GRAPH DIRECTED MARKOV SYSTEMS ON HILBERT SPACES

R. D. MAULDIN, T. SZAREK, AND M. URBAŃSKI

Abstract. We deal with contracting finite and countably infinite iterated function systems acting on Polish spaces, and we introduce conformal Graph Directed Markov Systems on Polish spaces. Sufficient conditions are provided for the closure of limit sets to be compact, connected, or locally connected. Conformal measures, topological pressure, and Bowen's formula (determining the Hausdorff dimension of limit sets in dynamical terms) are introduced and established. We show that, unlike the Euclidean case, the Hausdorff measure of the limit set of a finite iterated function system may vanish. Investigating this issue in greater detail, we introduce the concept of geometrically perfect measures and we proved sufficient conditions. Geometrical perfectness guarantees the Hausdorff measure of the limit set to be primitive. As a by–product of the mainstream of our investigations we prove $4r$–covering theorem for all metric spaces. It enables us to establish appropriate co–Frostman type theorems.

1. Introduction

Iterated function systems (abbreviated to IFSs henceforth) arise in many natural contexts. They are often used to encode and generate fractal images, such as landscapes and skyscapes, in computer games. They each generate, via a recursive procedure, a unique fractal set called attractor or limit set. IFSs also play an important role in the theory of dynamical systems. By dynamical system we mean a continuous map T from a metric space X to itself, where, given $x \in X$, one aims at describing the eventual behavior of the sequence of iterates $(T^n x)_{n=0}^{\infty}$. IFSs are in fact a generalization of the process of looking at the backward trajectories of dynamical systems. A repeller in a dynamical system sometimes coincides with the attractor of an associated IFS. For example, the middle-third Cantor set attracts all the backward orbits of the tent map: $T(x) = 3x$ if $x \leq 1/2$ and $T(x) = 3(1-x)$ if $x \geq 1/2$. This follows from the fact that the inverse branches of T are precisely the generators φ_1 and φ_2 of an IFS. As dynamical systems often model physical processes, the study of IFSs frequently turns out to be instrumental in describing real systems.

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The systematic development of modern theory of iterated function systems began with the works of Hutchinson [5], Falconer [4], Barnsley [2], Bandt and Graf [1], and Schief [17], just to name a few. Their pioneering works in the 1980's and 1990's on finite IFSs concerned systems consisting of similarities. In particular, the theory of finite IFSs was used in the study of the complex dynamics of rational functions.

In the middle of the 1990's, the need to investigate finite, and even infinite, IFSs that comprise more general conformal maps arose. The foundations of that theory were laid out by Mauldin and Urbański in [9]. Many new applications were also discovered. In 1999, Mauldin and Urbański [10] applied the theory of infinite conformal IFSs to continued fractions with restricted entries. Few years later, Kotus and Urbański [6] applied that theory to obtain a lower estimate for the Hausdorff dimension of the Julia set of elliptic functions, while Urbański and Zdunik [25] obtained similar results in the complex dynamics of the exponential function. They further showed that the framework of conformal IFSs is the right one to study the harmonic measure of various Cantor sets. Applications of conformal IFSs to number theory, more precisely to the theory of continued fractions and Diophantine approximations, were also developed by Urbański in [21], [22], [23], and [24]. Finally, Stratmann and Urbański [18] established extremality in the sense of Kleinbock, Lindenstrauss and Barak Weiss, of conformal measures for convex co-compact Kleinian groups.

A further generalization of the theory was achieved by Mauldin and Urbański [12] in 2003. This generalization relies on graphs. Indeed, every IFS can be thought of as a graph with a single vertex and a countable set of self-loops. To the unique vertex is attached a space $X \subset \mathbb{R}^d$. The self-loops represent the generators of the IFS, each generator being a contracting self-map of X. The limit set generated by all infinite paths on this graph is also attached to the vertex. Mauldin and Urbański extended the theory of IFSs to graph directed Markov systems (GDMSs). The graphs associated to these systems generally have more than one, though finitely many, vertices. Moreover, they can have a countably infinite set of edges between any two of their vertices. It turns out that a unique fractal set can be associated with each vertex, and the limit set is the disjoint union of those sets. In 2007, Stratmann and Urbański [19] went one step further by laying the foundations to the theory of pseudo-Markov systems. In contradistinction with GDMSs, the underlying graph of these systems can have infinitely many vertices. They applied their work to infinitely generated Schottky groups.

In all those works, the metric structure of the limit set J is usually described by its Hausdorff, packing and/or box-counting dimension(s). The Hausdorff dimension is particularly interesting as it is characterized by the pressure function. The pressure function plays a central role in thermodynamic formalism, which arose from statistical physics. For finite systems which satisfy a certain separation property (the famous open set condition (OSC)), the Hausdorff dimension of the limit set is simply the zero of the pressure function $P(t)$, that is, the unique $t_0 > 0$ such that $P(t_0) = 0$. This latter equation is sometimes called Moran-Bowen formula. For infinite systems satisfying the OSC and a bounded distortion property, Mauldin and Urbański [9] showed that a variant of Moran-Bowen formula holds: the Hausdorff dimension of the limit set is the infimum of all $t \geq 0$ for which $P(t) \leq 0$.

They also showed that a t-conformal measure exists if and only if $P(t) = 0$, and there is at most one such t.

In this paper we propose to extend the theory of conformal iterated function systems and their generalization, graph directed Markov systems to the context of Hilbert spaces. Hilbert spaces carry enough geometric structure to allow us to define and meaningfully consider conformal homeomorphisms. In particular a version of Liouville's classification theorem holds. To define conformal graph directed systems on Hilbert spaces is natural and not too difficult. However when we investigate such systems a number of new geometric and dynamical phenomena appear that differentiate them drastically from their finite–dimensional counterparts. We now want to discuss these phenomena in greater detail providing simultaneously a description of the content of our paper.

As a matter of fact we start our paper with even more general setting of iterated function systems acting on complete metric spaces. Introducing the concept of asymptotically compact systems we provide necessary and sufficient conditions for the closure of the limit set to be compact. Several examples are given illustrating the case with compact and non–compact closures $\overline{J}$ of the limit set J as well as various relations between $\overline{J}$ and J.

Sticking with topological issues we then turned our attention to the concept of connectedness. We established sufficient conditions for $\overline{J}$ to be a continuum. This condition is significant even in the case of a compact phase space X or a finite alphabet. Our next step was to look at local connectivity. If the alphabet is finite, then it is well–known that connectivity of the limit set implies its local connectivity. As we demonstrate, in the case of an infinite alphabet it is not the case any more. However we provide a sufficient conditions for the closure of a limit set to be a locally connected continuum, and, using it we easily conclude that the closure of the limit set of complex continued fractions iterated function system, (see [9] for definition), is locally connected.

In Section 7 we introduce conformal Graph Directed Markov Systems in Hilbert spaces. We define for them the concept of topological pressure, conformal measures, and thermodynamical formalism. We are here in a position to apply the symbolic thermodynamic formalism developed in [11] and [12]. In order to investigate geometric properties of limits sets we need a covering theorem. We have been able to find in the literature only results that pertain to Euclidean, or at best, to compactly bounded metric space. We prove the $4r$–covering Theorem which implies the existing $5r$–covering Theorem by reducing 5 to 4, and what is much more important, by replacing boundedly compact metric spaces just by metric spaces alone. Making substantial use of this covering theorem we prove co–Frostman lemmas for both Hausdorff and packing measures on metric space. They provide sufficient conditions for a metric space to have a positive, finite or infinite (generalized) Hausdorff or packing measure, and in some literature, in the context of Euclidean spaces are referred to as mass redistribution principle.

In the next section we establish Bowen's formula. Following Schief's paper [17] it has been proved [13] that in the case of finite alphabet on Euclidean space the Open Set Condition (OSC), the Strong Open Set Condition (SOSC) and positivity of Hausdorff measure (with exponent equal to the Bowen's parameter, the only zero of the pressure function) are equivalent. Being still in Euclidean spaces but allowing the alphabet to be infinite all the equivalences fail. Example of infinite alphabet similarity systems with the SOSC but vanishing Hausdorff measure were given in [9], whereas infinite similarity system with the OSC but without the SOSC were constructed in [20]. In this paper we actually demonstrate that the OSC is of rather marginal value in Hilbert space even in the case of finite alphabet similarities. Namely, Bowen's formula fails. However, with the SOSC Bowen's formula holds even if the alphabet is infinite.

The last section is devoted to investigate the behaviour of the Hausdorff measure $H_h(J)$. In the case of Euclidean spaces it is well–known that we have always $0 < H_h(J) < +\infty$. However, as Example 10.1 shows, this is not longer the case in Hilbert spaces. In order to deal with positivity of the Hausdorff measure we introduce the concept of geometrical perfectness of a limit set, and we provide effective sufficient condition for IFS to be geometrically perfect.

2. General setup

Graph directed Markov systems (GDMS) are based upon a directed multigraph and an associated incidence matrix (V, E, i, t, A). The multigraph consists of a finite set V of vertices and a countable (either finite or infinite) set of directed edges E and two functions $i, t : E \to V$. For each e, $i(e)$ is the initial vertex of the edge e and $t(e)$ is the terminal vertex of e. The edge goes from $i(e)$ to $t(e)$. Also a function $A : E \times E \to \{0, 1\}$ is given, called an incidence matrix. The matrix A is an edge incidence matrix. It determines which edges may follow a given edge. So, the matrix has the property that if $A_{uv} = 1$, then $t(u) = i(v)$. We will consider finite and infinite walks through the vertex set consistent with the incidence matrix. Thus, we define the set of infinite admissible words

$$E_A^\infty = \{\omega \in E^\infty : A_{\omega_i\omega_{i+1}=1} \text{ for all } i \geq 1\},$$

by E_A^* we denote the set of all finite subwords of E_A^∞. Additionally, define the set E_A^0 as the set composed of the empty word. We will drop the subscribe A when the matrix is clear from context.

All elements of $E_A^* \cup E_A^\infty$ are called A–admissible words. If ω is A–admissible, then there exists $0 \leq n \leq \infty$ such that the length of ω is equal to n. We write then $|\omega| = n$. The set of all subwords of E_A^∞ of length $n \geq 1$ is denoted by E_A^n. If $k \leq |\omega|$, then $\omega_{|_k} = \omega_1 \cdots \omega_k$. We put $[\omega] = \{\tau \in E_A^\infty : \tau_{|_{|\omega|}} = \omega\}$. Let $\omega, \tau \in E_A^* \cup E_A^\infty$. If $\tau_{|_{|\omega|}} = \omega$, then we write $\omega \leq \tau$. By $\omega \wedge \tau$ we denote the word $\upsilon \in E_A^* \cup E_A^\infty$ with maximal length such that $\upsilon \leq \omega$ and $\upsilon \leq \tau$. If neither $\omega \leq \tau$ nor $\tau \leq \omega$, then ω, τ are called incomparable.

As usual, $\sigma : E_A^\infty \to E_A^\infty$ denotes the left shift, i.e. $\sigma(\omega_1\omega_2 \ldots) = \omega_2\omega_3 \ldots$

Definition 2.1. *The incidence matrix A is called* finitely irreducible *provided that there exists a finite set $\Lambda \subset E_A^*$ such that:*

$$\forall\, a \in E \ \ \forall\, b \in E \ \ \exists\, \gamma \in \Lambda \qquad a\gamma b \in E_A^*.$$

The matrix A is called finitely primitive *if one can find Λ as above consisting of words with the same length.*

Suppose now that for every $v \in V$ is given a complete metrizable space X_v. Assume $\{X_v\}_{v\in V}$ to be mutually disjoint and consider $X = \oplus_{v\in V} X_v$, the disjoint union of X_v, $v \in V$, endowed with a complete metric ρ. We assume that the space (X, ρ) is bounded.

Definition 2.2. *A collection* $\mathcal{S} = \{\varphi_e : X_{t(e)} \to X_{i(e)} : e \in E\}$ *is a* complete contracting Markov system *provided that all the maps* φ_e*,* $e \in E$*, are injective and closed and there is* $s \in (0,1)$ *such that:*

$$\rho(\varphi_e(x), \varphi_e(y)) \le s\rho(x,y) \qquad \forall\, e \in E,\ \forall\, x, y \in X_{t(e)}.$$

The Markov system $\mathcal{S}$ is called *rapidly decreasing* provided that

$$\lim_{e\to\infty} \operatorname{diam}(\varphi_e(X_{t(e)})) = 0, \tag{2.1}$$

meaning that for every $\varepsilon > 0$ there exists a finite subset F of E such that

$$\operatorname{diam}(\varphi_e(X_{t(e)})) < \varepsilon$$

for all $e \in E \setminus F$.

For every $\omega \in E_A^*$, say $\omega \in E_A^n$, set

$$\varphi_\omega = \varphi_{\omega_1} \circ \cdot \circ \varphi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)}$$

and note that this composition is well–defined.

The Markov system $\mathcal{S}$ is called *pointwise finite* if and only if

$$\#\{\omega \in E_A^n : x \in \varphi_\omega(X_{t(\omega)})\} < \infty \qquad \forall\, n \ge 1, \forall\, x \in X.$$

The system $\mathcal{S}$ is called *finitely irreducible (primitive)* if and only if the matrix A is finitely irreducible (primitive). The system $\mathcal{S}$ is called an *iterated function system* if and only if the set of vertices V is a singleton and the incidence matrix A has no zeros.

Now, fix $\omega \in E_A^\infty$. Then the sequence $(\varphi_{\omega_{|n}}(X_{t(\omega_n)}))_1^\infty$ is descending, consists of closed sets and $\operatorname{diam}(\varphi_{\omega_{|n}}(X_{t(\omega_n)})) \le s^n \operatorname{diam}(X_{t(\omega_n)}) \le s^n \operatorname{diam}(X)$. Hence $\bigcap_{n=1}^\infty \varphi_{\omega_{|n}}(X_{t(\omega_n)})$ is a singleton, and denotes its only element by $\pi(\omega)$. We have thus defined a Lipschitz map

$$\pi : E_A^\infty \to X,$$

if E_A^∞ is endowed with the metric $d_s(\omega,\tau) = s^{|\omega\wedge\tau|}$. The image of π, i.e. the set $\pi(E_A^\infty)$, is called the *limit set* of the system $\mathcal{S}$ and is denoted by $J_{\mathcal{S}}$. Note that if the system $\mathcal{S}$ is pointwise finite, then

$$J_{\mathcal{S}} = \bigcap_{n=1}^{\infty} \bigcup_{\omega\in E_A^n} \varphi_\omega(X_{t(\omega_n)}). \tag{2.2}$$

We now pass to describe $\overline{J_{\mathcal{S}}}$, the closure of $J_{\mathcal{S}}$. Recall that for any indexed family $\mathcal{F} = \{F_t\}_{t\in T}$ of subsets of X, $\overline{\lim}\,\mathcal{F}$, the topological lim sup of $\mathcal{F}$ is defined as follows

$$x \in \overline{\lim}\,\mathcal{F} \Leftrightarrow \forall\, \varepsilon > 0\ \exists\, T_\varepsilon \subset T\ \#T_\varepsilon \ge \aleph_0\ \forall\, t \in T_\varepsilon\ F_t \cap B(x,\varepsilon) \ne \emptyset.$$

We set

$$\partial S = \overline{\lim}\,\{\varphi_e(X_{t(e)})\}_{e\in E}$$

and for every $a \in E$,

$$\partial S(a) = \overline{\lim}\,\{\varphi_e(X_{t(e)}) : A_{ae} = 1\}_{e\in E} \subset X_{t(a)}$$

and

$$J_a = \pi([a]) \subset X_{i(a)},\ J_a^- = \pi(\bigcup_{b:A_{ab}=1} [b]) = \bigcup_{b:A_{ab}=1} J_b \subset X_{t(a)}.$$

We also put

$$\partial S(\omega) = \partial S(\omega_{|\omega|}),\ J_\omega^- = J_{\omega_{|\omega|}}^-,\ \omega \in E_A^*,$$

and for every $v \in V$,

$$J_v = \pi(\{\omega \in E_A^\infty : i(\omega_1) = v\}) \text{ and } \partial S(v) = X_v \cap \partial S.$$

As a direct consequence of the rapid decreasing condition (2.1), we get the following.

Lemma 2.1. *If $\mathcal{S}$ is rapidly decreasing, then for any $a \in E$ and $\omega \in E_A^*$ we have*

$$\varphi_a(J_a^-) = J_a, \quad \partial S(a) = \overline{\lim}\,\{\varphi_e(J_e^-) : A_{ae} = 1\}_{e\in E} \subset \overline{J_a^-}, \quad \partial S(\omega) \subset \overline{J_\omega^-}.$$

From now onwards each Markov system considered is assumed to be finitely primitive. We shall prove the following.

Proposition 2.1. *If the system $\mathcal{S}$ is rapidly decreasing, then for every vertex $v \in V$, we have*

$$\overline{J_v} = J_v \cup \bigcup_{\omega\in E_v^*} \varphi_\omega(\partial S(\omega)) \subseteq J_v \cup \partial S(v),$$

where $E_v^* = \{\omega \in E_A^* : i(\omega) = v\}$.

