HAUSDORFF DIMENSION OF RADIAL AND ESCAPING POINTS FOR TRANSCENDENTAL MEROMORPHIC FUNCTIONS

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ABSTRACT. We consider a class of transcendental meromorphic functions $f : \mathbb{C} \mapsto \overline{\mathbb{C}}$ with infinitely many poles. Under some regularity assumption on the location of poles and the behavior of the function near the poles, we provide explicite lower bounds for the hyperbolic dimension (Hausdorff dimension of radial points) of the Julia set and upper bounds for the Hausdorff dimension of the set of escaping points in the Julia set. In particular the Hausdorff dimension of the latter set is less than the Hausdorff dimension of the former set. Consequently, the Hausdorff dimension of the set of escaping points is less than 2 and the area of this set is equal to zero. The functions under consideration may have infinitely many singular values, and we do not even assume them to belong to the class $\mathcal{B}$. We only require the distance between the set of poles and the set of finite singular values to be positive.

1. INTRODUCTION AND GENERAL PRELIMINARIES

The *Fatou set* $F(f)$ of a meromorphic function $f : \mathbb{C} \mapsto \overline{\mathbb{C}}$ is defined in exactly the same manner as for rational functions; $F(f)$ is the set of points $z \in \mathbb{C}$ such that all the iterates are defined and form a normal family on a neighborhood of z. The *Julia set* $J(f)$ is the complement of $F(f)$ in $\overline{\mathbb{C}}$. Thus, $F(f)$ is open, $J(f)$ is closed, $F(f)$ is completely invariant while $f^{-1}(J(f)) \subset J(f)$ and $f(J(f) \setminus \{\infty\}) = J(f)$. For a general description of the dynamics of meromorphic functions see e.g. [3]. It follows from Montel's criterion of normality that if $f : \mathbb{C} \mapsto \overline{\mathbb{C}}$ has at least one pole which is not an omitted value then

$$J(f) = \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}. \tag{1.1}$$

(cf. [2]). By $\mathrm{Sing}(f^{-1})$ we denote the set of *singular values* of f i.e. $c \in \mathrm{Sing}(f^{-1})$ if $c \in \mathbb{C}$ and c is a critical or an asymptotic value of f. We want to point out that we do not consider multiple poles as critical points. We also recall that $f \in \mathcal{B}$ if $\mathrm{Sing}(f^{-1})$ is bounded. Let

$$I_\infty(f) := \{z \in J(f) : \lim_{n \to \infty} f^n(z) = \infty\}$$

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be the subset of the Julia set consisting the points escaping to infinity under iterates of f. We also define the radial Julia set $J_r(f)$ as the set of points z in $J(f)$ for which there exists a family of neighborhoods $D(z, r_j)$, $r_j \to 0$, which can be mapped by f with bounded distortion until the diameter of the image reaches of fixed size. The Hausdorff dimension of $J_r(f)$ is called the hyperbolic dimension of the Julia set $J(f)$, which we denote by $\mathrm{HypDim}(J(f))$. Let H^h and l_2 denote the h-dimensional Hausdorff measure and the 2-dimensional Lebesgue measure, respectively, $\mathrm{HD}(X)$ denote the Hausdorff dimension of the set X.

It was shown by Baker [1] that, if f is a transcendental entire function, then $J(f)$ must contain continua and so the Hausdorff dimension of $J(f)$ satisfies $1 \le \mathrm{HD}(f) \le 2$. The result of Baker was extended recently by Stallard and Rippon to the class $\mathcal{M}_F$ of meromorphic functions with finitely many poles. In [11] they showed that if $f \in \mathcal{M}_F$ then $J(f)$ contains continua, so $1 \le \mathrm{HD}(f) \le 2$. Note that, for transcendental meromorphic functions with infinitely many poles, the Hausdorff dimension of the Julia set is positive but can be arbitrarily small - see [17]. If f is in the class $\mathcal{B}$, then one can get a better estimate on the lower bound of the Hausdorff dimension of the Julia set. First, in [16] Stallard proved that for entire $f \in \mathcal{B}$ one has $\mathrm{HD}(f) > 1$, next Stallard and Rippon proved the same for $f \in \mathcal{M}_F \cap \mathcal{B}$ (see [12]).

