Ergodic properties of sub-hyperbolic functions with polynomial Schwarzian derivative

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Abstract. The ergodic theory and geometry of the Julia set of meromorphic functions on the complex plane with polynomial Schwarzian derivative is investigated under the condition that the forward trajectory of asymptotic values in the Julia set is bounded and the map f restricted to its closure is expanding, the property refered to as sub-expanding. We first show the existence, uniqueness, conservativity and ergodicity of a conformal measure m with minimal exponent h; furthermore, we show weak metrical exactness of this measure. Then we prove the existence of a σ–finite invariant measure μ absolutely continuous with respect to m. Our main result states that μ is finite if and only if the order ρ of the function f satisfies the condition $h > 3\frac{\rho}{\rho+1}$. When finite, this measure is shown to be metrically exact. We also establish a version of Bowen's formula showing that the exponent h equals the Hausdorff dimension of the Julia set of f.

1. Introduction

The study of the ergodic theory and geometry of the Julia set of transcendental meromorphic functions appears to be a delicate task due to the infinite degree of these functions. For example, even the existence of conformal measures, on which the whole theory relies and which is by now completely standard in the realm of rational functions or Kleinian groups, is not known in general. Employing Nevanlinna's theory and a convenient change of the Riemannian metric we provided a complete treatise for a very general class of hyperbolic meromorphic functions in the papers [MU1] and [MU2]. In the present paper we relax the hyperbolicity assumption and allow the Julia set to contain singularities. Clearly one can adopt the arguments developed in the theory of rational iteration to deal with certain type of critical points. More challenging is to analyze the contribution of logarithmic singularities and, as we will see, this gives quite surprising results. The class of meromorphic functions with polynomial Schwarzian derivatives fit best to such a project since they have logarithmic singularities but they do not have critical points. We therefore restrict our considerations to this class of functions which, in particular, contains the tangent family; definitions and other examples are given in Section 2.

In the context of ergodic theory and fractal geometry, meromorphic and entire functions with logarithmic singularities have been investigated in [Sk1], [Sk2], [UZ3] (see also [KU] for a more complete historical outline and list of references) and, more recently, in [KS]. In [Sk1] and [Sk2] these singularities landed at poles an, in [UZ3], they were escaping to infinity extremely (like the trajectory of zero under the exponential function) fast. In both of these cases the forward trajectory of images of logarithmic singularities experienced a

Date: October 25, 2007.
1991 *Mathematics Subject Classification.* Primary: 30D05; Secondary:
Key words and phrases. Holomorphic dynamics, Hausdorff dimension, Meromorphic functions.
Research of the second author supported in part by the NSF Grant DMS 0400481.

large expansion neutralizing the contracting effect of singularities themselves. Assuming that a meromorphic map is subhyperbolic, the postsingular set is bounded, the Julia set is an entire sphere, and the reference conformal measure is the Lebesgue measure, the paper [KS] addressed the role of logarithmic singularities (algebraic singularities were also allowed).

In the present paper we consider subexpanding meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$ with polynomial Schwarzian derivative. By subexpanding we understand that the postsingular set $\mathcal{P}_f$ is bounded, and that the map f restricted to $\mathcal{P}_f$ is expanding. Employing the full power of Nevanlinna theory we first prove the existence of an atomless conformal measure via the Patterson-Sullivan construction. This measure is proved to be weakly metrically exact, which implies its ergodicity and conservativity. We then show the following result in which the existence of the σ–finite measure μ is obtained by employing M. Martens general method.

Theorem 1.1. *Let f be a subhyperbolic meromorphic function f of polynomial Schwarzian derivative and let m be the h–conformal measure of f obtained via the Patterson–Sullivan construction. Then there exists a σ–finite invariant measure μ absolutely continuous with respect to m. Moreover, the measure*

$$\mu \;\text{ is finite } \quad \text{if and only if} \quad h > 3\frac{\rho}{\rho+1}$$

where $\rho = \rho(f)$ is the order of the function f. If μ is finite, then the dynamical systems (f, μ) it generates is metrically exact and, in consequence, its Rokhlin's natural extension is K-mixing.

Notice that $3\frac{\rho}{\rho+1} \geq 2$ if and only if the order $\rho \geq 2$. Consequently the measure μ is most often infinite. However, in the case of the tangent family, which is just one specific example among others, this invariant measure can be finite. Curiously, finiteness of the invariant measure for the strictly preperiodic function $z \mapsto 2\pi i e^z$ is not known as yet.

Let us mention that we do not assume that the Julia set is the entire sphere nor that the conformal measure is the Lebesgue measure. In fact we do not assume that any conformal measure exists at all. But in the special situation when the Julia set is the entire sphere (in which case the spherical Lebesque measure is automatically a conformal measure) and if in addition $h = 2 > 3\frac{\rho}{\rho+1}$, i.e. if the order of the function $\rho < 2$, then the existence of a probability invariant measure absolutely continuous with respect to the Lebesgue measure follows also from [KS]. Indeed, in that situation our necessary and sufficient condition $\rho < 2$ conincides with the sufficient condition (Z3) from the paper by Kotus and Światek. Concerning the reciprocal statement, [KS] simply provides a counterexample.

The most involved part of the proof of Theorem 1.1 is to show finiteness. In the case the measure μ is finite, the dynamical system it generates is shown to be K–mixing which, in particular, implies mixing of all orders.

We also investigate the Hausdorff dimension of the Julia set and show that this dimension coincides with h, the exponent of the conformal measure m. Notice that this holds despite that the h–dimensional Hausdorff measure is shown to vanish on the Julia set.

2. The class of functions and definitions

2.1. **Definitions.** The reader may consult, for example, [Nev1], [Nev2] or [H1] for a detailed exposition on meromorphic functions and [Bw] for their dynamical aspects. We collect here the properties of interest for our concerns. The *Julia set* of a meromorphic function $f:\mathbb{C}\to\hat{\mathbb{C}}$ is denoted by J_f and the *Fatou set* by $\mathcal{F}_f$. Note that, in contrast to [MU1, MU2], we include here $\infty\in J_f$ since we are dealing with spherical geometry. However, $O^-(\infty)$ is a very special subset of the Julia set.

Let $\mathcal{A}_f$ be the set of *asymptotic values.* Note that the functions we consider do not have critical values. Therefore $\mathcal{A}_f$ coincides with the so called set of *singular values* $sing(f^{-1})$. The *post-singular set* $\mathcal{P}_f$ is the closure (in the sphere) of the set $\bigcup_{n>0} f^n(\mathcal{A}_f)$.

Concerning the singularities of a meromorphic function f, we dispose of Iversen's classification (see e.g. [Bw]): let $a\in sing(f^{-1})$ and, for every $r>0$, U_r be a component of $f^{-1}(D(a,r))$ in such a way that $r_1<r_2$ implies $U_{r_1}\subset U_{r_2}$. Then there are two possibilities:

a) $\bigcap_{r>0} U_r=\{c\}$ consists of one point, or
b) $\bigcap_{r>0} U_r=\emptyset$.

In the latter case we say that our choice $r\mapsto U_r$ defines a *transcendental singularity of* f^{-1} *over* a. Such a singularity is called *logarithmic* if the restriction $f:U_r\to D(a,r)\setminus\{a\}$ is a universal cover for some $r>0$. If this is the case, then the component U_r is called *logarithmic tract.* For the functions we consider all the transcendental singularities are logarithmic.

In case a), the point c can be regular or it is a critical point $c\in\mathcal{C}_f$.

We will always denote by
$$d\sigma(z)=\frac{|dz|}{1+|z|^2}$$
the spherical metric and by
$$|f'(z)|_\sigma=|f'(z)|\frac{1+|z|^2}{1+|f(z)|^2}$$
the derivative of f with respect to the spherical metric. The following direct consequence of Koebe's distortion theorem will be used.

Lemma 2.1. *Let $f:\mathbb{C}\to\hat{\mathbb{C}}$ be a meromorphic function and suppose that $D(w,2\delta)\subset\hat{\mathbb{C}}\setminus\mathcal{P}_f$. Then, for every $n\geq 1$, $z\in f^{-n}(w)$ and all $x,y\in D(w,\delta)$ we have that*
$$K^{-1}\leq\frac{|(f_z^{-n})'(y)|_\sigma}{|(f_z^{-n})'(x)|_\sigma}\leq K$$
for some universal constant $K\geq 1$.

Here and in the rest of the paper f_z^{-n} signifies the inverse branch of f^n defined near $f^n(z)$ mapping back $f^n(z)$ to z. An other convention will be that $D(z,r)$ stands for the spherical metric centered at z and of radius r. To indicate a spherical r–neighborhood of a set X we write $B(X,r)$.

2.2. **Meromorphic functions with polynomial Schwarzian derivative.** We consider meromorphic functions $f:\mathbb{C}\to\hat{\mathbb{C}}$ for which the Schwarzian derivative

$$S(f)=\left(\frac{f''}{f'}\right)'-\frac{1}{2}\left(\frac{f''}{f'}\right)^2=2P \qquad (2.1)$$

is a polynomial and for which the set of asymptotic values $\mathcal{A}_f$ does not contain infinity. Nevanlinna [Nev3] established that meromorphic functions with polynomial Schwarzian derivative are exactly the functions that have only finitely many asymptotical values and no critical values. Moreover, if such a function has a pole, then it is of order one. Consequently the maps of this class are locally injective. We also mention that any solution of (2.1) is of order $\rho = p/2$, where $p = deg(P) + 2$, and it is of normal type of its order (cf. [H2]).

Standard examples are furnished by the tangent family $f(z) = \lambda \tan(z)$ for which $S(f)$ is constant. By Möbius invariance of $S(f)$, functions like

$$\frac{e^z}{\lambda e^z + e^{-z}} \quad and \quad \frac{\lambda e^z}{e^z - e^{-z}}$$

have also constant Schwarzian derivative. Examples for which $S(f)$ is a polynomial are

$$f(z) = \frac{a\,Ai(z) + b\,Bi(z)}{c\,Ai(z) + d\,Bi(z)} \qquad with \quad ad - bc \neq 0 \tag{2.2}$$

where Ai and Bi are the Airy functions of the first and second kind. These a linear independent solutions of $g'' - zg = 0$ and, in general, if g_1, g_2 are linear independent solutions of

$$g'' + Pg = 0\ , \tag{2.3}$$

then $f = \frac{g_1}{g_2}$ is a solution of the Schwarzian equation (2.1). Conversely, every solution of (2.1) can be written locally as a quotient of two linear independent solutions of the linear differential equation (2.3). The asymptotic properties of these solutions are well known due to work of Hille ([H3], see also [H2]). They give a precise description of the function f near infinity. We now collect the facts that are important for our needs (more details and references are for example in [MU2]).