Proof. The inclusion is obvious. Let us prove the equality part. Indeed, by Lemma 2.1, for any $\omega \in E_v^*$ we have

$$\varphi_\omega(\partial S(\omega)) \subseteq \varphi_\omega(\overline{J_\omega^-}) \subseteq \overline{J_{i(\omega)}} = \overline{J_v}.$$

Thus the "$\supseteq$" part of the equality claimed is proved. In order to prove the reverse inclusion, fix $x \in \overline{J_v}$. Then there exists a sequence $(\omega^{(n)})_1^\infty$ of elements of E_A^∞ with $i(\omega^{(n)}) = v$, such that $x = \lim_{n\to\infty} \pi(\omega^{(n)})$. For each $k \geq 0$, define

$$E_k(x) = \{\omega^{(n)}_{|_k} : n \geq k\} \qquad (E_0(x) = \{\emptyset\})$$

and note that if $\tau \in E_{k+1}(x)$, then $\tau_{|_k} \in E_k(x)$. Thus the set $\{E_k(x) : k \geq 0\}$ is a tree rooted at the vertex $E_0(x)$. We now distinguish two cases. First, suppose that there exists $k \geq 1$ such that $E_k(x)$ has infinitely many elements. Then put

$$q := \min\{k \geq 1 : E_k(x)\,\text{is infinite}\}.$$

So, the set $E_{q-1}(x)$ is finite and non-empty (although it may be equal to the singleton $\{\emptyset\}$). There thus exists $\tau \in E_{q-1}(x) \subset E_A^{q-1}$ and an infinite sequence $(\omega_q^{(n_j)})_{j=1}^\infty$ of distinct elements of E such that $\tau\omega_q^{(n_j)} = \omega^{(n_j)}_{|_q}$ for all $j \geq 1$. Now note that

$$\pi(\sigma^{(q-1)}(\omega^{(n_j)})) = \varphi_\tau^{-1}(\pi(\omega^{(n_j)}))$$

and $\pi(\omega^{(n_j)}) \in X_{i(\tau)}$. Since the map $\varphi_\tau^{-1} : \varphi_\tau(X_{t(\tau)}) \to X_{t(\tau)}$ is a homeomorphism (as φ_τ is closed), the sequence $(\pi(\sigma^{(q-1)}(\omega^{(n_j)})))_{j=1}^\infty$ converges to $\varphi_\tau^{-1}(x)$. Note then that $\varphi_\tau^{-1}(x) \in \partial S(\tau)$, and therefore $x \in \varphi_\tau(\partial S(\tau))$. As $\tau \in E_v^*$, we are done. Now suppose that $E_k(x)$ is finite for each $k \geq 1$. Since, as mentioned above, these sets form a tree rooted at $E_0(x)$, it follows from König's Infinity Lemma that there exists an infinite path $\omega \in E^{\mathbb{N}}$ such that $\omega_{|_k} \in E_k(x)$ for all $k \geq 0$. So, $\omega \in E_A^\infty$, $i(\omega) = v$ and there exists an increasing sequence $(n_k)_1^\infty$ such that $\omega_{|_k} = \omega_{|_k}^{(n_k)}$ for all $k \geq 0$. Hence $\pi(\omega), \pi(\omega^{(n_k)}) \in \varphi_{\omega_{|_k}}(X_{t(\omega_k)})$ for $k \geq 0$. Since $\lim_{n\to\infty} \operatorname{diam} \varphi_{\omega_{|_k}}(X_{t(\omega_k)}) = 0$, we conclude that $x = \lim_{k\to\infty} \pi(\omega^{(n_k)}) = \pi(\omega) \in J_v$. We are done. □

Corollary 2.1. *If the system $\mathcal{S}$ is rapidly decreasing, then*

$$\overline{J_{\mathcal{S}}} = J_{\mathcal{S}} \cup \bigcup_{\omega \in E_A^*} \varphi_\omega(\partial S(\omega)) \subseteq J_{\mathcal{S}} \cup \partial S.$$

3. Compactness of $\overline{J_{\mathcal{S}}}$.

Since $J_{\mathcal{S}} = \pi(E_A^\infty)$ and $\pi : E_A^\infty \to X$ is continuous, if the alphabet E is finite, then the limit set (equal to its closure) is compact. In the case when the alphabet E is infinite neither $J_{\mathcal{S}}$ nor even $\overline{J_{\mathcal{S}}}$ need to be compact (see Examples 2.1 and 2.2). We provide in this section verifiable necessary and sufficient conditions for the closure $\overline{J_{\mathcal{S}}}$ to be compact.

A set $F \subset X$ is called *attracting* for $\mathcal{S}$ iff for every $\varepsilon > 0$, $\varphi_e(X_{t(e)}) \subseteq \mathcal{N}(F, \varepsilon)$ for all but finitely many indices $e \in E$, where $\mathcal{N}(F, \varepsilon) = \{y \in X : \exists\, x \in X,\ \rho(x, y) < \varepsilon\}$. Clearly

$$\partial S \subset F. \tag{3.1}$$

Let us record the following straightforward fact.

Observation 3.1. *If $\mathcal{S}$ is a rapidly decreasing sequence, then $\overline{J_{\mathcal{S}}}$ is an attracting set for $\mathcal{S}$.*

Given two subsets A, B of X, we set

$$\operatorname{dist}(A, B) = \inf\{\rho(a, b) : (a, b) \in A \times B\},$$
$$\operatorname{Dist}(A, B) = \inf\{\varepsilon > 0 : A \subseteq \mathcal{N}(B, \varepsilon)\}$$

and

$$\rho_H(A, B) = \max\{\operatorname{Dist}(A, B), \operatorname{Dist}(B, A)\}.$$

The number $\rho_H(A, B)$ is called the *Hausdorff distance* from A to B. Restricted to the family $\mathcal{K}(X)$, the compact subsets of X, ρ_H becomes a metric. The topology induced by ρ_H coincides with the compact–open (Vietoris) topology. With this terminology we can say that $F \subseteq X$ is an attracting set for $\mathcal{S}$ if and only if $\lim_{e\to\infty} \operatorname{Dist}(\varphi_e(X_{t(e)}), F) = 0$.

The system $\mathcal{S}$ is called *asymptotically compact* iff it admits a compact attracting set. Then

$$\partial S \neq \emptyset. \tag{3.2}$$

We shall prove the followinig.

Theorem 3.1. *A rapidly decreasing system $\mathcal{S}$ is asymptotically compact iff $\overline{J_\mathcal{S}}$ is a compact set.*

Proof. Suppose first that the system $\mathcal{S}$ is asymptotically compact. We are going to show that $\overline{J_\mathcal{S}}$ is compact. The proof parallels mostly the proof of Proposition 2.1 but we provide it here in full for the convenience of the reader and since some details are different. It suffices to show that any sequence $(\omega^{(n)})_1^\infty$ has a subsequence $(\omega^{(n_j)})_{j=1}^\infty$ such that $(\pi(\omega^{(n_j)}))_{j=1}^\infty$ converges. For each $k \geq 0$ define

$$E_k = \{\omega^{(n)}|_k : n \geq k\}, \ \ (E_0 = \{\emptyset\})$$

and note that if $\tau \in E_{k+1}$, then $\tau_{|_k} \in E_k$. Then the set $\{E_k : k \geq 0\}$ is a tree rooted at the vertex $E_0 = \{\emptyset\}$. We now distinguish two cases. First suppose that there exists $k \geq 1$ such that E_k has infinitely many elements. Then put

$$q = \min\{k \geq 1 : E_k \text{ is infinite}\}.$$

So, the set E_{q-1} is finite and non-empty (although it may be equal to the singleton $\{\emptyset\}$). There thus exists $\tau \in E_{q-1} \subseteq E_A^{q-1}$ and an infinite sequence $(\omega_q^{(n_j)})_{j=1}^\infty$ of distinct elements of E such that $\tau\omega_q^{(n_j)} = \omega^{(n_j)}|_q$ for all $j \geq 1$. Since the attracting set F is compact and since $\lim_{j\to\infty} \mathrm{Dist}(\varphi_{\omega_q^{(n_j)}}(X_{t(\omega_q^{(n_j)})}), F) = 0$, passing to a subsequence, we may assume without loss of generality that there exists a point $z \in F$ such that $\lim_{j\to\infty} \mathrm{Dist}(\varphi_{\omega_q^{(n_j)}}(X_{t(\omega_q^{(n_j)})}), \{z\}) = 0$. But since $\pi(\sigma^{q-1}(\omega^{(n_j)})) \in \varphi_{\omega_q^{(n_j)}}(X_{t(\omega_q^{(n_j)})})$, we thus conclude that $\lim_{j\to\infty} \pi(\sigma^{q-1}(\omega^{(n_j)})) = z$. Since the map $\varphi_\tau : X_{t(\tau)} \to X_{i(\tau)}$ is continuous, we therefore get that

$$\lim_{j\to\infty} \pi(\omega^{(n_j)}) = \lim_{j\to\infty} \varphi_\tau(\pi(\sigma^{q-1}(\omega^{(n_j)}))) = \varphi_\tau(z).$$

In particular, the sequence $(\pi(\omega^{(n_j)})_{j=1}^\infty$ converges, and we are done. Now, suppose that the set E_k is finite for each $k \geq 1$. Since, as mentioned above, these sets form a tree rooted at E_0, it follows from König's Infinite Lemma that there exists an infinite path $\omega \in E^\mathbb{N}$ such that $\omega_{|_k} \in E_k$ for all $k \geq 0$. So, $\omega \in E_A^\infty$ and there exists an infinite sequence $(n_k)_1^\infty$ such that $\omega_{|_k} = \omega_{|_k}^{n_k}$ for all $k \geq 0$. Hence, $\pi(\omega), \pi(\omega^{(n_k)}) \in \varphi_{\omega_{|_k}}(X_{t(\omega_k)})$. Since $\lim_{k\to\infty} \mathrm{diam}(\varphi_{\omega_{|_k}}(X_{t(\omega_k)}) = 0$, we conclude that $(\pi(\omega^{(n_k)}))_{k=1}^\infty$ converges to $\pi(\omega)$. We are done. The implication "$\Leftarrow$" follows directly from Observation 3.1. The proof is complete. $\square$

Examples: We shall describe three simple examples about various relations between $J_\mathcal{S}$ and $\overline{J_\mathcal{S}}$. Let $\mathcal{H}$ be a separable Hilbert space. Let $B(0,1)$ be the open ball in $\mathcal{H}$ centered at 0 and with radius 1. Let $S^\infty = \partial B(0,1)$ and let $\overline{B}(0,1) = \overline{B(0,1)}$.

Example 3.1. Fix a countable set $(z_n)_1^\infty$ of points in $B(0,1)$ whose closure contains S^∞. One can then easily construct by induction a sequence $(\varphi_n)_1^\infty$ of bijective similarities of $\mathcal{H}$ with the following properties:

(a) $\varphi_n(0) = z_n$;
(b) $\varphi_n(\overline{B}(0,1)) \subseteq B(0,1)$;

(c) $\lim_{n\to\infty} \operatorname{diam}(\varphi_n(\overline{B}(0,1))) = 0$;
(d) $\varphi_i(\overline{B}(0,1)) \cap \varphi_j(\overline{B}(0,1)) = \emptyset$ whenever $i \neq j$;
(e) $\|\varphi_n'\| \le 1/2$ for all $n \ge 1$.

Then $\mathcal{S} = \{\varphi_n : \overline{B}(0,1) \to \overline{B}(0,1), n \ge 1\}$ forms an IFS satisfying all the requirements of this section and $\overline{J_\mathcal{S}} \supseteq \partial S \supseteq S^\infty$. Since S^∞ is not compact, it follows that $\overline{J_\mathcal{S}}$ is not compact and $\mathcal{S}$ is not asymptotically compact. Also $J_\mathcal{S} \subsetneq \overline{J_\mathcal{S}}$.

Example 3.2. Let $(e_n)_1^\infty$ be an orthonormal basis in $\mathcal{H}$. We can easily construct a sequence $(\varphi_n)_1^\infty$ of bijective similarities of $\mathcal{H}$ with the following properties:
(a) $\varphi_n(0) = \frac{1}{2}e_n$ for all $n \ge 1$;
(b) $\varphi_n(\overline{B}(0,1)) \subseteq B(0,1)$;
(c) $\lim_{n\to\infty} \operatorname{diam}(\varphi_n(\overline{B}(0,1))) = 0$;
(d) $\varphi_i(\overline{B}(0,1)) \cap \varphi_j(\overline{B}(0,1)) = \emptyset$ whenever $i \neq j$;
(e) $\|\varphi_n'\| \le 1/2$ for all $n \ge 1$.

Then $\mathcal{S} = \{\varphi_n : \overline{B}(0,1) \to \overline{B}(0,1), n \ge 1\}$ forms an IFS satisfying all our requirements. It follows from (a) and (c) that $\partial S = \emptyset$. It then follows from (3.2) that $\mathcal{S}$ is not asymptotically compact, and, by Theorem 3.1, $\overline{J_\mathcal{S}}$ is not compact. It however follows from Corollary 2.1 that $\overline{J_\mathcal{S}} = J_\mathcal{S}$.

Example 3.3. The same as Example 3.1 but $(z_n)_1^\infty$ is an arbitrary sequence of distinct points such that $\lim_{n\to\infty} z_n$ exists, say is equal to z_∞. Then $\{z_\infty\}$ is a compact attracting set, our system $\mathcal{S}$ is asymptotically compact, and therefore, in virtue of Theorem 3.1, $\overline{J_\mathcal{S}}$ is a compact set.

4. Connectedness and local connectedness

4.1. **Finite alphabet case.** We assume here that that the set E is finite. Through the section our graph directed system is assumed to be an iterated function system, that is there is only one vertex and all entries of the matrix A are equal to 1. For all $x, y \in J_\mathcal{S}$ let $\mathrm{n}(x,y) \ge 0$ be the largest integer $n \ge 0$ with the property that there exists $\omega \in E^n$ such that $x, y \in \varphi_\omega(J_\mathcal{S})$.

Lemma 4.1. *For every $x \in J_\mathcal{S}$, $\lim_{y\to x} \mathrm{n}(x,y) = +\infty$.*

Proof. Suppose for the contrary that for some $x \in J_\mathcal{S}$ we have $l := \liminf_{y\to x} \mathrm{n}(x,y) < \infty$. Then there exists a sequence $(y_k)_1^\infty$ converging to x and such that $\mathrm{n}(x,y_k) = l$ for all $k \ge 1$. For every $k \ge 1$ there thus exists a word $\omega^{(k)}$ of length $l+1$ such that $x \notin \varphi_{\omega^{(k)}}(J_\mathcal{S})$ and $y_k \in \varphi_{\omega^{(k)}}(J_\mathcal{S})$. Since there are only finitely many words of length $l+1$, passing to a subsequence, we may assume without loss of generality, that all words $\omega^{(k)}$ are equal, say, to ω. So, $x \notin \varphi_\omega(J_\mathcal{S})$ and $y_k \in \varphi_\omega(J_\mathcal{S})$ for all $k \ge 1$. Since $y_k \to x$ and since the set $\varphi_\omega(J_\mathcal{S})$ is closed, we conclude that $x \in \varphi_\omega(J_\mathcal{S})$. This contradiction finishes the proof. □

Theorem 4.1. *The following conditions are equivalent:*

(a) *$J_{\mathcal{S}}$ has finitely many connected components;*
(b) *$J_{\mathcal{S}}$ has finitely many arcwise connected components;*
(c) *$J_{\mathcal{S}}$ is locally connected;*
(d) *$J_{\mathcal{S}}$ is locally arcwise connected.*

Proof. The strategy of the proof is to establish the following equivalences: $(a) \Leftrightarrow (c)$, $(a) \wedge (c) \Leftrightarrow (b) \wedge (d)$, $(b) \Rightarrow (a)$ and $(d) \Rightarrow (c)$.

$(a) \Rightarrow (c)$ Let $J_1, \ldots, J_k$ be all the connected components of $J_{\mathcal{S}}$. Then all $J_1, \ldots, J_k$ are continua and $J_{\mathcal{S}} = J_1 \cup \ldots \cup J_k$. Fix $\varepsilon > 0$ and take $n \geq 0$ so large that $\operatorname{diam} \varphi_\omega(X) < \varepsilon$ for all $\omega \in E^n$. Since $J_{\mathcal{S}} = \bigcup_{j=1}^k \bigcup_{|\omega| \in E^n} \varphi_\omega(J_j)$, since E^n is finite and since all sets are compact connected, (c) follows directly from the Hahn–Mazurkiewicz–Sierpiński Theorem (see [7])

$(c) \Rightarrow (a)$ Since $J_{\mathcal{S}}$ is locally connected, all its connected components are open, and since $J_{\mathcal{S}}$ is compact, there can be only finitely many of them.

$(b) \wedge (d) \Rightarrow (a) \wedge (c)$ is obvious.