Restricting the class of functions considered, further progress has been done in [5] and [7]. Then in [8]. In [5] and [7] explicite estimates for lower bounds of $\mathrm{HypDim}(J(f))$, the hyperbolic dimension of the Julia set and upper bounds for the Hausdorff dimension of $I_\infty(f)$, the set of escaping points in the Julia set, have been obtained for the class of elliptic functions. Mayer in [8] has also obtained the explicite lower bound for $\mathrm{HypDim}(J(f))$. In the present paper, developing the methods from [7] and getting rid of periodicity assumptions, we provide explicite bounds for a much wider class of meromorphic functions. It follows as an immediate corollary that for this class of meromorphic functions $\mathrm{HD}(I_\infty(f)) < 2$, which in turn readily implies that ∞ is not a metric attractor, meaning that the area of $I_\infty(f)$ vanishes. Let

$$A := f^{-1}(\infty)$$

be the set of poles. For every pole a of f, by $b(a)$ we denote the residuum of f at a. The following theorems are the main results of our paper.

Theorem A. *Let $f : \mathbb{C} \mapsto \overline{\mathbb{C}}$ be a transcendental meromorphic function of finite order $\rho > 0$ satisfying*

(a) *∞ is not an asymptotic value of f, A is infinite and*
  (i) *there exist $\alpha \ge 0$ such that for $a \in A$ one has $|b(a)| \asymp |a|^{-\alpha}$*
  (ii) *there exist $M \in \mathbb{N}$ and $\kappa \ge 0$ such that*

$$|f'(z)| \asymp \frac{m(a)b(a)}{|z-a|^{m(a)+1}} \quad \text{and} \quad |f(z)| \asymp \frac{b(a)}{|z-a|^{m(a)}}$$

  for $z \in D(a, r(a))$, where $m(a) \in \mathbb{N}$, $m(a) \le M$ and $r(a) \asymp |a|^{-\kappa}$.

(b) $\mathrm{dist}(\overline{\mathrm{Sing}(f^{-1})}, a) > 2r(a)$ *for $a \in A$.*

Then

$$\mathrm{HD}(I_\infty(f)) \leq \frac{\rho M}{\alpha + M + 1}.$$

The comparability sign e.g. $|b(a)| \asymp |a|^{-\alpha}$ means that

$$C^{-1} \leq |b(a)|/|a|^{-\alpha} \leq C$$

for some constant $C > 0$ and all $a \in A$. Roughly speaking the condition (ii) enables us to replace f' by its principal parts in $r(a)$-neighborhood of the pole a uniformly with respect to a. The condition on f given in (ii) implies

$$f(z) \asymp c(a) + b(a)(z-a)^{-m(a)} + \dots$$

in $D(a, r(a))$ with $c(a)$ bounded uniformly in a. It says that when we reconstruct f from f' in $D(a, r(a))$ the 'constants of integration' are not too large. If $w \in \overline{\mathbb{C}}$ is not an omitted value, then by w-points we call $f^{-1}(w) = \{z_n(w);\, n \in \mathbb{N}\}$. The exponent of convergence $\rho_c(f, w)$ of the series

$$\sum(u, w) = \sum_n |z_n(w)|^{-u}$$

is defined by $\rho_c(f, w) = \inf\{u > 0 : \sum(u, w) < \infty\}$. Theorem of Borel says that, if f is a meromorphic function of finite order ρ then $\rho_c(f, w) = \rho$ for all values $w \in \overline{\mathbb{C}} \setminus \mathcal{E}_f$, where $\mathcal{E}_f$ is the set of Picard exceptional values. It follows from Borel-Picard Theorem that f is of finite order if and only if

$$\sum(u, w) < \infty \quad \text{if} \quad u > \rho \quad \text{and}$$
$$\sum(u, w) = \infty \quad \text{if} \quad u < \rho$$

for $w \in \overline{\mathbb{C}} \setminus \mathcal{E}_f$. The meromorphic function of finite order ρ is of divergent type if

$$\sum(u, w) = \sum_n |z_n(w)|^{-\rho} = \infty$$

for $w \in \overline{\mathbb{C}} \setminus \mathcal{E}_f$. Notice also that in our case f does not even have to belong to the class $\mathcal{B}$. We only need to know that $\mathrm{dist}(\overline{\mathrm{Sing}(f^{-1})}, a) > 2r(a)$ for $a \in A$.

