Some remarks on topological entropy of a semigroup of continuous maps

Andrzej Biś
Uniwersytet Łódzki,
Łódź, Poland

Mariusz Urbański
University of North Texas,
Denton, USA

December 1, 2003

Abstract

We investigate the notion of topological entropy of a semigroup of continuous maps and provide several of its basic properties.

1 Introduction

The concept of entropy of a transformation plays a central role in topological dynamics. The notion of topological entropy was introduced by Adler, Konheim and McAndrew in [1] as an invariant of topological conjugacy. Later, Bowen [4] and Dinaburg [5] presented an equivalent approach to the notion of entropy in the case when the domain of the considered transformation is a metrizable space. The topological entropy $h(f)$, of an endomorphism f, measures the complexity of the transformation acting on a compact topological space in the sense that it shows the rate at which the action of the transformation disperses points.

Since the entropy appeared to be a very useful invariant in ergodic theory and dynamical systems, there were several attemps to find its suitable generalizations for other systems such like groups, pseudogroups, graphs, foliations. Among the others, Ghys, Langevin and Walczak in [7] proposed a definition of a topological entropy for finitely generated groups and pseudogroups of continous transformations. Biś and Walczak in [3] applied the notion of entropy of a group to hyperbolic groups in the sense of Gromov to study its geometry and dynamics. Friedland in [6] used the notion of entropy to study some aspects of dynamics of graphs and semigroups.

Also, there have been attemps to introduce several entropy-like invariants for noninvertable maps. Langevin and Walczak in [10], Hurley in [8], Langevin and Przytycki [9], Nitecki and Przytycki ([13]) studied different entropy-like invariants. Nitecki in [12] investigated topological entropy and preimage structure of maps. Mihailescu and Urbański in [11] focused on inverse topological pressure and the Hausdorff dimension of the intersection between the local stable manifold and the basic set. Hurley ([8]) established relations between topological entropy, preimage relation entropy, preimage branch entropy and point entropy of a single transformation. Biś in ([2]) generalised Hurley's results entropies of a single transformations to the case of finitely generated semigroup of transformations acting on a compact space.

In this paper we examine in detail the concept of topological entropy of semigroups introduced in [2]. Our article is organized as follows. In Section 1, we recall the notion of topological entropy for a finitely generated semigroup and, for the convenience of the reader, provide some results. In the Sections 2 and 3 we formulate analogues of properties of topological entropy in the context of an action of any finitely generated semigroup of continous maps on a compact metric space. In the last two sections we state some sufficient conditions for a finitely generated semigroup to have zero, positive or finite entropy.

2 Topological entropy of a semigroup

Many useful properties of the concept of entropy of a single transformation can be found in [14]. Let X be a compact metric space with a distance function d. Consider a semigroup G of continuous transformations of X into itself. The semigroup G is assumed to be finitely generated, e.g. there exists a finite set $G_1 = \{f_1, ..., f_k\}$ such that

$$G = \bigcup_{n \in N} G_n,$$

where

$$G_n = \{g_1 \circ ... \circ g_n : X \to X\}_{g_1,...,g_n \in G_1}.$$

We always assume that id_X, the identity map on X is in G_1. This implies that $G_m \subset G_n$ for all $m \leq n$. Following [7] we say that two points $p, q \in X$ are $(n, \varepsilon)-$separated by G (with respect to the metric d^n_{max}) if there exists $g \in G_n$ such that $d(g(p), g(q)) \geq \varepsilon$, e.g.

$$d^n_{max}(p, q) = \max\{d(g(p), g(q)) : g \in G_n\} \geq \varepsilon.$$

We say that a subset A of X is $(n, \varepsilon)-$separated if any two distinct points of A have this property. All $(n, \varepsilon)-$separated subsets of X are always finite, since X is compact. Therefore, we can write

$$s(n, \varepsilon, X) := \max\{card(A) : A \text{ is } (n, \varepsilon)-\text{separated subset of X}\}.$$

The following definition has appeared in [2].

Definition 1. Let

$$h(G, G_1, X) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log(s(n, \varepsilon, X)).$$

The quantity $h(G, G_1, X)$ is called the topological entropy of a semigroup G generated by G_1.