$(a) \wedge (c) \Rightarrow (b) \wedge (d)$ Since each connected component of $J_{\mathcal{S}}$ is locally connected, in virtue of Hahn–Mazurkiewicz–Sierpiński's Theorem, it is arcwise connected and locally arcwise connected. So $(b) \wedge (d)$ is established. The implications $(b) \Rightarrow (a)$ and $(d) \Rightarrow (c)$ are obvious. □

Theorem 4.2. *Suppose $J_{\mathcal{S}}$ has only finitely many connected components, say, $J_1, \ldots, J_q$. Then $q \leq \#E^n/2$, where $n \geq 1$ is so large that $s^n \operatorname{diam} J_{\mathcal{S}} < \min\{\operatorname{dist}(J_i, J_j) : i \neq j\}$.*

Proof. Let $J_1, \ldots, J_q$ be all the connected components of $J_{\mathcal{S}}$. If $q = 1$, there is nothing to prove. So, suppose that $q \geq 2$. Let $n \geq 1$ be such that $s^n \operatorname{diam} J_{\mathcal{S}} < \min\{\operatorname{dist}(J_i, J_j) : i \neq j\}$. Observe that for every $\omega \in E^n$ there exists $k \in \{1, \ldots, q\}$ such that $\varphi_\omega(J_{\mathcal{S}}) \subset J_k$. Since J_k, $k \in \{1, \ldots, q\}$, is a connected component, for every $k \in \{1, \ldots, q\}$ there exists at least two distinct words, say, $\omega, \omega' \in E^n$ such that the above condition holds. Hence $2q \leq \#E^n$, which finishes the proof. □

4.2. **Infinite alphabet case.** In this section $\mathcal{S} = \{\varphi_i : X \to X, i \in I\}$ is a contracting pointwise finite rapidly decreasing iterated function system and X is assumed to be a complete connected space. Define inductively the sequence of $(Y_n)_0^\infty$ of closed subsets of X as follows:

$$Y_0 = X, \qquad Y_{n+1} = \overline{\bigcup_{i \in I} \varphi_i(Y_n)}. \tag{4.1}$$

Lemma 4.2. *The sequence $(Y_n)_0^\infty$ is descending and $\overline{J_{\mathcal{S}}} = \bigcap_{n=0}^\infty Y_n$.*

Proof. First, let us prove by induction that the sequence $(Y_n)_0^\infty$ is descending. Indeed, $Y_1 \subseteq X = Y_0$. So, suppose that $Y_n \subseteq Y_{n-1}$ for some $n \geq 1$. Then

$$Y_{n+1} = \overline{\bigcup_{i \in I} \varphi_i(Y_n)} \subseteq \overline{\bigcup_{i \in I} \varphi_i(Y_{n-1})} = Y_n.$$

So, we are done by induction. Now, let us establish the equality $\overline{J_{\mathcal{S}}} = \bigcap_{n=0}^\infty Y_n$. Indeed, it follows from (2.2), that $J_{\mathcal{S}} \subseteq Y_n$ for all $n \geq 0$. So, $J_{\mathcal{S}} \subseteq \bigcap_{n=0}^\infty Y_n$, and, as $\bigcap_{n=0}^\infty Y_n$ is closed, $\overline{J_{\mathcal{S}}} \subseteq \bigcap_{n=0}^{\infty} Y_n$. In order to prove the reverse inclusion for every $k \geq 0$ and every $x \in Y_k \setminus \overline{J_{\mathcal{S}}}$ consider the set

$$E_k(x) = \{\omega \in I^k : x \in \varphi_\omega(X)\}.$$

We shall show that

$$E_k(x) \neq \emptyset \tag{4.2}$$

for all $x \in Y_k \setminus \overline{J_{\mathcal{S}}}$ and all $k \geq 0$. Indeed $E_0(x) = \{\emptyset\} \neq \emptyset$. In order to proceed further by induction, suppose that $E_k(x) \neq \emptyset$ for some $k \geq 0$ and all $x \in Y_k \setminus \overline{J_{\mathcal{S}}}$. Take an arbitrary $x \in Y_{k+1} \setminus \overline{J_{\mathcal{S}}}$. Then $x \in \overline{\bigcup_{i \in I} \varphi_i(Y_k)}$. So, either $x \in \partial S$ or there exists $i \in I$ such that $x \in \varphi_i(Y_k)$. The former case is ruled out since $x \notin \overline{J_{\mathcal{S}}}$. In the latter case, there exists $z \in Y_k$ such that $x = \varphi_i(z)$. Since $\varphi_i(\overline{J_{\mathcal{S}}}) \subset \overline{J_{\mathcal{S}}}$ and since $x \notin \overline{J_{\mathcal{S}}}$, we conclude that $z \notin \overline{J_{\mathcal{S}}}$. Thus $z \in Y_k \setminus \overline{J_{\mathcal{S}}}$. Hence, by the induction hypothesis, $E_k(z) \neq \emptyset$. Take $\omega \in E_k(z)$. Then $x = \varphi_i(z) \in \varphi_i(\varphi_\omega(X)) = \varphi_{i\omega}(X)$. This means that $i\omega \in E_{k+1}(x)$, in particular $E_{k+1}(x) \neq \emptyset$. Our inductive proof is complete. Now, suppose for a contradiction that $\bigcap_{n=0}^{\infty} Y_n \setminus \overline{J_{\mathcal{S}}} \neq \emptyset$ and fix $y \in \bigcap_{n=0}^{\infty} Y_n \setminus \overline{J_{\mathcal{S}}}$. It then follows from (4.2) that $y \in \bigcup_{|\omega|=k} \varphi_\omega(X)$ for all $k \geq 0$. So, in view of (2.2), $y \in J_{\mathcal{S}}$, which is impossible. The proof is finished. $\square$

Proposition 4.1. *If $\overline{J_{\mathcal{S}}}$ is compact and all the sets Y_n, $n \geq 1$, are connected, then $\overline{J_{\mathcal{S}}}$ is connected.*

Proof. Suppose that $\overline{J_{\mathcal{S}}}$ is not connected. Then $\overline{J_{\mathcal{S}}} = A \cup B$, where A and B are disjoint non-empty closed subsets of $\overline{J_{\mathcal{S}}}$, hence of X. By the Urysohn Lemma there exist two open (in X) disjoint sets U, V such that $A \subseteq U$ and $B \subseteq V$. Hence $\overline{J_{\mathcal{S}}} \subseteq U \cup V$. Since $\overline{J_{\mathcal{S}}}$ is compact, there exists $\varepsilon > 0$ such that

$$\overline{J_{\mathcal{S}}} \subseteq \mathcal{N}(\overline{J_{\mathcal{S}}}, \varepsilon) \subseteq U \cup V. \tag{4.3}$$

Since the diameters of $\varphi_\omega(X)$ converge to zero uniformly (exponentially) fast, and since all of them intersect $\overline{J_{\mathcal{S}}}$, there exists $n \geq 1$ so large that $Y_n \subseteq B(\overline{J_{\mathcal{S}}}, \varepsilon)$. It then follows from (4.3) that $Y_n = (U \cap Y_n) \cup (V \cap Y_n)$. Since, both $U \cap Y_n$ and $V \cap Y_n$ are non-empty ($A \subseteq U \cap Y_n$, $B \subseteq V \cap Y_n$) open subsets of Y_n, we conclude that Y_n is disconnected, contrary to our hypothesis. We are done. $\square$

The main result concerning connectedness of $\overline{J_{\mathcal{S}}}$ is the following.

Theorem 4.3. *If $\overline{J_{\mathcal{S}}}$ is compact and there exists a transitive bijection $^* : I \to I$ such that $\varphi_{i^*}(\overline{J_{\mathcal{S}}}) \cap \varphi_i(\overline{J_{\mathcal{S}}}) \neq \emptyset$ for all $i \in I$, then $\overline{J_{\mathcal{S}}}$ is connected.*

Proof. We shall show by induction that each set Y_n (see (4.1)), $n \geq 0$, is connected. Indeed Y_0 is connected since $Y_0 = X$. Suppose Y_n is connected for some $n \geq 0$. Then all the sets $\varphi_i(Y_n)$, $i \in I$, are also connected, and, as $\varphi_i(Y_n) \supseteq \varphi_i(\overline{J_{\mathcal{S}}})$, it follows from our hypothesis that $\bigcup_{i \in I} \varphi_i(Y_n)$ is connected. Thus $Y_{n+1} = \overline{\bigcup_{i \in I} \varphi_i(Y_n)}$ is connected. The inductive proof is finished. Now, the proof of our theorem follows directly from Proposition 4.1. $\square$

Corollary 4.1. *If $I = \mathbb{N}$, $\overline{J_{\mathcal{S}}}$ is compact, and $\varphi_n(\overline{J_{\mathcal{S}}}) \cap \varphi_{n+1}(\overline{J_{\mathcal{S}}}) \neq \emptyset$ for all $n \geq 1$, then $\overline{J_{\mathcal{S}}}$ is connected.*

Corollary 4.2. *If $\overline{J_{\mathcal{S}}}$ is compact and there exists a transitive bijection $^* : I \to I$ such that $\varphi_{i^*}(\partial S) \cap \varphi_i(\partial S) \neq \emptyset$ for all $i \in I$, then $\overline{J_{\mathcal{S}}}$ is connected.*

Passing to local connectedness, we shall prove the following.

Theorem 4.4. *Suppose that $\overline{J_{\mathcal{S}}}$ is compact, ∂S is a singleton, and there exists a transitive bijection $^* : I \to I$ such that $\varphi_{i^*}(\overline{J_{\mathcal{S}}}) \cap \varphi_i(\overline{J_{\mathcal{S}}}) \neq \emptyset$ for all $i \in I$, Then $\overline{J_{\mathcal{S}}}$ is a Peano continuum (arcwise connected, locally arcwise connected compactum).*

Proof. Notice that the hypothesis of our theorem equivalently means that we may bijectively parameterize the alphabet I by the set of natural numbers $\mathbb{N}$ so that the transitive bijection $^* : I \to I$ becomes the map $\mathbb{N} \ni n \to n+1 \in \mathbb{N}$. So, we may assume that $I = \mathbb{N}$ and

$$\varphi_n(\overline{J_{\mathcal{S}}}) \cap \varphi_{n+1}(\overline{J_{\mathcal{S}}}) \neq \emptyset.$$

In virtue of Corollary 4.1, $\overline{J_{\mathcal{S}}}$ is a continuum. We have,

$$\overline{J_{\mathcal{S}}} = \overline{\bigcup_{i \in I} \varphi_i(\overline{J_{\mathcal{S}}})}. \tag{4.4}$$

Fix $\varepsilon > 0$. Since $\overline{J_{\mathcal{S}}}$ is compact, there exists a finite set $I_\varepsilon = \{1, 2, \ldots, q_\varepsilon\} \subseteq I$ such that

$$\overline{\bigcup_{i \in I \setminus I_\varepsilon} \varphi_i(\overline{J_{\mathcal{S}}})} = \partial S \cup \bigcup_{i \in I \setminus I_\varepsilon} \varphi_i(\overline{J_{\mathcal{S}}}) \subseteq \mathcal{N}(\partial S, \varepsilon/2). \tag{4.5}$$

Set

$$J_\varepsilon = \overline{\bigcup_{i \in I \setminus I_\varepsilon} \varphi_i(\overline{J_{\mathcal{S}}})}. \tag{4.6}$$

We shall show by induction that for every $n \geq 1$ there are two finite sets $I_n^{(1)} \subseteq \bigcup_{0 \leq k \leq n-1} I^k$ and $I_n^{(2)} \subset I^n$ such that

$$\overline{J_{\mathcal{S}}} = \bigcup_{\omega \in I_n^{(1)}} \varphi_\omega(J_\varepsilon) \cup \bigcup_{\omega \in I_n^{(2)}} \varphi_\omega(\overline{J_{\mathcal{S}}}). \tag{4.7}$$

Indeed, for $n = 1$ take $I_1^{(1)} = \emptyset$ and $I_1^{(2)} = I_\varepsilon$; (4.7) follows then from (4.4) and (4.6). So suppose that (4.7) holds for some $n \geq 1$. Then, it follows from (4.4) and (4.6) that

$$\begin{aligned}
\bigcup_{\omega \in I_n^{(2)}} \varphi_\omega(\overline{J_{\mathcal{S}}}) &= \bigcup_{\omega \in I_n^{(2)}} \varphi_\omega\left(J_\varepsilon \cup \bigcup_{i \in I_\varepsilon} \varphi_i(\overline{J_{\mathcal{S}}}) \right) \\
&= \bigcup_{\omega \in I_n^{(2)}} \varphi_\omega(J_\varepsilon) \cup \bigcup_{\tau \in I_n^{(2)} \times I_\varepsilon} \varphi_\tau(\overline{J_{\mathcal{S}}}).
\end{aligned}$$

Combining this and (4.7), we conclude the inductive reasoning by setting $I_{n+1}^{(1)} = I_n^{(1)} \cup I_n^{(2)}$ and $I_{n+1}^{(2)} = I_n^{(2)} \times I_\varepsilon$. Now, take $n \geq 1$ so large that $\operatorname{diam}(\varphi_\omega(X)) < \varepsilon$ for all $\omega \in I^n$. Then all the terms in (4.7) corresponding to the set $I_n^{(2)}$ are continua with diameters less than

ε. All the sets $\varphi_\omega(J_\varepsilon)$, $\omega \in I_n^{(1)}$ have also diameters less than ε in virtue of (4.5) and the fact that ∂S is a singleton. But since J_ε is a continuum (since all the sets $\varphi_i(\overline{J_\mathcal{S}})$ are and $I \setminus I_\varepsilon = \{q_\varepsilon + 1, q_\varepsilon + 2, \ldots\}$), so are all the sets $\varphi_\omega(J_\varepsilon)$, $\omega \in I_n^{(1)}$. In consequence, (4.7) provides a representation of $\overline{J_\mathcal{S}}$ as a union of finitely many continua with diameters less than ε. Therefore $\overline{J_\mathcal{S}}$ is a locally connected continuum by Hahn–Mazurkiewicz–Sierpiński's Theorem and it is arcwise connected and locally arcwise connected by Mazurkiewicz–Moore–Menger's Theorem. The proof is complete. $\square$

Example: We shall consider complex continued fractions (see [9]). Let $I = \{m + ni : (m,n) \in \mathbb{N} \times \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ is the set of positive integers. Let $X \subset \mathbb{C}$ be the closed disc centered at the point $1/2$ with radius $1/2$ and let $V = B(1/2, 3/4)$. For $b \in I$ we define $\varphi_b : V \to V$ putting

$$\varphi_b(z) = \frac{1}{b+z}$$

Then $\overline{J_\mathcal{S}}$ is a Peano continuum. Indeed, observe that $0, 1 \in \overline{J_\mathcal{S}}$. Set $A = \varphi_{1+i}(0)$ and $B = \varphi_{1-i}(0)$. Obviously $A, B \in \overline{J_\mathcal{S}}$. Since for every $b, b' \in \{1, -1, i, -i\}$, there exists $u \in \varphi(\{0, 1, A, B\})$ such that $u \in \varphi_b(\overline{J_\mathcal{S}}) \cap \varphi_{b'}(\overline{J_\mathcal{S}})$, from Theorem 4.4 it follows that $\overline{J_\mathcal{S}}$ is a Peano continuum.

Now, we shall prove the following partial converse of the last theorem.

Theorem 4.5. *Set $I = \mathbb{N}$. Suppose $\overline{J_\mathcal{S}}$ is compact, ∂S contains at least two points, $\partial S \cap \bigcup_{n=1}^\infty \varphi_n(\overline{J_\mathcal{S}}) = \emptyset$ and $\varphi_i(\overline{J_\mathcal{S}}) \cap \varphi_j(\overline{J_\mathcal{S}}) \neq \emptyset$ if and only if $|i-j| \leq 1$. Then $\overline{J_\mathcal{S}}$ is not a locally connected continuum, more precisely, $\overline{J_\mathcal{S}}$ fails to be locally connected at all points of ∂S.*

Proof. $\overline{J_\mathcal{S}}$ is connected by Corollary 4.1. Our hypothesis imply that for every $k \geq 1$, the sets $F_k^+ = \bigcup_{n \geq k+1} \varphi_n(\overline{J_\mathcal{S}})$ and $F_k^- = \bigcup_{n=1}^{k-1} \varphi_n(\overline{J_\mathcal{S}})$ are separated. Thus we have the following.

Claim. If $\Gamma \subseteq \overline{J_\mathcal{S}}$ is a connected set intersecting $\varphi_i(\overline{J_\mathcal{S}})$ and $\varphi_j(\overline{J_\mathcal{S}})$, then $\Gamma \cap \varphi_k(\overline{J_\mathcal{S}}) \neq \emptyset$ for all $\min\{i,j\} \leq k \leq \max\{i,j\}$.

