
$$\mathrm{HD}(W_{c,\xi}) = \sup\left\{s \in (0, \delta) : P_\xi(s) \leq \frac{\delta - s}{c}\right\}$$
$$= \inf\left\{s \in (0, \delta) : P_\xi(s) \geq \frac{\delta - s}{c}\right\} \leq \frac{\delta}{c + 1}.$$

In particular

(7.2) $$\mathrm{HD}(\mathrm{VWA}_\xi) = \sup\{s \in (0, \delta) : P_\xi(s) < +\infty\} = \inf\{s \in (0, \delta) : P_\xi(s) = +\infty\}$$

and $\mathrm{HD}(\mathrm{Liouville}_\xi) = 0$. In these formulas we let $\sup \varnothing = 0$ and $\inf \varnothing = \delta$.

Remark 7.2. *If $\delta = +\infty$, then for each $c > 0$, our proof shows that*

$$\mathrm{HD}(W_{c,\xi}) \geq \sup\{s \in (0, +\infty) : P_\xi(s) < +\infty\}.$$

However, we cannot prove any upper bound in this case.

Remark 7.3. For the remainder of this section, we assume $\delta < +\infty$.

7.1. The function P_ξ.

Proposition 7.4. *The function $P_\xi : (0, \delta) \to (0, \infty]$ is nondecreasing and satisfies $P_\xi(s) \geq s$.*

Proof. Fix $s, t \in (0, \delta)$ with $s < t$ and notice that for all $A \subseteq \partial X$

$$\mathrm{Diam}^s(A) \leq \mathrm{Diam}^{s-t}(\partial X) \mathrm{Diam}^t(A).$$

Running this inequality through formulas (1.5), (1.6), and (1.7) yields $P_\xi(s) \geq P_\xi(t)$, i.e. P_ξ is nondecreasing.

For all $s \in (0, \delta)$ and for all $r > 0$,

$$H_{\xi,\sigma,s}(r) \leq \mathcal{H}_\infty^s(B(\xi, r)) \leq \mathrm{Diam}^s(B(\xi, r)) \leq (2r)^s;$$

running this inequality through (1.7) yields $P_\xi(s) \geq s$. $\qquad\square$

Corollary 7.5.

(A) $$\sup\left\{s \in (0, \delta) : P_\xi(s) \leq \frac{\delta - s}{c}\right\}$$

(B) $$= \inf\left\{s \in (0, \delta) : P_\xi(s) \geq \frac{\delta - s}{c}\right\}$$

(C) $$= \sup\left\{s \in (0, \delta) : P_\xi(s) < \frac{\delta - s}{c}\right\}$$

(D) $$= \inf\left\{s \in (0, \delta) : P_\xi(s) > \frac{\delta - s}{c}\right\}$$

(E) $$\leq \frac{\delta}{c + 1}.$$

Furthermore, the common value of (A), (B), (C), and (D) is continuous as a function of c on $(0, +\infty)$.

Proof. The three equalities are an easy consequence of the fact that the map $s \mapsto P_\xi(s) - \frac{\delta - s}{c}$ is strictly increasing.[34] The inequality is demonstrated by noting that

$$P_\xi\left(\frac{\delta}{c+1}\right) \geq \frac{\delta}{c+1} = \frac{\delta - \frac{\delta}{c+1}}{c},$$

and so $s = \frac{\delta}{c+1}$ is a member of the infimum defining (B).

To demonstrate the continuity, fix $0 < c_1 < c_2 < +\infty$, and let $s_1, s_2 \in [0, \delta]$ be the corresponding values of (A)-(D). Then $s_1 \geq s_2$; let us suppose that $s_1 > s_2$. Then for every $\varepsilon > 0$ sufficiently small

$$\frac{\delta - (s_2 + \varepsilon)}{c_2} \leq P_\xi(s_2 + \varepsilon) \leq P_\xi(s_1 - \varepsilon) \leq \frac{\delta - (s_1 - \varepsilon)}{c_1};$$

letting ε tend to zero we have

$$\frac{\delta - s_2}{c_2} \leq \frac{\delta - s_1}{c_1};$$

rearranging gives

$$s_1 - s_2 \leq \frac{(c_2 - c_1)(\delta - s_2)}{c_2} \leq \delta \frac{c_2 - c_1}{c_2}.$$

This in fact demonstrates that for every $c_0 > 0$, the function $c \mapsto s$ is Lipschitz continuous on $[c_0, +\infty)$ with a corresponding constant of δ/a. $\qquad\square$

Corollary 7.5 reduces the proof of Theorem 7.1 to showing that $\mathrm{HD}(W_{c,\xi})$ is equal to the common value (A) = (B) = (C) = (D) of Corollary 7.5, and then demonstrating the statements about $\mathrm{HD}(\mathrm{VWA}_\xi)$ and $\mathrm{HD}(\mathrm{Liouville}_\xi)$. The former will be demonstrated in Subsections 7.2 - 7.4, and the latter will be left to the reader.

Corollary 7.6. *The function $c \mapsto \mathrm{HD}(W_{c,\xi})$ is continuous on $(0, +\infty)$.*

Proof. This follows directly from Theorem 7.1 and Corollary 7.5. $\qquad\square$

7.2. $\leq$ direction. Fix $s \in (0, \delta)$ which is a member of the infimum of Corollary 7.5(D), i.e. so that $P_\xi(s) > \frac{\delta - s}{c}$. Fix $1 < \widetilde{c} < c$ so that

$$(7.3) \qquad P_\xi(s) > \frac{\delta - s}{\widetilde{c}}.$$

For simplicity of exposition we will assume that $P_\xi(s) < +\infty$. Fix $\sigma > 0$, and let $\tau > 0$ be given by Lemma 2.39.

For each $g \in G$ let

$$(7.4) \qquad B_{g,\widetilde{c}} := B\big(g(\xi), b^{-(1+\widetilde{c})d(o,g(o))}\big).$$

Claim 7.7. *For all $g \in G$ and for all $\eta_1, \eta_2 \in g^{-1}(B_{g,\widetilde{c}})$,*

$$(7.5) \qquad g'(\eta_1) \asymp_\times g'(\eta_2) \asymp_\times g'(\xi)$$

$$(7.6) \qquad \frac{D(g(\eta_1), g(\eta_2))}{D(\eta_1, \eta_2)} \asymp_\times g'(\xi)$$

Proof. Fix $\eta \in g^{-1}(B_{g,\widetilde{c}})$. We have

$$(7.7) \qquad \left|\log_b\left(\frac{g'(\eta)}{g'(\xi)}\right)\right| \asymp_+ |\mathcal{B}_\eta(o, g^{-1}(o)) - \mathcal{B}_\xi(o, g^{-1}(o))| \asymp_+ |\langle g(o)|g(\eta)\rangle_o - \langle g(o)|g(\xi)\rangle_o|.$$

---

[34]Note however that we don't know whether this function is continuous, and so in particular we don't know whether $P_\xi(s) = 0$ if s is the common value of (A)-(D).

Let

$$(A) = \langle g(o)|g(\eta)\rangle_o$$
$$(B) = \langle g(o)|g(\xi)\rangle_o$$
$$(C) = \langle g(\xi)|g(\eta)\rangle_o.$$

By Gromov's inequality, at least two of the expressions (A), (B), and (C) must be asymptotic. On the other hand, since $g(\eta) \in B_{g,\widetilde{c}}$, we have

$$\langle g(\xi)|g(\eta)\rangle_o - d(o,g(o)) \asymp_+ -\log_b(D(g(\xi),g(\eta))) - d(o,g(o)) \gtrsim_+ \widetilde{c}\, d(o,g(o)) \underset{g}{\to} +\infty,$$

Since (A) and (B) are both less than $d(o,g(o))$, this demonstrates that for all but finitely many g, (C) is not asymptotic to either (A) or (B). Thus by Gromov's inequality, (A) and (B) are asymptotic (for the finitely many exceptions we just let the implied constant be $d(o,g(o))$), and so (7.7) is asymptotic to zero. This demonstrates (7.5). Now (7.6) follows from (7.5) and the geometric mean value theorem. $\qquad\square$

Let $C > 0$ be the implied constant of (7.6), and let

$$r_g = Cb^{-(\widetilde{c}+1)d(o,g(o))}/g'(\xi).$$

Then (7.6) implies that

$$g^{-1}(B_{g,\widetilde{c}}) \subseteq B(\xi, r_g).$$

Fix $\varepsilon > 0$ small to be determined; by the definition of $P_\xi(s)$ we have

$$H_{\xi,\tau,s}(r_g) \lesssim_\times r_g^{P_\xi(s)-\varepsilon}.$$

On the other hand, by the definition of $H_{\xi,\tau,s}(r_g)$, there exists a cover $\mathcal{C}_g$ of $B(\xi,r_g) \cap \Lambda_{\mathrm{r},\tau}$ such that

$$\sum_{A \in \mathcal{C}_g} \mathrm{Diam}^s(A) \leq H_{\xi,\tau,s}(r_g) + r_g^{P_\xi(s)-\varepsilon};$$

thus

$$\sum_{A \in \mathcal{C}_g} \mathrm{Diam}^s(A) \lesssim_\times r_g^{P_\xi(s)-\varepsilon}.$$

For each $A \in \mathcal{C}_g$, Claim 7.7 implies that

$$\mathrm{Diam}(g(A) \cap B_{g,\widetilde{c}}) \lesssim_\times g'(\xi)\,\mathrm{Diam}(A).$$

Let $\mathcal{C} = \bigcup_{g \in G}\{g(A) \cap B_{g,\widetilde{c}} : A \in \mathcal{C}_g\}$.

Claim 7.8.

$$\sum_{A \in \mathcal{C}} \mathrm{Diam}^s(A) < +\infty.$$

Proof.

$$\sum_{A \in \mathcal{C}} \mathrm{Diam}^s(A) = \sum_{g \in G}\sum_{A \in \mathcal{C}_g} \mathrm{Diam}^s(g(A) \cap B_{g,\widetilde{c}})$$
$$\lesssim_\times \sum_{g \in G}[g'(\xi)]^s \sum_{A \in \mathcal{C}_g} \mathrm{Diam}^s(A)$$
$$\lesssim_\times \sum_{g \in G}[g'(\xi)]^s r_g^{P_\xi(s)-\varepsilon}$$
$$\asymp_\times \sum_{g \in G}[g'(\xi)]^{s-(P_\xi(s)-\varepsilon)}b^{-(P_\xi(s)-\varepsilon)(\widetilde{c}+1)d(o,g(o))}.$$

By Proposition 7.4, $P_\xi(s) \geq s$. Let $B_g = \log_b(g'(\xi))$. Since $|B_g| \leq d(o,g(o))$, we have

$$(P_\xi(s) - s - \varepsilon)B_g \leq |P_\xi(s) - s - \varepsilon| \cdot |B_g| \leq \max(P_\xi(s) - s - \varepsilon, \varepsilon)d(o,g(o))$$

and thus

$$\sum_{A \in \mathcal{C}} \operatorname{Diam}^s(A) \lesssim_\times \sum_{g \in G} b^{-\min(\widetilde{c}(P_\xi(s) - \varepsilon) + s, (\widetilde{c} + 1)P_\xi(s) - (\widetilde{c} + 2)\varepsilon)d(o, g(o))}.$$

The right hand side is the Poincaré series of G with respect to the exponent

$$\min(\widetilde{c}(P_\xi(s) - \varepsilon) + s, (\widetilde{c} + 1)P_\xi(s) - (\widetilde{c} + 2)\varepsilon).$$

It can be seen by rearranging (7.3) that this exponent exceeds δ for all ε sufficiently small. Therefore the series converges. $\qquad \square$

Claim 7.9. *For all $\eta \in W_{c,\xi} \cap \Lambda_{\mathrm{r},\sigma}$, $\eta \in A$ for infinitely many $A \in \mathcal{C}$.*

Proof. Since $\omega_\xi(\eta) \geq c > \widetilde{c}$, for infinitely many $g \in G$, we have $\eta \in B_{g,\widetilde{c}}$. Thus $g^{-1}(\eta) \in B(\xi, r_g)$. On the other hand, by Lemma 2.39, $g^{-1}(\eta) \in \Lambda_{\mathrm{r},\tau}$. Thus $g^{-1}(\eta) \in \bigcup \mathcal{C}_g$; fix $A_g \in \mathcal{C}_g$ for which $g^{-1}(\eta) \in A_g$. Then $\eta \in g(A_g) \cap B_{g,\widetilde{c}} \in \mathcal{C}$. $\qquad \square$

Now by the Hausdorff-Cantelli lemma (see e.g. Lemma 3.10 of [6]), we have $\mathcal{H}^s(W_{c,\xi} \cap \Lambda_{\mathrm{r},\sigma}) = 0$, and thus $\operatorname{HD}(W_{c,\xi} \cap \Lambda_{\mathrm{r},\sigma}) \leq s$. But σ was arbitrary; taking the union over all $\sigma \in \mathbb{N}$ gives $\operatorname{HD}(W_{c,\xi}) \leq s$.

7.3. Preliminaries for $\geq$ direction.

7.3.1. *Multiplying numbers and sets.*

Definition 7.10. Let (Z, D) be a metric space. For each $S \subseteq Z$ and for each $a \geq 1$, let

$$aS := S^{\left(\frac{a-1}{2} \operatorname{Diam}(S)\right)},$$

where

$$S^{(r)} := \{z : D(z, S) \leq r\}$$

is the *r-thickening* of S.

Observation 7.11.

$$\operatorname{Diam}(aS) \leq a \operatorname{Diam}(S)$$
$$aB(z, r) \subseteq B(z, ar)$$
$$a(bS) \subseteq (ab)S.$$

For each inequality, equality holds if (Z, D) is a Banach space.

Proposition 7.12 (Variant of the Vitali covering lemma)**.** *Let (Z, D) be a metric space, and let $\mathcal{C}$ be a collection of subsets of Z with the property that*

$$\sup\{\operatorname{Diam}(S) : S \in \mathcal{C}\} < +\infty.$$

Then there exists a disjoint subcollection $\widetilde{\mathcal{C}} \subseteq \mathcal{C}$ so that

$$\bigcup \mathcal{C} \subseteq \bigcup_{A \in \widetilde{\mathcal{C}}} 5A.$$

The proof of this proposition is a straightforward generalization of the standard proof of the Vitali covering lemma (see e.g. Theorem 1.5.1 of [25]).

Lemma 7.13. *Let X be a hyperbolic metric space. For each $C, \sigma > 0$, there exists $\tau > 0$ such that for all $x \in X$*

$$C \operatorname{Shad}(x, \sigma) \subseteq \operatorname{Shad}(x, \tau).$$

Proof. Suppose $\eta \in C \operatorname{Shad}(x, \sigma)$; then $\eta \in B(\xi, \frac{C-1}{2} \operatorname{Diam}(\operatorname{Shad}(x, \sigma)))$ for some $\xi \in \operatorname{Shad}(x, \sigma)$. Now by the Diameter of Shadows Lemma

$$D(\xi, \eta) \lesssim_{\times, C} \operatorname{Diam}(\operatorname{Shad}(x, \sigma)) \lesssim_{\times, \sigma} b^{-d(o,x)};$$

thus

$$\langle \xi | \eta \rangle_o \gtrsim_{+, C, \sigma} d(o, x);$$

on the other hand since $\xi \in \operatorname{Shad}(x, \sigma)$

$$\langle x | \xi \rangle_o \asymp_{+, \sigma} d(o, x)$$

and so by Gromov's inequality

$$\langle x | \eta \rangle_o \asymp_{+, C, \sigma} d(o, x).$$

$\square$

7.3.2. *Fuzzy metric spaces.*

Definition 7.14. A *fuzzy metric space* consists of a metric space (Y, D) together with a function $f : Y \to (0, +\infty)$ satisfying

(7.8) $$f(y) \leq D(y, Y \setminus \{y\}).$$

Let (Y, f) be a fuzzy metric space. A ball $B(y, r) \subseteq Y$ is called *admissible* (with respect to f) if $r \geq f(y)$.

The idea is that each point $y \in Y$ represents some subset of a larger metric space Z which is contained in the set $B(y, f(y))$. An admissible ball is supposed to be one that contains this set.

Observation 7.15. Let $B(y, r) \subseteq Y$ be an admissible ball. Then $r \geq f(\widetilde{y})$ for every $\widetilde{y} \in B(y, r)$.

Proof. Since the case $\widetilde{y} = y$ holds by definition, let us suppose that $\widetilde{y} \neq y$. But then by (7.8) we have

$$r \geq D(\widetilde{y}, y) \geq D(\widetilde{y}, Y \setminus \{\widetilde{y}\}) \geq f(\widetilde{y}).$$

$\square$

Now fix $s > 0$. A measure $\mu \in \mathcal{M}(Y)$ is called *s-admissible* if $\mu(B(y, r)) \leq r^s$ for every admissible ball $B(y, r) \subseteq Y$. Consider the quantities

$$\alpha_{Y,f}(s) := \sup\{\mu(Y) : \mu \in \mathcal{M}(Y) \text{ is } s\text{-admissible}\}$$

$$\beta_{Y,f}(s) := \inf \left\{ \sum_{k=1}^{\infty} r_k^s : \big(B(y_k, r_k)\big)_1^{\infty} \text{ is a collection of admissible balls which covers } Y \right\}.$$

Observation 7.16.

$$\alpha_{Y,f}(s) \leq \beta_{Y,f}(s).$$

Proof. If $\mu \in \mathcal{M}(Y)$ is s-admissible and if $\big(B(y_k, r_k)\big)_1^{\infty}$ is a collection of admissible balls covering Y, then

$$\mu(Y) \leq \sum_{k=1}^{\infty} \mu(B(y_k, r_k)) \leq \sum_{k=1}^{\infty} r_k^s.$$

$\square$

Claim 7.17.

$$\beta_{Y,f}(s) \leq 32^s \alpha_{Y,f}(s).$$

Proof. Let $(y_n)_1^{\infty}$ be an indexing of Y. Fix $N \in \mathbb{N}$, and let $Y_N = \{y_1, \ldots, y_N\}$. The collection of s-admissible measures on Y_N is compact, and so there exists an s-admissible measure $\mu_N \in \mathcal{M}(Y_N)$ satisfying

(7.9) $$\mu_N(Y_N) = \sup_{\substack{\mu \in \mathcal{M}(Y_N) \\ s\text{-admissible}}} \mu(Y_N) \leq \alpha_{Y,f}(s).$$

Now fix $n \leq N$. It follows from (7.9) that the measure $\mu_N + \frac{1}{N}\delta_{y_n}$ is not s-admissible. By definition, this means that there is some admissible ball $B(y_{n,N}, r_{n,N}) \subseteq Y$ such that

$$(7.10) \qquad \left(\mu_N + \frac{1}{N}\delta_{y_n}\right)\left(B(y_{n,N}, r_{n,N})\right) > r_{n,N}^s \geq \mu_N(B(y_{n,N}, r_{n,N})).$$

Clearly, this implies

$$(7.11) \qquad\qquad\qquad\qquad y_n \in B(y_{n,N}, r_{n,N}).$$

Thus by Observation 7.15 we have

$$(7.12) \qquad\qquad\qquad\qquad r_{n,N} \geq f(y_n).$$

On the other hand, by (7.9) and (7.10) we have

$$r_{n,N} \leq \left(\alpha_{Y,f}(s) + 1\right)^{1/s}.$$

Thus, for each $n \in \mathbb{N}$ the sequence $(r_{n,N})_{N=n}^{\infty}$ is bounded from above and below.

Choose an increasing sequence $(N_k)_1^{\infty}$ such that for every $n \in \mathbb{N}$,

$$r_n^{(k)} := r_{n,N_k} \xrightarrow[k]{} r_n \in [f(y_n), +\infty).$$

Now fix $n \in \mathbb{N}$, and notice that for all $k \in \mathbb{N}$ sufficiently large we have

$$(7.13) \qquad\qquad\qquad\qquad \frac{1}{2}r_n \leq r_n^{(k)} \leq 2r_n$$

and thus by (7.11)

$$B(y_n^{(k)}, r_n^{(k)}) \subseteq B(y_n, 2r_n^{(k)}) \subseteq B(y_n, 4r_n).$$

Thus by (7.10) and (7.13) we have

$$\left(\mu_{N_k} + \frac{1}{N_k}\delta_{y_n}\right)\left(B(y_n, 4r_n)\right) > \left(r_n/2\right)^s$$

i.e.

$$(7.14) \qquad\qquad \mu_{N_k}\left(B(y_n, 4r_n)\right) > \left(r_n/2\right)^s - \frac{1}{N_k}.$$

Now by the Vitali covering lemma, there exists a set $A \subseteq \mathbb{N}$ such that the collection

$$\left(B(y_n, 4r_n)\right)_{n \in A}$$

is disjoint and such that

$$Y \subseteq \bigcup_{n=1}^{\infty} B(y_n, 4r_n) \subseteq \bigcup_{n \in A} B(y_n, 16r_n).$$

In particular, for each $k \in \mathbb{N}$ we have

$$\alpha_{Y,f}(s) \geq \mu_{N_k}(Y) \geq \sum_{n \in A} \mu_{N_k}\left(B(y_n, 4r_n)\right) \geq \sum_{\substack{n \in A \\ (7.13)\text{ holds}}} \left(\left(r_n/2\right)^s - \frac{1}{N_k}\right)_+,$$

where $x_+ = \max(0, x)$. Now let us take the limit as k approaches infinity. By the monotone convergence theorem, we may consider the limit of each summand separately. But if n is fixed, then (7.13) holds for all sufficiently large k. Thus

$$\alpha_{Y,f}(s) \geq \sum_{n \in A} \left(\left(r_n/2\right)^s - 0\right)_+ = \frac{1}{2^s}\sum_{n \in A} r_n^s.$$

On the other hand, note that for each $n \in A$, the ball $B(y_n, 16r_n)$ is admissible by (7.12) and (7.13). Thus the collection $\left(B(y_n, 16r_n)\right)_{n \in A}$ is a collection of admissible balls which covers Y, and so

$$\beta_{Y,f}(s) \leq \sum_{n \in A} (16r_n)^s = 16^s \sum_{n \in A} r_n^s.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7.4. $\geq$ direction. Now let X be a hyperbolic metric space, let G be a strongly discrete subgroup of $\mathrm{Isom}(X)$ of general type, and fix distinguished points $o \in X$ and $\xi \in \partial X$. Fix $c > 0$. Fix $s \in (0, \delta)$ which is a member of the supremum of Corollary 7.5(C), i.e. so that

$$(7.15) \qquad P_\xi(s) < \frac{\delta - s}{c}.$$

Fix $\tau \geq \sigma > 0$ large to be announced below. τ will depend on σ.