First of all, there are p critical directions $\theta_1, ..., \theta_p$ which are given by

$$arg\, c + p\theta = 0\, (mod\, 2\pi)$$

where c is the leading coefficient of $P(z) = cz^{p-2} + ...$ In a sector

$$S_j = \left\{ |argz - \theta_j| < \frac{2\pi}{p} - \delta\ ;\ |z| > R \right\},$$

$R > 0$ is sufficiently large and $\delta > 0$, the equation (2.3) has two linear independent solutions

$$\begin{aligned} g_1(z) &= P(z)^{-\frac{1}{4}} exp\big(iZ + o(1)\big) \quad and \\ g_2(z) &= P(z)^{-\frac{1}{4}} exp\big(-iZ + o(1)\big) \end{aligned} \tag{2.4}$$

where

$$Z = \int_{2Re^{i\theta_j}}^{z} P(t)^{\frac{1}{2}}\, dt = \frac{2}{p} c^{\frac{1}{2}} z^{\frac{p}{2}} \big(1 + o(1)\big) \quad for \quad z \to \infty \ \ in\ \ S_j. \tag{2.5}$$

If f is a meromorphic solution of the Schwarzian equation (2.1), then there are $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ such that

$$f(z) = \frac{ag_1(z) + bg_2(z)}{cg_1(z) + dg_2(z)} \quad , \ \ z \in S_j. \tag{2.6}$$

Observe that $f(z) \to a/c$ if $z \to \infty$ on any ray in $S_j \cap \{arg\, z < \theta_j\}$ and that $f(z) \to b/d$ if $z \to \infty$ on any ray in $S_j \cap \{arg\, z > \theta_j\}$. The asymptotic values of f are given by all the a/b, c/d corresponding to all the sectors S_j, $j = 1, ..., p$.

With this precise description of the asymptotic behavior of f, one can show ([MU2]) that

$$|f'(z)| \asymp |z|^{\rho - 1} \big|\alpha + \beta f(z) + \gamma f^2(z)\big| \quad for \ \ z \in S_j \tag{2.7}$$

where $\alpha = -ab/\delta$, $\beta = (ad + bc)/\delta$, $\gamma = -cd/\delta$ and $\delta = ad - bc$. Notice that $\gamma \neq 0$ if all the asymptotic values of f are finite.

2.3. **Sub-hyperbolic functions.**

Definition 2.2. *The function f is called boundedly non-recurrent if $\infty \notin \mathcal{A}_f \cup \mathcal{P}_f$, if $\mathcal{A}_f \cap \mathcal{P}_f \cap J_f = \emptyset$ and if every asymptotic value that belongs to the Fatou set is in an attracting component. If in addition, f is expanding on $\mathcal{P}_f$, the map f is called sub-expanding.*

Notice that this definition implies that all the asymptotic values of the function f are finite and that the post-singular set is bounded and nowhere dense in the Julia set. From now on we fix a number $T > 0$ such with the following properties.

- (T_1) $4T < |a_1 - a_2|$ for all distinct $a_1, a_2 \in \mathcal{A}_f$.
- (T_2) $B(\mathcal{P}_f, 4T) \cap \mathcal{A}_f = \emptyset$.
- (T_3) $f^{-1}(\mathcal{A}_f \cup \mathcal{P}_f) \cap \big(B(\mathcal{A}_f \cup \mathcal{P}_f, 4T) \setminus (\mathcal{A}_f \cup \mathcal{P}_f)\big) = \emptyset$.

To every asymptotic value $a \in \mathcal{A}_f$ there correspond (finitely many) logarithmic tracts U_a. In the following such a tract U_a will always be a component of $D(a, T)$ and we may suppose that $U_a \cap B(\mathcal{A}_f \cup \mathcal{P}_f, 4T) = \emptyset$.

Notice that being boundedly non-recurrent implies sub-expanding whenever the Julia set is equal to $\hat{\mathbb{C}}$ ([GKS]). We do always assume this property. From now on we also require $T > 0$ to be so small that $|(f^p)'|_\sigma > 2$ on $B(\mathcal{P}_f, T)$ for some $p \geq 1$ and that there are open neighborhoods Ω_1, Ω_0 of $\mathcal{P}_f$ such that $\overline{\Omega_1} \subset \Omega_0 \subset B(\mathcal{P}_f, T)$ and $g = f^p_{|\Omega_1} : \Omega_1 \to \Omega_0$ is a proper mapping. Denote

$$\Omega_n = g^{-n}(\Omega_0) \quad \textit{and} \quad \Gamma_n = \Omega_n \setminus \Omega_{n+1}. \tag{2.8}$$

Using the facts that repelling periodic points are dense in J_f and that J_f contains poles, one can easily prove the following.

Observation 2.3. *(topological exactness of f) For every non-empty open set U intersecting J_f, there exists $n \geq 0$ such that $f^n(U) \supset \hat{\mathbb{C}} \setminus \mathcal{A}_f$. In particular, for every $r > 0$ there exists $q_r \geq 0$ such that $f^{q_r}(D(z, r)) \supset \hat{\mathbb{C}} \setminus \mathcal{A}_f$ for all $z \in J_f$.*

Since $\mathcal{P}_f$ is a closed forward-invariant set and the map $f|_{\mathcal{P}_f}$ is expanding, following inverse trajectory of a point near $\mathcal{P}_f$, one can prove the following.

Observation 2.4. *(repeller) The set $\mathcal{P}_f$ is a repeller for f, precisely, assuming $T > 0$ to be small enough, we have*

$$\bigcap_{n=0}^{\infty} f^{-n}(B(\mathcal{P}_f, 2T)) = \mathcal{P}_f.$$

2.4. **First observations and transfer operator.** If one choses the right metric space $(\mathbb{C}, d\sigma)$, then the ergodic theory of meromorphic functions can be well developped. This has been done in great generality and in the hyperbolic case in [MU1, MU2]. For the functions we consider here the right geometry is simply the spherical one (which result from (2.7). Indeed, the functions satisfy the balanced growth condition of [MU1] with $\alpha_1 = \rho - 1$ and with $\alpha_2 = 2$ the later meaning that one has to work with the spherical metric).

Lemma 2.5. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be of polynomial Schwarzian derivative with $\infty \notin \mathcal{A}_f$. Then, if z belongs to a logarithmic tract $U_{2T} \subset f^{-1}(D(a, 2T))$ over an asymptotic value $a \in \mathcal{A}_f$, we have that*

$$|f'(z)|_\sigma \asymp (1 + |z|^{\rho+1})|f(z) - a|$$

and otherwise

$$|f'(z)|_\sigma \asymp 1 + |z|^{\rho+1}$$

where $\rho = \rho(f) < \infty$ is the order of f.

Proof. Follows from asymptotic description of f near infinity, in particular (2.7), together with the fact that f has only simple poles. □

Let us consider the transfer operator with respect to the spherical geometry.

$$\mathcal{L}_t\varphi(w) = \sum_{z \in f^{-1}(w)} |f'(z)|_\sigma^{-t} \varphi(z) \quad \varphi \in C(J_f). \tag{2.9}$$

It follows from Lemma 2.5 that

$$\mathcal{L}_t 1\!\!1(w) \preceq \max\{1, dist(w, \mathcal{A}_f)^{-t}\} \sum_{z \in f^{-1}(w)} (1 + |z|^{\rho+1})^{-t} \tag{2.10}$$

for every $w \in \hat{\mathbb{C}} \backslash \mathcal{A}_f$. This last sum is very well known in the theory of meromorphic functions and a Theorem of Borel [Nev2] together with the divergence property of f established in Theorem 3.2 of [MU2] implies that

$$\mathcal{L}_t 1\!\!1(w) < \infty \quad \text{if and only if} \quad t > \frac{\rho}{\rho + 1}. \tag{2.11}$$

We need the following additional properties.

Proposition 2.6. *For every $t > \frac{\rho}{\rho+1}$, there exists a constant M_t such that*

$$\Sigma(t, w) = \sum_{z \in f^{-1}(w)} (1 + |z|^{\rho+1})^{-t} \le M_t \quad \textit{for every} \quad w \in \hat{\mathbb{C}}.$$

The proof of this result uses parts of [MU1, MU2] and relies heavily on Nevanlinna theory. Good references for this are [Nev1] or [CY]. Let us simply recall that $n_f(r, a)$ stands for the number of a–points of modulus at most t, that the integrated counting number $N_f(r, a)$ is defined by $dN_f(r, a) = n_f(r, a)/r$ and that $T_f(r)$ denotes the characteristic function of f.

Proof. Fix $\varepsilon = 1$ and let $A > 0$ a constant that will be precised later. We may suppose that the origin is not a pole of f.

Case 1. $w \notin D(f(0), \varepsilon)$. Then we have that

$$\Sigma(t,w) \preceq C_A + \sum_{\substack{f(z)=w\\|z|>A}} (1+|z|^{\rho+1})^{-t} \preceq C_A + \sum_{\substack{f(z)=w\\|z|>A}} |z|^{-u}$$

with $C_A = \sup_{w\in\hat{\mathbb{C}}} n_f(w,A) < \infty$ and with $u = (\rho+1)t$. Since f is of finite order ρ we can make the following two integrations by part:

$$\sum_{\substack{f(z)=w\\|z|>A}} |z|^{-u} = -\frac{n_f(A,w)}{A^u} - u\frac{N_f(A,w)}{A^{u+1}} + u^2\int_A^\infty \frac{N_f(s,w)}{s^{u+1}} \le u^2 \int_A^\infty \frac{N_f(s,w)}{s^{u+1}}.$$

The First Main Theorm of Nevanlinna theorem (see [MU1, Corollary 4.2]) gives that

$$N_f(r,w) \le T_f(r) - \log[f(0),w],$$

where $[a,b]$ denotes the chordal distance on the Riemann sphere (with in particular $[a,b] \le 1$ for all $a,b \in \hat{\mathbb{C}}$). Since in this first case $w \notin D(f(0),\varepsilon)$ there is $\Theta < \infty$ such that

$$N_f(r,w) \le T_f(r) + \Theta \quad for\ \ every\ \ w \notin D(f(0),\varepsilon).$$

Therefore

$$\sum_{\substack{f(z)=w\\|z|>A}} (1+|z|^{\rho+1})^{-t} \preceq u^2 \int_A^\infty \frac{T_f(s)+\Theta}{su+1} ds = \tilde{M}_u < \infty \quad for\ \ every\ \ w \notin D(f(0),\varepsilon).$$

All in all, there exists $M_u^{(1)} < \infty$ such that

$$\Sigma(t,w) \le M_u^{(1)} \quad for\ \ every\ \ w \notin D(f(0),\varepsilon).$$

Case 2. $w \in D(f(0),\varepsilon)$. We are let to find a uniform bound for

$$\sum_{\substack{f(z)=w\\|z|>A}} |z|^{-u} \quad , \quad w \in J_f \cap D(f(0),\varepsilon).$$

Let $v \in \mathbb{C}$ be a point that is not a pole of f and such that $|f(-v)-f(0)| > 2\varepsilon$. Set $A = 3|v|$ and define the meromorphic function $g(\xi) = f(\xi - v) + v$. If $\xi = z + v$, then $f(z) = w$ is equivalent to $g(\xi) = w + v$.