Fix an arbitrary point $x \in \partial S$ and then a point $y \in \partial S \setminus \{x\}$. In order to show that $\overline{J_\mathcal{S}}$ is not locally connected at x it suffices to prove that for every $\varepsilon > 0$ there are two points $a, b \in \overline{J_\mathcal{S}} \cap B(x,\varepsilon)$ such that every connected set $F \subseteq \overline{J_\mathcal{S}}$ containing a and b, intersects $B(y,\varepsilon)$. And indeed, since $x \in \partial S$, there exists $i \geq 1$ such that $\varphi_i(\overline{J_\mathcal{S}}) \subseteq B(x,\varepsilon)$. Since $y \in \partial S$, there exists $k > i$ such that $\varphi_k(\overline{J_\mathcal{S}}) \subseteq B(y,\varepsilon)$. Again, since $x \in \partial S$, there exists $j > k$ such that $\varphi_j(\overline{J_\mathcal{S}}) \subseteq B(x,\varepsilon)$. Now, fix $a \in \varphi_i(\overline{J_\mathcal{S}})$ and $b \in \varphi_j(\overline{J_\mathcal{S}})$. If now $\Gamma \subseteq \overline{J_\mathcal{S}}$ is an arbitrary connected set containing a and b, then $\Gamma \cap \varphi_i(\overline{J_\mathcal{S}}) \neq \emptyset$ and $\Gamma \cap \varphi_j(\overline{J_\mathcal{S}}) \neq \emptyset$. Hence, in view of the Claim, $\Gamma \cap \varphi_k(\overline{J_\mathcal{S}}) \neq \emptyset$. Thus $\Gamma \cap B(y,\varepsilon) \neq \emptyset$. The proof is complete. $\square$

5. Strongly bounded multiplicity and F–conformal measures

In this entire section $\mathcal{S} = \{\varphi_e : X_{t(e)} \to X_{i(e)}, e \in E\}$ is rapidly decreasing finitely primitive graph directed system. We do not mention bounded multiplicity here since we will formulate and we will need in the sequel a stronger property. A finite word $\omega\tau \in E^*$ is called a *p*seudo–code of an element $x \in X$ if and only if $\omega, \tau \in E_A^*$, $\varphi_\tau(X_{t(\tau)}) \subseteq X_{t(\omega)}$ and $x \in \varphi_\omega(\varphi_\tau(X_{t(\tau)}))$. Note that the word $\omega\tau$ is not required to belong to E_A^*. If we do not want to specify the element x we simply say that $\omega\tau$ is a pseudo–code.

Now, the system $\mathcal{S}$ is said to be of *strongly bounded multiplicity* provided that there exists $M \geq 1$ such that the number of elements of any collection of mutually incomparable pseudo–codes of a point $x \in X$, is bounded above by M. Obviously, any system of strongly bounded multiplicity is of bounded multiplicity. We now pass to describe and to examine in some detail summable Hölder continuous families of functions. A family

$$F = \{f^{(e)} : X_{t(e)} \to \mathbb{C}\}_{e \in E}$$

is called *Hölder continuous with an exponent* $\beta > 0$ provided that all its members are Hölder continuous with exponent β and the same positive constant. Precisely, there exists $\vartheta_\beta(F) > 0$ such that for all $e \in E$ and all $x, y \in X_{t(e)}$,

$$|f^{(e)}(x) - f^{(e)}(y)| \leq \vartheta_\beta(F)\rho^\beta(x, y).$$

Note that each Hölder continuous family with $\mathbb{C}$ replaced by $\mathbb{R}$ is a Hölder family in the sense of Section 3.1 from [12].

The Hölder family F is called *summable* if F is real, i.e. $f^{(e)}(X_{t(e)}) \subseteq \mathbb{R}$ for all $e \in E$ and

$$\sum_{e \in E} \| \exp(f^{(e)})\|_\infty = \sum_{e \in E} \exp(\sup_{x \in X_{t(e)}} (f^{(e)})) < \infty.$$

For every $\omega \in E_A^n$, $n \geq 1$ define the function $S_\omega F : X_{t(\omega)} \to \mathbb{R}$ by

$$S_\omega F = \sum_{j=1}^n f^{(\omega_j)} \circ \varphi_{\sigma^j \omega}.$$

Note that the sequence $(n \to \log \sum_{\omega \in E_A^n} \| \exp(S_\omega F)\|_\infty)_1^\infty$ is subadditive and define the topological pressure $P(F)$ of F by setting

$$P(F) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in E_A^n} \| \exp(S_\omega F)\|_\infty.$$

The following lemma can be proved in exactly the same way as Lemma 3.1.2 in [12] with slight obvious modifications.

Lemma 5.1. *For all $\omega \in E_A^*$ and all $x, y \in X_{t(\omega)}$, we have*

$$|S_\omega F(x) - S_\omega F(y)| \leq (1 - s^\beta)^{-1}\vartheta_\beta(F)\rho^\beta(x, y).$$

($s \in (0, 1)$ is the contraction rate of the system $\mathcal{S}$).

The amalgamated function $f : E_A^\infty \to \mathbb{C}$ induced by the family F is defined as follows

$$f(\omega) = f^{(\omega_1)}(\pi(\sigma\omega)).$$

Having the function f we may define the *topological pressure* of f by the formula

$$P(f) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(f),$$

where

$$Z_n(f) = \sum_{\omega\in E_A^n} \exp\Big(\sup_{\tau\in[\omega]\cap E_A^\infty} \sum_{j=0}^{n-1} f(\sigma^j(\tau))\Big).$$

Our convention will be to use lower case letters for amalgamated functions induced by Hölder continuous families of functions (referred to by upper case letters). A crucial fact for us is the following straightforward lemma.

Lemma 5.2. *If F is a Hölder continuous family of functions with an exponent $\beta > 0$, then the amalgamated function $f : E_A^\infty \to \mathbb{C}$ is Hölder continuous with exponent β (assuming that the symbol space E_A^∞ is endowed with the metric d_s).*

This lemma permits us to apply the machinery of symbolic thermodynamic formalism developed in Chapter 2 of [12]. Also, most of the arguments from Chapter 3 of this book go through almost unaltered. Firstly, the same proof as that of Proposition 3.1.4 in [12] gives the following.

Proposition 5.1. *If F is a real Hölder continuous family, then $P(F) = P(f)$.*

A Borel probability measure m on X is said to be F–conformal provided that $m(J_{\mathcal{S}}) = 1$ and the following two conditions are satisfied. For every $e \in E$ and for every Borel set $A \subseteq X_{t(e)}$,

$$m(\varphi_e(A)) = \int_A \exp(f^{(e)}(x) - P(F))m(\mathrm{d}x)$$

and

$$m(\varphi_a(X_{t(a)}) \cap \varphi_b(X_{t(b)})) = \emptyset$$

whenever $a, b, e \in E$ and $a \neq b$. A straightforward argument shows that

$$m(\varphi_\omega(A)) = \int_A \exp(S_\omega F(x) - P(F)|\omega|)m(\mathrm{d}x)$$

for all $\omega \in E_A^*$ and every Borel set $A \subseteq X_{t(\omega)}$, and

$$m(\varphi_\tau(X_{t(\tau)}) \cap \varphi_\omega(X_{t(\omega)}) = \emptyset$$

for all incomparable $\omega, \tau \in E_A^*$. Recall that $f : E_A^\infty \to \mathbb{R}$ is the amalgamated function induced by the summable Hölder continuous family F. Let $\mathcal{L}_f : C_b(E_A^\infty) \to C_b(E_A^\infty)$ be the corresponding Perron–Frobenius operator, i.e. the operator given by the formula

$$\mathcal{L}_f(g(\omega) = \exp(-P(f)) \sum_{\substack{e\in E\\ A_{e\omega_1}=1}} g(e\omega)\exp(f(e\omega)).$$

Let $\mathcal{L}_f^* : C_b^*(E_A^\infty) \to C_b^*(E_A^\infty)$ be the corresponding dual operator. The following result (for all Hölder continuous summable potentials defined on the symbol space E_A^∞) was proved in Chapter 2 of [12].

Theorem 5.1. *There exists exactly one Borel probability measure m_f on E_A^∞ such that $\mathcal{L}_f^* m_f = m_f$. There also exists exactly one Borel probability shift–invariant measure μ_f on E_A^* that is absolutely continuous with respect to m_f. In addition μ_f is ergodic and the Radon–Nikodym derivative $\frac{\mathrm{d}\mu_f}{\mathrm{d}m_f}$ is a Hölder continuous function whose supremum is finite and infimum is positive. The invariant measure μ_f is characterized in the space of Borel probability measures on E_A^∞ by the property that $\exists\, C \geq 1\ \forall\, \omega \in E_A^\infty\ \forall\, n \geq 1$*

$$C^{-1} \leq \frac{\mu_f([\omega_{|n}])}{\exp(S_n f(\omega) - P(f)n)} \leq C.$$

The same inequalities hold, perhaps with a different $C \geq 1$, with μ_f replaced by m_f. The measure m_f is referred to as the standard Gibbs state of f and μ_f as the invariant Gibbs state of f. The main fact that we will need from this section is the following.

Theorem 5.2. *Suppose that $\mathcal{S}$ is a contracting rapidly decreasing graph directed Markov system with strongly bounded multiplicity. Suppose also that F is a summable Hölder continuous family. Then $m_F := m_f \circ \pi^{-1}$ is the only F–conformal measure on X.*

The proof of this theorem goes through exactly as the proof of Theorem 3.2.3 p. 58 in [12]. One only needs to observe that each element of E_n has at least two different pseudo–codes $\omega\rho$ and $\omega\tau$ (with some $\omega \in E_A^n$) of length $n+q$ and then to apply strongly bounded multiplicity to conclude that $\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n = \emptyset$.

6. Conformal mappings in Hilbert spaces

Let $\mathcal{H}$ be a separable Hilbert space with $\dim(\mathcal{H}) \geq 3$. Let $U \subseteq \mathcal{H}$ be a non–empty open set. A C^4–differentiable 1–to–1 map $\varphi : U \to \mathcal{H}$ is called conformal provided that for every $x \in \mathcal{H}$, the derivative $\varphi'(x) : \mathcal{H} \to \mathcal{H}$ is a bijective similarity map (a bijective isometry followed by multiplication by a non–zero scalar). The (positive) similarity factor of $\varphi'(x)$ is denoted by $|\varphi'(x)|$. There are two distinguished classes of conformal maps: similarity maps and inversions with respect to spheres. Given $a \in \mathcal{H}$ and $r > 0$, the inversion $I_{a,r} : \mathcal{H} \setminus \{a\} \to \mathcal{H} \setminus \{a\}$ with respect to sphere $S(a,r) := \{x \in \mathcal{H} : \|x-a\| = r\}$ is given by the formula

$$I_{a,r}(x) = a + r^2 \|x-a\|^{-2}(x-a),$$

that is $I_{a,r}(x)$ is the only point different from x on the ray emanating from a and passing through x such that $\|I_{a,r}(x) - a\| \cdot \|x-a\| = r^2$. It is easy to check that $I_{a,r}$ is a conformal map, that $I_{a,r}^2 = I_{a,r}$, meaning that $I_{a,r}$ is an involution, and that

$$|I'_{a,r}(x)| = r^2 \|x-a\|^{-2}. \tag{6.1}$$

In the case when $r = 1$, we frequently write I_a for $I_{a,1}$. We also write I_∞ for the identity map on $\mathcal{H}$. Inspecting the proof of Liouville's Theorem for (finite dimensional) Euclidean spaces (see [3], Sec. 5.2), we see that the following representation theorem holds.

Theorem 6.1. *If $U \subseteq \mathcal{H}$ is a non–empty open connected set and $\varphi : U \to \mathcal{H}$ is a conformal map, then there is a unique quadruple (M, a, λ, b) such that*

$$\varphi = S_{\lambda,b}^M \circ I_a,$$

where $S^M_{\lambda,b}(z) = \lambda M(z)+b$ *is the similarity map determined by* λ, b *and a bijective isometry* M.

Denote the space of all conformal mappings on $\mathcal{H}$ by $C_f(\mathcal{H})$. It then follows from Theorem 6.1 and (6.1) that

$$|\varphi'(x)| = \lambda\|x-a\|^{-2}$$

for all $x \in \mathcal{H}\setminus\{a\}$. Copying word by word the proof of Theorem 4.1.3 from [12], p. 65, we get the following.

Theorem 6.2. *Suppose that* Y *is a bounded subset of the Hilbert space* $\mathcal{H}$ *and that* $W \subseteq \mathcal{H}$ *is an open connected set containing* Y *such that* $\mathrm{dist}(Y, \partial W) = \mathrm{dist}(Y, W^c) > 0$. *Then there exists a constant* $\hat{K} \geq 1$ *such that for every conformal map* $\varphi : W \to \mathcal{H}$, *we have*

$$||\varphi'(x)| - |\varphi'(y)|| \leq \hat{K}|\varphi'(x)|\|x-y\|$$

for all $x, y \in Y$. *In particular,*

$$K^{-1} \leq \frac{|\varphi'(y)|}{|\varphi'(x)|} \leq K,$$

where $K = 1 + \hat{K}\,\mathrm{diam}(Y)$.

7. Conformal Graph Directed Markov Systems

A subset F of a Banach space B is called Q*–quasi–convex* ($Q \geq 1$) provided that for all points $x, y \in F$ there exists a polygonal line $\gamma \subseteq F$ with end points x and y whose length is bounded above by $Q\|x-y\|$. A subset of a Banach space B is called *quasi–convex* if it is Q–quasi convex with some $Q \geq 1$

Fix a separable Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) \geq 3$. A contracting (with contraction rate $s \in (0,1)$ graph directed Markov system $\mathcal{S} = \{\varphi_e : X_{t(e)} \to X_{i(e)}, e \in E\}$ is called *conformal* if the following conditions are satisfied:

(a) X_v is a closed bounded quasi–convex subset of $\mathcal{H}$ for all $v \in V$;
(b) The incidence matrix $A : E \times E \to \{0,1\}$ is finitely primitive;
(c) $\mathcal{S}$ is rapidly decreasing;
(d) The Open Set Condition (OSC) holds, i.e. if $i \neq j$, then

$$\varphi_i(\mathrm{Int}(X_{t(i)})) \cap \varphi_j(\mathrm{Int}(X_{t(j)})) = \emptyset;$$

(e) $\mathcal{S}$ is of strongly bounded multiplicity;
(f) For every $v \in V$ there exist an open connected bounded set W_v containing X_v and $\hat{W}_v$ containing $\mathcal{N}(W_v, \varepsilon)$ for some $\varepsilon > 0$ such that $\mathrm{dist}(X_v, \partial W_v) > 0$, $\varphi_e(W_{t(e)}) \subseteq W_{i(e)}$ and $\hat{\varphi}_e(W_{t(e)}) \subseteq \hat{W}_{i(e)}$ for all $e \in E$;
(g) For every $e \in E$, $\varphi_e : W_{t(e)} \to W_{i(e)}$ is a conformal map;
(h) For every $e \in E$, $\|\varphi'_e\| = \sup\{|\varphi'_e(x)| : x \in W_{t(e)}\} \leq s$.

Applying Theorem 6.2, we immediately get the following Bounded Distortion Property.

Lemma 7.1. *There exists a constant $K \geq 1$ such that*

$$K^{-1} \leq \frac{|\varphi'_\omega(x)|}{|\varphi'_\omega(y)|} \leq K$$

for all $\omega \in E_A^*$ and all $x, y \in W_{t(\omega)}$ (decreasing W_v slightly if necessary).

Furthermore, the proof of Lemma 4.2.3 from [12] p. 72 repeats to give the following.

Lemma 7.2. *There exists $L \geq 1$ such that*

$$|\log|\varphi'_\omega(y)| - \log|\varphi'_\omega(x)|| \leq L\|x-y\|$$

for all $\omega \in E_A^*$ and all $x, y \in W_{t(\omega)}$.

For every $e \in E$ and every $t \geq 0$ let

$$tl^{(e)}(x) = t\log|\varphi'_e(x)|, \quad x \in X_{t(e)}.$$

Let

$$tL = \{tl^{(e)}\}_{e\in E}.$$

As an immediate consequence of Lemma 7.2 we obtain the following.

Lemma 7.3. *Fore every $t \geq 0$, tL is a Hölder continuous family with exponent 1. Its amalgamated function $tl : E_A^\infty \to \mathbb{R}$ is given by the formula $tl(\omega) = t\log|\varphi'_{\omega_1}(\pi(\sigma\omega))|$.*

Since $\|\varphi'_e\| \leq s < 1$ for all $e \in E$, the set

$$\mathcal{F}_\mathcal{S} = \{t \geq 0 : tL \text{ is summable}\}$$

is of the form $[\Theta_s, +\infty)$ or $(\Theta_s, +\infty)$ with some $\Theta_s \in [0, +\infty]$. Let $P(t) := P(tL)$, i.e.

$$P(t) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{\omega\in E_A^n}\|\varphi'_\omega\|^t.$$

In view of Lemma 7.3, Theorem 5.1 produces a unique standard Gibbs state $\hat{m}_t$ and a unique invariant Gibbs state $\hat{\mu}_t$ for all $t \in \mathcal{F}_\mathcal{S}$. As an immediate consequence of Theorem 5.2 we then get the following.

