Bowen parameter and Hausdorff dimension for expanding rational semigroups

Hiroki Sumi and Mariusz Urbański

Abstract. We estimate the Bowen parameters and the Hausdorff dimensions of the Julia sets of expanding finitely generated rational semigroups. We show that the Bowen parameter is larger than or equal to the ratio of the entropy of the skew product map $\tilde{f}$ and the Lyapunov exponent of $\tilde{f}$ with respect to the maximal entropy measure for $\tilde{f}$. Moreover, we show that the equality holds if and only if the generators are simultaneously conjugate to the form $a_j z^{\pm d}$ by a Möbius transformation. Furthermore, we show that there are plenty of expanding finitely generated rational semigroups such that the Bowen parameter is strictly larger than 2.

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1. Introduction

A **rational semigroup** is a semigroup generated by a family of non-constant rational maps $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere, with the semigroup operation being functional composition. A polynomial semigroup is a semigroup generated by a family of non-constant polynomial maps on $\hat{\mathbb{C}}$. The work on the dynamics of rational semigroups was initiated by A. Hinkkanen and G. J. Martin ([7]), who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups of Möbius transformations, and by F. Ren's group ([35]), who studied such semigroups from the perspective of random dynamical systems.

The theory of the dynamics of rational semigroups on $\hat{\mathbb{C}}$ has developed in many directions since the 1990s ([7, 35, 15, 17, 18, 19, 20, 21, 22, 23, 30, 32, 25, 27, 16, 28, 29]). Since the

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Julia set $J(G)$ of a rational semigroup generated by finitely many elements $f_1, \ldots, f_s$ has **backward self-similarity** i.e.

$$(1.1) \qquad J(G) = f_1^{-1}(J(G)) \cup \cdots \cup f_s^{-1}(J(G)),$$

(see [17, 19]), it can be viewed as a significant generalization and extension of both the theory of iteration of rational maps (see [11]) and conformal iterated function systems (see [10]). Indeed, because of (1.1), the analysis of the Julia sets of rational semigroups somewhat resembles "backward iterated functions systems", however since each map f_j is not in general injective (critical points), some qualitatively different extra effort in the cases of semigroups is needed. The theory of the dynamics of rational semigroups borrows and develops tools from both of these theories. It has also developed its own unique methods, notably the skew product approach (see [19, 20, 21, 22, 25, 26, 27, 29, 30, 31, 32]).

The theory of the dynamics of rational semigroups is intimately related to that of the random dynamics of rational maps. For the study of random complex dynamics, the reader may consult [5, 3, 4, 2, 1, 6, 29]. The deep relation between these fields (rational semigroups, random complex dynamics, and (backward) IFS) is explained in detail in the subsequent papers ([23, 25, 26, 27, 24, 28, 29]) of the first author.

In this paper, we deal at length with Bowen's parameter δ (the unique zero of the pressure function) of expanding finitely generated rational semigroups $\langle f_1, \ldots, f_s \rangle$ (see Definition 2.12). In the usual iteration dynamics of a single expanding rational map, it is well known that the Hausdorff dimension of the Julia set is equal to the Bowen's parameter. For a general expanding finitely generated rational semigroup $\langle f_1, \ldots, f_s \rangle$, it was shown that the Bowen's parameter is larger than or equal to the Hausdorff dimension of the Julia set ([18, 21]). If we assume further that the semigroup satisfies the "open set condition" (see Definition 3.2), then it was shown that they are equal ([21]). However, if we do not assume the open set condition, then there are a lot of examples such that the Bowen's parameter is strictly larger than the Hausdorff dimension of the Julia set. In fact, the Bowen's parameter can be strictly larger than two. Thus, it is very natural to ask when we have this situation and what happens if we have such a case. We will show the following.

Theorem 1.1 (Theorem 3.1). *For an expanding rational semigroup $G = \langle f_1, \ldots, f_m \rangle$, the Bowen's parameter δ satisfies*

$$(1.2) \qquad \delta \geq \frac{\log(\sum_{j=1}^{s} \deg(f_j))}{\int \log \|\tilde{f}'\| d\mu},$$

where $\tilde{f}$ denotes the skew product map associated with the multi-map $f = (f_1, \ldots, f_s)$ (see section 2), and μ denotes the unique maximal entropy measure for $\tilde{f}$ (see [12, 19]). Moreover, the equality in the (1.2) holds if and only if we have a very special condition, i.e., there exists a Möbius transformation φ and a positive integer d_0 such that for each j, $\varphi f_j \varphi^{-1}(z)$ is of the form $a_j z^{\pm d_0}$.

Note that $\log(\sum_{j=1}^{s} \deg(f_j))$ is equal to the entropy of $\tilde{f}$. The above result (Theorem 3.1) generalizes a weak form of A. Zdunik's theorem ([34]), which is a result for the usual iteration of a single rational map. In fact, in the proof of the main result of our paper,

Zdunik's theorem is one of the key ingredients. We emphasize that in the main result of our paper, we can take the Möbius map φ which does not depend on j.

If each f_j is a polynomial with $\deg(f_j) \geq 2$, then by using potential theory, we can calculate $\int \log \|\tilde{f}'\| d\mu$ in (1.2) in terms of $\deg(f_j)$ and an integral related to fiberwise Green's functions (see Lemmas 3.13, 3.14). From this calculation, we can prove the following.

Theorem 1.2 (Theorem 3.17). *Let $s \in \mathbb{N}$ and for each $j = 1, \ldots, s$, let f_j be a polynomial with $\deg(f_j) \geq 2$. If $G = \langle f_1, \ldots, f_s \rangle$ is an expanding polynomial semigroup, the postcritical set of G in $\mathbb{C}$ is bounded, $(\log d)/(\sum_{j=1}^{s} \frac{d_j}{d} \log d_j) \geq 2$ where $d_j := \deg(f_j)$ and $d = \sum_{j=1}^{s} d_j$, and $\delta \leq 2$, then there exists a Möbius transformation φ such that for each j, $\varphi f_j \varphi^{-1}(z)$ is of the form $a_j z^s$.*

Thus, if the postcritical set of G in $\mathbb{C}$ is bounded and $(\log d)/(\sum_{j=1}^{s} \frac{d_j}{d} \log d_j) \geq 2$, then typically we have that $\delta > 2$. Note that in the usual iteration dynamics of a single rational map, we always have $\delta \leq 2$.

Therefore, we can say that there are plenty of expanding finitely generated polynomial semigroups for which the Bowen's parameter is strictly larger than 2.

Note that combining these estimates of Bowen's parameter and the "transversal family" type arguments, we will show that we have a large amount of expanding 2-generator polynomial semigroups G such that the Julia set of G has positive 2-dimensional Lebesgue measure ([33]).

We remark that, as illustrated in [24, 29], estimating the Hausdorff dimension of the Julia sets of rational semigroups plays an important role when we investigate random complex dynamics and its associated Markov process on $\hat{\mathbb{C}}$. For more details, see Remark 4.5 and [24, 29].