As it was shown in [2] the topological entropy of a semigroup G depends on the generating set G_1. It may however still serves as a natural generalisation of the notion of the topological entropy of a continous mapping $f : X \to X$. Indeed, Let $f : X \to X$ be a continous transformation of a compact metric space X and $G(f)$ a semigroup generated by $G_1(f) = \{id_X, f\}$. Then, we get that

$$h(f) = h(G(f), G_1(f)),$$

where $h(f)$ is the entropy of f. We can also describe the entropy of a semigroup G generated by G_1 in terms of $(n, \varepsilon)-$spanning sets. Namely, a subset A of X is called $(n, \varepsilon)-$spanning if for every $x \in X$ there exists $a \in A$ such that

$$d^n_{max}(x, a) = \max\{d(g(x), g(a)) : g \in G_n\} < \varepsilon.$$

The minimum of cardinalities of all $(n, \varepsilon)-$spanning sets is denoted by $r(n, \varepsilon, X)$. The following characterization of the topological entropy of a semigropup G generated by a finite set G_1 has been established in [2].

Lemma 1. *For any semigroup G generated by a finite set G_1 the following equality holds*

$$h(G, G_1, X) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log(r(n, \varepsilon, X)).$$

The notion of topological entropy of a semigroup of transformations shares many-common features with the concept of the topological entropy of a single transformation. We examine them in detail in the following sections.

3 First results

A classical result concerning the entropy of a single transformation $f : X \to X$ states that for any integer $n \geq 1$ we have $h(f^n) = n \cdot h(f)$. A corresponding result holds for the entropy of a semigroup of transformations.

Theorem 1. *If (G, G_1) and (G^*, G_1^*) are finitely generated semigroups generated respectively by $G_1 = \{id_X, g_1, ..., g_k\}$ and $G_1^* = \{id_X, g_1^m, ..., g_k^m\}$, $m \in N$, then*

$$h(G^*, G_1^*) = m \cdot h(G, G_1).$$

Proof. Denote by (X, d_X) the compact metric space the semigroup G acts on. Consider two points $x, y \in X$ that are $(n, \varepsilon)-$separated by G^*. This means that there exists $p \in G_n^*$ such that $d_X(p(x), p(y)) \geq \varepsilon$. Since g is the composition $g_1^m \circ ... \circ g_n^m$, with some $g_1, ..., g_n \in G_1$, we have that $d_X(g_{i_1}^m \circ, ..., \circ g_{i_n}^m(x), g_{i_1}^m \circ, ..., \circ g_{i_n}^m(y)) \geq \varepsilon$. Thus, the points x, y are $(m \cdot n, \varepsilon)-$separated with respect to (G, G_1). So, $s(n, \varepsilon, (G^*, G_1^*)) \leq s(m \cdot n, \varepsilon, (G, G_1))$ and taking the appropriate limit, we obtain that

$$h(G^*, G_1^*) \leq m \cdot h(G, G_1).$$

Starting to prove the opposite inequality, let $A \subset X$ be an $(m \cdot n, \varepsilon)-$spanning subset of X, with respect to (G, G_1), with minimal cardinality. Then, for any $x \in X$ there exists $a \in A$ such that for any $g \in G_{m \cdot n}$ we have $d_X(g(x), g(a)) < \varepsilon$. So, in particular, for any $x \in X$ there exists $a \in A$ such that for any n-tuple $g_1^m, ..., g_n^m$, where all elements g_i are in G_1,, we have

$$d_X(g_{i_1}^m \circ, ..., \circ g_{i_n}^m(x), g_{i_1}^m \circ, ..., \circ g_{i_n}^m(a)) < \varepsilon.$$

Therefore, A is $(n, \varepsilon)-$spanning subset of X with respect to (G^*, G_1^*) and

$$card(A) = r(m \cdot n, \varepsilon, (G, G_1)) \geq r(n, \varepsilon, (G^*, G_1^*)).$$

Passing to the appropriate limit, we obtain

$$m \cdot h(G, G_1) \geq h(G^*, G_1^*),$$

which completes the proof. ■

If semigroups (G, G_1) and (H, H_1) act respectively on compact metric spaces (X, d_1) and (Y, d_2), then $(G \times H, G_1 \times H_1)$ is a finitely generated semigroup, acting on the compact space $X \times Y$. We shall prove in this context the following.

