RANDOM GRAPH DIRECTED MARKOV SYSTEMS

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Abstract. We introduce and explore random conformal graph directed Markov systems governed by measure-preserving ergodic dynamical systems. We first develop the symbolic thermodynamic formalism for random finitely primitive subshifts of finite type with a countable alphabet (by establishing tightness in a narrow topology). We then construct fibrewise conformal and invariant measures along with fibrewise topological pressure. This enables us to define the expected topological pressure $\mathcal{E}P(t)$ and to prove a variant of Bowen's formula which identifies the Hausdorff dimension of almost every limit set fiber with $\inf\{t : \mathcal{E}P(t) \leq 0\}$, and is the unique zero of the expected pressure if the alphabet is finite or the system is regular. We introduce the class of essentially random systems and we show that in the realm of systems with finite alphabet their limit set fibers are never homeomorphic in a bi-Lipschitz fashion to the limit sets of deterministic systems; they thus make up a drastically new world. We also provide a large variety of examples, with exact computations of Hausdorff dimensions, and we study in detail the small random perturbations of an arbitrary elliptic function.

1. Introduction

In this paper we introduce and systematically develop the theory of random conformal graph directed Markov systems satisfying the open set condition, which comprise the random conformal iterated function systems satisfying the open set condition. Our main emphasis is on infinite systems, i.e. systems that have a countably infinite alphabet. Our approach builds on the following three main sources of motivation: random distance expanding dynamical systems (cf. [8]), random measures (cf. [2]), and deterministic conformal graph directed Markov systems (cf. [6]).

In section 2 we first deal with purely symbolic systems, namely random shifts of finite type. We prove that, under a Hölder continuous potential, these systems admit fibrewise "conformal" measures, fibrewise "invariant" measures and fibrewise topological pressure. In the case of finite systems, i.e. systems whose alphabet is finite, this directly follows from [1] and [8]. The infinite case is tackled by exhausting the alphabet with its finite subalphabets, and by proving tightness in the narrow topology of random measures (cf. [2]). In particular, this requires showing that the limit objects resulting from compactness (tightness) satisfy the requirements of conformality and invariantness.

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In section 3 we define random conformal graph directed Markov systems (abbr. RCGDMSs) and, as an application of our symbolic dynamics results, we demonstrate that RCGDMSs admit fibrewise conformal measures and topological pressure (see Theorem 3.7). Particularly involved is the argument establishing measurewise disjointness of the first-level sets (see (3.14)). We are then in a position to define the expected pressure $\mathcal{E}P(t)$ (see Proposition 3.12), and its related features, among others the finiteness set $\mathcal{F}in$ and the left endpoint θ of this latter, also known as finiteness parameter of the system. Motivated by the deterministic case (see [5], [6]) we classify RCGDMSs in regular and irregular ones, and further divide regular systems into critically regular, strongly regular and cofinitely regular, all of this in terms of the shape of the expected pressure function $\mathcal{E}P(t)$. Our main geometric result is a variant of Bowen's formula (see Theorem 3.18 and its extension, Theorem 3.26). It affirms that the Hausdorff dimension of almost every limit set fiber of the system is $\inf\{t \geq 0 : \mathcal{E}P(t) \leq 0\}$, which coincides with the only zero of the expected pressure function when the system is regular. Inspired by definitions from [8] we coin the concept of essentially random systems (see Definition 3.27) and we prove that these systems have limit set fibers with almost surely zero Hausdorff measure but infinite packing measure (see Theorem 3.28). This result has much more striking consequences in the case of a finite alphabet than in the infinite one. Indeed, the limit sets of deterministic systems with infinite alphabets may have zero Hausdorff measure and/or infinite packing measure (see [5], [6] for appropriate examples), whereas the limit sets of finite deterministic systems have Hausdorff and packing measures which are both finite and positive. Our result implies that almost no limit set fiber of a finite RCGDMS is bi-Lipschitz homeomorphic to the limit set of a finite deterministic CGDMS. Hence, random CGDMSs form a new realm, drastically different from the deterministic one.

The last two sections of our paper are devoted to examples. In section 4 we provide general methods of how to naturally construct random systems out of deterministic ones. We also provide examples of random systems built from scratch. In most examples, we further give an exact formula for the almost sure Hausdorff dimension of the limit set fibers. In the fifth and last section we deal with small random perturbations of an arbitrary non-constant elliptic function. Motivated by the construction from [4] we associate to such a random dynamical system of elliptic functions a random conformal iterated function system. We then estimate from below the Hausdorff dimension of the limit set fibers of the corresponding random IFSs by showing that they are evenly varying and by computing their θ number.

There are several ways to generate and study random fractal sets. However, in the attempts made so far, the proposed constructions either dealt with similarity maps only or demanded identically distributed randomly independent choices of maps. We, on the other hand, assume only that the generators are conformal, and that the random choice of generators is governed by a measure-preserving ergodic dynamical system. Furthermore, even if this dynamical system is a Bernoulli shift preserving a Bernoulli measure (in other words, this means that our random process is identically distributed and independent), our limit sets are different than those generated via (in some sense) parallel constructions in [7], the reason for this being that the Hausdorff dimensions of corresponding limit sets are different.

Finally, we would like to mention that the paper [3] also deals with the thermodynamic formalism for random shifts with a countably infinite alphabet, and produces objects like fibrewise pressure, and fibrewise conformal and invariant measures. The primary hypothesis in [3] is that the potential is positive recurrent, a concept involving the asymptotic behavior of partition functions. In our paper, we prefer simply assuming the Hölder continuity of the potential and we take a direct path to develop the thermodynamic formalism. However, there is also a second reason, albeit a less important one, why we avoided making use of [3]. Namely, in the proof of Proposition 6.3, just after formula (6.5), the authors conclude the existence of fibrewise weak limits on the ground of tightness in the random narrow topology. In general, this is not true, as shows a counterexample in [2].

2. Random Shifts over a Countable Alphabet

Let E be a countable (finite or infinite) alphabet. Without loss of generality, we may assume that $E \subset I\!N$. Let $A : E \times E \to \{0,1\}$ be a matrix whose entries are indexed by the elements of E. This matrix determines the set of all one-sided infinite A-admissible words

$$E_A^\infty := \Big\{\omega \in E^\infty : A_{\omega_k \omega_{k+1}} = 1, \forall\, k \in I\!N\Big\}.$$

Equip this set with the topology generated by the one-cylinders $[e]_k := \{\omega \in E_A^\infty : \omega_k = e\}$, $e \in E$, $k \in I\!N$. This topology coincides with the topology induced on E_A^∞ by Tychonov's product topology on E^∞ when E is endowed with the discrete topology. The space E_A^∞ is a closed subspace of E^∞. The space E_A^∞ is sometimes called coding space. When endowed with the Borel σ-algebra $\mathcal{B}$, the coding space E_A^∞ becomes a measurable space.

The set of all subwords of length $k \in I\!N$ of words in E_A^∞ will be denoted by E_A^k, whereas the set of all finite subwords of words in E_A^∞ will be denoted by $E_A^* = \cup_{k \in I\!N} E_A^k$. For every $\omega \in E_A^* \cup E_A^\infty$, the length of ω, i.e. the unique $k \in I\!N \cup \{\infty\}$ such that $\omega \in E_A^k$, shall be denoted by $|\omega|$. If $\omega \in E_A^* \cup E_A^\infty$ and $k \in I\!N$ does not exceed the length of ω, we shall denote the initial subword $\omega_1\omega_2 \dots \omega_k$ by $\omega|_k$. Moreover, for every $\omega \in E_A^*$ we shall denote the open set of all infinite A-admissible words beginning with ω by $[\omega] := \{\tau \in E_A^\infty : \tau|_{|\omega|} = \omega\}$. Note that $[e] = [e]_1$ for all $e \in E$. Furthermore, for every $\omega, \tau \in E_A^\infty$, let $\omega \wedge \tau \in E_A^* \cup E_A^\infty$ the longest prefix of ω and τ such that $\omega|_k = \tau|_k$. From a dynamical point of view, we will be interested in the (left) shift map $\sigma : E^\infty \to E^\infty$ which drops the first letter of each word. The shift map is obviously continuous, and thereby measurable.

Let $F \subset E$. We now briefly describe some spaces of functions on F_A^∞. Denote by $C(F_A^\infty)$ the space of all continuous real-valued functions on F_A^∞ and by $C^b(F_A^\infty)$ the subspace of all bounded continuous functions on F_A^∞, i.e. those $g \in C(F_A^\infty)$ such that $\|g\|_\infty := \sup\{|g(\omega)| : \omega \in F_A^\infty\} < \infty$. This subspace is a Banach space. Let $0 < s < 1$. For every $g \in C(F_A^\infty)$, set

$$v_{s,k}(g) := \inf_{C \geq 0}\Big\{|g(\omega) - g(\tau)| \leq Cs^k : \omega, \tau \in F_A^\infty \text{ such that } |\omega \wedge \tau| \geq k\Big\}$$

and

$$v_s(g) := \sup\Big\{v_{s,k}(g) : k \in I\!N\Big\}.$$

A function $g \in C(F_A^\infty)$ is called Hölder continuous with exponent s if $v_s(g) < \infty$. The constant $v_s(g)$ is the smallest Hölder constant such a g admits. We shall denote by $H_s(F_A^\infty)$ the vector space of all Hölder continuous functions with exponent s. We shall further denote by $H_s^b(F_A^\infty)$ the vector subspace of all Hölder continuous functions with exponent s which are bounded, i.e. $H_s^b(F_A^\infty) := H_s(F_A^\infty) \cap C^b(F_A^\infty)$. Endowed with the norm

$$\|g\|_s := \|g\|_\infty + v_s(g),$$

the space $H_s^b(F_A^\infty)$ becomes a Banach space.

The randomness of the graph directed Markov systems we shall study later will be based on a probability space $(\Lambda, \mathcal{F}, \nu)$ and an invertible ergodic map $T : \Lambda \to \Lambda$ preserving a complete measure ν. The Cartesian product $E_A^\infty \times \Lambda$ becomes a measurable space when equipped with the product σ-algebra $\mathcal{B} \otimes \mathcal{F}$, i.e. the σ-algebra generated by the countable unions of Cartesian products of the form $B \times F$ with $B \in \mathcal{B}$ and $F \in \mathcal{F}$. Let $p_{E_A^\infty} : E_A^\infty \times \Lambda \to E_A^\infty$ and $p_\Lambda : E_A^\infty \times \Lambda \to \Lambda$ be the canonical projections onto E_A^∞ and Λ, respectively, i.e. $p_{E_A^\infty}(\omega, \lambda) = \omega$ and $p_\Lambda(\omega, \lambda) = \lambda$. Both projections are trivially measurable. In fact, $\mathcal{B} \otimes \mathcal{F}$ is the smallest σ-algebra with respect to which both projections are measurable.

The product map $\sigma \times T : E_A^\infty \times \Lambda \to E_A^\infty \times \Lambda$, defined as

$$(\sigma \times T)(\omega, \lambda) := \big(\sigma(\omega), T(\lambda)\big),$$

is obviously measurable. Indeed, $(\sigma \times T)^{-1}(B \times F) = \sigma^{-1}(B) \times T^{-1}(F)$ for all $B \in \mathcal{B}$ and all $F \in \mathcal{F}$.

We now turn our attention to spaces of random functions. Again, let $F \subset E$.

Definition 2.1. *A function $f : F_A^\infty \times \Lambda \to I\!R$ is said to be a random continuous function on F_A^∞ if*

- *for all $\omega \in F_A^\infty$ the ω-section $\lambda \mapsto f_\omega(\lambda) := f(\omega, \lambda)$ is measurable; and*
- *for all $\lambda \in \Lambda$ the λ-section $\omega \mapsto f_\lambda(\omega) := f(\omega, \lambda)$ is continuous on F_A^∞.*

We shall denote the vector space of all random continuous functions on F_A^∞ by $C_\Lambda(F_A^\infty)$. Note that by Lemma 1.1 in [2], any random continuous function f is jointly measurable.

It is then natural to make the following definitions.

Definition 2.2. *A random continuous function $f \in C_\Lambda(F_A^\infty)$ is said to be bounded if*

$$\|f_\lambda\|_\infty < \infty, \ \forall \lambda \in \Lambda \qquad \text{and} \qquad \|f\|_\infty := \text{ess sup}\{\|f_\lambda\|_\infty : \lambda \in \Lambda\} < \infty.$$

Bounded random continuous functions, as defined above, are random continuous in the sense of Crauel [2] (cf. Definition 3.9). The space of all bounded random continuous functions on F_A^∞ shall be denoted by $C_\Lambda^b(F_A^\infty)$. When equipped with the norm $\|f\|_\infty$, this space is Banach.

Definition 2.3. *A random continuous function $f \in C_\Lambda(F_A^\infty)$ is said to be Hölder with exponent s if*

$$v_s(f_\lambda) < \infty, \ \forall \lambda \in \Lambda \qquad \text{and} \qquad v_s(f) := \text{ess sup}\{v_s(f_\lambda) : \lambda \in \Lambda\} < \infty.$$

We shall denote by $H_{s,\Lambda}(F_A^\infty)$ the vector space of all random Hölder continuous functions with exponent s and by $H_{s,\Lambda}^b(F_A^\infty)$ the subspace of all bounded random Hölder continuous functions with exponent s. Endowed with the norm

$$\|f\|_s := \|f\|_\infty + v_s(f),$$

the space $H_{s,\Lambda}^b(F_A^\infty)$ becomes a Banach space.

We now introduce the concept of summability for random continuous functions.

Definition 2.4. *A random continuous function $f \in C_\Lambda(F_A^\infty)$ is called summable if*

$$M_f := \sum_{e\in F} \exp\Big(\text{ess sup}\{\sup(f_\lambda|_{[e]}) : \lambda \in \Lambda\}\Big) < \infty. \tag{2.1}$$

Note that no bounded random continuous function on F_A^∞ is summable whenever F is infinite. Henceforth, we shall denote by a superscript Σ spaces of summable functions. For instance, the vector space of all summable random Hölder continuous functions with exponent s will be denoted by $H_{s,\Lambda}^\Sigma(F_A^\infty)$.

Now, we shall describe properties of random measures which will play a crucial role later. Denote by $P_\Lambda(\nu)$ the space of all probability measures $\tilde{m}$ on $(E_A^\infty \times \Lambda, \mathcal{B}\otimes\mathcal{F})$ whose marginal is ν, i.e. all probability measures $\tilde{m}$ such that $\tilde{m}\circ p_\Lambda^{-1} = \nu$. By Propositions 3.3($ii$) and 3.6 in [2], this space is isomorphic to the space of random probabiblity measures $\tilde{m}$ on E_A^∞, i.e. the space of all functions $(B,\lambda)\mapsto \tilde{m}_\lambda(B)\in[0,1]$ such that

- for every $B\in\mathcal{B}$, the function $\lambda\mapsto\tilde{m}_\lambda(B)$ is measurable; and
- for ν-a.e. $\lambda\in\Lambda$, the function $B\mapsto\tilde{m}_\lambda(B)$ is a Borel probability measure.

We then write $\tilde{m} = \tilde{m}_\lambda\otimes\nu$. Endow $P_\Lambda(\nu)$ with the narrow topology. Recall that a sequence $(\tilde{m}_n)_{n=1}^\infty$ of measures in $P_\Lambda(\nu)$ converges to a measure $\tilde{m}\in P_\Lambda(\nu)$ in the narrow topology if

$$\lim_{n\to\infty} \tilde{m}_n(f) = \tilde{m}(f)$$

for all $f\in C_\Lambda^b(E_A^\infty)$, where

$$\tilde{\mu}(f) := \int_{E_A^\infty\times\Lambda} f(\omega,\lambda)\,d\tilde{\mu}(\omega,\lambda) = \int_\Lambda\int_{E_A^\infty} f_\lambda(\omega)\,d\tilde{\mu}_\lambda(\omega)\,d\nu(\lambda).$$

for any $\tilde{\mu} = \tilde{\mu}_\lambda\otimes\nu\in P_\Lambda(\nu)$. Prohorov's Theorem for random measures (see Theorem 4.4 in [2]) states that a subset $\Gamma\subset P_\Lambda(\nu)$ is relatively compact if and only if Γ is tight, and that such a set Γ is relatively sequentially compact.

Finally, we introduce Perron-Frobenius operators. Let $f\in H_{s,\Lambda}^\Sigma(E_A^\infty)$ and $F\subset E$. For ν-a.e. $\lambda\in\Lambda$ the Perron-Frobenius operator $\mathcal{L}_{f,F,\lambda} : C^b(E_A^\infty)\to C^b(E_A^\infty)$ defined by

$$\mathcal{L}_{f,F,\lambda} g(\omega) = \sum_{e\in F: A_{e\omega_1}=1} \exp\Big(f(e\omega,\lambda)\Big)\, g(e\omega)$$

is well defined. For every $k\geq 2$, we may thereafter define for ν-a.e. $\lambda\in\Lambda$ the operators

$$\mathcal{L}_{f,F,\lambda}^k g := \mathcal{L}_{f,F,T^{k-1}(\lambda)}\circ\mathcal{L}_{f,F,T^{k-2}(\lambda)}\circ\cdots\circ\mathcal{L}_{f,F,\lambda} g.$$

It is easy to show that for ν-a.e. $\lambda \in \Lambda$, we have

$$\mathcal{L}^k_{f,F,\lambda} g(\omega) = \sum_{\tau \in F^k_A : A_{\tau_k \omega_1} = 1} \exp\Big(S_k f(\tau\omega, \lambda)\Big)\, g(\tau\omega), \tag{2.2}$$

where

$$S_k f(\rho, \lambda) = \sum_{j=0}^{k-1} f((\sigma \times T)^j(\rho, \lambda)) = \sum_{j=0}^{k-1} f(\sigma^j \rho, T^j \lambda). \tag{2.3}$$

It is also easy to check that all $\mathcal{L}^k_{f,F,\lambda}$, $k \in I\!N$, preserve the Banach spaces $C^b(E^\infty_A)$, $C^b(F^\infty_A)$, $H^b_s(E^\infty_A)$ and $H^b_s(F^\infty_A)$ for ν-a.e. $\lambda \in \Lambda$. Let $\mathcal{L}^{k*}_{f,F,\lambda}$ be the operator dual to $\mathcal{L}^k_{f,F,\lambda}$ which acts on either $(C^b(E^\infty_A))^*$ or $(C^b(F^\infty_A))^*$, depending on whether the operator $\mathcal{L}^k_{f,F,\lambda}$ is seen acting on $C^b(E^\infty_A)$ or $C^b(F^\infty_A)$. From this point on, we shall omit the subscript F when $F = E$ and write $\mathcal{L}^k_{f,\lambda}$ for $\mathcal{L}^k_{f,E,\lambda}$. Also, when no confusion may arise, we shall frequently drop the subscript f.

When $F \subset E$ is a finite set, our setting reduces to the random distance expanding maps studied in [8] and in which the following theorem has been proved (see Theorem 3.1, Lemma 4.5 and Lemma 4.3 in [8]).

Theorem 2.5. *Let $F \subset E$ be a finite subalphabet such that $A|_{F\times F}$ is irreducible. If $f \in H^b_{s,\Lambda}(F^\infty_A)$, then for ν-a.e. $\lambda \in \Lambda$ there exist a unique $P_{F,\lambda}(f) \in I\!R$ and a unique Borel probability measure $\tilde{m}^{f,F}_\lambda$ on F^∞_A with* supp $\tilde{m}^{f,F}_\lambda = F^\infty_A$ *such that*

$$\mathcal{L}^*_{f,F,\lambda} \tilde{m}^{f,F}_{T(\lambda)} = e^{P_{F,\lambda}(f)} \tilde{m}^{f,F}_\lambda$$

and such that the function $\lambda \mapsto P_{F,\lambda}(f)$ is ν-integrable while the function $\lambda \mapsto \tilde{m}^{f,F}_\lambda(B)$ is measurable for every $B \in \mathcal{B} \cap F^\infty_A$.

In particular, this result shows that $\tilde{m}^{f,F}(B,\lambda) := \tilde{m}^{f,F}_\lambda(B)$, i.e. $\tilde{m}^{f,F}_\lambda \otimes \nu$, is a random probability measure on F^∞_A and so is its extension to E^∞_A. This extension shall be denoted by the same notation as the original random measure. Moreover, for ν-a.e. $\lambda \in \Lambda$ we deduce by recurrence that

$$\mathcal{L}^{k*}_{f,F,\lambda} \tilde{m}^{f,F}_{T^k(\lambda)} = e^{P^k_{F,\lambda}(f)} \tilde{m}^{f,F}_\lambda, \tag{2.4}$$

where

$$P^k_{F,\lambda}(f) := \sum_{j=0}^{k-1} P_{F,T^j(\lambda)}(f). \tag{2.5}$$

The main technical fact proved in this section is the following. It concerns sequences of random probability measures which arise from ascending sequences of finite subalphabets $(F_n)_{n=1}^\infty$ that cover the entire alphabet E. In order to allege notation, for all $\lambda \in \Lambda$ for which they are defined, we shall henceforth denote $P^k_{F_n,\lambda}(f)$ by $P^k_n(\lambda)$ and $\tilde{m}^{f,F_n}_\lambda$ by $\tilde{m}^n_\lambda$. Moreover, note that the following result does not require that the random Hölder continuous function $f \in H_{s,\Lambda}(E^\infty_A)$ be bounded, as this is a property that the natural potentials $t\zeta$ for random graph directed Markov systems do not fulfill (cf. section 3). Instead, we demand that f be summable and bounded over finite subalphabets. We thus make the following definition.

Definition 2.6. *A random continuous function $f \in C_\Lambda(E_A^\infty)$ is said to be bounded over finite subalphabets if $f|_{F_A^\infty \times \Lambda} \in C_\Lambda^b(F_A^\infty)$ for every finite set $F \subset E$.*

Now, the result. Recall that a matrix A is finitely irreducible if there exists a finite set $\Omega \subset E_A^*$ such that for all $e, f \in E$ there is a word $\omega \in \Omega$ for which $e\omega f \in E_A^*$.

