The transfinite Hausdorff dimension

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Abstract. Making an extensive use of small transfinite topological dimension trind, we ascribe to every metric space X an ordinal number (or -1 or Ω) $\mathrm{tHD}(X)$, and we call it the transfinite Hausdorff dimension of X. This ordinal number shares many common features with Hausdorff dimension. It is monotone with respect to subspaces, it is invariant under bi-Lipschitz maps (but in general not under homeomorphisms), in fact like Hausdorff dimension, it does not increase under Lipschitz maps, and it also satisfies the intermediate dimension property (Theorem 2.7). The primary goal of transfinite Hausdorff dimension is to classify metric spaces with infinite Hausdorff dimension. Indeed, if $\mathrm{tHD}(X) \geq \omega_0$, then $\mathrm{HD}(X) = +\infty$. We prove that $\mathrm{tHD}(X) \leq \omega_1$ for every separable metric space X, and, as our main theorem, we show that for every ordinal number $\alpha < \omega_1$ there exists a compact metric space X_α (a subspace of the Hilbert space l_2) with $\mathrm{tHD}(X_\alpha) = \alpha$ and which is a topological Cantor set, thus of topological dimension 0. In our proof we develop a metric version of Smirnov topological spaces and we establish several properties of transfinite Hausdorff dimension, including its relations with classical Hausdorff dimension.

1. Introduction

In [5] Felix Hausdorff has defined the concept of Hausdorff dimension. It ascribes to each metric space either a real non-negative number or $+\infty$. Hausdorff dimension is naturally invariant under isometries but is not, in general, invariant under homeomorphisms. Isometries form however a rather narrow class of maps. Fortunately Hausdorff dimension is invariant under bi-Lipschitz maps, which provide a much bigger variety of mappings. This is primarily why the class of Lipschitz maps seems to be most appropriately suited to deal with the issues related to Hausdorff dimension. As matter of fact the situation is even better since bi-Lipschitz maps preserve measure classes of Hausdorff measures, and the corresponding Radon-Nikodym derivatives are uniformly bounded above and uniformly separated from zero. A good modern account of the theory of Hausdorff dimension can be found n [3], [4], and [9]; the reader may also consult Chapter 7 of [11].

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P. Urysohn in [13] and K. Menger in [10] have introduced the concept of (small inductive) topological dimension, and in [14] P. Urysohn has indicated a possibility of defining transfinite topological dimensions. The formal definition appeared in [6]. An excellent account of the theory of topological dimensions, both finite and infinite, can be found in [2].

All existing transfinite dimensions are topological invariants. E. Marczewski has proved in [8] that for any separable metric space (X, ρ) its Hausdorff dimension is greater than or equal to its topological dimension (in the class of separable metric spaces all three classical dimensions ind, Ind, and dim coincide). A proof and more details can be found in [7]. In fact $\mathrm{ind}(X) = \inf\{\mathrm{HD}((X, \rho))\}$ where the supremum is taken over all metrics ρ compatible with topology on X. In general the two dimensions, Hausdorff and topological, are therefore really different, in fact B. Mandelbrot proposed to call a metric space X a fractal if its Hausdorff dimension is larger than the topological dimension. The Hausdorff and topological dimensions differ in one important aspect more. Namely, as we have already indicated, the topological spaces with infinite topological dimension can be further classified by ascribing to them transfinite topological dimensions trind and trInd. In contrast, for the the spaces with infinite Hausdorff dimension there seems to have been no step further. In this paper we propose to fill in this gap. Namely, making an extensive use of small transfinite topological dimension trind, we ascribe to every metric space X an ordinal number (or -1 or Ω) $\mathrm{tHD}(X)$, and we call it the transfinite Hausdorff dimension of X. This ordinal number shares many common features with Hausdorff dimension. It is monotone with respect to subspaces, it is invariant under bi-Lipschitz maps (but in general not under homeomorphisms), in fact, like Hausdorff dimension, it does not increase under Lipschitz maps, and it also satisfies the intermediate dimension property (Theorem 2.7). The primary goal of transfinite Hausdorff dimension is to classify metric spaces with infinite Hausdorff dimension. Indeed, if $\mathrm{tHD}(X) \geq \omega_0$, then $\mathrm{HD}(X) = +\infty$. We prove that $\mathrm{tHD}(X) \leq \omega_1$ for every separable metric space X, and, as our main theorem, we show that for every ordinal number $\alpha < \omega_1$ there exists a compact metric space X_α (a subspace of the Hilbert space l_2) with $\mathrm{tHD}(X_\alpha) = \alpha$ and which is a topological Cantor set, thus of topological dimension 0. In our proof we develop a metric version of Smirnov topological spaces and we establish several properties of transfinite Hausdorff dimension, including its relations with classical Hausdorff dimension.

2. Definition and Basic Properties of tHD

We first recall the definition of the small transfinite dimension trind.

Definition 2.1. *To every topological regular space X assigned is the small transfinite dimension of X, denoted by* $\mathrm{trind}(X)$*, which is the integer -1, an ordinal number, or the symbol Ω. The value of* $\mathrm{trind}(X)$ *is uniquely determined by the following conditions.*

- $\mathrm{trind}(X) = -1$ *if and only if $X = \emptyset$.*

- $\operatorname{trind}(X) \leq \alpha$, *where* α *is an ordinal number, if for every point* $x \in X$ *and each neighbourhood* V *of* x, *there exists an open set* $U \subset X$ *such that*

$$x \in U \subset V \quad \textit{and} \quad \operatorname{trind}(\partial U) < \alpha.$$

- $\operatorname{trind}(X) = \alpha$ *if* $\operatorname{trind}(X) \leq \alpha$ *and* $\operatorname{trind}(X) \leq \beta$ *for no ordinal* $\beta < \alpha$.
- $\operatorname{trind}(X) = \Omega$ *if there is no ordinal* α *such that* $\operatorname{trind}(X) \leq \alpha$.

We keep the convention that $\alpha < \Omega$ for every ordinal α.