2. Preliminaries

In this section we introduce notation and basic definitions. Throughout the paper, we frequently follow the notation from [19] and [21].

Definition 2.1 ([7, 35]). *A "rational semigroup" G is a semigroup generated by a family of non-constant rational maps $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere, with the semigroup operation being functional composition. A "polynomial semigroup" is a semigroup generated by a family of non-constant polynomial maps on $\hat{\mathbb{C}}$. For a rational semigroup G, we set*

$$F(G) := \{z \in \hat{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\}$$

*and we call $F(G)$ the **Fatou set** of G. Its complement,*

$$J(G) := \hat{\mathbb{C}} \setminus F(G)$$

*is called the **Julia set** of G. If G is generated by a family $\{f_i\}_i$, then we write $G = \langle f_1, f_2, \ldots \rangle$.*

For the papers dealing with dynamics of rational semigroups, see for example [7, 35, 15, 17, 18, 19, 20, 21, 22, 23, 30, 32, 25, 26, 27, 16, 28, 29, 24], etc.

We denote by Rat the set of all non-constant rational maps on $\hat{\mathbb{C}}$ endowed with the topology induced by uniform convergence on $\hat{\mathbb{C}}$. Note that Rat has countably many connected

components. In addition, each connected component U of Rat is an open subset of Rat and U has a structure of a finite dimensional complex manifold. Similarly, we denote by $\mathcal{P}$ the set of all polynomial maps $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $\deg(g) \geq 2$ endowed with the relative topology from Rat. Note that $\mathcal{P}$ has countably many connected components. In addition, each connected component U of $\mathcal{P}$ is an open subset of $\mathcal{P}$ and U has a structure of a finite dimensional complex manifold.

Let V be an open subset of $\hat{\mathbb{C}}$ and let $z \in V$. For a holomorphic map $g : V \to \hat{\mathbb{C}}$, we denote by $\|g'(z)\|$ the norm of the derivative of $g : V \to \hat{\mathbb{C}}$ at z with respect to the spherical metric on $\hat{\mathbb{C}}$.

Definition 2.2. *For each $s \in \mathbb{N}$, let $\Sigma_s := \{1, \ldots, s\}^{\mathbb{N}}$ be the space of one-sided sequences of s-symbols endowed with the product topology. This is a compact metrizable space. For each $f = (f_1, \ldots, f_s) \in (\mathrm{Rat})^s$, we define a map*

$$\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$$

by the formula

$$\tilde{f}(\omega, z) = (\sigma(\omega), \ f_{\omega_1}(z)),$$

*where $(\omega, z) \in \Sigma_s \times \hat{\mathbb{C}}$, $\omega = (\omega_1, \omega_2, \ldots)$, and $\sigma : \Sigma_s \to \Sigma_s$ denotes the shift map. The transformation $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ is called the **skew product map** associated with the multi-map $f = (f_1, \ldots, f_s) \in (\mathrm{Rat})^s$. We denote by $\pi_1 : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s$ the projection onto Σ_s and by $\pi_2 : \Sigma_s \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the projection onto $\hat{\mathbb{C}}$. That is, $\pi_1(\omega, z) = \omega$ and $\pi_2(\omega, z) = z$. For each $n \in \mathbb{N}$ and $(\omega, z) \in \Sigma_s \times \hat{\mathbb{C}}$, we put*

$$(\tilde{f}^n)'(\omega, z) := (f_{\omega_n} \circ \cdots \circ f_{\omega_1})'(z).$$

Moreover, we denote by $\|(\tilde{f}^n)'(\omega, z)\|$ the norm of the derivative of $f_{\omega_n} \circ \cdots \circ f_{\omega_1}$ at z with respect to the spherical metric on $\hat{\mathbb{C}}$. We define

$$J_\omega(\tilde{f}) := \{z \in \hat{\mathbb{C}} \mid \{f_{\omega_n} \circ \cdots \circ f_{\omega_1}\}_{n \in \mathbb{N}} \text{ is not normal in each neighborhood of } z\}$$

for each $\omega \in \Sigma_s$ and we set

$$J(\tilde{f}) := \overline{\cup_{w \in \Sigma_s} \{\omega\} \times J_\omega(\tilde{f})},$$

*where the closure is taken with respect to the product topology on the space $\Sigma_s \times \hat{\mathbb{C}}$. $J(\tilde{f})$ is called the **Julia set** of the skew product map $\tilde{f}$. In addition, we set $F(\tilde{f}) := (\Sigma_s \times \hat{\mathbb{C}}) \setminus J(\tilde{f})$. and $\deg(\tilde{f}) := \sum_{j=1}^s \deg(f_j)$.*

Remark 2.3. *By definition, the set $J(\tilde{f})$ is compact. Furthermore, if we set $G = \langle f_1, \ldots, f_s \rangle$, then, by [19, Proposition 3.2], the following hold:*

(1) *$J(\tilde{f})$ is completely invariant under $\tilde{f}$;*
(2) *$\tilde{f}$ is an open map on $J(\tilde{f})$;*
(3) *if $\sharp J(G) \geq 3$ and $E(G) := \{z \in \hat{\mathbb{C}} \mid \sharp \cup_{g \in G} g^{-1}\{z\} < \infty\}$ is contained in $F(G)$, then the dynamical system $(\tilde{f}, J(\tilde{f}))$ is topologically exact;*
(4) *$J(\tilde{f})$ is equal to the closure of the set of repelling periodic points of $\tilde{f}$ if $\sharp J(G) \geq 3$, where we say that a periodic point (ω, z) of $\tilde{f}$ with period n is repelling if $\|(\tilde{f}^n)'(\omega, z)\| > 1$.*

(5) $\pi_2(J(\tilde{f})) = J(G)$.

Definition 2.4 ([21]). *A finitely generated rational semigroup $G = \langle f_1, \ldots, f_s \rangle$ is said to be expanding provided that $J(G) \neq \emptyset$ and the skew product map $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ associated with $f = (f_1, \ldots, f_s)$ is expanding along fibers of the Julia set $J(\tilde{f})$, meaning that there exist $\eta > 1$ and $C \in (0, 1]$ such that for all $n \geq 1$,*

(2.1) $$\inf\{\|(\tilde{f}^n)'(z)\| : z \in J(\tilde{f})\} \geq C\eta^n.$$

Definition 2.5. *Let G be a rational semigroup. We put*

$$P(G) := \overline{\cup_{g \in G}\{all\ critical\ values\ of\ g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}}\ (\subset \hat{\mathbb{C}})$$

*and we call $P(G)$ the **postcritical set** of G. A rational semigroup G is said to be **hyperbolic** if $P(G) \subset F(G)$.*

Definition 2.6. *Let G be a polynomial semigroup. We set $P^*(G) := P(G) \setminus \{\infty\}$. We say that G is postcritically bounded if $P^*(G)$ is bounded in $\mathbb{C}$.*

Remark 2.7. *Let $G = \langle f_1, \ldots, f_s \rangle$ be a rational semigroup such that there exists an element $g \in G$ with $\deg(g) \geq 2$ and such that each Möbius transformation in G is loxodromic. Then, it was proved in [18] that G is expanding if and only if G is hyperbolic.*

Definition 2.8. *We define*

$$\mathrm{Exp}(s) := \{(f_1, \ldots, f_s) \in (\mathrm{Rat})^s \mid \langle f_1, \ldots, f_s \rangle\ is\ expanding\}.$$

We also set $\Sigma_s^* := \cup_{j=1}^{\infty}\{1, \ldots, s\}^j$ (disjoint union). For every $\omega \in \Sigma_s \cup \Sigma_s^*$ let $|\omega|$ be the length of ω. For each $f = (f_1, \ldots, f_s) \in (\mathrm{Rat})^s$ and each $\omega = (\omega_1, \ldots, \omega_n) \in \Sigma_s^*$, we put $f_\omega := f_{\omega_n} \circ \cdots \circ f_{\omega_1}$.