Lemma 2.7. *Let E be a countably infinite alphabet and A a finitely irreducible matrix. Let $f \in H_{s,\Lambda}^\Sigma(E_A^\infty)$ be bounded over finite subalphabets. If $(F_n)_{n=1}^\infty$ is an ascending sequence of finite subalphabets whose union is E, then the sequence of random probability measures $(\tilde{m}_\lambda^n \otimes \nu)_{n=1}^\infty$ is tight in $P_\Lambda(\nu)$.*

Proof. Since A is finitely irreducible, there exists a finite set $F \subset E$ that witnesses the finite irreducibility of A, that is, such that for any pair of letters $e, \tilde{e} \in E$ there is $\tau \in F^*$ such that $e\tau\tilde{e} \in E_A^*$. Without loss of generality, we may assume that $F \subset F_1$. Thus, $A|_{F_n \times F_n}$ is irreducible for all $n \in I\!N$. In virtue of Theorem 2.5, we have for ν-a.e. $\lambda \in \Lambda$ and all $n \in I\!N$ that

$$e^{P_n(\lambda)} = \tilde{m}_{T(\lambda)}^n(\mathcal{L}_{f,F_n,\lambda}(1\!\!1_{(F_n)_A^\infty})) \geq \tilde{m}_{T(\lambda)}^n(\mathcal{L}_{f,F,\lambda}(1\!\!1_{F_A^\infty})) \geq Q_f > 0, \tag{2.6}$$

where $Q_f := \text{ess inf}\{\mathcal{L}_{f,F,\lambda}(1\!\!1_{F_A^\infty})(\omega) : \lambda \in \Lambda, \omega \in F_A^\infty\}$. Note that $Q_f > 0$ since $f|_{F_A^\infty \times \Lambda} \in C_\Lambda^b(F_A^\infty)$ and $A|_{F \times F}$ is irreducible. By Theorem 2.5 and relations (2.1) to (2.6), we get for ν-a.e. $\lambda \in \Lambda$, every $n \in I\!N$, every $k \in I\!N$ and every $e \in E$, that

$$\begin{aligned}
\tilde{m}_\lambda^n([e]_k) &= \sum_{\omega \in (F_n)_A^k : \omega_k = e} \tilde{m}_\lambda^n([\omega]) = \sum_{\omega \in (F_n)_A^k : \omega_k = e} \exp(-P_n^k(\lambda)) \mathcal{L}_{f,F_n,\lambda}^{k*} \tilde{m}_{T^k(\lambda)}^n([\omega]) \\
&= \exp(-P_n^k(\lambda)) \sum_{\omega \in (F_n)_A^k : \omega_k = e} \int_{(F_n)_A^\infty} \mathcal{L}_{f,F_n,\lambda}^k 1\!\!1_{[\omega]}(\tau) \, d\tilde{m}_{T^k(\lambda)}^n(\tau) \\
&= \exp(-P_n^k(\lambda)) \sum_{\omega \in (F_n)_A^k : \omega_k = e} \int_{(F_n)_A^\infty} \exp\Big(S_k f(\omega\tau, \lambda)\Big) \, d\tilde{m}_{T^k(\lambda)}^n(\tau) \\
&\leq Q_f^{-k} \sum_{\omega \in (F_n)_A^k : \omega_k = e} \exp\Big(\sup_{\rho \in [\omega]} S_k f(\rho, \lambda)\Big) \\
&\leq Q_f^{-k} \sum_{\omega \in (F_n)_A^{k-1} : A_{\omega_{k-1}e} = 1} \exp\Big(\sup_{\rho \in [\omega]} S_{k-1} f(\rho, \lambda) + \sup(f_{T^{k-1}(\lambda)}|_{[e]})\Big) \\
&\leq Q_f^{-k} \sum_{\omega \in (F_n)_A^{k-1}} \exp\Big(\sup_{\rho \in [\omega]} S_{k-1} f(\rho, \lambda)\Big) \exp\Big(\sup(f_{T^{k-1}(\lambda)}|_{[e]})\Big) \\
&\leq Q_f^{-k} M_f^{k-1} \exp\Big(\sup(f_{T^{k-1}(\lambda)}|_{[e]})\Big).
\end{aligned}$$

Therefore, for ν-a.e. $\lambda \in \Lambda$, every $n \in I\!N$, every $k \in I\!N$ and every $e \in E$, we have

$$\tilde{m}_\lambda^n\Big(\bigcup_{j > e} [j]_k\Big) \leq Q_f^{-k} M_f^{k-1} \sum_{j=e+1}^\infty \exp\Big(\sup(f_{T^{k-1}(\lambda)}|_{[j]})\Big). \tag{2.7}$$

Now, fix $\varepsilon > 0$. It follows from (2.1) and (2.7) that for all $k \in I\!N$ there exists $e_k \in E$ such that for ν-a.e. $\lambda \in \Lambda$ and all $n \in I\!N$, we have

$$\tilde{m}_\lambda^n\Big(\bigcup_{j>e_k}[j]_k\Big) \leq \frac{\varepsilon}{2^k}.$$

Consequently,

$$\tilde{m}_\lambda^n\Big(E_A^\infty \cap \prod_{k=1}^\infty \{1,\dots,e_k\}\Big) \geq 1 - \sum_{k=1}^\infty \tilde{m}_\lambda^n\Big(\bigcup_{j>e_k}[j]_k\Big) \geq 1 - \sum_{k=1}^\infty \frac{\varepsilon}{2^k} = 1-\varepsilon$$

for ν-a.e. $\lambda \in \Lambda$ and all $n \in I\!N$. Thus,

$$\int_\Lambda \tilde{m}_\lambda^n\Big(E_A^\infty \cap \prod_{k=1}^\infty \{1,\dots,e_k\}\Big)\,d\nu(\lambda) \geq 1-\varepsilon$$

for all $n \in I\!N$. Since $E_A^\infty \cap \prod_{k=1}^\infty\{1,\dots,e_k\}$ is a compact subset of E_A^∞, the sequence $(\tilde{m}_\lambda^n \otimes \nu)_{n=1}^\infty$ is tight according to Proposition 4.3 in [2]. □

Using (2.1), (2.6) and Theorem 2.5, we also observe that

$$Q_f \leq e^{P_n(\lambda)} \leq M_f \tag{2.8}$$

for ν-a.e. $\lambda \in \Lambda$ and all $n \in I\!N$. Therefore, the following is an immediate consequence of Lemma 2.7.

Corollary 2.8. *Under the assumptions of Lemma 2.7, the sequence $((e^{P_n(\lambda)}\tilde{m}_\lambda^n) \otimes \nu)_{n=1}^\infty$ is tight in $P_\Lambda(\nu)$.*

We shall now prove that there is a relationship between the accumulation point(s) of the sequences $(\tilde{m}_\lambda^n \otimes \nu)_{n=1}^\infty$ and $((e^{P_n(\lambda)}\tilde{m}_\lambda^n) \otimes \nu)_{n=1}^\infty$.

Lemma 2.9. *Let $(n_j)_{j=1}^\infty$ be a sequence of natural numbers. If the sequences $(\tilde{m}_j)_{j=1}^\infty := (\tilde{m}_\lambda^{n_j} \otimes \nu)_{j=1}^\infty$ and $(\tilde{\mu}_j)_{j=1}^\infty := ((e^{P_{n_j}(\lambda)}\tilde{m}_\lambda^{n_j}) \otimes \nu)_{j=1}^\infty$ converge in the narrow topology of $P_\Lambda(\nu)$ to $\tilde{m} = \tilde{m}_\lambda \otimes \nu$ and $\tilde{\mu} = \tilde{\mu}_\lambda \otimes \nu$, respectively, then for ν-a.e. $\lambda \in \Lambda$ there exists $\gamma_\lambda \in [Q_f, M_f]$ such that $\tilde{\mu}_\lambda = \gamma_\lambda \tilde{m}_\lambda$ and the function $\lambda \mapsto \gamma_\lambda$ is measurable.*

Proof. Fix a non-negative $g \in C_\Lambda^b(E_A^\infty)$. Thanks to (2.8), we have

$$\begin{aligned}
\int_{E_A^\infty \times \Lambda} g\,d\tilde{\mu} = \lim_{j\to\infty}\int_{E_A^\infty \times \Lambda} g\,d\tilde{\mu}_j &= \lim_{j\to\infty}\int_\Lambda\int_{E_A^\infty} g_\lambda(\omega)\,d(e^{P_{n_j}(\lambda)}\tilde{m}_\lambda^{n_j})(\omega)\,d\nu(\lambda)\\
&= \lim_{j\to\infty}\int_\Lambda e^{P_{n_j}(\lambda)}\Big(\int_{E_A^\infty} g_\lambda(\omega)\,d\tilde{m}_\lambda^{n_j}(\omega)\Big)d\nu(\lambda)\\
&\leq M_f \lim_{j\to\infty}\int_\Lambda\int_{E_A^\infty} g_\lambda(\omega)\,d\tilde{m}_\lambda^{n_j}(\omega)\,d\nu(\lambda)\\
&= M_f \lim_{j\to\infty}\int_{E_A^\infty\times\Lambda} g\,d\tilde{m}_j\\
&= M_f \int_{E_A^\infty\times\Lambda} g\,d\tilde{m}.
\end{aligned}$$

Similarly, we have

$$\int_{E_A^\infty\times\Lambda} g\,d\tilde{\mu} \geq Q_f \int_{E_A^\infty\times\Lambda} g\,d\tilde{m}.$$

Therefore, $\tilde{\mu}$ is equivalent to $\tilde{m}$ and the Radon-Nikodym derivative satisfies $Q_f \leq d\tilde{\mu}/d\tilde{m} \leq M_f$. Hence, for ν-a.e. $\lambda \in \Lambda$ the measure $\tilde{\mu}_\lambda$ is equivalent to $\tilde{m}_\lambda$ and the Radon-Nikodym derivative $\gamma_\lambda := d\tilde{\mu}_\lambda/d\tilde{m}_\lambda : E_A^\infty \to [0,\infty)$ is bounded below by Q_f and above by M_f. We shall now prove that each function γ_λ is constant. Indeed, suppose that $g^{(1)}, g^{(2)} \in C_\Lambda^b(E_A^\infty)$ are two $\tilde{m}$-integrable functions such that

$$\int_{E_A^\infty} g_\lambda^{(1)}\,d\tilde{m}_\lambda = \int_{E_A^\infty} g_\lambda^{(2)}\,d\tilde{m}_\lambda$$

for ν-a.e. $\lambda \in \Lambda$. Then, by an argument similar to the one above, we have

$$\begin{aligned}\int_{E_A^\infty\times\Lambda} g^{(1)}\,d\tilde{\mu} &= \lim_{j\to\infty}\int_\Lambda e^{P_{n_j}(\lambda)}\int_{E_A^\infty} g_\lambda^{(1)}\,d\tilde{m}_\lambda^{n_j}\,d\nu(\lambda) = \lim_{j\to\infty}\int_\Lambda e^{P_{n_j}(\lambda)}\int_{E_A^\infty} g_\lambda^{(2)}\,d\tilde{m}_\lambda^{n_j}\,d\nu(\lambda)\\ &= \int_{E_A^\infty\times\Lambda} g^{(2)}\,d\tilde{\mu}.\end{aligned}$$

Now, let $g_\lambda^{(1)} = 1/\gamma_\lambda$ and $g_\lambda^{(2)} = \int_{E_A^\infty}(1/\gamma_\lambda)d\tilde{m}_\lambda = \tilde{m}_\lambda(1/\gamma_\lambda)$. It is clear that $\int_{E_A^\infty} g_\lambda^{(1)}\,d\tilde{m}_\lambda = \int_{E_A^\infty} g_\lambda^{(2)}\,d\tilde{m}_\lambda$, and thus $\int_{E_A^\infty\times\Lambda} g^{(1)}\,d\tilde{\mu} = \int_{E_A^\infty\times\Lambda} g^{(2)}\,d\tilde{\mu}$. It follows that

$$\begin{aligned}\int_\Lambda \tilde{m}_\lambda(1/\gamma_\lambda)\tilde{m}_\lambda(\gamma_\lambda)\,d\nu(\lambda) &= \int_\Lambda \tilde{m}_\lambda(1/\gamma_\lambda)\int_{E_A^\infty}\gamma_\lambda(\omega)\,d\tilde{m}_\lambda(\omega)\,d\nu(\lambda)\\ &= \int_\Lambda \tilde{m}_\lambda(1/\gamma_\lambda)\int_{E_A^\infty} d\tilde{\mu}_\lambda(\omega)\,d\nu(\lambda)\\ &= \int_\Lambda\int_{E_A^\infty} \tilde{m}_\lambda(1/\gamma_\lambda)\,d\tilde{\mu}_\lambda(\omega)\,d\nu(\lambda)\\ &= \int_{E_A^\infty\times\Lambda} g^{(2)}\,d\tilde{\mu}\\ &= \int_{E_A^\infty\times\Lambda} g^{(1)}\,d\tilde{\mu}\\ &= \int_\Lambda\int_{E_A^\infty} \frac{1}{\gamma_\lambda(\omega)}\,d\tilde{\mu}_\lambda(\omega)\,d\nu(\lambda)\\ &= \int_\Lambda\int_{E_A^\infty} \frac{1}{\gamma_\lambda(\omega)}\cdot\gamma_\lambda(\omega)\,d\tilde{m}_\lambda(\omega)\,d\nu(\lambda)\\ &= \int_\Lambda\int_{E_A^\infty} d\tilde{m}_\lambda(\omega)\,d\nu(\lambda)\\ &= 1.\end{aligned}$$

By Cauchy-Schwartz inequality, we also have that

$$\tilde{m}_\lambda(1/\gamma_\lambda)\tilde{m}_\lambda(\gamma_\lambda) \geq \left(\tilde{m}_\lambda(\sqrt{1/\gamma_\lambda\cdot\gamma_\lambda})\right)^2 = (\tilde{m}_\lambda(1))^2 = 1.$$

Therefore $\tilde{m}_\lambda(1/\gamma_\lambda)\tilde{m}_\lambda(\gamma_\lambda) = 1$ for ν-a.e. $\lambda \in \Lambda$. Hence $\tilde{m}_\lambda(1/\gamma_\lambda) = 1/\tilde{m}_\lambda(\gamma_\lambda)$ for ν-a.e. $\lambda \in \Lambda$. By Jensen's inequality, we deduce that $\gamma_\lambda = d\tilde{\mu}_\lambda/d\tilde{m}_\lambda$ is constant for ν-a.e. $\lambda \in \Lambda$. □

Now, we can prove the first main result of this section. It is a generalization of Theorem 2.5.

Theorem 2.10. *Let E be a countably infinite alphabet and A a finitely irreducible matrix. Let $f \in H^{\Sigma}_{s,\Lambda}(E_A^\infty)$ be bounded over finite subalphabets. For every such potential f there exists a unique random probability measure $\tilde{m} \in P_\Lambda(\nu)$ and a unique bounded measurable function $\lambda \mapsto P_\lambda(f) \in I\!R$ such that*

$$\mathcal{L}^*_{f,\lambda}\tilde{m}_{T(\lambda)} = e^{P_\lambda(f)}\tilde{m}_\lambda \tag{2.9}$$

for ν-a.e. $\lambda \in \Lambda$. Moreover, for ν-a.e. $\lambda \in \Lambda$,

$$\operatorname{supp}\tilde{m}_\lambda = E_A^\infty, \quad Q_f \le e^{P_\lambda(f)} \le M_f \quad \textit{and} \quad P_\lambda(f) = \lim_{n\to\infty} P_n(\lambda). \tag{2.10}$$

Proof. Take an arbitrary ascending sequence $(F_n)_{n=1}^\infty$ of finite subalphabets such that $\cup_{n=1}^\infty F_n = E$. By Lemma 2.7, Corollary 2.8 and Prohorov's Theorem for random measures (see Theorem 4.4 in [2]), there exists an unbounded increasing sequence $(n_j)_{j=1}^\infty$ such that both sequences $(\tilde{m}_\lambda^{n_j} \otimes \nu)_{j=1}^\infty$ and $((e^{P_{n_j}(\lambda)}\tilde{m}_\lambda^{n_j}) \otimes \nu)_{j=1}^\infty$ converge in the narrow topology of $P_\Lambda(\nu)$ to, say, $\tilde{m}_\lambda \otimes \nu$ and $\tilde{\mu}_\lambda \otimes \nu$, respectively. By Lemma 2.9, there exists a measurable function $\lambda \mapsto \gamma_\lambda \in [Q_f, M_f]$ such that $\tilde{\mu}_\lambda = \gamma_\lambda\tilde{m}_\lambda$ for ν-a.e. $\lambda \in \Lambda$. Set $\mathcal{L}_{j,\lambda} := \mathcal{L}_{f,F_{n_j},\lambda}$ and $\mathcal{L}_\lambda := \mathcal{L}_{f,\lambda}$. Since $f \in H^{\Sigma}_{s,\Lambda}(E_A^\infty)$, the operators $\mathcal{L}_{j,\lambda}$ converge to $\mathcal{L}_\lambda$ as $j \to \infty$, and this uniformly in λ. Pick any $g \in C^b_\Lambda(E_A^\infty)$. Then

$$\begin{aligned}
\int_\Lambda \int_{E_A^\infty} g_\lambda(\omega)\, d(\mathcal{L}^*_\lambda \tilde{m}_{T(\lambda)})(\omega)\, d\nu(\lambda) &= \int_\Lambda \int_{E_A^\infty} \mathcal{L}_\lambda g_\lambda(\omega)\, d\tilde{m}_{T(\lambda)}(\omega)\, d\nu(\lambda) \\
&= \lim_{j\to\infty} \int_\Lambda \int_{E_A^\infty} \mathcal{L}_\lambda g_\lambda(\omega)\, d\tilde{m}^{n_j}_{T(\lambda)}(\omega)\, d\nu(\lambda) \\
&= \lim_{j\to\infty} \int_\Lambda \int_{E_A^\infty} \mathcal{L}_{j,\lambda} g_\lambda(\omega)\, d\tilde{m}^{n_j}_{T(\lambda)}(\omega)\, d\nu(\lambda) \\
&= \lim_{j\to\infty} \int_\Lambda \int_{E_A^\infty} g_\lambda(\omega)\, d(\mathcal{L}^*_{j,\lambda}\tilde{m}^{n_j}_{T(\lambda)})(\omega)\, d\nu(\lambda) \\
&= \lim_{j\to\infty} \int_\Lambda \int_{E_A^\infty} g_\lambda(\omega)\, d(e^{P_{n_j}(\lambda)}\tilde{m}^{n_j}_\lambda)(\omega)\, d\nu(\lambda) \\
&= \int_\Lambda \int_{E_A^\infty} g_\lambda(\omega)\, d\tilde{\mu}_\lambda(\omega)\, d\nu(\lambda) \\
&= \int_\Lambda \int_{E_A^\infty} g_\lambda(\omega)\, d(\gamma_\lambda\tilde{m}_\lambda)(\omega)\, d\nu(\lambda),
\end{aligned}$$

where the third inequality sign follows from the fact that $\mathcal{L}_{j,\lambda}g_\lambda$ converges to $\mathcal{L}_\lambda g_\lambda$ uniformly with respect to $\lambda \in \Lambda$, while the fifth inequality is an application of Theorem 2.5. Therefore, $\mathcal{L}^*_\lambda\tilde{m}_{T(\lambda)} = \gamma_\lambda\tilde{m}_\lambda$ for ν-a.e. $\lambda \in \Lambda$ and we are done with the existence part if we set $P_\lambda(f) := \log\gamma_\lambda$.

We shall now prove that equation (2.9), which holds for ν-a.e. $\lambda \in \Lambda$, determines the measures $\tilde{m}_\lambda$ and the numbers $P_\lambda(f)$ uniquely for ν-a.e. $\lambda \in \Lambda$. To ease notation, we write P_λ instead of $P_\lambda(f)$. Take an element $\omega \in E_A^{2n}$, $n \in I\!N$. If the family $\{\tilde{m}_\lambda\}_{\lambda\in\Lambda}$ satisfies (2.9), then

$$\tilde{m}_\lambda([\omega]) = \tilde{m}_\lambda(1\!\!1_{[\omega]}) = e^{-P_\lambda^n}\tilde{m}_{T^n(\lambda)}(\mathcal{L}_\lambda^n(1\!\!1_{[\omega]})) = e^{-P_\lambda^n}\int_{[\sigma^n\omega]}\Big(\sum_{\tau\in E_A^n} 1\!\!1_{[\omega]}(\tau\rho)e^{S_nf(\tau\rho,\lambda)}\Big)d\tilde{m}_{T^n(\lambda)}(\rho)$$
$$= e^{-P_\lambda^n}\int_{[\sigma^n\omega]} e^{S_nf(\omega\sigma^n\rho,\lambda)}d\tilde{m}_{T^n(\lambda)}(\rho). \tag{2.11}$$

Now, fix $\lambda \in \Lambda$ and suppose that two sequences of Borel probability measures $(\tilde{m}^{(1)}_{T^n(\lambda)})_{n=-\infty}^\infty$ and $(\tilde{m}^{(2)}_{T^n(\lambda)})_{n=-\infty}^\infty$ on E_A^∞ are given along with two corresponding sequences of real numbers $(P_{1,T^n(\lambda)})_{n=-\infty}^\infty$ and $(P_{2,T^n(\lambda)})_{n=-\infty}^\infty$ such that

$$\mathcal{L}^*_{f,T^n(\lambda)}\tilde{m}^{(i)}_{T^{n+1}(\lambda)} = e^{P_{i,T^n(\lambda)}}\tilde{m}^{(i)}_{T^n(\lambda)}$$

holds for all $i = 1, 2$ and all $n \in \boldsymbol{Z}$. By the bounded variation of the ergodic sums S_nf, we have for all $\omega \in E_A^\infty$ that

$$\lim_{n\to\infty}\frac{\int_{[\sigma^n(\omega|_{2n})]} e^{S_nf(\omega|_{2n}\sigma^n\rho,\lambda)}d\tilde{m}^{(2)}_{T^n(\lambda)}(\rho)}{\int_{[\sigma^n(\omega|_{2n})]} e^{S_nf(\omega|_{2n}\sigma^n\rho,\lambda)}d\tilde{m}^{(1)}_{T^n(\lambda)}(\rho)} = 1$$

for ν-a.e. $\lambda \in \Lambda$. Since the sequence $(e^{-P^n_{2,\lambda}}/e^{-P^n_{1,\lambda}})_{n=1}^\infty$ is independent of ω, we conclude from (2.11) that the sequence $(P^n_{2,\lambda} - P^n_{1,\lambda})_{n=1}^\infty$ must converge and its limit must equal 0. This simultaneously shows that the measures $\tilde{m}^{(2)}_\lambda$ and $\tilde{m}^{(1)}_\lambda$ are equivalent and the Radon-Nikodym derivative $d\tilde{m}^{(2)}_\lambda/d\tilde{m}^{(1)}_\lambda$ is identically equal to 1. But this means that $\tilde{m}^{(2)}_\lambda = \tilde{m}^{(1)}_\lambda$ and in particular the uniqueness of the fiber measures $\{\tilde{m}_\lambda\}_{\lambda\in\Lambda}$ is established. Since, by (2.9), we have $P_\lambda = \log(\mathcal{L}^*_\lambda(\tilde{m}_{T(\lambda)})(1\!\!1))$, we deduce that $P_{1,\lambda} = P_{2,\lambda}$ and, in particular, the uniqueness of the pressure parameters P_λ, $\lambda \in \Lambda$, follows from the uniqueness of the fiber measures.