Let $\mathcal{M}$ be the category of all metric spaces, and let $\mathcal{M}_0$ be the category of all separable metric spaces. If $X \in \mathcal{M}$ and $E \subset X$, then the set E is considered as a metric subspace of X endowed with the metric inherited from X. The collection of all metric subspaces of X is denoted by $\mathcal{P}_m(X)$. Let (X, ρ_X) and (Y, ρ_Y) be two arbitrary metric spaces. Recall that a map $f : X \to Y$ is called Lipschitz (or Lipschitz continuous) if there exists a real number $L \geq 0$ such that

$$\rho_Y(f(x_2), f(x_1)) \leq L\rho_X(x_2, x_1)$$

for all $x_1, x_2 \in X$. The number L is referred to as a Lipschitz constant of the map f. Denote the least Lipschitz constant of the map f by $\operatorname{Lip}(f)$. Note that the composition of two Lipschitz maps $f : X \to Y$ and $g : Y \to Z$ is a Lipschitz map, and $\operatorname{Lip}(g \circ f) \leq \operatorname{Lip}(f)\operatorname{Lip}(g)$. Given a Lipschitz map $f : X \to Y$ we denote its domain by $\operatorname{Dom}(f)$ (in our case equal to X), and the image $f(X)$ by $\operatorname{Im}(f)$. A bijective Lipschitz map $f : X \to Y$ is said to be bi-Lipschitz if its inverse $f^{-1} : Y \to X$ is also Lipschitz continuous. Denote by $L(X, Y)$ the collection of all Lipschitz maps from X to Y, and by $\mathcal{L}(X, Y)$ the collection of all Lipschitz maps $f : E \to Y$, where $E \in \mathcal{P}_m$. Set

$$L(X) = \bigcup_{Y \in \mathcal{M}} L(X, Y), \; L_0(X) = \bigcup_{Y \in \mathcal{M}_0} L(X, Y),$$

and

$$\mathcal{L}(X) = \bigcup_{Y \in \mathcal{M}} \mathcal{L}(X, Y), \; \mathcal{L}_0(X) = \bigcup_{Y \in \mathcal{M}_0} \mathcal{L}(X, Y).$$

Let $L^s(X)$, $L_0^s(X)$, $\mathcal{L}^s(X)$, and $\mathcal{L}_0^s(X)$ be the subcollections respectively of $L(X)$, $L_0(X)$, $\mathcal{L}(X)$, and $\mathcal{L}_0(X)$ consisting of surjective maps. The basic concept introduced in this paper is provided by the following.

Definition 2.2. *The transfinite Hausdorff dimension* $\operatorname{tHD}(X)$ *of a metric space* X *is equal to* -1 *if and only if* $X = \emptyset$, *and is less than or equal to* ($\leq$) *an ordinal* α *if and only if* $\operatorname{trind}(\operatorname{Im}(f)) \leq \alpha$ *for every map* $f \in \mathcal{L}(X)$. *Then we define the transfinite Hausdorff dimension of the space* X *by setting*

$$\operatorname{tHD}(X) := \sup\{\operatorname{trind}(\operatorname{Im}(f)) : f \in \mathcal{L}(X)\} \geq \operatorname{trind}(X).$$

Otherwise, we set $\operatorname{tHD}(X) = \Omega$, *and in any case we write*

$$\operatorname{tHD}(X) = \sup\{\operatorname{trind}(\operatorname{Im}(f)) : f \in \mathcal{L}(X)\} \geq \operatorname{trind}(X).$$

Directly from this definition we get the following.

Theorem 2.3. *(monotonicity theorem) If $X \in \mathcal{M}$ and $E \in \mathcal{P}_m$, then* $\mathrm{tHD}(E) \le \mathrm{tHD}(X)$.

Since, as we already mentioned, the composition of two Lipschitz maps is Lipschitz, we get the following.

Theorem 2.4. *If X and Y are two metric spaces and $f : X \to Y$ is a Lipschitz map, then* $\mathrm{tHD}(f(X)) \le \mathrm{tHD}(X)$. *So, if $f : X \to Y$ is bi-Lipschitz, then* $\mathrm{tHD}(Y) = \mathrm{tHD}(X)$.

Since the image of a separable metric space under a Lipschitz continuous map is a separable metric space, we get the following.

Theorem 2.5. *If $X \in \mathcal{M}_0$, then*

$$\mathrm{tHD}(X) = \sup\{\mathrm{trind}(\mathrm{Im}(f)) : f \in \mathcal{L}_0(X)\} \ge \mathrm{trind}(X).$$

We shall prove the following.

Theorem 2.6. *If $X \in \mathcal{M}$, then*

$$\mathrm{tHD}(X) = \sup\{\mathrm{trind}(\mathrm{Im}(f))\},$$

where the supremum is taken over all surjective maps $f \in \mathcal{L}(X)$, with closed domains and complete codomains.

Proof. Suppose $f : M \to Y$ is a Lipschitz map with $M \in \mathcal{P}_m$. Let $\hat{Y}$ be the metric completion of Y. Since f is Lipschitz continuous, it extends (uniquely) to a Lipschitz continuous map (with the same Lipschitz constant) $\hat{f} : \bar{M} \to \hat{Y}$. Then the map $\hat{f}|_{\bar{M}} : \bar{M} \to \hat{f}(\bar{M})$ belongs to $\mathcal{L}(X)$, $\bar{M}$ is a closed subspace of X, $\hat{f}|_{\bar{M}}(\bar{M})$ is a complete metric space, and $\mathrm{trind}(\hat{f}|_{\bar{M}}(\bar{M})) \ge \mathrm{trind}(f(M))$. We are done. □

Theorem 2.7. *(intermediate dimension property) If X is a compact metric space and* $\mathrm{tHD}(X) < \Omega$, *then for every* $\beta \le \mathrm{tHD}(X)$ *there exists a closed subspace M_β of X such that* $\mathrm{tHD}(M_\beta) = \beta$.

Proof. The theorem is trivially obvious if $X = \emptyset$. So, in what follows we may assume that $X \ne \emptyset$. We shall prove first the following.

Claim. For every $\beta < \mathrm{tHD}(X)$ there exists $\beta \le \gamma < \mathrm{tHD}(X)$ such that $\gamma = \mathrm{tHD}(M)$ for some closed subspace M of X.