Theorem 7.1. *For every $t \in \mathcal{F}_\mathcal{S}$, $m_t := \hat{m}_t \circ \pi^{-1}$ is the unique tL–conformal measure for $\mathcal{S}$, meaning that*

$$m_t(\varphi_\omega(A)) = \int_A e^{-P(t)|\omega|}\|\varphi'_\omega\|^t \mathrm{d}m_t$$

for all $\omega \in E_A^*$ and all Borel sets $A \subseteq X_{t(\omega)}$. If in addition $\tau \in E_A^*$ is incomparable with ω, then

$$m_t(\varphi_\omega(X_{t(\omega)}) \cap \varphi_\tau(X_{t(\tau)})) = 0.$$

We also set $\mu_t := \hat{\mu}_t \circ \pi^{-1}$. If $P(t) = 0$, the tL–conformal measure m_t is simply refered to as t–conformal. The basic (purely symbolic) properties of the pressure function $P(t)$ are comprised in the following (compare Proposition 4.2.8 in [12]).

Proposition 7.1. *It holds*

(a) $\mathcal{F}_\mathcal{S} = \{t \geq 0 : P(t) < +\infty\}$,

(b) *The topological pressure function $t \to P(t)$ is non–increasing on $[0,+\infty)$, strictly decreasing on $[\Theta_{\mathcal{S}},+\infty)$ to $-\infty$, convex and continuous on $\mathcal{F}_{\mathcal{S}}$ (in fact real–analytic on $(\Theta_{\mathcal{S}},+\infty)$ but this is more involved and proved in Chapter 2 of [12]),*

(c) *$P(0)=+\infty$ iff the set E is infinite.*

Set

$$h = h_{\mathcal{S}} := \inf\{t \geq \Theta_{\mathcal{S}} : P(t) < 0\} \geq \Theta_{\mathcal{S}}.$$

The system $\mathcal{S}$ is called *finitary* iff $h_{\mathcal{S}} < +\infty$, otherwise it is called *infinitary*. As a direct consequence of Proposition 7.1 and Theorem 7.1, we obtain the following.

Proposition 7.2. *It holds*

(a) *There is at most one $t \geq 0$ such that $P(t) = 0$. If such a t exists, then $t = h_{\mathcal{S}}$.*

(b) *There is at most one $h_{\mathcal{S}}$–conformal measure.*

If $P(h_{\mathcal{S}}) = 0$, equivalently if an $h_{\mathcal{S}}$–conformal measure exists, the system $\mathcal{S}$ is called *regular*. Otherwise ($P(h_{\mathcal{S}}) < 0$), it is called *irregular*.

8. A digression to geometric measure theory

In this section we collect, with proofs, some facts from the geometric measure theory showing how geometric properties of "abstract" measures affect the behavior of Hausdorff and packing measures. We start with the $4r$–covering theorem. In the literature we were able to find the $5r$–covering theorem, which in addition was formulated and proved for compactly bounded metric spaces (see [8]). We prove it for all metric spaces. For every ball $B := B(x,r)$, we put $r(B) = r$ and $c(B) = x$.

Theorem 8.1. *($4r$–Covering Theorem) Suppose (X,ρ) is a metric space and $\mathcal{B}$ is a family of open balls in X such that $\sup\{r(B) : B \in \mathcal{B}\} < +\infty$. Then there is a family $\mathcal{B}' \subseteq \mathcal{B}$ consisting of mutually disjoint balls such that $\bigcup_{B\in\mathcal{B}} B \subseteq \bigcup_{B\in\mathcal{B}'} 4B$. In addition, if the metric space X is separable, then $\mathcal{B}'$ is countable.*

Proof. Fix an arbitrary $M > 0$. Suppose that there is given a family $\mathcal{B}'_M \subseteq \mathcal{B}$ consisting of mutually disjoint balls such that

(a) $r(B) > M$ for all $B \in \mathcal{B}'_M$,

(b) $\bigcup_{B\in\mathcal{B}'_M} 4B \supseteq \bigcup\{B : B \in \mathcal{B} \text{ and } r(B) > M\}$.

We shall show that there exists a family $\mathcal{B}''_M \subseteq \mathcal{B}$ with the following properties:

(c) $\mathcal{B}''_M \subseteq \mathcal{F} := \{B \in \mathcal{B} : 3M/4 < r(B) \leq M\}$,

(d) $\mathcal{B}'_M \cup \mathcal{B}''_M$ consists of mutually disjoint balls,

(e) $\bigcup_{B\in\mathcal{B}'_M\cup\mathcal{B}''_M} 4B \supseteq \bigcup\{B : B \in \mathcal{B} \text{ and } r(B) > 3M/4\}$.

Indeed, put

$$\mathcal{B}'''_M = \{B \in \mathcal{F} : B \cap \bigcup_{D\in\mathcal{B}'_M} D = \emptyset\}. \tag{8.1}$$

Consider $B \in \mathcal{F} \setminus \mathcal{B}'''_M$. Then there exists $D \in \mathcal{B}'_M$ such that $B \cap D \neq \emptyset$. Hence, $r(B) \leq M < r(D)$, and in consequence,

$$\rho(c(B), c(D)) < r(B) + r(D) \leq M + r(D) < r(D) + r(D) = 2r(D),$$

and

$$B \subseteq B(c(D), r(B) + 2r(D)) \subseteq B(c(D), 3r(D)) = 3D \subseteq 4D.$$

Therefore,

$$\bigcup_{B \in \mathcal{F} \setminus \mathcal{B}'''_M} B \subseteq \bigcup_{B \in \mathcal{B}'_M} 4B. \tag{8.2}$$

So, if $\mathcal{B}'''_M = \emptyset$ we are done with the proof by setting $\mathcal{B}''_M = \emptyset$. Otherwise, fix an arbitrary $B_0 \in \mathcal{B}'''_M$, and further by transfinite induction, $\mathcal{B}_\alpha \in \mathcal{B}'''_M$ such that

$$c(B_\alpha) \in c(\mathcal{B}'''_M) \setminus \bigcup_{\gamma < \alpha} \frac{8}{3} B_\gamma$$

as long as the difference on the right-hand side above is nonempty. This procedure terminates at some ordinal number λ. First, we claim that the balls $(B_\alpha)_{\alpha<\lambda}$ are mutually disjoint. Indeed, fix $0 \leq \alpha < \beta < \lambda$. Then $c(B_\beta) \notin \frac{8}{3} B_\alpha$. So, $\rho(c(B_\beta), c(B_\alpha)) \geq \frac{8}{3} r(B_\alpha) > \frac{8}{3} \cdot \frac{3}{4} M = 2M$ and $r(B_\beta) + r(B_\alpha) \leq M + M = 2M$. Thus $B_\beta \cap B_\alpha = \emptyset$. Now, if $B \in \mathcal{B}'_M$ and $0 \leq \alpha < \lambda$, then $B_\alpha \in \mathcal{B}'''_M$, and by (8.1), $B_\alpha \cap B = \emptyset$. Thus we proved item (d) with $\mathcal{B}''_M = \{B_\alpha\}_{\alpha<\lambda}$. Item (c) is obvious since $B_\alpha \in \mathcal{B}'''_M \subseteq \mathcal{F}$ for all $0 \leq \alpha < \lambda$. It remains to prove item (e). By the definition of λ, $c(\mathcal{B}'''_M) \subset \bigcup_{\gamma<\lambda} \frac{8}{3} B_\gamma = \bigcup_{B \in \mathcal{B}''_M} \frac{8}{3} B$. Hence, if $x \in B$ and $B \in \mathcal{B}'''_M$, then there exists $D \in \mathcal{B}''_M$ such that $c(B) \in \frac{8}{3} D$. Therefore,

$$\begin{aligned} \rho(x, c(D)) &\leq \rho(x, c(B)) + \rho(c(B), c(D)) \leq r(B) + \frac{8}{3} r(D) \\ &\leq M + \frac{8}{3} r(D) < \frac{4}{3} r(D) + \frac{8}{3} r(D) = 4r(D). \end{aligned}$$

Thus, $x \in 4D$, and consequently, $\bigcup \mathcal{B}'''_M \subseteq \bigcup_{D \in \mathcal{B}''_M} 4D$. Combining this and (8.2), we get that $\bigcup_{B \in \mathcal{F}} B \subseteq \bigcup_{B \in \mathcal{B}'_M \cup \mathcal{B}''_M} 4B$. This and (b) immediately imply (e). The properties (c), (d) and (e) are established. Now, take $S = \sup\{r(B) : B \in \mathcal{B}\} + 1 < +\infty$, and define inductively the sequence $(\mathcal{B}'_{(3/4)^n S})_{n=0}^\infty$ by declaring $\mathcal{B}'_S = \emptyset$ and $\mathcal{B}'_{(3/4)^{n+1} S} = \mathcal{B}'_{(3/4)^n S} \cup \mathcal{B}''_{(3/4)^n S}$. Then

$$\mathcal{B}' = \bigcup_{n=0}^{\infty} \mathcal{B}'_{(3/4)^n S}.$$

It then follows directly from (d) and our inductive definition that $\mathcal{B}'$ consists of mutually disjoint balls. It follows from (e) that $\bigcup_{B \in \mathcal{B}'} 4B \supseteq \bigcup \{B \in \mathcal{B} : r(B) > 0\} = \bigcup \mathcal{B}$. The first part of our theorem is thus proved. The last part follows immediately from the fact that any family of mutually disjoint open subsets of a separable space is countable. $\square$

Remark 1. *Assume the same as in Theorem 8.1 (separability of X is not needed) and suppose that there exists a finite Borel measure μ on X such that $\mu(B) > 0$ for all $B \in \mathcal{B}'$. Then $\mathcal{B}'$ is countable.*

A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called an *evenly varying* function if for all $K > 0$ there exists $c_\varphi(K) \geq 1$ such that $(c_\varphi(K))^{-1}\varphi(t) \leq \varphi(Kt) \leq c_\varphi(K)\varphi(t)$ for all $t \geq 0$.

We shall now derive several geometrical consequences of the $4r$–Covering Theorem. In the finite–dimensional case they are actually known in the literature (see [14] for instance). They were formulated and proved for subsets of Euclidean spaces only (for further applications in forthcoming sections we extended them to Hilbert spaces) and with the help of the Besicovitch Covering Theorem, which holds for finite–dimensional Euclidean spaces only and whose proof is substantially more complicated than the proof of the $4r$–Covering Theorem.

Theorem 8.2. *Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a continuous evenly varying function. Suppose (X, ρ) to be an arbitrary metric space and μ a Borel probability measure on X. Fix $A \subseteq X$. Assume that there exists $c \in (0, +\infty]$ $(1/+\infty = 0)$ such that one of the following holds.*

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{\varphi(r)} \geq c \tag{1}$$

for all $x \in A$ except for countably many perhaps. Then the Hausdorff measure H_φ corresponding to the function φ satisfies $\mathrm{H}_\varphi(E) \leq c_\varphi(8)\mu(E)/c$ for every Borel set $E \subseteq A$. In particular $\mathrm{H}_\varphi(A) < +\infty$ ($\mathrm{H}_\varphi(A) = 0$ if $c = \infty$)

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{\varphi(r)} \leq c < +\infty \tag{2}$$

for all $x \in A$. Then $\mu(E) \leq \mathrm{H}_\varphi(E)$ for every Borel set $E \subseteq A$. In particular $\mathrm{H}_\varphi(A) > 0$ whenever $\mu(E) > 0$.

Proof. Since H_φ on any countable set equals 0, we may assume without loss of generality that E does not intersect the exceptional countable set. Fix $\varepsilon > 0$ and $r > 0$. Since μ is regular there exists an open set $G \supseteq E$ such that $\mu(G) \leq \mu(E)+\varepsilon$. Further, for every $x \in E$ there exists $r(x) \in (0, r)$ such that $B(x, r(x)) \subseteq G$ and $(c^{-1}+\varepsilon)\mu(B(x, r(x)) \geq \varphi(r(x)) > 0$. By $4r$–covering theorem and Remark 1 there exists a sequence $\{x_k\}_{k=1}^\infty$ such that

$$B(x_i, r(x_i)) \cap B(x_j, r(x_j)) = \emptyset \qquad \text{for } i \neq j$$

and

$$\bigcup_{k=1}^\infty B(x_k, 4r(x_k)) \supseteq \bigcup_{x\in E} B(x, r(x)) \supseteq E.$$

Hence

$$\begin{aligned}
\mathrm{H}_\varphi^{2r}(E) &\le \sum_{k=1}^\infty \varphi(2\cdot 4r(x_k)) \le \sum_{k=1}^\infty c_\varphi(8)\varphi(r(x_k)) \le c_\varphi(8)\sum_{k=1}^\infty (c^{-1}+\varepsilon)\mu(B(x_k,r(x_k))) \\
&= c_\varphi(8)(c^{-1}+\varepsilon)\mu(\bigcup_{k=1}^\infty B(x_k,r(x_k))) \le c_\varphi(8)(c^{-1}+\varepsilon)\mu(G) \\
&\le c_\varphi(8)(c^{-1}+\varepsilon)(\mu(E)+\varepsilon).
\end{aligned}$$

Passing with $r\to 0$ we get,

$$\mathrm{H}_\varphi(E) \le c_\varphi(8)(c^{-1}+\varepsilon)(\mu(E)+\varepsilon)$$

and since $\varepsilon>0$ was arbitrary, we finally get

$$\mathrm{H}_\varphi(E) \le c_\varphi(8)c^{-1}\mu(E),$$

which finishes the first part of the proof.

Now, let us deal with the second part. Fix an arbitrary $s>c$. The function (for every $r>0$)

$$X \ni x \mapsto \frac{\mu(B(x,r))}{\varphi(r)} \qquad \text{Borel measurable.}$$

For every $k\ge 1$ consider the function

$$\varphi_k(x) = \sup\left\{\frac{\mu(B(x,r)}{\varphi(r)} :\ r\in Q\cap(0,1/k]\right\},$$

where Q are rational numbers. The above function is Borel–measurable as the supremum of countably many measurable functions. Let

$$A_k = \varphi_k^{-1}((0,s]) \qquad \text{for } k\ge 1.$$

Fix an arbitrary $r\in(0,1/k)$. Then pick $r_j \searrow r$, $r_j\in Q$. Since the function $t\mapsto \mu(B(x,t))$ is non–decreasing and the function φ is continuous, we get for every $x\in A_k$ that

$$\frac{\mu(B(x,r))}{\varphi(r)} \le \lim_{j\to\infty}\frac{\mu(B(x,r_j))}{\varphi(r_j)} \le s.$$

Fix $F\subseteq A_k$, $r<1/k$ and $\{F_i\}_1^\infty$, a countable cover of F by sets contained in F and with diameter less than $r/2$. For every $i\ge 1$ pick $x_i\in F_i$. Then $F_i\subset B(x_i,\mathrm{diam}(F_i))$. Hence

$$\sum_{i=1}^\infty \varphi(\mathrm{diam}(F_i)) \ge s^{-1}\sum_{i=1}^\infty \mu(B(x_i,\mathrm{diam}(F_i))) \ge s^{-1}\sum_{i=1}^\infty \mu(F_i) \ge s^{-1}\mu(F).$$

Hence

$$\mathrm{H}_\varphi(F) \ge s^{-1}\mu(F).$$

By our hypothesis

$$\bigcup_{k=1}^\infty A_k\cap A = A.$$

Define inductively

$$B_1 = A_1 \cap A$$

and

$$B_{k+1} = A_{k+1} \cap (A \setminus \bigcup_{j=1}^{k} A_j \cap A).$$

Obviously $\{B_k\}_1^\infty$ consists of mutually disjoint sets and

$$\bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k = A.$$

Hence, if a measurable set $E \subseteq A$, then

$$\mathrm{H}_\varphi(E) = \bigcup_{k=1}^{\infty} \mathrm{H}_\varphi(E \cap B_k) \geq s^{-1} \sum_{k=1}^{\infty} \mu(E \cap B_k) = s^{-1}\mu(E).$$

Letting $s \searrow c$ finishes the proof. $\square$

Theorem 8.3. *Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous evenly varying function. Suppose (X, ρ) to be a metric space and μ a Borel probability measure on X. Fix $A \subset X$ and assume that there exists $c \in (0, +\infty]$, $(1/+\infty = 0)$ such that one of the following holds.*

(1) $$\liminf_{r \to 0} \frac{\mu(B(x, r))}{\varphi(r)} \leq c \qquad \text{for all } x \in A.$$

Then $\mu(E) \leq \mathrm{P}_\varphi(E)$ for every Borel set $E \subseteq A$, where P_φ denotes the packing measure corresponding to the gauge function φ. In particular, if $\mu(E) > 0$, then $\mathrm{P}_\varphi(E) > 0$.