Notice that $g(0) = f(-v) + v$. If we set $a = w + v$, then

$$|a - g(0)| = |w - f(-v)| \ge |f(-v) - f(0)| - |f(0) - w| > \varepsilon. \tag{2.12}$$

On the other hand, if $|\xi - v| \ge A$, then $|\xi| \ge A - |v| \ge 2|v|$ and $\frac{1}{|\xi - v|} \le \frac{2}{|\xi|}$. It follows that

$$\sum_{\substack{f(z)=w\\|z|>A}} |z|^{-u} = \sum_{\substack{g(\xi)=a\\|\xi-v|>A}} |\xi - v|^{-u} \le \sum_{\substack{g(\xi)=a\\|\xi|>2|v|}} \left(\frac{2}{|\xi|}\right)^u.$$

In the same way as before we can now use again the First Main Theorem of Nevanlinna theory, this time applied to the function g. Remember that by (2.12) we have $a \notin D(g(0), \varepsilon)$ whenever $w = a - v \in D(f(0), \varepsilon)$. Therefore,

$$\sum_{\substack{f(z)=w \\ |z|>A}} |z|^{-u} \leq 2^u \sum_{\substack{g(\xi)=a \\ |\xi|>2|v|}} |\xi|^{-u} \leq \tilde{\tilde{M}}_u$$

for every $a = w + v$, $w \in D(f(0), \varepsilon)$. It follows that there is $M_u^{(2)} < \infty$ such that

$$\Sigma(t, w) \leq M_u^{(2)} \quad \textit{for every} \ \ w \in D(f(0), \varepsilon).$$

All in all we showed that there is $M_t < \infty$ such that $\|\Sigma(t, .)\|_\infty \leq M_t$. □

3. Conformal measures

A probability measure m is a $t-$conformal measure for the meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ if $\frac{dm\circ f}{dm} = |f'|_\sigma^t$ or, equivalently, if for every mesurable set $E \subset \mathbb{C}$ for which the restriction $f_{|E}$ is injective we have

$$m(f(E)) = \int_E |f'|_\sigma^t \, dm.$$

Notice that if $J_f = \hat{\mathbb{C}}$, which is the case when all the asymptotic values are strictly preperiodic, the spherical Lebesgue measure is a 2–conformal measure. Therefore we restrict in the following subsection to functions with non-empty Fatou set.

3.1. **Existence of conformal measures and the pressure function.** Conformal measures are usually obtained via the standard Patterson-Sullivan method (see for example [McM] which contains a nice description of this procedure). For meromorphic functions however one must very carefully check what is going on at infinity because at this point the function is not defined. We ignore this for a moment and work on the compact set $J_f \subset \hat{\mathbb{C}}$. Let us first consider the Poincaré series of f at ∞

$$\mathcal{P}(t) = \mathcal{P}(t, \infty) = \sum_{n>0} \sum_{z\in f^{-n}(\infty)} |(f^n)'(z)|_\sigma^{-t} = \sum_{n>0} \mathcal{L}_t^n 1\!\!1(\infty) \tag{3.1}$$

which is well defined for $t > \frac{\rho}{\rho+1}$. Define

$$h = h_f = \inf\left\{ t > \frac{\rho}{\rho+1} \ ; \quad \mathcal{P}(t) < \infty \right\}. \tag{3.2}$$

We have the following.

Lemma 3.1. *For a sub-hyperbolic meromorphic function f of polynomial Schwarzian derivative with $\mathcal{F}_f \neq \emptyset$ we have that*

$$\frac{\rho}{\rho+1} < h_f \leq 2.$$

Proof. The function f cannot have Baker nor rotation domains (see [Bw]). Therefore one can find a disc $D \subset \mathcal{F}_f$ arbitrarily close to ∞ such that all the inverse images $f^{-n}(D)$, $n \geq 0$, are disjoint. It follows then from Koebe's distortion theorem that

$$\mathcal{P}(2) = \sum_{n>0} \sum_{z\in f^{-n}(\infty)} |(f^n)'(z)|_\sigma^{-2} \asymp \sum_{n>0} meas(f^{-n}(D)) < \infty,$$

where $meas$ stands for the spherical Lebesgue measure. The fact that $h > \frac{\rho}{\rho+1}$ is an immediate consequence of [My] together with the divergence property of f given in Theorem 3.2 of [MU2]. □

The Patterson-Sullivan method as described in [McM] applies now and furnishes a h–conformal measure say m. Using this measure we can make the following important improvement of the above estimate for h_f.

Proposition 3.2. *For a sub-hyperbolic meromorphic function f of polynomial Schwarzian derivative with $\mathcal{F}_f \neq \emptyset$ we have that*

$$2\frac{\rho}{\rho+1} < h_f.$$

Proof. Let $a \in \mathcal{A}_f$ and let $a' = f(a)$. The function f is expanding on $\mathcal{P}_f$: there is $p > 0$ such that $|(f^p)'|_\sigma > 2$ on $B(\mathcal{P}_f, T)$. Let again $g = f^p_{\Omega_1}$ (see 2.8). Notice that $g(D(g^n(a'), T)) \supset D(g^{n+1}(a'), T)$. Denote

$$A_n = D(g^n(a'), T) \setminus g^{-1}_{g^n(a')}\Big(D(g^{n+1}(a'), T)\Big). \tag{3.3}$$

Increasing p if necessary we have from the fact that $\mathcal{P}_f$ is compact and nowhere dense in J_f that

$$\inf_{n\geq 0} m(A_n) \geq c > 0.$$

If $V_n = f_a^{-1} \circ g_{a'}^{-n}(A_n)$ then

$$m(V_n) \asymp \Big(|f'(a)|_\sigma |(g^n)'(a')|_\sigma\Big)^{-h} m(A_n) \asymp diam(V_n)^h \quad , \quad n \geq 1. \tag{3.4}$$

The preimages of V_n to a logarithmic tract U over a can be labeled by $U_{n,k} = f_k^{-1}(A_n)$. Let $z_{n,k} \in U_{n,k}$ be any point. Then

$$\begin{aligned} m(U_{n,k}) &\asymp |f'(z_{n,k})|_\sigma^{-h} diam(V_n)^h \\ &\asymp |z_{n,k}|^{-(\rho+1)h} |f(z_{n,k}) - a|^{-h} diam(V_n)^h \asymp |z_{n,k}|^{-(\rho+1)h}, \end{aligned} \tag{3.5}$$

where the relation

$$|z_{n,k}| \asymp (n^2 + k^2)^{\frac{1}{2\rho}} \tag{3.6}$$

follows from an elementary calculation based on (2.5) and (2.6). Hence,

$$1 \geq m(U) = \sum_{n,k} m(U_{n,k}) \asymp \sum_{n,k} |z_{n,k}|^{-(\rho+1)h} \asymp \sum_{n,k} \left(n^2 + k^2\right)^{-\frac{\rho+1}{2\rho}h} \tag{3.7}$$

The assertion of the Lemma follows now since this last sum is convergent if and only if $\frac{\rho+1}{2\rho}h > 1$. □

As a first application of this estimate on h_f we can now prove that the measure m does not charge infinity.

Lemma 3.3. *For the h–conformal measure m obtained from the Patterson-Sullivan construction we have $m(\{\infty\}) = 0$.*

Proof. The measure m is obtained in the following way. If the Poincaré series $\mathcal{P}(t)$ diverges for $t = h$, then m is a limit of measures of the form

$$\nu_s = \frac{1}{\mathcal{P}(s)} \sum_n \sum_{z \in f^{-n}(\infty)} |(f^n)'(z)|_\sigma^{-s} \delta_z = \frac{1}{\mathcal{P}(s)} \sum_n (\mathcal{L}_s^n)^* \delta_\infty .$$

If ever $\mathcal{P}(h) < \infty$ then one must add artificially a divergence behavior. A simple way of doing this is to follow the exposition of [McM] and to replace the exponent $s = h + \delta$ by $s' = h - \delta$ in a finite number (depending on s) of terms in this expression. To take into account these modifications we write

$$\nu_s = \frac{1}{\tilde{\mathcal{P}}(s)} \sum_n (\tilde{\mathcal{L}}_s^n)^* \delta_\infty .$$

Notice that $\delta \to 0$. We can therefore suppose that $s' = h - \delta \geq \tau > 2\frac{\rho}{\rho+1}$. For every $E \subset \mathbb{C}$ for which $f_{|E}$ is injective one has

$$\int_E \min\left\{|f'|_\sigma^s, |f'|_\sigma^{s'}\right\} d\nu_s \leq \nu_s(f(E)) \leq \int_E \max\left\{|f'|_\sigma^s, |f'|_\sigma^{s'}\right\} d\nu_s \tag{3.8}$$

(see [McM] for details). By classical arguments (that the reader can find in [DU1]) it follows that

$$\nu_s(E) \leq \int_{\mathbb{C}} \sum_{z \in f^{-1}(w)} \max\left\{|f'(z)|_\sigma^{-s}, |f'(z)|_\sigma^{-s'}\right\} 1\!\!1_E(z)\, d\nu_s(w) \tag{3.9}$$

for every mesurable set $E \subset \mathbb{C}$.

By definition $\nu_s(\{\infty\}) = 0$. We have to show that the sequence $(\nu_s)_s$ is tight at infinity which means that for every $\varepsilon > 0$ there is $R > 0$ such that $\nu_s(W_R) < \varepsilon$ for every $s > h$ where $W_R = \{|z| > R\}$. This set can be written as

$$W_R = \bigcup_{a \in \mathcal{A}_f} (U_a \cap W_R) \cup \tilde{W}_R$$

where the union is taken over all the (finitely many) logarithmic tracts $U_a \subset f^{-1}(D(a, T)$ over the asymptotic values $a \in \mathcal{A}_f$.