Proof. Suppose on the contrary that there exists $\beta < \mathrm{tHD}(X)$ such that for every closed subspace M of X either $\mathrm{tHD}(M) < \beta$ or $\mathrm{tHD}(M) = \mathrm{tHD}(X)$. By Theorem 2.6

there exists a closed subspace F of X and a Lipschitz continuous surjection $f : F \to Y$ such that $\mathrm{trind}(Y) > \beta$. Since $\mathrm{trind}(Y) \le \mathrm{tHD}(X) < \Omega$, it follows from Theorem 7.1.8 in [2] that Y is countable dimensional, meaning that

$$Y = \bigcup_{n=1} Y_n, \tag{2.1}$$

where $\mathrm{ind}(Y_n) = 0$ for all $n \ge 1$. Suppose that for every point $y \in Y$ and every open neighbourhood U of y there exists a partition L between y and ∂U such that $\mathrm{tHD}(f^{-1}(L)) < \mathrm{tHD}(X)$. Then we would have $\mathrm{trind}(L) \le \mathrm{tHD}(L) \le \mathrm{tHD}(f^{-1}(L)) < \beta$. But this would imply that $\mathrm{trind}(Y) \le \beta$. The contradiction obtained shows that there exist $y_1 \in Y$ and an open neighbourhood U_1 of y_1 in y such that $\mathrm{tHD}(f^{-1}(L)) = \mathrm{tHD}(X)$ for every partition L between y_1 and ∂U_1. By Theorem 4.1.13 in [2] there now exists a partition L_1 between y_1 and ∂U_1 such that $L_1 \cap Y_1 = \emptyset$. Since $\mathrm{tHD}(f^{-1}(L)) = \mathrm{tHD}(X)$ and $f^{-1}(L) \subset X$, we can proceed by induction to produce a sequence $(y_n)_1^\infty$, of points in Y, a sequence $(U_n)_1^\infty$ of open subsets of Y, and a sequence $(L_n)_1^\infty$ of closed of Y with the following properties holding for all $n \ge 1$:

(a) $y_n \in U_n \cap L_{n-1}$.
(b) L_n is a partition between y_n and $L_{n-1} \cap \partial U_n$ in L_{n-1}.
(c) $\mathrm{tHD}(f^{-1}(L_n)) = \mathrm{tHD}(X)$
(d) $L_n \cap Y_n = \emptyset$,

where $L_0 = Y$. It follows from (b) that $L_n \subset L_{n-1}$. Also, because of (c), we have for each $n \ge 0$ that $L_n \ne \emptyset$. Thus (all the sets L_n are compact) $\bigcap_{n=0}^\infty L_n \ne \emptyset$. This however contradicts (d) and (2.1). The claim is proved.

Now, the conclusion of the proof is a consequence of Claim. Indeed, denote by $\mathcal{F}_X$ the collection of all closed subsets of X. Let

$$V = \{- \le \alpha \le \mathrm{tHD}(X) : \forall_{(-1 \le \beta \le \alpha)} \exists_{(M \in \mathcal{F}_X)} \mathrm{tHD}(M) = \beta\}.$$

Then $\sup(V) \le \mathrm{tHD}(X)$, and if $\sup(V) = \mathrm{tHD}(X)$, we are done. Otherwise, put

$$W = \{- \le \alpha \le \mathrm{tHD}(X) : \alpha \notin V \ \text{ and } \ \exists_{(M \in \mathcal{F}_X)} \mathrm{tHD}(M) = \alpha\}.$$

Then $W \ne \emptyset$ (as $\mathrm{tHD}(X) \in W$) and $\sup(V) < \min(W)$. Take $M \in \mathcal{F}_X$ such that $\mathrm{tHD}(M) = \min(W)$. Applying now our claim to the space M and ordinal $\beta = \sup(V)$, we get a closed subset K of M such that $\sup(V) \le \mathrm{tHD}(K)) < \min(W)$. But then $\mathrm{tHD}(K) = \sup(V) \in V$. If $\sup(V) + 1 = \min(W)$, we would have $[0, \sup(V) + 1] \subset V$, which is a contradiction. Thus $\sup(V) + 1 < \min(W)$, and therefore, applying Claim with the space M and ordinal $\beta = \sup(V) + 1$, we would get a closed subspace L of M such that $\sup(V) + 1 \le \mathrm{tHD}(L) < \min(W)$. This however contradicts the definition of W and finishes the proof. □

The last theorem in this section is this.

Theorem 2.8. *If X is a metric space and its Hausdorff dimension is finite, then* $\mathrm{trind}(X) \le \mathrm{tHD}(X) \le E(\mathrm{HD}(X))$, *where $E(t)$ is the integer value of the real number t. Consequently,* $\mathrm{HD}(X) = +\infty$ *whenever* $\mathrm{tHD}(X) \ge \omega_0$.

Proof. The left-hand side inequality is already stated in the definition of transfinite Hausdorff dimension. Since $\mathrm{HD}(X) < +\infty$, it follows from Marczewski's Theorem that $\mathrm{trind}(X) < +\infty$ and, applying this theorem once more, we get that,

$$\mathrm{HD}(X) \geq \mathrm{HD}(D(f)) \geq \mathrm{HD}(\mathrm{Im}(f)).$$

So, taking the supremum, we obtain that $\mathrm{HD}(X) \geq \mathrm{tHD}(X)$, and, as $\mathrm{tHD}(X)$ is now an integer, we are done. □

3. Further Properties of Transfinite Hausdorff Dimension

As an immediate consequence of Theorem 7.1.6 and Theorem 7.1.17 in [2], we get the following.

Theorem 3.1. *If a metric space X has a topological base of cardinality $\leq \aleph_\alpha$ and $\mathrm{tHD}(X) < \Omega$, then $\mathrm{tHD}(X) \leq \omega_{\alpha+1}$. In particular, if X is separable, then $\mathrm{tHD}(X) \leq \omega_1$.*

Given two ordinal numbers α_1, α_2 write them in the canonical form $\alpha_i = \lambda_i + n_i$, $i = 1, 2$, where λ_i is a limit ordinal number and $n_i \geq 0$ is a finite ordinal number. Set

$$\Sigma(\alpha_1, \alpha_2) = \begin{cases} \lambda_i + n_i & \text{if } \lambda_i > \min\{\lambda_1, \lambda_2\} \\ \lambda_1 + n_2 + n_1 + 1 & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Theorem 3.2. *If a compact metric space X is a union of two closed subspaces X_1 and X_2, then*

$$\mathrm{tHD}(X) \leq \Sigma(\mathrm{tHD}(X_1), \mathrm{tHD}(X_2)).$$

Proof. Let M be a closed subspace of X and let $f : M \to Y$ be a Lipschitz continuous map. Then $f(M \cap X_1)$ and $f(M \cap X_2)$ are closed subspaces of $f(M)$, and, by Theorem 7.2.6 in [2], we get

$$\begin{aligned} \mathrm{trind}(f(M)) &\leq \Sigma(\mathrm{trind}(f(M \cap X_1)), \mathrm{trind}(f(M \cap X_2)) \\ &\leq \Sigma(\mathrm{tHD}(X_1), \mathrm{tHD}(X_2)). \end{aligned}$$

We are thus done by applying Theorem 2.6. □

For Γ, any set of ordinals, let $l\sup(\Gamma)$ be the least ordinal greater than all elements of Γ. Corollary 7.2.8 from [2] can be restated as follows.