Remark 1.1. *In Theorem A it suffices if the assumption*

$$\mathrm{dist}(\overline{\mathrm{Sing}(f^{-1})}, a) > 2r(a)$$

holds for all but finitely many poles $a \in A$.

Now, we give the lower bound on the hyperbolic dimension of Julia set for the functions under consideration.

Theorem B. *Let $f : \mathbb{C} \mapsto \overline{\mathbb{C}}$ be a transcendental meromorphic function satisfying the assumptions of Theorem A except for that concerning ∞. Then*

$$\mathrm{HypDim}(J(f)) \geq \frac{\rho M}{\alpha + M + 1}.$$

If, in addition, the function is of divergent type, then this inequality becomes strict.

The proof of Theorem B does not depend on the assumption that ∞ is or not an asymptotic value of f. Theorems A and B imply the following corollary.

Corollary 1.2. *Let $f : \mathbb{C} \mapsto \overline{\mathbb{C}}$ be a transcendental meromorphic function satisfying the assumptions of Theorem A and $h := \mathrm{HD}(J(f))$. Then $\mathrm{H}^h(I_\infty(f)) = 0$, and consequently $l_2(I_\infty(f)) = 0$.*

The transcendental meromorphic functions considered in Theorem A are not entire nor have finitely many poles. In those cases ∞ is an asymptotic value, so there is an asymptotic tract associated with ∞. Therefore, if z escapes to infinity, its forward trajectory stays in that tract. In our case the escaping points must come arbitrarily close to poles. This difference is reflected in the estimates of the Hausdorff dimension of escaping points. For entire functions of finite order e.g. the exponential or cosine family, C. McMullen proved $\mathrm{HD}(I_\infty(f)) = \mathrm{HD}(J(f)) = 2$, while in our case $\mathrm{HD}(I_\infty(f)) < \mathrm{HD}(J(f)) \leq 2$.

In Section 2 we prove Theorem A and in Section 3 Theorem B. In Section 4 we provide some examples of non-periodic functions for which the assumptions of Theorems A and B are satisfied.

In the sequel $f^\sharp$ and diam_s denote the derivatives and diameters defined by means of the spherical metric. By $B(x,r)$ and $B_s(x,r)$, respectively, we mean the open ball centered at x and with the Euclidean (resp. spherical) radius r.

2. Proof of Theorem A

Let $B_R = \{z \in \overline{\mathbb{C}} : |z| > R\}$. Take R_0 such that

$$R_0 > 2\max\{r(a) : \ a \in A\}. \tag{2.1}$$

The hypothesis (ii) means that the sets $D(a, r(a))$ are mutually disjoint. Let $a \in A$ and $z \in D(a, r(a))$, then

$$|f(z)| \asymp \frac{|b(a)|}{|z-a|^{m(a)}} \qquad \text{and} \quad |f'(z)| \asymp \frac{m(a)|b(a)|}{|z-a|^{m(a)+1}}, \tag{2.2}$$

where $m(a) \leq M$, $b(a) \asymp |a|^{-\alpha}$ and $r(a) \asymp |a|^{-\kappa}$ for all $a \in A$. A straightforward calculation based on (2.2) shows that $f(D(a, r(a)) \supset B_R$ for all except finitely many poles. Indeed,