Tightness on $\tilde{W}_R$ can be shown like in [MU1]. The key point is the following. If $z \in \tilde{W}_R$ then $w = f(z) \notin B(\mathcal{A}_f, T)$. Therefore, if $\gamma > 0$ is any small number such that $(\rho+1)\tau - \gamma > \rho$, then it follows from (3.4) and from Proposition 2.11 that

$$\begin{aligned} \nu_s(\tilde{W}_R) &\preceq \int_{\mathbb{C}} \sum_{z \in f^{-1}(w)} 1\!\!1_{\tilde{W}_R}(z) |z|^{-(\rho+1)\tau} d\nu_s(w) \\ &\leq \frac{1}{R^\gamma} \int_{\mathbb{C}} \sum_{z \in f^{-1}(w)} (1 + |z|)^{\gamma - (\rho+1)\tau} d\nu_s(w) \leq \frac{M}{R^\gamma}. \end{aligned}$$

Let us now consider what happens on a logarithmic tract $U = U_a \subset f^{-1}(D(a, T))$ over $a \in \mathcal{A}_f$. With the notations of the proof of Proposition 3.2 and with the same arguments, one has $\nu_s(V_n) \preceq diam(V_n)^\tau$ and

$$\nu_s(U_{n,k}) \preceq |f'(z_{n,k})|_\sigma^{-\tau} diam(V_n)^\tau \asymp |z_{n,k}|^{-(\rho+1)\tau} \asymp \left(n^2 + k^2\right)^{-\frac{\rho+1}{2\rho}\tau} .$$

Notice that these estimates do not depend on $s > h$ and imply that

$$\nu_s(U) = \sum_{n,k} \nu_s(U_{n,k}) \preceq \sum_{n,k} \left(n^2 + k^2\right)^{-\frac{\rho+1}{2\rho}\tau} \leq c < \infty$$

since $\tau > 2\frac{\rho}{\rho+1}$. Therefore $\nu_s(U \cap W_R) \to 0$ as $R \to \infty$ uniformly in $s > h$. □

3.2. **Additional properties.** Recall the definition of the annuli Γ_n are given in (2.8). We start with the following.

Lemma 3.4. *There exists $0 < \gamma < 1$ such that $m(\Gamma_n) \preceq \gamma^n$ for every $n \geq 0$.*

Proof. Let $\mathcal{P}_f \subset \bigcup_{j=1}^N D_j$ where the discs $D_j = D(x_j, 2T)$, $x_j \in \mathcal{P}_f$, built a Besicovitch covering of $\mathcal{P}_f$. We may suppose that $\Omega_0 \subset \bigcup_{j=1}^N D_j$. Fix $q \geq 1$ such that for every $j = 1, ..., N$

$$m(\Gamma_q \cap D_j) \leq \eta\, m(\Gamma_0 \cap D_j)$$

with $\eta > 0$ some small number to be determined later on. Remember that $g = f^p_{|\Omega_1}$. Clearly all the inverse branches of g^n are well defined and of bounded distortion on every disc D_j. Let us denote these by g_*^{-n}. With this notation we can calculate, for every $n \geq 1$, that

$$\begin{aligned} m(g^{-n}(D_j \cap \Gamma_q)) &= \sum_* m(g_*^{-n}(D_j \cap \Gamma_q)) = \sum_* \frac{m(g_*^{-n}(D_j \cap \Gamma_q))}{m(g_*^{-n}(D_j \cap \Gamma_0))} m(g_*^{-n}(D_j \cap \Gamma_0)) \\ &\preceq \sum_* \frac{m(D_j \cap \Gamma_q)}{m(D_j \cap \Gamma_0)} m(g_*^{-n}(D_j \cap \Gamma_0)) \leq \eta \sum_* m(g_*^{-n}(D_j \cap \Gamma_0)) \\ &= \eta\, m(g^{-n}(D_j \cap \Gamma_0)). \end{aligned}$$

Summing over j and using the Besicovitch property of the covering we get that

$$m(\Gamma_{q+n}) \leq C\eta\, m(\Gamma_n) \quad for\;\; every\; n \geq 0.$$

The assertion follows provided η has been chosen such that $C\eta < 1/2$. □

In the rest of this section we denote ν any h–conformal measure (and keep the letter m for the conformal measure that has been constructed above). Note that for any Borel probability measure ν on a compact metric space (X, ρ),

$$M_\nu(r) := \inf\{\nu(B(x, r) : x \in \mathrm{supp}(\nu)\} > 0$$

for every $r > 0$. Let us also prove the following.

Lemma 3.5. *For any h–conformal measure ν we have $\nu(\mathcal{P}_f) = 0$.*

Proof. Recall that one condition imposen on T was that for every $z \in \mathcal{P}_f$ and every $n \geq 0$, there exists a holomorphic inverse branch $f_z^{-n} : D(f^n(z), 2T) \to \hat{\mathbb{C}}$ of f^n sending $f^n(z)$ to z. It then follows from the bounded distortion property (Lemma 2.1) that

$$\begin{aligned} \nu(D(z, K^{-1}T|(f^n)'(z)|_\sigma^{-1})) &\leq \nu\big(f_z^{-n}(D(f^n(z), T))\big) \\ &\preceq |(f^n)'(z)|_\sigma^{-h} \nu(D(f^n(z), T)) \leq |(f^n)'(z)|_\sigma^{-h}. \end{aligned} \tag{3.10}$$

Since $\mathcal{P}_f$ is a nowhere dense subset of J_f, there exists $\gamma > 0$ such that for every $y \in \mathcal{P}_f$ there exists $\hat{y} \in J_f$ such that

$$D_y := D(\hat{y}, \gamma) \subset D(y, K^{-2}T) \setminus \mathcal{P}_f.$$

Then

$$f_z^{-n}(D_{f^n(z)}) \subset f_z^{-n}(D(f^n(z), K^{-2}T)) \setminus \mathcal{P}_f \subset D(z, K^{-1}T|(f^n)'(z)|_\sigma^{-1}) \setminus \mathcal{P}_f \tag{3.11}$$

and

$$\nu\big(f_z^{-n}(D_{f^n(z)})\big) \succeq |(f^n)'(z)|_\sigma^{-h}\nu(D_{f^n(z)}) \geq M_\nu(\gamma)|(f^n)'(z)|_\sigma^{-h}.$$

Combining this, (3.10), (3.11), and noting that $\mathrm{supp}(\nu) = J_f$, we get that

$$\frac{\nu\big(D(z, K^{-1}T|(f^n)'(z)|_\sigma^{-1}) \setminus \mathcal{P}_f\big)}{\nu\big(D(z, K^{-1}T|(f^n)'(z)|_\sigma^{-1})\big)} \succeq M_\nu(\gamma) \quad \text{for every} \quad n \geq 1.$$

Therefore,

$$\limsup_{r\to 0} \frac{\nu(D(z,r) \setminus \mathcal{P}_f)}{\nu(D(z,r))} \succeq M_\nu(\gamma) > 0.$$

So, z is not a Lebesgue density point of ν, and therefore $\nu(\mathcal{P}_f) = 0$. □

3.3. **Metric exactness, conservativity and ergodicity.** Suppose that $(X, \mathcal{F}, \nu)$ is a probability space and $T : X \to X$ is a measurable map such that $T(A) \in \mathcal{F}$ whenever $A \in \mathcal{F}$. The map $T : X \to X$ is said to be *weakly metrically exact* provided that $\overline{\lim}_{n\to\infty}\nu(T^n(A)) = 1$ whenever $A \in \mathcal{F}$ and $\nu(A) > 0$. A straightforward observation concerning weak metrical exactness is this.

Observation 3.6. *If a measurable transformation $T : X \to X$ of a probability space $(X, \mathcal{F}, \nu)$ is weakly metrically exact, then it is ergodic and conservative.*

In the context of invariant measures there is the following, more involved fact, also indicating a dynamical significance of weak metrical exactness (see e.g. [CFS, PU]).

Fact 3.7. *A measure-preserving transformation $T : X \to X$ of a probability space $(X, \mathcal{F}, \mu)$ is weakly metrically exact if and only if it is exact, which means that $\lim_{n\to\infty} \mu(T^n(A)) = 1$ whenever $A \in \mathcal{F}$ and $\mu(A) > 0$, or equivalently, the σ-algebra $\bigcap_{n\geq 0} T^{-n}(\mathcal{F})$ consists of sets of measure 0 and 1 only. Then the Rokhlin's natural extension $(\tilde{T}, \tilde{X}, \tilde{\mu})$ of (T, X, μ) is K-mixing.*

The main result of this subsection is this.

Theorem 3.8. *m is a unique h-conformal measure. The dynamical system $f : J_f \to J_f$ is weakly metrically exact with respect to m. In particular it is ergodic and conservative.*

Proof. Let

$$\mathcal{P}_f^* = \{z \in J_f : \mathrm{dist}_\sigma(z, \mathcal{A}_f \cup \mathcal{P}_f) > 2T\}.$$

By Observation 2.4,

$$J_f^* = \{z \in J_f \setminus O^-(\infty) : \omega(z) \cap \mathcal{P}_f^* \neq \emptyset\} = J_f \setminus \bigcup_{n=0}^{\infty} f^{-n}(\mathcal{P}_f \cup \{\infty\}). \tag{3.12}$$

Take $z \in J_f^*$. Then there exists a strictly increasing sequence $(n_j = n_j(z))_{j=1}^\infty$ of positive integers such that

$$f^{n_j}(z) \in \mathcal{P}_f^* \setminus \{\infty\}$$

for all $j \geq 1$. Then for every $j \geq 1$, there exists a meromorphic inverse branch $f_z^{-n_j} : D(f^{n_j}(z), 2T) \to \hat{\mathbb{C}}$ of f^{n_j} sending $f^{n_j}(z)$ to z. It then follows from Lemma 2.1 (bounded distortion property) that for every h-conformal measure ν on J_f,

$$\begin{aligned}\nu(D(z,K^{-1}T|(f^{n_j})'(z)|_\sigma^{-1})) &\le \nu\big(f_z^{-n_j}(D(f^{n_j}(z),T))\big)\\ &\preceq |(f^{n_j})'(z)|_\sigma^{-h}\nu(D(f^{n_j}(z),T)) \le |(f^{n_j})'(z)|_\sigma^{-h}.\end{aligned} \tag{3.13}$$

Put

$$r_j(z) = (4K)^{-1}T|(f^{n_j})'(z)|_\sigma^{-1}.$$

The above formula rewrites then as follows.

$$\nu(D(z,4r_j(z))) \preceq r_j^h(z). \tag{3.14}$$

It also follows from Lemma 2.1 that

$$\begin{aligned}\nu(D(z,r_j(z))) &\ge \nu\big(f_z^{-n_j}(D(f^{n_j}(z),(4K^2)^{-1}T))\big)\\ &\succeq |(f^{n_j})'(z)|_\sigma^{-h}\nu(D(f^{n_j}(z),4K^2)^{-1}T))\\ &\ge M_\nu((4K^2)^{-1}T)|(f^{n_j})'(z)|_\sigma^{-h}\\ &\asymp r_j^h(z).\end{aligned} \tag{3.15}$$

Now fix E, an arbitrary Borel set contained in J_f^*. Fix also $\varepsilon > 0$. Since the measure m is regular, for every $z \in E$ there exists $j(z) \ge 1$ such that, with $r(z) = r_{j(z)}(z)$, we will have

$$m\left(\bigcup_{z\in E} D(z,r(z))\right) \le m(E) + \varepsilon. \tag{3.16}$$

By the $(4r)$-covering theorem there exists now a countable set $\hat{E} \subset E$ such that the balls $\{D(z,r(z))\}_{z\in\hat{E}}$ are mutually disjoint and

$$\bigcup_{z\in\hat{E}} D(z,4r(z)) \supset \bigcup_{z\in E} D(z,r(z)) \supset E.$$