Finally, we shall prove that $P_\lambda(f) = \lim_{n\to\infty} P_n(\lambda)$ for ν-a.e. $\lambda \in \Lambda$. Because of the uniqueness part it suffices to show that if $\lambda \in \Lambda$ is such that for all $n \geq 1$ and all $k \in \boldsymbol{Z}$ there are measures $\tilde{m}^{(n)}_{T^k(\lambda)}$ satisfying

$$\mathcal{L}^*_{f,n,T^k(\lambda)}\tilde{m}^{(n)}_{T^{k+1}(\lambda)} = e^{P_n(T^k(\lambda))}\tilde{m}^{(n)}_{T^k(\lambda)}, \tag{2.12}$$

and if $(n_j)_{j=1}^\infty$ is an arbitrary increasing sequence of positive integers for which the sequences $(P_{n_j}(T^k(\lambda)))_{j=1}^\infty$ converge for all $k \in \boldsymbol{Z}$ (denote their limits by $R(T^k(\lambda))$), then for every $k \in \boldsymbol{Z}$ there exists a Borel probability measure $\tilde{m}_{T^k(\lambda)}$ on E_A^∞ such that

$$\mathcal{L}^*_{f,T^k(\lambda)}\tilde{m}_{T^{k+1}(\lambda)} = e^{R(T^k(\lambda))}\tilde{m}_{T^k(\lambda)} \tag{2.13}$$

holds for all $k \in \boldsymbol{Z}$. But passing to a subsequence of $(n_j)_{j=1}^\infty$ and using the standard diagonal procedure, we may assume without loss of generality that all the sequences $(\tilde{m}^{(n_j)}_{T^k(\lambda)})_{j=1}^\infty$, $k \in \boldsymbol{Z}$, converge weakly to some Borel probability measures on E_A^∞; denote them respectively by

$\tilde{m}_{T^k(\lambda)}$, $k \in \boldsymbol{Z}$. Now, fix $g \in C_b(E_A^\infty)$. Since all involved Perron-Frobenius operators are continuous and since for each $k \in \boldsymbol{Z}$, we have that $\mathcal{L}_{f,n,T^k(\lambda)}g$ converges uniformly to $\mathcal{L}_{f,T^k(\lambda)}g$ as $n \to \infty$, we infer from (2.12) that

$$\begin{aligned}
\mathcal{L}^*_{f,T^k(\lambda)}\tilde{m}_{T^{k+1}(\lambda)}(g) &= \tilde{m}_{T^{k+1}(\lambda)}(\mathcal{L}_{f,T^k(\lambda)}g) \\
&= \lim_{j\to\infty} \tilde{m}^{(n_j)}_{T^{k+1}(\lambda)}(\mathcal{L}_{f,n_j,T^k(\lambda)}g) \\
&= \lim_{j\to\infty} \mathcal{L}^*_{f,n_j,T^k(\lambda)}\tilde{m}^{(n_j)}_{T^{k+1}(\lambda)}(g) \\
&= \lim_{j\to\infty} e^{P_{n_j}(T^k(\lambda))}\tilde{m}^{(n_j)}_{T^k(\lambda)}(g) \\
&= e^{R(T^k(\lambda))}\tilde{m}_{T^k(\lambda)}(g).
\end{aligned}$$

Hence, $\mathcal{L}^*_{f,T^k(\lambda)}\tilde{m}_{T^{k+1}(\lambda)} = e^{R(T^k(\lambda))}\tilde{m}_{T^k(\lambda)}$ and we are done. $\square$

The next result follows from the proof of Theorem 2.10.

Lemma 2.11. *Let E be a countably infinite alphabet and A a finitely irreducible matrix. Let $f \in H^\Sigma_{s,\Lambda}(E_A^\infty)$ be bounded over finite subalphabets. If $(F_n)_{n=1}^\infty$ is an ascending sequence of finite subalphabets of E such that $\cup_{n=1}^\infty F_n = E$, then*

$$P_\lambda(f) = \lim_{n\to\infty} P_\lambda(f|_{(F_n)_A^\infty\times\Lambda})$$

for ν-a.e. $\lambda \in \Lambda$.

We shall now prove the second main result of this section. This result concerns invariant measures. Recall that a matrix A is finitely primitive if there exists $p \in I\!N$ and a finite set $\Omega \subset E_A^p$ such that for all $e, f \in E$ there is a word $\omega \in \Omega$ for which $e\omega f \in E_A^*$.

Theorem 2.12. *Let E be a countably infinite alphabet and A a finitely primitive matrix. Let $f \in H^\Sigma_{s,\Lambda}(E_A^\infty)$ be bounded over finite subalphabets, and let $\tilde{m} \in P_\Lambda(\nu)$ and $\lambda \mapsto P_\lambda(f) \in I\!R$ be the unique random probability measure and bounded measurable function such that*

$$\mathcal{L}^*_{f,\lambda}\tilde{m}_{T(\lambda)} = e^{P_\lambda(f)}\tilde{m}_\lambda$$

for ν-a.e. $\lambda \in \Lambda$. Then there exists a non-negative $q \in C^b_\Lambda(E_A^\infty)$ with the following properties:

- *(a)* $\int_{E_A^\infty} q_\lambda(\omega)\, d\tilde{m}_\lambda(\omega) = 1$ *for ν-a.e. $\lambda \in \Lambda$;*
- *(b)* $0 < C^{-1} \le \inf\{q_\lambda(\omega) : \omega \in E_A^\infty, \lambda \in \Lambda\} \le \sup\{q_\lambda(\omega) : \omega \in E_A^\infty, \lambda \in \Lambda\} \le C < \infty$ *for some constant $C \ge 1$;*
- *(c)* $(q_\lambda\tilde{m}_\lambda)\circ\sigma^{-1} = q_{T(\lambda)}\tilde{m}_{T(\lambda)}$ *for ν-a.e. $\lambda \in \Lambda$;*
- *(d)* $((q_\lambda\tilde{m}_\lambda)\otimes\nu)\circ(\sigma\times T)^{-1} = (q_\lambda\tilde{m}_\lambda)\otimes\nu$, *that is, the measure $(q_\lambda\tilde{m}_\lambda)\otimes\nu$ is $(\sigma\times T)$-invariant.*

Proof. Since the matrix A is finitely primitive, there is an ascending sequence $(F_n)_{n=1}^\infty$ of finite subalphabets such that $\cup_{n=1}^\infty F_n = E$ and such that for each $n \in I\!N$ the matrix $A|_{F_n\times F_n}$ is (finitely) primitive with the same finite set of finite words yielding (finite) primitivity.

Inspecting the proof of Proposition 3.7 in [8] (which consists of Lemma 3.8 followed by a short argument) and using Lemma 3.9, we see that there exists a constant $C \geq 1$ such that for every $n \in I\!N$ there is a non-negative $q^{(n)} \in C^b_\Lambda(E^\infty_A)$ with the following properties:

(a_n) $\int_{(F_n)^\infty_A} q^{(n)}_\lambda \, d\tilde{m}^n_\lambda = 1$ for ν-a.e. $\lambda \in \Lambda$;

(b_n) $C^{-1} \leq \inf\{q^{(n)}_\lambda(\omega) : \omega \in (F_n)^\infty_A, \lambda \in \Lambda\} \leq \sup\{q^{(n)}_\lambda(\omega) : \omega \in (F_n)^\infty_A, \lambda \in \Lambda\} \leq C$;

(c_n) $(q^{(n)}_\lambda \tilde{m}^n_\lambda) \circ \sigma^{-1} = q^{(n)}_{T(\lambda)} \tilde{m}^n_{T(\lambda)}$;

(d_n) $((q^{(n)}_\lambda \tilde{m}^n_\lambda) \otimes \nu) \circ (\sigma \times T)^{-1} = (q^{(n)}_\lambda \tilde{m}^n_\lambda) \otimes \nu$, that is, the measure $(q^{(n)}_\lambda \tilde{m}^n_\lambda) \otimes \nu$ is $(\sigma \times T)$-invariant.

Note that property (c_n) is equivalent to property (d_n) and we will thus only need (d_n) in the forthcoming proof. For every $n \in I\!N$ let $\tilde{\mu}^n_\lambda := q^{(n)}_\lambda \tilde{m}^n_\lambda$. Let also $\tilde{m}^n := \tilde{m}^n_\lambda \otimes \nu$ and $\tilde{\mu}^n := \tilde{\mu}^n_\lambda \otimes \nu$. Note that each $\tilde{\mu}^n \in P_\Lambda(\nu)$ by (a_n). By (b_n) and in virtue of Lemma 2.7, the sequence $(\tilde{\mu}^n)^\infty_{n=1}$ is tight. By passing to a subsequence if necessary, we may thus assume that this sequence converges in the narrow topology of $P_\Lambda(\nu)$ to a random measure, say $\tilde{\mu}$. Fix a non-negative $g \in C^b_\Lambda(E^\infty_A)$. Using (b_n), we obtain

$$\begin{aligned}
\int_{E^\infty_A \times \Lambda} g \, d\tilde{\mu} &= \lim_{n\to\infty} \int_{E^\infty_A \times \Lambda} g \, d\tilde{\mu}^n = \lim_{n\to\infty} \int_\Lambda \int_{E^\infty_A} g_\lambda(\omega) \, d\tilde{\mu}^n_\lambda(\omega) \, d\nu(\lambda) \\
&= \lim_{n\to\infty} \int_\Lambda \int_{E^\infty_A} g_\lambda(\omega) \, q^{(n)}_\lambda(\omega) \, d\tilde{m}^n_\lambda(\omega) \, d\nu(\lambda) \\
&\leq C \lim_{n\to\infty} \int_\Lambda \int_{E^\infty_A} g_\lambda(\omega) \, d\tilde{m}^n_\lambda(\omega) \, d\nu(\lambda) = C \lim_{n\to\infty} \int_{E^\infty_A \times \Lambda} g \, d\tilde{m}^n = C \int_{E^\infty_A \times \Lambda} g \, d\tilde{m}.
\end{aligned}$$

This implies that $\tilde{\mu} \ll \tilde{m}$ and $d\tilde{\mu}/d\tilde{m} \leq C$. Similarly,

$$\int_{E^\infty_A \times \Lambda} g \, d\tilde{\mu} \geq C^{-1} \int_{E^\infty_A \times \Lambda} g \, d\tilde{m}.$$

This yields $d\tilde{\mu}/d\tilde{m} \geq C^{-1}$. Hence, $\tilde{\mu}_\lambda \ll \tilde{m}_\lambda$ for ν-a.e. $\lambda \in \Lambda$ and $d\tilde{\mu}_\lambda/d\tilde{m}_\lambda \in [C^{-1}, C]$. With $q(\omega, \lambda) := q_\lambda(\omega) := d\tilde{\mu}_\lambda/d\tilde{m}_\lambda$, statement (b) is proved. Moreover, statement (a) holds since q_λ is a Radon-Nikodym derivative.

Now, fix an arbitrary $g \in C^b_\Lambda(E^\infty_A)$. Using (d_n), we get

$$\begin{aligned}
\int_{E^\infty_A \times \Lambda} g \, d(\tilde{\mu} \circ (\sigma \times T)^{-1}) &= \int_{E^\infty_A \times \Lambda} g \circ (\sigma \times T) \, d\tilde{\mu} = \lim_{n\to\infty} \int_{E^\infty_A \times \Lambda} g \circ (\sigma \times T) \, d\tilde{\mu}^n \\
&= \lim_{n\to\infty} \int_{E^\infty_A \times \Lambda} g \, d(\tilde{\mu}^n \circ (\sigma \times T)^{-1}) = \lim_{n\to\infty} \int_{E^\infty_A \times \Lambda} g \, d\tilde{\mu}^n \\
&= \int_{E^\infty_A \times \Lambda} g \, d\tilde{\mu}.
\end{aligned}$$

This shows that $\tilde{\mu}\circ(\sigma\times T)^{-1}=\tilde{\mu}$, that is, $\tilde{\mu}$ is $(\sigma\times T)$-invariant. As $\tilde{\mu}=\tilde{\mu}_\lambda\otimes\nu=(q_\lambda\tilde{m}_\lambda)\otimes\nu$, statement (d) is proved. Furthermore, as ν is T-invariant, we have

$$\begin{aligned}\int_\Lambda\int_{E_A^\infty} g_\lambda(\omega)\,d\tilde{\mu}_\lambda(\omega)\,d\nu(\lambda)=\int_{E_A^\infty\times\Lambda} g\,d\tilde{\mu} &= \int_{E_A^\infty\times\Lambda} g\circ(\sigma\times T)\,d\tilde{\mu}\\ &= \int_\Lambda\int_{E_A^\infty} g_{T(\lambda)}\circ\sigma(\omega)\,d\tilde{\mu}_\lambda(\omega)\,d\nu(\lambda)\\ &= \int_\Lambda\int_{E_A^\infty} g_\lambda\circ\sigma(\omega)\,d\tilde{\mu}_{T^{-1}(\lambda)}(\omega)\,d\nu(\lambda)\\ &= \int_\Lambda\int_{E_A^\infty} g_\lambda(\omega)\,d(\tilde{\mu}_{T^{-1}(\lambda)}\circ\sigma^{-1})(\omega)\,d\nu(\lambda).\end{aligned}$$

We deduce from this that $\tilde{\mu}_{T^{-1}(\lambda)}\circ\sigma^{-1}=\tilde{\mu}_\lambda$ for ν-a.e. $\lambda\in\Lambda$. Since $\tilde{\mu}_\lambda=q_\lambda\tilde{m}_\lambda$, statement (c) is proved. □

3. Random Graph Directed Markov Systems

Like deterministic graph directed Markov systems, random graph directed Markov systems are based on a directed multigraph (V,E,i,t) and an edge incidence matrix $A:E\times E\to\{0,1\}$, together with a set of non-empty compact subsets $\{X_v\}_{v\in V}$ of a common Euclidean space $\mathbb{R}^d$. From this point on, we shall assume that A is finitely primitive.

In contradistinction with a deterministic GDMS, a random GDMS (RGDMS) $\Phi=(T:\Lambda\to\Lambda,\{\lambda\mapsto\varphi_e^\lambda\}_{e\in E})$ is generated by an invertible ergodic measure-preserving map $T:(\Lambda,\mathcal{F},\nu)\to(\Lambda,\mathcal{F},\nu)$ of a complete probability space $(\Lambda,\mathcal{F},\nu)$ and one-to-one contractions $\varphi_e^\lambda:X_{t(e)}\to X_{i(e)}$ with Lipschitz constant at most a common number $0<s<1$. Thereafter, the maps $x\mapsto\varphi_e^\lambda(x)$ are continuous for each $\lambda\in\Lambda$. We further assume that the maps $\lambda\mapsto\varphi_e^\lambda(x)$ are measurable for every $x\in X_{t(e)}$. According to Lemma 1.1 in [2], this implies that the map $(x,\lambda)\mapsto\varphi_e(x,\lambda):=\varphi_e^\lambda(x)$ is jointly measurable. As in the deterministic case, a RGDMS is a random iterated function system (RIFS) if V is a singleton and the matrix $A:E\times E\to\{0,1\}$ takes on the value 1 only.

For every $\omega\in E_A^*$, set

$$\lambda\in\Lambda\mapsto\varphi_\omega^\lambda:=\varphi_{\omega_1}^\lambda\circ\varphi_{\omega_2}^{T(\lambda)}\circ\ldots\circ\varphi_{\omega_{|\omega|}}^{T^{|\omega|-1}(\lambda)}.$$

Observe that for each $\omega\in E_A^*$ the map $(x,\lambda)\in X_{t(\omega)}\times\Lambda\mapsto\varphi_\omega(x,\lambda):=\varphi_\omega^\lambda(x)\in X_{i(\omega)}$ is jointly measurable. Indeed, for each $\omega\in E_A^*$ the map $x\mapsto\varphi_\omega(x,\lambda)$ is continuous for each $\lambda\in\Lambda$. Moreover, the map $\lambda\mapsto\varphi_\omega(x,\lambda)$ is measurable for each $x\in X$. For instance, for the word $\omega=\omega_1\omega_2\in E_A^*$ the map $\lambda\mapsto\varphi_\omega(x,\lambda)$ is measurable for each $x\in X$ since $\varphi_\omega(x,\lambda)=\varphi_{\omega_1}(\varphi_{\omega_2}(x,T(\lambda)),\lambda)$ and thus the map $\lambda\mapsto\varphi_\omega(x,\lambda)$ is the composition of the measurable map $\lambda\mapsto T(\lambda)$, followed by the measurable map $\lambda\mapsto\varphi_{\omega_2}(x,\lambda)$, followed by the measurable map $y\mapsto(y,\lambda)$, followed by the measurable map $\lambda\mapsto\varphi_{\omega_1}(x,\lambda)$.

The main object of interest in a RGDMS Φ is its associated 'random limit set' J. However, in contradistinction with the deterministic case, this 'set' is in fact a set function: to each $\lambda\in\Lambda$ is associated the image of the symbolic space E_A^∞ under a coding map π_λ. Indeed, given any $\lambda \in \Lambda$ and any $\omega \in E_A^\infty$, the sets $\varphi^\lambda_{\omega|_n}(X_{t(\omega_n)})$, $n \in I\!N$, form a decreasing sequence of non-empty compact sets whose diameters do not exceed s^n and hence converge to zero. Therefore their intersection

$$\bigcap_{n=1}^{\infty} \varphi^\lambda_{\omega|_n}(X_{t(\omega_n)})$$

is a singleton. Denote its element by $\pi_\lambda(\omega)$. For every $\lambda \in \Lambda$, this defines the coding map $\pi_\lambda : E_A^\infty \to X$, where $X := \bigoplus_{v\in V} X_v$ is the disjoint union of the compact sets X_v. It is easy to see that each π_λ is a Hölder continuous map with respect to the metric $d(\omega,\tau) = s^{|\omega\wedge\tau|}$ on E_A^∞ which induces Tychonov's topology. In particular, this implies that the map $\omega \mapsto \pi(\omega,\lambda) := \pi_\lambda(\omega)$ is continuous for each $\lambda \in \Lambda$. The map $\lambda \mapsto \pi(\omega,\lambda)$ is measurable for each $\omega \in E_A^\infty$ since the map $(x,\lambda) \mapsto \varphi_{\omega|n}(x,\lambda)$ is jointly measurable for every $n \in I\!N$, and thus for any sequence $(x_n \in X_{t(\omega_n)})_{n=1}^\infty$ we deduce that $\lambda \mapsto \pi(\omega,\lambda) = \lim_{n\to\infty} \varphi_{\omega|_n}(x_n,\lambda)$ is measurable for each $\omega \in E_A^\infty$. Thus, by Lemma 1.1 in [2], the map $(\omega,\lambda) \mapsto \pi(\omega,\lambda)$ is jointly measurable.

Now, for every $\lambda \in \Lambda$ set

$$J_\lambda := \pi_\lambda(E_A^\infty).$$

The set J_λ is called the limit set corresponding to the parameter λ while the function

$$\lambda \in \Lambda \mapsto J_\lambda \subset X$$

is called the random limit set of the RGDMS Φ. In this paper, we will mainly be interested in the geometric properties of the limit sets J_λ, primarily in their Hausdorff dimensions. Note that each J_λ is compact when E is finite, but this property usually fails to hold when E is infinite. Furthermore, notice that $\varphi^\lambda_\omega(J_{T^{|\omega|}(\lambda)}) = J_\lambda$ for every $\lambda \in \Lambda$ and every $\omega \in E_A^*$.

A RGDMS Φ is called conformal (and thereafter a RCGDMS) if the following conditions are satisfied.