7.4.1. A fuzzy metric space for each $r > 0$. Let

$$\mathcal{P}_x := \mathrm{Shad}(x, \sigma) \qquad\qquad\qquad \widetilde{\mathcal{P}}_x := \mathrm{Shad}(x, \tau)$$
$$D_x := \mathrm{Diam}(\mathcal{P}_x) \qquad\qquad\qquad \widetilde{D}_x := \mathrm{Diam}(\widetilde{\mathcal{P}}_x).$$

Now fix $r > 0$ and let

$$S_{\xi,\tau}(r) := \{x \in G(o) : 10\widetilde{\mathcal{P}}_x \subseteq B(\xi, r)\}.$$

Observation 7.18.

$$\Lambda_{\mathrm{r},\sigma} \cap B(\xi, r) \subseteq \bigcup_{x \in S_{\xi,\tau}(r)} \mathcal{P}_x.$$

Proof. Fix $\eta \in \Lambda_{\mathrm{r},\sigma} \cap B(\xi, r)$. Then there exists a sequence $x_n \underset{n}{\to} \eta$ so that $\eta \in \bigcap_{n=1}^\infty \mathcal{P}_{x_n}$. Now since $\mathrm{Diam}(\widetilde{\mathcal{P}}_{x_n}) \underset{n}{\to} 0$, we have

$$10\widetilde{\mathcal{P}}_{x_n} \subseteq B(\xi, r)$$

for all n sufficiently large. Thus $x_n \in S_{\xi,\tau}(r)$ for all n sufficiently large, which completes the proof. $\qquad\square$

Now applying Proposition 7.12, we see that there exists a set $A_r \subseteq S_{\xi,\tau}(r)$ such that the collection

$$(7.16) \qquad\qquad \left(2\widetilde{\mathcal{P}}_x\right)_{x \in A_r}$$

is disjoint, and such that

$$\bigcup_{x \in S_{\xi,\tau}(r)} \mathcal{P}_x \subseteq \bigcup_{x \in S_{\xi,\tau}(r)} 2\widetilde{\mathcal{P}}_x \subseteq \bigcup_{x \in A_r} 10\widetilde{\mathcal{P}}_x.$$

Combining with Observation 7.18 and (1.6) yields

$$(7.17) \qquad\qquad H_{\xi,\sigma,s}(r) \leq \mathcal{H}_\infty^s\left(\bigcup_{x \in A_r} 10\widetilde{\mathcal{P}}_x\right).$$

Now for each $x \in A_r$, choose $\pi(x) \in \mathcal{P}_x$, and let

$$(7.18) \qquad\qquad f(\pi(x)) := \widetilde{D}_x \leq D(\pi(x), \partial X \setminus 2\widetilde{\mathcal{P}}_x).$$

Let

$$Y_r = \pi(A_r).$$

Then by the inequality of (7.18) and the disjointness of the collection (7.16), the pair (Y_r, f) is a fuzzy metric space.

Claim 7.19.

$$H_{\xi,\sigma,s}(r) \leq 7^s \beta_{Y_r,f}(s) \leq 224^s \alpha_{Y_r,f}(s).$$

Proof. Note that the second inequality is just Claim 7.17. To prove the first inequality, we prove the following:

Subclaim 7.20. *Let $\left(B(y_k, r_k)\right)_{k=1}^\infty$ be a collection of admissible balls which covers Y_r. Then*

$$\bigcup_{x \in A_r} 10\widetilde{\mathcal{P}}_x \subseteq \bigcup_{k=1}^\infty B(y_k, 7r_k).$$

Proof. Fix $\eta \in 10\widetilde{\mathcal{P}}_x$ for some $x \in A_r$. Then $\pi(x) \in Y_r$, so $\pi(x) \in B(y_k, r_k)$ for some $k \in \mathbb{N}$. Since $B(y_k, r_k)$ is admissible, by Observation 7.15 we have

$$r_k \geq f(\pi(x)) = \widetilde{D}_x$$

and thus

$$10\widetilde{\mathcal{P}}_x = \widetilde{\mathcal{P}}_x^{\left(\frac{9}{2}\widetilde{D}_x\right)} \subseteq \widetilde{\mathcal{P}}_x^{(5\widetilde{D}_x)} \subseteq B(\pi(x), 6\widetilde{D}_x) \subseteq B(\pi(x), 6r_k) \subseteq B(y_k, 7r_k).$$

◁

Thus

$$7^s \beta_{Y_r, f}(s) = \inf\left\{\sum_{k=1}^{\infty}(7r_k)^s : \left(B(y_k, r_k)\right)_1^{\infty} \text{ is a collection of admissible balls which covers } Y_r\right\}$$

$$\geq \inf\left\{\mathcal{H}_{\infty}^s\left(\bigcup_{k=1}^{\infty} B(y_k, 7r_k)\right) :\right.$$

$$\left. \left(B(y_k, r_k)\right)_1^{\infty} \text{ is a collection of admissible balls which covers } Y_r\right\}$$

$$\geq \mathcal{H}_{\infty}^s\left(\bigcup_{x \in A_r} 10\widetilde{\mathcal{P}}_x\right) \qquad\qquad \text{(by Subclaim 7.20)}$$

$$\geq H_{\xi, \sigma, s}(r) \qquad\qquad \text{(by (7.17))}.$$

□

Reading out the definition of $\alpha_{Y_r, f}(s)$, we come to the following conclusion:

Corollary 7.21. *There exists a measure $\nu_r \in \mathcal{M}(A_r)$ with*

$$(7.19) \qquad\qquad \nu_r(A_r) \geq \frac{1}{225^s} H_{\xi, \sigma, s}(r),$$

and such that the measure $\pi[\nu_r]$ is s-admissible.

7.4.2. *Construction of a tree.* By (7.15), there exist $t \in (s, \delta)$ and $\varepsilon > 0$ such that

$$(7.20) \qquad\qquad P_\xi(s) + \varepsilon \leq \frac{t - s - \varepsilon}{c}.$$

Now by (1.7), if σ is large enough, then there exists a sequence $r_k \underset{k}{\to} 0$ such that

$$\frac{\log_b(H_{\xi, \sigma, s}(r_k))}{\log_b(r_k)} \leq P_\xi(s) + \varepsilon$$

for all $k \in \mathbb{N}$. Rearranging yields

$$H_{\xi, \sigma, s}(r_k) \geq r_k^{P_\xi(s) + \varepsilon}$$

which together with (7.20) yields

$$(7.21) \qquad\qquad H_{\xi, \sigma, s}(r_k) \geq r_k^{(t - s - \varepsilon)/c}$$

for all $k \in \mathbb{N}$.

Now, since G is of general type, we have $h_1(\xi) \neq \xi$ for some $h_1 \in G$ (Observation 1.8). Choose $\varepsilon_2 > 0$ small enough so that

$$D\big(B(\xi, \varepsilon_2), h_1(B(\xi, \varepsilon_2))\big) > 0.$$

Let $\widetilde{B} = B(\xi, \varepsilon_2)$. Let $h_0 = \text{id}$.

By the Big Shadows Lemma, we may suppose that σ is large enough so that for all $z \in X$,

$$\text{Diam}(\partial X \setminus \text{Shad}_z(o, \sigma)) < D(\widetilde{B}, h_1(\widetilde{B})).$$

Thus, there exists $i = i_z = 0, 1$ such that $h_i(\widetilde{B}) \subseteq \text{Shad}_z(o, \sigma)$.

Fix $C > 0$ large and $\lambda > 0$ small to be announced below.

We will define a set $T \subseteq G(o)$,[35] a binary relation $\in_T \subseteq T \times T$, and a map $\mu : T \to [0,1]$ recursively as follows:[36]

1. Each time a new point is put into T, call it x and go to step 3.
2. Set $\mu(o) = 1$ and put o into T; it is the root node.
3. Recall that x is a point which has just been put into T, with $\mu(x) > 0$ defined. We will follow one of two different possible procedures, using the following test:
   i. If $\mu(x) \geq D_x^{t-\varepsilon}$, then go to step 4.
   ii. If there is no $k \in \mathbb{N}$ such that

$$(7.22) \qquad\qquad C^{-1} r_k \leq b^{-cd(o,x)} < r_k \leq \varepsilon_2,$$

   then go to step 4.
   iii. Otherwise, go to step 5.
4.   i. Label x as type 1.
   ii. Apply Lemma 5.14 (with $s = t$) to get a finite subset $T(x) \subseteq G(o)$.
   iii. For each $y \in T(x)$, set

$$\mu(y) = \frac{D_y^t}{\sum_{z \in T(x)} D_z^t} \mu(x),$$

   and then put y into T, setting $y \in_T x$.
5.   i. Label x as type 2.
   ii. Let $j = j_x \in G$ be the unique element so that $x = j_x(o)$.
   iii. Let $i = i_x = 0, 1$ be such that

$$h_i(\widetilde{B}) \subseteq \mathrm{Shad}_{j^{-1}(o)}(o, \sigma),$$

   and let $h = h_x = h_{i_x}$.
   iv. Let $g = g_x = j \circ h$. Note that applying j to the above equation we get

$$(7.23) \qquad\qquad g(\widetilde{B}) \subseteq \mathcal{P}_x.$$

   v. Choose $k = k_x \in \mathbb{N}$ so that (7.22) holds.
   vi. Let $\widetilde{B}_k = B(\xi_i, r_k)$; by the last inequality of (7.22), we have $\widetilde{B}_k \subseteq \widetilde{B}$. In particular, by (7.23) we have

$$(7.24) \qquad\qquad g(\widetilde{B}_k) \subseteq \mathcal{P}_x.$$

   vii. Let

$$\widetilde{T}(x) = g(A_{r_k})$$
$$\nu_x = g[\nu_{r_k}].$$

   where A_{r_k}, ν_{r_k} are as in §7.4.1.
   viii. For each $y \in \widetilde{T}(x)$, set

$$\mu(y) = \frac{\nu_x(y)}{\nu_x(\widetilde{T}(x))} \mu(x),$$

   and put y into T, setting $y \in_T x$.

Claim 7.22. *Let $x \in T$ be a type 2 point. Then the collection*

$$\left(3\mathcal{P}_y\right)_{y \in \widetilde{T}(x)}$$

is a disjoint collection of shadows which are contained in $g(\widetilde{B}_k)$, where $k = k_x$, assuming τ is sufficiently large.

Note that by (7.24), we therefore have $3\mathcal{P}_y \subseteq \mathcal{P}_x$.

---

[35]We will think of T as being a "tree"; however, it is not a tree in the strict sense according to Definition 5.2.

[36]You can imagine this as a program being run by an infinite computer which has infinitely many parallel processors. Each time a command of the form "for all X, do Y" is performed, the computation splits up, giving each X its own processor. This parallel processing is necessary, because otherwise the program would only compute one branch of the tree.

Proof of Claim 7.22. It is equivalent to require that the collection

$$\left(g^{-1}(3\mathcal{P}_{g(y)})\right)_{y \in A_{r_k}}$$

is a disjoint collection of sets which are contained in $\widetilde{B}_k$. But by the construction of A_{r_k}, the collection

$$\left(2\widetilde{\mathcal{P}}_y\right)_{y \in A_{r_k}}$$

is such a collection. Thus to complete the proof it suffices to demonstrate the following:

Subclaim 7.23.

(7.25)
$$g^{-1}\left(3\mathcal{P}_{g(y)}\right) \subseteq 2\widetilde{\mathcal{P}}_y$$

for every $y \in A_{r_k}$, assuming τ is sufficiently large.

Proof. Fix such a y. Let $z = g^{-1}(o)$, and note that

$$\begin{aligned}
\zeta := \pi(y) \in \mathcal{P}_y \subseteq 10\widetilde{\mathcal{P}}_y \subseteq \widetilde{B}_k \subseteq \widetilde{B} \subseteq g^{-1}(\mathcal{P}_x) \\
= \mathrm{Shad}_z(h_x(o), \sigma) \\
\subseteq \mathrm{Shad}_z(o, \sigma + d(o, h_x(o))),
\end{aligned}$$

and thus by Gromov's inequality

$$0 \asymp_{+,\sigma} \langle z|\zeta\rangle_o \gtrsim_+ \min(\langle y|z\rangle_o, \langle y|z\rangle_o).$$

On the other hand, since $\zeta \in \mathcal{P}_y$, we have

$$\langle y|\zeta\rangle_o \asymp_{+,\sigma} d(o, y) \geq \langle y|z\rangle_o,$$

and so

(7.26)
$$\langle y|z\rangle_o \asymp_{+,\sigma} 0.$$

Now let us demonstrate (7.25). Fix $\eta \in g^{-1}\left(3\mathcal{P}_{g(y)}\right)$; then

$$\langle y|\eta\rangle_z \asymp_{+,\sigma} d(y, z) \asymp_{+,\sigma} d(o, y) + d(o, z)$$

by (7.26). Now by (d) of Proposition 2.3

$$\langle y|\eta\rangle_o \gtrsim_+ \langle y|\eta\rangle_z - d(o, z) \asymp_{+,\sigma} d(o, y).$$

Letting τ be the implied constant of this asymptotic, we have $\eta \in \widetilde{\mathcal{P}}_y$. ◁

□

Let $T(\infty)$ be the set of all branches through T, i.e. the set of all sequences $\mathbf{x} = (x_n)_0^\infty$ for which

$$\cdots \in_T x_2 \in_T x_1 \in_T x_0 = o.$$

We have a map

$$\pi : T(\infty) \to \Lambda_{\mathrm{r},\sigma}(G)$$

defined as follows: If $\mathbf{x} = (x_n)_0^\infty \in T(\infty)$, then the sequence

$$(\mathcal{P}_{x_n})_0^\infty$$

is a decreasing sequence of closed sets whose diameters tend to zero. We define $\pi(\mathbf{x})$ to be the unique intersection point and we let $\mathcal{J} = \pi(T(\infty)) \subseteq \Lambda_{\mathrm{r},\sigma}$. By the Kolmogorov consistency theorem, there exists a unique measure $\mu \in \mathcal{M}(\mathcal{J})$ satisfying

$$\mu(\mathcal{P}_x) = \mu(x)$$

for all $x \in T$.

Claim 7.24.

$$\mathcal{J} \subseteq W_{c,\xi}.$$

Proof. Fix $\mathbf{x} = (x_n)_0^\infty \in T(\infty)$ and let $\eta = \pi(\mathbf{x}) \in \mathcal{J}$. For each $n \in \mathbb{N}$ let

(7.27) $$\mathcal{P}_n = \mathcal{P}_{x_n}; \quad D_n = D_{x_n}; \quad g_n = g_{x_n}.$$

Subclaim 7.25. *Infinitely many of the points $(x_n)_0^\infty$ are type 2.*

Proof. Suppose not; suppose that the points $x_N, x_{N+1}, \ldots$ are all type 1. Then for each $n \geq N$ we have

$$\mu(x_{n+1}) = \frac{D_{n+1}^t}{\sum_{z \in T(x_n)} D_z^t} \mu(x_n);$$

by (iii) of Lemma 5.14 we have

$$\sum_{z \in T(x_n)} D_z^t \geq D_n^t$$

and thus

$$\frac{\mu(x_{n+1})}{\mu(x_n)} \leq \frac{D_{n+1}^t}{D_n^t}.$$

Iterating gives

$$\frac{\mu(x_n)}{\mu(x_N)} \leq \frac{D_n^t}{D_N^t}.$$

It follows that for all n sufficiently large say $n \geq N_2 \geq N$, we have

(7.28) $$\mu(x_n) < D_n^{t-\varepsilon}.$$

Let $k \in \mathbb{N}$ be large enough so that

$$r_k < \min\left(b^{-cd(o,x_{N_2})}, \varepsilon_2\right).$$

Let $n > N_2$ be minimal so that

$$b^{-cd(o,x_n)} \leq r_k;$$

by the Diameter of Shadows Lemma and (ii) of Lemma 5.14 we have

$$b^{-cd(o,x_n)} \asymp_{\times,\sigma} r_k.$$

Let $C > 0$ be the implied constant of this asymptotic; then (7.22) holds. But we also have (7.28); it follows that x_n is type 2, contradicting our hypothesis that the points $x_N, x_{N+1}, \ldots$ are all type 1. ◁

The idea now is to associate each type 2 point in the sequence $(x_n)_0^\infty$ to a good approximation of η. Fix $n \in \mathbb{N}$ so that x_n is type 2. Write $i_n = i_{x_n}$ and $k_n = k_{x_n}$.

Now by Claim 7.22 we have

$$\eta \in \mathcal{P}_{x_{n+1}} \subseteq g_n(\widetilde{B}_{k_n}),$$

i.e.

$$D(\xi, g_n^{-1}(\eta)) \leq r_{k_n}.$$

Note that $g_n(\xi) \in \mathcal{P}_{x_n}$ by (7.23), and so

$$\xi, g_n^{-1}(\eta) \in g_n^{-1}(\mathcal{P}_{x_n}) \subseteq \mathrm{Shad}_{g_n^{-1}(o)}(o, \sigma + d(o, h_1(o)))$$

Thus by (ii) of the Bounded Distortion Lemma we have

$$D(g_n(\xi), \eta) \asymp_{\times,\sigma} b^{-d(o,g_n(o))} D(\xi, g_n^{-1}\eta)$$
$$\leq b^{-d(o,g_n(o))} r_{k_n}$$
$$\asymp_{\times,\sigma} b^{-d(o,g_n(o))} b^{-cd(o,x_n)}$$
$$\asymp_\times b^{-(c+1)d(o,g_n(o))}.$$

Thus

$$\omega_\xi(\eta) \geq \limsup_{\substack{n \to \infty \\ x_n \text{ type 2}}} \frac{-\log_b D(g_n(\xi), \eta)}{d(o, g_n(o))} \geq 1 + c,$$

i.e. $\eta \in W_{c,\xi}$. □

It is left to show that $\mathrm{HD}(\mathcal{J}) \geq s$.

Claim 7.26. $\mu(B(\eta, r)) \lesssim_{\times, \sigma} r^s$ for all $r > 0$ and for all $\eta \in \mathcal{J}$.

Proof. Fix $\mathbf{x} = (x_n)_0^\infty \in T(\infty)$ so that $\eta = \pi(\mathbf{x})$. As before we use the abbreviations (7.27). Let $n = n(\eta, r) \in \mathbb{N}$ be maximal so that $D_n \geq r$. Then

$$B(\eta, r) \subseteq B(\eta, D_n) \subseteq 3\mathcal{P}_n.$$

Since $3\mathcal{P}_n \cap \mathcal{J} \subseteq \mathcal{P}_n$, we have $B(\eta, r) \cap \mathcal{J} \subseteq \mathcal{P}_n$.

On the other hand, the maximality of n implies that

(7.29) $$r > D_{n+1}.$$

Now we divide into cases depending on whether $x = x_n$ is a type 1 point or a type 2 point.

Case 1: x is type 2. In this case, let

$$i = i_n = i_{x_n}, \quad k = k_n = k_{x_n}, \text{ and } g = g_n = g_{x_n}.$$

Then

$$\mu(B(\eta, r)) \leq \sum_{\substack{y \in \widetilde{T}(x) \\ \mathcal{P}_y \cap B(\eta, r) \neq \varnothing}} \mu(y) = \sum_{\substack{y \in A_{r_k} \\ \mathcal{P}_{g(y)} \cap B(\eta, r) \neq \varnothing}} \frac{\nu_{r_k}(y)}{\nu_{r_k}(A_{r_k})} \mu(x).$$

On the other hand, since x is type 2 we have

$$\mu(x) \leq D_x^{t-\varepsilon} \asymp_{\times, \sigma} b^{-(t-\varepsilon)d(o,x)};$$

moreover, by (7.19) and (7.21) we have

$$\nu_{r_k}(A_{r_l}) \gtrsim_\times H_{\xi, \sigma, s}(r_k) \geq r_k^{(t-s-\varepsilon)/c} \asymp_{\times, \sigma} b^{-(t-s-\varepsilon)d(o,x)}.$$

Combining gives

(7.30) $$\mu(B(\eta, r)) \lesssim_{\times, \sigma} b^{-sd(o,x)} \sum_{\substack{y \in A_{r_k} \\ \mathcal{P}_{g(y)} \cap B(\eta, r) \neq \varnothing}} \nu_{r_k}(y).$$

Subclaim 7.27. *For every $y \in A_{r_k}$ for which $\mathcal{P}_{g(y)} \cap B(\eta, r) \neq \varnothing$, we have*

$$g(\mathcal{P}_y) \subseteq B(\eta, C_1 r),$$

where $C_1 > 0$ is independent of n.

Proof. We claim first that $r \geq D_{g(y)}$.

Case 1: $g(y) = x_{n+1}$. Then

$$r > D_{n+1} = D_{g(y)}.$$

Case 2: $g(y) \neq x_{n+1}$. Then

$$r \geq D(\mathcal{P}_{g(y)}, \eta) \geq D(\mathcal{P}_{g(y)}, \partial X \setminus 3\mathcal{P}_{g(y)}) \geq D_{g(y)}.$$

Subclaim 7.28. *For all $\zeta \in \mathcal{P}_y$ we have*

(7.31) $$\langle o | g(\zeta) \rangle_{g(y)} \asymp_{+, \sigma} 0.$$

Proof. Since

$$\zeta \in \mathcal{P}_y \subseteq \widetilde{B}_k \subseteq \widetilde{B} \subseteq g^{-1}(\mathcal{P}_x),$$

we have

$$\langle g^{-1}(o) | \zeta \rangle_o = \langle o | g(\zeta) \rangle_{g(o)} \asymp_+ \langle o | g(\zeta) \rangle_x \asymp_{+, \sigma} 0.$$

On the other hand, since $\zeta \in \mathcal{P}_y$

$$\langle y | \zeta \rangle_o \asymp_{+, \sigma} d(o, y) \underset{n}{\to} +\infty.$$

Gromov's inequality yields

$$\langle g^{-1}(o) | y \rangle_o \asymp_{+, \sigma} 0.$$

Thus
$$\langle o|g^{-1}(o)\rangle_y \asymp_{+,\sigma} d(o,y) \underset{n}{\to} +\infty.$$
On the other hand
$$\langle o|\zeta\rangle_y \asymp_{+,\sigma} 0;$$
Gromov's inequality yields
$$\langle o|g(\zeta)\rangle_{g(y)} = \langle g^{-1}(o)|\zeta\rangle_y \asymp_{+,\sigma} 0.$$

◁

Letting $\widetilde{\sigma} \geq \sigma$ be the implied constant of the asymptotic (7.31), we have $g(\mathcal{P}_y) \subseteq \mathrm{Shad}(g(y),\widetilde{\sigma})$. Thus
$$D(\zeta,\eta) \leq r + \mathrm{Diam}(\mathrm{Shad}(g(y),\widetilde{\sigma})) \asymp_\times r + D_{g(y)} \lesssim_\times r,$$
completing the proof.