Hence, using (3.14), (3.15) (with ν replaced by m) and (3.16), we get

$$\begin{aligned}\nu(E) &\le \sum_{z\in\hat{E}} \nu(D(z,4r(z))) \le (4K^2/T)^h \sum_{z\in\hat{E}} r^h(z)\\ &\le K^{2h}M_m((4K^2)^{-1}T)\sum_{z\in\hat{E}} m(D(z,r(z)))\\ &\asymp m\left(\bigcup_{z\in\hat{E}} D(z,r(z)\right) \le m(E)+\varepsilon.\end{aligned}$$

Thus, letting $\varepsilon \searrow 0$, we get $\nu(E) \preceq m(E)$. Hence, $\nu_{|J_f^*}$ is absolutely continuous with respect to $m_{|J_f^*}$. Exchanging the roles of ν and m we get $m_{|J_f^*} \preceq \nu_{|J_f^*}$, and finally that $\nu_{|J_f^*}$ is equivalent to $m_{|J_f^*}$. Since, in view of Lemma 3.5,

$$m\left(\bigcup_{n=0}^\infty f^{-n}(\mathcal{P}_f)\right) = \nu\left(\bigcup_{n=0}^\infty f^{-n}(\mathcal{P}_f)\right) = 0,$$

we thus conclude that ν and m are equivalent on $J_f \setminus O^-(\infty)$, $O^-(\infty) = \bigcup_{n=0}^\infty f^{-n}(\{\infty\})$. Finally, if $\nu\left(O^-(\infty)\right) > 0$ then $\nu^* = \nu_{|O^-(\infty)}$ would be a conformal measure without mass on $J_f \setminus O^-(\infty)$. But then we would have a contradiction since we have just seen that m and ν^*

are equivalent on $J_f \setminus O^-(\infty)$. Therefore $\nu(O^-(\infty)) = 0$ and both measures are equivalent on the whole Julia set.

Passing to the proof of weak metrical exactness of f with respect to the measure m, suppose that $E \subset J_f$ and

$$\limsup_{n\to\infty} \sup\{m(f^n(E) \cap D(y, K^{-2}T))/m(D(y, K^{-2}T)) : y \in \mathcal{P}_f^*\} = 1. \tag{3.17}$$

We shall show that

$$\limsup_{n\to\infty} m(f^n(E)) = 1. \tag{3.18}$$

In virtue of Observation 2.3 there exists $q \geq 0$ such that

$$f^q(D(y, K^{-2}T)) \supset \hat{\mathbb{C}} \setminus \mathcal{A}_f$$

for all $y \in J_f$. Clearly, by conformality of m, for every $\varepsilon > 0$ there then exists $\delta > 0$ such that if $y \in J_f$, $G \subset D(y, K^{-2}T)$, and $m(G)/m(D(y, K^{-2}T)) \geq 1 - \delta$, then $m(f^q(G)) \geq 1 - \varepsilon$. Combining this with (3.17) yields (3.18). In order to the weak metrical exactness of m, suppose by contrapositive that $E \subset J_f$ and $\limsup_{n\to\infty} m(f^n(E)) < 1$. By (3.17) and (3.18), this implies that

$$2\kappa := \liminf_{n\to\infty} \inf\left\{m(D(y, K^{-2}T \setminus f^n(E))/m(D(y, K^{-2}T)) : y \in \mathcal{P}_f^*\right\} > 0.$$

So, for all $n \geq 1$ large enough, say $n \geq p$,

$$\inf\left\{m(D(y, K^{-2}T \setminus f^n(E))/m(D(y, K^{-2}T)) : y \in \mathcal{P}_f^*\right\} \geq \kappa > 0.$$

Fix $z \in E \cap J_f^*$. We shall show that z is not a Lebesgue density point for the measure m. Let $n_j = n_j(z) \geq p$, $j \geq 1$, have the same meaning as in the first part of the proof. Then

$$f_z^{-n_j}\big(D(f^{n_j}(z), K^{-2}T) \setminus f^{n_j}(E)\big) \subset D(z, K^{-1}T|(f^{n_j})'(z)|_\sigma^{-1}) \setminus E \tag{3.19}$$

and

$$\begin{aligned} m\big(f_z^{-n_j}\big(D(f^{n_j}(z), K^{-2}T) \setminus f^{n_j}(E)\big)\big) \geq \\ &\geq K^{-h}|(f^{n_j})'(z)|_\sigma^{-h}) m\big(D(f^{n_j}(z), K^{-2}T) \setminus f^{n_j}(E)\big) \\ &\geq \kappa K^{-h}|(f^{n_j})'(z)|_\sigma^{-h}) m\big(D(f^{n_j}(z), K^{-2}T)\big) \\ &\geq \kappa K^{-h} M_m(K^{-2}T)|(f^{n_j})'(z)|_\sigma^{-h}). \end{aligned}$$

Combining this along with (3.19) and (3.13), we get that

$$\frac{m\big(D(z, K^{-1}T|(f^{n_j})'(z)|_\sigma^{-1}) \setminus E\big)}{m\big(D(z, K^{-1}T|(f^{n_j})'(z)|_\sigma^{-1})\big)} \geq \kappa K^{-2h} M_m(K^{-2}T).$$

Therefore,

$$\lim_{r\to 0} \frac{m(D(z,r) \setminus E)}{m(D(z,r))} \geq \kappa K^{-2h} M_m(K^{-2}T) > 0.$$

So, z is not a Lebesgue density point for m. Thus $m(E \cap J_f^*) = 0$. Since $m(J_f^*) = 1$ (see (Lemma 3.5 and (3.12), we finally get that $m(E) = 0$. The weak metrical exactness of f with respect to m is established. Ergodicity and conservativity follow from Observation 3.6. Since ν (introduced in the first part of the proof) is equivalent to m, the equality $\nu = m$ follows from ergodicity of m. We are done. ∎

4. Invariant Measures

We now consider $f : \mathbb{C} \to \hat{\mathbb{C}}$ a sub-hyperbolic meromorphic function f of polynomial Schwarzian derivative and investigate invariant measures equivalent to the conformal measure m obtained in the previous section. In particular we show in the course of this section Theorem 1.1.

4.1. **Existence of σ–finite invariant measures.** Since we already established conservativity of the conformal measure m we can use the method of M. Martens [M] (see also [KU] for a description of this method) in order to obtain the following.

Proposition 4.1. *Let f be a sub-hyperbolic meromorphic function f of polynomial Schwarzian derivative and let m be the conservative h–conformal measure of f with $m(\mathcal{P}_f) = m(\{\infty\}) = 0$. Then there exists μ a σ–finite invariant measure absolutely continuous with respect to m.*

Proof. Using a Whitney decomposition of $\mathbb{C} \setminus (\mathcal{A}_f \cup \mathcal{P}_f)$ it is easy to construct a countable partition $\{A_n;\, n \geq 0\}$ of $X = J_f \setminus (\{\infty\} \cup \mathcal{A}_f \cup \mathcal{P}_f)$ such that for every $n, m \geq 0$ there is $k \geq 0$ such that

$$m(f^{-k}(A_m) \cap A_n) > 0.$$

Since m has no mass on $J_f \setminus X$ and since m is conservative M. Martens result [M] applies and gives the σ–finite invariant measure absolutely continuous with respect to m. Notice that for every Borel set $A \subset X$ we have that

$$\mu(A) = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} m(f^{-k}(A))}{\sum_{k=0}^{n} m(f^{-k}(A_0))}. \tag{4.1}$$

For the choice of the set A_0 there is much freedom. We will use in particular that $A_0 \subset X$ is such that all the inverse of the iterates of f are well defined and have bounded distortion. □

Let $\Delta = \hat{\mathbb{C}} \setminus B(\mathcal{A}_f \cup \mathcal{P}_f, T)$.

Lemma 4.2. *There is $K > 1$ such that $1/K \leq \varphi < K$ on Δ where $\varphi = d\mu/dm$.*

Proof. Let $z \in \Delta$. From the expression (4.1) follows that

$$\varphi(z) = \lim_{r\to 0} \frac{\mu(D(z,r))}{m(D(z,r))} \asymp \lim_{r\to 0} \frac{1}{m(D(z,r))} \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \mathcal{L}_h^k 1\!\!1(z) m(D(z,r))}{\sum_{k=0}^{n} \mathcal{L}_h^k 1\!\!1(z_0) m(A_0)}$$

where $z_0 \in A_0$ is any point. Now, if $z_1, z_2 \in \Delta$ are any two points, then they can be joined by a chain of at most $N = N(\Delta)$ spherical discs of radius T. On each of these discs all the inverse branches of every iterate of f is well defined and have distortion bounded by some universal constant. Therefore

$$\mathcal{L}_h^k 1\!\!1(z) \asymp \mathcal{L}_h^k 1\!\!1(z_0) \quad \textit{for every } k \geq 0.$$

The Lemma is proven. □

This simple observation on the density h has several important applications starting with the following.

Proposition 4.3. $\mu(B(\mathcal{A}_f, T)) < \infty$.

Proof. It suffices to show that $\mu(D(a,T)) < \infty$, $a \in \mathcal{A}_f$. The measure μ being invariant, $\mu(D(a,T)) = \mu(f^{-1}(D(a,T)))$. By the choice of the constant $T > 0$ (see (T_3)), $f^{-1}(D(a,T)) \cap B(\mathcal{A}_f \cup \mathcal{P}_f, T) = \emptyset$. It therefore follows from Lemma 4.2 that

$$\mu(D(a,T)) = \mu(f^{-1}(D(a,T))) \asymp m(f^{-1}(D(a,T)) < \infty.$$

□

4.2. **When is the σ–finite invariant measure finite?** To our big surprise it turns out that finiteness of the invariant measure μ does depend on the order of the function.

Theorem 4.4. *Let f be a sub-hyperbolic meromorphic function of polynomial Schwarzian derivative and let m be the (unique) h_f–conformal measure of f. Then there is a finite f–invariant measure μ absolutely continuous with respect to m if and only if $h > 3\frac{\rho}{\rho+1}$.*

Consequently the invariant measure μ is finite in the particular case of the tangent family and also for the examples of (2.2) that involve the Airy functions. Notice that $3\frac{\rho}{\rho+1} \geq 2$ as soon as the order $\rho = deg(P)/2 + 1 \geq 2$.