$|b(a)||r(a)|^{-m(a)} \asymp |a|^{\kappa m(a)-\alpha} \succeq R_0$. Thus there exists $R_1 > R_0$ such that $f(D(a,r(a))) \supset B_{R_0}$ for all $a \in A \cap B_{R_1}$. For every $a \in A$ by $B_a(R)$ we denote the connected component of $f^{-1}(B_R)$ containing a. Thus if $R \geq R_1$, then for all a with $|a| > R_1$, we have

$$B_a(R) \subset D(a, r(a)). \tag{2.3}$$

Also (2.2) implies that there is a constant $L \geq 1$ such that for all poles a and all $R \geq R_0$, we have

$$\begin{aligned} \operatorname{diam}(B_a(R)) &\leq LR^{-\frac{1}{m(a)}}|a|^{-\alpha/m(a)}, \\ \operatorname{diam}_s(B_a(R)) &\leq LR^{-\frac{1}{m(a)}}|a|^{-2-\alpha/m(a)}. \end{aligned} \tag{2.4}$$

If

$$U \subset B_R \setminus \{\infty\} \cap \bigcup_{a \in A} D(a, 2r(a))$$

is an open simply-connected set, then all holomorphic inverse branches $f^{-1}_{a,U,1}, \dots, f^{-1}_{a,U,m(a)}$ of f are well-defined on U, and for every $1 \leq j \leq m(a)$ and all $z \in U$ we have

$$|(f^{-1}_{a,U,j})'(z)| \asymp |z|^{-\frac{m(a)+1}{m(a)}}|a|^{-\frac{\alpha}{m(a)}}. \tag{2.5}$$

Therefore

$$|(f^{-1}_{a,U,j})^\sharp(z)| \asymp |z|^{-\frac{m(a)+1}{m(a)}} \frac{1+|z|^2}{1+|(f^{-1}_{a,U,j})(z)|^2} \asymp \frac{|z|^{\frac{m(a)-1}{m(a)}}}{|a|^{2+\alpha/m(a)}}, \tag{2.6}$$

where the second comparability sign we wrote assuming in addition that $|a|$ is large enough, say $|a| \geq R_2 > R_1$. Let K be an upper bound of the ratios of $|(f^{-1}_{a,U,j})^\sharp(z)|$ and $|z|^{\frac{m(a)-1}{m(a)}}/|a|^{2+\alpha/m(a)}$ with a, U, j as above. Given two poles $a_1, a_2 \in B_{2R_2}$, we denote by $f^{-1}_{a_1,a_2,j} : B(a_2, 2r(a_2)) \mapsto \mathbb{C}$, $j = 1, \dots, m(a_1)$, all holomorphic inverse branches of f. It follows from (2.1) and (2.3) that

$$f^{-1}_{a_2,a_1,j}\Big(B(a_1, r(a_1))\Big) \subset B_{a_2}(2R_2 - r(a_1)) \subset B_{a_2}(R_2) \subset B(a_2, r(a_2)) \tag{2.7}$$

for $j = 1, \dots, m(a_1)$. Set

$$I_R(f) = \{z \in \mathbb{C} : \quad \forall_{n \geq 0} \quad |f^n(z)| > R\}.$$

Since the series

$$\sum_{a \in A} |a|^{-u}$$

converges for all $u > \rho$, given $t > \frac{\rho M}{\alpha+M+1}$, there exists $R_3 > R_2$ such that

$$MK^t \sum_{a \in A \cap B_{R_3}} |a|^{-t(\frac{\alpha+M+1}{M})} \leq 1, \tag{2.8}$$

where a constant $K>0$ comes from the comparability signs in (2.6). Consider $R_4>4R_3$. Define $I=A\cap B_{R_3}$. It follows from (2.3) and (2.7) that for every $l\ge 1$, and $R>2R_4$ the family of sets

$$W_l := \Big\{ f^{-1}_{a_l,a_{l-1},j_l}\circ f^{-1}_{a_{l-1},a_{l-2},j_{l-1}}\ldots\circ f^{-1}_{a_2,a_1,j_2}\circ f^{-1}_{a_1,a_0,j_1}\Big(B_{a_0}(R/2)\Big) : a_i\in I,\ 1\le j_i\le m(a_i),\ i=0,1,\ldots,l\Big\}$$

is well-defined and covers $I_R(f)$. Applying (2.6) and (2.4), we may now estimate as follows.