(2) $$\liminf_{r \to 0} \frac{\mu(B(x, r))}{\varphi(r)} \geq c \qquad \text{for all } x \in A.$$

Then $\mathrm{P}_\varphi(E) \leq C^{-1}\mu(E)$ for every Borel set $E \subseteq A$. In particular, if $\mu(E) < +\infty$, then $\mathrm{P}_\varphi(E) < +\infty$.

The proof, analogous to the proof of the previous theorem, is omitted. As a consequence of the above theorems we get this.

Theorem 8.4. *Suppose (X, ρ) to be a metric space and μ a Borel probability measure on X. Let A be a subset of X.*

If $\mu(A) > 0$ and there exists $\theta_1 \geq 0$ such that

$$\liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \geq \theta_1 \qquad \text{for all } x \in A,$$

then $\mathrm{HD}(A) \geq \theta_1$, where HD denotes the Hausdorff dimension.

If there exists $\theta_2 \geq 0$ such that

$$\liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \leq \theta_2 \qquad \text{for all } x \in A,$$

then $\mathrm{HD}(A) \leq \theta_2$.

Proof. Fix an arbitrary $0 \le \theta < \theta_1$ ($\theta_1 = 0$ is obvious). Then for every $x \in A$ there exists $R \in (0,1)$ such that

$$\frac{\log \mu(B(x,r))}{\log r} \ge \theta \qquad \text{for } r < R.$$

Hence we have

$$\mu(B(x,r)) \le r^\theta$$

and consequently

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^\theta} \le 1.$$

This in virtue of Theorem 7.2 (2) gives $\mathrm{H}_\theta(A) > 0$, which, in turn, implies $\mathrm{HD}(A) \ge \theta$. Letting $\theta \searrow \theta_1$ gives $\mathrm{HD}(A) \ge \theta_1$.

Now fix $\theta > \theta_2$. Then for every $x \in A$ there exists a sequence $(r_n)_1^\infty$, $r_n \to 0$, such that

$$\frac{\log \mu(B(x,r_n))}{\log r_n} < \theta.$$

Hence we have

$$\mu(B(x,r_n)) > r_n^\theta$$

and consequently

$$\limsup_{n\to \infty} \frac{\mu(B(x,r_n))}{r_n^\theta} \ge 1.$$

From Theorem 8.2 (1) we then get $\mathrm{H}_\theta(A) < +\infty$. Hence $\mathrm{HD}(A) \le \theta$ and since $\theta > \theta_2$ was arbitrary, we finally get $\mathrm{HD}(A) \le \theta_2$. □

Recall that if ν is a finite Borel measure on X, then $\mathrm{HD}(\nu)$, the *Hausdorff dimension* of ν, is the minimum of Hausdorff dimensions of sets of full ν measure.

Corollary 8.1. *Suppose that μ is a Borel probability measure on a metric space X. If there exists $\theta_1 \ge 0$ such that for μ-a.e. $x \in X$*

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \ge \theta_1,$$

then $\mathrm{HD}(\mu) \ge \theta_1$.

If there exists $\theta_2 \ge 0$ such that for μ-a.e. $x \in X$

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \le \theta_2,$$

then $\mathrm{HD}(\mu) \le \theta_2$.

Proof. Fix $Y \subseteq X$ with $\mu(Y) = 1$. By the assumption there exists a Borel set $A \subseteq Y$ $\mu(A) > 0$ and

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \ge \theta_1, \qquad \text{for all } x \in A.$$

From Theorem 8.4 it follows then that $\mathrm{HD}(Y) \ge \mathrm{HD}(A) \ge \theta_1$. Hence $\mathrm{HD}(\mu) \ge \theta_1$.

To prove the second part note that by the assumption there exists $Y \subseteq X$ such that $\mu(Y) = 1$ and

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \le \theta_2, \qquad \text{for all } x \in Y.$$

From Theorem 8.4 it follows then that $\mathrm{HD}(Y) \le \theta_2$. Since $\mu(Y) = 1$, we thus get $\mathrm{HD}(\mu) \le \mathrm{HD}(Y) \le \theta_2$, which finishes the proof. □

Corollary 8.2. *If μ is a Borel probability measure on a metric space X, then*

$$\mathrm{HD}(\mu) = \operatorname{essup} \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

In particular, if there exists $\theta \ge 0$ such that

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \theta \qquad \textit{for } \mu\textit{-a.e. } x \in X,$$

then $\mathrm{HD}(\mu) = \theta$.

A Borel probability measure on a metric space X is called *geometric with exponent t* iff there exists $C \ge 1$ such that

$$C^{-1} \le \frac{\mu(B(x,r))}{r^t} \le C$$

for all $x \in X$ and $0 < r \le 1$.

As usually PD and BD denote the *packing* and *box* dimension, respectively. We have the following theorem.

Theorem 8.5. *Suppose X is a metric space and μ is a Borel probability geometric measure on X with exponent t. Then*

(a) *μ, H_t and P_t are mutually equivalent. Furthermore,*

$$0 < \inf \frac{d\mathrm{H}_t}{d\mu} \le \sup \frac{d\mathrm{H}_t}{d\mu} < +\infty \quad \textit{and} \quad 0 < \inf \frac{d\mathrm{P}_t}{d\mu} \le \sup \frac{d\mathrm{P}_t}{d\mu} < +\infty$$

and, in particular, $0 < \mathrm{H}_t(X), \mathrm{P}_t(X) < +\infty$.

(b) $\mathrm{HD}(X) = \mathrm{PD}(X) = \mathrm{BD}(X) = t$.

Proof. (a) follows immediately from Theorems 8.2 and 8.3 (with $A = X$). Consequently $t = \mathrm{HD}(X) = \mathrm{PD}(X)$. To finish the proof it is enough to show that $\overline{\mathrm{BD}}(X) \le t$. And indeed, let $\{(x_i, r)\}_1^k$ be a packing of X $(r \le 1)$. Then

$$kr^t = \sum_{i=1}^k r^t \le C \sum_{i=1}^k \mu(B(x_i, r)) = C\mu\left(\bigcup_{i=1}^k B(x_i, r)\right) \le C\mu(X) = C.$$

Hence $k \le Cr^{-t}$ and $P(X, r) \le Cr^{-t}$, where $P(X, r)$ denotes the maximal number of balls with radius r which are mutually disjoint. This gives (see Theorem from [14] for the first equality sign)

$$\overline{\mathrm{BD}}(X) = \limsup_{r\to 0} \frac{\log P(X,r)}{\log(1/r)} \le \limsup_{r\to 0} \frac{\log C - t\log r}{\log(1/r)} = t,$$

which finishes the proof. □

9. Hausdorff dimension

Again in this section $\mathcal{S} = \{\varphi_e : X_{t(e)} \to X_{i(e)}, e \in E\}$ is a conformal GDMS. We shall prove the following.

Lemma 9.1. *If $t \ge h_{\mathcal{S}}$, then (with $m_t = m_{tL}$)*

$$\inf_{x\in J}\left\{\limsup_{r\to 0}\frac{m_t(B(x,r))}{r^t}\right\}\begin{cases} = +\infty & \text{if } \ P(t) < 0; \\ > 0 & \text{if } \ P(t) = 0 \ (t = h_{\mathcal{S}}). \end{cases}$$

Proof. Fix $x \in J$ and write $x = \pi(\omega)$, $\omega \in E_A^\infty$. Then

$$B(x, D^{-1}\|\varphi'_{\omega_{|n}}\|) \supseteq \varphi_{\omega_{|n}}(X_{t(\omega_{|n})})$$

for all $n \ge 1$, where D comes from Bounded Distortion Property and quasi–convexity of $X_{t(\omega_{|n})}$. Hence (as $P(t) \le 0$),

$$\begin{aligned} m_t(B(x, D^{-1}\|\varphi'_{\omega_{|n}}\|)) &\ge m_t(\varphi_{\omega_{|n}}(X_{t(\omega_{|n})})) \ge e^{-P(t)n}K^{-t}\|\varphi'_{\omega_{|n}}\|^t m_t(X_{t(\omega_{|n})}) \\ &= (DK^{-1})^t e^{-P(t)n}(D^{-1}\|\varphi'_{\omega_{|n}}\|)^t m_t(X_{t(\omega_{|n})}) \\ &\ge (DK^{-1})^t e^{-P(t)n}(D^{-1}\|\varphi'_{\omega_{|n}}\|)^t \inf_{v\in V}\{m_t(X_v)\}, \end{aligned}$$

where the constant K comes from Bounded Distortion Property. Thus

$$\limsup_{r\to 0}\frac{m_t(B(x,r))}{r^t}\begin{cases} = +\infty & \text{if } \ P(t) < 0; \\ \ge (DK^{-1})^{h_{\mathcal{S}}}\inf_{v\in V}\{m_t(X_v)\} & \text{if } \ t = h_{\mathcal{S}}, \end{cases}$$

and we are done. □

As a direct consequence of this lemma, Theorem 8.4, and Theorem 8.2, we get the following.

Proposition 9.1. *We have* $\mathrm{HD}(J_{\mathcal{S}}) \le h_{\mathcal{S}}$, $\mathrm{H}_h|_{J_{\mathcal{S}}} << m_h$ *and* $\mathrm{H}_h(J_{\mathcal{S}}) < +\infty$.

The fact that $\mathrm{HD}(J_{\mathcal{S}}) \le h$ and $H_h(J_{\mathcal{S}}) < +\infty$ (if $h \in \mathcal{F}_{\mathcal{S}}$) could have been obtained directly, without invoking Theorems 8.4 and 8.2. As a direct consequence of Lemma 9.1 and Theorem 8.2, we also get the following.

Proposition 9.2. *If the system $\mathcal{S}$ is irregular, then $H_{h_{\mathcal{S}}}(J_{\mathcal{S}}) = 0$.*

We shall prove the following.

Lemma 9.2. *If the system $\mathcal{S}$ satisfies the strong open set condition, then for all $t \in \mathcal{F}_{\mathcal{S}} \cap [0, h_{\mathcal{S}}]$, we have*

$$\liminf_{r\to 0}\frac{m_t(B(x,r))}{r^t}\begin{cases} = 0 & \text{if } \ t < h; \\ < +\infty & \text{if } \ t = h \text{ and } \mathcal{S} \text{ is regular}, \end{cases}$$

for all $x \in J_t$, where J_t is some Borel subset of $J_{\mathcal{S}}$ with $m_t(J_t) = 1$.

Proof. Since $\mathcal{S}$ satisfies the strong open set condition, there exists $x \in J_{\mathcal{S}}$ and $R > 0$ such that $B(x, 2R) \subseteq \mathrm{Int}_{\mathcal{H}} X$. Since $\mathrm{supp}(\mu_t) = \mathrm{supp}(m_t) = \overline{J_{\mathcal{S}}}$, we have $m_t(B(x,R)) \cdot \mu_t(B(x,R)) > 0$. Hence $\hat{\mu}_t(\pi^{-1}(B(x,R))) > 0$. Since the measure $\hat{\mu}_t$ is ergodic (with respect to the shift map $\sigma : E_A^\infty \mapsto E_A^\infty$), there exists a Borel set $\hat{E}_t \subseteq E_A^\infty$ such that $\hat{\mu}_t(\hat{E}_t) = 1$ and for every $\omega \in \hat{E}_t$ there exists an unbounded increasing sequence $(n_j = n_j(\omega))_{j=1}^\infty$ of positive integers such that $\sigma^{n_j}(\omega) \in \pi^{-1}(B(x,R))$. Fix $\omega \in \hat{E}_t$. Then for all $j \geq 1$, we have

$$\varphi_{\omega_{|n_j}}(B(\pi(\sigma^{n_j}(\omega)), R)) \supseteq B(\pi(\omega), K^{-1}R\|\varphi'_{\omega_{|n_j}}\|).$$

Since $B(\pi(\sigma^{n_j}(\omega)), R) \subseteq X_{t(\omega_{n_j})}$, we thus get from conformality of m_t,

$$\begin{aligned} m_t(B(\pi(\omega), K^{-1}R\|\varphi'_{\omega_{|n_j}}\|)) &\leq e^{-P(t)n_j}\|\varphi'_{\omega_{|n_j}}\|^t m_t(B(\pi(\sigma^{n_j}(\omega), R))) \\ &\leq e^{-P(t)n_j}\|\varphi'_{\omega_{|n_j}}\|^t = e^{-P(t)n_j}(KR^{-1})^t(K^{-1}R\|\varphi'_{\omega_{|n_j}}\|)^t. \end{aligned}$$

Therefore,

$$\liminf_{r\to 0} \frac{m_t(B(\pi(\omega), r))}{r^t} \begin{cases} = 0 & \text{if} \quad t < h_{\mathcal{S}} \text{ (equivalently } P(t) > 0); \\ \leq (KR^{-1})^h & \text{if} \quad t = h_{\mathcal{S}} \text{ and } \mathcal{S} \text{ is regular } (P(h_{\mathcal{S}}) = 0). \end{cases}$$

So, we are done by taking $J_t = \pi(\hat{E}_t)$. $\square$

As an immediate consequence of this lemma and Theorem 8.3, we get the following.

Proposition 9.3. *If the system $\mathcal{S}$ is regular and satisfies the strong open set condition, then $m_h << P_h$ and $P_h(J_{\mathcal{S}}) > 0$.*

10. Bowen's Formula

In this section we prove a version of Bowen's formula, which for finite alphabet system says that if a system satisfies the SOSC, then Hausdorff dimension of the limit set is equal to its Bowen's parameter, i.e. the only solution h to the equation $P(t) = 0$. We want to stress that this formula holds even though the corresponding Hausdorff measure $H_h(J)$ (even in the case of finite alphabet) may vanish (see Example 11.1)

Although h–dimensional Hausdorff measure may vanish, motivated by Theorem 2.6 in [17] we shall prove the following.

Theorem 10.1. *(Bowen's formula for finite SOSC) If $\mathcal{S} = \{\varphi_e : X_{t(e)} \to X_{i(e)}, e \in E\}$ is a finitely primitive graph directed Markov system satisfying the SOSC, then* $\mathrm{HD}(J_{\mathcal{S}}) = h$*, the only zero of the pressure function $t \to P(t)$, $t \in \mathbb{R}$.*

Proof. Because of the SOSC there exists $x \in J_{\mathcal{S}} \cap \mathrm{Int}\, X$ Then there exists $\tau \in E_A^*$ such that

$$x \in \varphi_\tau(J_{t(\tau)}) \subseteq \varphi_\tau(X_{t(\tau)}) \subseteq \mathrm{Int}\, X_{i(\tau)}. \tag{10.1}$$

Consider the iterated function system

$$\mathcal{S}_n = \{\varphi_{\tau\omega} : \varphi_\tau(X_{t(\tau)}) \to \varphi_\tau(X_{t(\tau)}), \omega, |\omega| = n, \tau\omega\tau \in E_A^*\} \quad \text{for } n \geq 1.$$

Because of (10.1) $\mathcal{S}_n$ satisfies the separation condition, and, because of primitivity of $\mathcal{S}$, $\mathcal{S}_n \neq \emptyset$ for all $n \geq 1$ large enough, say $n \geq q$. Let J_n be the limit set of $\mathcal{S}_n$. Clearly, $J_n \subseteq J_{\mathcal{S}}$, and therefore, $h_n \leq \mathrm{HD}(J_{\mathcal{S}}) \leq h$, where $h_n = \mathrm{HD}(J_n)$. It therefore suffices to show that $\gamma := \sup_{n\geq 1}(h_n) \geq h$. Since the system $\mathcal{S}$ is primitive, there exists a finite set $\Lambda \subseteq E_A^*$, consisting of words of the same length, say $p \geq 1$, such that for all $\omega \in E_A^*$ there are $\alpha_\omega, \beta_\omega \in \Lambda$ such that $\tau\alpha_\omega\omega\beta_\omega\tau \in E_A^*$. Put

$$\xi = \min\{\|\varphi_\alpha'\| : \alpha \in \Lambda\} > 0.$$

For every $n \geq q$ let m_n be the h_n–conformal measure for the iterated function system $\mathcal{S}_n$. Fix $n \geq \max\{p, q\}$. We then have,

$$\begin{aligned}
1 &= m_{n+2p}(J_{n+2p}) \geq m_{n+2p}\Big(\bigcup_{\omega\in E_A^n} \varphi_{\tau\alpha_\omega\omega\beta_\omega}(\varphi_\tau(X_{t(\tau)}))\Big) \\
&= \sum_{\omega\in E_A^n} m_{n+2p}(\tau\varphi_{\alpha_\omega\omega\beta_\omega}(\varphi_\tau(X_{t(\tau)})) \\
&\geq K^{-h_n}\sum_{\omega\in E_A^n} \|\varphi_{\tau\alpha_\omega\omega\beta_\omega}'\|^{h_n} m_{n+2p}(\varphi_\tau(X_{t(\tau)})) = K^{-h_n}\sum_{\omega\in E_A^n} \|\varphi_{\tau\alpha_\omega\omega\beta_\omega}'\|^{h_n} \\
&\geq K^{-4h_n}\sum_{\omega\in E_A^n} \|\varphi_\tau'\|^{h_n}\|\varphi_{\alpha_\omega}'\|^{h_n}\|\varphi_\omega'\|^{h_n}\|\varphi_{\beta_\omega}'\|^{h_n} \geq (\xi^2\|\varphi_\tau'\|K^{-4})^{h_n}\sum_{\omega\in E_A^n}\|\varphi_\omega'\|^{h_n} \\
&\geq (\xi^2\|\varphi_\tau'\|K^{-4})^{\gamma}\sum_{\omega\in E_A^n}\|\varphi_\omega'\|^{\gamma}.
\end{aligned}$$

Hence $\sum_{\omega\in E_A^n}\|\varphi_\omega'\|^\gamma \leq (K^4\|\varphi_\tau'\|^{-1}\xi^{-2})^\gamma$, and therefore,

$$P(\gamma) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{\omega\in E_A^n}\|\varphi_\omega'\|^\gamma \leq \lim_{n\to\infty}\frac{1}{n}\log((K^4\|\varphi_\tau'\|^{-1}\xi^{-2})^\gamma) \leq 0.$$

So, $\gamma \geq h$ and we are done. □

We finish this section with the proof of Bowen's formula in the case when alphabet is infinite.