Theorem 3.3. *If a metric space X is a union of finitely many closed subsets $X_1, X_2, \ldots, X_n$, then $\mathrm{trind}(X) < l\sup\{\mathrm{trind}(X_1), \ldots, \mathrm{trind}(X_n)\}$.*

As an immediate consequence of Theorem 3.2, we get the following.

Theorem 3.4. *If a metric space X is a union of finitely many closed subspaces $X_1, X_2, \ldots, X_n$, then $\mathrm{tHD}(X) < l\sup\{\mathrm{tHD}(X_1), \ldots, \mathrm{tHD}(X_n)\}$.*

We shall now prove a purely topological lemma which will be used in the sequel.

Lemma 3.5. *Suppose a metric space $X = \bigcup_{i \in J} X_i$, where $\{X_j\}_{j\in J}$ is a locally finite family of closed subsets of X, then* $\operatorname{trind}(X) \le l\sup\{\operatorname{trind}(X_j) : j \in J\}$.

Proof. Set $\lambda = l\sup\{\operatorname{trind}(X_j) : j \in J\}$. Let $x \in X$ and let V be an arbitrary open neighbourhood of x. Since the family $\{X_i\}_{j\in J}$ is locally finite, there exist an open neighbourhood U of x and a finite subset F of J such that $\bar{U} \subset V$ and $\bar{U} \subset \bigcup_{j\in F} X_j$. It then follows from Corollary 3.3 that

$$\operatorname{trind}(\partial U) \le \operatorname{trind}(\bar{U}) \le \operatorname{trind}\left(\bigcup_{j\in F} X_j\right) < l\sup\{\operatorname{trind}(X_j) : j \in F\} \le \lambda.$$

Hence, $\operatorname{trind}(X) \le \lambda$, and we are done. □

Let α and λ be two arbitrary ordinals. Define $\alpha^*(\lambda)$ by the following transfinite recursion.

$$0^*(\lambda) = \lambda \text{ and } \alpha^*(\lambda) = l\sup\{\beta^*(\lambda) : \beta < \alpha\}.$$

We shall prove the following.

Lemma 3.6. *Suppose a metric space $X = X_0 \cup \bigcup_{j\in J} X_j$, where X_0 is a closed set, and $\{X_i\}_{j\in J}$ is a family of closed subsets of X, locally finite at each point of $X \setminus X_0$. Then*

$$\operatorname{trind}(X) \le (1 + \operatorname{trind}(X_0))^*(l\sup\{\operatorname{trind}(X_j) : j \in J\}).$$

Proof. The proof is by transfinite induction with respect to the ordinal number $\operatorname{trind}(X_0)$. Indeed, if $\operatorname{trind}(X_0) = -1$, the statement reduces to Lemma 3.5. So, suppose for the inductive step that the lemma is true for all $\beta < \alpha = \operatorname{trind}(X_0)$. Fix $x \in X$ and then V, an open neighbourhood of x in X. If $x \notin X_0$, then there exist an open neighbourhood U of x and a finite subset F of J such that $\bar{U} \subset V$ and $\bar{U} \subset \bigcup_{j\in F} X_j$. It then follows from Theorem 3.3 that

$$\begin{aligned}
\operatorname{trind}(\partial U) &\le \operatorname{trind}(\bar{U}) \le \operatorname{trind}\big(\bigcup_{j\in F} X_j\big) < l\sup\{\operatorname{trind}(X_j) : j \in F\} \\
&\le l\sup\{\operatorname{trind}(X_j) : j \in J\} \\
&\le (1+\alpha)^*(l\sup\{\operatorname{trind}(X_j) : j \in J\}.
\end{aligned} \tag{3.1}$$

So, suppose that $x \in X_0$. By the very definition of the dimension trind there exists an open (with respect to the relative topology on X_0) neighbourhood U' of x in X_0 contained with closure in $X_0 \cap V$ and such that $\operatorname{trind}(\partial U') < \alpha$. Then, by the last assertion in Lemma 1.2.9 in [2], there exists a partition L between x and ∂V such that $L \cap X_0 \subset \partial U'$. We have $L = (L \cap X_0) \cup \bigcup_{j\in J} L \cap X_j$. Since $\operatorname{trind}(L \cap X_0) \le$

$\mathrm{trind}(\partial U') < \alpha$, applying the inductive assumption, we get that

$$\begin{aligned}\mathrm{trind}(L) &\le (1+\mathrm{trind}(L\cap X_0))^*(l\sup\{\mathrm{trind}(L\cap X_j) : j\in J\}) \\ &\le (1+\mathrm{trind}(L\cap X_0))^*(l\sup\{\mathrm{trind}(X_j) : j\in J\}) \\ &< (1+\alpha)^*(l\sup\{\mathrm{trind}(X_j) : j\in J\}).\end{aligned}$$

Looking at this and (3.1), we conclude that

$$\mathrm{trind}(X)) \le (1+\mathrm{trind}(X_0))^*(l\sup\{\mathrm{trind}(X_j) : j\in J\}).$$

The inductive proof is complete. □

Drawing conclusions for the transfinite Hausdorff dimension, we shall prove the following.

Theorem 3.7. *Suppose a compact metric space $X = X_0 \cup \bigcup_{j\in J} X_j$, where X_0 is a closed set and $\{X_j\}_{j\in J}$ is a family of closed subsets of X, locally finite at each point of $X\setminus X_0$. Then*

$$\mathrm{tHD}(X) \le (1+\mathrm{tHD}(X_0))^*(l\sup\{\mathrm{tHD}(X_j) : j\in J\}).$$

In particular, if the set J is countable and $\mathrm{tHD}(X_j) < \omega_1$ for all $j\in J\cup\{0\}$, then $\mathrm{tHD}(X) < \omega_1$.