Then we have the following.

Lemma 2.9 ([17, 31]). $\mathrm{Exp}(s)$ *is an open subset of $(\mathrm{Rat})^s$.*

Definition 2.10. *We set*

$$\mathrm{Epb}(s) := \{f = (f_1, \ldots, f_s) \in \mathrm{Exp}(s) \cap \mathcal{P}^s \mid \langle f_1, \ldots, f_s \rangle\ is\ postcritically\ bounded\},$$

Lemma 2.11 ([27, 29]). $\mathrm{Epb}(s)$ *is open in $\mathcal{P}^s$.*

Definition 2.12. *Let $f = (f_1, \ldots, f_s) \in \mathrm{Exp}(s)$ and let $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ be the skew product map associated with $f = (f_1, \ldots, f_s)$. For each $t \in \mathbb{R}$, let $P(t, f)$ be the topological pressure of the potential $\varphi(z) := -t \log \|\tilde{f}'(z)\|$ with respect to the map $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$. (For the definition of the topological pressure, see [12].) We denote by $\delta(f)$ the unique zero of $t \mapsto P(t, f)$. (Note that the existence and the uniqueness of the zero of $P(t, f)$ was shown in [21].) The number $\delta(f)$ is called the **Bowen parameter** of the semigroup $f = (f_1, \ldots, f_s) \in \mathrm{Exp}(s)$.*

We have the following fact, which is one of the main results of [31].

Theorem 2.13 ([31]). *The function $\mathrm{Exp}(s) \ni f \mapsto \delta(f) \in \mathbb{R}$ is real-analytic and plurisubharmonic.*

Definition 2.14. *For a subset A of $\hat{\mathbb{C}}$, we denote by $\mathrm{HD}(A)$ the Hausdorff dimension of A with respect to the spherical metric. For a Riemann surface S, we denote by $\mathrm{Aut}(S)$ the set of all holomorphic isomorphisms of S. For a compact metric space X, we denote by $C(X)$ the space of all continuous complex-valued functions on X, endowed with the supremum norm.*

3. Results

In this section, we prove our main results. Note that for any $f = (f_1, \ldots, f_s) \in \mathrm{Exp}(s)$, by Remark 2.3, [12], and [19], there exists a unique maximal entropy measure μ for $\tilde{f}$: $J(\tilde{f}) \to J(\tilde{f})$ and $h_\mu(\tilde{f}) = h(\tilde{f}) = \log(\deg(\tilde{f}))$. We start with the following.

Theorem 3.1. *Let $f = (f_1, \ldots, f_s) \in \mathrm{Exp}(s)$. Let $d := \deg(\tilde{f})$ and let $d_j = \deg(f_j)$ for each $j = 1, \ldots, s$. Let μ be the maximal entropy measure for $\tilde{f} : J(\tilde{f}) \to J(\tilde{f})$. Then the following statements (1) and (2) hold.*

(1)
$$\delta(f) \geq \frac{\log d}{\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\mu}.$$

(2) *Suppose $d_1 \geq 2$. If*
$$\delta(f) = \frac{\log d}{\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\mu},$$

then, the following items (a),(b),(c) hold.

(a) *$d_1 = \cdots = d_s$. We set $d_0 := d_1 = \cdots = d_s$.*

(b) *There exist an automorphism $\varphi \in \mathrm{Aut}(\hat{\mathbb{C}})$ and complex numbers $a_1, \ldots, a_s$ with $a_1 = 1$ such that for each $j = 1, \ldots, s$,*
$$\varphi f_j \varphi^{-1}(z) = a_j z^{\pm d_0}.$$

(c) *$\delta(f) = 1 + \frac{\log s}{\log d_0}$.*

Proof. We have that $\mathbb{R} \ni t \mapsto P(t, f)$ is convex and real-analytic ([21], [31]). Also,
$$\frac{\partial P(t, f)}{\partial t}\Big|_{t=0} = -\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\mu.$$

From the convexity of $P(t, f)$, we obtain that
$$\delta(f) \geq \frac{\log d}{\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\mu}.$$

We now assume that $d_1 \geq 2$ and
$$\delta(f) = \frac{\log d}{\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\mu}.$$

Because of the convexity of $P(t, f)$ again, we infer that

$$\frac{\partial P(t, f)}{\partial t} = -\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\mu$$

for all $t \in \mathbb{R}$. Let ν be the unique $\delta(f)$-conformal measure on $J(\tilde{f})$ for $(\tilde{f}, J(\tilde{f}))$ (see [21]). Let

$$L_\nu : C(J(\tilde{f})) \to C(J(\tilde{f}))$$

be the operator, called the transfer operator, defined by the following formula

$$L_\nu(\varphi)(z) = \sum_{\tilde{f}(y)=z} \varphi(y) \|\tilde{f}'(y)\|^{-\delta(f)}.$$

In virtue of [21], the limit $\alpha := \lim_{l \to \infty} L_\nu^l(1) \in C(J(\tilde{f}))$ exists, where 1 denotes the constant function taking its only value 1. Let $\tau := \alpha\nu$. Then

$$-\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\tau = \frac{\partial P(t, f)}{\partial t}\Big|_{t=\delta(f)} = -\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\mu.$$

Thus

$$\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\tau = \int_{J(\tilde{f})} \log \|\tilde{f}'\| d\mu.$$

Since

$$\delta(f) = \frac{h_\tau(\tilde{f})}{\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\tau}$$

(see [21]), it follows that $h_\tau(\tilde{f}) = \log d$. By the uniqueness of maximal entropy measure of $(\tilde{f}, J(\tilde{f}))$, we obtain that

(3.1) $$\tau = \mu.$$

Let $L_\mu : C(J(\tilde{f})) \to C(J(\tilde{f}))$ be the operator defined as follows

$$L_\mu(\varphi)(z) = \frac{1}{d} \sum_{\tilde{f}(y)=z} \varphi(y).$$

Since $L_\mu^*(\mu) = \mu$, (3.1) implies that $L_\mu^*(\alpha\nu) = \alpha\nu$. Thus, for any open subset A of $J(\tilde{f})$ such that $\tilde{f} : A \to \tilde{f}(A)$ is injective, if B is a Borel subset of A, then $(\alpha\nu)(\tilde{f}(B)) = \int_B d\,d(\alpha\nu)$.