◁

Thus

$$(7.32) \qquad \sum_{\substack{y \in A_{r_k} \\ \mathcal{P}_{g(y)} \cap B(\eta,r) \neq \varnothing}} \nu_{r_k}(y) \leq \sum_{\substack{y \in A_{r_k} \\ g(\mathcal{P}_y) \subseteq B(\eta,C_1 r)}} \nu_{r_k}(y) \leq \pi[\nu_{r_k}](g^{-1}(B(\eta,C_1 r))).$$

Now,

$$\begin{aligned}
\widetilde{D}_{g^{-1}(x_{n+1})} &\asymp_{\times,\sigma} b^{-d(o,g^{-1}(x_{n+1}))} \asymp_\times b^{-d(x_n,x_{n+1})} \\
&\asymp_{\times,\sigma} b^{d(o,x_n)-d(o,x_{n+1})} && \text{(by the Intersecting Shadows Lemma)} \\
&\asymp_{\times,\sigma} b^{d(o,x_n)} D_{n+1} && \text{(by the Diameter of Shadows Lemma)} \\
&\leq b^{d(o,x_n)} r. && \text{(by (7.29))}
\end{aligned}$$

Let $C_2 > 0$ be the implied constant of this asymptotic; then

$$(7.33) \qquad \widetilde{D}_{g^{-1}(x_{n+1})} \leq C_2 b^{d(o,x)} r.$$

Subclaim 7.29.
$$g^{-1}\big(B(\eta,C_1 r)\big) \subseteq U := B(\pi(g^{-1}(x_{n+1})), C_3 b^{d(o,x)} r).$$
for some $C_3 \geq 2C_2$ independent of n.

Proof. Fix $\zeta \in g^{-1}(B(\eta,C_1 r))$. Then by (d) of Proposition 2.3, we have
$$D(g^{-1}(\eta),\zeta) \lesssim_{\times,\sigma} b^{d(o,g(o))} D(\eta,g(\zeta)) \leq C_1 b^{d(o,g(o))} r \asymp_{\times,\sigma} b^{d(o,x)} r.$$
On the other hand, since $\eta \in \mathcal{P}_{x_{n+1}}$, Subclaim 7.23 shows that
$$g^{-1}(\eta) \in 2\widetilde{\mathcal{P}}_{g^{-1}(x_{n+1})}$$
and thus
$$D\big(g^{-1}(\eta),\pi(g^{-1}(x_{n+1}))\big) \leq 2\widetilde{D}_{g^{-1}(x_{n+1})} \leq 2C_2 b^{d(o,x)} r.$$
This completes the proof.

◁

Combining (7.30), (7.32), and Subclaim 7.29 gives

$$(7.34) \qquad \mu(B(\eta,r)) \lesssim_{\times,\sigma} b^{-sd(o,x)} \pi[\nu_{r_k}](U).$$

Now by (7.33) we have
$$f(\pi(g^{-1}(x_{n+1}))) = D_{g^{-1}(x_{n+1})} \leq 2\widetilde{D}_{g^{-1}(x_{n+1})} \leq 2C_2 b^{d(o,x)} r \leq C_3 b^{d(o,x)},$$
and so the ball U is admissible. On the other hand, the measure $\pi[\nu_{r_k}]$ is s-admissible; it follows that
$$\pi[\nu_{r_k}](U) \leq \big(C_3 b^{d(o,x)} r\big)^s.$$

Combining with (7.34), we have

$$\mu(B(\eta, r)) \lesssim_\times b^{-sd(o,x)} \big(b^{d(o,x)}r\big)^s = r^s.$$

Case 2: x is type 1. In this case, by (7.29) and (ii) of Lemma 5.14 we have

$$r \geq D_{n+1} \asymp_\times D_n.$$

On the other hand, since $B(\eta, r) \cap \mathcal{J} \subseteq \mathcal{P}_n$, we have

$$\mu(B(\eta, r)) \leq \mu(x_n).$$

Thus to complete the proof of the claim it suffices to show

(7.35)
$$\mu(x_n) \lesssim_\times D_n^s.$$

Let $m < n$ be the largest number such that x_m is a type 2 point. (If no such number exists, let $m = 0$.) Let

$$\widetilde{r} = \min(2D_{m+1}, D_m) > D_{m+1};$$

then $n(\eta, \widetilde{r}) = m$. Thus since x_m is type 2, the ball $B(\eta, \widetilde{r})$ falls into Case 1: So by our earlier argument, we have

(7.36)
$$\mu(B(\eta, \widetilde{r})) \lesssim_\times \widetilde{r}^s.$$

(If $m = 0$, then (7.36) holds just because the right hand side is asymptotic to 1.) On the other hand, $\mathcal{P}_{m+1} \subseteq B(\eta, D_{m+1}) \subseteq B(\eta, \widetilde{r})$, and so

$$\mu(x_{m+1}) \leq \mu(B(\eta, \widetilde{r})) \lesssim_\times D_{m+1}^s,$$

i.e. (7.35) is satisfied for $n = m + 1$. On the other hand, since the points $x_{m+1}, \ldots, x_{n-1}$ are all type 1, the argument used in the proof of Subclaim 7.25 shows that

$$\frac{\mu(x_n)}{\mu(x_{m+1})} \leq \frac{D_n^t}{D_{m+1}^t} \leq \frac{D_n^s}{D_{m+1}^s},$$

since $t > s$ by (7.20). This demonstrates (7.35).

$\square$

Now by the mass distribution principle we have $\mathrm{HD}(\mathcal{J}) \geq s$, completing the proof.

7.5. **Special cases.** In this subsection we prove the assertions made in §1.4.1 concerning special cases of Theorem 7.1, except for Theorem 7.33 which will be proven in Subsection 7.6 below.

Example 7.30. If $\xi \in \Lambda_\mathrm{r}$ then $P_\xi(s) = s$. Thus in this case (7.1) reduces to

(7.37)
$$\mathrm{HD}(W_{c,\xi}) = \frac{\delta}{c+1},$$

i.e. the inequality of (7.1) is an equality.

Proof. In fact, this is a special case of Example 7.31 below, since if $\xi \in \Lambda_\mathrm{r}$ then $\beta(\xi) = 1$. $\square$

Example 7.31. If ξ is a horospherical limit point of G then $P_\xi(s) \leq 2s$. Thus in this case we have

$$\frac{\delta}{2c+1} \leq \mathrm{HD}(W_{c,\xi}) \leq \frac{\delta}{c+1}.$$

More generally, let $\langle \cdot, \cdot \rangle$ denote the Gromov product. If we let

(7.38)
$$\beta = \beta(\xi) := \limsup_{\substack{g \in G \\ d(o, g(o)) \to \infty}} \frac{\langle g(o) | \xi \rangle_o}{d(o, g(o))}$$

then $P_\xi(s) \leq \beta^{-1}s$ and so

$$\frac{\delta}{\beta^{-1}c+1} \leq \mathrm{HD}(W_{c,\xi}) \leq \frac{\delta}{c+1}.$$

In particular, if $\beta > 0$ then $\mathrm{HD}(\mathrm{VWA}_\xi) = \delta$.

Proof. Fix $s \in (0, \delta)$. Since $\mathrm{HD}(\Lambda_{\mathrm{r}}) = \delta > s$, there exists $\sigma > 0$ so that $\mathrm{HD}(\Lambda_{\mathrm{r},\sigma}) > s$. Then $\mathcal{H}^s_\infty(\Lambda_{\mathrm{r},\sigma}) > 0$. Let $\tau > 0$ be guaranteed by Lemma 2.39.

Let $h_1, h_2 \in G$ be as in the definition of general type. Then there exists $\varepsilon > 0$ so that for all $\xi \in \partial X$,

$$(7.39) \qquad \max_{i=1}^{2} D(\xi, h_i(\xi)) > \varepsilon.$$

Let $\widetilde{\sigma} > 0$ be given by the Big Shadows Lemma.

Now fix $g \in G$. By the Big Shadows Lemma, we have

$$\mathrm{Diam}(\partial X \setminus \mathrm{Shad}_{g^{-1}(o)}(o, \widetilde{\sigma})) \leq \varepsilon,$$

and thus by (7.39) we have

$$\partial X = S_g \cup h_1^{-1}(S_g) \cup h_2^{-1}(S_g), \text{ where } S_g = \mathrm{Shad}_{g^{-1}(o)}(o, \widetilde{\sigma}).$$

Let $h_0 = \mathrm{id}$. Then there exists $i = i_g = 0, 1, 2$ such that

$$\mathcal{H}^s_\infty(\Lambda_{\mathrm{r},\sigma} \cap h_i^{-1}(S_g)) \geq \frac{1}{3}\mathcal{H}^s_\infty(\Lambda_{\mathrm{r},\sigma}) \asymp_{\times,\sigma} 1.$$

Thus

$$
\begin{aligned}
b^{-sd(o,g(o))} \asymp_\times \quad & b^{-sd(o,g \circ h_i(o))} \\
\lesssim_{\times,\sigma}\ & \mathcal{H}^s_\infty\big(g \circ h_i\big(\Lambda_{\mathrm{r},\sigma} \cap h_i^{-1}(S_g)\big)\big) \\
\leq\ & \mathcal{H}^s_\infty(\Lambda_{\mathrm{r},\tau} \cap g(S_g)) \qquad\qquad \text{(by Lemma 2.39)} \\
=\ & \mathcal{H}^s_\infty(\Lambda_{\mathrm{r},\tau} \cap \mathrm{Shad}(g(o), \widetilde{\sigma})).
\end{aligned}
$$

Now fix $\eta \in \mathrm{Shad}(g(o), \widetilde{\sigma})$. We have

$$
\begin{aligned}
\langle \eta | \xi \rangle_o \gtrsim_+ \quad & \min(\langle g(o) | \eta \rangle_o, \langle g(o) | \xi \rangle_o) \\
\asymp_{+,\widetilde{\sigma}}\ & \min(d(o, g(o)), \langle g(o) | \xi \rangle_o) \qquad\qquad \text{(since } \eta \in \mathrm{Shad}(g(o), \widetilde{\sigma})) \\
=\ & \langle g(o) | \xi \rangle_o,
\end{aligned}
$$

i.e.

$$\mathrm{Shad}(g(o), \widetilde{\sigma}) \subseteq B(\xi, Cb^{-\langle g(o) | \xi \rangle_o})$$

for some $C > 0$ independent of G. Thus

$$b^{-sd(o,g(o))} \lesssim_{\times,\sigma} \mathcal{H}^s_\infty\big(\Lambda_{\mathrm{r},\tau} \cap B(\xi, Cb^{-\langle g(o),\xi\rangle_o})\big) = H_{\xi,\tau,s}(Cb^{-\langle g(o),\xi\rangle_o}).$$

Let $(g_n)_1^\infty$ be a sequence so that

$$\frac{\langle g_n(o) | \xi \rangle_o}{d(o, g_n(o))} \xrightarrow[n]{} \beta \text{ and } d(o, g_n(o)) \xrightarrow[n]{} +\infty.$$

In particular, $\langle g_n(o) | \xi \rangle_o \xrightarrow[n]{} +\infty$, and thus

$$
\begin{aligned}
P_\xi(s) &\leq \liminf_{r \to 0} \frac{\log_b(H_{\xi,\tau,s}(r))}{\log_b(r)} \\
&\leq \liminf_{n \to \infty} \frac{\log_b(H_{\xi,\tau,s}(Cb^{-\langle g_n(o),\xi\rangle_o}))}{\log_b(Cb^{-\langle g_n(o),\xi\rangle_o})} \\
&\leq \liminf_{n \to \infty} \frac{-sd(o, g_n(o))}{-\langle g_n(o), \xi \rangle_o} = \beta^{-1} s.
\end{aligned}
$$

Finally, if ξ is a horospherical limit point, let $(g_n(o))_1^\infty$ be a sequence tending horospherically to ξ; then

$$
\begin{aligned}
\beta(\xi) &\geq \limsup_{n\to\infty} \frac{\langle g_n(o) | \xi \rangle_o}{d(o, g_n(o))} \\
&= \limsup_{n\to\infty} \frac{1}{2} \frac{d(o, g_n(o)) + \mathcal{B}_\xi(o, g_n(o))}{d(o, g_n(o))} && \text{(by (g) of Proposition 2.3)} \\
&\geq \limsup_{n\to\infty} \frac{1}{2} \frac{d(o, g_n(o))}{d(o, g_n(o))} && \text{(since } x_n \underset{n}{\to} \xi \text{ horospherically)} \\
&= 1/2.
\end{aligned}
$$

$\square$

Proposition 7.32. *There exists a (discrete) nonelementary Fuchsian group G and a point $\xi \in \Lambda$ such that* $\mathrm{HD}(\mathrm{VWA}_\xi) = 0$.

Proof. Let D_e denote the Euclidean metric on $\mathbb{C}$, and let D_s denote the spherical metric on $\partial\mathbb{H}^2$, i.e. the metric $|\mathrm{d}z|/(|z|^2 + 1)$. For each $n \in \mathbb{N}$, let $a_n = 2^{n!}$ and let $g_n \in \mathrm{Isom}(\mathbb{H}^2)$ be a Möbius transformation such that

$$
g_n\left(\widehat{\mathbb{C}} \setminus B_\mathrm{e}(-a_n, 1/a_{n+2})\right) = B_\mathrm{e}(a_n, 1/a_{n+2}).
$$

The group G generated by the sequence $(g_n)_1^\infty$ is a Schottky group and is therefore discrete. Clearly, ∞ is a limit point of G.

We claim that $\mathrm{HD}(\mathrm{VWA}_\infty) = 0$. By (7.2), it suffices to show that

$$
P_\infty(s) = +\infty \quad \forall s \in (0, \delta).
$$

Fix $s \in (0, \delta)$ and $0 < r \leq 1$. Computation shows that

$$
B_\mathrm{s}(\infty, r) \subseteq \widehat{\mathbb{C}} \setminus B_\mathrm{e}(0, 1/(2r)).
$$

There exists $n = n_r \in \mathbb{N}$ so that

$$
a_n \leq \frac{1}{2r} < a_{n+1}.
$$

Then

$$
B_\mathrm{s}(\infty, r) \cap \Lambda \subseteq \{\infty\} \cup \bigcup_{m \geq n} \left[B_\mathrm{e}(a_m, 1/a_{m+2}) \cup B_\mathrm{e}(-a_m, 1/a_{m+2}) \right]
$$

and thus for all $\sigma > 0$

$$
\begin{aligned}
H_{\xi,\sigma,s}(r) &\leq \sum_{m \geq n} \left[\mathrm{Diam}_\mathrm{s}(B_\mathrm{e}(a_m, 1/a_{m+2})) + \mathrm{Diam}_\mathrm{s}(B_\mathrm{e}(-a_m, 1/a_{m+2})) \right] \\
&\asymp_\times \sum_{m \geq n} \frac{1}{a_{m+2} a_m^2} \\
&\lesssim_\times \frac{1}{a_{n+2}} \\
&= \frac{1}{a_{n+1}^{n+2}} \leq (2r)^{n_r+2}.
\end{aligned}
$$

Thus

$$
P_\infty(s) \geq \liminf_{r\to 0} \frac{\log((2r)^{n_r+2})}{\log(r)} = \liminf_{r\to 0} [n_r + 2] = +\infty.
$$

$\square$

7.6. **Proof of Theorem 7.33.** In this subsection we prove Theorem 7.33. The proof requires several lemmas.

Theorem 7.33. *Let (X, d, o, b, G) be as in §1.1.4, and let ξ be a bounded parabolic point of G. Let δ and δ_ξ be the Poincaré exponents of G and G_ξ, respectively. Then for all $s \in (0, \delta)$*

$$(7.40) \qquad P_\xi(s) = \max(s, 2s - 2\delta_\xi) = \begin{cases} s & s \geq 2\delta_\xi \\ 2s - 2\delta_\xi & s \leq 2\delta_\xi \end{cases}.$$

Thus for each $c > 0$

$$(7.41) \qquad \mathrm{HD}(W_{c,\xi}) = \begin{cases} \delta/(c+1) & c+1 \geq \delta/(2\delta_\xi) \\ \dfrac{\delta + 2\delta_\xi c}{2c+1} & c+1 \leq \delta/(2\delta_\xi) \end{cases}.$$

In this subsection, D_e denotes the Euclidean metametric $D_{\xi,o}$, and D_s denotes the spherical metametric D_o.

Definition 7.34. Let G_ξ be the stabilizer of ξ relative to G. We define the *orbital counting function* of G_ξ with respect to the metametric D_e to be the function $f_\xi : [0, +\infty) \to \mathbb{N}$ defined by

$$(7.42) \qquad f_\xi(R) := \#\{g \in G_\xi : D_\mathrm{e}(o, g(o)) \leq R\}.$$

Before proceeding further, let us remark on the function (7.42). In the case where X is a real, complex, or quaternionic hyperbolic space, f_ξ satisfies a power law, i.e. $f_\xi(R) \asymp_\times R^{2\delta_\xi}$, where δ_ξ is the Poincaré exponent of G_ξ (Lemma 3.5 of [54], see also Proposition 2.10 of [64]). However, this is not the case for more general spaces, including the infinite-dimension space $\mathbb{H}^\infty$ and proper $\mathbb{R}$-trees; indeed, there are essentially no restrictions on the orbital counting function of a parabolic group acting on such spaces. See Appendix A for details.

Let us now give a rough outline of the proof of Theorem 7.33. Let

$$H^*_{\xi,\sigma,s}(R) = \mathcal{H}^s_{\mathrm{s},\infty}(\Lambda_{\mathrm{r},\sigma} \setminus B_\mathrm{e}(o, R)).$$

Then by Lemma 2.20, (1.7) can be rewritten as

$$(7.43) \qquad P_\xi(s) = \lim_{\sigma \to \infty} \liminf_{R \to \infty} \frac{\log(H^*_{\xi,\sigma,s}(R))}{\log(1/R)}.$$

So to compute $P_\xi(s)$, we will compute $H^*_{\xi,\sigma,s}(R)$ for all $R \geq 1$ sufficiently large. Since the spherical metric is locally comparable to the Euclidean metric via (2.20), we will compute $H^*_{\xi,\sigma,s}(R)$ by first computing $\mathcal{H}^s_{\mathrm{e},\infty}$ of appropriately chosen subsets of $\Lambda_{\mathrm{r},\sigma} \setminus B_\mathrm{e}(o, R)$. Specifically, we will consider the sets

$$S_{R,\alpha} := \bigcup_{\substack{g \in G_\xi \\ \alpha R < D_\mathrm{e}(o, g(o)) \leq R}} g(S \cap \Lambda_{\mathrm{r},\sigma}),$$

where $R \geq 1$ and $\alpha \in [0, 1]$. Our first step is to estimate $\mathcal{H}^s_{\mathrm{e},\infty}(S_{R,\alpha})$:

Lemma 7.35. *Let S be a ξ-bounded set containing o, and let $\sigma > 0$. Fix $R \geq C_0 \mathrm{Diam}_\mathrm{e}(S)$ and $\alpha \in [0, 1]$. Then*

(i)

$$(7.44) \qquad \mathcal{H}^s_{\mathrm{e},\infty}(S_{R,0}) \lesssim_{\times,S} \frac{f_\xi((C_0+1)R)}{\sup_{Q \in [1,R]}(f_\xi(Q)/Q^s)}.$$

(ii) *If*

$$(7.45) \qquad \mathcal{H}^s_{\mathrm{e},\infty}(S \cap \Lambda_{\mathrm{r},\sigma}) > 0,$$

then for all $\alpha \in [0, 1]$

$$(7.46) \qquad \mathcal{H}^s_{\mathrm{e},\infty}(S_{R,\alpha}) \gtrsim_{\times,S,\sigma} \frac{f_\xi(R) - f_\xi(\alpha R)}{\sup_{Q \in [1,12C_0R]}(f_\xi(Q)/Q^s)}.$$

Remark 7.36. In the proof of Lemma 7.35, we will not indicate the dependence of implied constants on the ξ-bounded set S.

Proof of (i). Fix $Q \in [1, R]$, and let A_Q be a maximal $2C_0Q$-separated subset of $B_e(o, R) \cap G_\xi(o)$. Then the sets $\big(B_e(x, C_0Q) \cap G_\xi(o)\big)_{x \in A_Q}$ are disjoint, so

$$
\begin{aligned}
\#(B_e(o, R + C_0Q) \cap G_\xi(o)) &\geq \sum_{x \in A_Q} \#(B_e(x, C_0Q) \cap G_\xi(o)) \\
&\geq \sum_{x \in A_Q} \#(B_e(o, Q) \cap G_\xi(o)) \qquad \text{(by Observation 2.42)} \\
&= f_\xi(Q)\#(A_Q),
\end{aligned}
$$

and thus

$$
\#(A_Q) \leq \frac{f_\xi(R + C_0Q)}{f_\xi(Q)} \leq \frac{f_\xi((C_0 + 1)R)}{f_\xi(Q)}.
$$

On the other hand, the maximality of A_Q implies

$$
B_e(o, R) \cap G_\xi(o) \subseteq \bigcup_{x \in A_Q} B_e(x, 2C_0Q)
$$

and thus by Observation 2.42

$$
S_R := S_{R,0} \subseteq \bigcup_{x \in B_e(o,R) \cap G_\xi(o)} B_e(x, C_0 \operatorname{Diam}_e(S)) \subseteq \bigcup_{x \in A_Q} B_e(x, 2C_0Q + C_0 \operatorname{Diam}_e(S)),
$$

i.e. $\big(B_e(x, 2C_0Q + C_0 \operatorname{Diam}_e(S))\big)_{x \in A_Q}$ is a cover of S_R. Thus

$$
\begin{aligned}
\mathcal{H}^s_{e,\infty}(S_R) &\leq \sum_{x \in A_Q} \big(2C_0Q + C_0 \operatorname{Diam}_e(S)\big)^s \\
&= \#(A_Q)\big(2C_0Q + C_0 \operatorname{Diam}_e(S)\big)^s \\
&\lesssim_\times \frac{f_\xi((C_0 + 1)R)}{f_\xi(Q)}Q^s,
\end{aligned}
$$

since $Q \geq 1$ by assumption, and C_0 and $\operatorname{Diam}_e(S)$ are constants. Thus

$$
R^{-2s}\mathcal{H}^s_{e,\infty}(S_R) \lesssim_\times \frac{f_\xi((C_0 + 1)R)/((C_0 + 1)R)^{2s}}{f_\xi(Q)/Q^s},
$$

and taking the infimum over all $Q \in [1, R]$ finishes the proof. $\qquad\square$

Proof of (ii). Let $\mathcal{C}$ satisfy $\bigcup(\mathcal{C}) = S_{R,\alpha}$.