Proof. Let M be a closed subspace of X and let $f : M \to Y$ be a Lipschitz map. Then

$$f(M) = f(M\cap X_0) \cup \bigcup_{j\in J} f(M\cap X_j),$$

and constituents of this union are closed subsets of $f(M)$. We shall show that the family $\{f(M\cap X_j)_{j\in J}$ is locally finite at each point of $f(M)\setminus f(M\cap X_0)$. Indeed, suppose for the contrary that there exists $y\in f(M)\setminus f(M\cap X_0)$ such that the family $\{f(M\cap X_j)_{j\in J}$ is not locally finite at y. This means that there exist an infinite countable subset $\{j_n\}_{n=1}^\infty$ of J, and for each $n\ge 1$ a point $x_n\in M\cap X_{j_n}$ such that $\lim_{n\to\infty} f(x_n) = y$. Since M is a compact set, passing to a subsequence, we may assume without loss of generality that $\lim_{n\to\infty} x_n = x$ for some $x\in M$. But then the family $\{X_j\}_{j\in J}$ is not locally finite at x. Hence $x\in X_0$. Then, $y = f(x)\in f(M\cap X_0)$, contrary to the choice of y. Therefore, we may apply Lemma 3.6 to get that

$$\begin{aligned}\mathrm{trind}(f(M)) &\le (1+\mathrm{trind}(f(M\cap X_0)))^* l\sup\{\mathrm{trind}(f(M\cap X_j) : j\in J\}) \\ &\le (1+\mathrm{tHD}(M\cap X_0))^* l\sup\{\mathrm{tHD}(M\cap X_j) : j\in J\}) \\ &\le (1+\mathrm{tHD}(X_0))^* l\sup\{\mathrm{tHD}(X_j) : j\in J\}).\end{aligned}$$

Applying Theorem 2.6, we therefore get that

$$\mathrm{tHD}(X) \le (1+\mathrm{tHD}(X_0))^*(l\sup\{\mathrm{tHD}(X_j) : j\in J\}).$$

We are done. □

Toward the end of the section, we shall prove the following little fact from the theory of topological transfinite dimension.

Proposition 3.8. *If X is a metric space and $X = X_* \cup X_0$, where X_* is closed and X_0 is a F_σ set with* $\operatorname{ind}(X_0) \le 0$, *then*

$$\operatorname{trind}(X) = \max\{\operatorname{trind}(X_*), \operatorname{trind}(X_0)\}.$$

Proof. Replacing X_0 by $X_0 \setminus X_*$ we may assume without loss of generality that $X_* \cap X_0 = \emptyset$. We will proceed by transfinite induction with respect to $\alpha = \operatorname{trind}(X_*)$. Indeed, if $\alpha < \omega_0$, this is a special case of the Sum Theorem for the dimension ind. So, suppose that $\alpha \ge \omega_0$ and that theorem is true if $\operatorname{trind}(X_*) < \alpha$. Fix a point $x \in X$ and a closed set F not containing x. If $x \in X_0$, then (as $X_* \cap X_0 = \emptyset$) there exists $r > 0$ such that $F \cap B(x, 2r) = \emptyset$ and $B(x, 2r) \cap X_* = \emptyset$. But then $\partial B(x, r) \subset X_0$, and therefore, $\operatorname{trind}(\partial B(x, r)) \le 0$. So, $\partial B(x, r)$ is a partition between x and F whose trind dimension is ≤ 0. If $x \in X_*$, then there exists a partition L' in the space X_* between x and $F \cap X_*$ such that $\operatorname{trind}(L') < \alpha = \operatorname{trind}(X_*)$. By Lemma 1.2.9 in [2] there then exists a partition L in X between x and F such that $X_* \cap L \subset L'$. Writing $L = (X_* \cap L) \cup (X_0 \cap L)$ and noting that $\operatorname{trind}(X_* \cap L) \le \operatorname{trind}(L') < \alpha$, we may apply the inductive assumption to conclude that $\operatorname{trind}(L) = \max\{\operatorname{trind}(X_* \cap L), \operatorname{trind}(X_0 \cap L)\} < \alpha$. Thus, $\operatorname{trind}(X) \le \alpha$, and we are done. □

Corollary 3.9. *If X is a compact metric space and $X = X_* \cup X_0$, where X_* is closed and X_0 is a F_σ set with* $\operatorname{tHD}(X_0) \le 0$, *then*

$$\operatorname{tHD}(X) = \max\{\operatorname{tHD}(X_*), \operatorname{tHD}(X_0)\}.$$

In particular, if $X_* \neq \emptyset$, then $\operatorname{tHD}(X) = \operatorname{tHD}(X_*)$.

Proof. Let M be a closed subspace of X and let $f : X \to Y$ be a Lipschitz continuous surjection. Then $Y = f(X_*) \cup f(X_0)$, where $f(X_*)$ is a closed set and $f(X_0)$ is a F_σ set. But $\operatorname{trind}(f(X_*)) \le \operatorname{tHD}(X_*)$ and $\operatorname{trind}(f(X_0)) \le \operatorname{tHD}(X_0) \le 0$. So, applying Proposition 3.8, we get that

$$\operatorname{trind}(f(X)) = \max\{\operatorname{trind}(f(X_*)), \operatorname{trind}(f(X_0))\} \le \max\{\operatorname{tHD}(X_*), \operatorname{tHD}(X_0)\}.$$

Taking the supremum we thus get that

$$\operatorname{tHD}(X) \le \max\{\operatorname{tHD}(X_*), \operatorname{tHD}(X_0)\}.$$

Since the opposite inequality holds because of the monotonicity theorem, we are done. □

4. Operations on Metric Spaces

If (X_1, ρ_1) and (X_2, ρ_2) are two arbitrary metric spaces, then $X_1 \times X_2$ the metric space with the metric ρ given by the formula

$$\rho((a_1, a_2), (b_1, b_2)) = \sqrt{\rho_1^2(a_1, b_1) + \rho_2^2(a_2, b_2)}.$$

Obviously, we have the following.

Observation 4.1. *If the metric spaces X_1 and X_2 are both respectively isometrically embedded in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, then the Cartesian product $X_1 \times X_2$ embeds isometrically in the Hilbert space $\mathcal{H}_1 \times \mathcal{H}_2$.*

The Cartesian product of any finite number of metric spaces is defined analogously, and if all factors are isometrically embedded in Hilbert spaces, then so is the product.

Given two sets A and B in a metric space (X, ρ) we define

$$\mathrm{dist}_\rho(A, B) := \inf\{\rho(a, b) : (a, b) \in A \times B\}$$

and

$$\mathrm{Dist}_\rho(A, B) := \sup\{\rho(a, b) : (a, b) \in A \times B\}.$$

Let now J be a countable infinite set and let $\{(X_j, \rho_j)\}_{j\in J}$ be a collection of compact metric spaces. Let $\omega(\oplus_{j\in J} X_j)$ be the topological one point (Alexandrov) compactification of the topological disjoint union $\oplus_{j\in J} X_j$. A metric space $(\omega(\oplus_{j\in J} X_j), \rho)$ is called a metric one point (Alexandrov) compactification of $\oplus_{j\in J} X_j$ if ρ induces on $\omega(\oplus_{j\in J} X_j)$ the Alexandrov compactification topology, and for each $j \in J$ the restriction $\rho|_{X_j}$ is proportional to ρ_j. The metric ρ is then referred to as an Alexandrov metric. An Alexandrov metric ρ on $\omega(\oplus_{j\in J} X_j)$ is called balanced if

$$D_\rho := \max\left\{\sup_{i,j\in J}\left\{\frac{\mathrm{Dist}_\rho(X_i, X_j)}{\mathrm{dist}_\rho(X_i, X_j)}\right\}, \sup_{j\in J}\left\{\frac{\mathrm{Dist}_\rho(\omega, X_j)}{\mathrm{dist}_\rho(\omega, X_j)}\right\}\right\} < +\infty.$$

The number D_ρ is referred to as the balance constant of the metric ρ. We have the following, actually obvious.