Moreover, we have

$$\begin{aligned}
(\alpha\nu)(\tilde{f}(B)) &= \int_{\tilde{f}(B)} \alpha \, d\nu \\
&= \int_{\tilde{f}(B)} (\alpha \circ \tilde{f}) \circ (\tilde{f}|_A)^{-1} d\nu \\
&= \int_B \alpha \circ \tilde{f} \, d((\tilde{f}|_A^{-1})_* \nu) \\
&= \int_B \alpha \circ \tilde{f} \cdot \frac{d((\tilde{f}|_A^{-1})_* \nu)}{d\nu} d\nu \\
&= \int_B (\alpha \circ \tilde{f}) \cdot \|\tilde{f}'\|^{\delta(f)} d\nu.
\end{aligned}$$

Thus $(\alpha \circ \tilde{f}(z)) \cdot \|\tilde{f}'(z)\|^{\delta(f)} = \alpha(z)d$ for ν-a.e. $z \in J(\tilde{f})$. Since $\operatorname{supp}\tau = J(\tilde{f})$ (see [21]), it follows that

$$(3.2) \qquad (\alpha \circ \tilde{f}(z)) \cdot \|\tilde{f}'(z)\|^{\delta(f)} = \alpha(z)d \quad \text{for every } z \in J(\tilde{f}).$$

Hence

$$(3.3) \qquad \log\|\tilde{f}'(z)\| = \frac{1}{\delta(f)}(\log\alpha(z) - \log\alpha \circ \tilde{f}(z) + \log d) \quad \text{for every } z \in J(\tilde{f}).$$

Therefore, for each $w \in \Sigma_s^*$ there exists a continuous function $\alpha_w : J(f_w) \to \mathbb{R}$ such that

$$(3.4) \qquad \log\|f_w'(z)\| = \frac{1}{\delta(f)}(\log\alpha_w(z) - \log\alpha_w \circ f_w(z) + |w|\log d) \quad \text{for every } z \in J(f_w).$$

Thus, for each f_w-invariant Borel probability measure β on $J(f_w)$, we have

$$\int_{J(f_w)} \log\|f_w'\| d\beta = |w|\frac{\log d}{\delta(f)}.$$

Let $p(t, w)$ be the topological pressure of $f_w : J(f_w) \to J(f_w)$ with respect to the potential function $-t\log\|f_w'\|$. It follows that for each $w \in \Sigma_s^*$ with $\deg(f_w) \geq 2$,

$$(3.5) \qquad \frac{\partial p(t, w)}{\partial t} = -|w|\frac{\log d}{\delta(f)} \quad \text{for each } t \in \mathbb{R}.$$

In particular, $t \mapsto p(t, w)$ is linear. Hence,

$$\mathrm{HD}(J(f_w)) = \frac{\log(\deg(f_w))}{\int_{J(f_w)} \log\|f_w'\| d\mu_w},$$

where μ_w denotes the maximal entropy measure for $f_w : J(f_w) \to J(f_w)$. Therefore, by Zdunik's theorem ([34]), it follows that for each $w \in \Sigma_s^*$ with $\deg(f_w) \geq 2$, there exists an $n_w \in \mathbb{Z} \setminus \{0, \pm 1\}$ and an element $\psi_w \in \operatorname{Aut}(\hat{\mathbb{C}})$ such that

$$(3.6) \qquad \psi_w \circ f_w \circ \psi_w^{-1}(z) = z^{n_w} \quad \text{for every } z \in \hat{\mathbb{C}}.$$

In particular, there exists an element $\varphi \in \operatorname{Aut}(\hat{\mathbb{C}})$ such that $\varphi \circ f_1 \circ \varphi^{-1}(z) = z^{\pm d_1}$ for each $z \in \hat{\mathbb{C}}$. Suppose that there exists a $j \in \{1, \ldots, s\}$ such that $(\varphi \circ f_j \circ \varphi^{-1})^{-1}(\{0, \infty\}) \neq \{0, \infty\}$. If $d_j \geq 2$, then since each point of $(\varphi \circ f_j \circ \varphi^{-1})^{-1}(\{0, \infty\})$ is a critical point of $\varphi \circ f_1 \circ f_j \circ \varphi^{-1}$

and $\sharp(\varphi \circ f_j \circ \varphi^{-1})^{-1}(\{0, \infty\}) \geq 3$, it contradicts (3.6). If $d_j = 1$, then since each point of $A := (\varphi \circ f_1 \circ \varphi^{-1})^{-1}((\varphi \circ f_j \circ \varphi^{-1})^{-1}(\{0, \infty\}))$ is a critical point of $\varphi \circ f_1 \circ f_j \circ f_1 \circ \varphi^{-1}$ and $\sharp A \geq 3$, it contradicts (3.6) again. Therefore, for each $j \in \{1, \ldots, s\}$,

$$(3.7) \qquad\qquad \varphi \circ f_j \circ \varphi^{-1}(z) = a_j z^{\pm d_j}$$

for some $a_j \in \mathbb{C} \setminus \{0\}$. Since G is expanding and $d_1 \geq 2$, it follows that $d_j \geq 2$ for each $j = 1, \ldots, s$. By (3.5) and (3.7), it follows that for each j,

$$\log d_j = \int_{J(f_j)} \log \|f_j'\| d\mu_j = \frac{\log d}{\delta(f)}.$$

Therefore, $d_1 = \cdots = d_s$. Thus, we have completed the proof. $\qquad\square$

Regarding Theorem 3.1, we give several remarks. In order to relate the Bowen parameter to the geometry of the Julia set we need the concept of the open set condition. We define it now.

Definition 3.2. *Let $f = (f_1, \ldots, f_s) \in (\mathrm{Rat})^s$ and let $G = \langle f_1, \ldots, f_s \rangle$. Let also U be a non-empty open set in $\hat{\mathbb{C}}$. We say that f (or G) satisfies the open set condition (with U) if*

$$\cup_{j=1}^s f_j^{-1}(U) \subset U \quad and \quad f_i^{-1}(U) \cap f_j^{-1}(U) = \emptyset$$

for each (i, j) with $i \neq j$. There is also a stronger condition. Namely, we say that f (or G) satisfies the separating open set condition (with U) if

$$\cup_{j=1}^s f_j^{-1}(U) \subset U \quad and \quad f_i^{-1}(\overline{U}) \cap f_j^{-1}(\overline{U}) = \emptyset$$

for each (i, j) with $i \neq j$.

We remark that the above concept of "open set condition" (for "backward IFS's") is an analogue of the usual open set condition in the theory of IFS's.

We introduce two other analytic invariants.