$$\begin{aligned}
\Sigma_l &= \\
&= \sum_{a_l\in I}\sum_{j_l=1}^{m(a_l)}\cdots\sum_{a_1\in I}\sum_{j_1=1}^{m(a_1)}\sum_{a_0\in I}\mathrm{diam}_s^t\Big(f^{-1}_{a_l,a_{l-1},j_l}\circ f^{-1}_{a_{l-1},a_{l-2},j_{l-1}}\ldots\circ f^{-1}_{a_2,a_1,j_2}\circ f^{-1}_{a_1,a_0,j_1}\Big(B_{a_0}(R/2)\Big)\Big)\\
&\le \sum_{a_l\in I}\sum_{j_l=1}^{m(a_l)}\cdots\sum_{a_1\in I}\sum_{j_1=1}^{m(a_1)}\sum_{a_0\in I}\Big|\Big|\Big(f^{-1}_{a_l,a_{l-1},j_l}\circ f^{-1}_{a_{l-1},a_{l-2},j_{l-1}}\ldots\circ f^{-1}_{a_2,a_1,j_2}\circ f^{-1}_{a_1,a_0,j_1}\Big)^{\sharp}_{|B_{a_0}(R/2)}\Big|\Big|_\infty^t\\
&\qquad\qquad \cdot\,\mathrm{diam}_s^t\Big(B_{a_0}(R/2)\Big)\\
&\le \sum_{a_l\in I}\sum_{j_l=1}^{m(a_l)}\cdots\sum_{a_1\in I}\sum_{j_1=1}^{m(a_1)}\sum_{a_0\in I}K^{lt}\left(\frac{|a_{l-1}|^{(m(a_l)-1)/m(a_l)}}{|a_l|^{2+\alpha/m(a_l)}}\right)^t\cdot\left(\frac{|a_{l-2}|^{(m(a_{l-1})-1)/m(a_{l-1})}}{|a_{l-1}|^{2+\alpha/m(a_{l-1})}}\right)^t\cdots\\
&\qquad\cdots\left(\frac{|a_0|^{(m(a_1)-1)/m(a_1)}}{|a_1|^{2+\alpha/m(a_1)}}\right)^t L^t\left(\frac{R}{2}\right)^{-\frac{t}{m(a_0)}}\frac{1}{|a_0|^{(2+\alpha/m(a_0))t}}\\
&\le L^t\left(\frac{2}{R}\right)^{\frac{t}{M}}K^{lt}\sum_{a_l\in I}\sum_{j_l=1}^{m(a_l)}\cdots\sum_{a_1\in I}\sum_{j_1=1}^{m(a_1)}\sum_{a_0\in I}|a_l|^{-t(2+\alpha/M)}\Big(|a_{l-1}|^{-t\frac{\alpha+M+1}{M}}\ldots|a_0|^{-t\frac{\alpha+M+1}{M}}\Big)\\
&= L^t\left(\frac{2}{R}\right)^{\frac{t}{M}}K^{lt}\sum_{a_l\in I}\sum_{j_l=1}^{m(a_l)}\cdots\sum_{a_1\in I}\sum_{j_1=1}^{m(a_1)}\sum_{a_0\in I}\Big(|a_l|^{-t\frac{\alpha+M+1}{M}}|a_{l-1}|^{-t\frac{\alpha+M+1}{M}}\ldots|a_0|^{-t\frac{\alpha+M+1}{M}}\Big)\\
&\le L^t\left(\frac{2}{R}\right)^{\frac{t}{M}}K^{lt}\left(\sum_{a\in A\cap B_{R_3}}|a|^{-t\frac{\alpha+M+1}{M}}\right)^l M^l\\
&\le L^t\left(\frac{2}{R}\right)^{\frac{t}{M}}\left(MK^t\sum_{a\in A\cap B_{R_3}}|a|^{-t\frac{\alpha+M+1}{M}}\right)^l.
\end{aligned}$$