Corollary 4.5. *A sub-hyperbolic solution f of the polynomial Schwarzian equation $S(f) = 2P$ has finite invariant measure absolutely continuous with respect to the h_f–conformal measure if and only if $deg(P) = 0$ or $deg(P) = 1$.*

We now prove Theorem 4.4 in several steps and use again the notations given in (2.8). Let us consider in the following f a sub-hyperbolic meromorphic function of polynomial Schwarzian derivative and let m be again the h_f–conformal measure of f.

Lemma 4.6. *If $h_f \leq 3\frac{\rho}{\rho+1}$ then there is no finite invariant measure absolutely continuous with respect to m.*

Proof. Suppose to the contrary that such a finite invariant measure μ exists. Remember that $g = f^p_{|\Omega_1} : \Omega_1 \to \Omega_0$. Let $a \in \mathcal{A}_f$, set $a'' = f^p(a)$ and $D'' = D(a'', T)$. By invariance of μ we have that

$$\mu(\Omega_n) = \mu(f^{-p}(\Omega_n)) \geq \mu(f_a^{-p}(\Omega_n \cap D'')) + \mu(\Omega_{n+1}) \ , \ n \geq 0.$$

If we denote $W_n = f_a^{-p}(\Omega_n \cap D'')$ then we get inductively that

$$\mu(\Omega_0) \geq \sum_{n \geq 0} \mu(W_n).$$

Since $|(f^p)'|_\sigma$ is bounded on a $B(\{a\} \cup \mathcal{P}_f, 2T)$, there exists $L > 1$ such that $W_n \supset D(a, L^{-n})$ for every $n \geq 1$. Therefore

$$\mu(\Omega_0) \geq \sum_{n \geq 0} \mu(D(a, L^{-n})) \geq \sum_{n \geq 0} \mu(f^{-1}(D(a, L^{-n})) \cap U_a)$$

with U_a a logarithmic tract over the asymptotic value a. But on U_a μ is equivalent to the conformal measure m (Lemma 4.2) and, with the same calculations that lead to (3.7), we get that

$$\sum_{n \geq 0} m(f^{-1}(D(a, L^{-n})) \cap U_a) \asymp \sum_{n \geq 0} \left(\sum_{N \geq n} \sum_{k} \left(N^2 + k^2\right)^{-\frac{\rho+1}{2\rho}h} \right) \tag{4.2}$$

which is finite if and only if $h > 3\frac{\rho}{\rho+1}$. □

It remains to investigate the case $h > 3\frac{\rho}{\rho+1}$. In order to do so we write

$$f^{-p}(\Gamma_n) = \Gamma_{n+1} \cup W_n \cup S_n \tag{4.3}$$

where

$$W_n = \bigcup_{a\in\mathcal{A}_f} W_n^a \quad with \quad W_n^a = f_a^{-p}(D_a'' \cap \Gamma_n), \;\; D_a'' = D(f^p(a), T)$$

and where S_n is the remaining set. The measure μ being f–invariant, the sequence $(\mu(\Gamma_n))_n$ is decreasing. We need the following additional property.

Lemma 4.7. *For the σ–finite invariant measure μ we have that* $\lim_{n\to\infty}\mu(\Gamma_n) = 0$*.*

Proof. Let $l = \lim_{n\to\infty}\mu(\Gamma_n)$. From (4.3) follows inductively that

$$\mu(\Gamma_0) = l + \sum_{n=0}^{\infty}(\mu(W_n) + \mu(S_n)).$$

It is therefore natural to consider the set $B = \bigcup_{n=0}^{\infty}(W_n \cup S_n)$. Define $\Gamma_\infty = \Gamma_0 \cup B$ and let f_∞ be the induced map, i.e. the first return map, of f^p on the set Γ_∞. Since μ is conservative, the conditional measure $\mu_\infty = \frac{\mu}{\mu(\Gamma_\infty)}$ is f_∞ invariant (see [Aa]). Hence

$$\mu_\infty(B) = \mu_\infty(f_\infty^{-1}(\Gamma_0)) = \mu_\infty(\Gamma_0)$$

which implies that $\mu(B) = \mu(\Gamma_0)$. But this is only possible if $l = 0$. □

The last step of the proof of Theorem 4.4 is the following.

Lemma 4.8. *If $h_f > 3\frac{\rho}{\rho+1}$ then the measure μ is finite.*

Proof. We have to show that $\mu(\Omega_0) < \infty$. Since $\lim_{n\to\infty}\mu(\Gamma_n) = 0$ it follows from induction that

$$\mu(\Omega_0) = \sum_{N=0}^{\infty}\mu(\Gamma_N) = \sum_{N=0}^{\infty}\left(\sum_{n=N}^{\infty}\mu(W_n) + \mu(S_n)\right).$$

Let us first consider the term corresponding to S_n.

Choose again a Besicovitch covering of Ω_0 by discs $D_j = D(x_j, 2T)$, $x_j \in \mathcal{P}_f$. Let D be one of these discs and denote by f_*^{-p} the inverse branches of f^p defined on D such that

$$S_n = \bigcup_{D\in\{D_j\}}\bigcup_{*} f_*^{-p}(\Gamma_n \cap D) \quad for \;\; every \;\; n \geq 0.$$

Since there is $c > 0$ for which the sets $S_n \subset \Delta = \hat{\mathbb{C}} \setminus B(\mathcal{A}_f \cup \mathcal{P}_f, cT)$ we have $\mu(S_n) \asymp m(S_n)$ (Lemma 4.2). Therefore we can do the following estimation.

$$\mu\left(\bigcup_{*} f_*^{-p}(\Gamma_n \cap D)\right) \asymp \sum_{*} m(f_*^{-p}(\Gamma_n \cap D)) \asymp \sum_{*} |(f^p)'(z_*)|_\sigma^{-h} m(\Gamma_n \cap D)$$

where, for every $*$, z_* is any fixed point in $f_*^{-p}(D)$. Since $D\cap B(\mathcal{A}_f,T)=\emptyset$ it follows from Lemma 2.5 together with Proposition 2.6 that

$$\mu\left(\bigcup_* f_*^{-p}(\Gamma_n\cap D)\right)\preceq\sum_*(1+|z|^{\rho+1})^{-h}m(\Gamma_n\cap D)\preceq m(\Gamma_n\cap D).$$

Summing now over the discs of the Besicovitch covering and using the exponential decay of the m-mass of the sets Γ_n given in Lemma 3.4 we finally get

$$\mu(S_n)\preceq m(\Gamma_n)\preceq\gamma^n,$$

and thus

$$\sum_{N=0}^\infty\sum_{n\ge N}^\infty\mu(S_n)<\infty.$$

It suffices now to obtain the corresponding statements for the sets W_n. Notice again that $\mu(W_n)=\mu(f^{-1}(W_n))\asymp m(f^{-1}(W_n))$. The set $f^{-1}(W_n)$ contains a subset that lies in parabolic tracts and a remaining set say S_n'. The m–mass of the later can be estimated exactly like we just did for S_n. It therefore suffices to see what happens in just one tract U_a and to estimate the mass of $U_a\cap f^{-1}(W_n)$. Clearly there is $c>0$ such that $W_n\subset D(a,c2^{-n})$. We therefore can conclude precisely like in (4.2) that

$$\sum_{N=0}^\infty\sum_{n\ge N}^\infty\mu(W_n)<\infty$$

if and only if $h>3\frac{\rho}{\rho+1}$. □

5. Bowen's formula, Hausdorff dimension and Hausdorff measures

We start with the following fact concerning the h-dimensional Hausdorff measure $\mathcal{H}^h$ on J_f.

Proposition 5.1. *If $h<2$, then h-dimensional Hausdorff measure of J_f vanishes, $\mathcal{H}^h(J_f)=0$. If $h=2$, then $J_f=\hat{\mathbb{C}}$. In either case $\mathrm{HD}(J_f)\le h$.*

Proof. Fix an arbitrary $z\in J_f\setminus\bigcup_{n=0}^\infty f^{-n}(\mathcal{A}_f\cup\{\infty\})$. Then there exists an increasing unbounded sequence $(n_j)_{j=1}^\infty$ such that for every $j\ge1$ there exists a meromorphic inverse branch $f_z^{-n_j}:D(f^{n_j}(z),2T)\to\hat{\mathbb{C}}$ sending $f^{n_j}(z)$ to z. Then $f_z^{-n_j}\big(D(f^{n_j}(z),T)\big)\subset D\big(z,KT|(f^{n_j})'(z)|^{-1}\big)$, and therefore

$$\begin{aligned}m\big(D\big(z,KT|(f^{n_j})'(z)|_\sigma^{-1}\big)\big)&\ge K^{-h}|(f^{n_j})'(z)|_\sigma^{-h}m\big(D(f^{n_j}(z),T)\big)\\&\ge M_m(T)(K^2T)^{-h}\big(KT|(f^{n_j})'(z)|_\sigma^{-1}\big)^h.\end{aligned}$$

Hence,

$$\limsup_{r\to0}\frac{m(D(z,r))}{r^h}\ge\liminf_{j\to\infty}\frac{m\big(D\big(z,KT|(f^{n_j})'(z)|_\sigma^{-1}\big)\big)}{\big(KT|(f^{n_j})'(z)|_\sigma^{-1}\big)^h}\ge M_m(T)(K^2T)^{-h}>0.$$

Thus

$$\mathcal{H}^h|_{J_f}\le Cm\tag{5.1}$$

with some universal constant $C > 0$. Proceeding further, suppose first that $h < 2$. Recall that $W_R = \{z \in \hat{\mathbb{C}} : |z| > R\}$. It follows from (3.5) that

$$m(W_R) \succeq R^{2\rho-(\rho+1)h}. \tag{5.2}$$

Due to conservativity and ergodicity of the measure m, there exists a Borel set $Y \subset J_f \setminus \bigcup_{n\geq 0} f^{-n}(\infty)$ such that $m(Y) = 1$ and $\infty \in \omega(z)$ for all $z \in Y$. Fix one $z \in Y$. There then exists an unbounded increasing sequence $(n_j)_1^\infty$ such that

$$\lim_{j\to\infty} |(f^{n_j})(z)| = +\infty \quad \text{and} \quad |(f^{n_j})(z)| \geq 4T^{-1} \tag{5.3}$$

for all $j \geq 1$. So there exist meromorphic inverse branches $f_z^{-n_j} : W_{|(f^{n_j}(z)|} \to \hat{\mathbb{C}}$ sending $f^{n_j}(z)$ to z. Put $r_j = 2K|(f^{n_j})'(z)|_\sigma^{-1}|f^{n_j}(z)|^{-1}$. Looking at (5.2), we get

$$\begin{aligned}
\frac{m(D(z,r_j))}{r_j^h} &\geq \frac{m\big(f_z^{-n_j}\big(D(f^{n_j})(z), 2|(f^{n_j})(z)|^{-1})\big)\big)}{r_j^h} \\
&\geq \frac{K^{-h}|(f^{n_j})'(z)|_\sigma^{-h} m\big(D(f^{n_j})(z), 2|(f^{n_j})(z)|^{-1})\big)}{K^h|(f^{n_j})'(z)|_\sigma^{-h}|(f^{n_j})(z)|^{-h}} \\
&\succeq K^{-2h}|(f^{n_j})(z)|^h m\big(W_{|(f^{n_j})(z)|}\big) \geq K^{-2h}|(f^{n_j})(z)|^h|(f^{n_j})(z)|^{2\rho-(\rho+1)h} \\
&= K^{-2h}|(f^{n_j})(z)|^{\rho(2-h)}.
\end{aligned}$$