(*i*) For every $v \in V$, the set X_v is a compact connected subset of $I\!R^d$ which is the closure of its interior (i.e., $X_v = \overline{\mathrm{Int}_{I\!R^d}(X_v)}$);

(*ii*) (Open set condition (OSC)) For ν-a.e. $\lambda \in \Lambda$ and all $e, f \in E$, $e \neq f$,

$$\varphi^\lambda_e(\mathrm{Int}(X_{t(e)})) \cap \varphi^\lambda_f(\mathrm{Int}(X_{t(f)})) = \emptyset;$$

(*iii*) For every vertex $v \in V$, there exists a bounded open connected set W_v such that $X_v \subset W_v \subset I\!R^d$ and such that for every $e \in E$ with $t(e) = v$ and ν-a.e. $\lambda \in \Lambda$, the map φ^λ_e extends to a C^1 conformal diffeomorphism of W_v into $W_{i(e)}$. Moreover, for every $e \in E$ the map $\lambda \in \Lambda \mapsto \varphi^\lambda_e(x)$ is measurable for every $x \in W_v$;

(*iv*) (Cone property) There exist $\gamma, l > 0$ such that for every $v \in V$ and every $x \in X_v$ there is an open cone $\mathrm{Con}(x,\gamma,l) \subset \mathrm{Int}(X_v)$ with vertex x, central angle γ, and altitude l;

(*v*) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\Big||(\varphi^\lambda_e)'(y)| - |(\varphi^\lambda_e)'(x)|\Big| \leq L\|((\varphi^\lambda_e)')^{-1}\|^{-1}_{W_{i(e)}}\|y-x\|^\alpha$$

for ν-a.e. $\lambda \in \Lambda$, every $e \in E$ and every pair of points $x, y \in W_{t(e)}$, where $|\varphi'(x)|$ denotes the norm of the derivative of φ at x and $\|(\varphi')^{-1}\|_W$ is the supremum norm taken over W.

Remark. According to Proposition 4.2.1 in [6], condition (v) is automatically satisfied with $\alpha = 1$ when $d \geq 2$. This condition is also fulfilled if $d = 1$, the alphabet E is finite and all the φ_e^λ's are of class $C^{1+\varepsilon}$ for some $\varepsilon > 0$.

The following useful fact has essentially been proved in Lemma 4.2.2 of [6].

Lemma. *For ν-a.e. $\lambda \in \Lambda$, all $\omega \in E_A^*$ and all $x, y \in W_{t(\omega)}$, we have*

$$\Big|\log|(\varphi_\omega^\lambda)'(y)| - \log|(\varphi_\omega^\lambda)'(x)|\Big| \leq L(1-s)^{-1}\|y-x\|^\alpha.$$

An immediate consequence of this lemma is the famous bounded distortion property.

(v') (Bounded distortion property (BDP)) There exists a constant $K \geq 1$ such that

$$|(\varphi_\omega^\lambda)'(y)| \leq K|(\varphi_\omega^\lambda)'(x)|$$

for ν-a.e. $\lambda \in \Lambda$, every $\omega \in E_A^*$ and every $x, y \in W_{t(\omega)}$.

Let us now collect some geometric consequences of (BDP). For ν-a.e. $\lambda \in \Lambda$, all words $\omega \in E_A^*$ and all convex subsets C of $W_{t(\omega)}$, we have

$$\operatorname{diam}(\varphi_\omega^\lambda(C)) \leq K\|(\varphi_\omega^\lambda)'\|\operatorname{diam}(C) \tag{3.1}$$

and

$$\operatorname{diam}(\varphi_\omega^\lambda(W_{t(\omega)})) \leq KD\|(\varphi_\omega^\lambda)'\| \tag{3.2}$$

for some constant $D \geq 1$ which depends only on the X_v and W_v. Moreover,

$$\operatorname{diam}(\varphi_\omega^\lambda(X_{t(\omega)})) \geq (KD)^{-1}\|(\varphi_\omega^\lambda)'\|, \tag{3.3}$$

$$\varphi_\omega^\lambda(B(x,r)) \subset B(\varphi_\omega^\lambda(x), K\|(\varphi_\omega^\lambda)'\|r), \tag{3.4}$$

and

$$\varphi_\omega^\lambda(B(x,r)) \supset B(\varphi_\omega^\lambda(x), K^{-1}\|(\varphi_\omega^\lambda)'\|r) \tag{3.5}$$

for ν-a.e. $\lambda \in \Lambda$, every $x \in X_{t(\omega)}$, every $0 < r \leq \operatorname{dist}(X_{t(\omega)}, \partial V_{t(\omega)})$, and every word $\omega \in E_A^*$.

Finally, we define special classes of systems.

Definition 3.1. *We say that a RCGDMS Φ satisfies the Strong Open Set Condition (SOSC) if*

$$\nu\Big(\{\lambda \in \Lambda : J_\lambda \cap \operatorname{Int}(X) \neq \emptyset\}\Big) > 0.$$

Definition 3.2. *We say that a RCGDMS Φ satisfies the Strong Separation Condition if*

$$\operatorname{dist}(I\!\!R^d \backslash X, \cup_{\lambda\in\Lambda} \cup_{e\in E} \varphi_e^\lambda(X)) > 0.$$

3.1. **Pseudo-codes.** We now derive a property of pseudo-codes. Pseudo-codes have been introduced in [9]. We extend their definition to our setting to take into account the dependence on $\lambda \in \Lambda$.

Definition 3.3. *A finite word $\omega\tau \in E^*$ is called a pseudo-code of an element $(x, \lambda) \in X \times \Lambda$ if the following three conditions are satisfied.*

(i) $\omega, \tau \in E_A^*$;
(ii) $\varphi_\tau^{T^{|\omega|}(\lambda)}(X_{t(\tau)}) \subset X_{t(\omega)}$; *and*
(iii) $x \in \varphi_\omega^\lambda(\varphi_\tau^{T^{|\omega|}(\lambda)}(X_{t(\tau)}))$.

Note that the word $\omega\tau$ need not belong to E_A^*. Whenever we do not need to specify the element (x, λ), we simply say that $\omega\tau$ is a pseudo-code. As for finite admissible words, two pseudo-codes are called comparable if one of them is an extension of the other. Also, two pseudo-codes $\omega\tau$ and $\omega\rho$ are said to form an essential pair of pseudo-codes if $\tau \neq \rho$ and $|\tau| = |\rho|$. Finally, the essential length of an essential pair of pseudo-codes $\omega\tau$ and $\omega\rho$ is defined to be $|\omega|$.

Lemma 3.4. *No element of $X \times \Lambda$ admits essential pairs of pseudo-codes of arbitrary long essential lengths.*

Proof. On the contrary, suppose that there exists a point $(x, \lambda) \in X \times \Lambda$ so that for each $k \in I\!N$, there are words $\omega^{(k)}, \tau^{(k)}, \rho^{(k)} \in E_A^*$ such that $\tau^{(k)} \neq \rho^{(k)}$, $|\tau^{(k)}| = |\rho^{(k)}|$,

$$\lim_{k\to\infty} |\omega^{(k)}| = \infty, \tag{3.6}$$

$$\varphi_{\tau^{(k)}}^{T^{|\omega^{(k)}|}(\lambda)}(X_{t(\tau^{(k)})}) \subset X_{t(\omega^{(k)})} \quad \text{and} \quad \varphi_{\rho^{(k)}}^{T^{|\omega^{(k)}|}(\lambda)}(X_{t(\rho^{(k)})}) \subset X_{t(\omega^{(k)})},$$

and

$$x \in \left[\varphi_{\omega^{(k)}}^\lambda \circ \varphi_{\tau^{(k)}}^{T^{|\omega^{(k)}|}(\lambda)}(X_{t(\tau^{(k)})})\right] \bigcap \left[\varphi_{\omega^{(k)}}^\lambda \circ \varphi_{\rho^{(k)}}^{T^{|\omega^{(k)}|}(\lambda)}(X_{t(\rho^{(k)})})\right].$$

We shall construct inductively for each $n \in I\!N$ a finite set C_n which contains at least $n+1$ mutually incomparable pseudo-codes of (x, λ). The existence of such a set for large n's will contradict Corollary 4.6 in [9], and this will finish the proof. Define $C_1 := \{\omega^{(1)}\tau^{(1)}, \omega^{(1)}\rho^{(1)}\}$, and suppose that the finite set C_n has been constructed with at least $n+1$ mutually incomparable pseudo-codes of (x, λ). In view of (3.6), there exists $k_n \in I\!N$ such that

$$|\omega^{(k_n)}| > \max\{|\xi| : \xi \in C_n\}. \tag{3.7}$$

If $\omega^{(k_n)}\rho^{(k_n)}$ does not extend any word from C_n, it follows from (3.7) that $\omega^{(k_n)}\rho^{(k_n)}$ is not comparable with any element of C_n. The set C_{n+1} can then be constructed by simply adding the word $\omega^{(k_n)}\rho^{(k_n)}$ to C_n. Similarly, if $\omega^{(k_n)}\tau^{(k_n)}$ does not extend any word from C_n, form C_{n+1} by adding $\omega^{(k_n)}\tau^{(k_n)}$ to C_n. However, if $\omega^{(k_n)}\rho^{(k_n)}$ extends an element $\alpha \in C_n$ and $\omega^{(k_n)}\tau^{(k_n)}$ extends an element $\beta \in C_n$, then $\alpha = \omega^{(k_n)}|_{|\alpha|}$ and $\beta = \omega^{(k_n)}|_{|\beta|}$. Since C_n consists of mutually incomparable words, this implies that $\alpha = \beta$. In this case, form C_{n+1} by removing $\alpha(= \beta)$ from C_n while adding both $\omega^{(k_n)}\rho^{(k_n)}$ and $\omega^{(k_n)}\tau^{(k_n)}$. Note that no element $\gamma \in C_n \setminus \{\alpha\}$

is comparable with $\omega^{(k_n)}\rho^{(k_n)}$ or $\omega^{(k_n)}\tau^{(k_n)}$; otherwise, $\gamma = \omega^{(k_n)}|_{|\gamma|}$ and thus γ would be comparable with α. Since $\omega^{(k_n)}\rho^{(k_n)}$ and $\omega^{(k_n)}\tau^{(k_n)}$ are not comparable, the set C_{n+1} consists also in this case of at least $n+2$ mutually incomparable pseudo-codes of (x,λ). This completes our inductive construction, and hence finishes the proof. □

3.2. **Gibbs states for the potentials $t\zeta$.** Define the potential $\zeta : E_A^\infty \times \Lambda \to I\!\!R$ as follows:

$$\zeta(\omega,\lambda) = \log\Big|(\varphi_{\omega_1}^\lambda)'(\pi_{T(\lambda)}(\sigma\omega))\Big|.$$

The map $\omega \mapsto \zeta(\omega,\lambda)$ is continuous for each $\lambda \in \Lambda$, while the map $\lambda \mapsto \zeta(\omega,\lambda)$ is measurable for each $\omega \in E_A^\infty$. Thus, the map ζ is jointly measurable.

Definition 3.5. *For a given RCGDMS Φ, we say that $t \in \mathcal{F}in$ if*

$$M_t := \sum_{e\in E} \text{ess sup}\{\|(\varphi_e^\lambda)'\|^t : \lambda \in \Lambda\} < \infty.$$

Note that the potential $t\zeta$ is summable if and only if $t \in \mathcal{F}in$. In fact, $t\zeta \in H_{s,\Lambda}^\Sigma(E_A^\infty)$ and $t\zeta$ is bounded over finite subalphabets for every $t \in \mathcal{F}in$. Therefore, the thermodynamic formalism for random dynamical systems (see [1] and [8] if E is finite; see Theorem 2.12 with $f = t\zeta$ when E is infinite) gives the following: If $t \in \mathcal{F}in$, then for ν-a.e. $\lambda \in \Lambda$ there are a unique bounded measurable function $\lambda \mapsto P_\lambda(t) := P_\lambda(t\zeta)$ and a unique random probability measure $\tilde{m}^t \in P_\Lambda(\nu)$ such that

$$\mathcal{L}_{t,\lambda}^* \tilde{m}_{T(\lambda)}^t = e^{P_\lambda(t)} \tilde{m}_\lambda^t \tag{3.8}$$

for ν-a.e. $\lambda \in \Lambda$, i.e. $\lambda \mapsto P_\lambda(t)$ and $\tilde{m}^t$ are uniquely determined by the condition that for all $e \in E$ and $\omega \in E_A^*$ such that $e\omega \in E_A^*$ we have

$$\tilde{m}_\lambda^t([e\omega]) = e^{-P_\lambda(t)} \int_{[\omega]} \Big|(\varphi_e^\lambda)'(\pi_{T(\lambda)}(\tau))\Big|^t \, d\tilde{m}_{T(\lambda)}^t(\tau). \tag{3.9}$$

for ν-a.e. $\lambda \in \Lambda$. Furthermore, there exists a unique non-negative $q^t \in C_\Lambda^b(E_A^\infty)$ with the following properties:

(a) $\int_{E_A^\infty} q_\lambda^t(\omega)\, d\tilde{m}_\lambda^t(\omega) = 1$ for ν-a.e. $\lambda \in \Lambda$;
(b) $0 < C(t)^{-1} \le \inf\{q_\lambda^t(\omega) : \omega \in E_A^\infty, \lambda \in \Lambda\} \le \sup\{q_\lambda^t(\omega) : \omega \in E_A^\infty, \lambda \in \Lambda\} \le C(t) < \infty$ for some constant $C(t) \ge 1$;
(c) $(q_\lambda^t \tilde{m}_\lambda^t) \circ \sigma^{-1} = q_{T(\lambda)}^t \tilde{m}_{T(\lambda)}^t$ for ν-a.e. $\lambda \in \Lambda$;
(d) $((q_\lambda^t \tilde{m}_\lambda^t) \otimes \nu) \circ (\sigma \times T)^{-1} = (q_\lambda^t \tilde{m}_\lambda^t) \otimes \nu$, that is, the measure $(q_\lambda^t \tilde{m}_\lambda^t) \otimes \nu$ is $(\sigma \times T)$-invariant.

Letting $\tilde{\mu}_\lambda^t = q_\lambda^t \tilde{m}_\lambda^t$, we can rewrite (c) and (d) in the more compact form

$$\tilde{\mu}_\lambda^t \circ \sigma^{-1} = \tilde{\mu}_{T(\lambda)}^t, \qquad \nu\text{-a.e. } \lambda \in \Lambda \tag{3.10}$$

and

$$(\tilde{\mu}_\lambda^t \otimes \nu) \circ (\sigma \times T)^{-1} = \tilde{\mu}_\lambda^t \otimes \nu. \tag{3.11}$$

Let $\tilde{\mu}^t := \tilde{\mu}_\lambda^t \otimes \nu$ be the integration of the measures $\{\tilde{\mu}_\lambda^t\}_{\lambda\in\Lambda}$ with respect to the measure ν. Property (3.11) then says the following.

Proposition 3.6. *$\tilde{\mu}^t \circ (\sigma \times T)^{-1} = \tilde{\mu}^t$, i.e. the random probability measure $\tilde{\mu}^t$ is $(\sigma \times T)$-invariant. Moreover, $\tilde{\mu}^t \circ p_\Lambda^{-1} = \nu$, where $p_\Lambda : E_A^\infty \times \Lambda \to \Lambda$ is the canonical projection onto Λ. That is, $\tilde{\mu}^t \in P_\Lambda(\nu)$.*

Set also $\mu_\lambda^t := \tilde{\mu}_\lambda^t \circ \pi_\lambda^{-1}$ for all $\lambda \in \Lambda$ and $\mu^t := \mu_\lambda^t \otimes \nu$.

Finally, note that by a straightforward induction, relation (3.9) gives the following: for all $\omega, \tau \in E_A^*$ such that $\omega\tau \in E_A^*$, we have

$$\tilde{m}_\lambda^t([\omega\tau]) = e^{-P_\lambda^{|\omega|}(t)} \int_{[\tau]} \left|(\varphi_\omega^\lambda)'(\pi_{T^{|\omega|}(\lambda)}(\eta))\right|^t d\tilde{m}_{T^{|\omega|}(\lambda)}^t(\eta) \tag{3.12}$$

for ν-a.e. $\lambda \in \Lambda$, where $P_\lambda^n(t) = \sum_{j=0}^{n-1} P_{T^j(\lambda)}(t)$.

The next result asserts that the push-down of the measures $\{\tilde{m}_\lambda^t\}_{\lambda\in\Lambda}$ from E_A^∞ to X, i.e. the measures $\{m_\lambda^t := \tilde{m}_\lambda^t \circ \pi_\lambda^{-1}\}_{\lambda\in\Lambda}$, are t-conformal measures.

Theorem 3.7. *Let $t \in \mathcal{F}in$. Set $m_\lambda^t := \tilde{m}_\lambda^t \circ \pi_\lambda^{-1}$ for all $\lambda \in \Lambda$. Then for ν-a.e. $\lambda \in \Lambda$, every $\omega \in E_A^*$ and every Borel set $B \subset X_{t(\omega)}$ we have*

$$m_\lambda^t(\varphi_\omega^\lambda(B)) = e^{-P_\lambda^{|\omega|}(t)} \int_B |(\varphi_\omega^\lambda)'(x)|^t \, dm_{T^{|\omega|}(\lambda)}^t(x). \tag{3.13}$$

Moreover, for ν-a.e. $\lambda \in \Lambda$ we have

$$m_\lambda^t\Big(\varphi_\rho^\lambda(X_{t(\rho)}) \cap \varphi_\tau^\lambda(X_{t(\tau)})\Big) = 0 \tag{3.14}$$

whenever $\rho, \tau \in E_A^*$ are incomparable. Furthermore, $m_\lambda^t(J_\lambda) = 1$ for ν-a.e. $\lambda \in \Lambda$.

Proof. First, note that $m_\lambda^t(J_\lambda) = \tilde{m}_\lambda^t \circ \pi_\lambda^{-1}(J_\lambda) = \tilde{m}_\lambda^t(E_A^\infty) = 1$ for ν-a.e. $\lambda \in \Lambda$.

In order to show that $\{m_\lambda^t\}_{\lambda\in\Lambda}$ satisfies (3.14), assume for a contradiction that (3.14) fails, i.e. that there are two incomparable words $\rho, \tau \in E_A^\infty$ such that

$$m^t(Z) := (m_\lambda^t \otimes \nu)(Z) > 0, \tag{3.15}$$

where

$$Z = \bigcup_{\lambda\in\Lambda} V_\lambda \times \{\lambda\} \qquad \text{and} \qquad V_\lambda = \varphi_\rho^\lambda(X_{t(\rho)}) \cap \varphi_\tau^\lambda(X_{t(\tau)}).$$

Without loss of generality, we may assume that $|\rho| = |\tau|$. For every $n \geq 0$, set

$$\begin{aligned} Z_n &:= \bigcup_{\lambda\in\Lambda}\Big(\Big(\bigcup_{\omega\in E_A^n} \varphi_\omega^{T^{-n}(\lambda)}(V_\lambda)\Big) \times \{T^{-n}(\lambda)\}\Big) = \bigcup_{\lambda\in\Lambda}\bigcup_{\omega\in E_A^n} \varphi_\omega^{T^{-n}(\lambda)}(V_\lambda) \times \{T^{-n}(\lambda)\} \\ &= \bigcup_{\lambda\in\Lambda}\bigcup_{\omega\in E_A^n} \varphi_\omega^{T^{-n}(\lambda)}\Big(\varphi_\rho^\lambda(X_{t(\rho)}) \cap \varphi_\tau^\lambda(X_{t(\tau)})\Big) \times \{T^{-n}(\lambda)\}. \end{aligned}$$

Since each element of Z_n admits at least one essential pair of pseudo-codes of essential length n, we conclude from Lemma 3.4 that

$$\bigcap_{j=0}^\infty \bigcup_{n=j}^\infty Z_n = \emptyset. \tag{3.16}$$

On the other hand, we have

$$Z_n \supset \pi_*\Big((\sigma \times T)^{-n}(\pi_*^{-1}(Z))\Big)$$

for each $n \geq 0$, where $\pi_* : E_A^\infty \times \Lambda \to X \times \Lambda$ is given by the formula $(\omega, \lambda) \mapsto (\pi_\lambda(\omega), \lambda) = (\pi(\omega, \lambda), \lambda)$. This implies

$$\pi_*^{-1}(Z_n) \supset (\sigma \times T)^{-n}(\pi_*^{-1}(Z)). \tag{3.17}$$

Since $\tilde{\mu}^t = \tilde{\mu}_\lambda^t \otimes \nu$ is $(\sigma \times T)$-invariant, we get from (3.17) that

$$\begin{aligned}\tilde{\mu}^t(\pi_*^{-1}(Z_n)) \geq \tilde{\mu}^t\Big((\sigma \times T)^{-n}(\pi_*^{-1}(Z))\Big) = \tilde{\mu}^t(\pi_*^{-1}(Z)) = (\tilde{\mu}_\lambda^t \otimes \nu)(\pi_*^{-1}(Z)) &= (\mu_\lambda^t \otimes \nu)(Z) \\ &= \mu^t(Z)\end{aligned}$$

for every $n \geq 0$. As $\tilde{\mu}^t$ is equivalent to $\tilde{m}^t$ according to Theorem 2.12, we deduce by means of (3.15) that

$$\tilde{\mu}^t\Big(\pi_*^{-1}\Big(\bigcap_{j=0}^\infty \bigcup_{n=j}^\infty Z_n\Big)\Big) \geq \mu^t(Z) \asymp m^t(Z) > 0.$$

Hence $\cap_{j=0}^\infty \cup_{n=j}^\infty Z_n \neq \emptyset$. This contradicts (3.16). Thus, there exists a measurable set $\Lambda_* \subset \Lambda$ such that $\nu(\Lambda_*) = 1$ and

$$m_\lambda^t\Big(\varphi_\rho^\lambda(X_{t(\rho)}) \cap \varphi_\tau^\lambda(X_{t(\tau)})\Big) = 0$$

for all $\lambda \in \Lambda_*$ and all incomparable words $\rho, \tau \in E_A^*$.