Definition 3.3 ([21])**.** *Let G be a countable rational semigroup. For any $t \geq 0$ and $z \in \hat{\mathbb{C}}$, we set*

$$S_G(z, t) := \sum_{g \in G} \sum_{g(y)=z} \|g'(y)\|^{-t}$$

counting multiplicities. We also set

$$S_G(z) := \inf\{t \geq 0 : S_G(z, t) < \infty\}$$

(if no t exists with $S_G(z, t) < \infty$, then we set $S_G(z) := \infty$). Furthermore, we put

$$s_0(G) := \inf\{S_G(z) : z \in \hat{\mathbb{C}}\}$$

*The number $s_0(G)$ is called the **critical exponent of the Poincaré series** of G.*

Definition 3.4 ([21])**.** *Let $f = (f_1, \ldots, f_s) \in (\mathrm{Rat})^s$, $t \geq 0$, and $z \in \hat{\mathbb{C}}$. We put*

$$T_f(z, t) := \sum_{\omega \in \Sigma_s^*} \sum_{f_\omega(y)=z} \|f_\omega'(y)\|^{-t},$$

counting multiplicities. Moreover, we set

$$T_f(z) := \inf\{t \geq 0 : T_f(z, t) < \infty\}$$

(if no t exists with $T_f(z,t) < \infty$, then we set $T_f(z) = \infty$). Furthermore, we set

$$t_0(f) := \inf\{T_f(z) : z \in \hat{\mathbb{C}}\}.$$

*The number $t_0(f)$ is called the **critical exponent of the Poincaré series** of $f = (f_1, \ldots, f_s) \in (\mathrm{Rat})^s$.*

Remark 3.5. *Let $f = (f_1, \ldots, f_s) \in (\mathrm{Rat})^s$, $t \geq 0$, $z \in \hat{\mathbb{C}}$ and let $G = \langle f_1, \ldots, f_s \rangle$. Then, $S_G(t,z) \leq T_f(t,z), S_G(z) \leq T_f(z)$, and $s_0(G) \leq t_0(f)$. Note that for almost every $f \in (\mathrm{Rat})^s$ with respect to the Lebesgue measure, $G = \langle f_1, \ldots, f_s \rangle$ is a free semigroup and so we have $S_G(t,z) = T_f(t,z), S_G(z) = T_f(z)$, and $s_0(G) = t_0(f)$.*

Lemma 3.6 ([31]). *Let $f = (f_1, \ldots, f_s) \in \mathrm{Exp}(s)$. Then $\delta(f) = t_0(f)$.*

Definition 3.7. *Let G be a rational semigroup. Then, we define*

$$A(G) := \overline{\cup_{g \in G} g(\{z \in \hat{\mathbb{C}} : \exists u \in G, u(z) = z, \|u'(z)\| < 1\})}.$$

Let us record the following fact proved in [21].

Theorem 3.8 ([21]). *Let $f = (f_1, \ldots, f_s) \in \mathrm{Exp}(s)$ and let $G = \langle f_1, \ldots, f_s \rangle$. Then, by [21] and Lemma 3.6, we have $\mathrm{HD}(J(G)) \leq s_0(G) \leq S_G(z) \leq \delta(f) = T_f(z) = t_0(f)$, for each $z \in \hat{\mathbb{C}} \setminus (A(G) \cup P(G))$. If in addition to the above assumption, f satisfies the open set condition, then*

$$\mathrm{HD}(J(G)) = s_0(G) = S_G(z) = \delta(f) = T_f(z) = t_0(f),$$

for each $z \in \hat{\mathbb{C}} \setminus (A(G) \cup P(G))$.

In order to prove our second main theorem (see Theorem 3.17), we need some notation and lemmas from [29]. We shall provide the full proofs of these lemmas for the sake of completeness of our exposition and convenience of the readers.

Definition 3.9. *For each $s \in \mathbb{N}$, we set $\mathcal{W}_s := \{(p_1, \ldots, p_s) \in (0,1)^s \mid \sum_{j=1}^s p_j = 1\}$.*

Definition 3.10 ([14, 8, 9, 29]). *Let $f = (f_1, \ldots, f_s) \in \mathcal{P}^s$. Let $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ be the skew product map associated with f. For any $\omega \in \Sigma_s$, we set*

$$A_{\infty, \omega} := \{z \in \hat{\mathbb{C}} : f_{\omega_n} \circ \cdots \circ f_{\omega_1}(z) \to \infty \text{ as } n \to \infty\}.$$

For any $(\omega, y) \in \Sigma_s \times \mathbb{C}$, let

$$G_\omega(y) := \lim_{n \to \infty} \frac{1}{\deg(f_{\omega_n} \circ \cdots \circ f_{\omega_1})} \log^+ |f_{\omega_n} \circ \cdots \circ f_{\omega_1}(y)|,$$

where $\log^+ a := \max\{\log a, 0\}$ for each $a > 0$. By the arguments in [14], for each $\omega \in \Sigma_s$, the limit $G_\omega(y)$ exists, the function G_ω is subharmonic on $\mathbb{C}$, and $G_\omega|_{A_{\infty,\omega}}$ is equal to the Green's function on $A_{\infty,\omega}$ with pole at ∞. Moreover, $(\omega, y) \mapsto G_\omega(y)$ is continuous on $\Sigma_s \times \mathbb{C}$. Let $\mu_\omega := dd^c G_\omega$, where $d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$. Note that by the argument in [8, 9], μ_ω is a Borel probability measure on $J_\omega(\tilde{f})$ such that $\mathrm{supp}\,\mu_\omega = J_\omega(\tilde{f})$. Furthermore, for each $\omega \in \Sigma_s$, let $\Omega(\omega) = \sum_c G_\omega(c)$, where c runs over all critical points of f_{ω_1} in $\mathbb{C}$, counting multiplicities.

Remark 3.11 ([19]). *Let* $f = (f_1, \ldots, f_s) \in (\mathrm{Rat})^s$. *Let* $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ *be the skew product map associated with* f. *Also, let* $p = (p_1, \ldots, p_s) \in \mathcal{W}_s$ *and let* τ *be the Bernoulli measure on* Σ_s *with respect to the weight* p. *Suppose that* $\deg(f_j) \geq 2$ *for each* $j = 1, \ldots, s$. *Then, there exists a unique* $\tilde{f}$-*invariant Borel probability ergodic measure* μ *on* $\Sigma_s \times \hat{\mathbb{C}}$ *such that* $(\pi_1)_*(\mu) = \tau$ *and*

$$h_\mu(\tilde{f}|\sigma) = \max_{\rho \in \mathcal{E}_1(\Sigma_s \times \hat{\mathbb{C}}) : \tilde{f}_*(\rho) = \rho, (\pi_1)_*(\rho) = \tau} h_\rho(\tilde{f}|\sigma) = \sum_{j=1}^s p_j \log(\deg(f_j)),$$

where $h_\rho(\tilde{f}|\sigma)$ *denotes the relative metric entropy of* $(\tilde{f}, \rho)$ *with respect to* (σ, τ), *and* $\mathcal{E}_1(\cdot)$ *denotes the space of ergodic measures for* $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ *(see [19]). The measure* μ *is called the* **maximal relative entropy measure** *for* $\tilde{f}$ *with respect to* (σ, τ).