Claim 7.37. *For all $A \subseteq \mathcal{E}_\xi$,*

$$
\#\{g \in G_\xi : g(S) \cap A \neq \varnothing\} \leq f_\xi(C_0 \operatorname{Diam}_e(A) + 2C_0^2 \operatorname{Diam}_e(S)).
$$

Proof. Suppose that the left hand side is nonzero; then there exists $g_0 \in G_\xi$ such that $g_0(S) \cap A \neq \varnothing$. Then

$$
\begin{aligned}
\#\{g \in G_\xi : g(S) \cap A \neq \varnothing\} &\leq \#\Big\{g \in G_\xi : D_e(g_0(o), g(o)) \leq \operatorname{Diam}_e(A) + 2C_0 \operatorname{Diam}_e(S)\Big\} \\
&\leq f_\xi(C_0 \operatorname{Diam}_e(A) + 2C_0^2 \operatorname{Diam}_e(S)).
\end{aligned}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\triangleleft$

Let

$$
\mathcal{C}_1 = \{A \in \mathcal{C} : \operatorname{Diam}_e(A) \leq C_0 \operatorname{Diam}_e(S)\},
$$

and let $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1$. Then by Claim 7.37, we have

$$
A \in \mathcal{C}_1 \Rightarrow \#\{g \in G_\xi : g(S) \cap A \neq \varnothing\} \leq f_\xi(3C_0^2 \operatorname{Diam}_e(S)) \asymp_\times 1,
$$

and thus

$$
\begin{aligned}
\sum_{A \in \mathcal{C}_1} \mathrm{Diam}_{\mathrm{e}}^s(A) &\gtrsim_{\times} \sum_{\substack{g \in G_\xi}} \sum_{\substack{A \in \mathcal{C}_1 \\ g(S) \cap A \neq \varnothing}} \mathrm{Diam}_{\mathrm{e}}^s(A) \\
&\geq \sum_{\substack{g \in G_\xi \\ g(S \cap \Lambda_{\mathrm{r},\sigma}) \subseteq \bigcup(\mathcal{C}_1)}} \mathcal{H}_{\mathrm{e},\infty}^s(g(S \cap \Lambda_{\mathrm{r},\sigma})) \\
&\asymp_{\times} \sum_{\substack{g \in G_\xi \\ g(S \cap \Lambda_{\mathrm{r},\sigma}) \subseteq \bigcup(\mathcal{C}_1)}} \mathcal{H}_{\mathrm{e},\infty}^s(S \cap \Lambda_{\mathrm{r},\sigma}) \qquad \text{(by Observation 2.42)} \\
&\asymp_{\times,\sigma} \#\left\{ g \in G_\xi : g(S \cap \Lambda_{\mathrm{r},\sigma}) \subseteq \bigcup(\mathcal{C}_1) \right\}. \qquad \text{(by (7.45))}
\end{aligned}
$$

On the other hand, Claim 7.37 implies that for all $A \in \mathcal{C}_2$, we have

$$
\#\{ g \in G_\xi : g(S) \cap A \neq \varnothing \} \leq f_\xi(3C_0 \, \mathrm{Diam}_{\mathrm{e}}(A))
$$

Moreover, since $\bigcup(\mathcal{C}) = S_R$, we have $A \subseteq S_R$ and so

$$
3C_0 \, \mathrm{Diam}_{\mathrm{e}}(A) \leq 3C_0 \, \mathrm{Diam}_{\mathrm{e}}(S_R) \leq 3C_0(2R + 2C_0 \, \mathrm{Diam}_{\mathrm{e}}(S)) \leq 12C_0 R,
$$

since $R \geq C_0 \, \mathrm{Diam}_{\mathrm{e}}(S)$. Thus

$$
\begin{aligned}
\#\{ g \in G_\xi : g(S) \cap A \neq \varnothing \} &\leq f_\xi(3C_0 \, \mathrm{Diam}_{\mathrm{e}}(A)) \\
&\leq (3C_0 \, \mathrm{Diam}_{\mathrm{e}}(A))^s \sup_{Q \in [1, 12C_0 R]} (f_\xi(Q)/Q^s) \\
&\asymp_{\times} \mathrm{Diam}_{\mathrm{e}}^s(A) \sup_{Q \in [1, 12C_0 R]} (f_\xi(Q)/Q^s).
\end{aligned}
$$

Thus

$$
\begin{aligned}
f_\xi(R) - f_\xi(\alpha R) &\leq \#\left\{ g \in G_\xi : g(S \cap \Lambda_{\mathrm{r},\sigma}) \subseteq \bigcup(\mathcal{C}) \right\} \qquad \text{(since } \bigcup(\mathcal{C}) = S_{R,\alpha}) \\
&\leq \#\left\{ g \in G_\xi : g(S \cap \Lambda_{\mathrm{r},\sigma}) \subseteq \bigcup(\mathcal{C}_1) \right\} + \sum_{A \in \mathcal{C}_2} \#\{ g \in G_\xi : g(S) \cap A \neq \varnothing \} \\
&\lesssim_{\times,\sigma} \sum_{A \in \mathcal{C}_1} \mathrm{Diam}_{\mathrm{e}}^s(A) + \sum_{A \in \mathcal{C}_2} \mathrm{Diam}_{\mathrm{e}}^s(A) \sup_{Q \in [1, 12C_0 R]} (f_\xi(Q)/Q^s) \\
&\lesssim_{\times} \sum_{A \in \mathcal{C}} \mathrm{Diam}_{\mathrm{e}}^s(A) \sup_{Q \in [1, 12C_0 R]} (f_\xi(Q)/Q^s).
\end{aligned}
$$

In the last inequality, we have used the inequality

$$
1 \asymp_{\times} f_\xi(1)/1^2 \leq \sup_{Q \in [1, 12C_0 R]} (f_\xi(Q)/Q^s).
$$

Taking the infimum over all collections $\mathcal{C}$ such that $\bigcup(\mathcal{C}) = S_{R,\alpha}$ gives

$$
f_\xi(R) - f_\xi(\alpha R) \lesssim_{\times,\sigma} \mathcal{H}_{\mathrm{e},\infty}^s(S_{R,\alpha}) \sup_{Q \in [1, 12C_0 R]} (f_\xi(Q)/Q^s),
$$

and rearranging finishes the proof. $\qquad \square$

Having estimated $\mathcal{H}_{\mathrm{e},\infty}^s(S_{R,\alpha})$, we proceed to estimate $H_{\xi,\sigma,s}^*(R)$:

Lemma 7.38. *There exist $C_5, \alpha > 0$ such that for all $\sigma > 0$ sufficiently large, for all $R > 0$ sufficiently large, and for all $s \geq 0$,*

(i)

$$
(7.47) \qquad H_{\xi,\sigma,s}^*(R) \lesssim_{\times} \sum_{\ell=0}^{\infty} \frac{f_\xi\big(2^\ell(C_0+1)R\big)(2^\ell R)^{-2s}}{\sup_{Q \in [1, 2^\ell R]}(f_\xi(Q)/Q^s)}.
$$

(ii) *If $f_\xi(\alpha R) > f_\xi(\alpha R/2)$, then*

(7.48)
$$H^*_{\xi,\sigma,s}(R) \gtrsim_\times \frac{f_\xi(\alpha R)R^{-2s}}{\sup_{Q\in[1,C_5R]}(f_\xi(Q)/Q^s)}.$$

Proof of (i). Since ξ is a bounded parabolic point, by Proposition 2.49 there exists a ξ-bounded set $S \subseteq \mathcal{E}_\xi$ such that $\Lambda_r \subseteq \Lambda \setminus \{\xi\} \subseteq G_\xi(S)$. Without loss of generality, we may assume that $o \in S$.

Fix $R \geq 4C_0 \operatorname{Diam}_e(S)$. Then

$$
\begin{aligned}
\Lambda_r \setminus B_e(o,R) &\subseteq \bigcup_{g\in G_\xi} g(S) \setminus B_e(o,R) \\
&\subseteq \bigcup_{\substack{g\in G_\xi \\ g(S)\setminus B_e(o,R)\neq\varnothing}} g(S) \\
&\subseteq \bigcup_{\substack{g\in G_\xi \\ D_e(o,g(o))\geq R-C_0 \operatorname{Diam}_e(S)}} g(S) \\
&\subseteq \bigcup_{\substack{g\in G_\xi \\ D_e(o,g(o))\geq R/2}} g(S).
\end{aligned}
$$

Thus for all $\sigma > 0$ and for all $s \geq 0$,

(7.49)
$$
\begin{aligned}
H^*_{\xi,\sigma,s}(r) = \mathcal{H}^s_{s,\infty}(\Lambda_{r,\sigma} \setminus B_e(o,R)) &\leq \mathcal{H}^s_{s,\infty}\left(\bigcup_{\substack{g\in G_\xi \\ D_e(o,g(o))\geq R/2}} g(S)\right) \\
&\leq \sum_{\ell=0}^\infty \mathcal{H}^s_{s,\infty}\left(\bigcup_{\substack{g\in G_\xi \\ 2^{\ell-1}R\leq D_e(o,g(o))<2^\ell R}} g(S)\right).
\end{aligned}
$$

Fix $\ell \in \mathbb{N}$, and let

$$A_\ell = B_e(o, 2^{\ell+1}R) \setminus B_e(o, 2^{\ell-2}R).$$

Fix $g \in G_\xi$ such that $2^{\ell-1}R \leq D_e(o,g(o)) < 2^\ell R$. Then

$$D_e(g(o), \mathcal{E}_\xi \setminus A_\ell) \geq \frac{R}{4} \geq C_0 \operatorname{Diam}_e(S),$$

and so

$$g(S) \subseteq A_\ell.$$

Now, for $y_1, y_2 \in A_\ell$, by (2.20) and (2.18) we have

$$D_s(y_1,y_2) \asymp_\times (2^\ell R)^{-2} D_e(y_1,y_2),$$

and so

$$\mathcal{H}^s_{s,\infty}\left(\bigcup_{\substack{g \in G_\xi \\ 2^{\ell-1}R \leq D_e(o,g(o)) < 2^\ell R}} g(S)\right)$$

$$\asymp_\times (2^\ell R)^{-2s} \mathcal{H}^s_{e,\infty}\left(\bigcup_{\substack{g \in G_\xi \\ 2^{\ell-1}R \leq D_e(o,g(o)) < 2^\ell R}} g(S)\right)$$

$$\leq (2^\ell R)^{-2s} \mathcal{H}^s_{e,\infty}\left(\bigcup_{\substack{g \in G_\xi \\ D_e(o,g(o)) < 2^\ell R}} g(S)\right)$$

$$\lesssim_\times (2^\ell R)^{-2s} \frac{f_\xi(2^\ell(C_0+1)R)}{\sup_{Q \in [1,2^\ell R]}(f_\xi(Q)/Q^s)}. \qquad \text{(by Lemma 7.35)}$$

Combining with (7.49) finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of (ii). Since G is of general type, there exists $h \in G$ so that $h(\xi) \neq \xi$. Let S be a ξ-bounded set so that $\overline{X} = S \cup h(S)$. Without loss of generality, we may suppose that $o \in S$.

Claim 7.39. *There exists $\sigma > 0$ satisfying* (7.45).

Proof. Since $\mathrm{HD}(\Lambda_r) = \delta > s$, there exists $\sigma > 0$ so that $\mathrm{HD}(\Lambda_{r,\sigma}) > s$. Letting $\widetilde{h} = \mathrm{id}$ or h, we have $\mathrm{HD}(\Lambda_{r,\sigma} \cap \widetilde{h}(S)) > s$. By Lemma 2.39, we have $\mathrm{HD}(\Lambda_{r,\tau} \cap S) > s$ for some $\tau > 0$, finishing the proof. (Hausdorff dimension is the same in the Euclidean and spherical metrics.) $\qquad\qquad\triangleleft$

Let $\alpha = 4(C_0 + 1)$. Fix $R \geq C_0 \mathrm{Diam}_e(S)$, and assume that $f_\xi(\alpha R) > f_\xi(\alpha R/2)$. Then there exists $g_0 \in G_\xi$ such that

$$(7.50) \qquad\qquad \alpha R/2 = 2(C_0+1)R < D(o,g_0(o)) \leq \alpha R = 4(C_0+1)R.$$

Now for every $x \in B_e(o, 2R) \cap G_\xi(o)$,

$$D_e(g_0(o), g_o(x)) \leq 2C_0 R$$

and thus

$$2R < D_e(o, g_0(x)) \leq (6C_0 + 4)R.$$

So

$$(7.51) \qquad\qquad\qquad\qquad f_\xi((6C_0+4)R) \geq 2f_\xi(2R),$$

and thus by Lemma 7.35,

$$\mathcal{H}^s_{e,\infty}(S_{(6C_0+4)R,1/(3C_0+2)}) \gtrsim_\times \frac{f_\xi((6C_0+4)R) - f_\xi(\beta R)}{\sup_{Q \in [1,12C_0(6C_0+4)R]}(f_\xi(Q)/Q^s)}$$

$$\asymp_\times \frac{f_\xi((6C_0+4)R)}{\sup_{Q \in [1,12C_0(6C_0+4)R]}(f_\xi(Q)/Q^s)}$$

$$\geq \frac{f_\xi(\alpha R)}{\sup_{Q \in [1,12C_0(6C_0+4)R]}(f_\xi(Q)/Q^s)}$$

On the other hand,

$$(7.52) \qquad S_{(6C_0+4)R,1/(3C_0+2)} \subseteq B_e(o, (6C_0+4)R + C_0 \mathrm{Diam}_e(S)) \setminus B_e(o, 2R - C_0 \mathrm{Diam}_e(S))$$
$$\subseteq B_e(o, (6C_0+4)R + C_0 \mathrm{Diam}_e(S)) \setminus B_e(o, R).$$

Let τ be as in Lemma 2.39. Then

$$
\begin{aligned}
H^*_{\xi,\tau,s}(R) = \mathcal{H}^s_{s,\infty}(\Lambda_{r,\tau} \setminus B_e(o,R)) \geq \; & \mathcal{H}^s_{s,\infty}(S_{(6C_0+4)R,1/(3C_0+2)}) && \text{(by (7.52))} \\
\asymp_\times \; & R^{-2s}\mathcal{H}^s_{e,\infty}(S_{(6C_0+4)R,1/(3C_0+2)}) && \text{(by (7.52))} \\
\gtrsim_\times \; & R^{-2s}\frac{f_\xi(\alpha R)}{\sup_{Q\in[1,12C_0(6C_0+4)R]}(f_\xi(Q)/Q^s)}.
\end{aligned}
$$

$\square$

We now begin the proof of (7.40). It suffices to consider the two cases $s \in (0,2\delta_\xi)$ and $s \in (2\delta_\xi,\delta)$; the case $s = 2\delta_\xi$ follows by continuity.

Remark 7.40. By Observation 2.44,

$$
\text{(7.53)} \qquad\qquad 2\delta_\xi = \limsup_{R\to\infty}\frac{\log f_\xi(R)}{\log(R)}.
$$

Case 1: $s > 2\delta_\xi$. In this case, fix $\varepsilon > 0$ so that $2\delta_\xi + \varepsilon < s$; then by (7.53), for all $R \geq 1$ we have

$$
\text{(7.54)} \qquad\qquad f_\xi(R) \lesssim_{\times,\varepsilon} R^{2\delta_\xi+\varepsilon},
$$

so by (7.47),

$$
\begin{aligned}
H^*_{\xi,\sigma,s}(R) \lesssim_\times \; & \sum_{\ell=0}^\infty \frac{f_\xi(2^\ell C_5 R)(2^\ell R)^{-2s}}{\sup_{Q\in[1,2^\ell R]}(f_\xi(Q)/Q^s)} \\
\lesssim_{\times,\varepsilon} \; & \sum_{\ell=0}^\infty (2^\ell C_5 R)^{2\delta_\xi+\varepsilon}(2^\ell R)^{-2s} \\
\asymp_\times \; & R^{-(2s-2\delta_\xi-\varepsilon)}\sum_{\ell=0}^\infty (2^\ell)^{2\delta_\xi+\varepsilon-2s} \\
\asymp_{\times,\varepsilon} \; & R^{-(2s-2\delta_\xi-\varepsilon)}.
\end{aligned}
$$

Applying (7.43) gives

$$
P_\xi(s) \geq 2s - 2\delta_\xi - \varepsilon,
$$

and taking the limit as $\varepsilon \to 0$ gives $P_\xi(s) \geq 2s - 2\delta_\xi$.

To demonstrate the other direction, note that by (7.54) we have

$$
\sup_{Q\in[1,R]}(f_\xi(Q)/Q^s) \asymp_{\times,\varepsilon} 1
$$

and so by (7.48) we have

$$
H^*_{\xi,\sigma,s}(R) \gtrsim_\times \frac{f_\xi(\alpha R)R^{-2s}}{\sup_{Q\in[1,C_5 R]}(f_\xi(Q)/Q^s)} \asymp_{\times,\varepsilon} f_\xi(\alpha R)R^{-2s}
$$

whenever $f_\xi(\alpha R) > f_\xi(\alpha R/2)$. Thus

$$
\begin{aligned}
P_\xi(s) &= \lim_{\sigma \to \infty} \liminf_{R \to \infty} \frac{\log(H^*_{\xi,\sigma,s}(R))}{\log(1/R)} \\
&\le \lim_{\sigma \to \infty} \liminf_{\substack{R \to \infty \\ f_\xi(\alpha R) > f_\xi(\alpha R/2)}} \frac{\log(R^{-2s} f_\xi(\alpha R))}{\log(1/R)} \\
&= 2s + \liminf_{\substack{R \to \infty \\ f_\xi(R) > f_\xi(R/2)}} \frac{\log f_\xi(R)}{\log(1/R)} \\
&= 2s - \limsup_{\substack{R \to \infty \\ f_\xi(R) > f_\xi(R/2)}} \frac{\log f_\xi(R)}{\log(R)} \\
&= 2s - \limsup_{R \to \infty} \frac{\log f_\xi(R)}{\log(R)} \qquad \text{(see below)} \\
&= 2s - 2\delta_\xi. \qquad\qquad\qquad\qquad\quad \text{(by (7.53))}
\end{aligned}
$$

To justify removing the restriction on the limsup when moving to the penultimate line, simply notice that for each R, if we let $\widetilde{R}$ be the smallest value so that $f_\xi(\widetilde{R}) = f_\xi(R)$, then $\widetilde{R}$ satisfies the restriction, and moreover

$$
\frac{\log f_\xi(\widetilde{R})}{\log(\widetilde{R})} = \frac{\log f_\xi(R)}{\log(\widetilde{R})} \ge \frac{\log f_\xi(R)}{\log(R)}.
$$

So letting $R_n \underset{n}{\to} \infty$ be a sequence such that $\frac{\log f_\xi(R_n)}{\log(R_n)}$ tends to the lim sup, the sequence $(\widetilde{R}_n)_1^\infty$ also tends to the lim sup; moreover, $\widetilde{R}_n \underset{n}{\to} \infty$ because otherwise f_ξ would be eventually constant, G_ξ would be finite, and ξ would fail to be a parabolic point.

Case 2: $s < 2\delta_\xi$. In this case, by (7.53), we have

(7.55)
$$
\limsup_{R \to \infty} f_\xi(R)/R^s = +\infty.
$$

Claim 7.41. *There exists a constant $C_7 > 0$ and a sequence $R_n \underset{n}{\to} \infty$ such that for all $n \in \mathbb{N}$,*

$$
\sup_{Q \in [1, C_5 R_n]} (f_\xi(Q)/Q^s) \le C_7 f_\xi(\alpha R_n)/R_n^s
$$

and

$$
f_\xi(\alpha R_n) > f_\xi(\alpha R_n/2).
$$

Proof. For convenience let $h_{\xi,s}(R) = f_\xi(R)/R^s$ for all $R \ge 1$ and let $C_8 = C_5/\alpha$. Fix $C_7 > 0$ large to be announced below, and by contradiction suppose that there exists R_0 such that for all $R \ge R_0$, either

(7.56)
$$
\sup_{[1, C_8 R]} h_{\xi,s} > C_7 h_{\xi,s}(R)
$$

or

(7.57)
$$
f_\xi(R) = f_\xi(R/2).
$$

(In each equation, we have changed variables replacing R by R/α.) By (7.55), there exists $R_1 \ge R_0$ so that

(7.58)
$$
h_{\xi,s}(R_1) = \sup_{[1, R_1]} h_{\xi,s},
$$

and in particular $h_{\xi,s}(R_1) \ge h_{\xi,s}(R_2)$, which implies $f_\xi(R_1) > f_\xi(R_1/2)$. Thus R_1 does not satisfy (7.57), so it does satisfy (7.56); there exists $R_2 \in [1, C_8 R_1]$ such that

(7.59)
$$
h_{\xi,s}(R_2) \ge C_7 h_{\xi,s}(R_1).
$$

By (7.58), such an R_2 must be larger than R_1. Letting R_2 be minimal satisfying (7.59), we see that

$$h_{\xi,s}(R_2) = \sup_{[1,R_2]} h_{\xi,s}.$$

We can continue this process indefinitely to get a sequence $R_1 < R_2 < \cdots$ satisfying

$$h_{\xi,s}(R_{n+1}) \geq C_7 h_{\xi,s}(R_n) \text{ and } R_{n+1} \leq C_8 R_1.$$

By iterating, we see that

$$\frac{h_{\xi,s}(R_n)}{h_{\xi,s}(R_1)} \geq C_7^{n-1} = C_8^{\log_{C_8}(C_7)(n-1)} \geq \left(\frac{R_n}{R_1}\right)^{\log_{C_8}(C_7)}$$

and thus

$$f_\xi(R_n) \gtrsim_{\times, C_7, R_1} R_n^{\log_{C_8}(C_7)+s}.$$

Applying (7.53) gives

$$\delta_\xi \geq \log_{C_8}(C_7) + s,$$

but if C_7 is large enough, then this is a contradiction, since $\delta_\xi \leq \delta < +\infty$ by assumption. $\quad\triangleleft$

Applying (7.48) gives

$$H^*_{\xi,\sigma,s}(R_n) \gtrsim_\times R_n^{-s}$$

and applying (7.43) shows that $P_\xi(s) \leq s$. On the other hand, the direction $P_\xi(s) \geq s$ holds in general (Proposition 7.4).

This completes the proof of (7.40). Deducing (7.41) from (7.1) and (7.40) is trivial, so this completes the proof of Theorem 7.33.