In order to prove (3.13), fix $\omega \in E_A^*$, say $\omega \in E_A^n$, and for any set $F \subset E_A^\infty$ let

$$[\omega] := \{\omega\tau \in E_A^\infty : \tau \in F\}.$$

Fix an arbitrary Borel set $B \subset X_{t(\omega)}$. In view of the just proven property (3.14), we have

$$\begin{aligned}\tilde{m}_\lambda^t\Big(\Big\{\tau \in E_A^\infty : \tau|_n \neq \omega,\, \pi_\lambda(\tau) \in \varphi_\omega^\lambda(B)\Big\}\Big) &= \tilde{m}_\lambda^t\Big(\bigcup_{\tau \in E_A^n \setminus \{\omega\}} [\tau] \cap \pi_\lambda^{-1}(\varphi_\omega^\lambda(B))\Big) \\ &\leq \tilde{m}_\lambda^t\Big(\bigcup_{\tau \in E_A^n \setminus \{\omega\}} \pi_\lambda^{-1}\Big(\varphi_\tau^\lambda(X_{t(\tau)}) \cap \varphi_\omega^\lambda(B)\Big)\Big) \\ &= \tilde{m}_\lambda^t \circ \pi_\lambda^{-1}\Big(\bigcup_{\tau \in E_A^n \setminus \{\omega\}} \varphi_\tau^\lambda(X_{t(\tau)}) \cap \varphi_\omega^\lambda(B)\Big) \\ &= m_\lambda^t\Big(\bigcup_{\tau \in E_A^n \setminus \{\omega\}} \varphi_\tau^\lambda(X_{t(\tau)}) \cap \varphi_\omega^\lambda(B)\Big) \\ &\leq \sum_{\tau \in E_A^n \setminus \{\omega\}} m_\lambda^t\Big(\varphi_\tau^\lambda(X_{t(\tau)}) \cap \varphi_\omega^\lambda(B)\Big) \\ &\leq \sum_{\tau \in E_A^n \setminus \{\omega\}} m_\lambda^t\Big(\varphi_\tau^\lambda(X_{t(\tau)}) \cap \varphi_\omega^\lambda(X_{t(\omega)})\Big) \\ &= 0\end{aligned}$$

for ν-a.e. $\lambda \in \Lambda$. Using this and a generalization of (3.12), we conclude that

$$\begin{aligned}
m_\lambda^t(\varphi_\omega^\lambda(B)) &= \tilde{m}_\lambda^t \circ \pi_\lambda^{-1}(\varphi_\omega^\lambda(B)) = \tilde{m}_\lambda^t\Big(\Big\{\tau \in E_A^\infty : \pi_\lambda(\tau) \in \varphi_\omega^\lambda(B)\Big\}\Big) \\
&= \tilde{m}_\lambda^t\Big(\Big\{\tau \in E_A^\infty : \tau|_n \neq \omega,\, \pi_\lambda(\tau) \in \varphi_\omega^\lambda(B)\Big\}\dot\cup\Big\{\omega\rho \in E_A^\infty : \pi_\lambda(\omega\rho) \in \varphi_\omega^\lambda(B)\Big\}\Big) \\
&= \tilde{m}_\lambda^t\Big(\Big\{\tau \in E_A^\infty : \tau|_n \neq \omega,\, \pi_\lambda(\tau) \in \varphi_\omega^\lambda(B)\Big\}\Big) \\
&\qquad\qquad\qquad + \tilde{m}_\lambda^t\Big(\Big\{\omega\rho \in E_A^\infty : \pi_\lambda(\omega\rho) \in \varphi_\omega^\lambda(B)\Big\}\Big) \\
&= 0 + \tilde{m}_\lambda^t\Big(\Big\{\omega\rho \in E_A^\infty : \varphi_\omega^\lambda(\pi_{T^n(\lambda)}(\rho)) \in \varphi_\omega^\lambda(B)\Big\}\Big) \\
&= \tilde{m}_\lambda^t\Big(\Big\{\omega\rho \in E_A^\infty : \pi_{T^n(\lambda)}(\rho) \in B\Big\}\Big) \\
&= \tilde{m}_\lambda^t\Big(\Big\{\omega\rho \in E_A^\infty : \rho \in \pi_{T^n(\lambda)}^{-1}(B)\Big\}\Big) \\
&= \tilde{m}_\lambda^t\Big([\omega\pi_{T^n(\lambda)}^{-1}(B)]\Big) \\
&= e^{-P_\lambda^n(t)} \int_{\pi_{T^n(\lambda)}^{-1}(B)} \Big|(\varphi_\omega^\lambda)'(\pi_{T^n(\lambda)}(\rho))\Big|^t \, d\tilde{m}_{T^n(\lambda)}^t(\rho) \\
&= e^{-P_\lambda^n(t)} \int_B |(\varphi_\omega^\lambda)'(x)|^t \, d(\tilde{m}_{T^n(\lambda)}^t \circ \pi_{T^n(\lambda)}^{-1})(x) \\
&= e^{-P_\lambda^n(t)} \int_B |(\varphi_\omega^\lambda)'(x)|^t \, dm_{T^n(\lambda)}^t(x)
\end{aligned}$$

for ν-a.e. $\lambda \in \Lambda$. We are done. $\square$

Before presenting our next result, we will address the measurability of the sets Z_n and Z that were defined in the proof of the previous theorem. This is a question we deliberately avoided in order to not digress from the crux of the proof. To establish the measurability of the sets Z and Z_n, we make a brief incursion in the theory of random sets. For the basic notions in this theory, see [2].

In the following, the set of all subsets of a set X shall be denoted by 2^X.

Definition 3.8. *Let X be a Polish space and let $C_\alpha : \Lambda \to 2^X$, $\alpha \in A$, be closed random sets, where A is any index set. We define the set-valued map $\cap_{\alpha \in A} C_\alpha : \Lambda \to 2^X$ by the formula*

$$\Big(\bigcap_{\alpha \in A} C_\alpha\Big)(\lambda) := \bigcap_{\alpha \in A} C_\alpha(\lambda).$$

We call it the intersection of the closed random sets C_α, $\alpha \in A$.

Similarly, we define the set-valued map $\cup_{\alpha \in A} C_\alpha : \Lambda \to 2^X$ by the formula

$$\Big(\bigcup_{\alpha \in A} C_\alpha\Big)(\lambda) := \bigcup_{\alpha \in A} C_\alpha(\lambda).$$

We call it the union of the closed random sets C_α, $\alpha \in A$.

We shall prove the following simple but useful lemma. Note that this is the only place where we use the standing assumption that the σ-algebra $\mathcal{F}$ is complete with respect to the measure ν.

Lemma 3.9. *A countable intersection of closed random sets is a closed random set. A finite union of closed random sets is a closed random set.*

Proof. Let $C_\alpha : \Lambda \to 2^X$, $\alpha \in A$, be a countable family of closed random sets. Then for every $\lambda \in \Lambda$, the sets $C_\alpha(\lambda)$ are all closed and therefore so is the set $(\cap_{\alpha\in A} C_\alpha)(\lambda) = \cap_{\alpha\in A} C_\alpha(\lambda)$. Moreover, it follows from Proposition 2.4 in [2] that the graphs $\mathrm{graph}(C_\alpha) = \cup_{\lambda\in\Lambda} C_\alpha(\lambda)\times\{\lambda\}$, $\alpha \in A$, are all measurable in $X \times \Lambda$. Hence

$$\mathrm{graph}(\cap_{\alpha\in A} C_\alpha) = \cup_{\lambda\in\Lambda}(\cap_{\alpha\in A} C_\alpha)(\lambda)\times\{\lambda\} = \cup_{\lambda\in\Lambda}(\cap_{\alpha\in A} C_\alpha(\lambda))\times\{\lambda\} = \cap_{\alpha\in A}\cup_{\lambda\in\Lambda} C_\alpha(\lambda)\times\{\lambda\}$$

is a measurable set as it is a countable intersection of measurable sets. It then follows from Proposition 2.4 in [2] that $\cap_{\alpha\in A} C_\alpha$ is a closed random set.

Similarly, if $C_\alpha : \Lambda \to 2^X$, $\alpha \in A$, is a finite family of closed random sets, then for every $\lambda \in \Lambda$, the sets $C_\alpha(\lambda)$ are all closed and therefore so is the set $(\cup_{\alpha\in A} C_\alpha)(\lambda) = \cup_{\alpha\in A} C_\alpha(\lambda)$. Moreover, it follows from Proposition 2.4 in [2] that the graphs $\mathrm{graph}(C_\alpha)$, $\alpha \in A$, are all measurable in $X \times \Lambda$. Hence

$$\mathrm{graph}(\cup_{\alpha\in A} C_\alpha) = \cup_{\lambda\in\Lambda}(\cup_{\alpha\in A} C_\alpha)(\lambda)\times\{\lambda\} = \cup_{\lambda\in\Lambda}(\cup_{\alpha\in A} C_\alpha(\lambda))\times\{\lambda\} = \cup_{\alpha\in A}\cup_{\lambda\in\Lambda} C_\alpha(\lambda)\times\{\lambda\}$$

is a measurable set as a finite union of measurable sets. It then follows from Proposition 2.4 in [2] that $\cup_{\alpha\in A} C_\alpha$ is a closed random set. □

We deduce from this the following result about each level set of a RCGDMS.

Lemma 3.10. *For every $\rho \in E_A^*$ and any $k \in \mathbb{Z}$, the map $\lambda \in \Lambda \mapsto \varphi_\rho^{T^k(\lambda)}(X_{t(\rho)}) \in 2^X$ is a closed random set.*

Proof. Let $\rho \in E_A^*$ and $k \in \mathbb{N}$. Obviously, all the sets $\varphi_\rho^{T^k(\lambda)}(X_{t(\rho)})$, $\lambda \in \Lambda$, are closed. Let $\{x_n\}_{n=1}^\infty$ be a countable dense subset of $X_{t(\rho)}$. Since each map $\lambda \in \Lambda \mapsto \varphi_\rho^{T^k(\lambda)}(x_n)$, $n \in \mathbb{N}$, is measurable and since $\varphi_\rho^{T^k(\lambda)}(X_{t(\rho)}) = \overline{\{\varphi_\rho^{T^k(\lambda)}(x_n) : n \in \mathbb{N}\}}$, we conclude from Theorem 2.6 in [2] that the map $\lambda \in \Lambda \mapsto \varphi_\rho^{T^k(\lambda)}(X_{t(\rho)})$ is a closed random set. □

The measurability of the sets Z and Z_n in the proof of Theorem 3.7 follows directly from Lemma 3.10, Lemma 3.9, and Proposition 2.4 in [2].

We can now turn our attention to the pressure function.

Definition 3.11. *Let $\Omega \subset E_A^p$ be a finite set of finite words that witnesses the finite primitivity of the matrix A. Let F be the set of all letters appearing in words in Ω. For every $t \in \mathcal{F}in$, let $(\mathcal{L}_{t,F,\lambda})_{\lambda\in\Lambda}$ be the Perron-Frobenius operators associated to the function $t\zeta_F : F_A^\infty \times \Lambda \to \mathbb{R}$. Thereafter, let*

$$Q_t := \text{ess inf}\{\mathcal{L}_{t,F,\lambda}(1\!\!1_{F_A^\infty})(\omega) : \lambda \in \Lambda,\, \omega \in F_A^\infty\}.$$

Note that $Q_t > 0$ for every $t \in \mathcal{F}in$ since $t\zeta_F \in C_\Lambda^b(F_A^\infty)$ and $A|_{F\times F}$ is irreducible.

Proposition 3.12. *For every ascending sequence $(F_n)_{n=1}^\infty$ of finite subalphabets of E such that $\cup_{n=1}^\infty F_n = E$ and for all $t \in \mathcal{F}in$, we have*

(a) $P_{n,\lambda}(t) \in [\log Q_t, \log M_t]$ *for* ν*-a.e.* $\lambda \in \Lambda$ *and all* $n \in I\!N$*;*
(b) $P_\lambda(t) = \lim_{n\to\infty} P_{n,\lambda}(t) \in [\log Q_t, \log M_t]$ *for* ν*-a.e.* $\lambda \in \Lambda$*;*
(c) *the function* $\lambda \in \Lambda \mapsto P_\lambda(t)$ *is* ν*-integrable and*

$$\mathcal{E}P(t) = \lim_{n\to\infty} \mathcal{E}P_n(t) \in [\log Q_t, \log M_t],$$

where

$$\mathcal{E}P(t) = \int_\Lambda P_\lambda(t)\, d\nu(\lambda) \qquad \text{and} \qquad \mathcal{E}P_n(t) = \int_\Lambda P_{n,\lambda}(t)\, d\nu(\lambda).$$

The number $\mathcal{E}P(t)$ *is called the expected pressure of the system at the parameter* t*.*

Proof. Let $(F_n)_{n=1}^\infty$ be an ascending sequence of finite subsets of E such that $\cup_{n=1}^\infty F_n = E$. Fix $t \in \mathcal{F}in$. According to Lemma 2.11, we know that $P_\lambda(t) = \lim_{n\to\infty} P_{n,\lambda}(t)$ for ν-a.e. $\lambda \in \Lambda$. Without loss of generality, we may assume that $F \subset F_1$, where F arises from Definition 3.11. Thus, $A|_{F_n\times F_n}$ is irreducible for all $n \in I\!N$. For every $n \in I\!N$, let $(\mathcal{L}_{t,n,\lambda})_{\lambda\in\Lambda}$ be the Perron-Frobenius operators associated to the function $t\zeta_n : (F_n)_A^\infty \times \Lambda \to I\!R$. In virtue of Theorem 2.5, we have for ν-a.e. $\lambda \in \Lambda$ and all $n \in I\!N$ that

$$e^{P_{n,\lambda}(t)} = \tilde{m}^{t,n}_{T(\lambda)}(\mathcal{L}_{t,n,\lambda}(1\!\!1_{(F_n)_A^\infty})) \geq \tilde{m}^{t,n}_{T(\lambda)}(\mathcal{L}_{t,F,\lambda}(1\!\!1_{F_A^\infty})) \geq Q_t.$$

On the other hand, we obtain from the last part of Theorem 2.10 that for ν-a.e. $\lambda \in \Lambda$ and all $n \in I\!N$

$$e^{P_{n,\lambda}(t)} \leq \sum_{e\in E} \exp\Big(\text{ess sup}\{\sup(t\zeta_\lambda|_{[e]}) : \lambda \in \Lambda\}\Big) \leq \sum_{e\in E} \text{ess sup}\Big\{\|(\varphi_e^\lambda)'\|^t : \lambda \in \Lambda\Big\} \;=\; M_t.$$

Hence $P_{n,\lambda}(t) \in [\log Q_t, \log M_t]$ for ν-a.e. $\lambda \in \Lambda$ and all $n \in I\!N$. This establishes statement (a). Statement (b) then follows from Lemma 2.11. Moreover, statement (c) follows from the above and Lebesgue's Dominated Convergence Theorem. □

We shall now establish some basic properties of the expected pressure. Let

$$\theta = \inf(\mathcal{F}in).$$

The number $\theta \geq 0$ is called finiteness parameter of the system Φ.

Proposition 3.13. *The function* $\mathcal{E}P : \mathcal{F}in \to I\!R$ *has the following properties:*

(a) *it is convex and continuous;*
(b) *it is strictly decreasing;*
(c) $\lim_{t\to\infty} \mathcal{E}P(t) = -\infty$.

Proof. Let $(F_n)_{n=1}^\infty$ be an ascending sequence of finite subsets of E such that $\cup_{n=1}^\infty F_n = E$. Let $t \in \mathcal{F}in$. Lemma 10.5 in [8] gives convexity of all the functions $t \in I\!R \mapsto \mathcal{E}P_n(t) \in I\!R$, $n \in I\!N$. Hence, by Proposition 3.12(c), the function $t \in \mathcal{F}in \mapsto \mathcal{E}P(t) \in I\!R$ is convex as a pointwise limit of convex functions. To get statement (a), it only remains to show the right-continuity at θ when $\theta \in \mathcal{F}in$. This is postponed to the end of the proof.

As derived in the proof of Proposition 3.12,

$$\exp(P_\lambda(t)) \leq \sum_{e\in E} \text{ess sup}\Big\{\|(\varphi_e^\lambda)'\|^t : \lambda \in \Lambda\Big\} = M_t < \infty$$

for ν-a.e. $\lambda \in \Lambda$ and for every $t \in \mathcal{F}in$. Since M_t tends to 0 as $t \to \infty$ and since it does so uniformly over a subset of Λ of full measure, we conclude that (c) holds.

Moreover, in view of Lemma 10.6 in [8], all the functions $t \in I\!\!R \mapsto \mathcal{E}P_n(t) \in I\!\!R$ are strictly decreasing, and therefore the function $t \in \mathcal{F}in \mapsto \mathcal{E}P(t) \in I\!\!R$ is (weakly) decreasing by Proposition 3.12. If this function were not strictly decreasing, say $\mathcal{E}P(t_2) = \mathcal{E}P(t_1)$ for some $t_1 < t_2$, it would be constant on the interval $[t_1, \infty)$ because of its convexity. This would however contradict the just proven statement (c). This proves statement (b).

We are only left to show the right-continuity at the point θ when $\theta \in \mathcal{F}in$. Since the function $\mathcal{E}P : \mathcal{F}in \to I\!\!R$ is decreasing, it is enough to show that

$$\limsup_{t\to\theta^+} \mathcal{E}P(t) \geq \mathcal{E}P(\theta).$$

Since each function $\mathcal{E}P_n : I\!\!R \to I\!\!R$ is continuous, for each $n \in I\!\!N$ there exists $t_n \in (\theta, \theta + 1/n)$ such that

$$\mathcal{E}P_n(\theta) \leq \mathcal{E}P_n(t_n) + \frac{1}{n} \leq \mathcal{E}P(t_n) + \frac{1}{n}.$$

By Proposition 3.12(c), we obtain

$$\mathcal{E}P(\theta) = \lim_{n\to\infty} \mathcal{E}P_n(\theta) \leq \liminf_{n\to\infty}\Big(\mathcal{E}P(t_n) + \frac{1}{n}\Big) = \liminf_{n\to\infty} \mathcal{E}P(t_n) \leq \limsup_{t\to\theta^+} \mathcal{E}P(t).$$

We are done. $\square$

This result suggests the following classification of RCGDMS. It is inspired from the well-known classification of deterministic CGDMS.

Definition 3.14. *A RCGDMS is called regular if there exists $t \geq 0$ such that $\mathcal{E}P(t) = 0$. A RCGDMS which is not regular is called irregular.*

Regular RCGDMS can be further divided into subclasses.

Definition 3.15. *A regular RCGDMS is called critically regular if $\mathcal{E}P(\theta) = 0$. A regular RCGDMS is called strongly regular if $0 < \mathcal{E}P(t) < \infty$ for some $t \geq 0$. A strongly regular RCGDMS is called cofinitely regular if $\lim_{t\to\theta^+} \mathcal{E}P(t) = \infty$.*

We now want to investigate what happens at the finiteness parameter of the system θ when $\theta \notin \mathcal{F}in$. We then set $\mathcal{E}P(\theta) = \infty$.

Definition 3.16. *A RCGDMS Φ is said to be evenly varying if*

$$\Delta := \sup_{e\in E} \frac{\text{ess sup}\{\|(\varphi_e^\lambda)'\| : \lambda \in \Lambda\}}{\text{ess inf}\{\|(\varphi_e^\lambda)'\| : \lambda \in \Lambda\}} < \infty.$$

We shall prove the following.

Lemma 3.17. *Let $(F_n)_{n=1}^\infty$ be an ascending sequence of finite subsets of E such that $\cup_{n=1}^\infty F_n = E$. If a RCGDMS Φ is evenly varying, then*

$$\mathcal{E}P(\theta) = \lim_{n\to\infty} \mathcal{E}P_n(\theta).$$

Hence, evenly varying RCGDMS have an expected pressure function which is right-continuous at their finiteness parameter θ, and thus continuous on $[\theta, \infty)$ by Proposition 3.13.