Theorem 2. *If (G, G_1) and (H, H_1) are finitely generated semigroups, then*

$$h(G \times H, G_1 \times H_1) = h(G, G_1) + h(H, H_1)$$

Proof. In the topological space $X \times Y$ we consider the metric $d_{X \times Y} = d_X + d_Y$ inducing the product topology. Thus, the finitely generated semigroup $(G \times H, G_1 \times H_1)$ acts on the compact metric space $(X \times Y, d_{X \times Y})$. Fix now an $(n, \varepsilon/2)-$separated set (with respect to (G, G_1)) $A = \{a_1, ..., a_p\} \subset X$ with maximal cardinality, and an $(n, \varepsilon/2)-$separated set $B = \{b_1, ..., b_r\} \subset Y$ (with respect to (H, H_1)) with maximal cardinality. Then, for any two distinct elements $(a_{i_1}, b_{j_1}), (a_{i_2}, b_{j_2}) \in A \times B$ we get that $d_{X \times Y}((a_{i_1}, b_{j_1}), (a_{i_2}, b_{j_2})) \geq \varepsilon$. This means that the points $(a_{i_1}, b_{j_1}), (a_{i_2}, b_{j_2})$ are $(n, \varepsilon)-$separated with respect to $(G \times H, G_1 \times H_1)$. So, $card(A \times B) = card(A) \cdot card(B) = s(X, n, \varepsilon/) \cdot s(Y, n, \varepsilon/)$ and $s(X \times Y, n, \varepsilon) \geq card(A \times B) = s(X, n, \varepsilon) \cdot s(Y, n, \varepsilon)$. Passing to the appropriate limit, we get that

$$h(G \times H, G_1 \times H_1) \geq h(G, G_1) + h(H, H_1).$$

In order to prove the opposite inequality, we consider a set $C \subset X$ with minimal cardinality which is $(n, \varepsilon/2)-$spaning with respect to (G, G_1) and a set $D \subset Y$ with minimal cardinality which is $(n, \varepsilon/2)-$spaning with respect to (H, H_1). Then, $card(C \times D) = r(X, n, \varepsilon/2) \cdot r(Y, n, \varepsilon/2)$ and $C \times D$ is $(n, \varepsilon)-$spaning with respect to $(G \times H, G_1 \times H_1)$. Thus, $r(X \times Y, n, \varepsilon) \leq r(X, n, \varepsilon/2) \cdot r(Y, n, \varepsilon/2)$. Taking now the appropriate limit, we get that

$$h(G \times H, G_1 \times H_1) \leq h(G, G_1) + h(H, H_1).$$

The proof is complete. ■

Theorem 3. *If (G, G_1) is a finitely generated semigroup acting on a compact metric space (X, d), generated by $G_1 = \{id_X, g_1, ..., g_k\}$, and there exists a compact subset M of X such that for every $g_i \in G_1$*

$$g_i : M \to M$$

then,

$$h((G, G_1), X) \geq h((G^M, G_1^M), M),$$

where (G^M, G_1^M) is a semigroup acting on M, generated by

$$G_1^M = \{id_M, g_1|_M, ..., g_k|_M\}.$$

Proof. Denote by A an $(n, \varepsilon)-$ separated subset of M (with maximal cardinality) with respect to (G^M, G_1^M). This means that for two distinct points $x, y \in A$ there exists $g \in G_n$ such that $d_M(g|_M(x), g|_M(y)) \geq \varepsilon$. But this means that $d_X(g(x), g(y)) \geq \varepsilon$. Therefore, the set A is $(n, \varepsilon)-$ separated with respect to (G, G_1), and consequently $s(n, \varepsilon, (G^M, G_1^M)) \leq s(n, \varepsilon, (G, G_1))$. We are thus done by passing the appropriate limit when $n \to \infty$. ∎

Theorem 4. *Let (G, G_1) be a finitely generated semigroup acting on a compact metric space (X, d), generated by $G_1 = \{id_X, g_1, ..., g_k\}$. Assume that there exist compact subsets M_1 and M_2 of X such that $X = M_1 \cup M_2$ and that for every $j \in \{1, 2, \ldots, k\}$ $g_j(M_1) = M_1$ and $g_j(M_2) = M_2$ Then*

$$h((G, G_1), X) = \max\{h((G^{M_1}, G_1^{M_1}), M_1), h((G^{M_2}, G_1^{M_2}), M_2)\}.$$