8. Proof of Theorem 8.1 (Generalization of Khinchin's Theorem)

Theorem 8.1 (Generalization of Khinchin's Theorem). *Let (X, d, o, b, G) be as in §1.1.4, and fix a distinguished point $\xi \in \partial X$. Let δ be the Poincaré exponent of G, and suppose that $\mu \in \mathcal{M}(\Lambda)$ is an ergodic δ-quasiconformal probability measure satisfying $\mu(\xi) = 0$. Let*

$$\Delta_{\mu,\xi}(r) = \mu(B(\xi, r)).$$

Let $\Phi : [0, +\infty) \to (0, +\infty)$ be a function such that the function $t \mapsto t\Phi(t)$ is nonincreasing. Then:

   (i) *If the series*

$$\text{(8.1)} \qquad \sum_{g \in G} b^{-\delta d(o,g(o))} \Delta_{\mu,\xi}\left(b^{d(o,g(o))} \Phi(Kb^{d(o,g(o))})\right)$$

       diverges for all $K > 0$, then $\mu(\text{WA}_{\Phi,\xi}) = 1$.

  (iiA) *If the series*

$$\text{(8.2)} \qquad \sum_{g \in G} (g'(\xi))^\delta \Delta_{\mu,\xi}\left(\Phi(Kb^{d(o,g(o))})/g'(\xi)\right)$$

       converges for some $K > 0$, then $\mu(\text{WA}_{\Phi,\xi}) = 0$.

  (iiB) *If ξ is a bounded parabolic point of G (see Definition 2.48), and if the series (8.1) converges for some $K > 0$, then $\mu(\text{WA}_{\Phi,\xi}) = 0$.*

Let us begin by introducing some notation. First of all, we will assume that the map $G \ni g \mapsto g(o)$ is injective; avoiding this assumption would only change the notation and not the arguments. For each $x \in G(o)$, let $g_x \in G$ be the unique element so that $g_x(o) = x$, and for each $K > 0$ let

$$\text{(8.3)} \qquad B_{x,K} := B(g_x(\xi), \Phi(Kb^{d(o,x)}));$$

then

$$\text{(8.4)} \qquad \text{WA}_{\Phi,\xi} = \bigcap_{K>0} \bigcup_{x \in G(o)} B_{x,K}.$$

We will also need the following:

Lemma 8.2 (Sullivan's Shadow Lemma). *Let X be a hyperbolic metric space, and let G be a subgroup of* Isom(X). *Fix $s \geq 0$, and let $\mu \in \mathcal{M}(\partial X)$ be an s-quasiconformal measure which is not a pointmass. Then for all $\sigma > 0$ sufficiently large and for all $x \in G(o)$,*

$$(8.5) \qquad \mu(\mathrm{Shad}(x, \sigma)) \asymp_{\times, \sigma} b^{-sd(o,x)}.$$

The proof of this theorem is similar to the proof in the case of standard hyperbolic space (Proposition 3 of [72]), and can be found in [16] or [21].

8.1. **The convergence case.** Suppose that either

(1) the series (8.2) converges for some $K > 0$, or that
(2) ξ is a bounded parabolic point, and the series (8.1) converges for some $K > 0$.

By (8.4) and by the easy direction of the Borel-Cantelli lemma,[37] to show that $\mu(\mathrm{WA}_{\Phi, \xi}) = 0$ it suffices to show that there exists $K > 0$ so that the series

$$(8.6) \qquad \sum_{x \in G(o)} \mu(B_{x,K})$$

converges. Fix $K > 0$ large to be announced below. Fix $x \in G(o)$ and let $g = g_x$. Note that

$$\mu(B_{x,K}) \asymp_{\times} \int_{g^{-1}(B_{x,K})} (g')^{\delta} \mathrm{d}\mu.$$

Claim 8.3. *By choosing K sufficiently large, we may ensure that for all $x \in G(o)$ and for all $\eta \in g^{-1}(B_{x,K})$,*

$$(8.7) \qquad g'(\eta) \asymp_{\times} g'(\xi)$$

$$(8.8) \qquad D(\xi, \eta) \lesssim_{\times} \frac{\Phi(Kb^{d(o,x)})}{g'(\xi)}.$$

In both equations $g = g_x$.

Proof. (cf. proof of Claim 7.7) Fix $\eta \in g^{-1}(B_{x,K})$. We have

$$(8.9) \qquad \left| \log_b \left(\frac{g'(\eta)}{g'(\xi)} \right) \right| \asymp_{+} |\mathcal{B}_\eta(o, g^{-1}(o)) - \mathcal{B}_\xi(o, g^{-1}(o))| \asymp_{+} |\langle g(o)|g(\eta)\rangle_o - \langle g(o)|g(\xi)\rangle_o|.$$

Let

$$(\mathrm{A}) = \langle g(o)|g(\eta)\rangle_o$$
$$(\mathrm{B}) = \langle g(o)|g(\xi)\rangle_o$$
$$(\mathrm{C}) = \langle g(\xi)|g(\eta)\rangle_o.$$

By Gromov's inequality, at least two of the expressions (A), (B), and (C) must be asymptotic. On the other hand, since $g(\eta) \in B_{x,K}$, we have

$$\langle g(\xi)|g(\eta)\rangle_o - d(o, g(o)) \asymp_{+} -\log_b(D(g(\xi), g(\eta))) - d(o, g(o))$$
$$\geq -\log_b(\Phi(Kb^{d(o,x)})) - d(o, x) \qquad \qquad \text{(by (8.3))}$$
$$\geq -\log_b(\Phi(1)/(Kb^{d(o,x)})) - d(o, x) \qquad \text{(since } t \mapsto t\Phi(t) \text{ is nonincreasing)}$$
$$= \log_b(K/\Phi(1))$$

Since (A) and (B) are both less than $d(o, x)$, this demonstrates that (C) is not asymptotic to either (A) or (B) if K is sufficiently large. Thus by Gromov's inequality, (A) and (B) are asymptotic, and so (8.9) is asymptotic to zero. This demonstrates (8.7). Finally, (8.8) follows from (8.7), (8.3), and the geometric mean value theorem. ◁

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[37]The condition $\mu(\xi) = 0$ guarantees that for μ-almost every η, $\eta \in \mathrm{WA}_{\Phi, \xi}$ if and only if $\forall K > 0 \, \exists^{\infty} x \in G(o) \, \eta \in B_{x,K}$.

Let $C > 0$ be the implied constant in (8.8). Then

$$
\begin{aligned}
\mu(B_{x,K}) &\asymp_\times \int_{g^{-1}(B_{x,K})} (g')^\delta \mathrm{d}\mu \\
&\asymp_\times (g'(\xi))^\delta \mu\big(g^{-1}(B_{x,K})\big) \\
&\leq (g'(\xi))^\delta \mu\left(B\big(\xi, C\frac{\Phi(Kb^{d(o,x)})}{g'(\xi)}\big)\right) \\
&= (g'(\xi))^\delta \Delta_{\mu,\xi}\left(C\Phi(Kb^{d(o,x)})/g'(\xi)\right) \\
&\leq (g'(\xi))^\delta \Delta_{\mu,\xi}\left(\Phi(C^{-1}Kb^{d(o,x)})/g'(\xi)\right),
\end{aligned}
$$

(8.10)

the last inequality due to the fact that the function $t \mapsto t\Phi(t)$ is nonincreasing. Now the argument splits according to the cases (1) and (2):

(1) In this case, let $\widetilde{K} > 0$ be a value of K for which (8.2) converges, and choose $K > 0$ large enough so that $K \geq C\widetilde{K}$ and so that Claim 8.3 is satisfied. Then we have (8.6) $\lesssim_\times$ (8.2) $< +\infty$, completing the proof.

(2) In this case, let us call a point $x \in G(o)$ *minimal* if

$$
d(o,x) = \min\{d(o,\widetilde{g}(o)) : \widetilde{g}(\xi) = g_x(\xi)\}.
$$

Then

$$
\mathrm{WA}_{\Phi,\xi} = \bigcap_{K > 0} \bigcup_{\substack{x \in G(o) \\ \text{minimal}}} B_{x,K}.
$$

Again, by the easy direction of the Borel-Cantelli lemma, to show that $\mu(\mathrm{WA}_{\Phi,\xi}) = 0$ it suffices to show that there exists $K > 0$ so that the series

(8.11)
$$
\sum_{\substack{x \in G(o) \\ \text{minimal}}} \mu(B_{x,K})
$$

converges.

Since ξ is a bounded parabolic point, there exists a ξ-bounded set S so that $G(o) \subseteq G_\xi(S)$, where G_ξ is the stabilizer of ξ relative to G. Fix $x \in G(o)$ minimal. Then there exists $h \in G_\xi$ such that $h \circ g^{-1}(o) \in S$; by Observation 2.47 we have

(8.12)
$$
\langle h \circ g^{-1}(o)|\xi\rangle_o \asymp_+ 0.
$$

Then

$$
\begin{aligned}
d(o,x) &\leq d(o, g \circ h^{-1}(o)) && \text{(since x is minimal)} \\
&= d(h \circ g^{-1}(o), o) \\
&\asymp_+ \mathcal{B}_\xi(h \circ g^{-1}(o), o) && \text{(by (8.12))} \\
&\asymp_+ \mathcal{B}_\xi(g^{-1}(o), o); && \text{(since ξ is parabolic)}
\end{aligned}
$$

exponentiating gives

$$
g'(\xi) \asymp_\times b^{-d(o,x)}.
$$

Let $C_2 > 0$ be the implied constant of this asymptotic. Then by (8.10), we have

$$
\mu(B_{x,K}) \lesssim_\times b^{-\delta d(o,x)} \Delta_{\mu,\xi}\left(b^{d(o,x)}\Phi((CC_2)^{-1}Kb^{d(o,x)})\right).
$$

Let $\widetilde{K} > 0$ be a value of K for which (8.1) converges, and choose $K > 0$ large enough so that $K \geq CC_2\widetilde{K}$ and so that Claim 8.3 is satisfied. Then we have (8.11) $\lesssim_\times$ (8.1) $< +\infty$, completing the proof.

8.2. **The divergence case.** [38] By contradiction, suppose that $\mu(\mathrm{WA}_{\Phi,\xi}) = 0$. Fixing $\lambda > 0$ small to be announced below, we have

$$\mu\left(\bigcup_{x \in G(o)} B_{x,K}\right) \le \lambda$$

for some $K > 0$. For convenience of notation let

$$U := \bigcup_{x \in G(o)} B_{x,K}.$$

Let $\sigma > 0$ be large enough so that (8.5) holds for all $x \in G(o)$, and fix $\varepsilon > 0$ small to be announced below. Let $\phi(t) = t\Phi(t)$, so that the function $\phi : [0, +\infty) \to (0, +\infty)$ is nonincreasing. For each $x \in G(o)$ let

$$\mathcal{P}_x = \mathrm{Shad}(x, \sigma)$$
$$B_x = g_x(B(\xi, \varepsilon\phi(Kb^{d(o,x)}))).$$

Let

(8.13) $$A := \{x \in G(o) : B_x \subseteq \mathcal{P}_x\}.$$

In a specific sense A comprises "at least half" of $G(o)$; see Lemma 8.10 below.

Let $d : A \to \mathbb{N}$ be a bijection with the following property: $d_x \le d_y \Rightarrow d(o, x) \le d(o, y)$. Such a bijection is possible since G is strongly discrete. Note that $d_o = 0$.

We will define three sets $T_1, T_2, T_3 \subseteq A$, two binary relations $\in_1 \subseteq T_1 \times T_1$ and $\in_2 \subseteq T_2 \times T_1$, and one ternary relation $\in_3 \subseteq T_3 \times T_1 \times T_2$ via the following procedure. Points in T_i will be called *type i*.[39]

1. Each time a new point is put into T_1, call it x and go to step 3.
2. Put o into T_1; it is the root node.
3. Recall that x is a point which has just been put into T_1. Let

(8.14) $$S_x = \{y \in A : \mathcal{P}_x \cap \mathcal{P}_y \ne \varnothing \text{ and } d_y > d_x\}$$

   and

(8.15) $$\widetilde{S}_x = \{y \in S_x : \mathcal{P}_y \subseteq g_x(U)\}.$$

4. Construct a sequence of points $z_n \in \widetilde{S}_x$ recursively as follows: If $z_1, \ldots, z_{n-1}$ have been chosen, then let $z_n \in \widetilde{S}_x$ be such that

(8.16) $$\mathcal{P}_{z_n} \cap \mathcal{P}_{z_j} = \varnothing \ \ \forall j = 1, \ldots, n-1.$$

   If more than one such z_n exists, then choose z_n so as to minimize d_{z_n}. Note that the sequence may terminate in finitely many steps if (8.16) is not satisfied for any $z_n \in \widetilde{S}_x$.
5. For each n, put z_n into T_1, setting $z \in_1 x$.
6. Construct a sequence of points $y_n \in S_x \setminus \widetilde{S}_x$ recursively as follows: If $y_1, \ldots, y_{n-1}$ have been chosen, then let $y_n \in \widetilde{S}_x$ be such that

(8.17) $$B_{y_n} \cap B_{y_j} = \varnothing \ \ \forall j = 1, \ldots, n-1.$$

   If more than one such y_n exists, then choose y_n so as to minimize d_{y_n}.
7. For each n,
   i. Put $y = y_n$ into T_2, setting $y \in_2 x$.
   ii. Let

$$S_{x,y} = \{z \in S_x \setminus \widetilde{S}_x : B_y \cap B_z \ne \varnothing \text{ and } d_z > d_y\}.$$

   iii. For each $z \in S_{x,y}$, put z into T_3, setting $z \in_3 (x, y)$.

---

[38]This proof was heavily influenced by the proof of Theorem 4 of Section VII of [1].
[39]See footnote 36.

Notation 8.4. Write $y > x$ if $y \in S_x$. Note that by construction $y > x \Rightarrow d_y > d_x$.

Let us warn that the relation $>$ is not necessarily transitive, due to the fact that the relation of having nonempty intersection is not an equivalence relation.

Lemma 8.5.

$$T_1 \cup T_2 \cup T_3 = A.$$

Proof. By contradiction, suppose that $w \in A \setminus (T_1 \cup T_2 \cup T_3)$. We claim that there is an infinite sequence of type 1 points $(x_n)_0^\infty$ so that $\cdots \in_1 x_2 \in_1 x_1 \in_1 x_0 = o$ and $x_n < w$ for all n. Then $d_{x_0} < d_{x_1} < \cdots < d_w$, which is absurd since these are natural numbers.

Let $x_0 = o$. By induction, suppose that $x = x_n < w$ is a type 1 point.

Case 1: $w \in \widetilde{S}_x$. In this case, note that w could have been chosen in step 4, but was not (otherwise it would have been put into T_1). The only explanation for this is that $\mathcal{P}_w \cap \mathcal{P}_z \neq \varnothing$ for some $z \in_1 x$ for which $d_z < d_w$. But then $x_n < z < w$; letting $x_{n+1} = z$ completes the inductive step.

Case 2: $w \notin \widetilde{S}_x$. In this case, note that w could have been chosen in step 6, but was not (otherwise it would have been put into T_2). The only explanation for this is that $B_w \cap B_z \neq \varnothing$ for some $y \in_2 x$ for which $d_y < d_w$. But then $w \in_3 (x, y)$, which demonstrates that $w \in T_3$, contradicting our hypothesis. $\square$

Lemma 8.6. *There exists $\tau > 0$ (depending on σ) so that for all $x, y \in A$ with $y > x$ we have*

$$\mathcal{P}_y \subseteq \mathrm{Shad}(x, \tau).$$

Proof. This follows directly from the Intersecting Shadows Lemma, (8.14), and the fact that $d_y > d_x \Rightarrow d(o, y) \geq d(o, x)$. $\square$

Lemma 8.7. *For all $x, y \in A$ such that $y > x$, we have*

$$3B_y \subseteq g_x(U),$$

if ε is chosen small enough.

Proof. By Lemma 8.6 and (8.13) we have

$$B_y \subseteq \mathcal{P}_y \subseteq \mathrm{Shad}(x, \tau),$$

where τ is as in Lemma 8.6. Thus by Lemma 7.13, there exists $\widetilde{\tau} > 0$ such that

$$3B_y \subseteq \mathrm{Shad}(x, \widetilde{\tau}).$$

Suppose $\eta \in B_y$; then

$$D(\xi, g_y^{-1}(\eta)) \leq \varepsilon \phi(K b^{d(o,y)});$$

since $B_y \subseteq \mathcal{P}_y$, the Bounded Distortion Lemma gives

$$D(g_y(\xi), \eta) \lesssim_{\times, \sigma} b^{-d(o,y)} \varepsilon \phi(K b^{d(o,y)}).$$

Thus

$$\mathrm{Diam}(3B_y) \leq 3 \, \mathrm{Diam}(B_y) \lesssim_{\times, \sigma} b^{-d(o,y)} \varepsilon \phi(K b^{d(o,y)}).$$

Now fix $\eta \in 3B_y$; then

$$D(g_y(\xi), \eta) \lesssim_{\times, \sigma} b^{-d(o,y)} \varepsilon \phi(K b^{d(o,y)})$$

$$D(g_x^{-1} \circ g_y(\xi), g_x^{-1}(\eta)) \lesssim_{\times, \widetilde{\tau}} b^{d(o,x) - d(o,y)} \varepsilon \phi(K b^{d(o,y)}) \qquad \text{(by the Bounded Distortion Lemma)}$$

$$\asymp_{\times, \sigma} b^{-d(x,y)} \varepsilon \phi(K b^{d(o,y)}) \qquad \text{(by the Intersecting Shadows Lemma)}$$

$$\leq \quad b^{-d(x,y)} \varepsilon \phi(K b^{d(x,y)}) \qquad \text{(since } \phi \text{ is nonincreasing)}$$

$$= \quad \varepsilon K \Phi(K b^{d(x,y)}) \asymp_{\times, K} \varepsilon \Phi(K b^{d(x,y)}).$$

If we choose ε to be less than the reciprocal of the implied constant, then we have

$$D(g_x^{-1} \circ g_y(\xi), g_x^{-1}(\eta)) \leq \Phi(Kb^{d(x,y)})$$

and thus

$$g_x^{-1}(\eta) \in B\big(g_x^{-1} \circ g_y(\xi), \Phi(Kb^{d(o,g_x^{-1} \circ g_y(o))})\big) = B_{g_x^{-1}(y),K} \subseteq U,$$

i.e. $\eta \in g_x(U)$. Since $\eta \in 3B_y$ was arbitrary, this demonstrates that $3B_y \subseteq g_x(U)$. $\square$

Lemma 8.8. *If $x \in T_1$, $y \in_2 x$, and $z \in_3 (x,y)$, then $B_y \subseteq (z, \rho)$ for some $\rho > 0$ large independent of z. In particular $g_y(\xi) \in \mathrm{Shad}(z, \rho)$.*

Proof. Since $z \in_3 (x,y)$, we have $B_y \cap B_z \neq \varnothing$, but by (8.13), $B_z \subseteq \mathcal{P}_z$, so

$$(8.18) \qquad\qquad B_y \cap \mathcal{P}_z \neq \varnothing.$$

Since $z \in S_{x,y} \subseteq S_x \setminus \widetilde{S}_x$, we have

$$\mathcal{P}_z \not\subseteq g_x(U),$$

but on the other hand, $3B_y \subseteq g_x(U)$ by Lemma 8.7, so

$$\mathcal{P}_z \not\subseteq 3B_y.$$

Combining with (8.18), we have

$$\mathrm{Diam}(\mathcal{P}_z) \geq D(B_y, \partial X \setminus 3B_y) \geq \mathrm{Diam}(B_y)$$

and thus by Lemma 7.13, we have

$$B_y \subseteq 3\mathcal{P}_z \subseteq \mathrm{Shad}(z, \rho)$$

if $\rho > 0$ is large enough. $\square$

Lemma 8.9. *For each $x \in T_1$, If $x \in T_1$, then*

$$(8.19) \qquad\qquad \sum_{z \in_1 x} \mu(\mathcal{P}_z) \lesssim_{\times,\sigma} \lambda \mu(\mathcal{P}_x)$$

$$(8.20) \qquad\qquad \sum_{y \in_2 x} \mu(B_y) \lesssim_{\times,\sigma} \mu(\mathcal{P}_x).$$

For each $x \in T_1$ and for each $y \in_2 x$,

$$(8.21) \qquad\qquad \sum_{z \in_3 (x,y)} \mu(B_z) \lesssim_{\times,\sigma} \mu(B_y).$$

Proof.

(8.19): By Lemma 8.6, (8.15), and (8.16), the collection $(\mathcal{P}_z)_{z \in_1 x}$ is a disjoint collection of shadows contained in $g_x(U) \cap \mathrm{Shad}(x, \tau)$. By the Bounded Distortion Lemma,

$$\begin{aligned}
\sum_{z \in_1 x} \mu(\mathcal{P}_z) &\leq \quad \mu(g_x(U) \cap \mathrm{Shad}(x, \tau)) \\
&\asymp_{\times,\tau} b^{-\delta d(o,x)} \mu(U \cap g^{-1}(\mathrm{Shad}(x, \tau))) \\
&\leq \quad \lambda b^{-\delta d(o,x)} \asymp_{\times,\sigma} \lambda \mu(\mathcal{P}_x).
\end{aligned}$$

(8.20): By Lemma 8.6, (8.13), and (8.17), the collection $(B_y)_{y \in_2 x}$ is a disjoint collection of balls contained in $\mathrm{Shad}(x, \tau)$. Thus

$$\sum_{y \in_2 x} \mu(B_y) \leq \mu(\mathrm{Shad}(x, \tau)) \asymp_{\times,\tau} b^{-\delta d(o,x)} \asymp_{\times,\sigma} \mu(\mathcal{P}_x).$$

(8.21): By the Bounded Distortion Lemma, we have for each $z \in A$

$$\mu(B_z) \asymp_{\times,\sigma} b^{-\delta d(o,z)} \mu(B(\xi, \varepsilon\phi(Kb^{d(o,z)}))).$$

Since ϕ is nonincreasing, we have for each $z \in_3 (x, y)$

$$\mu(B(\xi, \varepsilon\phi(Kb^{d(o,z)}))) \leq \mu(B(\xi, \varepsilon\phi(Kb^{d(o,y)})))$$

and thus

$$\mu(B_z) \lesssim_{\times,\sigma} b^{-\delta \, \mathcal{B}_o(z,y)} \mu(B_y),$$

and so

(8.22)
$$\sum_{z \in_3(x,y)} \mu(B_z) \lesssim_{\times,\sigma} \mu(B_y) b^{\delta d(o,y)} \sum_{z \in_3(x,y)} b^{-\delta d(o,z)}.$$

Recall that by Lemma 8.8, there exists $\rho > 0$ so that $g_y(\xi) \in \mathrm{Shad}(z, \rho)$ for every $z \in_3 (x, y)$; moreover, $d(o, z) \geq d(o, y)$ for every such z. Thus, modifying the proof of Lemma 6.4, we see that

$$\sum_{z \in_3(x,y)} b^{-\delta d(o,z)} \leq M b^{\delta\tau} \sum_{n=\lfloor d(o,y) \rfloor}^{\infty} b^{-\delta n},$$

where $\tau > 0$ is large enough so that (6.3) is satisfied for every $n \in \mathbb{N}$, and where $M = \#\{g \in G : g(o) \in B(o, \tau)\}$. Summing the geometric series, we see

$$\sum_{z \in_3(x,y)} b^{-\delta d(o,z)} \lesssim_{\times,\rho} b^{-\delta d(o,y)}.$$

Applying this to (8.22) yields (8.21).