Proof. If $\theta \in \mathcal{F}in$, then the results follows from the proof of Proposition 3.13. So, suppose that $\theta \notin \mathcal{F}in$. Then, by definition, $\mathcal{E}P(\theta) = \infty$ and we have to prove that

$$\lim_{n\to\infty} \mathcal{E}P_n(\theta) = \infty.$$

Let $\Omega \subset E_A^p$ be a finite set of finite words that witnesses the finite primitivity of the matrix A. Let F be the set of all letters appearing in words in Ω. Let $M_F = \min\{M_e : e \in F\}$, where $M_e := \text{ess inf}\{\|(\varphi_e^\lambda)'\| : \lambda \in \Lambda\}$. Let also $M_n = \min\{M_e : e \in F_n\}$. Without loss of generality, we may assume that each F_n contains F. Let $k \geq 2$. For every $\omega \in E^k$ there exists elements $\alpha_1, \alpha_2, \ldots, \alpha_{k-1} \in \Omega$ such that

$$\overline{\omega} := \omega_1\alpha_1\omega_2\alpha_2 \ldots \omega_{k-2}\alpha_{k-2}\omega_{k-1}\alpha_{k-1}\omega_k \in E_A^*.$$

Note that the map $\omega \in E^k \mapsto \overline{\omega} \in E_A^{(p+1)k-p}$ is injective, and therefore for all $k \geq 1$, all $\tau \in E_A^\infty$, and ν-a.e. $\lambda \in \Lambda$, we have, using (3.21),

$$\begin{aligned}
\mathcal{L}_{\theta,n,\lambda}^{(p+1)k-p}\mathbb{1}_{E_A^\infty}(\tau) &= \sum_{\beta \in E_A^{(p+1)k-p} : \beta\tau \in E_A^\infty} |(\varphi_\beta^\lambda)'(\pi_{T^{(p+1)k-p}(\lambda)}(\tau))|^\theta \\
&\geq \sum_{\omega \in F_n^k : A_{\omega_k\tau_1}=1} |(\varphi_{\overline{\omega}}^\lambda)'(\pi_{T^{(p+1)k-p}(\lambda)}(\tau))|^\theta \\
&\geq K^{-\theta((p+1)k-p)} M_F^{\theta pk} \sum_{\omega \in F_n^k : A_{\omega_k\tau_1}=1} \|(\varphi_{\omega_1}^\lambda)'\|^\theta \|(\varphi_{\omega_2}^{T^{p+1}(\lambda)})'\|^\theta \cdot \ldots \cdot \|(\varphi_{\omega_k}^{T^{(p+1)(k-1)}(\lambda)})'\|^\theta \\
&\geq K^{\theta(p-(p+1)k)} M_F^{\theta pk} M_n^\theta \sum_{\omega \in F_n^{k-1}} \|(\varphi_{\omega_1}^\lambda)'\|^\theta \|(\varphi_{\omega_2}^{T^{p+1}(\lambda)})'\|^\theta \cdot \ldots \cdot \|(\varphi_{\omega_{k-1}}^{T^{(p+1)(k-2)}(\lambda)})'\|^\theta \\
&\geq K^{\theta(p-(p+1)k)} M_F^{\theta pk} M_n^\theta \Big(\sum_{e \in F_n} \text{ess inf}\{\|(\varphi_e^\lambda)'\|^\theta : \lambda \in \Lambda\}\Big)^{k-1} \\
&\geq K^{\theta(p-(p+1)k)} M_F^{\theta pk} M_n^\theta \Delta^{-\theta(k-1)} \Big(\sum_{e \in F_n} \text{ess sup}\{\|(\varphi_e^\lambda)'\|^\theta : \lambda \in \Lambda\}\Big)^{k-1} \\
&= \Big(K^{(p-(p+1)k)} M_F^{pk} M_n \Delta^{1-k}\Big)^\theta M_{\theta,n}^{k-1},
\end{aligned}$$

where $M_{\theta,n} := \sum_{e\in F_n} \text{ess sup}\{\|(\varphi_e^\lambda)'\|^\theta : \lambda\in\Lambda\}$. Therefore, using Lemma 4.6 in [8],

$$\begin{aligned}
\mathcal{E}P_n(\theta) &= \lim_{k\to\infty}\frac{1}{(p+1)k-p}\log\mathcal{L}_{\theta,n,\lambda}^{(p+1)k-p}1\!\!1_{E_A^\infty}(\tau)\\
&\geq \lim_{k\to\infty}\frac{1}{(p+1)k-p}\Big[\theta\Big((p-(p+1)k)\log K + pk\log M_F + (1-k)\log\Delta\Big)\\
&\qquad\qquad + \theta\log M_n + (k-1)\log M_{\theta,n}\Big]\\
&= \theta\Big(-\log K + \frac{p}{p+1}\log M_F - \frac{1}{p+1}\log\Delta\Big) + \frac{1}{p+1}\log M_{\theta,n}.
\end{aligned}$$

The result follows since $\lim_{n\to\infty} M_{\theta,n} = M_\theta = \infty$. $\square$

Let

$$h = \inf\{t\geq 0 : \mathcal{E}P(t)\leq 0\}.$$

Our next goal is to prove a variant of Bowen's formula for RCGDMS.

Theorem 3.18. *(Bowen's formula) For ν-a.e. $\lambda\in\Lambda$,*

$$\mathrm{HD}(J_\lambda) = h \geq \theta \geq 0,$$

where $\mathrm{HD}(J_\lambda)$ is the Hausdorff dimension of J_λ. Moreover, $h>\theta$ if the system Φ is strongly regular.

The proof will be given in several steps. We start with the following.

Lemma 3.19. *If $\mathcal{E}P(t)\leq 0$, then*

$$\liminf_{r\to 0}\frac{\log m_\lambda^t(B(x,r))}{\log r}\leq t$$

for ν-a.e. $\lambda\in\Lambda$ and all $x\in J_\lambda$.

Proof. Suppose that $t\in\mathcal{F}in$ is so that $\mathcal{E}P(t)\leq 0$. From (3.2), there is a set $\Lambda_1\subset\Lambda$ such that $\nu(\Lambda_1)=1$ and such that for all $\lambda\in\Lambda_1$, all $x\in J_\lambda$, all $\omega\in E_A^\infty$ such that $\pi_\lambda(\omega)=x$, and all $n\in I\!\!N$, we have

$$\varphi_{\omega|_n}^\lambda(X_{t(\omega_n)})\subset B(x, KD\|(\varphi_{\omega|_n}^\lambda)'\|).$$

From (BDP) and (3.13), there is a set $\Lambda_2\subset\Lambda_1$ such that $\nu(\Lambda_2)=1$ and such that for all $\lambda\in\Lambda_2$, all $x\in J_\lambda$, all $\omega\in E_A^\infty$ such that $\pi_\lambda(\omega)=x$, and all $n\in I\!\!N$, we have

$$\begin{aligned}
m_\lambda^t\Big(B(x,KD\|(\varphi_{\omega|_n}^\lambda)'\|)\Big)\geq m_\lambda^t\Big(\varphi_{\omega|_n}^\lambda(X_{t(\omega_n)})\Big) &\geq e^{-P_\lambda^n(t)}K^{-t}\|(\varphi_{\omega|_n}^\lambda)'\|^t m_{T^n(\lambda)}^t(X_{t(\omega_n)})\\
&= K^{-t}e^{-P_\lambda^n(t)}\|(\varphi_{\omega|_n}^\lambda)'\|^t\\
&= K^{-2t}D^{-t}e^{-P_\lambda^n(t)}(KD\|(\varphi_{\omega|_n}^\lambda)'\|)^t,
\end{aligned}$$

and hence

$$\liminf_{r\to 0}\frac{\log m_\lambda^t(B(x,r))}{\log r} \leq \limsup_{n\to\infty}\frac{\log\Big(K^{-2t}D^{-t}e^{-P_\lambda^n(t)}(KD\|(\varphi_{\omega|_n}^\lambda)'\|)^t\Big)}{\log(KD\|(\varphi_{\omega|_n}^\lambda)'\|)}$$

$$= \limsup_{n\to\infty} \frac{-P_\lambda^n(t)}{\log \|(\varphi_{\omega|_n}^\lambda)'\|} + t. \tag{3.18}$$

for all $\lambda \in \Lambda_2$, all $x \in J_\lambda$, and all $\omega \in E_A^\infty$ such that $\pi_\lambda(\omega) = x$.

Now, if $\mathcal{E}P(t) < 0$ then by Birkhoff's Ergodic Theorem there is $\Lambda_3 \subset \Lambda_2$ such that $\nu(\Lambda_3) = 1$ and such that for each $\lambda \in \Lambda_3$ there is $N(\lambda) \geq 1$ for which $-P_\lambda^n(t) > 0$ for all $n \geq N(\lambda)$. Then

$$\limsup_{n\to\infty} \frac{-P_\lambda^n(t)}{\log \|(\varphi_{\omega|_n}^\lambda)'\|} \leq 0 \tag{3.19}$$

for all $\lambda \in \Lambda_3$ and all $\omega \in E_A^\infty$. Combining (3.18) with (3.19), we conclude that

$$\liminf_{r\to 0} \frac{\log m_\lambda^t(B(x,r))}{\log r} \leq t$$

for all $\lambda \in \Lambda_3$ and all $x \in J_\lambda$.

On the other hand, if $\mathcal{E}P(t) = 0$ we obtain

$$\limsup_{n\to\infty} \frac{-P_\lambda^n(t)}{\log \|(\varphi_{\omega|_n}^\lambda)'\|} \leq \limsup_{n\to\infty} \frac{|P_\lambda^n(t)|}{\left|\log \|(\varphi_{\omega|_n}^\lambda)'\|\right|} \leq \limsup_{n\to\infty} \frac{|P_\lambda^n(t)|}{-n\log s} = 0, \tag{3.20}$$

where the last equality follows from Birkhoff's Ergodic Theorem for all λ in a set $\tilde{\Lambda}_3 \subset \Lambda_2$ such that $\nu(\tilde{\Lambda}_3) = 1$ and for all $\omega \in E_A^\infty$. Combining (3.18) with (3.20), we conclude that

$$\liminf_{r\to 0} \frac{\log m_\lambda^t(B(x,r))}{\log r} \leq t$$

for all $\lambda \in \tilde{\Lambda}_3$ and all $x \in J_\lambda$. We are done. □

As an immediate consequence of this lemma, we get the following.

Corollary 3.20. $\mathrm{HD}(J_\lambda) \leq h$ *for* ν*-a.e.* $\lambda \in \Lambda$.

Remark 3.21. *Note that the OSC has not been used in establishing Lemma 3.19 and Corollary 3.20. Indeed, relation (3.13) holds even in the absence of the OSC.*

Henceforth we shall assume that the following condition is fulfilled by the RCGDMS under scrutiny. This condition seems indispensable and in the case of random self-similar IFSs in the sense of Mauldin and Williams [7] this condition is always assumed. Moreover, it has the same form no matter whether E is finite or infinite.

For all $e \in E$ there exists $M_e \in (0, s]$ such that

$$\|(\varphi_e^\lambda)'\| \geq M_e, \quad \nu\text{-a.e. } \lambda \in \Lambda. \tag{3.21}$$

In order to demonstrate that $\mathrm{HD}(J_\lambda) \geq h$ under this condition, we shall first prove the following.

Lemma 3.22. *If the alphabet E is finite, then*

$$\liminf_{r\to 0} \frac{\log m_\lambda^h(B(x,r))}{\log r} \geq h$$

for ν-a.e. $\lambda \in \Lambda$ and all $x \in J_\lambda$.

Proof. Fix $\lambda \in \Lambda$ in a set of full measure ν which will be specified later. Fix also $x \in J_\lambda$. Set

$$M := \min\{M_e \,|\, e \in E\} > 0.$$

For every $r \in (0, M)$, set

$$G_x(r) := \Big\{\omega \in E_A^* : \varphi_\omega^\lambda(X_{t(\omega)}) \cap B(x,r) \neq \emptyset,\ \|(\varphi_\omega^\lambda)'\| \leq r \text{ and } \|(\varphi^\lambda_{\omega|_{|\omega|-1}})'\| > r\Big\}.$$

Obviously, $G_x(r)$ is an anti-chain (i.e. its elements are pairwise incomparable) and

$$\bigcup_{\omega\in G_x(r)} [\omega] \supset \pi_\lambda^{-1}(B(x,r)).$$

By (3.12), there is $\Lambda_1 \subset \Lambda$ with $\nu(\Lambda_1) = 1$ such that

$$\begin{aligned} m_\lambda^h(B(x,r)) = \tilde{m}_\lambda^h\Big(\pi_\lambda^{-1}(B(x,r))\Big) &\leq \tilde{m}_\lambda^h\Big(\bigcup_{\omega\in G_x(r)} [\omega]\Big) \\ &= \sum_{\omega\in G_x(r)} \tilde{m}_\lambda^h([\omega]) \\ &= \sum_{\omega\in G_x(r)} e^{-P_\lambda^{|\omega|}(h)} \int_{E_A^\infty} \Big|(\varphi_\omega^\lambda)'(\pi_{T^{|\omega|}(\lambda)}(\tau))\Big|^h \, d\tilde{m}^h_{T^{|\omega|}(\lambda)}(\tau) \\ &\leq \sum_{\omega\in G_x(r)} e^{-P_\lambda^{|\omega|}(h)} \|(\varphi_\omega^\lambda)'\|^h \\ &\leq r^h \sum_{\omega\in G_x(r)} e^{|P_\lambda^{|\omega|}(h)|} \end{aligned} \tag{3.22}$$

for each $\lambda \in \Lambda_1$ and each $x \in J_\lambda$. Moreover, if $\omega \in G_x(r)$ then $M^{|\omega|} \leq r < s^{|\omega|-1}$, which means that

$$\frac{\log(sr)}{\log s} > |\omega| \geq \frac{\log r}{\log M}. \tag{3.23}$$

By Birkhoff's Ergodic Theorem, there exists $\Lambda_0 \subset \Lambda_1$ such that $\nu(\Lambda_0) = 1$ and

$$\lim_{n\to\infty} \frac{1}{n} P_\lambda^n(h) = \mathcal{E}P(h) = 0$$

for all $\lambda \in \Lambda_0$. Fix $\varepsilon > 0$. Fix also an arbitrary $\lambda \in \Lambda_0$ and $x \in J_\lambda$. There thus exists $n_0 \in I\!N$ such that

$$|P_\lambda^n(h)| \leq \varepsilon n$$

for all $n \geq n_0$. Assume $r > 0$ to be so small that $\log r / \log M \geq n_0$. If $\omega \in G_x(r)$ we obtain from (3.23) that

$$\exp\Big(|P_\lambda^{|\omega|}(h)|\Big) \leq \exp(\varepsilon|\omega|) \leq \exp\Big(\varepsilon\frac{\log(sr)}{\log s}\Big) = (sr)^{\varepsilon/\log s}.$$

Hence, we can deduce from (3.22) that

$$m_\lambda^h(B(x,r)) \leq \#G_x(r) s^{\varepsilon/\log s} r^{h+\varepsilon/\log s}.$$

It remains to estimate $\#G_x(r)$ from above. To do this, note that all the sets $\varphi_\omega^\lambda(\mathrm{Int}(X_{t(\omega)}))$, $\omega \in G_x(r)$, are mutually disjoint, contained in the ball $B(x, KDr)$, and each of them contains a ball of radius at least $D^{-1}K^{-2}Mr$. A straightforward volume argument shows that $\#G_x(r)$ is bounded above by a constant C depending only on D, K and M. Therefore,

$$m_\lambda^h(B(x,r)) \leq C s^{\varepsilon/\log s} r^{h+\varepsilon/\log s}.$$

Consequently, we conclude that

$$\liminf_{r\to 0} \frac{\log m_\lambda^h(B(x,r))}{\log r} \geq h + \frac{\varepsilon}{\log s},$$

and letting $\varepsilon \to 0$, the result follows for each $\lambda \in \Lambda_0$ and each $x \in J_\lambda$. □

As an immediate consequence of this lemma, we have the following.

Corollary 3.23. *If the alphabet E is finite, then* $\mathrm{HD}(J_\lambda) \geq h$ *for ν-a.e. $\lambda \in \Lambda$.*

Combining together Corollaries 3.20 and 3.23, as well as the fact that finite systems have finite expected pressures, i.e. $-\infty < \mathcal{E}P(t) < \infty$ for all $t \in I\!R$, we get the following.

Corollary 3.24. *If Φ is a RCGDMS with finite alphabet E, then* $\mathrm{HD}(J_\lambda) = h$ *for ν-a.e. $\lambda \in \Lambda$, where $h = \inf\{t \geq 0 : \mathcal{E}P(t) \leq 0\}$. In fact, h is the unique number such that $\mathcal{E}P(h) = 0$.*

For the first part of this result to carry over to an infinite alphabet E, we will need the following.

Theorem 3.25. *If Φ is a RCGDMS, then for all $t \geq 0$*

$$\mathcal{E}P(t) = \sup\{\mathcal{E}P_F(t) : F \in \mathcal{F}in(E)\},$$

where $\mathcal{F}in(E)$ is the family of all finite subsets of E.

Proof. By Lemma 4.6 in [8], we have $\mathcal{E}P_A(t) \leq \mathcal{E}P_B(t)$ whenever $A \subset B \subset E$. Our theorem therefore follows from Proposition 3.12(c). □

Now, we can prove the following.

Theorem 3.26. *If Φ is a RCGDMS, then for ν-a.e. $\lambda \in \Lambda$,*

$$\mathrm{HD}(J_\lambda) = \sup\{\mathrm{HD}(J_{F,\lambda}) : F \in \mathcal{F}in(E)\} = h = \inf\{t \geq 0 : \mathcal{E}P(t) \leq 0\} \geq \theta \geq 0.$$

Moreover, $h > \theta$ when the system Φ is strongly regular.

Proof. It is obvious that $\eta := \sup\{\mathrm{HD}(J_{F,\lambda}) : F \in \mathcal{F}in(E)\} \leq \mathrm{HD}(J_\lambda)$ for every $\lambda \in \Lambda$ and, by Corollary 3.20, $\mathrm{HD}(J_\lambda) \leq h = \inf\{t \geq 0 : \mathcal{E}P(t) \leq 0\}$ for ν-a.e. $\lambda \in \Lambda$. It therefore suffices to show that $h \leq \eta$. But, by Theorem 3.25 and Proposition 3.13, $\mathcal{E}P(\eta) = \sup\{\mathcal{E}P_F(\eta) : F \in \mathcal{F}in(E)\} \leq \sup\{\mathcal{E}P_F(h_F) : F \in \mathcal{F}in(E)\} = 0$. Hence $\eta \geq h$ and we are done. □

Definition 3.27. *A regular RCGDMS Φ is called essentially random if*

$$\liminf_{n\to\infty} P_\lambda^n(h) = -\infty \qquad \text{and} \qquad \limsup_{n\to\infty} P_\lambda^n(h) = \infty$$

for ν-a.e. $\lambda \in \Lambda$, where

$$P_\lambda^n(h) = \sum_{j=0}^{n-1} P_{T^j(\lambda)}(h).$$

Note that if the sequence of random variables $(P_{T^n(\lambda)}(h))_{n=0}^\infty$ (with respect to the probability measure ν) satisfies the Law of Iterated Logarithms, then the system Φ is essentially random.

Theorem 3.28. *The following statements hold.*

(a) *If Φ is an essentially random RCGDMS or an irregular RCGDMS, then $\mathcal{H}^h(J_\lambda) = 0$ for ν-a.e. $\lambda \in \Lambda$.*

(b) *If Φ is essentially random and satisfies the strong separation condition, then $\mathcal{P}^h(J_\lambda) = \infty$ for ν-a.e. $\lambda \in \Lambda$.*

Proof. (a) Suppose that Φ is an essentially random RCGDMS or an irregular RCGDMS. It follows from formula (3.13) of Theorem 3.7 that

$$m_\lambda^h(\varphi_\omega^\lambda(J_{T^n(\lambda)})) \asymp e^{-P_\lambda^n(h)} \|(\varphi_\omega^\lambda)'\|^h m_{T^n(\lambda)}^h(J_{T^n(\lambda)}) = e^{-P_\lambda^n(h)} \|(\varphi_\omega^\lambda)'\|^h$$

for ν-a.e. $\lambda \in \Lambda$, for all $n \geq 0$ and all $\omega \in E_A^n$. Thus,

$$\begin{aligned}\sum_{\omega\in E_A^n} \Big(\mathrm{diam}(\varphi_\omega^\lambda(J_{T^n(\lambda)}))\Big)^h &\asymp \sum_{\omega\in E_A^n} \|(\varphi_\omega^\lambda)'\|^h \asymp e^{P_\lambda^n(h)} \sum_{\omega\in E_A^n} m_\lambda^h(\varphi_\omega^\lambda(J_{T^n(\lambda)})) \\ &= e^{P_\lambda^n(h)} m_\lambda^h\Big(\bigcup_{\omega\in E_A^n} \varphi_\omega^\lambda(J_{T^n(\lambda)})\Big) = e^{P_\lambda^n(h)} m_\lambda^h(J_\lambda) = e^{P_\lambda^n(h)}, \end{aligned} \tag{3.24}$$

where we used formula (3.14) of Theorem 3.7 to establish the first equality sign. Now, in either case (essentially random or irregular alike),

$$\liminf_{n\to\infty} e^{P_\lambda^n(h)} = 0 \tag{3.25}$$

for ν-a.e. $\lambda \in \Lambda$. Indeed, in the essentially random case, this is an immediate consequence of its definition, and in the irregular case, this follows directly from Birkhoff's Ergodic Theorem. It follows from (3.24) and (3.25) that $\mathcal{H}^h(J_\lambda) = 0$.

Now, suppose Φ is essentially random and that the strong separation condition holds. Let $R = \mathrm{dist}(X^c, \cup_{\lambda\in\Lambda} \cup_{e\in E} \varphi_e^\lambda(X)) > 0$. Fix $\lambda \in \Lambda$ and $\omega \in E_A^\infty$. Then for every $k \geq 0$ we have

$B(\pi_{T^n(\lambda)}(\sigma^n(\omega)), R) \subset X$. Therefore $\varphi^\lambda_{\omega|_n}(B(\pi_{T^n(\lambda)}(\sigma^n(\omega)), R)) \supset B(\pi_\lambda(\omega), K^{-1}\|(\varphi^\lambda_{\omega|_n})'\|)$ and hence

$$\begin{aligned} m^h_\lambda(B(\pi_\lambda(\omega), K^{-1}\|(\varphi^\lambda_{\omega|_n})'\|)) &\leq e^{-P^n_\lambda(h)}\|(\varphi^\lambda_{\omega|_n})'\|^h m^h_{T^n(\lambda)}(B(\pi_{T^n(\lambda)}(\sigma^n(\omega)), R)) \\ &\leq K^h e^{-P^n_\lambda(h)}\Big(K^{-1}\|(\varphi^\lambda_{\omega|_n})'\|\Big)^h. \end{aligned}$$

Since Φ is essentially random, we deduce for ν-a.e. $\lambda \in \Lambda$ and all $\omega \in E^\infty_A$ that

$$\liminf_{r\to 0} \frac{m^h_\lambda(B(\pi_\lambda(\omega), r))}{r^h} \leq \liminf_{n\to\infty} \frac{m^h_\lambda(B(\pi_\lambda(\omega), K^{-1}\|(\varphi^\lambda_{\omega|_n})'\|))}{(K^{-1}\|(\varphi^\lambda_{\omega|_n})'\|)^h} \leq K^h \liminf_{n\to\infty} e^{-P^n_\lambda(h)} = 0.$$

Thus, $\mathcal{P}^h(J_\lambda) = \infty$ for ν-a.e. $\lambda \in \Lambda$ and the proof is complete. □

Corollary 3.29. *Almost no limit set fiber J_λ of an essentially random system is bi-Lipschitz homeomorphic to the limit set of any deterministic system with a finite alphabet.*

Proof. This directly follows from Theorem 3.28 and the fact that the limit sets of finite deterministic systems have Hausdorff and packing measures which are positive and finite. □

This corollary asserts that in the realm of systems with finite alphabet, essentially random systems and deterministic systems form drastically different, non-overlapping subworlds.

As another immediate consequence of Theorem 3.28, we get the following remarkable geometric statements.