Theorem 10.2. *Suppose $\mathcal{S} = \{\varphi_e : X_{t(e)} \to X_{i(e)}, e \in E\}$ is a finitely primitive (the set E can be infinite) graph directed Markov system satisfying the SOSC. Then*

$$\mathrm{HD}(J_{\mathcal{S}}) = \sup\{\mathrm{HD}(J_F) : E \supset F - \textit{finite}\} = \inf\{t \geq 0 : P(t) \leq 0\}.$$

Proof. Put $\eta = \sup\{\mathrm{HD}(J_F) : E \supset F - \text{finite}\}$, $h = h_{\mathcal{S}} = \inf\{t \geq 0; P(t) \leq 0\}$ and $\xi = \mathrm{HD}(J_{\mathcal{S}})$. Obviously, $\eta \leq \xi$ and, by Proposition 8.1, $\xi \leq h$. It thus suffices to show that $h \leq \eta$. To do this, fix $t > \eta$. Then $P_F(t) < 0$ for every finite subset F of E, and therefore, by Theorem 2.1.5 in [12], $P(t) = \sup\{P_F(t) : E \supset F - \text{finite}\} \leq 0$. Hence $t \geq h$, and consequently $\eta \geq h$. We are done. □

Theorem 10.3. *We have*

$$\mathrm{HD}(J_{\mathcal{S}}) = +\infty \Longleftrightarrow \Theta_{\mathcal{S}} = +\infty.$$

If a system $\mathcal{S} = \{\varphi_i\}_{i\in I}$ consists of similarities, then $P(t) = \log \sum_{i\in I} |\varphi_i'|^t$. Therefore in this case Theorem 10.3 takes on the following form.

Theorem 10.4. *If $\mathcal{S} = \{\varphi_i\}_{i\in I}$ consists of similarities, then we have*

$$\mathrm{HD}(J_{\mathcal{S}}) = \inf\{t \geq 0 : \sum_{i\in I} |\phi_i'|^t \leq 1\}.$$

Example 10.1. *We shall construct an infinite conformal iterated function system $\mathcal{S}$ satisfying the OSC with the following properties.*

(a) *$\mathcal{S}$ satisfies the strong separation condition, so $\overline{J}_{\mathcal{S}}$ is a topological Cantor set, in particular the topological dimension of $\overline{J}_{\mathcal{S}}$ is equal to zero.*
(b) *The system $\mathcal{S}$ is topologically thin.*
(c) *$\mathcal{S}$ consists of similarities.*
(d) $\mathrm{HD}(J_{\mathcal{S}}) = +\infty$.

In order to start the construction take $\{r_n\}_1^\infty$, a sequence of positive numbers less than $1/10$ such that

$$\sum_{n=1}^{\infty} r_n^t = +\infty \tag{10.2}$$

for all $t \geq 0$. Let $B = B(0,1)$ be the open unit ball in the Hilbert space l_2, and let $\{e_n\}_1^\infty$ be an orthonormal basis. Define the maps $\varphi_n : l_2 \mapsto l_2$

$$\varphi_n(x) = r_n x + 4r_n e_n.$$

If $x \in \overline{B}$, then $\|\varphi_n(x)\| \leq \|r_n x\| + \|4r_n\| \leq 5\|r_n\| \leq 1/2$. So

$$\varphi_n(\overline{B}) \subseteq \overline{B}(0,1/2) \subseteq B(0,1/2). \tag{10.3}$$

Since in addition $|\varphi_n'(x)| = r_n$, we see that $\{\varphi_n : \overline{B} \mapsto \overline{B}\}_1^\infty$ is a system of similarities. We shall check that $\mathcal{S}$ satisfies the strong separation condition. Indeed, if $m \neq n$, then for all $x, y \in \overline{B}$, $\|\varphi_m(x) - 4r_m e_m\| \leq r_m$ and $\|\varphi_n(y) - 4r_n e_n\| \leq r_n$. Therefore,

$$\begin{aligned}\|\varphi_n(y) - \varphi_m(x)\| &\geq \|4r_n e_n - 4r_m e_m\| - (r_m + r_n) = 4\sqrt{r_n^2 + r_m^2} - (r_m + r_n) \\ &\geq (2(r_n + r_m) - (r_n + r_m) = r_n + r_m.\end{aligned}$$

Therefore, $\mathrm{dist}(\varphi_n(\overline{B}), \varphi_m(\overline{B})) \geq r_n + r_m > 0$. Together with (10.3) this gives that $\mathcal{S}$ satisfies the strong separation condition. Combining now Theorem 10.4 with (10.2), we see that we are only left to prove item (b). But $\varphi_n(\overline{B}) \subset \overline{B}(0, 5r_n)$ for all $n \geq 1$. Thus $\mathcal{S}$ is asymptotically compact and $\partial S(\infty) = \{0\}$. This means that $\mathcal{S}$ is topologically thin and we are done.

This is an example, but in fact it gives a huge class of mutually distinct systems. Indeed, the collection of sequences $(r_n)_1^\infty$ satisfying (10.2) and such that $r_n \leq 1/10$ for all $n \geq 1$ is of cardinality $\mathfrak{c}$. The set of systems $\mathcal{S}$ for which $\mathrm{HD}(J_{\mathcal{S}}) = +\infty$ is also big in a topological sense. Indeed, in [URoy] the space of $\mathrm{CIFS}(X)$ of all conformal iterated function systems on X was topologized in the following way. A sequence $(\Phi^{(n)})_1^\infty$ of conformal iterated function systems on X is said to converge to a system Φ provided that the following conditions are satisfied.

(a) $\forall\, i \in I$ $\Phi_i^{(n)}$ converges uniformly to Φ_i on X.
(b) $\forall\, i \in I$ $(\Phi_i^{(n)})'$ converges uniformly to $(\Phi_i)'$ on X.
(c) There exists $C \geq 1$ such that for all $n \geq 1$ large enough and all $i \in I$,

$$C^{-1} \leq \frac{\|(\Phi_i^{(n)})'\|}{\|\Phi_i'\|} \leq C.$$

This endows CIFS(X) with the topology, called λ–topology, by declaring that a subset A of CIFS(X) is closed if and only if all limits of sequences in A belong to A. In [15] the set X was assumed to be a compact subset of a finite dimensional Euclidean space but it is irrelevant for the definition of λ–topology and most of its properties established in [16] such as normality, lack of metrizability, or even stronger, lack of the first axiom of countability. Perhaps most important feature of λ–topology is that the Hausdorff dimension function $\mathrm{CIFS}(X) \ni \mathcal{S} \mapsto \mathrm{HD}(J_{\mathcal{S}})$ is continuous. Again, this was established in [15] for X lying in a finite–dimensional Euclidean space, but because of Theorem 10.3 is true for a Hilbert space as well. We shall prove the following.

Theorem 10.5. *The set* $\mathrm{CIFS}_\infty(X) = \{\mathcal{S} \in \mathrm{CIFS}(X) : \mathrm{HD}(J_{\mathcal{S}}) = +\infty\}$ *is closed and open in the λ–topology.*

Proof. Because of Theorem 10.3 and item (c) above the set $\mathrm{CIFS}_\infty(X)$ is closed. To prove its openness, suppose that $\mathcal{S}_n \in \mathrm{CIFS}(X) \setminus \mathrm{CIFS}_\infty(X)$ and that $\mathcal{S}_n \to \mathcal{S}$ in the λ–topology. Then $\Theta_{\S_n}$ is finite for all $n \geq 1$ and, because of (c), $\Theta_{\mathcal{S}} = \Theta_{\mathcal{S}_n}$ for all n large enough. Thus $\Theta_{\mathcal{S}}$ is finite, and by Theorem 10.3, $\mathcal{S} \in \mathrm{CIFS}(X) \setminus \mathrm{CIFS}_\infty(X)$. We are done. $\square$

Motivated by the complex continued fraction system described in [9], we would like to describe some special system whose elements are not similarities. We call the *ccf–like systems.* Fix $d = 1, 2, \ldots, \infty$. Let $X = \{x : \|x \leq 1\}$ be a subset of $\mathbb{R}^d$ if d is finite and a subset of the Hilbert separable space l_2 if $d = \infty$. Let $I_d = \mathbb{Z}^d \setminus \{0\}$ if $d < \infty$ and let I_d be the set of all non–zero sequences $(m_n)_1^\infty \in \mathbb{Z}^{\mathbb{N}}$ such that $m_n \neq 0$ for finitely many n's only. Let i be the inversion with respect to the sphere $\partial X = \{x : \|x\| = 1\}$, i.e. $i(x)$ is the only vector different from x lying on the line passing through 0 and x and such that $\|i(x)\| \cdot \|x\| = 1$, ($i(0) = \infty$, $i(\infty) = 0$). For every $b \in I_d$ let φ_b be defined by the formula

$$\varphi_b(x) = i(x + b).$$

Clearly $\varphi_b(X) \subseteq X$, $\|\varphi_b'(x)\| \approx \|b\|^{-2}$ and $\|\varphi_b(x)\| \approx \|b\|^{-1}$ for all $b \in I_d$. We shall prove the following.

Theorem 10.6. *For every d, $\mathcal{S}_d = \{\varphi_b : X \mapsto X\}_{b \in I_d}$ is a conformal iterated function system satisfying the strong open set condition with the following properties.*

(a) *$\mathcal{S}_d$ is co–finitely (hereditarily) regular and $\Theta_{\mathcal{S}_d} = d/2$.*
(b) *$\mathrm{HD}(J_{\mathcal{S}_d}) > d/2$ if d is finite and $\mathrm{HD}(J_{\mathcal{S}_\infty}) = \infty$.*
(c) *$\overline{J_{\mathcal{S}_d}}$ is a Peano (locally connected) continuum.*
(d) *Each system $\mathcal{S}_d$ is topologically thin and $\mathcal{S}_d(\infty) = \{0\}$.*

Proof. It is immediate from the definition that each $\mathcal{S}_d$ is a conformal iterated function system satisfying the strong open set condition. Property (d) follows from the fact that $\varphi_b(X) \subseteq B(0, c\|b\|^{-1})$ with some constant $C \geq 1$. Property (c) results from Theorem 3.4. In order to prove (a) and (b) note that

$$\Phi(t) = \sum_{b \in I_d} \|\varphi_b'\|^t \approx \sum_{b \in I_d} \|b\|^{-2t} = \sum_{k=0}^{\infty} \sum_{2^k \leq \|b\| < 2^{k+1}} \|b\|^{-2t}$$
$$\approx \sum_{k=0}^{\infty} 2^{-2tk} \#\{b \in I_d : 2^k \leq \|b\| < 2^{k+1}\}.$$

Now, if $d = \infty$, the the set $\{b \in I_d : 2^k \leq \|b\| < 2^{k+1}\}$ is infinite for all $k \geq 1$. Thus, $\Phi(t) = \infty$ for all $t \geq 0$. This means that $\Theta_{\mathcal{S}_d} = +\infty$ and (a) and (b) follow in this case from applying Theorem 10.3. If d is finite, $\{b \in I_d : 2^k \leq \|b\| < 2^{k+1}\} \approx 2^{dk}$, and therefore,

$$\Phi(t) \approx \sum_{k=1}^{\infty} 2^{k(d-2t)}.$$

Hence, $\Theta_{\mathcal{S}_d} = d/2$ and $\Phi(\Theta_{\mathcal{S}_d}) = +\infty$. Items (a) and (b) are established. □

11. Geometrically Perfect Property

In this section we investigate the behavior of the Hausdorff measure $H_h(J)$. In the case of Euclidean spaces it is well–known that we have always $0 < H_h(J) < +\infty$. However, as Example 10.1 shows, this is not longer the case in Hilbert spaces. In order to deal with positivity of the Hausdorff measure we introduce the concept of geometrical perfectness of a limit set. The issue is complex enough already in the case of finite alphabet (in Hilbert spaces) and throughout the whole section we assume the set of edges (alphabet) E is finite. Unless specifically stated, we do not assume any GDMS appearing in this section to be of strongly bounded multiplicity. The measure m_h is the projection of $m_{-h \log|\varphi'_{\omega_1} \circ \sigma|}$ from the symbol space, where h is uniquely determined by the requirement that $P(h) = 0$. Further $m_h(B(x,r)) \geq r^h$ always holds.

Assume that the open set condition holds. We would like now to find sufficient conditions for

$$m_h(B(x,r)) \leq Cr^h.$$

One sufficient condition is that the separation condition is satisfied, i.e. $\varphi_i(X) \cap \varphi_j(X) = \emptyset$ if $i \neq j$.

Theorem 11.1. *There exists a finite alphabet IFS (consisting of similarities) in a Hilbert space satisfying the OSC but failing to satisfy the SOSC.*

Proof. This is a slight modification of Schief's example (see [17]). Let $l_2 = \{(x_n)_{-\infty}^{\infty} \in \mathbb{R}^{\mathbb{Z}} : \sum_{-\infty}^{+\infty} x_n^2 < +\infty\}$. Let $\pi_1, \pi_2 : \mathbb{Z} \to \mathbb{Z}$ be bijections such that $\pi_i(n) = 2n + i$, $i = 1, 2$, for all $n \geq 0$. Further, let $(e_n)_{-\infty}^{+\infty}$ be the standard orthonormal basis in l_2. By $\varphi_i : l_2 \to l_2$, $i = 1, 2$, we denote the similarities given by the formulas

$$\varphi_i\left(\sum_{n\in\mathbb{Z}} x_n e_n\right) = \sum_{n\in\mathbb{Z}} \frac{x_n}{2} e_{\pi_i(n)} \qquad \text{for } i = 1, 2.$$

For every $n \in \mathbb{Z}$, let

$$U_n = \left\{x \in l_2 : \|x\| < 2,\ \langle x, e_n\rangle > 0,\ \frac{\langle x, e_n\rangle}{\|x\|} > 1 - \frac{1}{10}\right\}.$$

Further, let

$$G = \bigcup_{k=0}^{\infty} U_k.$$

Since $\varphi_i(U_n) \subseteq U_{\pi_i(n)}$, we have $\varphi_i(G) \subseteq G$. Since $U_n \cap U_k = \emptyset$ for $n \neq k$, we thus get $\varphi_1(G) \cap \varphi_2(G) = \emptyset$. Hence the open set condition is satisfied. But since φ_1 and φ_2 have a common fixed point, namely $0 = (\ldots, 0, 0, 0, \ldots)$, the SOSC fails. $\square$

In a sharp contrast to the case of finite–dimensional Euclidean spaces, the Hausdorff measure of the limit set of finite alphabet conformal, even similarity, IFS may vanish. The example below shows it.

Example 11.1. *There exists a finite alphabet IFS (consisting of similarities) in a Hilbert space satisfying the SOSC with $H_h(J) = 0$, where $h = \mathrm{HD}(J)$.*

Proof. Let $\varphi_2, \varphi_2 : l_1 \to l_2$ be as in Theorem 11.1. Add $\varphi_3 : l_2 \to l_2$ given by the formula $\varphi_3(x) = e_0 + \beta(x - e_0) = (1 - \beta)e_0 + \beta x$, $0 < \beta << 1$. If β is sufficiently small, then $\varphi_3(U_n) = (1 - \beta)e_0 + \beta U_n \subseteq U_0 \subseteq G$ and hence $\varphi_3(G) \subseteq U_0 \subseteq G$. Notice, $\varphi_1(G) \cup \varphi_2(G) \subseteq \bigcup_{n=1}^{\infty} U_n$. Hence $\varphi_3(G) \cap (\varphi_1(G) \cup \varphi_2(G)) = \emptyset$. Thus, the OSC is satisfied. Now, $0 \in J$ since $\varphi_1(0) = 0$. Hence $J \ni \varphi_3(0) = (1 - \beta)e_0 \in U_0 \subseteq G$ if β is small enough. Thus $J \cap G \neq \emptyset$ and the SOSC is satisfied.