$\square$

Next, choose λ small enough so that (8.19) becomes

$$\sum_{z \in_1 x} \mu(\mathcal{P}_z) \leq \frac{1}{2}\mu(\mathcal{P}_x).$$

Iterating yields

$$\sum_{z \in T_1} \mu(\mathcal{P}_z) \leq 2\mu(\mathcal{P}_o) = 2.$$

Combining with (8.20), (8.21), (8.13), and Lemma 8.5 yields

(8.23)
$$\sum_{x \in A} \mu(B_x) < +\infty.$$

Recall that we are trying to derive a contradiction by finding a value of K for which the series (8.1) converges. (It won't necessarily be the same K that we chose earlier.) For each $\widetilde{K} > 0$ and $x \in X$ let

$$f_{\widetilde{K}}(x) := b^{-\delta d(o,x)} \Delta_{\mu,\xi}(\varepsilon\phi(Kb^{d(o,x)})).$$

Then if $x \in A$, by the Bounded Distortion Lemma we have

$$\mu(B_x) \asymp_{\times,\sigma} f_K(x).$$

Thus by (8.23), $\sum_A f_K < +\infty$. To complete the proof we need to show that $\sum_{G(o)} f_{\widetilde{K}} < +\infty$ for some $\widetilde{K} > 0$.

By Observation 1.8, there exists $h \in G$ so that $h(\xi) \neq \xi$.

Lemma 8.10. *For every $g \in G$, either $g(o) \in A$ or $g \circ h(o) \in A$, assuming $\sigma > 0$ is chosen large enough and $\varepsilon > 0$ is chosen small enough.*

Proof. By contradiction, suppose $g(o), g \circ h(o) \notin A$. Then

$$B(\xi, \varepsilon\phi(Kb^{d(o,g(o))})) \nsubseteq \mathrm{Shad}_{g^{-1}(o)}(o, \sigma)$$

$$h(B(\xi, \varepsilon\phi(Kb^{d(o,g\circ h(o))}))) \nsubseteq \mathrm{Shad}_{g^{-1}(o)}(h(o), \sigma).$$

On the other hand,

$$\mathrm{Shad}_{g^{-1}(o)}(h(o), \sigma) \supseteq \mathrm{Shad}_{g^{-1}(o)}(o, \sigma - d(o, h(o)))$$

and thus

$$h(B(\xi, \varepsilon\phi(Kb^{d(o,g\circ h(o))}))) \nsubseteq \mathrm{Shad}_{g^{-1}(o)}(o, \sigma - d(o, h(o))).$$

But by the Big Shadows Lemma, we may choose σ large enough so that

$$\mathrm{Diam}\left(\partial X \setminus \mathrm{Shad}_{g^{-1}(o)}(o, \sigma - d(o, h(o)))\right) \leq \frac{1}{4}D(\xi, h(\xi)),$$

and so

$$D(B(\xi, \varepsilon\phi(Kb^{d(o,g(o))})), h(B(\xi, \varepsilon\phi(Kb^{d(o,g\circ h(o))})))) \leq \frac{1}{4}D(\xi, h(\xi)).$$

On the other hand, we may choose ε small enough so that

$$\varepsilon\phi(K) \leq \frac{1}{4}D(\xi, h(\xi))$$

$$\left(\max h'\right)\varepsilon\phi(K) \leq \frac{1}{4}D(\xi, h(\xi)),$$

which implies

$$D(\xi, h(\xi)) \leq \frac{3}{4}D(\xi, h(\xi)),$$

a contradiction. $\square$

Fix $\widetilde{K} \geq K$ to be announced below, and note that $f_{\widetilde{K}} \leq f_K$ since ϕ and $\Delta_{\mu,\xi}$ are both monotone (nonincreasing and nondecreasing, respectively). Now

$$\sum_{G(o)} f_{\widetilde{K}} \leq \sum_{\substack{g \in G \\ g(o) \in A}} f_{\widetilde{K}}(g(o)) + \sum_{\substack{g \in G \\ g\circ h(o) \in A}} f_{\widetilde{K}}(g(o))$$

$$= \sum_A f_{\widetilde{K}} + \sum_{\substack{g \in G \\ g\circ h(o) \in A}} b^{-\delta d(o,g(o))}\Delta_{\mu,\xi}(\varepsilon\phi(\widetilde{K}b^{d(o,g(o))})).$$

Let $\widetilde{K} = Kb^{d(o,h(o))}$; then

$$\widetilde{K}b^{d(o,g(o))} \geq Kb^{d(o,g\circ h(o))}$$

and so

$$\sum_{G(o)} f_{\widetilde{K}} \leq \sum_A f_K + b^{\delta d(o,h(o))} \sum_{\substack{g \in G \\ g\circ h(o) \in A}} b^{-\delta d(o,g\circ h(o))}\Delta_{\mu,\xi}(\varepsilon\phi(Kb^{d(o,g\circ h(o))}))$$

$$\leq \sum_A f_K + b^{\delta d(o,h(o))} \sum_A f_K < +\infty.$$

8.3. Special cases. In this subsection we prove the assertions made in §1.5.1-§1.5.2 concerning special cases of Theorem 8.1. We will be interested in the following asymptotic formulas:

$$(8.24) \qquad \qquad \Delta_{\mu,\xi}(r) \asymp_\times r^\delta \ \ \forall r > 0$$

$$(8.25) \qquad f_G(t) := \#\{g \in G : d(o, g(o)) \leq t\} \asymp_\times b^{\delta t} \ \ \forall t > 0.$$

Proposition 8.11 (Reduction of (8.1) and (8.2) assuming power law asymptotics)**.** *If* (8.25) *holds then for each* $K > 0$

$$(8.26) \qquad (8.1) \asymp_{+,\times} \int_{t=0}^{\infty} \Delta_{\mu,\xi}(K^{-1}e^t \Phi(e^t)) \mathrm{d}t.$$

If (8.24) *holds then for each* $K > 0$

$$(8.27) \qquad (8.1) \asymp_{\times} (8.2) \asymp_{\times} \sum_{g \in G} \Phi^{\delta}(Kb^{d(o,g(o))}).$$

If both (8.25) *and* (8.24) *hold then for each* $K > 0$

$$(8.28) \qquad (8.1) \asymp_{\times} (8.2) \asymp_{+,\times} \int_{t=0}^{\infty} e^{\delta t} \Phi^{\delta}(e^t) \mathrm{d}t,$$

and so by Theorem 8.1

$$(8.29) \qquad \mu(\mathrm{WA}_{\Phi,\xi}) = 0 \iff \int_{t=0}^{\infty} e^{\delta t} \Phi^{\delta}(e^t) \mathrm{d}t < +\infty.$$

Proof. We prove only (8.26), as (8.27) is obvious, and (8.28) is obvious given (8.26).

If $f : [0, +\infty) \to (0, +\infty)$ is any $\mathcal{C}^1$ decreasing function for which $f(t) \underset{t}{\to} 0$, then

$$
\begin{aligned}
\sum_{g \in G} f(d(o, g(o))) &= \sum_{g \in G} \int_{t=d(o,g(o))}^{\infty} (-f'(t)) \mathrm{d}t && (\text{since } f(t) \underset{t}{\to} 0) \\
&= \sum_{g \in G} \int_{t=0}^{\infty} (-f'(t)) \chi(t \geq d(o, g(o))) \mathrm{d}t \\
&= \int_{t=0}^{\infty} \sum_{g \in G} (-f'(t)) \chi(d(o, g(o)) \leq t) \mathrm{d}t && (\text{by Tonelli's theorem}) \\
&= \int_{t=0}^{\infty} (-f'(t)) \#\{g \in G : d(o, g(o)) \leq t\} \mathrm{d}t \\
&\asymp_{\times} \int_{t=0}^{\infty} (-f'(t)) b^{\delta t} \mathrm{d}t && (\text{by (8.25)}) \\
&= f(0) + \log(b^{\delta}) \int_{t=0}^{\infty} f(t) b^{\delta t} \mathrm{d}t. && (\text{integration by parts})
\end{aligned}
$$

Thus

$$(8.30) \qquad \sum_{g \in G} f(d(o, g(o))) \asymp_{\times} f(0) + \int_{t=0}^{\infty} f(t) b^{\delta t} \mathrm{d}t.$$

The implied constant is independent of f. Now (8.30) makes sense even if f is not $\mathcal{C}^1$. An approximation argument shows that (8.30) holds for any nonincreasing function $f : [0, +\infty) \to (0, +\infty)$ for which $f(t) \underset{t}{\to} 0$. Now let

$$f(t) = b^{-\delta t} \Delta_{\mu,\xi}(b^t \Phi(Kb^t)).$$

Since by assumption $t \mapsto t\Phi(t)$ is nonincreasing, it follows that f is also nonincreasing. On the other hand, clearly $f(t) \leq e^{-\delta t}$, so $f(t) \underset{t}{\to} 0$. Thus (8.30) holds; substituting yields

$$(8.1) \asymp_{+,\times} \int_{t=0}^{\infty} \Delta_{\mu,\xi}(b^t \Phi(Kb^t)) \mathrm{d}t.$$

Substituting $s = t\log(b) + \log(K)$, we have

$$(8.1) \asymp_{+,\times} \int_{t=0}^{\infty} \Delta_{\mu,\xi}(b^t \Phi(Kb^t))\mathrm{d}t = \int_{s=\log(K)}^{\infty} \Delta_{\mu,\xi}(K^{-1}e^s\Phi(e^s))\log(b)\mathrm{d}s$$

$$\asymp_+ \int_{t=0}^{\infty} \Delta_{\mu,\xi}(K^{-1}e^t\Phi(e^t))\mathrm{d}t.$$

$\square$

Corollary 8.12 (Measures of BA_ξ and VWA_ξ assuming power law asymptotics). *Assume $\xi \in \Lambda_G$. If (8.24) holds, then $\mu(\mathrm{VWA}_\xi) = 0$, and $\mu(\mathrm{BA}_\xi) = 0$ if and only if G is of divergence type. If (8.25) holds, then $\mu(\mathrm{BA}_\xi) = 0$. If both hold, then $\mu(\mathrm{BA}_\xi) = \mu(\mathrm{VWA}_\xi) = 0$.*

Proof. Fix $c > 0$, and let $\Phi_c(t) = t^{-(1+c)}$. If (8.24) holds, then

$$(8.2) \asymp_{\times} \Sigma_{\delta(1+c)}(G) < +\infty,$$

so by Theorem 8.1 we have $\mu(\mathrm{WA}_{\Phi_c,\xi}) = 0$. Since c was arbitrary we have $\mu(\mathrm{VWA}_\xi) = 0$.
   Now let $\Psi(t) = t^{-1}$. If (8.24) holds, then

$$(8.1) \asymp_{\times} (8.2) \asymp_{\times} \Sigma_\delta(G),$$

so by Theorem 8.1, $\mu(\mathrm{BA}_\xi) = \mu(\mathrm{BA}_{\Phi,\xi}) = 0$ if and only if G is of divergence type. If (8.25) holds rather than (8.24), then

$$(8.1) \asymp_{+,\times} \int_{t=0}^{\infty} \Delta_{\mu,\xi}(K^{-1})\mathrm{d}t = +\infty$$

since $\xi \in \Lambda \implies \Delta_{\mu,\xi}(1/K) > 0$. Thus $\mu(\mathrm{BA}_\xi) = \mu(\mathrm{BA}_{\Phi,\xi}) = 0$. $\square$

Example 8.13. If ξ is a uniformly radial limit point, then (8.24) holds.

   The proof will show that if $\xi \in \Lambda_{\mathrm{ur},\sigma}$, then the implied constant of (8.24) depends only on σ.

Proof of Example 8.13. Let $(g_n)_1^\infty$ be a sequence such that $g_n(o) \xrightarrow[n]{} \xi$ σ-uniformly radially. Fix $0 < r < 1$, and let $n = n_r \in \mathbb{N}$ be maximal such that

$$b^{-d(o,g_n(o))} \geq r.$$

Now for each $\eta \in B(\xi, r)$, we have

$$\langle g_n(o)|\eta\rangle_o \gtrsim_+ \min(\langle g_n(o)|\xi\rangle_o, \langle\eta|\xi\rangle_o) \gtrsim_{+,\sigma} \min(d(o, g_n(o)), -\log_b(r)) = d(o, g_n(o)).$$

Letting $\tau > 0$ be the implied constant of this asymptotic, we have

$$B(\xi, r) \subseteq \mathrm{Shad}(g_n(o), \tau).$$

Now by Sullivan's Shadow Lemma,

$$\mu(B(\xi, r)) \leq \mu(\mathrm{Shad}(g_n(o), \tau)) \asymp_{\times,\tau} b^{-\delta d(o,g_n(o))}.$$

On the other hand, by (2.31), we have

$$b^{-d(o,g_n(o))} \asymp_{\times,\sigma} b^{-d(o,g_{n+1}(o))} < r,$$

and so

$$\mu(B(\xi, r)) \lesssim_{\times,\sigma} r^\delta.$$

The opposite direction is similar. $\square$

Example 8.14 (Quasiconvex-cocompact group). Let X be a proper and geodesic hyperbolic metric space, and let $G \leq \mathrm{Isom}(X)$. G is called *quasiconvex-cocompact* if the set $G(o)$ is *quasiconvex*, i.e. there exists $\rho > 0$ such that for all $x, y \in G(o)$,

$$(8.31) \qquad\qquad [x, y] \subseteq B(G(o), \rho).$$

Any quasiconvex-cocompact group is perfect.

Proof. By Lemma 8.20 below, we have $\Lambda = \Lambda_{\mathrm{ur},\sigma}$ for some $\sigma > 0$; without loss of generality, assume that σ is large enough so that Sullivan's Shadow Lemma holds. In particular, (8.24) follows from Example 8.13. To obtain (8.25), we generalize an argument of Sullivan (p. 180 of [72]). Fix $t \geq 0$, and let $A_t = \{x \in G(o) : (t - \sigma) \leq d(o, x) < t\}$. Then for all $\xi \in \Lambda = \Lambda_{\mathrm{ur},\sigma}$, there exists $x \in A_t$ such that $\xi \in \mathrm{Shad}(x, \sigma)$. Let $A_{t,\xi} := \{x \in A_t : \xi \in \mathrm{Shad}(x, \sigma)\}$, so that $A_{t,\xi} \neq \varnothing$. On the other hand, by the Intersecting Shadows Lemma, we have for $x_1, x_2 \in A_{t,\xi}$

$$d(x_1, x_2) \asymp_{+,\sigma} |\mathcal{B}_o(x_1, x_2)| \asymp_{+,\sigma} 0,$$

i.e. the set $A_{t,\xi} := \{x \in A_t : \xi \in \mathrm{Shad}(x, \sigma)\}$ has diameter bounded depending only on σ. Since G is strongly discrete, this means that the cardinality of $A_{t,\xi}$ is bounded depending only on σ; since $A_{t,\xi} \neq \varnothing$, we have

$$\#(A_{t,\xi}) \asymp_{\times,\sigma} 1.$$

Thus

$$1 = \mu(X) \asymp_{\times,\sigma} \int \#(A_{t,\xi}) \mathrm{d}\mu(\xi)$$

$$= \sum_{x \in A_t} \mu(\mathrm{Shad}(x, \sigma))$$

$$\asymp_{\times,\sigma} \sum_{x \in A_t} b^{-\delta d(o,x)} \qquad \text{(by Sullivan's Shadow Lemma)}$$

$$\asymp_{\times,\sigma} b^{-\delta t} \#(A_t).$$

So $\#(A_t) \asymp_{\times,\sigma} b^{\delta t}$; thus for all $n \in \mathbb{N}$

$$f_G(n\sigma) = \sum_{i=1}^{n} \#(A_{i\sigma}) \asymp_{\times,\sigma} \sum_{i=1}^{n} b^{\delta i \sigma}$$

$$\asymp_{\times,\sigma} b^{\delta n \sigma},$$

the last asymptotic following from the fact that $\sigma\delta > 0$. This proves (8.25) in the case $t \in \mathbb{N}\sigma$; the general case follows from an easy approximation argument. $\qquad\square$

Proposition 8.15. *Let $X = \mathbb{H}^{d+1}$ for some $d \in \mathbb{N}$, and let $G \leq \mathrm{Isom}(X)$ be geometrically finite. If ξ is a bounded parabolic point whose rank is strictly greater than $4\delta/3$, then there exists a function Φ such that $t \mapsto t\Phi(t)$ is nonincreasing and such that for every $K > 0$, (8.2) diverges but (8.1) converges.*

Proof. Let $k = k_\xi$ be the rank of ξ. Fix a function $\Phi : (0, +\infty) \to (0, +\infty)$ to be announced below and fix any $K > 0$. Then by Lemma 3.2 of [71] (a special case of the global measure formula),

$$(8.2) \asymp_{\times} \sum_{g \in G} (g'(\xi))^{\delta-k} \Phi^{2\delta-k}(Ke^{d(o,g(o))}).$$

Let G_ξ be the stabilizer of ξ in G, and let S be a ξ-bounded set satisfying $G(o) \subseteq G_\xi(S)$ (see Definition 2.48). Let A be a transversal of $G_\xi \backslash G$ satisfying $a(o) \in S$ for all $a \in A$.

Claim 8.16. *For $a \in A$ and $h \in G_\xi$,*

$$\langle a(o)|h(o)\rangle_o \asymp_+ 0.$$

Proof. By contradiction, suppose $\langle a_n(o)|h_n(o)\rangle_o \underset{n}{\to} +\infty$ with $a_n \in A$, $h_n \in G_\xi$. Then $d(o, h_n(o)) \underset{n}{\to} +\infty$, so by Observation 2.45 we have $h_n(o) \underset{n}{\to} \xi$. But then $a_n(o) \underset{n}{\to} \xi$, contradicting that S is ξ-bounded. $\qquad\triangleleft$

Clearly, the same holds with h replaced by h^{-1}, i.e.

$$0 \asymp_+ 2\langle a(o)|h^{-1}(o)\rangle_o = d(o, a(o)) + d(o, h(o)) - d(o, h \circ a(o)).$$

Writing $g = h \circ a$, we have

$$(8.32) \qquad\qquad d(o, g(o)) \asymp_+ d(o, a(o)) + d(o, h(o)).$$

Note that every $g \in G$ can be written uniquely as $h \circ a$ with $h \in G_\xi$, $a \in A$ by the transversality of A. On the other hand,

$$\mathcal{B}_\xi(g(o), o) = \mathcal{B}_\xi(h(o), o) + \mathcal{B}_{h^{-1}(\xi)}(a(o), o) = \mathcal{B}_\xi(a(o), o) = d(o, a(o)) - 2\langle a(o) | \xi \rangle_o \asymp_+ d(o, a(o)),$$

the last asymptotic being a restatement of the fact that $\xi \notin \overline{A(o)}$. Thus

$$(8.2) \asymp_\times \sum_{g \in G} e^{(\delta - k) \mathcal{B}_\xi(g(o), o)} \Phi^{2\delta - k}(K e^{d(o, g(o))})$$

$$\asymp_\times \sum_{a \in A} \sum_{h \in G_\xi} e^{(\delta - k) d(o, a(o))} \Phi^{2\delta - k}(K e^{d(o, h \circ a(o))})$$

$$\geq \sum_{a \in A} \sum_{h \in G_\xi} e^{(\delta - k) d(o, a(o))} \Phi^{2\delta - k}(K e^{d(o, a(o)) + d(o, h(o))}). \qquad \text{(since } \Phi \text{ is nonincreasing)}$$

Claim 8.17.

$$\#(A_t) := \#\{a \in A : d(o, a(o)) \leq t\} \asymp_\times e^{\delta t}.$$

Proof. By Example 1.45, we have $f_G(t) \asymp_\times e^{\delta t}$; in particular, since $\#(A_t) \leq f_G(t)$ we have

$$(8.33) \qquad\qquad\qquad\qquad \#(A_t) \lesssim_\times e^{\delta t}.$$

To establish the lower bound, let C be the implied constant of (8.32). We have

$$
\begin{aligned}
e^{\delta t} \asymp_\times e^{\delta(t-C)} \asymp_\times f_G(t-C) = {}& \#\{(h, a) \in G_\xi \times A : d(o, h \circ a(o)) \leq t - C\} \\
\leq {}& \#\{(h, a) \in G_\xi \times A : d(o, h(o)) + d(o, a(o)) \leq t\} \qquad \text{(by (8.32))} \\
\leq {}& \sum_{n=0}^{\lfloor t \rfloor} \#\{(h, a) \in G_\xi \times A : n \leq d(o, h(o)) < n+1, d(o, a(o)) \leq t - n\} \\
= {}& \sum_{n=0}^{\lfloor t \rfloor} \#\{h \in G_\xi : n \leq d(o, h(o)) < n+1\} \#(A_{t-n}) \\
\leq {}& \sum_{n=0}^{\lfloor t \rfloor} \#\{h \in G_\xi : d(o, h(o)) < n+1\} \#(A_{t-n}) \\
\asymp_\times {}& \sum_{n=0}^{\lfloor t \rfloor} e^{\delta_\xi n} \#(A_{t-n}).
\end{aligned}
$$

The last asymptotic follows from (1.12) and (2.35). Now fix $N \in \mathbb{N}$ large to be determined. We have

$$
\begin{aligned}
e^{\delta t} \lesssim_\times {}& \sum_{n=0}^{N-1} e^{\delta_\xi n} \#(A_{t-n}) + \sum_{n=N}^{\lfloor t \rfloor} e^{\delta_\xi n} \#(A_{t-n}) \\
\lesssim_\times {}& \sum_{n=0}^{N-1} e^{\delta_\xi n} \#(A_{t-n}) + \sum_{n=N}^{\lfloor t \rfloor} e^{\delta_\xi n} e^{\delta(t-n)} \qquad\qquad \text{(by (8.33))} \\
\leq {}& \sum_{n=0}^{N-1} e^{\delta_\xi n} \#(A_{t-n}) + e^{\delta t} \sum_{n=N}^{\infty} e^{(\delta_\xi - \delta) n} \\
\asymp_\times {}& \sum_{n=0}^{N-1} e^{\delta_\xi n} \#(A_{t-n}) + e^{\delta t} e^{(\delta_\xi - \delta) N}. \qquad\qquad \text{(since } \delta_\xi < \delta)
\end{aligned}
$$