Lemma 3.12 ([29]). *Let* $f = (f_1, \ldots, f_s) \in \mathcal{P}^s$ *and let* $G = \langle f_1, \ldots, f_s \rangle$. *Let* $p = (p_1, \ldots, p_s) \in \mathcal{W}_s$. *Let* $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ *be the skew product associated with* f. *Let* τ *be the Bernoulli measure on* Σ_s *with respect to the weight* p. *Let* μ *be a Borel probability measure on* $J(\tilde{f})$ *defined by*

$$\langle \mu, \varphi \rangle := \int_{\Sigma_s} \int_{\hat{\mathbb{C}}} \varphi(\omega, z) d\mu_\omega(z) \, d\tau(\omega)$$

for any continuous function φ *on* $\Sigma_s \times \hat{\mathbb{C}}$, *where* μ_ω *is the measure coming from Definition 3.10. Then,* μ *is an* $\tilde{f}$-*invariant ergodic measure,* $\pi_*(\mu) = \tau$, *and* μ *is the maximal relative entropy measure for* $\tilde{f}$ *with respect to* (σ, τ) *(see Remark 3.11).*

Proof. By the argument of the proof of [9, Theorem 4.2(i)], μ is $\tilde{f}$-invariant and ergodic, and $\pi_*(\mu) = \tau$. Moreover, the argument of the proof of [9, Theorem 5.2(i)], yields that

$$h_\mu(\tilde{f}|\sigma) \geq \int \log \deg(f_{\omega_1}) d\tau(\omega) = \sum_{j=1}^m p_j \log \deg(f_j).$$

Combining this with [19, Theorem 1.3(e)(f)], it follows that μ is the unique maximal relative entropy measure for $\tilde{f}$ with respect to (σ, τ). $\square$

Lemma 3.13 ([29]). *Let* $f = (f_1, \ldots, f_s) \in \mathcal{P}^s$. *Let* $p = (p_1, \ldots, p_s) \in \mathcal{W}_s$. *Let* τ *be the Bernoulli measure on* Σ_s *with respect to the weight* p. *Let* $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ *be the skew product associated with* f. *Let* μ *be the maximal relative entropy measure for* $\tilde{f}$ *with respect to* (σ, τ). *Then*

$$\int_{\Sigma_s \times \hat{\mathbb{C}}} \log \|\tilde{f}'\| d\mu = \sum_{j=1}^s p_j \log \deg(f_j) + \int_{\Sigma_s} \Omega(\omega) d\tau(\omega).$$

Proof. For each $\omega \in \Sigma_s$, let $d(\omega) = \deg(f_{\omega_1})$ and $R(\omega) := \lim_{z \to \infty}(G_\omega(z) - \log|z|)$. Also, we denote by $a(\omega)$ the coefficient of the highest order term of f_{ω_1}. Since $\frac{1}{d(\omega)} G_{\sigma(\omega)}(f_{\omega_1}(z)) = G_\omega(z)$, we obtain that $R(\sigma(\omega)) + \log|a(\omega)| = d(\omega)R(\omega)$ for each $\omega \in \Sigma_s$. Moreover, since $dd^c(\int_{\mathbb{C}} \log|w - z|d\mu_\omega(w)) = \mu_\omega$ and $\int_{\mathbb{C}} \log|w - z|d\mu_\omega(w) = \log|z| + o(1)$ as $z \to \infty$ (see [13]),

we have $\int_{\mathbb{C}} \log |w - z| d\mu_\omega(w) = G_\omega(z) - R(\omega)$ for each $\omega \in \Sigma_s$ and $z \in \mathbb{C}$. In particular, the function $\omega \mapsto R(\omega)$ is continuous on Σ_s. It follows from the above formula, that

$$\int_{\hat{\mathbb{C}}} \log |f'_{\omega_1}(z)| d\mu_\omega(z) = \log |a(\omega)| + \log d(\omega) - (d(\omega) - 1)R(\omega) + \Omega(\omega)$$

for each $\omega \in \Sigma_s$. In particular, the function $\omega \mapsto \int_\omega \log |f'_{\omega_1}(z)| d\mu_\omega(z)$ is continuous on Σ_s. Furthermore, $\sigma_*(\tau) = \tau$. From these arguments and Lemma 3.12, we obtain

$$\int_{\Sigma_s \times \hat{\mathbb{C}}} \log |\tilde{f}'| d\mu = \int_{\Sigma_s} d\tau(\omega) \int_{\hat{\mathbb{C}}} \log |f'_{\omega_1}(z)| d\mu_\omega(z)$$

$$= \int_{\Sigma_s} (\log |a(\omega)| + \log d(\omega) - (d(\omega) - 1)R(\omega) + \Omega(\omega)) \, d\tau(\omega)$$

$$= \int_{\Sigma_s} (R(\omega) - R(\sigma(\omega)) + \log d(\omega) + \Omega(\omega)) \, d\tau(\omega)$$

$$= \int_{\Sigma_s} (\log d(\omega) + \Omega(\omega)) d\tau(\omega) = \sum_{j=1}^s p_j \log \deg(f_j) + \int_{\Sigma_s} \Omega(\omega) d\tau(\omega).$$

Moreover, since μ is $\tilde{f}$-invariant, and since the Euclidian metric and the spherical metric are comparable on the compact subset $J(G)$ of $\mathbb{C}$, we have $\int_{\Sigma_s \times \hat{\mathbb{C}}} \log |\tilde{f}'| d\mu = \int_{\Sigma_s \times \hat{\mathbb{C}}} \log \|\tilde{f}'\| d\mu$. Thus, we have proved our lemma. $\qquad \square$

Lemma 3.14. *Let $f = (f_1, \ldots, f_s) \in \mathcal{P}^s$. Let $d_j = \deg(f_j)$ for each j and let $d = \sum_j d_j$. Let μ be the maximal entropy measure for $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ (see [19]). Let τ be the Bernoulli measure on Σ_s with respect to the weight $(\frac{d_1}{d}, \ldots, \frac{d_s}{d})$. Then, we have*

$$\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\mu = \sum_{j=1}^s \frac{d_j}{d} \log d_j + \int_{\Sigma_s} \Omega(\omega) d\tau(\omega).$$

In particular, if, in addition to the assumptions of our lemma, $\langle f_1, \ldots, f_m \rangle$ is postcritically bounded, then

$$\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\mu = \sum_{j=1}^s \frac{d_j}{d} \log d_j.$$

Proof. Let

$$p = \left(\frac{d_1}{d}, \ldots, \frac{d_s}{d} \right) \in \mathcal{W}_s.$$

Let τ be the Bernoulli measure on Σ_s with respect to the weight p. By [19], μ is equal to the maximal relative entropy measure for $\tilde{f}$ with respect to (σ, τ). By Lemma 3.13, the statement of our lemma holds. $\qquad \square$

We now give a lower estimate of the Hausdorff dimension of Julia sets of expanding finitely generated polynomial semigroups satisfying the open set condition.