Applying (2.8), we therefore get $\Sigma_l\le L^t(2/R)^{t/M}$. Since the diameters (in the spherical metric) of the sets of the covers W_l converge uniformly to 0 when $l\searrow\infty$, we infer that $\mathrm{H}_s^t(I_R(f))\le L^t(2/R)^{t/M}$, where the subscript s indicates that the Hausdorff measure is defined with respect to the spherical metric. Consequently $\mathrm{HD}(I_R(f))\le t$, and if we put

$$I_{R,e}(f) := \Big\{z\in\mathbb{C} : \liminf_{n\mapsto\infty}|f^n(z)|>R\Big\} = \bigcup_{k\ge1}f^{-k}(I_R(f)),$$

then also $\mathrm{HD}(I_\infty(f)) \le \mathrm{HD}(I_{R,e}(f)) = \mathrm{HD}(I_R(f)) \le t$. Letting now $t \searrow \frac{\rho M}{\alpha+M+1}$ finishes the proof. ■

3. Proof of Theorem B

Let R_2 be a constant defined above. Fix a pole $a_0 \in A \cap B_{2R_2}$ with $m(a_0) = M$. For every pole $a \in A$ satisfying $|a| > 2R_2$ and $m(a) = M$, we fix inverse branches of f:

$$f^{-1}_{a,a_0,1} : \overline{B(a, r(a))} \mapsto \mathbb{C} \quad \text{and} \quad f^{-1}_{a_0,a,1} : \overline{B(a, r(a))} \mapsto \mathbb{C}.$$

In view of (2.7), we have

$$f^{-1}_{a,a_0,1}\Big(\overline{B(a, r(a))}\Big) \subset \overline{B(a_0, r(a_0))} \quad \text{and} \quad f^{-1}_{a_0,a,1}\Big(\overline{B(a_0, r(a_0))}\Big) \subset \overline{B(a, r(a))}.$$

The family

$$S = \{f^{-k}_{a_0,a,1} \circ f^{-1}_{a,a_0,1} : \overline{B(a_0, r(a_0))} \mapsto \overline{B(a_0, r(a_0))}; \quad a_n \in A \cap B_{2R_2}\}$$

forms a conformal infinite iterated function system in the sense of [10]. We set

$$\phi_n = f^{-1}_{a_0,a,1} \circ f^{-1}_{a,a_0,1}$$

and, given $\omega \in (A \cap B_{2R_2})^n$, $n \ge 1$, we say that $|\omega| = n$, and we put

$$\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \ldots \circ \phi_{\omega_n}.$$

The set

$$J_S = \bigcap_{n\ge 0} \sum_{|\omega|=n} \phi_\omega(\overline{B(a_0, r_{n_0})})$$

is called the limit set of the iterated function system S. It was proved in [10] that J_S is contained in the closure of all fixed points of ϕ_ω, where $\omega \in \bigcup_{n\ge1}(A \cap B_{2R_2})^n$. Since these periodic points are repulsive, we conclude that $J_S \subset J(f)$. Given $t \ge 0$, we consider the Poincaré series associated to the system S,

$$\psi(t) = \sum_{a \in A \cap B_{2R_2}} ||(\phi_n)^\sharp||^t_\infty,$$

where $||(\phi_n)^\sharp||_\infty = \sup\{|(\phi_n)^\sharp(z)| : z \in \overline{B(a_0, r(a_0))}\}$ and the number

$$\theta_S = \inf\{t \ge 0 : \psi(t) < \infty\}.$$

We shall prove that $\theta_S < \frac{\rho M}{\alpha M+1}$ and $\psi(\theta_S) = \infty$. In view of (2.6), we can write

$$\begin{aligned}\psi(t) &\asymp \sum_{a \in A \cap B_{2R_2}} \left(\frac{|a|^{\frac{M-1}{M}}}{|a_0|^{2+\alpha/M}}\right)^t \left(\frac{|a_0|^{\frac{M-1}{M}}}{|a|^{2+\alpha/M}}\right)^t \\ &\asymp \sum_{a \in A \cap B_{2R_2}} |a|^{-t\frac{\alpha+M+1}{M}}.\end{aligned}$$