Let C_2 be the implied constant, and let N be large enough so that

$$e^{(\delta - \delta_\xi) N} \geq 2 C_2.$$

Then

$$e^{\delta t} \leq C_2 \sum_{n=0}^{N-1} e^{\delta_\xi (n+1)} \#(A_{t-n}) + \frac{1}{2} e^{\delta t};$$

rearranging gives

$$e^{\delta t} \lesssim_\times \sum_{n=0}^{N-1} e^{\delta_\xi (n+1)} \#(A_{t-n}) \lesssim_{\times, N} \#(A_t).$$

□

Thus by an argument similar to the argument used in the proof of Proposition 8.11, we have

$$\sum_{a \in A} f(d(o, a(o))) \asymp_\times f(0) + \int_{t=0}^{\infty} f(t) e^{\delta t} \mathrm{d}t$$

for every nonincreasing function $f \to 0$. Thus

$$(8.2) \gtrsim_\times \sum_{a \in A} \sum_{h \in G_\xi} e^{(\delta-k)d(o, a(o))} \Phi^{2\delta-k}(K e^{d(o, a(o)) + d(o, h(o))})$$

$$\gtrsim_\times \int_{t=0}^{\infty} e^{\delta t} \left[\sum_{h \in G_\xi} e^{(\delta-k)t} \Phi^{2\delta-k}(K e^{t + d(o, h(o))}) \right] \mathrm{d}t.$$

Substituting $s = t + d(o, h(o))$ (and letting $\|h\| = d(o, h(o))$),

$$(8.2) \gtrsim_{+, \times} \int_{s=0}^{\infty} \sum_{\substack{h \in G_\xi \\ \|h\| \leq s}} e^{(2\delta-k)[t - \|h\|]} \Phi^{2\delta-k}(K e^s) \mathrm{d}s.$$

Now

$$\sum_{\substack{h \in G_\xi \\ \|h\| \leq s}} e^{(2\delta-k)[-\|h\|]} \asymp_\times \int_{\substack{\mathbf{x} \in \mathbb{R}^k \\ \|\mathbf{x}\| \leq s}} (1 + \|\mathbf{x}\|^2)^{2\delta-k} \mathrm{d}\mathbf{x}$$

$$\asymp_\times \max(s^{3k-4\delta}, 1)$$

and so

$$(8.2) \gtrsim_{+, \times} \int_{s=0}^{\infty} \max(s^{3k-4\delta}, 1) e^{(2\delta-k)s} \Phi^{2\delta-k}(K e^s) \mathrm{d}s.$$

In particular, since $3k - 4\delta > 0$, Φ may be chosen in such a way so that this integral diverges whereas the integral

$$(8.1) \asymp_\times \int_{s=0}^{\infty} e^{(2\delta-k)s} \Phi^{2\delta-k}(K e^s) \mathrm{d}s$$

converges. For example, let

$$\Phi(e^s) = e^{-s} s^{[-1-(3k-4\delta)]/(2\delta-k)};$$

then the series reduce to

$$(8.2) \gtrsim_{+, \times} \int_{s=1}^{\infty} s^{-1} \mathrm{d}s = +\infty,$$

$$(8.1) \asymp_\times \int_{s=1}^{\infty} s^{-1-(3k-4\delta)} \mathrm{d}s < +\infty.$$

□

8.4. **The measure of Λ_{ur}.** We end this section by proving the following proposition, which was stated in the introduction:

Proposition 8.18. *Let (X, d, o, b, G) be as in §1.1.4, with X proper and geodesic. Let μ be a δ_G-quasiconformal measure on Λ. Then $\mu(\Lambda_{\mathrm{ur}}) = 0$ if and only if G is not quasiconvex-cocompact.*

To prove Proposition 8.18 we will need the following lemmas:

Lemma 8.19. *Let (X, d, o, b, G) be as in §1.1.4. If $\xi \in \partial X \setminus \Lambda_{\mathrm{h}}$, then*
$$\Lambda_{\mathrm{ur}} \subseteq \mathrm{BA}_\xi.$$

Proof. By contradiction fix $\eta \in \Lambda_{\mathrm{ur}} \cap \mathrm{WA}_\xi$. Let $g_m(o) \underset{m}{\longrightarrow} \eta$ uniformly radially, and let $(h_n)_1^\infty$ be such that
$$D(h_n(\xi), \eta) \leq b^{-d(o, h_n(o)) - n}.$$
For each n, there exists m_n such that $d(o, g_{m_n}(o)) \asymp_+ d(o, h_n(o)) + n$. Let $h = h_n$ and $g = g_{m_n}$; we have
$$\begin{aligned}
\mathcal{B}_\xi(o, h^{-1} \circ g(o)) &\asymp_+ \mathcal{B}_{h(\xi)}(h(o), g(o)) \\
&\asymp_+ \mathcal{B}_{h(\xi)}(h(o), o) + \mathcal{B}_{h(\xi)}(o, g(o)) \\
&\gtrsim_+ 2\langle h(\xi), g(o)\rangle_o - d(o, h(o)) - d(o, g(o)) \\
&\asymp_+ 2\langle h(\xi), g(o)\rangle_o - 2d(o, h(o)) - n.
\end{aligned}$$
Now,
$$\begin{aligned}
\langle h(\xi), g(o)\rangle_o &\gtrsim_+ \min(\langle h(\xi), \eta\rangle_o, \langle g(o), \eta\rangle_o) &&\text{(by Gromov's inequality)} \\
&\gtrsim_+ \min(d(o, h(o)) + n, d(o, g(o))) &&\text{(since } \eta \in \mathrm{Shad}(g(o), \sigma)) \\
&\asymp_+ d(o, h(o)) + n,
\end{aligned}$$
and so
$$\mathcal{B}_\xi(o, h^{-1} \circ g(o)) \gtrsim_+ n.$$
Thus $h_n^{-1} \circ g_{m_n}(o) \underset{n}{\to} \xi$ horospherically, contradicting that $\xi \notin \Lambda_{\mathrm{h}}$. $\qquad\square$

Lemma 8.20. *Let (X, d, o, b, G) be as in §1.1.4, with X proper and geodesic. The following are equivalent:*
  (A) *G is quasiconvex-cocompact.*
  (B) *$\Lambda = \Lambda_{\mathrm{ur}, \sigma}$ for some $\sigma > 0$.*
  (C) *$\Lambda = \Lambda_{\mathrm{h}}$.*

Proof. (B) $\Rightarrow$ (C) is obvious as $\Lambda_{\mathrm{ur}, \sigma} \subseteq \Lambda_{\mathrm{ur}} \subseteq \Lambda_{\mathrm{h}}$. To demonstrate (A) $\Rightarrow$ (B), suppose that G is quasiconvex-cocompact, let $\rho > 1$ be as in the definition of quasiconvex-cocompact, and fix $\xi \in \Lambda$. Fix $n \in \mathbb{N}$, and let $z_n \in G(o)$ be such that
$$\langle z_n | \xi\rangle_o \geq n;$$
such a z_n exists since $\xi \in \Lambda$. Let $y_n \in [o, z_n]$ be the unique point so that $d(o, y_n) = n$. Fix $x_n \in G(o) \cap B(y_n, \rho)$; such a point exists by (8.31). Without loss of generality, we let $x_1 = o$; this is possible since $o \in G(o) \cap B(y_1, \rho)$.

Claim 8.21. *$x_n \underset{n}{\to} \xi$ σ-uniformly radially, with $\sigma > 0$ independent of ξ.*

Proof.
$$\begin{aligned}
\langle x_n | \xi\rangle_o &\gtrsim_+ \min(\langle x_n | y_n\rangle_o, \langle y_n | z_n\rangle_o, \langle z_n | \xi\rangle_o) \\
&\geq \min(d(o, y_n) - d(o, x_n), d(o, y_n), n) \\
&\geq \min(n - \rho, n, n) = n - \rho \asymp_\rho n \asymp_\rho d(o, x_n).
\end{aligned}$$
This demonstrates that $x_n \underset{n}{\to} \xi$ radially. Now the Intersecting Shadows Lemma together with the asymptotic $d(o, x_n) \asymp_{+, \rho} n$ implies that $x_n \underset{n}{\to} \xi$ uniformly radially. The implied constants depend only on ρ and not on ξ. $\qquad\triangleleft$

Finally, we demonstrate (C) $\Rightarrow$ (A). Let

$$\mathcal{D}_{o,1} = \{x \in X : \mathcal{B}_x(o, g(o)) \leq 1 \;\; \forall g \in G\}$$

be the 1-*approximate Dirichlet domain centered at* o, and let

$$\mathcal{C}_G = \bigcup_{x,y \in G(o)} [x, y]$$

be the *approximate convex core* of G.

Claim 8.22. $\mathcal{C}_G \cap \mathcal{D}_{o,1}$ *is bounded.*

Proof. Since X is proper and geodesic, $\overline{X}$ is compact (Exercise III.H.3.18(4) of [11]), and so if $\mathcal{C}_G \cap \mathcal{D}_{o,1}$ is unbounded, then there is a point $\xi \in \overline{\mathcal{C}_G \cap \mathcal{D}_{o,1}} \cap \partial X$. We claim that $\xi \in \Lambda \setminus \Lambda_{\mathrm{h}}$, contradicting our hypothesis. Indeed, by Lemma 2.11, since $\xi \in \overline{\mathcal{D}_{o,1}}$ we have $\mathcal{B}_\xi(o, g(o)) \lesssim_+ 1 \;\; \forall g \in G$, which demonstrates that $\xi \notin \Lambda_{\mathrm{h}}$. On the other hand, since $\xi \in \overline{\mathcal{C}_G}$, there exists a sequence $x_n \xrightarrow[n]{} \xi$ in $\mathcal{C}_G$; for each $n \in \mathbb{N}$, there exist $y_n, z_n \in G(o)$ such that $x_n \in [y_n, z_n]$. Without loss of generality suppose that $\langle x_n | y_n \rangle_o \leq \langle x_n | z_n \rangle_o$ Now

$$\begin{aligned}
2\langle x_n | z_n \rangle_o &\geq \langle x_n | y_n \rangle_o + \langle x_n | z_n \rangle_o \\
&= d(o, x_n) + \langle y_n | z_n \rangle_o && \text{(since } \langle y_n | z_n \rangle_{x_n} = 0) \\
&\geq d(o, x_n) \xrightarrow[n]{} +\infty.
\end{aligned}$$

Thus $z_n \xrightarrow[n]{} \xi$, and so $\xi \in \Lambda$. $\triangleleft$

Write $\mathcal{C}_G \cap \mathcal{D}_{o,1} \subseteq B(o, \rho)$ for some $\rho > 0$. We claim that (8.31) holds for all $x, y \in G(o)$. Indeed, fix $z \in [x, y]$, and let $g \in G$ be such that

$$d(z, g(o)) < d(z, G(o)) + 1.$$

Then $g^{-1}(z) \in [g^{-1}(x), g^{-1}(y)] \subseteq \mathcal{C}_G$; moreover, for all $h \in G$

$$\begin{aligned}
\mathcal{B}_{g^{-1}(z)}(o, h(o)) &\leq d(o, g^{-1}(z)) - d(g^{-1}(z), G(o)) \\
&= d(z, g(o)) - d(z, G(o)) \leq 1,
\end{aligned}$$

i.e. $g^{-1}(z) \in \mathcal{D}_{o,1}$. Thus $g^{-1}(z) \in B(o,)$, and in particular

$$z \in B(g(o), \rho) \subseteq B(G(o), \rho).$$

$\square$

Proof of Proposition 8.18. Suppose first that G is quasiconvex-cocompact. Then by Lemma 8.20, $\Lambda = \Lambda_{\mathrm{ur}}$, so $\mu(\Lambda_{\mathrm{ur}}) = 1$.

Next, suppose that G is not quasiconvex-cocompact. By Lemma 8.20, there exists a point $\xi \in \Lambda \setminus \Lambda_{\mathrm{h}}$, and by Lemma 8.19 we have $\Lambda_{\mathrm{ur}} \subseteq \mathrm{BA}_\xi$. Now if G is of divergence type, we are done, since by (i) of Theorem 8.1 we have $\mu(\mathrm{BA}_\xi) = 0$. On the other hand, if G is of convergence type, then for each $\sigma > 0$ we have

$$\sum_{g \in G} \mu(\mathrm{Shad}(g(o), \sigma)) \lesssim_\times \sum_{g \in G} b^{-\delta d(o, g(o))} < +\infty,$$

and so by the Borel-Cantelli lemma, we have $\mu(\Lambda_{\mathrm{r},\sigma}) = 0$. Since σ was arbitrary we have $\mu(\Lambda_{\mathrm{r}}) = 0$, and in particular $\mu(\Lambda_{\mathrm{ur}}) = 0$. $\square$

9. Proof of Theorem 9.3 (BA_d has full dimension in $\Lambda_{\text{r}}(G)$)

In this section, fix $d \in \mathbb{N}$, let $X = \mathbb{H}^{d+1}$, let $Z = \mathbb{R}^d$, and let $G \leq \text{Isom}(X)$ be a discrete group. We are interested in approximating the points of $\Lambda_{\text{r}}(G)$ by rational vectors. We recall the following definitions:

Definition 9.1. A vector $\mathbf{x} \in \mathbb{R}^d$ is called *badly approximable* if for all $\mathbf{p}/q \in \mathbb{Q}^d$, we have

$$\left\| \mathbf{x} - \frac{\mathbf{p}}{q} \right\| \gtrsim_\times \frac{1}{q^{1+1/d}}.$$

The set of badly approximable vectors in $\mathbb{R}^d$ will be denoted BA_d.

Definition 9.2. We define the *geodesic span* of a set $S \subseteq \overline{\mathbb{H}^{d+1}}$ to be the smallest totally geodesic subspace of $\mathbb{H}^{d+1}$ whose closure contains S.

We say that G *acts irreducibly on* $\mathbb{H}^{d+1}$ if it is nonelementary and if there is no nonempty proper[40] totally geodesic subspace $V \subseteq \mathbb{H}^{d+1}$ such that $G(V) = V$. Equivalently, G acts irreducibly if there is no proper totally geodesic subspace of $\mathbb{H}^{d+1}$ whose closure contains Λ_G.

The remainder of this section will be devoted to proving the following theorem:

Theorem 9.3. *Fix $d \in \mathbb{N}$, let $X = \mathbb{H}^{d+1}$, and let $G \leq \text{Isom}(X)$ be a discrete group acting irreducibly on X. Then*

$$\text{HD}(\text{BA}_d \cap \Lambda_{\text{ur}}) = \delta = \text{HD}(\Lambda_{\text{r}}),$$

where BA_d denotes the set of badly approximable vectors in $\mathbb{R}^d$.

Notation 9.4. There are two natural metrics to put on $\mathbb{R}^d$: the usual Euclidean metric

$$D_{\text{e}}(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|,$$

and the *spherical metric*

$$D_{\text{s}} := D_{e,o},$$

where $o = (1, 0, \ldots, 0) \in \mathbb{H}^{d+1}$. From now on, we will use the subscripts e and s to distinguish between these metrics.

We recall the following definition and theorem:

Definition 9.5. A closed set $K \subseteq \mathbb{R}^d$ is said to be *hyperplane diffuse* if there exists $\gamma > 0$ such that for every $\mathbf{x} \in K$, for every $0 < r \leq 1$, and for every affine hyperplane $\mathcal{L} \subseteq \mathbb{R}^d$, we have

$$K \cap B_{\text{e}}(\mathbf{x}, r) \setminus \mathcal{L}^{(\gamma r)_{\text{e}}} \neq \varnothing.$$

Here $\mathcal{L}^{(r)_{\text{e}}}$ denotes the r-thickening of $\mathcal{L}$ with respect to the Euclidean metric D_{e}.

Theorem 9.6. *Let K be a subset of $\mathbb{R}^d$ which is Ahlfors regular and hyperplane diffuse. Then*

$$\text{HD}(\text{BA}_d \cap K) = \text{HD}(K).$$

Proof. See Theorems 2.5, 4.7, and 5.3 from [12]. $\qquad\qquad\square$

Remark 9.7. The conclusion of Theorem 9.6, and thus of Theorem 9.3, holds when BA_d is replaced by any subset of $\mathbb{R}^d$ which is hyperplane absolute winning (see [12] for the definition).

Fix $0 < s < \delta_G$. According to Corollary 5.8, there exists an Ahlfors s-regular set $\mathcal{J}_s \subseteq \Lambda_{\text{ur}}(G)$.

Claim 9.8. *The proof of Corollary 5.8 can be modified so that the resulting set $\mathcal{J}_s$ is hyperplane diffuse.*

Proof of Theorem 9.3 assuming Claim 9.8. By Theorem 9.6, we have $\text{HD}(\text{BA}_d \cap \Lambda_{\text{ur}}) \geq \text{HD}(\mathcal{J}_s) = s$ for every $0 < s < \delta_G$; letting $s \to \delta$ finishes the proof. $\qquad\qquad\square$

---

[40]In this section "proper" means "not equal to the entire space", not "bounded closed sets are compact".

Proof of Claim 9.8. The first modification to be made is in the proof of Lemma 5.14 assuming Sublemma 5.17. Instead of picking just two t-divergent points η_1 and η_2, let us notice that the set of t-divergent points is closed and invariant under the action of the group. Thus by a well-known theorem, the set of t-divergent points contains the entire limit set of G.

Recall that a collection of $(d+2)$ points $\eta_1, \ldots, \eta_{d+2} \in \partial X$ is in *general position* if there is no totally geodesic subspace $V \subseteq \mathbb{H}^{d+1}$ such that $\eta_1, \ldots, \eta_{d+2} \in \overline{V}$.

Claim 9.9. *There exist $(2d+4)$ points $\eta_1, \ldots, \eta_{2d+4} \in \Lambda_G$, such that for each $i = 1, \ldots, d+2$, the collection*

$$\tag{9.1} \eta_1, \ldots, \widehat{\eta_i}, \ldots, \eta_{d+2}, \eta_{d+2+i}$$

is in general position. (Here the hat indicates that η_i is omitted.)

Proof. First, pick $(d+2)$ points $\eta_1, \ldots, \eta_{d+2} \in \Lambda_G$ in general position; this is possible since G acts irreducibly on $\mathbb{H}^{d+1}$. Then, for each $i = 1, \ldots, d+2$, we may choose $\eta_{d+2+i} \in \Lambda_G$ so that the collection (9.1) is in general position by choosing $\eta_{d+2+i} \in \Lambda_G$ sufficiently close to η_i; this is possible since by a well-known theorem, the limit set Λ_G is perfect. $\qquad\square$

The following observation can be proven by an elementary compactness argument:

Observation 9.10. For each $i = 1, \ldots, d+2$, there exists $\varepsilon_i > 0$ such that if V is a proper geodesic subspace of X, if $\partial V := \overline{V} \cap \partial X$, and if $\partial V^{(\varepsilon_i)_s}$ is the ε_i-thickening of ∂V with respect to the spherical metric, then $\partial V^{(\varepsilon_i)_s}$ does not contain all $(d+2)$ of the points (9.1).

Let

$$\varepsilon = \frac{1}{2} \min_{i=1}^{d+2} \varepsilon_i.$$

We observe that $D_s(\eta_i, \eta_j) \geq 2\varepsilon$ for all $i \neq j$. For each $i = 1, \ldots, d+2$ let

$$B_i = B_s(\eta_i, \varepsilon) \cup \{x \in X : \mathrm{Shad}(x, \sigma) \subseteq B_s(\eta_i, \varepsilon)\};$$

the sets $(B_i)_1^{2d+4}$ are open and disjoint.

Now apply Sublemma 5.17 to each pair (η_i, B_i) for $i = 1, \ldots, 2d+4$ to get sets $S_i \subseteq G(o) \cap B_i$. For each $i = 1, \ldots, 2d+4$, fix a point $p_i \in S_i$. Suppose that $w = g_w(o) \in G(o)$, and let $z = g_w^{-1}(o)$. Let

$$T(w) = \bigcup_{\substack{i=1 \\ z \notin B_i}}^{2d+4} g_w(S_i).$$

Note that at most one index i is omitted from the union. Now (i)-(iii) of Sublemma 5.17 directly imply (i)-(iii) of Lemma 5.14. Note that here we need the fact that $\mathcal{P}_{z,x} \subseteq B_i$ for $x \in S_i$ to prove disjointness in (i) of Lemma 5.14 (whereas we did not need it in the earlier proof).

The next change to be made in the proof of Corollary 5.8 is in the application of Theorem 5.12 to Lemma 5.13. This time, we will want to use the second part of Theorem 5.12 - the tree $\widetilde{T}$ can be chosen so that for each $\omega \in \widetilde{T}$, we have that $\widetilde{T}(\omega)$ is an initial segment of $T(\omega)$. Obviously, the application of this additional information depends on the ordering which is given to the set $\widetilde{T}(\omega)$. In the original proof of Lemma 5.13, we used an ordering which had no meaning - any enumeration of $G(o)$. Now, we will choose some ordering so that the points $g_w(p_1), \ldots, g_w(p_{2d+4})$ (possibly minus one) mentioned above are first in the ordering.

By increasing $n_1, \ldots, n_{2d+4}$ if necessary, we can ensure that the points $g_w(p_1), \ldots, g_w(p_{2d+4})$ are all in $\widetilde{T}(w)$ for every w, since there is no way that (5.8) could hold if $N_w < 2d+4$.

Now we have constructed the set $\mathcal{J}_s$, and we come to the crucial point: with these modifications, the new set $\mathcal{J}_s$ is hyperplane diffuse.

Definition 9.11. A set $K \subseteq \widehat{\mathbb{R}^d}$ is *hyperbolically hyperplane diffuse* if there exists $\gamma > 0$ so that every $\eta \in K$, for every $0 < r \leq 1$, and for every totally geodesic subspace $V \subseteq X$, we have

$$K \cap B_s(\eta, r) \setminus \partial V^{(\gamma r)_s} \neq \varnothing.$$

Here (and from now on), balls and thickenings are all assumed to be with respect to the spherical metric D_s.

Proposition 9.12. *Hyperbolically hyperplane diffuse sets are hyperplane diffuse.*

(Proof left to the reader.)
Thus we are left with showing that $\mathcal{J}_s$ is hyperbolically hyperplane diffuse.
Fix $\gamma > 0$ to be announced below, $\eta \in \mathcal{J}_s$, $0 < r \le \varepsilon S_\varnothing$, and a totally geodesic subspace $V \subseteq X$.