Corollary 3.30. *If Φ is either essentially random or irregular and J_λ is not totally disconnected (i.e. contains a non-trivial connected component) for a set of positive measure ν of parameters $\lambda \in \Lambda$ (equivalently, for ν-a.e. $\lambda \in \Lambda$), then* $\mathrm{HD}(J_\lambda) > 1$ *for ν-a.e. $\lambda \in \Lambda$.*

Proof. This is an immediate consequence of Theorem 3.28 and the fact that $\mathcal{H}^1(Y) > 0$ whenever Y contains a non-degenerate connected component. □

Finally, Theorem 3.28 also has the following repercussion.

Corollary 3.31. *If Φ is an essentially random CGDMS acting on a phase space $X \subset I\!R^d$, then* $\mathrm{HD}(J_\lambda) = h < d$ *for ν-a.e. $\lambda \in \Lambda$.*

Proof. This is immediate from Theorem 3.28(b) since $\mathcal{P}^d(J_\lambda) \leq \mathcal{P}^d(X) < \infty$. (Recall that $\mathcal{P}^d$ is a multiple of Lebesgue measure on $I\!R^d$.) □

4. Examples of Random CGDMS

In this section, we give some examples of RCGDMS. In the following, a SIFS is a CIFS whose generators are all similarities.

Example 4.1. *Let $S = \{\varphi_e\}_{e\in E}$ be a deterministic CGDMS. For every $\lambda \in \Lambda$ and every $v \in V$, let $g_\lambda^v : W_v \to W_v$ be conformal injections such that $\|(g_\lambda^v)'\|_{X_v} \le 1$ for all $\lambda \in \Lambda$, such that the map $\lambda \mapsto g_\lambda^v(x)$ is measurable for every $x \in W_v$. When S is a one-dimensional system, we further require that the family $\{\varphi_e^\lambda := g_\lambda^{i(e)} \circ \varphi_e\}$ satisfy condition (v) of the definition of a RCGDMS. Let $T : \Lambda \to \Lambda$ be an invertible ergodic map preserving a measure ν. The family $\Phi = \{\varphi_e^\lambda := g_\lambda^{i(e)} \circ \varphi_e\}$ is then a random CGDMS.*

More specifically, one might have:

(1a) S is a deterministic SIFS in which X is a closed ball, and $g_\lambda : W \to W$ is a Euclidean isometry for each λ;

(1ab) S is a deterministic SIFS in which $X = \mathbb{D} := \{z \in \mathbb{C} : |z| \le 1\}$, and $g_\lambda(z) = e^{2\pi i\lambda}z$ for every $\lambda \in \Lambda := (0, 1]$;

(1b) S is a deterministic SIFS in which $X = \mathbb{D}$, and $g_\lambda(z) = \lambda z$ for every $\lambda \in \Lambda := \mathbb{D}\backslash\{0\}$;

(1c) S is a deterministic SIFS in which X is a star-shaped set centered at 0. Let $0 \le a < b \le 1$, $\Lambda = (a, b]^{\mathbb{Z}}$, $T : \Lambda \to \Lambda$ be the shift transformation, i.e. $T((\lambda_n)_{n=-\infty}^{\infty}) = (\lambda_{n+1})_{n=-\infty}^{\infty}$, ν_0 be the normalized Lebesgue measure on $(a, b]$, and $\nu = \nu_0^{\mathbb{Z}}$. Let $g_\lambda(z) = \lambda_0 z$.

Let us determine the Hausdorff dimension for example (1a). First, note that since the system is an IFS, we have $E_A^\infty = E^\infty$. Moreover, since the generators φ_e^λ are all similarities, their derivatives are independent of the point taken. Therefore, for every $\omega \in E_A^\infty$ we have

$$\begin{aligned}\mathcal{L}_{t,\lambda}1\!\!1_{E_A^\infty}(\omega) &= \sum_{e\in E: A_{e\omega_1}=1} \exp\Big(t\zeta(e\omega,\lambda)\Big)1\!\!1_{E_A^\infty}(e\omega)\\ &= \sum_{e\in E} |(\varphi_e^\lambda)'(\pi_{T(\lambda)}(\omega))|^t\\ &= \sum_{e\in E} |(\varphi_e^\lambda)'|^t.\end{aligned}$$

Since the g_λ are isometries, we obtain

$$\mathcal{L}_{t,\lambda}1\!\!1_{E_A^\infty}(\omega) = \sum_{e\in E} |(\varphi_e^\lambda)'|^t = \sum_{e\in E} |\varphi_e'|^t = Z_{1,S}(t),$$

where $Z_{1,S}(t)$ is the level 1 partition function of the pressure of the deterministic system S. Similarly,

$$\mathcal{L}_{t,\lambda}^n 1\!\!1_{E_A^\infty}(\omega) = \sum_{\tau\in E^n} |(\varphi_\tau^\lambda)'|^t = \sum_{\tau\in E^n} |\varphi_\tau'|^t = Z_{n,S}(t),$$

where $Z_{n,S}(t)$ is the nth-level partition function of the pressure of the deterministic system S. Since $\mathcal{L}_{t,\lambda}^n 1\!\!1_{E_A^\infty}$ does not depend on ω, we obtain from (2.9) that

$$P_\lambda^n(t) = \log \tilde{m}_{T^n(\lambda)}^t(\mathcal{L}_{t,\lambda}^n 1\!\!1_{E_A^\infty}) = \log \mathcal{L}_{t,\lambda}^n 1\!\!1_{E_A^\infty} = \log Z_{n,S}(t).$$

By Birkhoff's Ergodic Theorem, we conclude that

$$\mathcal{E}P(t) = \int_\Lambda P_\lambda(t)d\nu(\lambda) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} P_{T^j(\lambda)}(t) = \lim_{n\to\infty}\frac{1}{n}P_\lambda^n(t) = \lim_{n\to\infty}\frac{1}{n}\log Z_{n,S}(t) = P_S(t)$$

for ν-a.e. $\lambda \in \Lambda$, where $P_S(t)$ is the pressure of the deterministic system S. Thus,

$$\mathrm{HD}(J_\lambda) = \mathrm{HD}(J_S)$$

for ν-a.e. $\lambda \in \Lambda$ by Theorem 3.18, where $\mathrm{HD}(J_S)$ is the Hausdorff dimension of the limit set J_S of the deterministic system S.

We now turn our attention to (1b). As in (1a), observe that

$$\begin{aligned}
\mathcal{L}^n_{t,\lambda}1\!\!1_{E_A^\infty}(\omega) &= \sum_{\tau\in E^n} |(\varphi_\tau^\lambda)'|^t \\
&= \sum_{\tau\in E^n} |(\varphi_{\tau_1}^\lambda)'|^t |(\varphi_{\tau_2}^{T(\lambda)})'|^t \cdots |(\varphi_{\tau_n}^{T^{n-1}(\lambda)})'|^t \\
&= \sum_{\tau\in E^n} |\lambda|^t |\varphi'_{\tau_1}|^t |T(\lambda)|^t |\varphi'_{\tau_2}|^t \cdots |T^{n-1}(\lambda)|^t |\varphi'_{\tau_n}|^t \\
&= \sum_{\tau\in E^n} |\lambda|^t |T(\lambda)|^t \cdots |T^{n-1}(\lambda)|^t |\varphi'_{\tau_1}|^t |\varphi'_{\tau_2}|^t \cdots |\varphi'_{\tau_n}|^t \\
&= \prod_{j=0}^{n-1} |T^j(\lambda)|^t \sum_{\tau\in E^n} |\varphi'_\tau|^t \\
&= \left(\prod_{j=0}^{n-1} |T^j(\lambda)|\right)^t Z_{n,S}(t).
\end{aligned}$$

for every $\omega \in E_A^\infty$. By Birkhoff's Ergodic Theorem, we deduce that

$$\begin{aligned}
\mathcal{E}P(t) &= \int_\Lambda P_\lambda(t)d\nu(\lambda) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} P_{T^j(\lambda)}(t) = \lim_{n\to\infty}\frac{1}{n}P_\lambda^n(t) \\
&= \lim_{n\to\infty}\frac{1}{n}\log \mathcal{L}^n_{t,\lambda}1\!\!1_{E_A^\infty} = t\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log|T^j(\lambda)| + \lim_{n\to\infty}\frac{1}{n}\log Z_{n,S}(t) \\
&= t\int_\Lambda \log|\lambda| d\nu(\lambda) + P_S(t).
\end{aligned}$$

Hence the Hausdorff dimension of the random system Φ is

$$h = \inf\Big\{t \ge 0 : t\int_\Lambda \log|\lambda|d\nu(\lambda) + P_S(t) \le 0\Big\}.$$

Finally, in (1c), we obtain

$$\mathcal{L}^n_{t,\lambda}1\!\!1_{E_A^\infty}(\omega) = \sum_{\tau\in E^n} \lambda_0^t|\varphi'_{\tau_1}|^t\lambda_1^t|\varphi'_{\tau_2}|^t\cdots\lambda_{n-1}^t|\varphi'_{\tau_n}|^t = \left(\prod_{j=0}^{n-1}\lambda_j\right)^t Z_{n,S}(t).$$

for every $\omega \in E_A^\infty$. By Birkhoff's Ergodic Theorem, we deduce that

$$\begin{aligned}\mathcal{E}P(t) &= \int_\Lambda P_\lambda(t)d\nu(\lambda) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}P_{T^j(\lambda)}(t) = \lim_{n\to\infty}\frac{1}{n}P_\lambda^n(t)\\ &= \lim_{n\to\infty}\frac{1}{n}\log\mathcal{L}_{t,\lambda}^n 1\!\!1_{E_A^\infty} = t\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\lambda_j + \lim_{n\to\infty}\frac{1}{n}\log Z_{n,S}(t)\\ &= t\int_a^b \log\lambda\, d\nu_0(\lambda) + P_S(t).\end{aligned}$$

Hence the Hausdorff dimension of the random system Φ is

$$h = \inf\Big\{t \geq 0 : t\int_a^b \log\lambda\, d\nu_0(\lambda) + P_S(t) \leq 0\Big\}.$$

Example 4.2. *Let $S = \{\varphi_e\}_{e\in E}$ be a deterministic CGDMS. For every $\lambda \in \Lambda$ and every $v \in V$, let $f_\lambda^v : W_v \to W_v$ be conformal injections such that $\|(f_\lambda^v)'\|_{X_v} \leq 1$ for all $\lambda \in \Lambda$, such that the map $\lambda \mapsto f_\lambda^v(x)$ is measurable for every $x \in W_v$. When S is a one-dimensional system, we further require that the family $\{\varphi_e^\lambda := g_\lambda^{i(e)} \circ \varphi_e\}$ satisfy condition (v) of the definition of a RCGDMS. Let $T : \Lambda \to \Lambda$ be an invertible ergodic map preserving a measure ν. The family $\Phi = \{\varphi_e^\lambda := \varphi_e \circ f_\lambda^{t(e)}\}$ is then a random CGDMS.*

More specifically, one might have:

(2a) S is a deterministic SIFS in which X is a closed ball, and $f_\lambda : W \to W$ is a Euclidean isometry for each λ;

(2ab) S is a deterministic SIFS in which $X = \mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$, and $f_\lambda(z) = e^{2\pi i\lambda}z$ for every $\lambda \in \Lambda := (0, 1]$;

(2b) S is a deterministic SIFS in which $X = \mathbb{D}$, and $f_\lambda(z) = \lambda z$ for every $\lambda \in \Lambda := \mathbb{D}\backslash\{0\}$;

(2c) S is a deterministic SIFS in which X is a star-shaped set centered at 0. Let $0 \leq a < b \leq 1$, $\Lambda = (a, b]^{\boldsymbol{Z}}$, $T : \Lambda \to \Lambda$ be the shift transformation, i.e. $T((\lambda_n)_{n=-\infty}^\infty) = (\lambda_{n+1})_{n=-\infty}^\infty$, ν_0 be the normalized Lebesgue measure on $(a, b]$, and $\nu = \nu_0^{\boldsymbol{Z}}$. Let $f_\lambda(z) = \lambda_0 z$.

The results derived in Example 4.1 hold for Example 4.2.

Example 4.3. *Let $S = \{\varphi_e\}_{e\in E}$ be a deterministic CGDMS. For every $\lambda \in \Lambda$ and every $v \in V$, let $f_\lambda^v, g_\lambda^v : W_v \to W_v$ be conformal injections such that $\max\{\|(f_\lambda^v)'\|_{X_v}, \|(g_\lambda^v)'\|_{X_v}\} \leq 1$ for all $\lambda \in \Lambda$, such that the maps $\lambda \mapsto f_\lambda^v(x)$ and $\lambda \mapsto g_\lambda^v(x)$ are measurable for every $x \in W_v$. When S is a one-dimensional system, we further require that the family $\{\varphi_e^\lambda := g_\lambda^{i(e)} \circ \varphi_e\}$ satisfy condition (v) of the definition of a RCGDMS. Let $T : \Lambda \to \Lambda$ be an invertible ergodic map preserving a measure ν. The family $\Phi = \{\varphi_e^\lambda := g_\lambda^{i(e)} \circ \varphi_e \circ f_\lambda^{t(e)}\}$ is then a random CGDMS.*

More specifically, one might have:

(3a) S is a deterministic SIFS in which X is a closed ball, and $f_\lambda, g_\lambda : W \to W$ are Euclidean isometries for each λ;

(3ab) S is a deterministic SIFS in which $X = \mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$, and $f_\lambda(z) = g_\lambda(z) = e^{2\pi i\lambda}z$ for every $\lambda \in \Lambda := (0, 1]$;

(3b) S is a deterministic SIFS in which $X = I\!D$, and $f_\lambda(z) = g_\lambda(z) = \lambda z$ for every $\lambda \in \Lambda := I\!D\backslash\{0\}$;

(3c) S is a deterministic SIFS in which X is a star-shaped set centered at 0. Let $0 \leq a < b \leq 1$, $0 \leq c < d \leq 1$, $\Lambda = ((a,b] \times (c,d])^{\boldsymbol{Z}}$, $T : \Lambda \to \Lambda$ be the shift transformation, i.e. $T((\lambda_n)_{n=-\infty}^{\infty}) = (\lambda_{n+1})_{n=-\infty}^{\infty}$, where $\lambda = (\lambda_n^{(1)}, \lambda_n^{(2)})_{n=-\infty}^{\infty}$, ν_0 be the normalized Lebesgue measure on $(a,b] \times (c,d]$, and $\nu = \nu_0^{\boldsymbol{Z}}$. Let $f_\lambda(z) = \lambda_0^{(1)} z$ and $g_\lambda(z) = \lambda_0^{(2)} z$.

The results derived in Example 4.1(a),(ab) hold for Example 4.3(a),(ab), respectively. Inspiring ourselves from (1b), we obtain for (3b) that

$$\mathcal{E}P(t) \quad = \quad 2t \int_\Lambda \log|\lambda| d\nu(\lambda) + P_S(t).$$

Hence the Hausdorff dimension of the random system Φ is

$$h = \inf\Big\{t \geq 0 : 2t \int_\Lambda \log|\lambda| d\nu(\lambda) + P_S(t) \leq 0\Big\}.$$

Note the presence of an additional factor 2.

Finally, in (3c),

$$\mathcal{L}_{t,\lambda}^n 1\!\!1_{E_A^\infty}(\omega) = \Big(\prod_{j=0}^{n-1} \lambda_j^{(1)}\Big)^t \Big(\prod_{j=0}^{n-1} \lambda_j^{(2)}\Big)^t Z_{n,S}(t)$$

for every $\omega \in E_A^\infty$. Therefore

$$\mathcal{E}P(t) = t \int_a^b \log r \, dr + t \int_c^d \log r \, dr + P_S(t).$$

Hence the Hausdorff dimension of the random system Φ is

$$h = \inf\left\{t \geq 0 : t\left[\int_a^b \log r \, dr + \int_c^d \log r \, dr\right] + P_S(t) \leq 0\right\}.$$

Example 4.4. *Let $0 \leq a < b \leq 1$. Let $\Lambda = (a,b]^{\boldsymbol{Z}}$, and ν_0 a Borel probability measure on $(a,b]$. Let $\nu = \nu_0^{\boldsymbol{Z}}$, and $T : \Lambda \to \Lambda$ be the shift transformation, i.e. $T((\lambda_n)_{n=-\infty}^{\infty}) = (\lambda_{n+1})_{n=-\infty}^{\infty}$. We now define a one-dimensional random SIFS by first picking a strictly increasing sequence $(x_n)_{n=0}^{\infty}$ such that $x_0 = 0$ and $\lim_{n\to\infty} x_n = 1$. Moreover, we let $X = [0,1]$ and $E = \{0,1,2,3,\dots\}$. For every $n \in E$, let*

$$\varphi_n^\lambda(x) = x_n + \lambda_0(x_{n+1} - x_n)x.$$

The family $\{\varphi_e^\lambda : [0,1] \to [0,1]\}$ constitutes a one-dimensional random SIFS.

We will now find a formula for $\mathrm{HD}(J_\lambda)$. Let $E_q = \{0, 1, \ldots, q-1\}$. Observe that

$$\begin{aligned}
\mathcal{L}^n_{t,q,\lambda} 1\!\!1_{E_A^\infty}(\omega) &= \sum_{\tau \in E_q^n} |(\varphi^\lambda_\tau)'|^t \\
&= \sum_{\tau \in E_q^n} |(\varphi^\lambda_{\tau_1})'|^t |(\varphi^{T(\lambda)}_{\tau_2})'|^t \cdots |(\varphi^{T^{n-1}(\lambda)}_{\tau_n})'|^t \\
&= \sum_{\tau \in E_q^n} \lambda_0^t (x_{\tau_1+1} - x_{\tau_1})^t (T(\lambda))_0^t (x_{\tau_2+1} - x_{\tau_2})^t \cdots (T^{n-1}(\lambda))_0^t (x_{\tau_n+1} - x_{\tau_n})^t \\
&= \sum_{\tau \in E_q^n} \lambda_0^t (x_{\tau_1+1} - x_{\tau_1})^t \lambda_1^t (x_{\tau_2+1} - x_{\tau_2})^t \cdots \lambda_{n-1}^t (x_{\tau_n+1} - x_{\tau_n})^t \\
&= \lambda_0^t \lambda_1^t \cdots \lambda_{n-1}^t \sum_{\tau \in E_q^n} (x_{\tau_1+1} - x_{\tau_1})^t (x_{\tau_2+1} - x_{\tau_2})^t \cdots (x_{\tau_n+1} - x_{\tau_n})^t \\
&= \Big(\prod_{j=0}^{n-1} \lambda_j\Big)^t \Big(\sum_{e=0}^{q-1} (x_{e+1} - x_e)^t\Big)^n
\end{aligned}$$

for all $\omega \in E_A^\infty$. Using Proposition 3.12(c) and Birkhoff's Ergodic Theorem, we obtain

$$\begin{aligned}
\mathcal{E}P(t) &= \lim_{q\to\infty} \mathcal{E}P_q(t) = \lim_{q\to\infty} \int_\Lambda P_{q,\lambda}(t)\,d\nu(\lambda) = \lim_{q\to\infty}\lim_{n\to\infty} \frac{1}{n} P^n_{q,\lambda}(t) \\
&= \lim_{q\to\infty}\lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}^n_{t,q,\lambda} 1\!\!1_{E_A^\infty} = t \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \lambda_j + \lim_{q\to\infty} \log \sum_{e=0}^{q-1} (x_{e+1} - x_e)^t \\
&= t \int_a^b \log \lambda \, d\nu_0(\lambda) + \log \sum_{e=0}^{\infty} (x_{e+1} - x_e)^t.
\end{aligned}$$

By Theorem 3.18, the Hausdorff dimension of the system Φ is

$$h = \inf\left\{ t \geq 0 : t \int_a^b \log \lambda \, d\nu_0(\lambda) + \log \sum_{e=0}^{\infty} (x_{e+1} - x_e)^t \leq 0 \right\}.$$

If $x_{e+1} - x_e = 1/2^{e+1}$ for all $e \in E$, then

$$h \int_a^b \log \lambda \, d\nu_0(\lambda) - h \log 2 - \log(1 - 2^{-h}) = 0.$$

If, moreover, ν_0 is the Lebesgue measure on $(a, b]$, then

$$h\Big[(b \log b - a \log a) - (b - a)\Big] - h \log 2 - \log(1 - 2^{-h}) = 0.$$

Example 4.5. *Let $0 \leq a < b \leq 1$. Let $\Lambda = (a, b]^{\mathbf{Z}}$, and ν_0 a Borel probability measure on $(a, b]$. Let $\nu = \nu_0^{\mathbf{Z}}$, and $T : \Lambda \to \Lambda$ be the shift transformation, i.e. $T((\lambda_n)_{n=-\infty}^\infty) = (\lambda_{n+1})_{n=-\infty}^\infty$. We now define a one-dimensional random SIFS by first picking a strictly increasing sequence $(x_n)_{n=0}^\infty$ such that $x_0 = 0$ and $\lim_{n\to\infty} x_n = 1$. Moreover, we let $X = [0, 1]$ and $E = \{0, 1, 2, 3, \ldots\}$. For every $n \in E$, let*

$$\varphi^\lambda_n(x) = x_n + \lambda_n(x_{n+1} - x_n)x.$$

The family $\{\varphi_e^\lambda : [0,1] \to [0,1]\}$ *constitutes a one-dimensional random SIFS.*

Example 4.6. *Let* $0 \le a < b \le 1$*. Let* $\Delta_{a,b} = \{(s,t) \in I\!R^2 : a < s \le t < b\}$*, and* ν_0 *a Borel probability measure on* $\Delta_{a,b}$*. Let* $\Lambda = \Delta_{a,b}^{\mathbf{Z}}$*,* $\nu = \nu_0^{\mathbf{Z}}$*,* $T : \Lambda \to \Lambda$ *be the shift transformation, i.e.* $T((\lambda_n)_{n=-\infty}^\infty) = (\lambda_{n+1})_{n=-\infty}^\infty$*. We now define a one-dimensional random SIFS by setting* $X = [0,1]$*,* $E = \{0,1\}$*,* $\lambda = (\lambda_n^{(1)}, \lambda_n^{(2)})_{n=-\infty}^\infty$*, and*

$$\varphi_0^\lambda(x) = \lambda_0^{(1)} x, \qquad \varphi_1^\lambda(x) = 1 - (1 - \lambda_0^{(2)})(1-x).$$

The family $\Phi = \{\varphi_e^\lambda : [0,1] \to [0,1]\}$ *constitutes a one-dimensional random SIFS.*

We now obtain a formula for the Hausdorff dimension of this system. First, note that

$$\begin{aligned}
\mathcal{L}_{t,\lambda}^n 1\!\!1_{E_A^\infty}(\omega) = \sum_{\tau \in E^n} |(\varphi_\tau^\lambda)'|^t &= \sum_{\tau \in E^n} |(\varphi_{\tau_1}^\lambda)'|^t |(\varphi_{\tau_2}^{T(\lambda)})'|^t \cdots |(\varphi_{\tau_n}^{T^{n-1}(\lambda)})'|^t \\
&= \Big(\sum_{e \in E} |(\varphi_e^\lambda)'|^t\Big)\Big(\sum_{e \in E} |(\varphi_e^{T(\lambda)})'|^t\Big) \cdots \Big(\sum_{e \in E} |(\varphi_e^{T^{n-1}(\lambda)})'|^t\Big) \\
&= \prod_{j=0}^{n-1} \Big(\sum_{e \in E} |(\varphi_e^{T^j(\lambda)})'|^t\Big)
\end{aligned}$$

for every $\omega \in E_A^\infty$. Letting $g_t(\lambda) = \log \sum_{e \in E} |(\varphi_e^\lambda)'|^t$, we deduce that

$$\begin{aligned}
\mathcal{E}P(t) &= \int_\Lambda P_\lambda(t) d\nu(\lambda) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} P_{T^j(\lambda)}(t) = \lim_{n\to\infty} \frac{1}{n} P_\lambda^n(t) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_{t,\lambda}^n 1\!\!1_{E_A^\infty} \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \sum_{e \in E} |(\varphi_e^{T^j(\lambda)})'|^t = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g_t(T^j(\lambda)) = \int_\Lambda g_t(\lambda)\, d\nu(\lambda).
\end{aligned}$$

Since

$$g_t(\lambda) = \log\Big(|(\varphi_0^\lambda)'|^t + |(\varphi_1^\lambda)'|^t\Big) = \log\Big[(\lambda_0^{(1)})^t + (1 - \lambda_0^{(2)})^t\Big],$$

we infer that

$$\mathcal{E}P(t) = \int_{\Delta_{a,b}} \log\Big[(\lambda_0^{(1)})^t + (1 - \lambda_0^{(2)})^t\Big] d\nu_0(\lambda_0).$$

If ν_0 is the Lebesgue measure on $\Delta_{a,b}$, then

$$\mathcal{E}P(t) = \int_a^b \int_x^b \log\Big[x^t + (1-y)^t\Big] dy\, dx.$$

Hence the Hausdorff dimension of the random system Φ is the unique number h such that $\mathcal{E}P(h) = 0$, i.e. such that

$$\int_a^b \int_x^b \log\Big[x^h + (1-y)^h\Big] dy\, dx = 0.$$

Note the difference with Example 4.2 in [7].