Since $2 - h > 0$, we therefore conclude from this and (5.3) that

$$\limsup_{r\to 0} \frac{m(D(z,r))}{r^h} \geq \lim_{j\to\infty} \frac{m(D(z,r_j))}{r_j^h} \geq \lim_{j\to\infty} K^{-2h}|(f^{n_j})(z)|^{\rho(2-h)} = +\infty.$$

Thus, $\mathcal{H}^h(Y) = 0$. Since $\mathcal{H}^h(J_f \setminus Y) = 0$ by (5.1), we thus have $\mathcal{H}^h(J_f) = 0$. The case $h < 2$ is done. If $h = 2$, then for the sequence $(n_j)_1^\infty$ from the beginning of the proof, we will have $m(D(z,r_j)) \asymp r_j^2$, which implies that m and l_s, the spherical Lebesgue measure on $\hat{\mathbb{C}}$, are equivalent. So, $l_s(J_f) > 0$. Now, if $J_f \neq \hat{\mathbb{C}}$, then J_f would be nowhere dense in $\hat{\mathbb{C}}$, and in the same way as Lemma 3.5, making use of the Lebesgue Density Theorem, we would prove that $l_s(J_f) = 0$. This contradiction finishes the proof. □

Although $\mathcal{H}^h(J_f) = 0$ (if $h < 2$), we shall however show that $h = \mathrm{HD}(J_f)$. The proof will utilize the induced (first return) map we are going to describe now. Let

$$X = J_f \setminus \big(B(\mathcal{P}_f, T) \cup \bigcup_{a\in\mathcal{A}_f} f_a^{-1}(D(f(a), T))\big). \tag{5.4}$$

Let $f_* : X \to X$ be the first return map of f on X. That is

$$f_*(x) = f^{\tau(x)}(x),$$

where $\tau(x) = \min\{n \geq 1 : f^n(x) \in X\}$. Since $f : J_f \to J_f$ is conservative with respect to the measure μ (see Theorem 3.8), the map f_* is well-defined on the complement of a set of μ measure zere, in fact, as it is easy to see, it is well-defined on the complement of $\bigcup_{n\geq 0} f^{-n}(\mathcal{P}_f)$, which is of measure zero by Lemma 3.5 and by formula (4.1). Since the Radon-Nikodym derivative $d\mu/dm$ is uniformly bounded above pn X, $\mu(X) < +\infty$. For every $x \in X$ define

$$f_*'(x) = \big(f^{\tau(x)}\big)'(x) \quad \text{and} \quad |f_*'(x)|_\sigma = \Big|\big(f^{\tau(x)}\big)'(x)\Big|_\sigma .$$

We shall prove the following.

Lemma 5.2. $\beta := \inf\{|f_*'(z)|_\sigma : z \in X\} > 0$ *and there exists* $k \geq 1$ *so large that* $|(f_*^k)'(z)|_\sigma \geq 2$ *for all* $z \in X$.

Proof. In the course of the proof of this lemma Q stands for an appropriately large positive constant.

Suppose first that $z \in X \cap U_a$ where U_a is a logarithmic tract over some $a \in \mathcal{A}_f$ such that $f(U_a) = f_a^{-1}(D(f(a), T))$. Let $n \geq 0$ the least integer such that $f^{n+1}(z) \notin D(\mathcal{P}_f, T)$. Then

$$\begin{aligned} |f_*'(z)|_\sigma &= |(f^{n+1})'(z)|_\sigma \geq Q^{-1}|f(z) - a|(1 + |z|^{\rho+1})|(f^n)'(z)|_\sigma \\ &\geq Q^{-2}(1 + |z|^{\rho+1}) \geq Q^{-2} \end{aligned} \tag{5.5}$$

and

$$|f_*'(z)|_\sigma \geq 2Q^2 \quad \text{if, in addition, } |z| \geq R. \tag{5.6}$$

For all other $z \in X$, Lemma 2.5 implies that

$$|f_*'(z)|_\sigma \geq Q^{-1} \geq Q^{-2}. \tag{5.7}$$

If in addition $|z| > R$ with $R > 0$ large enough, then

$$|f_*'(z)|_\sigma \geq 2Q^2. \tag{5.8}$$

The first part of our lemma is thus proved. We shall now demonstrate the following.

Claim: There exists $l = l(R) \geq 1$ such that

$$|(f_*^n)'(z)|_\sigma \geq 2Q^2$$

for all $n \geq l$ and all $z \in D(0, R) \cap X$.

Indeed, suppose for the contrary that there exist an increasing sequence $n_j \to \infty$ and a sequence $z_j \in X \cap \overline{D}(0, R)$ such that

$$|(f_*^{n_j})'(z_j)|_\sigma < 2Q^2 \tag{5.9}$$

for all $j \geq 1$. Since $f_*^{n_j}(z_j) \in X$, there exists a unique meromorphic inverse branch $f_{z_j}^{-N_j} : D(f^{n_j}(z_j, 2T) \to \hat{\mathbb{C}}$ of f^{N_j}, sending $f_*^{n_j}(z_j)$ to z_j, where $N_j = \tau(z_j) + \tau(f_*(z_j)) + \ldots + \tau(f_*^{n_j-1}(z_j))$. It then follows from (2.1) and (5.9) that

$$f_{z_j}^{-N_j}\big(D(f_*^{n_j}(z_j, T/2))\big) \subset D(z_j, (4KTQ^2)^{-1}),$$

or equivalently,

$$f^{N_j}\big(D(z_j, (4KTQ^2)^{-1})\big) \subset D\big(f_*^{n_j}(z_j), T/2\big).$$

Passing to a subsequence we may assume without loss of generality that the sequnce $(z_j)_1^\infty$ converges to a point $z \in J_f \cap \overline{D}(0, R)$ and $|z_j - z| < (8KTQ^2)^{-1}$ for all $j \geq 1$. Since $D(f_*^{n_j}(z_j, T/2) \cap B(\mathcal{P}_f, T/2) = \emptyset$, it follows from Montel's Theorem that the family $\big\{f^{N_j}|_{D(z,(8KTQ^2)^{-1})}\big\}_{j=1}^\infty$ is normal, contrary to the fact that $z \in J_f$. The claim is proved.

Let $k = 2l$. If $|f_*^j(z)| \geq R$ for all $j = 1, 2, \ldots, l$, then by (5.7) - (5.6), we get

$$|(f_*^k)'(z)|_\sigma \geq (2Q^2)^l Q^{-2l} = 2^l \geq 2.$$

If $|f_*^j(z)| < R$ for some $0 \leq j \leq l$, then let j be minimal with this property. It then follows from (5.8), (5.6) and the claim, that

$$|(f_*^k)'(z)|_\sigma = |(f_*^j)'(z)|_\sigma |(f_*^{k-j})'(f^j(z))|_\sigma \geq |(f_*^{k-j})'(f^j(z))|_\sigma \geq 2Q^2 \geq 2.$$

We are done. □

Now, we shall prove the following.

Lemma 5.3. *The function $z \mapsto \log|f_*'(z)|_\sigma$ is integrable on X with respect to μ_X, the conditional measure on X induced by μ. In addition $\chi := \int \log|f_*'|_\sigma d\mu_X > 0$.*

Proof. Since the Radon Nikodym derivative $d\mu/dm$ is uniformly bounded on X, it suffices to demonstrate that the function $z \mapsto \log|f_*'(z)|_\sigma$ is integrable on X with respect to the measure m ($\chi > 0$ follows immediately from Lemma 5.2). For every $a \in \mathcal{A}_f$ let $A_n(a)$, $n \geq 0$, be the annuli defined by formula (3.3). Put $A_n = \bigcup_{a\in\mathcal{A}_f} A_n(a)$. Partition $X \setminus f(A_0)$ by disjoint Borel sets X_n, $n \geq 0$, such that $D(x_n, 2\text{diam}(X_n)) \cap (\mathcal{A}_f \cup \mathcal{P}_f) = \emptyset$ with some $x_n \in X_n$. Then, because of Lemma 2.5 and Proposition 2.6, we get that

$$
\begin{aligned}
\int_{X\cap f^{-1}(X\setminus f(A_0))} &|\log|f_*'|_\sigma|\, dm = \\
&= \sum_{n=1}^\infty \int_{X\cap f^{-1}(X_n)} |\log|f_*'|_\sigma|\, dm \\
&\asymp \sum_{n=1}^\infty m(X_n) \sum_{z\in X\cap f^{-1}(w_n)} |f_*'(z)|_\sigma^{-h} \, |\log|f_*'(z)|_\sigma| \\
&\asymp \sum_{n=1}^\infty m(X_n) \sum_{z\in X\cap f^{-1}(w_n)} (1+|z|^{\rho+1})^{-h} \left|\log(1+|z|^{\rho+1}) + O(1)\right| \\
&\preceq \sum_{n=1}^\infty m(X_n) \sum_{z\in X\cap f^{-1}(w_n)} (1+|z|^{\rho+1})^{-t} \\
&\leq M_t \sum_{n=1}^\infty m(X_n) \leq M_t < +\infty,
\end{aligned}
\tag{5.10}
$$

where w_n is an arbitrary point in X_n and t is a fixed number in $(\frac{\rho}{\rho+1}, h)$. Now, following notation from Prposition 3.2, for every $a \in \mathcal{A}_f$ and every $n \geq 0$, set

$$\Gamma_a = f^{-1}(f(a)) \setminus (\mathcal{A}_f \cup \mathcal{P}_f),$$

$$Y_n(a) = \bigcup_{b\in\Gamma} f_b^{-1} \circ g_{f(a)}^{-n}(A_n(a)) \cup \bigcup_{b\in f^{-1}(a)} f_b^{-1} \circ f_a^{-1} \circ g_{f(a)}^{-n}(A_n(a))$$

and

$$Y_a = \bigcup_{n=0}^\infty Y_n(a).$$

Keep $t \in (\frac{\rho}{\rho+1}, h)$. Again, in virtue of Lemma 2.5 and Proposition 2.6, and also of Lemma 3.4, we get that