Notation 9.13. For each $x \in G(o)$ let
$$\mathcal{P}_x := \operatorname{Shad}(x, \sigma).$$
Let $(x_n)_1^\infty$ be an enumeration of $G(o)$. Let
$$T = \bigcup_{n=1}^\infty \{\omega \in \mathbb{N}^n : x_{\omega_{j+1}} \in T(x_{\omega_j}) \ \forall j = 1, \dots, n-1\},$$
and for each $\omega \in T$ let
$$x_\omega = x_{\omega_{|\omega|}} \text{ and } \mathcal{P}_\omega = \mathcal{P}_{x_\omega}.$$
Let $\eta = \pi(\omega)$; for each $n \in \mathbb{N}$ let
$$x_n = x_{\omega_1^n}, \ \mathcal{P}_n = \mathcal{P}_{\omega_1^n}, \text{ and } D_n = D_{\omega_1^n}.$$

Let $n \in \mathbb{N}$ be the largest integer such that $r < \kappa D_n$. Then by the argument used in the proof of Theorem 5.12, we have
$$\kappa^2 D_n \le r \le \kappa D_n$$
and
$$\mathcal{P}_{n+k} \subseteq B(\eta, r) \subseteq \mathcal{P}_n,$$
where k is large enough so that $\lambda^k \le \kappa^2$. In particular
$$\mathcal{J}_s \cap B(\eta, r) \setminus \partial V^{(\gamma r)} \supseteq \mathcal{J}_s \cap \mathcal{P}_{n+k} \setminus \partial V^{(\gamma r)},$$
so to complete the proof it suffices to show
$$(9.2) \qquad\qquad \mathcal{J}_s \cap \mathcal{P}_{n+k} \nsubseteq \partial V^{(\gamma r)}.$$
By contradiction suppose not. Let $g \in G$ be such that $g(o) = x_{n+k}$, and let $z = g^{-1}(o)$. Then for some $j = 1, \dots, d+2$ we have
$$g(p_1), \dots, \widehat{g(p_j)}, \dots, g(p_{d+2}), g(p_{d+2+j}) \in \widetilde{T}(x_{n+k}).$$
For each $i = 1, \dots, \widehat{j}, \dots, d+2, d+2+j$, we have
$$\varnothing \ne \mathcal{J}_s \cap \mathcal{P}_{g(p_i)} \subseteq \mathcal{J}_s \cap \mathcal{P}_{n+k} \subseteq \partial V^{(\gamma r)},$$
which implies
$$\mathcal{P}_{g(p_i)} \cap \partial V^{(\gamma r)} \ne \varnothing$$
and thus
$$\operatorname{Shad}_z(p_i, \sigma) \cap g^{-1}\left(\partial V^{(\gamma r)}\right) \ne \varnothing.$$
On the other hand, by Sublemma 5.17 we have
$$\operatorname{Shad}_z(p_i, \sigma) \subseteq B_i,$$
so
$$B_i \cap g^{-1}\left(\partial V^{(\gamma r)}\right) \ne \varnothing.$$
Now,
$$g^{-1}\left(\partial V^{(\gamma r)}\right) \subseteq \left(\partial g^{-1}(V)\right)^{\left(e^{d(o, g(o))} \gamma r\right)};$$
on the other hand
$$r \asymp_\times D_n \asymp_\times D_{n+k} \asymp_{\times, \sigma} e^{-d(o, x_{n+k})} = e^{-d(o, g(o))};$$

letting $C_4 > 0$ be the implied asymptotic, we have

$$g^{-1}\left(\partial V^{(\gamma r)}\right) \subseteq \partial W^{(C_4\gamma)},$$

where $W = g^{-1}(V)$. Thus

$$B_i \cap \partial W^{(C_4\gamma)} \neq \varnothing.$$

Letting $\gamma = \varepsilon/C_4$, this contradicts Observation 9.10. $\qquad\square$

APPENDIX A. ANY FUNCTION IS AN ORBITAL COUNTING FUNCTION FOR SOME PARABOLIC GROUP

Let (X, d, o, b, G) be as in §1.1.4. In Subsection 7.6, we defined the *orbital counting function* of a parabolic fixed point ξ of G to be the function

$$(7.42) \qquad\qquad f_\xi(R) := \#\{g \in G_\xi : D_e(o, g(o)) \leq R\},$$

where G_ξ is the stabilizer of ξ in G and $D_e = D_{\xi,o}$. We remarked that when X is a real, complex, or quaternionic hyperbolic space, the orbital counting function f_ξ satisfies a power law, but that this is not true in general. Since the proof of Theorem 7.33 would be simplified somewhat if f_ξ were assumed to satisfy a power law, we feel that it is important to give some examples where it is not satisfied, to justify the extra work. Moreover, these examples corroborate the statement made in Remark 1.27 that the fact that (7.41) depends only on the Poincaré exponents δ and δ_ξ is surprising.

To begin with, let us give some background on the infinite-dimensional hyperbolic space $\mathbb{H}^\infty$. Its boundary $\mathcal{H} := \partial\mathbb{H}^\infty$ is itself a Hilbert space. As in finite dimensions, any isometry of $\mathcal{H}$ with respect to the Euclidean metric extends uniquely to an isometry of $\mathbb{H}^\infty$ which ∞. So immediately, there is a correspondence between parabolic subgroups of $\mathrm{Stab}(\mathrm{Isom}(\mathbb{H}^\infty); \infty)$ and subgroups of $\mathrm{Isom}(\mathcal{H})$ whose orbits are unbounded. However, unlike in finite dimensions, strongly discrete subgroups of $\mathrm{Isom}(\mathcal{H})$ are not necessarily virtually nilpotent. Groups which embed as strongly discrete subgroups of $\mathrm{Isom}(\mathcal{H})$ are said to have the *Haagerup property*, and they include both the amenable groups and the free groups. Moreover, even cyclic subgroups of $\mathrm{Isom}(\mathcal{H})$ are quite different from cyclic subgroups of $\mathrm{Isom}(\mathbb{R}^d)$ for $d < +\infty$; for example, a well-known example of M. Edelstein [23] is a cyclic subgroup of $\mathrm{Isom}(\mathcal{H})$ whose orbits are unbounded but which is not strongly discrete.

To relate the orbital counting function f_ξ with the intrinsic structure of the Hilbert space $\mathcal{H}$, we need the following lemma:

Lemma A.1. *For* $g \in \mathrm{Stab}(\mathrm{Isom}(\mathbb{H}^\infty); \infty)$*,*

$$D_e(\mathbf{e}_0, g(\mathbf{e}_0)) \asymp_\times \max(1, \|g(\mathbf{e}_0) - \mathbf{e}_0\|) = \max(1, \|g(\mathbf{0})\|).$$

Proof. By (2.35), we have

$$D_e(\mathbf{e}_0, g(\mathbf{e}_0)) \asymp_\times e^{(1/2)d(\mathbf{e}_0, g(\mathbf{e}_0))}.$$

The remaining asymptotic is a geometric calculation which is left to the reader, and the equality follows from the fact that g is an affine map whose linear part preserves $\mathbf{e}_0$. $\qquad\square$

For the remainder of the appendix, rather than working with f_∞, we will work with the modification

$$\widetilde{f}_\infty(R) := \{g \in G_\infty : \|g(\mathbf{0})\| \leq R\}.$$

The relation between f_∞ and $\widetilde{f}_\infty$ can be deduced easily from Lemma A.1.

Proposition A.2. *For any nondecreasing function* $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ *with* $f \to \infty$*, there exists a parabolic group* $G_\infty \leq \mathrm{Stab}(\mathrm{Isom}(\mathbb{H}^\infty); \infty)$ *such that*

$$\widetilde{f}_\infty(2^n) = f(2^n)$$

for all $n \in \mathbb{N}$*.*

Remark A.3. A similar statement holds for proper $\mathbb{R}$-trees.

Proof. For each $n \in \mathbb{N}$, let

$$m_n = \frac{f(2^{n+1})}{f(2^n)} \in 2^{\mathbb{N}}$$

and let $T_n : \mathbb{R}^{m_n} \to \mathbb{R}^{m_n}$ be an isometry of order m_n such that any two points in the orbit of $\mathbf{0}$ have a distance of exactly $2^{n-1/2}$. For example, we can let

$$T_n(\mathbf{x}) = \left(x_{k-1}\right)_{k=1}^{m_n} + 2^{n-1}(\mathbf{e}_2 - \mathbf{e}_1).$$

(Here by convention $x_0 = x_{m_n}$.) We will also denote by T_n the extension of T_n to the product

$$\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathbb{R}^{m_n}$$

obtained by letting T_n act as the identity in the other dimensions. By construction, the T_ns commute. Let $G = \langle T_n \rangle_{n \in \mathbb{N}}$. Then each element of G can be written in the form $T = \prod_{n=1}^{\infty} T_n^{q_n}$, where $(q_n)_1^{\infty}$ is a sequence of integers satisfying $0 \le q_n < m_n$, only finitely many of which are nonzero. Such an isometry satisfies

$$\|T(\mathbf{0})\|^2 = \sum_{n=1}^{\infty} \|T_n^{q_n}\|^2 = \sum_{\substack{n \in \mathbb{N} \\ q_n \ne 0}} 4^{n-1/2} \in \left[\frac{1}{2} 4^{n_{\max}}, 4^{n_{\max}}\right],$$

where

$$n_{\max} = \max\{n \in \mathbb{N} : q_n \ne 0\}.$$

In particular, for all $N \in \mathbb{N}$

$$\|T(\mathbf{0})\| \le 2^N \quad \Leftrightarrow \quad N \ge n_{\max},$$

and thus

$$\widetilde{f}_{\infty}(2^N) = \#\{(q_n)_1^{\infty} : n_{\max} \le N\} = \#\{(q_n)_1^N\} = \prod_{n=1}^{N} m_n = f(2^N).$$

$\square$

Corollary A.4. *For any nondecreasing function* $f : (0, +\infty) \to (0, +\infty)$ *with* $f \to \infty$, *there exists a parabolic group* $G \le \mathrm{Stab}(\mathrm{Isom}(\mathbb{H}^{\infty}); \infty)$ *such that*

$$\widetilde{f}_{\infty}(2^n) \asymp_{\times} f(2^n)$$

for all $n \in \mathbb{N}$.
   If the function f *satisfies* $f(2^{n+1}) \asymp_{\times} f(2^n)$, *then*

$$\widetilde{f}_{\infty}(R) \asymp_{\times} f(R) \ \ \forall R \ge 1.$$

Proof. The first assertion is immediate upon applying the above example to the function $\widetilde{f}(2^n) = 2^{\lfloor \log_2 f(2^n) \rfloor}$, and the second follows from the fact that the functions f and f_{ξ} are both nondecreasing. $\square$

Appendix B. Real, complex, and quaternionic hyperbolic spaces

In this appendix, we give some background on the "non-exceptional" rank one symmetric spaces of noncompact type, i.e. real, complex, and quaternionic hyperbolic spaces, and prove the assertions made about them in Remarks 1.29 and 1.47 and Example 1.41. Our presentation loosely follows §II.10 of [11].

Let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ denote the sets of real, complex, and quaternionic numbers, respectively. Fix $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $d \in \mathbb{N}$. Note that since the noncommutative $\mathbb{F} = \mathbb{H}$ is a possibility, we will have to be careful about the order products are written in. We define a sesquilinear inner product on $\mathbb{F}^{d+1}$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^{d+1} \overline{x_i} y_i.$$

Let $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

We define (cf. II.10.24 of [11]) $(d+1)$-*dimensional* $\mathbb{F}$-*hyperbolic space* to be the set

$$\mathbb{H}_{\mathbb{F}}^{d+1} = \{\mathbf{x} \in \mathbb{F}^{d+1} : \|\mathbf{x}\| < 1\}$$

equipped with the metric

$$\cosh d(\mathbf{x}, \mathbf{y}) = \frac{|1 - \langle \mathbf{x}, \mathbf{y} \rangle|}{\sqrt{1 - \|\mathbf{x}\|^2}\sqrt{1 - \|\mathbf{y}\|^2}}.$$

Proposition B.1 (Theorem II.10.10 of [11])**.** $\mathbb{H}_{\mathbb{F}}^{d+1}$ *is a CAT(-1) space, and in particular (by Proposition III.H.1.2 of [11]) is a hyperbolic metric space.*

Since $X := \mathbb{H}_{\mathbb{F}}^{d+1}$ is an open subset of $\mathbb{F}^{d+1}$, it has a natural topological boundary $\partial_T X = \{\mathbf{x} \in \mathbb{F}^{d+1} : \|\mathbf{x}\| = 1\}$. On the other hand, since X is a hyperbolic metric space, it has a Gromov boundary $\partial_G X$. The following relation between them is more or less a direct consequence of the fact that every Gromov sequence in X converges to a point in $\partial_T X$ (for details see [21]):

Proposition B.2. *There is a unique homeomorphism $j : \partial_G X \to \partial_T X$ so that the extension $\mathrm{id} \cup j : X \cup \partial_G X \to X \cup \partial_T X$ is a homeomorphism.*

From now on we will not distinguish between the two boundaries and will simply write ∂X for either one.

Let λ_X denote normalized Lebesgue measure on ∂X. Let $K = \dim_{\mathbb{R}}(\mathbb{F})$, and let

$$(\mathrm{B.1}) \qquad\qquad \delta_X := dK + 2(K - 1).$$

Proposition B.3.

    (i) λ_X *is an Ahlfors δ_X-regular measure with respect to the visual metric on ∂X.*
    (ii) *For any group $G \leq \mathrm{Isom}(X)$, λ_X is δ_X-conformal with respect to G.*

Remark B.4. If $\mathbb{F} \neq \mathbb{R}$, then (i) is very different from saying that λ_X is Ahlfors regular with respect to the metric inherited from $\mathbb{F}^{d+1}$, since the visual metric on ∂X is not bi-Lipschitz equivalent to the usual metric as we will see below (Remark B.6).

If $\mathbb{F} = \mathbb{R}$, then the visual metric on ∂X is bi-Lipschitz equivalent to the usual metric, so the proofs below can be simplified somewhat in this case.

Proof of Proposition B.3.

    (i) The first step is to write the visual metric on ∂X in terms of the sesquilinear form B. Indeed, direct calculation (for details see either [21] or p.32 of [46]) yields that for $\mathbf{x}, \mathbf{y} \in \partial X$,

$$D_{\mathbf{0}}(\mathbf{x}, \mathbf{y}) = D_{\mathbf{0},e}(\mathbf{x}, \mathbf{y}) = \sqrt{\frac{1}{2}|1 - \langle \mathbf{x}, \mathbf{y} \rangle|}.$$

We may define a coordinate chart on ∂X as follows: Let $W = \{\mathbf{x} \in \mathbb{F}^{d+1} : \mathrm{Re}[x_1] = 0\}$, and let $g : W \to \partial X$ be stereographic projection through $-\mathbf{e}_1$, i.e.

$$g(\mathbf{x}) = -\mathbf{e}_1 + \frac{2}{\|\mathbf{e}_1 + \mathbf{x}\|^2}(\mathbf{e}_1 + \mathbf{x}).$$

Then for $\mathbf{x} \in W$

$$
\begin{aligned}
D_{\mathbf{0}}(\mathbf{e}_1, g(\mathbf{x})) &= \sqrt{\frac{1}{2}|1 - g(\mathbf{x})_1|} \\
&= \sqrt{\frac{1}{2}\left|1 - \left(-1 + \frac{2}{\|\mathbf{e}_1 + \mathbf{x}\|^2}(1 + x_1)\right)\right|} \\
&= \sqrt{\left|1 - \frac{1}{1 + \|\mathbf{x}\|^2}(1 + x_1)\right|} \\
&= \sqrt{\left|\frac{1}{1 + \|\mathbf{x}\|^2}(\|\mathbf{x}\|^2 - x_1)\right|} \\
&= \frac{\sqrt[4]{\|\mathbf{x}\|^4 + |x_1|^2}}{\sqrt{1 + \|\mathbf{x}\|^2}}.
\end{aligned}
$$

Now fix $r > 0$ small enough so that $g^{-1}(B(\mathbf{e}_1, r))$ is bounded. Then for $\mathbf{x} \in g^{-1}(B(\mathbf{e}_1, r))$,

$$
r \geq D_{\mathbf{0}}(\mathbf{e}_1, g(\mathbf{x})) \asymp_\times \max(\|\mathbf{x}\|, \sqrt{|x_1|}).
$$

Letting C be the implied constant, we have

(B.2) $$g^{-1}(B(\mathbf{e}_1, r)) \subseteq \{\mathbf{x} \in W : |x_1| \leq (Cr)^2, \ |x_i| \leq Cr \text{ for } i \neq 1\}.$$

Now, since the Jacobian of g is bounded from above and below on $g^{-1}(B(\mathbf{e}_1, r))$, we have

$$
\lambda_X(g^{-1}(B(\mathbf{e}_1, r))) \asymp_\times \lambda_W(B(\mathbf{e}_1, r)),
$$

where λ_W is Lebesgue measure on W. Thus

$$
\lambda_X(B(\mathbf{e}_1, r)) \lesssim_\times \lambda_W(\{\mathbf{x} \in W : |x_1| \leq (Cr)^2, \ |x_i| \leq Cr \text{ for } i \neq 1\}) \asymp_\times r^{\delta_X}.
$$

On the other hand, we have

(B.3) $$\{\mathbf{x} \in W : |x_1| \leq (r/(d+2))^2, \ |x_i| \leq r/(d+2) \text{ for } i \neq 1\} \subseteq g^{-1}(B(\mathbf{e}_1, r))$$

since if $\mathbf{x}$ is a member of the left hand side, then

$$
D_{\mathbf{0}}(\mathbf{e}_1, g(\mathbf{x})) \leq \|\mathbf{x}\| + \sqrt{|x_1|} \leq (d+2)\frac{r}{d+2} = r.
$$

A similar argument yields

$$
\lambda_X(B(\mathbf{e}_1, r)) \gtrsim_\times r^{\delta_X}.
$$

Now since the group $\mathscr{O}(\mathbb{F}^{d+1})$ (i.e. the group of $\mathbb{F}$-linear isometries of $\mathbb{F}^{d+1}$) preserves both λ_X and $D_{\mathbf{0}}$ and acts transitively on ∂X, the expression $\lambda_X(B([\mathbf{x}], r))$ depends only on r and not on $[\mathbf{x}]$. Thus λ_X is Ahlfors δ_X-regular.

(ii) By (i), λ_X is Ahlfors δ_X-regular and so $\lambda_X \asymp_\times \mathcal{H}^{\delta_X}$, where $\mathcal{H}^{\delta_X}$ is Hausdorff δ_X-dimensional measure on ∂X. Let

$$
f = \frac{d\lambda_X}{d\mathcal{H}^{\delta_X}}
$$

be the Radon-Nikodym derivative of λ_X with respect to $\mathcal{H}^{\delta_X}$. Since both λ_X and $\mathcal{H}^{\delta_X}$ are invariant under $\mathscr{O}(\mathbb{F}^{d+1})$, for each $g \in \mathscr{O}(\mathbb{F}^{d+1})$, the equation $f = f \circ g$ holds λ_X-almost everywhere.

Claim B.5. *f is constant λ_X-almost everywhere.*

Proof. By contradiction, suppose that there exists α such that $\lambda_X(A_1), \lambda_X(A_2) > 0$ where $A_1 = \{\xi \in \partial X : f(\xi) > \alpha\}$ and $A_2 = \{\xi \in \partial X : f(\xi) < \alpha\}$. Let $\xi_1, \xi_2 \in \partial X$ be density points of A_1 and A_2, respectively. Since $\mathscr{O}(\mathbb{F}^{d+1})$ acts transitively on ∂X, there exists $g \in \mathscr{O}(\mathbb{F}^{d+1})$ such that $g(\xi_1) = \xi_2$. Since $f =_{\lambda_X} f \circ g$, we have $g(A_1) =_{\lambda_X} A_1$ and thus ξ_2 is a density point of A_1, which contradicts that it is a density point of A_2. ◁

Thus λ_X and $\mathcal{H}^{\delta_X}$ are proportional; since $\mathcal{H}^{\delta_X}$ is δ_X-conformal with respect to any group $G \leq \mathrm{Isom}(X)$ (by abstract metric space theory), it follows that λ_X is δ_X-conformal for any such group as well.

$\square$

Remark B.6. The inclusions (B.2) and (B.3) are reminiscent of the Ball-Box theorem of sub-Riemannian geometry (see e.g. Theorem 0.5.A of [33]). The difference is that while the Ball-Box theorem applies to a Carnot-Carathéodory metric, the argument is about visual metrics. We remark that it is possible to prove using (B.2)-(B.3) together with the Ball-Box theorem that the visual metric is bi-Lipschitz equivalent with the Carnot-Carathéodory metric of an appropriate sub-Riemannian metric on ∂X. This is irrelevant for our purposes since (B.2)-(B.3) already contain the information we need to analyze the visual metric.

Note further that (B.2)-(B.3) show that $D_\mathbf{0}$ is bi-Lipschitz equivalent to the Euclidean metric on ∂X if and only if $\mathbb{F} = \mathbb{R}$.

Now we are able to prove the assertions about real, complex, and quaternionic hyperbolic spaces made in Remarks 1.29 and 1.47 and Example 1.41.

Proposition B.7. *Let X be a real, complex, or quaternionic hyperbolic space, and let $G \leq \mathrm{Isom}(X)$ be a lattice. Then*

(i) *For all $\xi \in \Lambda_{\mathrm{bp}}(G)$, we have*

(B.4) $$\delta_\xi = \delta/2.$$

(ii) *The Patterson-Sullivan measure of G is Ahlfors regular.*

Proof.

(i) Let δ_X be defined by (B.1). Then

$$\begin{aligned} \delta &= \mathrm{HD}(\Lambda_{\mathrm{r}}(G)) &&\text{(by Theorem 5.9)} \\ &= \mathrm{HD}(\partial X) &&\text{(since G is a lattice)} \\ &= \delta_X. &&\text{(by Proposition B.3)} \end{aligned}$$

On the other hand, by Lemma 3.5 of [54] together with (7.53), we have

$$2\delta_\xi = k + 2\ell,$$

where (ℓ, k) is the *rank* of ξ (see p.226 of [54] for the definition). It is readily checked that if G is a lattice, then $(\ell, k) = (dK, K - 1)$ (in the notation of [54], we have $H = N$), and thus

$$2\delta_\xi = dK + 2(K - 1) = \delta_X = \delta,$$

demonstrating (B.4).

(ii) As demonstrated above, $\delta = \delta_X$, so by (ii) of Proposition B.3, λ_X is δ-conformal, and so it is the Patterson-Sullivan measure for G. On the other hand, λ_X is Ahlfors regular by (i) of Proposition B.3.

$\square$

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