It follows from Borel theorem that the series $\sum_{a\in A\cap B_{2R_2}}|a|^{-t\frac{\alpha+M+1}{M}}$ converges if $t\frac{\alpha+M+1}{M}>\rho$. Therefore the equalities $\theta_S<\frac{\rho M}{\alpha+M+1}$ and $\psi(\theta_S)=\infty$ are proved. It follows from Theorem 3.20 in [10] that $\mathrm{HD}(J_S)\geq\frac{\rho M}{\alpha+M+1}$. Since $J_S\subset J(f)$, we are done. If, in addition, f is of divergent type, then for $t\frac{\alpha+M+1}{M}=\rho$ the series $\sum_{a\in A\cap B_{2R_2}}|a|^{-t\frac{\alpha+M+1}{M}}$ diverges. It implies that $\theta_S=\frac{\rho M}{\alpha+M+1}$ and $\psi(\theta_S)=\infty$. Therefore, invoking again Theorem 3.20 in [10], we obtain that $\mathrm{HD}(J_S)>\frac{\rho M}{\alpha+M+1}$. ■

4. Examples

Example 1. We consider the first Painlevé equation (P_1)

$$f''=z+6f^2.$$

The solutions of (P_1) are meromorphic functions of order $\rho(f)=\frac{5}{2}$ of divergent type (cf. [4], [14], [15]). They have infinitely many poles with Laurent expansions series

$$f(z)=(z-p)^{-2}-\frac{p}{10}(z-p)^2-\frac{1}{6}(z-p)^3+h(z-p)^4+\frac{p^2}{300}(z-p)^4+\frac{p}{150}(z-p)^7+\sum_{k=8}^{\infty}a_k(z-p)^k$$

at every pole p, convergent at least for $0<|z-p|<c|p|^{-1/4}$, c is a constant independent of p. Thus the disks $D(p,\frac{c}{2}|p|^{-1/4})$ must be mutually disjoint for sufficiently large $|p|$. All the poles are double and have the same residuum equal to 1, so $M=2$ and $\alpha=0$. The estimate on $r(p)\asymp|p|^{-1/4}$ imply that $\kappa=\frac{1}{4}$. Since f is of divergence type, if for some first transcendent f the hypothesis (a) of Theorem A, modified as in Remark 1.1, is satisfied then

$$\mathrm{HypDim}(J(f))>\frac{5}{3}.$$

Example 2. Now we consider these solutions of the second Painlevé equation (P_2)

$$f''=\beta+zf+2f^3$$

which are meromorphic functions of order $\rho(f)=3$. We note that for most classes of second transcendents $3/2\leq\rho(f)\leq 3$ (cf. [4], [14], [15]). They have infinitely many poles with residue +1 and -1, except where $\beta=\pm 1/2$ and f solve the Riccati Equation. The Laurent development at the pole p

$$f(z)=\epsilon(z-p)^{-1}-\frac{\epsilon p}{6}(z-p)-\frac{\alpha+\epsilon}{4}(z-p)^2+h(z-p)^3+\frac{p^2}{300}(z-p)^4+\frac{p}{150}(z-p)^7+\sum_{k=8}^{\infty}a_k(z-p)^k$$

is convergent at least for $0<|z-p|<c|p|^{-1/2}$, c is a constant independent of p. Thus $M=1$, $\alpha=0$ and $\kappa=\frac{1}{2}$. If for some second transcendent f the hypothesis (a) of Theorem A, modified as in Remark 1.1, is satisfied then

$$\mathrm{HypDim}(J(f))\geq\frac{3}{2}.$$

It follows from the results of N. Steinmetz that for example the first transcendent has only one asymptotic value equals to ∞.