$$\begin{aligned}
&\int_{Y_a} |\log |f_*'|_\sigma|\, dm = \\
&\quad = \sum_{n=0}^{\infty} \int_{Y_n(a)} |\log |f_*'|_\sigma|\, dm \\
&\quad \asymp \sum_{n=0}^{\infty} \Bigg(\sum_{b\in\Gamma} m\big(f_b^{-1} \circ g_{f(a)}^{-n}(A_n(a))\big) \left|\log|f'(b)||(g^n)'(f(a))| + O(1)\right| + \\
&\qquad + \sum_{b\in f^{-1}(a)} m\big(f_b^{-1} \circ f_a^{-1} \circ g_{f(a)}^{-n}(A_n(a))\big) \left|\log|f'(b)||(g^n)'(f(a))| + O(1)\right| \Bigg) \\
&\quad \preceq \sum_{n=0}^{\infty} \sum_{b\in\Gamma} (1+|b|^{\rho+1})^{-h} \gamma^n \left|\log(1+|b|^{\rho+1}) + \log|(g^n)'(f(a))| + O(1)\right| + \\
&\qquad + \sum_{n=0}^{\infty} \sum_{b\in f^{-1}(a)} (1+|b|^{\rho+1})^{-h} \gamma^n \left|\log(1+|b|^{\rho+1}) + \log|(g^n)'(f(a))| + O(1)\right| \\
&\quad \le \sum_{n=0}^{\infty} \gamma^n \sum_{b\in\Gamma} (1+|b|^{\rho+1})^{-h} \gamma^n \left|\log(1+|b|^{\rho+1}) + O(n)\right| + \\
&\qquad + \sum_{n=0}^{\infty} \gamma^n \sum_{b\in f^{-1}(a)} (1+|b|^{\rho+1})^{-h} \gamma^n \left|\log(1+|b|^{\rho+1}) + O(n)\right| \\
&\quad \preceq \sum_{n=0}^{\infty} \gamma^n \sum_{b\in\Gamma\cup f^{-1}(a)} (1+|b|^{\rho+1})^{-h} \left|\log(1+|b|^{\rho+1})\right| + \sum_{n=0}^{\infty} n\gamma^n \sum_{b\in\Gamma\cup f^{-1}(a)} (1+|b|^{\rho+1})^{-h} \\
&\quad \preceq \sum_{n=0}^{\infty} \gamma^n \sum_{b\in\Gamma\cup f^{-1}(a)} (1+|b|^{\rho+1})^{-t} + M_h \sum_{n=0}^{\infty} n\gamma^n \\
&\quad < +\infty.
\end{aligned}$$

Hence,

$$\int_{\bigcup_{a\in\mathcal{A}_f} Y_a} |\log |f_*'|_\sigma|\, dm < +\infty. \tag{5.11}$$

Finally, for every $a \in \mathcal{A}_f$, let

$$U_a = \bigcup_{n,k\ge1} U_{n,k}(a).$$

In view of (3.5) and (3.6) we get that

$$\begin{aligned}\int_{U_a} |\log|f_*'|_\sigma|\,dm &= \sum_{n,k\ge 1}\int_{U_{n,k}(a)} |\log|f_*'|_\sigma|\,dm\\ &\asymp \sum_{n,k\ge 1} m(U_{n,k}(a))\left|\log\big(|z_{n,k}|^{\rho+1}|f(z_{n,k})-a||f(z_{n,k})-a|^{-1}\big)\right|\\ &\asymp \sum_{n,k\ge 1} |z_{n,k}|^{-h(\rho+1)}\left|\log\big(|z_{n,k}|^{\rho+1}\right|\\ &\asymp \sum_{n,k\ge 1}(n^2+k^2)^{-h\frac{\rho+1}{2\rho}}\log(n^2+k^2)\\ &\preceq \sum_{n,k\ge 1}(n^2+k^2)^{-t\frac{\rho+1}{2\rho}}\\ &< +\infty.\end{aligned}$$

Hence, $\int_{\bigcup_{a\in\mathcal{A}_f} U_a} |\log|f_*'|_\sigma|\,dm < +\infty$. Summing up this, (5.10), and (5.11), we conclude that $\int_X |\log|f_*'|_\sigma|\,dm < +\infty$, and the proof is complete. □

The main result of this section is this.

Theorem 5.4. *It holds* $\mathrm{HD}(J_f) = h$.

Proof. In view of Proposition 5.1 it suffices to show that $\mathrm{HD}(J_f)\ge h$. Let $X\subset J_f$ be the set defined by (5.4) and let $f_*: X\to X$ be the corresponding induced map. In virtue of Lemma 5.3, Lemma 5.2 and Birkhoff's Ergodic Theorem, there exists a Borel set $\hat X\subset X$ such that $\mu(\hat X)=1$ and

$$\lim_{n\to\infty}\frac{1}{n}\log|(f_*^n)'(z)|_\sigma = \chi > 0$$

for all $z\in\hat X$. In particular,

$$\lim_{n\to\infty}\frac{\log|(f_*^{k(n+1)})'(z)|_\sigma}{\log|(f_*^{kn})'(z)|_\sigma} = 1, \tag{5.12}$$

where $k\ge 1$ comes from Lemma 5.2. For every $z\in\hat X$ and every $n\ge 0$ define

$$r_n(z) = (2K)^{-1}|(f_*^{kn})'(z)|_\sigma^{-1}.$$

Fix $\varepsilon\in(0,h)$. In virtue of (5.12), for every $z\in\hat X$ we have

$$\frac{r_n(z)}{r_{n+1}(z)} \le r_n(z)^{-\frac{\varepsilon}{2}} \tag{5.13}$$

for all $n\ge 1$ large enough. It follows from (2.1) and conformality of m that

$$\begin{aligned} m(D(z,r_n)) &\le m\big(f_*^{-N_{kn}(z)}\big(D(f_*^{kn}(z),T/2)\big)\big)\\ &\le K^h\left|\left(f_*^{N_{kn}(z)}\right)'(z)\right|_\sigma^{-h} m\big(D(f_*^{kn}(z),T/2)\big)\\ &\le K^h\left|\left(f_*^{kn}\right)'(z)\right|_\sigma^{-h} = (2K^2T)^h r_n^h.\end{aligned} \tag{5.14}$$

Now, keeping $z \in \hat{X}$, take an arbitrary radius $r \in (0, (2K)^{-1}T)$. Since the sequence $(r_n)_0^\infty$ is strictly decreasing, there exists a unique $n \geq 0$ such that $r_{n+1} \leq r < r_n$. In view of (5.14) and (5.13), we get that

$$\lim_{r\to 0}\frac{m(D(z,r))}{r^{h-\varepsilon}} \leq \lim_{n\to\infty}\frac{m(D(z,r_n))}{r_{n+1}^{h-\varepsilon}} = \lim_{n\to\infty}\left(\frac{m(D(z,r_n))}{r_n^{h-\varepsilon}}\left(\frac{r_n}{r_{n+1}}\right)^{h-\varepsilon}\right) \leq \lim_{n\to\infty}\left(r_n^{\varepsilon} r_n^{-\frac{\varepsilon}{2}}\right) = \lim_{n\to\infty} r_n^{\frac{\varepsilon}{2}} = 0. \tag{5.15}$$

Since $m(\hat{X}) > 0$, we therefore conclude that $\mathcal{H}^{h-\varepsilon}(J_f) \geq \mathcal{H}^{h-\varepsilon}(\hat{X}) = +\infty$. Thus $\mathrm{HD}(J_f) \geq h - \varepsilon$, and eventually, letting $\varepsilon \searrow 0$, we get $\mathrm{HD}(J_f) \geq h$. We are done. □

References

[Aa] J. Aaronson, *An Introduction to Infinite Ergodic Theory,* Mathematical Surveys and Monographs, Vol. 50 (1997), 284 pp

[Bw] W. Bergweiler, *Iteration of meromorphic functions,* Bull. A.M.S. 29:2 (1993), 151-188.

[CFS]

[CY] W. Cherry, Z. Ye, *Nevanlinna's Theory of Value Distribution,* Spinger Monographs in Mathematics (2001).

[DU1] M. Denker, M. Urbański, *On the existence of conformal measures,* Trans. A.M.S. 328 (1991), 563-587.

[H1] E. Hille *Analytic function theory, Vol. II,* Ginn (1962).

[H2] E. Hille *Ordinary differential equations in the complex domain,* Dover Publications (1997).

[H3] E. Hille *On the zeroes of the Functions of the Parabolic Cylinder,* Ark. Mat. Astron. Fys., Vol. 18, No. 26 (1924).

[GKS] J.Graczyk, J.Kotus, G.Swiatek, *Non-recurrent meromorphic functions,* Fundamenta Mathematicae 182 (2004), 269-281.

[KS] J.Kotus, G.Swiatek, *Invariant measures for meromorphic Misiurewicz maps,* to appear Math. Proc. Camb. Phi. Soc.

[KU] J. Kotus, M. Urbański, *Fractal measures and ergodic theory of transcendental meromorphic functions (research expository article),* Preprint 2004, London Math. Soc. Lect. Notes., to appear.

[My] V. Mayer, *The size of the Julia set of meromorphic functions,* Preprint 2005.

[M] M. Martens, *The existence of σ–finite invariant measures, Applications to real one-dimensional dynamics,* Front for the Mathematics ArXiv, http://front.math.ucdavis.edu/math.DS/9201300.

[MU1] V. Mayer, M. Urbański, *Geometric thermodynamical formalism and real analyticity for meromorphic functions of finite order,* preprint 2006.

[MU2] V. Mayer, M. Urbański, *Thermodynamical formalism and multifractal analysis for meromorphic functions of finite order,* preprint 2007.

[McM] C.T. McMullen, *Hausdorff dimension and conformal dynamics II: Geometrically finite rational maps,* Comm. Math. Helv. 75 (2000), 535-593.

[Nev1] R. Nevanlinna, *Eindeutige analytische Funktionen,* Springer Verlag, Berlin (1953).

[Nev2] R. Nevanlinna, *Analytic functions,* Springer Verlag, Berlin (1970).

[Nev3] R. Nevanlinna, *Über Riemannsche Flächen mit endlich vielen Windungspunkten,* Acta Math. 58 (1932), 295-373.

[PU] F. Przytycki, M. Urbański, *Fractals in the Plane - the Ergodic Theory Methods,* available on Urbański's webpage, to appear Cambridge Univ. Press.

[Sk1] B. Skorulski, *Non-ergodic maps in the tangent family,* Indag. Math. 14 (2003), 103-118.

[Sk2] B. Skorulski, *Metric properties of the Julia set of some meromorphic functions with an asymptotic value eventually mapped onto a pole,* Math. Proc. Cambridge Phil. Soc. 138 (2005), 117-138.

[Su] D. Sullivan, *Seminar on conformal and hyperbolic geometry,* Preprint IHES (1982).

[UZ3] M. Urbański, A. Zdunik, *Geometry and ergodic theory of non-hyperbolic exponential maps,* Trans. AMS. 359 (2007), 3973-3997.

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