Theorem 3.15. *Let $f = (f_1, \ldots, f_s) \in \mathrm{Exp}(s) \cap \mathcal{P}^s$. Assume f satisfies the open set condition. Let $d_j = \deg(f_j)$ for each j and let $d = \sum_j d_j$. Let μ be the maximal entropy*

measure for $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$ *(see* [19]*). Let* τ *be the Bernoulli measure on* Σ_s *with respect to the weight* $(\frac{d_1}{d}, \ldots, \frac{d_s}{d})$. *Let* $G = \langle f_1, \ldots, f_s \rangle$. *Then, the following hold.*

(1)

$$(3.8) \qquad \mathrm{HD}(J(G)) = \delta(f) \geq \frac{\log d}{\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\mu} = \frac{\log d}{\sum_{j=1}^{s} \frac{d_j}{d} \log d_j + \int_{\Sigma_s} \Omega(\omega) d\tau(\omega)}.$$

(2) *If the inequality in* (3.8) *is replaced by the equality, then*
   (a) $d_1 = \cdots = d_s$. *We set* $d_0 = d_1 = \cdots = d_s$.
   (b) *There exists an element* $\varphi \in \mathrm{Aut}(\mathbb{C})$ *and complex numbers* $a_1, \ldots, a_s$ *with* $a_1 = 1$ *such that for each* $j = 1, \ldots, s$, $\varphi f_j \varphi^{-1}(z) = a_j z^{d_0}$.
   (c) $\delta(f) = 1 + \frac{\log s}{\log d_0}$.
(3) *If, in addition to the assumptions of our lemma,* $f \in \mathrm{Epb}(s)$, *then*

$$\mathrm{HD}(J(G)) = \delta(f) \geq \frac{\log d}{\sum_{j=1}^{s} \frac{d_j}{d} \log d_j}.$$

Proof. By Theorem 3.1, Lemma 3.14 and Theorem 3.8, we obtain the statement of our Theorem. $\qquad\square$

Remark 3.16. *If* $s > 1$, *then* $\frac{\log d}{\sum_{j=1}^{s} \frac{d_j}{d} \log d_j} > 1$.

We now formulate and prove our second main theorem.

Theorem 3.17. *Let* $f = (f_1, \ldots, f_s) \in \mathrm{Epb}(s)$. *Let* $d_j = \deg(f_j)$ *for each* j *and let* $d = \sum_j d_j$. *Suppose that* $(\log d)/(\sum_{j=1}^{s} \frac{d_j}{d} \log d_j) \geq 2$ *and* $\delta(f) \leq 2$. *Then, we have the following.*
   (1) *There exist a* $\varphi \in \mathrm{Aut}(\mathbb{C})$ *and non-zero complex numbers* $a_1, \ldots, a_s$ *such that for each* $j = 1, \ldots, s$, $\varphi \circ f_j \circ \varphi^{-1}(z) = a_j z^s$ *for all* $z \in \hat{\mathbb{C}}$.
   (2) $d_1 = \cdots = d_s = s$ *and*

$$\delta(f) = 2 = \frac{\log d}{\sum_{j=1}^{s} \frac{d_j}{d} \log d_j}.$$

Proof. By the assumptions of our theorem, Theorem 3.1 and Lemma 3.14, we obtain

$$2 \leq \frac{\log d}{\sum_{j=1}^{s} \frac{d_j}{d} \log d_j} \leq \delta(f) \leq 2.$$

Therefore

$$(3.9) \qquad\qquad 2 = \frac{\log d}{\sum_{j=1}^{s} \frac{d_j}{d} \log d_j} = \delta(f).$$

Thus, by Lemma 3.14, we obtain

$$\frac{\log d}{\int_{J(\tilde{f})} \log \|\tilde{f}'\| d\mu} = \delta(f),$$

where μ denotes the maximal entropy measure for $\tilde{f} : \Sigma_s \times \hat{\mathbb{C}} \to \Sigma_s \times \hat{\mathbb{C}}$. By Theorem 3.1, it follows that there exists a $\varphi \in \mathrm{Aut}(\mathbb{C})$, non-zero complex numbers $a_1, \ldots, a_s$, and a number

$d_0 \in \mathbb{N}$ such that $d_0 = d_1 = \cdots d_s$ and $\varphi \circ f_j \circ \varphi^{-1}(z) = a_j z^{d_0}$ for all $z \in \hat{\mathbb{C}}$. By (3.9), we obtain

$$2 = \frac{\log d}{\sum_{j=1}^{s} \frac{d_j}{d} \log d_j} = 1 + \frac{\log s}{\log d_0}.$$

Therefore, $d_0 = s$. Thus, we have completed the proof. □

Remark 3.18. *Let $f = (f_1, \ldots, f_s) \in \mathrm{Exp}(s)$. Suppose that f satisfies the open set condition. Then $\delta(f) = \mathrm{HD}(J(\langle f_1, \ldots, f_s \rangle)) \leq 2$ (see [21], [22]).*

Corollary 3.19. *Let $f = (f_1, f_2) \in \mathrm{Epb}(2)$. Suppose that $\deg(f_1) = \deg(f_2) = 2$. Then, the following statements (1),(2), (3), (4) are equivalent.*

(1) *$\delta(f) \leq 2$.*
(2) *$\delta(f) = 2$.*
(3) *There exists a $\varphi \in \mathrm{Aut}(\mathbb{C})$ and a non-zero complex number a such that*

$$\varphi \circ f_1 \circ \varphi^{-1}(z) = z^2, \varphi \circ f_2 \circ \varphi^{-1}(z) = az^2 \text{ for all } z \in \hat{\mathbb{C}}.$$

(4) *either*
    (a) *f satisfies the open set condition or*
    (b) *there exists a $\varphi \in \mathrm{Aut}(\mathbb{C})$ and a complex number a with $|a| = 1$ such that*

$$\varphi \circ f_1 \circ \varphi^{-1}(z) = z^2, \varphi \circ f_2 \circ \varphi^{-1}(z) = az^2 \text{ for all } z \in \hat{\mathbb{C}}.$$

Proof. "(1)⇒(2)" and "(2)⇒(3)" follow from Theorem 3.17. It is easy to see "(3)⇒(4)". "(4)⇒(1)" follows from Remark 3.18. Thus, we have completed the proof. □

4. Remarks and examples

In this section we collect some remarks and construct relevant examples illustrating our main theorems.