Let k be a positive integer and $\mathcal{I}_k = \{1, 2\}^k$. Let

$$\mathcal{I}_k^0 = \{(i_1, \ldots, i_p) : p \geq 1,\ i_p = 3,\ (i_l, \ldots, i_{l+k-1}) \notin \mathcal{I}_k \text{ for each } l\} \cup \{\emptyset\}.$$

We easily check that the following union is disjoint:

$$\Omega_k = \bigcup_{\mathbf{j}\in\mathcal{I}_k} \bigcup_{\mathbf{i}\in\mathcal{I}_k^0} [\mathbf{ij}].$$

Denote by r_1, r_2, r_3 the similarity ratios of $\varphi_1, \varphi_2, \varphi_3$, respectively. Let $\mathbb{P}$ be the unique probability measure on $\Omega = \{1, 2, 3\}^{\mathbb{N}}$ such that

$$\mathbb{P}([(i_1, \ldots, i_n]) = (r_{i_1} \cdots r_{i_n)})^h.$$

Since for $\mathbb{P}$–almost all elements $(i_1, i_2, \ldots)$ of Ω there is first sequence $(3, j_1, \ldots, j_k)$, where j_m are 1 or 2, we have $\mathbb{P}(\Omega_k) = 1$. This, in turn, gives

$$1 = \mathbb{P}(\Omega_k) = \sum_{\mathbf{j}\in\mathcal{I}_k} \sum_{\mathbf{i}\in\mathcal{I}_k^0} r_{\mathbf{i}}^h r_{\mathbf{j}}^h = 2^k (2^{-k})^h \sum_{\mathbf{i}\in\mathcal{I}_k^0} r_{\mathbf{i}}^h,$$

where $r_{\mathbf{i}} = r_{i_1} \cdots r_{i_n}$ for $\mathbf{i} = (i_1, \dots, i_n)$. Hence it follows that $m_h(L_k) = 1$, where

$$L_k = \bigcup_{\mathbf{j} \in \mathcal{I}_k} \bigcup_{\mathbf{i} \in \mathcal{I}_k^0} \varphi_{\mathbf{ij}}(J).$$

Since $m_h(J \setminus L_k) = 0$ and H_h, m_h are mutually equivalent, we get $H_h(J \setminus L_k) = 0$. Further, we see that

$$\bigcup_{\mathbf{j} \in \mathcal{I}_k} \varphi_{\mathbf{j}}(J) \subset \{x \in l_2 : \|x\| \leq 2^{1-k}\}.$$

Hence

$$L_k \subset \bigcup_{\mathbf{i} \in \mathcal{I}_k^0} \varphi_{\mathbf{i}}(\{x \in l_2 : \|x\| \leq 2^{1-k}\}),$$

which finally gives

$$H_h(J) = H_h(L_k) \leq \sum_{\mathbf{i} \in \mathcal{I}_k^0} (2^{2-k} r_{\mathbf{i}})^h = 2^{2-hk} 2^{-k} 2^{hk} = 2^{2-k},$$

which tends to zero as k tends to infinity. This completes the proof. $\square$

Remark 2. *Since $H_h(J) = 0$ and the SOSC holds, the limit set J from Example 11.1 is not bi–Lipschitz equivalent to any self–conformal subset of a finitely–dimensional Euclidean space.*

Definition 11.1. *A conformal graph directed system is called* geometrically perfect *provided $\exists\ C \geq 1$, $\forall\ x \in J$, $\forall\ r > 0$*

$$m_h(B(x, r)) \leq C r^h. \tag{11.1}$$

Since we always have $m_h(B(x, r)) \geq C^{-1} r^h, (0 < r \leq 1)$ we see that the conformal measure of every geometrically perfect system is geometric, and as an immediate consequence of Theorem 7.5 we get.

Theorem 11.2. *Suppose X is a metric space and m_h is the conformal measure of geometrically perfect system. Then*

(a) *m_h, H_h and P_h are mutually equivalent. Furthermore,*

$$0 < \inf \frac{\mathrm{dH}_h}{\mathrm{d}m_h} \leq \sup \frac{\mathrm{dH}_h}{\mathrm{d}m_h} < +\infty \quad \text{and} \quad 0 < \inf \frac{\mathrm{dP}_h}{\mathrm{d}m_h} \leq \sup \frac{\mathrm{dP}_h}{\mathrm{d}m_h} < +\infty$$

and, in particular, $0 < \mathrm{H}_h(X), \mathrm{P}_h(X) < +\infty$.

(b) $\mathrm{HD}(X) = \mathrm{PD}(X) = \mathrm{BD}(X) = h$.

Theorem 11.3. *Let $\mathcal{S} = \{\varphi_e : X_{i(e)} \to X_{t(e)}, e \in E\}$ be a primitive conformal graph directed system (on a separable Hilbert space) with a finite set of edges E satisfying the OSC. Suppose that $\varphi_a(X_{t(a)}) \cap \varphi_b(X_{t(b)}) \cap \partial X = \emptyset$ for all $a, b \in E$, $a \neq b$. Then the system $\mathcal{S}$ is geometrically perfect.*

Proof. Set $F_a = \varphi_a(J_{t(a)}) \cap \partial X$ for $a \in E$. Obviously F_a, $a \in E$, is a compact set. Since $F_a \cap F_b = \emptyset$ for $a \neq b, a, b \in E$ and the set E is finite, there exists $\eta_0 > 0$ such that

$$\mathcal{N}(F_a, 2\eta_0) \cap \mathcal{N}(F_b, 2\eta_0) = \emptyset \qquad \text{for } a \neq b.$$

Since

$$\varphi_a(J_{t(a)}) \cap \bigcup_{b \in E \setminus \{a\}} F_b = \emptyset \qquad \text{for } a \in E,$$

we may find $\eta \leq \eta_0$ such that for every $a \in E$

$$\varphi_a(J_{t(a)}) \cap \bigcup_{b \in E \setminus \{a\}} \mathcal{N}(F_b, \eta) = \emptyset. \tag{11.2}$$

Set $H_a = \mathcal{N}(F_a, \eta)$ for $a \in E$. Define $K_{\mathcal{S}} = J_{\mathcal{S}} \setminus \bigcup_{a \in E} H_a$ and observe that K is a compact set which satisfies

$$K_{\mathcal{S}} \cap X_v \subset \operatorname{Int} X_v \qquad \text{for } v \in V. \tag{11.3}$$

Since $K \cap X_v$ for $v \in V$ is compact, we may find $\varepsilon \in (0, \eta)$ such that

$$\mathcal{N}(K \cap X_v, \varepsilon) \subset \operatorname{Int} X_v.$$

Fix $r > 0$, $z \in J_{\mathcal{S}}$ and $a \in E$. Put $B := B(z, \varepsilon r/(2K))$, where K is given by Theorem 5.2. Let

$$\mathcal{F}_a = \{\tau \in E_A^* : \tau_{|\tau|} = a,\, \varphi_\tau(J_{t(\tau)}) \cap B \neq \emptyset,\, \|\varphi_\tau'\| < r \text{ and } \|\varphi'_{\tau_{|\tau|-1}}\| \geq r\}.$$

Observe that the family $\mathcal{F}_a$ is an antichain which means that no two elements in $\mathcal{F}_a$ are comparable. We are going to show that $\#\mathcal{F}_a \leq \#E$. For every $\tau \in \mathcal{F}_a$ choose $x_\tau \in J_{t(\tau)}$ such that $\varphi_\tau(x_\tau) \in B$. We will consider two cases.

Case 1: Assume that there exists $l \in \{2, \ldots, |\tau|\}$ such that $\varphi_{\tau_l} \circ \cdots \circ \varphi_{\tau_{|\tau|}}(x_\tau) \notin \bigcup_{a \in E} H_a$ for some $\tau \in \mathcal{F}_a$. Let p be the largest integer having the above property and let $\omega \in \mathcal{F}_a$ denote some word corresponding to p. We show that for every $\tau \in \mathcal{F}_a$ we have

$$\tau_1 = \omega_1, \tau_2 = \omega_2, \ldots, \tau_{p-1} = \omega_{p-1}.$$

Indeed, observe that

$$\varphi_{\omega_1} \circ \cdots \circ \varphi_{\omega_{p-1}}(\operatorname{Int} X_{t(\omega_{p-1})}) \supset \varphi_{\omega_1} \circ \cdots \circ \varphi_{\omega_{p-1}}(B(\varphi_{\omega_p} \circ \cdots \circ \varphi_{\omega_{|\omega|}}(x_\omega), \varepsilon)) \supset B.$$

On the other hand, from the OSC it follows that

$$\varphi_{\omega_1} \circ \cdots \circ \varphi_{\omega_{p-1}}(\operatorname{Int} X_{t(\omega_{p-1})}) \cap \varphi_{\tau_1} \circ \cdots \circ \varphi_{\tau_{p-1}}(X_{t(\tau_{p-1})}) \neq \emptyset$$

for $(\tau_1, \ldots, \tau_{p-1}) \neq (\omega_1, \ldots, \omega_{p-1})$. Hence

$$\varphi_\tau(J_{t(\tau)}) \cap B = \emptyset,$$

which is impossible. Suppose that $\#\mathcal{F}_a \geq \#E + 1$. Then there exist two sequences in $\mathcal{F}_a$, say, $\hat{\tau}$ and $\tilde{\tau}$ such that $\hat{\tau}|_p = \tilde{\tau}|_p$. Further, let $k \geq p$ be first such an integer that $\hat{\tau}_k \neq \tilde{\tau}_k$. By the definition of p we get that

$$\varphi_{\hat{\tau}_k} \circ \cdots \circ \varphi_{\hat{\tau}_{\hat{\tau}}}(x_{\hat{\tau}}) \in H_{\hat{\tau}_k}$$

and

$$\varphi_{\tilde{\tau}_k} \circ \cdots \circ \varphi_{\tilde{\tau}_{\tilde{\tau}}}(x_{\tilde{\tau}}) \in H_{\tilde{\tau}_k}.$$

Since $\operatorname{dist}(H_{\hat{\tau}_k}, H_{\tilde{\tau}_k}) \geq \eta > \varepsilon$ we get

$$\rho(\varphi_{\hat{\tau}}(x_{\hat{\tau}}), \varphi_{\tilde{\tau}}(x_{\tilde{\tau}})) > \varepsilon r/K$$

which is impossible by the fact that both $\varphi_{\hat{\tau}}(x_{\hat{\tau}})$ and $\varphi_{\tilde{\tau}}(x_{\tilde{\tau}})$ intersect B. Hence it follows that $\#\mathcal{F}_a \leq \#E$.

Case 2: Now assume that for every $\tau \in \mathcal{F}_a$ and every $l \in \{2, \dots, |\tau|\}$ we have $\varphi_{\tau_l} \circ \cdots \circ \varphi_{\tau_{|\tau|}}(x_\tau) \in \bigcup_{a\in E} H_a$. Denote by $\mathcal{F}_a^b$ for $b \in E$ the set of all $\tau \in \mathcal{F}_a$ such that $\tau_1 = b$. Suppose that there exist two different sequences in $\mathcal{F}_a^b$, say, $\hat{\tau}$ and $\tilde{\tau}$. Let let $k \geq 2$ be first such an integer that $\hat{\tau}_k \neq \tilde{\tau}_k$. Obviously,

$$\varphi_{\hat{\tau}_k} \circ \cdots \circ \varphi_{\hat{\tau}_{\hat{\tau}}}(x_{\hat{\tau}}) \in H_{\hat{\tau}_k},$$

$$\varphi_{\tilde{\tau}_k} \circ \cdots \circ \varphi_{\tilde{\tau}_{\tilde{\tau}}}(x_{\tilde{\tau}}) \in H_{\tilde{\tau}_k}$$

and, analogously as in Case 1 we get

$$\rho(\varphi_{\hat{\tau}}(x_{\hat{\tau}}), \varphi_{\tilde{\tau}}(x_{\tilde{\tau}})) > \varepsilon r/K,$$

by the fact that $H_{\hat{\tau}_k} \neq H_{\tilde{\tau}_k}$. From this it follows that only one of the sequences $\hat{\tau}, \tilde{\tau}$ may intersect B. Thus $\#\mathcal{F}_a^b \leq 1$ and $\#\mathcal{F}_a \leq \#E$.

Since

$$B(z, \varepsilon r/(2K)) \cap J_{\mathcal{S}} \subset \bigcup_{a\in E} \bigcup_{\tau\in\mathcal{F}_a} \varphi_\tau(J_{t(\tau)}),$$

we get

$$\begin{aligned} m_h(B(z, \varepsilon r/(2K))) &\leq \sum_{a\in E} \sum_{\tau\in\mathcal{F}_a} m_h(\varphi_\tau(J_{t(\tau)})) \leq \sum_{a\in E} \sum_{\tau\in\mathcal{F}_a} \|\varphi_\tau'\| \\ &\leq (\#E)^2 r^h. \end{aligned}$$

Since $r > 0$ was arbitrary, the proof is complete. $\square$

Remark 3. *If the system $\mathcal{S} = \{\varphi_e : X_{t(e)} \to X_{i(e)}, e \in E\}$ with the finite set of edges E satisfies the open set condition and $\varphi_a(X_{t(a)}) \cap \varphi_b(X_{t(b)}) \cap \partial X = \emptyset$ for all $a, b \in E$, $a \neq b$, then $\mathcal{S}$ is of strongly bounded multiplicity.*

Proof. We are going to show that the set C_x of mutually incomparable pseudo–codes of every point $x \in X$ is bounded above by $\#E$. Fix $x \in X$. Let $\omega_1\omega_2 \dots \omega_n \in C_x$ be a pseudo–code of x with minimal length. Let $y \in X$ be such that $x = \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \dots \circ \varphi_{\omega_n}(y)$. If $y \in \operatorname{Int} X$, then from the OSC it follows that every pseudo–code of x must be comparable with $\omega_1\omega_2 \dots \omega_n$.

On the other hand, assume now that $y \in \partial X$ and let $i \in \{1, \dots, n\}$ be such that $\varphi_{\omega_i} \circ \varphi_{\omega_{i+1}} \circ \dots \circ \varphi_{\omega_n}(y) \in \partial X$ and $\varphi_{\omega_{i-1}} \circ \varphi_{\omega_i} \circ \dots \circ \varphi_{\omega_n}(y) \in \operatorname{Int} X$. Then we see that $\omega_1 \dots \omega_{i-2}$ is comparable with every pseudo–code of C_x. Fix $\omega_{i-1} \in E$. Then $\omega_i \dots \omega_n$ is a pseudo–code of $z = \varphi_{\omega_{i-1}}^{-1} \circ \dots \circ \varphi_{\omega_1}^{-1}(x) \in \partial X$. From the condition $\varphi_a(X_{t(a)}) \cap \varphi_b(X_{t(b)}) \cap \partial X = \emptyset$ for all $a, b \in E$, $a \neq b$ we obtain that all two pseudo–codes of z are comparable. Since incomparable pseudo–codes differ only on ω_{i-1}, the number of elements of C_x is less than or equal to $\#E$. $\square$

As an immediate consequence of Theorem 9.1, we get the following.

Corollary 11.1. *Let $\mathcal{S} = \{\varphi_e : X_{t(e)} \to X_{i(e)}, e \in E\}$ be a primitive conformal graph directed Markov system satisfying the OSC with the sets X_v being closed balls and with a finite alphabet E. Then the system $\mathcal{S}$ is geometrically perfect.*

Remark 4. *Notice that the assumptions of the last corollary imply $\mathcal{S}$ to satisfy the SOSC.*

Corollary 11.2. *Let $\mathcal{S} = \{\varphi_e : X_{t(e)} \to X_{i(e)}, e \in E\}$ be a primitive conformal graph directed Markov system satisfying the separation condition ($\varphi_a(X_{t(a)}) \cap \varphi_b(X_{t(b)}) = \emptyset$ whenever $a \neq b$, $a, b \in E$). Then $\mathcal{S}$ is geometrically perfect.*

Proof. Replacing X_v, $v \in V$ with $B(X_v, \varepsilon)$ with $\varepsilon > 0$ so small that $B(X_v, \varepsilon) \subseteq B(X_v, 2\varepsilon) \subseteq W_v$ and $\varphi_a(B(X_{t(a)}, 2\varepsilon)) \cap \varphi_b(B(X_{t(b)}, 2\varepsilon)) = \emptyset$ whenever $a, b \in E$ and $a \neq b$, we will also have $\varphi_e(X_{t(e)}) \cap \partial X_{i(e)} = \emptyset$. Consequently, all the hypothesis of Theorem 9.1 will be satisfied, and we are done by an immediate application of this theorem. $\square$

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Department of Mathematics, University of North Texas, P.O. BOX 311430, Denton TX 76203–1430, USA; (RDM)

E-mail address: `mauldin@unt.edu`

Institute of Mathematics, University of Silesia, Bankowa 14, 40-007 Katowice, Poland; (TS)

E-mail address: `szarek@intertele.pl`

Department of Mathematics, University of North Texas, P.O. BOX 311430, Denton TX 76203–1430, USA; (MU)

E-mail address: `urbanski@unt.edu`