Proposition 4.2. *If J is a countable infinite set and if $\{(X_j, \rho_j)\}_{j\in J}$ is a collection of compact metric spaces, then there exists at least one balanced (even with balance constant equal to 1) Alexandrov metric on $\omega(\oplus_{j\in J} X_j)$.*

Proof. Let $\phi : J \to \mathbb{N}$ be an arbitrary bijection. Define a metric ρ on $\omega(\oplus_{j\in J} X_j)$ as follows.

$$\rho(x, y) = \begin{cases} 2^{-\phi(j)}\mathrm{diam}^{-1}(X_j)\rho_j(x, y) & \text{if } x, y \in X_j \\ 0 & \text{if } x = y = \omega \\ 2^{-\min\{\phi(i),\phi(j)\}} & \text{if } i \neq j \ \text{ and } \ x \in X_i, y \in X_j \\ 2^{-\phi(j)} & \text{if } x \in X_j \ \text{ and } \ y = \omega, \end{cases}$$

where we take the convention that $0^{-1} = \infty$ and $0 \cdot \infty = 0$. Clearly, ρ is an Alexandrov metric on $\omega(\oplus_{j\in J} X_j)$ with balanced constant $D_\rho = 1$. We are done. $\square$

Another obvious fact is the following.

Lemma 4.3. *Suppose that J is a countable infinite set, $\{(X_j, \rho_j)\}_{j\in J}$ is a collection of compact metric spaces, and ρ is a balanced Alexandrov metric on $\omega(\oplus_{j\in J} X_j)$. Suppose* *further that for each $j \in J$, A_j is a subset of X_j, and $f_j : A_j \to X_j$ is a Lipschitz continuous map with the Lipschitz constant bounded above by the same number L. Define the map $f : \{\omega\} \cup \bigcup_{j\in J} A_j \to \omega(\oplus_{j\in J} X_j)$ by requiring that $f(\omega) = \omega$ and $f|_{A_j} = f_j$ for all $j \in J$. Then f is a Lipschitz map with* $\mathrm{Lip}(f) \le \max\{L, D_\rho\}$. *Also* $\mathrm{Im}(f) = \{\omega\} \cup \bigcup_{j\in J} f_j(A_j)$.

We end this section with the following.

Proposition 4.4. *If J is a countable infinite set and $\{(X_j, \rho_j)\}_{j\in J}$ is a collection of compact metric spaces embedded in a separable Hilbert space, then there exists a balanced Alexandrov metric on $\omega(\oplus_{j\in J} X_j)$ embeddable in a separable Hilbert space and with its balanced constant bounded above by* 2.

Proof. We may assume without loss of generality that all the spaces X_j, $j \in J$, are contained in the Hilbert space l_2. Let $\phi : J \to \mathbb{N}$ be an arbitrary bijection. For every $x \in l_2$ let $T_x : l_2 \to l_2$ be the translation given by the formula $T_x(y) = y + x$. For every $\alpha > 0$ let $H_\alpha : l_2 \to l_2$ be the homothety given by the formula $H_\alpha(y) = \alpha y$. Since all the spaces X_j, $j \in J$, are bounded, for each $j \in J$ there exist $\alpha_j > 0$ and $x_j \in l_2$ such that

$$T_{x_j} \circ H_{\alpha_j}(X_j) \subset B(0, 2^{-\phi(j)}) \setminus \overline{B(0, 3 \cdot 2^{-(\phi(j)+2)})}.$$

Define the map $h : \omega(\oplus_{j\in J} X_j) \to l_2$ by requiring that $h(\omega) = 0$ and $h|_{X_j} = T_{x_j} \circ H_{\alpha_j}$ for all $j \in J$. For all $x, y \in \omega(\oplus_{j\in J} X_j)$ set then $\rho(x, y) = ||h(x) - h(y)||$. Clearly this is a balanced Alexandrov metric on $\omega(\oplus_{j\in J} X_j)$ with its balanced constant bounded above by 2. □

5. Smirnov's Cantor Sets

Let I be the interval $[0, 1]$ endowed with its standard Euclidean metric. Starting with the singleton $\{0\}$ we shall now define a transfinite sequence $((S_\alpha, \rho_\alpha))_{\alpha<\omega_1}$ consisting of compact metric spaces. We do it as follows. $S_0 = \{0\}$, $S_\alpha = S_\beta \times I$ if $\alpha = \beta + 1$, and (S_α, ρ_α) is a balanced Alexandrov metric compactification $\omega(\oplus_{\beta<\alpha} S_\beta)$ of $\oplus_{\beta<\alpha} S_\beta$. This is a well-defined sequence because of Proposition 4.2. We refer to it as a Smirnov's sequence. In view of Observation 4.1 and Proposition 4.4, we even get the following.

Proposition 5.1. *There exists a Smirnov's sequence $((S_\alpha, \rho_\alpha))_{\alpha<\omega_1}$ whose all elements are contained in the Hilbert space l_2.*

We recall that topological Smirnov spaces were introduced in [12]. A good account of their properties can be found in [2]. Now we pass to define Smirnov's Cantor sets and sequences. Suppose $C \subset I$ is a topological Cantor set (perfect, totally disconnected set) with positive (linear) Lebesgue measure $\lambda(C)$. Let $\phi : C \to I$ be the function given by the formula

$$\phi(t) = \lambda(C)^{-1} \lambda([\min(C), t]).$$

Clearly ϕ is a Lipschitz continuous map with Lipschitz constant equal to $\lambda(C)^{-1}$ and $\phi(C) = I$. Given Smirnov's sequence $((S_\alpha, \rho_\alpha))_{\alpha<\omega_1}$ define C_0 to be $\{0\}$, $C_\alpha = C_\beta \times C$ if $\alpha = \beta + 1$, and $C_\alpha = \{\omega\} \cup \bigcup_{\beta<\alpha} C_\beta$, if $\alpha < \omega_1$ is a limit ordinal number. $((C_\alpha, \rho_\alpha|_{C_\alpha}))_{\alpha<\omega_1}$ is refered to as a Smirnov's sequence of Cantor sets (associated to the Smirnov's sequnce $((S_\alpha, \rho_\alpha))_{\alpha<\omega_1}$ of Smirnov spaces). Clearly $C_\alpha \subset S_\alpha$ for all $\alpha < \omega_1$, and C_α is a topological Cantor set. Let us prove the following.