EXAMPLE 3. Let

$$f(z) = \frac{1}{z \sin z}.$$

So f is a meromorphic functions with infinitely many poles

$$A(f) = \{z_0 = 0\} \cup \{z_n = n\pi : n \in \mathbb{Z}^*\},$$

where all of them except for $z_0 = 0$ are simple. Notice that ∞ is not an asymptotic value of f. Thus we have $m = 1$, $\alpha = 1$, $\rho = 1$. $\mathrm{Sing}(f^{-1})$ consists of one asymptotic value 0 and infinitely many critical values $c_n \asymp \pm\Big(\big(n + \frac{1}{2}\big)\pi\Big)^{-1}$, $n \in \mathbb{Z}$. So $f \in \mathcal{B}$ and satisfies the hypothesis (a) of Theorem A, modified as in Remak 1.1. Consequently

$$\mathrm{HD}(I_\infty(f)) \le \frac{1}{3} < HypDim(J(f)).$$

EXAMPLE 4. Let

$$f(z) = \frac{1}{z \cos \sqrt{z}}.$$

So f is a meromorphic functions with infinitely many poles

$$A(f) = \{z_0 = 0\} \cup \left\{z_n = \Big(n + \frac{1}{2}\Big)^2 \pi^2 ; n \in \mathbb{N}\right\},$$

where all of them are simple. Notice that ∞ is not an asymptotic value of f. Thus we have $m = 1$, $\alpha = \frac{1}{2}$, $\rho = \frac{1}{2}$. $\mathrm{Sing}(f^{-1})$ consists of one asymptotic value 0 and infinitely many critical values $|c_n| = \left|\frac{1}{z_l \cos(z_l)}\right| \asymp \left|\frac{1}{z_l}\right| \to 0$, where $z_l = l\pi + \frac{1}{l\pi}$, $l \in \mathbb{Z}$. So $f \in \mathcal{B}$ and satisfies the hypothesis (a) of Theorem A, modified as in Remak 1.1. Consequently

$$\mathrm{HD}(I_\infty(f)) \le \frac{1}{5} < HypDim(J(f))$$

EXAMPLE 5. The Airy function

$$Ai(z) = \frac{1}{2\pi} \int_{\mathrm{Im}\zeta = \eta > 0} \exp\Big(\frac{1}{3} i\zeta^3 + i\zeta z\Big) d\zeta$$

is the solution of the equation $f'' - zf = 0$. The zeros of Ai are asymptotically $a_n = \Big(\frac{3}{2}\pi(n - \frac{1}{4})\Big)^{2/3} + O\Big(n^{-4/3}\Big)$. For $f_\lambda(z) := \frac{\lambda}{Ai(z)}$, $\lambda \in \mathbb{C}^*$, then $|b(a_n)| = \frac{1}{|(f_\lambda(a_n))'|}$ we have $m = 1$, $\rho = \frac{3}{2}$, $\alpha = \frac{1}{4}$. If for some λ the hypothesis (a) of Theorem A is satisfied then

$$\mathrm{HD}(I_\infty(f_\lambda)) \le \frac{2}{3} < \mathrm{HypDim}(J(f_\lambda)).$$

EXAMPLE 6. Let

$$f(z) = R(e^z),$$

where R is a rational function such that $R(0) \neq \infty$ and $R(\infty) \neq \infty$. So $f(z)$ is a simply-periodic meromorphic function with finitely many poles at each strip of periodicity. This class of functions contains for example, the tangent family $\lambda \tan(z), \lambda \in \mathbb{C}^*$. Let M denote the maximal multiplicity of the poles of R. Since $\mathrm{Sing}(f^{-1})$ is finite, the hypothesis (a) of Theorem A, modified as in Remark1.1, is always satisfied. It is easy to see that $\rho = 1$ and $\alpha = 0$, so

$$\mathrm{HD}(I_\infty(f)) \leq \frac{M}{M+1} < \mathrm{HypDim}(J(f)).$$

In this case one can get a better estimate on $\mathrm{HypDim}(J(f))$. It follows from [13] that $\mathrm{HypDim}(J(f)) > 1$.

Example 7. As we mentioned before, Theorems A and B can be applied to elliptic fictions (see [7]).

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