Remark 4.1 ([27, 25])**.** *Let $s \geq 2$ and let $d_2, \ldots, d_s \in \mathbb{N}$ be such that $d_j \geq 2$ for each $j = 2, \ldots, s$. Let $f_1 \in \mathrm{Epb}(1)$. Let $b_2, b_3, \ldots, b_s \in \mathrm{int}(K(f_1))$. Then, the following statements hold.*

(1) *There exists a number $c > 0$ such that for each $(a_2, a_3, \ldots, a_s) \in \mathbb{C}^{s-1}$ with $0 < |a_j| < c$ $(j = 2, \ldots, s)$, setting $f_j(z) = a_j(z - b_j)^{d_j} + b_j$ $(j = 2, \ldots, s)$, we have $(f_1, \ldots, f_s) \in \mathrm{Epb}(s)$.*
(2) *Suppose also that either (i) there exists a $j \geq 2$ with $d_j \geq 3$, or (ii) $\deg(f_1) = 3$, $b_2 = \cdots = b_s$. Then, there exist $a_2, a_3, \ldots, a_s > 0$ such that setting $f_j(z) = a_j(z-b_j)^{d_j}+b_j$ $(j = 2, \ldots, s)$, we have $(f_1, \ldots, f_s) \in \mathrm{Epb}(s)$ and $J(\langle f_1, \ldots, f_s \rangle)$ is disconnected.*

In [25, 27], the first author of this paper provided a lot of methods of constructing of examples of elements of $\mathrm{Epb}(s)$.

We give below concrete examples of expanding polynomial semigroups satisfying the open set condition.

Remark 4.2. *Let $f_1 \in \mathrm{Epb}(1)$ and let $b \in \mathrm{int}(K(f_1))$. Let $d_1 := \deg(f_1)$. Let $d_2 \in \mathbb{N}$ with $d_2 \geq 2$ and suppose that $(d_1, d_2) \neq (2, 2)$. Then there exists a number $c > 0$ such that for each $a \in \mathbb{C}$ with $0 < |a| < c$, setting $f_2(z) := a(z - b)^{d_2} + b$ and $f = (f_1, f_2) \in \mathcal{P}^2$, we have*

(a) $f \in \text{Epb}(2)$,

(b) f satisfies the separating open set condition,

(c) $\delta(f) < 2$, and

(d) setting $G = \langle f_1, f_2 \rangle$, we have $\text{HD}(J(G)) = \delta(f) < 2$.

For the proof of this result, see [31]. Moreover, by Theorem 3.15, setting $d := d_1 + d_2$, we have
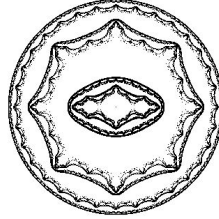
$$\text{HD}(J(G)) \geq \frac{\log d}{\sum_{j=1}^{2} \frac{d_j}{d} \log d_j} > 1.$$

If f_1 and f_2 are not simultaneously conjugate to the form az^2 by an element in $\text{Aut}(\mathbb{C})$, then by Theorem 3.1,

$$\text{HD}(J(G)) > \frac{\log d}{\sum_{j=1}^{2} \frac{d_j}{d} \log d_j} > 1.$$

See also Figure 1.

FIGURE 1. The Julia set of $G = \langle g_1^2, g_2^2 \rangle$, where $g_1(z) := z^2 - 1, g_2(z) := \frac{z^2}{4}$. $f := (g_1^2, g_2^2)$ satisfies (a)–(d) in Remark 4.2. Moreover, by Theorem 3.15, $\frac{\log 8}{\log 4} = \frac{3}{2} < \text{HD}(J(G)) < 2$.


We give examples of elements $f = (f_1, f_2) \in \text{Epb}(2)$ with $\delta(f) > 2$.

Example 4.3 ([27])**.** Let $f_1 \in \text{Epb}(1)$ with $\deg(f_1) = 2$. Let $b \in \text{int}(K(f_1))$, where $K(\cdot)$ denotes the filled-in Julia set. Then, by [27], there exists a number $c > 0$ such that for each $a \in \mathbb{C}$ with $0 < |a| < c$, setting $f_2(z) = a(z - b)^2 + b$, we have $f := (f_1, f_2) \in \text{Epb}(2)$. By Corollary 3.19, it follows that if f_1 and f_2 are not simultaneously conjugate to the form az^2 by an element in $\text{Aut}(\mathbb{C})$, then $\delta(f) > 2$. See Figure 2.

Example 4.4. Let $f_1(z) = z^2$. For each $c \in \mathbb{C}$, let $f_{2,c} = \frac{1}{4}z^2 + c$. Let $f_c := (f_1, f_{2,c})$ and $G_c := \langle f_1, f_{2,c} \rangle$. Then by Lemma 2.11, there exists a number $c_0 > 0$ such that for each $c \in \mathbb{C}$ with $|c| < c_0$, $f_c \in \text{Epb}(2)$. Moreover, by Corollary 3.19, $\delta(f_0) = 2$ and for each $c \in \mathbb{C}$ with $0 < |c| < c_0$, $\delta(f_c) > 2$. Let U_0 be the connected component of $\text{Exp}(2)$ with $(f_1, f_{2,0}) \in U_0$. Since $c \mapsto \delta(f_c)$ is real analytic on U_0 ([31]), it follows that $c \mapsto \delta(f_c)$ is not constant in any open subset of U_0.

Remark 4.5. We remark that, as illustrated in [24, 29], estimating the Hausdorff dimension of the Julia sets of rational semigroups plays an important role when we investigate random complex dynamics and its associated Markov process on $\hat{\mathbb{C}}$. For example, when we consider the random dynamics of a compact family Γ of polynomials of degree greater than or equal to two, then the function $T_\infty : \hat{\mathbb{C}} \to [0, 1]$ representing the probability of tending to $\infty \in \hat{\mathbb{C}}$

FIGURE 2. The Julia set of $G = \langle f_1, f_2 \rangle$, where $f_1(z) := z^2 - 1, f_2(z) := 0.09z^2$. $f := (f_1, f_2)$ belongs to Epb(2) (see [27]). By Corollary 3.19, $\delta(f) > 2$.
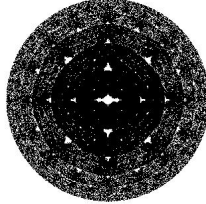
varies only on a subset of the Julia set of the polynomial semigroup generated by Γ, and under certain conditions, the function $T_\infty : \hat{\mathbb{C}} \to [0,1]$ is continuous on $\hat{\mathbb{C}}$. If the Hausdorff dimension of the Julia set is strictly less than two, then it means that $T_\infty : \hat{\mathbb{C}} \to [0,1]$ is a complex version of devil's staircase (Cantor function) ([23, 29]). For example, setting $g_1(z) := z^2 - 1, g_2(z) := \frac{z^2}{4}$, $f_1 := g_1^2$, and $f_2 := g_2^2$, we consider the random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a map f_j with probability $0 < p_j < 1$, where $p_1 + p_2 = 1$. Then the function T_∞ representing the probability of tending to ∞ is continuous on $\hat{\mathbb{C}}$ and varies exactly on the Julia set (Figure 1) of the polynomial semigroup $\langle f_1, f_2 \rangle$, whose Hausdorff dimension is strictly less than two (see [23, 29]).

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