Lemma 5.2. *If $(S_\alpha)_{\alpha<\omega_1}$ is a Smirnov's sequnce and $(C_\alpha)_{\alpha<\omega_1}$ is the corresponding Smirnov's sequence of Cantor sets, then* $\mathrm{tHD}(C_\alpha) \geq \mathrm{trind}(S_\alpha)$ *for all $\alpha < \omega_1$.*

Proof. We shall define by transfinite induction a sequence $(\phi_\alpha)_{\alpha<\omega_1}$ of Lipschitz continuous surjections from C_α onto S_α with Lipschitz constants bounded above by $\max\{2, \lambda(C)^{-1}\}$. Indeed, set ϕ_0 to be the identity map on $\{0\}$ and suppose that for some $0 \leq \alpha < \omega_1$ the claimed maps $\phi_\beta : C_\beta \to S_\beta$ are defined for all $0 \leq \beta < \alpha$. If $\alpha = \gamma + 1$, set $\phi_\alpha = \phi_\gamma \times \phi : C_\gamma \times I \to S_\gamma \times I$. Then $\mathrm{Im}(\phi_\alpha) = \mathrm{Im}(\phi_\gamma) \times \mathrm{Im}(\phi) = S_\gamma \times I = S_\alpha$. Also,

$$\mathrm{Lip}(\phi_\alpha) \leq \max\{\mathrm{Lip}(\phi_\gamma), \mathrm{Lip}(\phi)\} \leq \max\{2, \lambda(C)^{-1}, \lambda(C)^{-1}\} = \max\{2, \lambda(C)^{-1}\}.$$

If α is a limit number, let $\phi_\alpha : C_\beta \to S_\alpha$ be the Lipschitz continuous function constructed in Lemma 4.3 out of functions $\phi_\beta : C_\beta \to S_\beta$, $\beta < \alpha$. Then

$$\mathrm{Im}(\phi_\alpha) = \{\omega\} \cup \bigcup_{\beta<\alpha} \phi_\beta(C_\beta) = \{\omega\} \cup \bigcup_{\beta<\alpha} S_\beta = S_\alpha,$$

and, according to this lemma and because of our inductive assumption, the map $\phi_\alpha : C_\beta \to S_\alpha$ is Lipschitz continuous with

$$\mathrm{Lip}(\phi_\alpha) \leq \max\{\max\{2, \lambda(C)^{-1}\}, 2\} = \max\{2, \lambda(C)^{-1}\}.$$

The inductive construction of Lipschitz maps $(\phi_\alpha)_{\alpha<\omega_1}$ is complete. By the very definition of the transfinite Hausdorff dimension we thus have for all $\alpha < \omega_1$ that $\mathrm{tHD}(C_\alpha) \geq \mathrm{trind}(\mathrm{Im}(\phi_\alpha)) = \mathrm{trind}(S_\alpha)$. We are done. □

Now, we shall prove the following.

Lemma 5.3. *If $(S_\alpha)_{\alpha<\omega_1}$ is a Smirnov's sequnce, then* $\mathrm{tHD}(S_\alpha) < \omega_1$ *for all $\alpha < \omega_1$.*

Proof. Since S_0 is a singleton, the statement is true if $\alpha = 0$. Proceeding by transfinite induction suppose the lemma is true for all $\beta < \alpha$, where $\alpha < \omega_1$. Write $\alpha = \gamma + n$, where γ is a limit number and $n \geq 0$ is a finite number. Then S_α is isometric to $S_\gamma \times I^n$, where I^0 is a singleton. But $S_\gamma = \{\omega\} \cup \bigcup_{\beta<\gamma} S'_\beta$, where S'_β is a similar copy of S_β. So,

$$S_\gamma \times I^n = (\omega \times I^n) \cup \bigcup_{\beta<\gamma} S'_\beta \times I^n. \tag{5.1}$$

But $S'_\beta \times I^n$ is bi-Lipschitz equivalent to $S_\beta \times I^n = S_{\beta+n}$. Since $\beta + n < \gamma \leq \alpha$, the inductive hypothesis gives that $\mathrm{tHD}(S'_\beta \times I^n) = \mathrm{tHD}(S_{\beta+n}) < \omega_1$. We also know that

$\mathrm{tHD}(\{\omega\} \times I^n) = n$. Since, in addition, all the sets $\{\omega\} \times I^n$ and $S'_\beta \times I^n$, $\beta < \gamma$, are compact, and $S'_\beta \times I^n$ are also open subsets of $S_\gamma \times I^n$, we can apply Theorem 3.7 to the decomposition (5.1) to conclude that $\mathrm{tHD}(S_\alpha) = \mathrm{tHD}(S_\gamma \times I^n) < \omega_1$. We are done. □

Thus, by Theorem 2.3 (monotonicity of transfinite Hausdorff dimension), we have that $\mathrm{tHD}(C_\alpha) \le \mathrm{tHD}(S_\alpha) < \omega_1$. Since $\sup_{\alpha<\omega_1}\{\mathrm{trind}(S_\alpha)\} = \omega_1$ (see Example 7.2.12 in [2]), applying Lemma 5.2, we get the following.

Theorem 5.4. *If $(S_\alpha)_{\alpha<\omega_1}$ is a Smirnov's sequnce and and $(C_\alpha)_{\alpha<\omega_1}$ is the corresponding Smirnov's sequence of Cantor sets, then*

(a) $\mathrm{trind}(S_\alpha) \le \mathrm{tHD}(C_\alpha) < \omega_1$.
(b) $\sup_{\alpha<\omega_1}\{\mathrm{tHD}(C_\alpha\} = \omega_1$.
(c) $\#\{\mathrm{tHD}(C_\alpha) : \alpha < \omega_1\} = \aleph_1$.
(d) *The family $(C_\alpha)_{\alpha<\omega_1}$ contains uncountably many Cantor sets, no two of which are bi-Lipschitz equivalent.*
(e) *If $\alpha \ge \omega_0$, then* $\mathrm{HD}(C_\alpha) = +\infty$.

As a consequence of this theorem, Theorem 2.7, and Corollary 3.9, we get the following.