Example 4.7. *Let $0 \le a < b \le 1$. Let $\Delta_{a,b} = \{(s,t) \in I\!R^2 : a < s \le t < b\}$, and ν_0 a Borel probability measure on $\Delta_{a,b}$. Let $\Lambda = \Delta_{a,b}^{\mathbb{Z}}$, $\nu = \nu_0^{\mathbb{Z}}$, $T : \Lambda \to \Lambda$ be the shift transformation, i.e. $T((\lambda_n)_{n=-\infty}^{\infty}) = (\lambda_{n+1})_{n=-\infty}^{\infty}$. We now define a one-dimensional random SIFS by first picking a strictly increasing sequence $(x_n)_{n=0}^{\infty}$ such that $x_0 = 0$ and $\lim_{n\to\infty} x_n = 1$. Moreover, we let $X = [0,1]$ and $E = \{0,1,2,3,\dots\} \times \{0,1\}$. For all $n \ge 0$, let*

$$\varphi_{n,0}^{\lambda}(x) = x_n + \lambda_n^{(1)}(x_{n+1} - x_n)x \qquad \text{and} \qquad \varphi_{n,1}^{\lambda}(x) = x_{n+1} - (1 - \lambda_n^{(2)})(x_{n+1} - x_n)x.$$

The family $\{\varphi_e^{\lambda} : [0,1] \to [0,1]\}$ forms a one-dimensional random SIFS.

Calculations similar to the ones performed in Example 4.6 lead to

$$\begin{aligned}
\mathcal{E}P(t) &= \int_\Lambda \log \sum_{n=0}^{\infty} \Big(|(\varphi_{n,0}^{\lambda})'|^t + |(\varphi_{n,1}^{\lambda})'|^t \Big) d\nu(\lambda) \\
&= \int_\Lambda \log \sum_{n=0}^{\infty} \Big((\lambda_n^{(1)})^t (x_{n+1} - x_n)^t + (1 - \lambda_n^{(2)})^t (x_{n+1} - x_n)^t \Big) d\nu(\lambda) \\
&= \int_\Lambda \log \sum_{n=0}^{\infty} \Big[(x_{n+1} - x_n)^t \Big((\lambda_n^{(1)})^t + (1 - \lambda_n^{(2)})^t \Big) \Big] \, d\nu(\lambda).
\end{aligned}$$

Hence the Hausdorff dimension of the random system Φ is

$$h = \inf \left\{ t \ge 0 : \int_\Lambda \log \sum_{n=0}^{\infty} \Big[(x_{n+1} - x_n)^t \Big((\lambda_n^{(1)})^t + (1 - \lambda_n^{(2)})^t \Big) \Big] \, d\nu(\lambda) \le 0 \right\}.$$

5. Random elliptic functions

In this section, we provide an application of our results to the theory of random elliptic functions. In [4], the Hausdorff dimension of the Julia set $J(f)$ of a non-constant elliptic (meromorphic) function $f : \mathbb{C} \to \overline{\mathbb{C}}$ and the Hausdorff dimension of $I_\infty(f)$, the set of points escaping to infinity under iteration of f, were estimated as follows:

$$\mathrm{HD}(J(f)) > \frac{2q}{q+1} \qquad \text{whereas} \qquad \mathrm{HD}(I_\infty(f)) \le \frac{2q}{q+1},$$

where $q \ge 2$ is the maximal order of all poles of f. The idea to establish the former of these two inequalities was to associate to the function $f : \mathbb{C} \to \overline{\mathbb{C}}$ an (deterministic) infinite CIFS whose finiteness parameter θ is equal to $2q/(q+1)$. Now, we consider the situation where we randomly choose elliptic functions from a sufficiently small neighbourhood of f, and we thereafter generate the corresponding random Julia sets. We are going to estimate the Hausdorff dimensions of these random Julia sets by the same number $2q/(q+1)$. Random Julia sets can be defined in a more general context of arbitrary meromorphic functions.Indeed, let $T : \Lambda \to \Lambda$ be an invertible ergodic measurable transformation preserving a Borel probability measure ν, and to each $\lambda \in \Lambda$ ascribe a meromorphic function $f_\lambda : \mathbb{C} \to \overline{\mathbb{C}}$ such that the map $\lambda \in \Lambda \mapsto f_\lambda(z) \in \overline{\mathbb{C}}$ is measurable for all $z \in \mathbb{C}$. For every $n \ge 1$, let

$$f_\lambda^n := f_{T^{n-1}(\lambda)} \circ f_{T^{n-2}(\lambda)} \circ \cdots \circ f_\lambda : \mathbb{C} \to \overline{\mathbb{C}},$$

where we adopt the convention that $f_\lambda(\infty) = \infty$ for every $\lambda \in \Lambda$. The Julia set J_λ is said to consist of all points $z \in \mathbb{C}$ such that the family $\{f_\lambda^n : U \to \overline{\mathbb{C}}\}_{n\geq 1}$ is not normal on any open neighbourhood U of z. Clearly, $J_\lambda \subset \mathbb{C}$ is a closed set and $f_\lambda(J_\lambda) = J_{T(\lambda)}$.

We will restrict our attention to the more specific situation where $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a non-constant elliptic function, $q = q_f$ is the maximal order of all poles of f, and $f_\lambda = \lambda f$ for all $\lambda \in \mathbb{C}$ in a sufficiently small neighbourhood Λ of 1. Note that all poles of f_λ coincide with those of f. For every $\lambda \in \Lambda$, let

$$I_\lambda(\infty) = \{z \in \mathbb{C} : \lim_{n\to\infty} f_\lambda^n(z) = \infty\}.$$

We shall prove the following.

Theorem 5.1. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a non-constant elliptic function. If $\Lambda \subset \mathbb{C}$ is a sufficiently small neighbourhood of 1 and $T : \Lambda \to \Lambda$ is an arbitrary map (not necessarily measurable, not necessarily measure-preserving), then*

$$\mathrm{HD}(I_\lambda(\infty)) \leq \frac{2q}{q+1}$$

for **all** $\lambda \in \Lambda$.

Proof. Let $B_R = \{z \in \overline{\mathbb{C}} : |z| > R\}$. For every pole b of f, we denote by $B_\lambda(b, R)$ the connected component of $f_\lambda^{-1}(B_R)$ containing b. We also set $B_b(R) := B_1(b, R)$. Taking a neighbourhood Λ of 1 sufficiently small, we have that $B_\lambda(b, 2R) \subset B_b(R)$ for all $\lambda \in \Lambda$. There exists $R_0 > 1$ such that for all $\lambda \in \Lambda$ the set B_{R_0} contains no critical values of f_λ, such that for all $R \geq R_0$ and all $\lambda \in \Lambda$ the sets $\{B_\lambda(b, R)\}_{b\in f^{-1}(\infty)}$ are simply connected and mutually disjoint and, for $z \in B_b(R_0)$ and $\lambda \in \Lambda$,

$$f_\lambda(z) = \frac{\lambda G_b(z)}{(z-b)^{q_b}}, \tag{5.1}$$

where q_b is the order of pole b and $G_b : B_b(R_0) \to \mathbb{C}$ is a bounded holomorphic function such that $G_b(b) \neq 0$. If $U \subset B_{R_0}\backslash\{\infty\}$ is an open simply connected set, then all the holomorphic inverse branches $f^{-1}_{\lambda,b,U,1}, \dots, f^{-1}_{\lambda,b,U,q_b}$ of f_λ are well defined on U for all $\lambda \in \Lambda$, and for every $1 \leq j \leq q_b$, all $z \in U$ and all $\lambda \in \Lambda$ we have

$$\left|f^{-1}_{\lambda,b,U,j}(z) - b\right| \asymp |z|^{-1/q_b} \tag{5.2}$$

and

$$\left|(f^{-1}_{\lambda,b,U,j})'(z)\right| \asymp |z|^{-(q_b+1)/q_b}. \tag{5.3}$$

This means that

$$\begin{aligned} 0 < m \ &:= \ \inf\left\{\frac{|f^{-1}_{\lambda,b,U,j}(z) - b|}{|z|^{-1/q_b}} : b \in f^{-1}(\infty), 1 \leq j \leq q_b, z \in U, \lambda \in \Lambda\right\} \\ &\leq \ \sup\left\{\frac{|f^{-1}_{\lambda,b,U,j}(z) - b|}{|z|^{-1/q_b}} : b \in f^{-1}(\infty), 1 \leq j \leq q_b, z \in U, \lambda \in \Lambda\right\} =: M < \infty \end{aligned}$$

and

$$\begin{aligned} 0<\tilde m &:= \inf\left\{\frac{|(f^{-1}_{\lambda,b,U,j})'(z)|}{|z|^{-(q_b+1)/q_b}} : b\in f^{-1}(\infty), 1\le j\le q_b, z\in U, \lambda\in\Lambda\right\} \\ &\le \sup\left\{\frac{|(f^{-1}_{\lambda,b,U,j})'(z)|}{|z|^{-(q_b+1)/q_b}} : b\in f^{-1}(\infty), 1\le j\le q_b, z\in U, \lambda\in\Lambda\right\} =: \tilde M<\infty. \end{aligned}$$

Let $R_1>R_0$ be such that

$$\max\{M,\tilde M\}R_1^{-1/q}<R_0. \tag{5.4}$$

Given $b_1,b_2\in B_{2R_1}\cap f^{-1}(\infty)$, we denote by

$$f^{-1}_{\lambda,b_2,b_1,j}: B(b_1,R_0)\to \mathbb{C}$$

all the holomorphic inverse branches $f^{-1}_{\lambda,b_2,B(b_1,R_0),j}$, $1\le j\le q_{b_2}$. It follows from (5.2) and (5.4) that

$$f^{-1}_{\lambda,b_2,b_1,j}(B(b_1,R_0))\subset f^{-1}_{\lambda,b_2,B(b_1,R_1),j}(B(b_1,R_1))\subset B(b_2,MR_1^{-1/q})\subset B(b_2,R_0). \tag{5.5}$$

for all $b_1,b_2\in B_{2R_1}\cap f^{-1}(\infty)$ and all $\lambda\in\Lambda$. Since the series

$$\sum_{b\in f^{-1}(\infty)\setminus\{0\}}|b|^{-s}$$

converges for all $s>2$, given any $t>2q/(q+1)$ there exists $R_2>R_1$ such that

$$q\tilde M^t\sum_{b\in B_{R_2}\cap f^{-1}(\infty)}|b|^{-((q+1)/q)t}\le 1. \tag{5.6}$$

Set

$$I_\lambda(R):=\{z\in\mathbb{C}: |f^n_\lambda(z)|>R, \forall\, n\ge 0\}.$$

Let $R_3>2R_2$ be so that for every $z\in I_\lambda(R_3)$ and every $n\ge 0$ there exists a unique $z_n\in f^{-1}(\infty)$ such that $f^n_\lambda(z)\in B(z_n,R_0)\cap f^{-1}_{T^n(\lambda),z_n,z_{n+1}}(B(z_{n+1},R_0))$. Of course, $|z_n|>R_2$ by definition. Now, set

$$I_R=f^{-1}(\infty)\cap B_R.$$

Let $R>2R_3$. It follows from (5.5), (5.2) and (5.4) that for every $l\ge 1$, the family W_l defined as

$$\Big\{f^{-1}_{\lambda,b_l,b_{l-1},j_l}\circ f^{-1}_{T(\lambda),b_{l-1},b_{l-2},j_{l-1}}\circ\cdots \\ \circ f^{-1}_{T^{l-1}(\lambda),b_2,b_1,j_2}\circ f^{-1}_{T^l(\lambda),b_1,b_0,j_1}(B(b_0,R_0)): b_i\in I_R, 1\le j_i\le q_{b_i}, i=0,1,\dots,l\Big\}$$

is well defined and covers $I_\lambda(R)$. Using (5.3), (5.4) and (5.6), we obtain

$$\begin{aligned}
\Sigma_l^\lambda := &\sum_{b_l\in I}\sum_{j_l=1}^{q_{b_l}}\cdots\sum_{b_1\in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0\in I}\operatorname{diam}^t\Big(f^{-1}_{\lambda,b_l,b_{l-1},j_l}\circ f^{-1}_{T(\lambda),b_{l-1},b_{l-2},j_{l-1}}\circ\cdots \\
&\qquad\qquad \circ f^{-1}_{T^{l-1}(\lambda),b_2,b_1,j_2}\circ f^{-1}_{T^l(\lambda),b_1,b_0,j_1}(B(b_0,R_0))\Big) \\
\le &\sum_{b_l\in I}\sum_{j_l=1}^{q_{b_l}}\cdots\sum_{b_1\in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0\in I}\Big\|\Big(f^{-1}_{\lambda,b_l,b_{l-1},j_l}\circ f^{-1}_{T(\lambda),b_{l-1},b_{l-2},j_{l-1}}\circ\cdots \\
&\qquad\qquad \circ f^{-1}_{T^{l-1}(\lambda),b_2,b_1,j_2}\circ f^{-1}_{T^l(\lambda),b_1,b_0,j_1}\Big)'\Big|_{B(b_0,R_0)}\Big\|_\infty^t\operatorname{diam}^t(B(b_0,R_0)) \\
\le &\sum_{b_l\in I}\sum_{j_l=1}^{q_{b_l}}\cdots\sum_{b_1\in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0\in I}\tilde M^{(l+1)t}|b_{l-1}|^{-t(q_{b_l}+1)/q_{b_l}}\cdot|b_{l-2}|^{-t(q_{b_{l-1}}+1)/q_{b_{l-1}}}\cdots \\
&\qquad\qquad \cdot|b_0|^{-t(q_{b_1}+1)/q_{b_1}}(2R_0)^t \\
\le &\ (2R_0)^t\tilde M^{(l+1)t}\sum_{b_l\in I}\sum_{j_l=1}^{q_{b_l}}\cdots\sum_{b_1\in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0\in I}|b_{l-1}|^{-((q+1)/q)t}\cdots|b_0|^{-((q+1)/q)t} \\
\le &\ (2R_0)^t\tilde M^{(l+1)t}q^{l+1}\Big(\sum_{b\in I}|b|^{-((q+1)/q)t}\Big)^{l+1} \\
\le &\ (2R_0)^t\Big(q\tilde M^t\sum_{b\in B_{R_3}\cap f^{-1}(\infty)}|b|^{-((q+1)/q)t}\Big)^{l+1} \\
\le &\ (2R_0)^t.
\end{aligned}$$

Since the diameters of the sets of the covers W_l converge uniformly to 0 when $l\nearrow\infty$, we infer that $\mathcal{H}^t(I_\lambda(R))\le(2R_0)^t<\infty$ for all $\lambda\in\Lambda$. Consequently, $\mathrm{HD}(I_\lambda(R))\le t$ for all $\lambda\in\Lambda$. If we put

$$I_{\lambda,R}(f):=\Big\{z\in\mathbb{C}:\liminf_{n\to\infty}|f_\lambda^n(z)|>R\Big\}\subset\bigcup_{k\ge1}f_\lambda^{-k}(I_{T^k(\lambda)}(R)),$$

then $\mathrm{HD}(I_\lambda(\infty))\le\mathrm{HD}(I_{\lambda,R}(f))\le\max_{k\in\mathbb{N}}\mathrm{HD}(I_{T^k(\lambda)}(R))\le t$ for all $\lambda\in\Lambda$. Letting $t\searrow 2q/(q+1)$ finishes the proof. □

Applying the results proved in the previous sections, particularly Bowen's formula, we shall now demonstrate the following.

Theorem 5.2. *Let $f:\mathbb{C}\to\overline{\mathbb{C}}$ be a non-constant elliptic function. If Λ is a sufficiently small neighbourhood of 1 endowed with a Borel probability measure ν and $T:\Lambda\to\Lambda$ is an invertible ergodic map preserving the measure ν, whose second iterate $T^2:\Lambda\to\Lambda$ is ergodic. Then*

$$\mathrm{HD}(J_\lambda)>\frac{2q}{q+1}$$

for ν-a.e. $\lambda\in\Lambda$.

Proof. Choose constants R_0, R_1 and R_2 as in the proof of Theorem 5.1. Fix a pole $a \in B_{2R_2}$ with $q_a = q$. For every pole $b \in f^{-1}(\infty) \cap B_{2R_2}$ with $q_b = q$, consider for each $\lambda \in \Lambda$ the inverse branches of f_λ

$$f_{\lambda,b,a,1}^{-1} : \overline{B(a,R_0)} \to \mathbb{C} \qquad \text{and} \qquad f_{\lambda,a,b,1}^{-1} : \overline{B(b,R_0)} \to \mathbb{C}.$$

In view of (5.5), we have

$$f_{\lambda,b,a,1}^{-1}(\overline{B(a,R_0)}) \subset \overline{B(b,R_0)} \qquad \text{and} \qquad f_{\lambda,a,b,1}^{-1}(\overline{B(b,R_0)}) \subset \overline{B(a,R_0)}$$

for all $\lambda \in \Lambda$. Since, in addition, one can prove these last two inclusions in exactly the same way with R_0 replaced by $R_1 > R_0$, the family

$$\Phi = \left\{ f_{\lambda,a,b,1}^{-1} \circ f_{T(\lambda),b,a,1}^{-1} : \overline{B(a,R_0)} \to \overline{B(a,R_0)} \right\}_{b \in f^{-1}(\infty) \cap B_{2R_2},\ \lambda \in \Lambda}$$

forms an infinite random CIFS if we set $\varphi_b^\lambda = f_{\lambda,a,b,1}^{-1} \circ f_{T(\lambda),b,a,1}^{-1}$ and we consider the map $T^2 : \Lambda \to \Lambda$ rather than T. In view of (5.3), we can write

$$M_t \asymp \sum_{b \in f^{-1}(\infty) \cap B_{2R_2}} |a|^{-((q+1)/q)t} |b|^{-((q+1)/q)t} \asymp \sum_{b \in f^{-1}(\infty) \cap B_{2R_2}} |b|^{-((q+1)/q)t}.$$

But the series $\sum_{b \in f^{-1}(\infty) \cap B_{2R_2}} |b|^{-((q+1)/q)t}$ converges if and only if $t > 2q/(q+1)$, and therefore $\theta_\Phi = 2q/(q+1)$ and $\mathcal{E}P(\theta_\Phi) = \infty$. Hence $\mathrm{HD}((J_\Phi)_\lambda) = h > \theta_\Phi = 2q/(q+1)$ for ν-a.e. $\lambda \in \Lambda$ because of Theorem 3.18. Since $(J_\Phi)_\lambda \subset J(f_\lambda)$, we conclude that $\mathrm{HD}(J(f_\lambda)) \geq \mathrm{HD}((J_\Phi)_\lambda) > 2q/(q+1)$ for ν-a.e. $\lambda \in \Lambda$. $\square$

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