Theorem 5.5. *For every ordinal $0 \le \alpha < \omega_1$ there exists a topological Cantor set $X_\alpha \in \mathcal{M}_0$ (category of separable metric spaces), even a subspace of the Hilbert space l_2, such that* $\mathrm{tHD}(X_\alpha) = \alpha$. *In particular, no two distinct sets X_α are bi-Lipschitz equivalent.*

Proof. For $\alpha = 0$ take X_α to be the middle-third Cantor set. In view of Theorem 5.4(b), Theorem 2.7, and Proposition 5.1, for every ordinal $1 \le \alpha < \omega_1$ there exists a compact metric space $Y_\alpha \subset l_2$ such that $\mathrm{tHD}(Y_\alpha) = \alpha$ and $\mathrm{ind}(Y_\alpha) = 0$. In virtue of Cantor-Bendixon Theorem we can write $Y_\alpha = X_\alpha \cup X_0$ where X_a is a perfect set and X_0 is countable. Since $\mathrm{tHD}(Y_\alpha) \ge 1$, we have $X_\alpha \ne \emptyset$, whence X_α is a topological Cantor set, as $\in (X_\alpha) \le \mathrm{ind}(Y_\alpha) \le 0$. Since X_α is compact, and X_0 is F_σ and $\mathrm{tHD}(X_0) \le 0$ (as X_0 is countable), we get from Corollary 3.9 that $\mathrm{tHD}(X_\alpha) = \mathrm{tHD}(Y_\alpha) = \alpha$. We are done. □

6. Miscellanea

As we have shown in Theorem 2.8, if $\mathrm{HD}(X) < +\infty$, then $\mathrm{tHD}(X) \le E(\mathrm{HD}(X))$. It is however not true that always, if $\mathrm{HD}(X)$ is finite, then $\mathrm{tHD}(X) = E(\mathrm{HD}(X))$. For instance, if $C \subset [0,1]$ is a Cantor set whose Hausdorff dimension is equal to 1 but whose (linear) Lebesgue measure is equal to 0, then $\mathrm{tHD}(C) = 0$. We conjecture:

Conjecture 6.1. *If X is a metric space and* $\mathrm{HD}(X) < +\infty$, *then* $\mathrm{tHD}(X) \ge E(\mathrm{HD}(X)) - 1$. *Consequently,* $\mathrm{tHD}(X) \in \{E(\mathrm{HD}(X)) - 1, \mathrm{tHD}(X)\}$.

In Lemma 5.2 we have shown that for every Smirnov metric space S_α, $\alpha < \omega_1$, and any Cantor Smirnov's space C_α, we have $\mathrm{tHD}(S_\alpha) \ge \mathrm{tHD}(C_\alpha) \ge \mathrm{trind}(S_\alpha)$. Then Lemma 3.6 and the construction of Smirnov spaces allow us to get an explicite upper bound on $\mathrm{tHD}(S_\alpha)$. In fact we conjecture this.

Conjecture 6.2. *For every ordinal number $\alpha < \omega_1$, for every Smirnov metric space S_α, and any Cantor Smirnov's space C_α, we have* $\mathrm{tHD}(S_\alpha) = \mathrm{tHD}(C_\alpha) = \mathrm{trind}(S_\alpha)$.

Remark 6.3. It is easy to see that each separable metric space embeds in a Lipschitz continuous manner into the Hilbert space l_2. Therefore, if X is a separable metric space, then

$$\mathrm{tHD}(X) = \sup\{\mathrm{trind}(\mathrm{Im}(f) : f \in \mathcal{L}(X, l_2)\}.$$

Furthermore, if X is a subspace of a l_2, then (see [1]) each map in $\mathcal{L}(X, l_2)$ extends in a Lipschitz continuous fashion to a map from X to l_2. We then have the following.

$$\mathrm{tHD}(X) = \sup\{\mathrm{trind}(\mathrm{Im}(f) : f \in \mathcal{L}(X, l_2)\}.$$

Remark 6.4. We could have chosen the large transfinite topological dimension trInd to define the transfinite Hausdorff dimension. However, large transfinite dimension is monotone only with respect to closed subspaces, and not for all subspaces. This could affect monotonicity of the corresponding transfinite Hausdorff dimension, making it look less similar to the classical Hausdorff dimension.

Remark 6.5. If we defined the transfinite Hausdorff dimension as the supremum over all closed maps in $\mathcal{L}_c(X)$, where the subscript c indicates that we allow only closed domains and closed maps, we would get the same values for transfinite Hausdorff dimensions of compact metric spaces, and a theory behaving in some aspects better (for example the intermediate subspace theorem would hold for all complete metric spaces) for a larger classes of metric spaces. The transfinite Hausdorff dimension defined in such a way would be also invariant under bi-Lipschitz maps, however the property that $\mathrm{tHD}(f(X)) \le \mathrm{tHD}(X)$ would in general hold only for closed Lipschitz mappings f, which, like for the classical Hausdorff dimension, holds for the transfinite Hausdorff dimension, defined in this paper, for all Lipschitz continuous maps.

References

[1] L. Ambrosio and P. Tilli, Topics on Analysis in Metric Spaces, Oxford University Press 2004.
[2] R. Engelking, Theory of Dimensions, Finite and Infinite, Heldermann Verlag 1995.
[3] K. J. Falconer, Geometry of Fractal Sets, Cambridge University Press 1986.
[4] K. J. Falconer, Fractal Geometry, J. Wiley, New York 1999.
[5] F. Hausdorff, Dimension und äusseres Mass, Math. Ann. 79 (1919), 157-179.
[6] W. Hurewicz, Ueber unendlich-dimensionale Punktmengen, Proc. Akad. Amsterdam 31 (1928), 916-922.
[7] W. Hurewicz and H. Wallman, Dimension Theory, Princeton 1941.

[8] E. Marczewski (Szpilrajn), La dimension et la mesure, Fund. Math. 28 (1937), 81-89.
[9] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, Fractals and Rectifiability, Cambridge University Press 1995.
[10] K. Menger, Über die Dimensionalitätvon Punktmengen I, Monatsh. für Math. und Phys. 33 (1923), 148-160.
[11] F. Przytycki, M. Urbański, Fractals in the Plane - the Ergodic Theory Methods, available on the web:http://www.math.unt.edu/~urbanski, to appear Cambridge Univ. Press.
[12] Ju. M. Smirnov, On universal spaces for certain classes of infinite-dimensional spaces, Amer. Math. Soc. Transl. Ser. 2 21 (1962), 35-50.
[13] P. Urysohn, Les multiplicites Cantoriennes, C.R. Acad. Paris 175 (1922), 440-442.
[14] P. Urysohn, Memoire sur Les multiplicites Cantoriennes, Fund. Math. 7 (1925), 30-137.

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