Proof. Let $A_i \in M_i$, $i = 1, 2$, be $(n, \varepsilon)-$spanning sets with minimal cardinality, in the respective spaces M_i,$i = 1, 2$. Since

$$card(A_1 \cup A_2) \leq card(A_1) + card(A_2) \leq 2 \cdot \max\{card(A_1), card(A_2)\}$$

and since $A_1 \cup A_2$ forms an $(n, \varepsilon)-$spanning subset of X. We see that

$$\begin{aligned} r(n, \varepsilon, (G, G_1), X) &\leq 2 \cdot \max\{card(A_1), card(A_2)\} \\ &\leq 2 \max\{r(n, \varepsilon, (G^{M_1}, G_1^{M_1}), M_1), r(n, \varepsilon, (G^{M_2}, G_2^{M_2}), M_2)\}. \end{aligned}$$

Hence, passing to the appropriate limit, we obtain

$$h(G, G_1, X) \leq \max\{h((G^{M_1}, G_1^{M_1}), M_1), h((G^{M_2}, G_2^{M_2})M_2)\}.$$

Since, by Theorem 3, we have

$$h((G, G_1), X) \geq \max\{h((G^{M_1}, G_1^{M_1})M_1), h((G^{M_2}, G_2^{M_2})M_2)\},$$

we are done. ∎

4 Positive entropy

As before, let (X, d_X) be a compact metric space. We consider continous transformations of the space X into itself.

Theorem 5. *If $f_1, f_2 : X \to X$ are two surjective continuous maps of a compact metric space X and Y is a closed subset of X such that $f_1^{-1}(Y) \cap f_2^{-1}(Y) = \emptyset$ and $f_1^{-1}(Y) \cup f_2^{-1}(Y) \subset Y$, then $h(G, G_1) \geq \log 2 > 0$, where $G_1 = \{id_X, f_1, f_2\}$ and G is the semigroup generated by G_1.*

Proof. Since $f_1^{-1}(Y)$ and $f_2^{-1}(Y)$ are two disjoint compact sets, the distance δ between them is positive. Fix $\varepsilon \in (0, \delta)$. Since every map $g : X \to X$, $g \in G$ is surjective, one can select for every $g \in G$ exactly one point $z_g \in g^{-1}(Y)$. Now, for every $n \geq 0$ consider the set

$$A_n = \{z_g : g \in \hat{G}_n\}, \text{ where } \hat{G}_n = \{g_n \circ g_{n-1} \circ \ldots \circ g_2 \circ g_1 : g_1, \ldots, g_n \in \{f_1, f_2\}\}.$$

We shall show that A_n is an (n, ε)-separated set consisting of exactly 2^n elements. So, consider two arbitrary elements $g \neq h$ from $\hat{G}_n$. Write $g = g_n \circ g_{n-1} \circ \ldots \circ g_1$ and $h = h_n \circ h_{n-1} \circ \ldots \circ h_1$, where $g_j, h_j \in \{f_1, f_2\}$ for all $j = \{1, 2, \ldots, n\}$. Since $g \neq h$, there exist $k \in \{1, 2, \ldots, n\}$ such that $g_1 = h_1$, $g_2 = h_2, \ldots, g_{k-1} = h_{k-1}$, and $g_k \neq h_k$. Hence

$$g_{k-1} \circ \ldots \circ g_1(z_g) \in g_{k-1} \circ \ldots \circ g_1((g_n \circ \ldots \circ g_1)^{-1}(Y)) \subset (g_n \circ \ldots \circ g_k)^{-1}(Y) \subset g_k^{-1}(Y)$$

and similarly

$$\begin{aligned} g_{k-1} \circ \ldots \circ g_1(z_h) &\in g_{k-1} \circ \ldots \circ g_1((h_n \circ \ldots \circ h_1)^{-1}(Y)) \\ &= h_{k-1} \circ \ldots \circ h_1((h_n \circ \ldots \circ h_1)^{-1}(Y)) \subset h_k^{-1}(Y). \end{aligned}$$

Hence, $d(g_{k-1} \circ \ldots \circ g_1(z_g), g_{k-1} \circ \ldots \circ g_1(z_h)) \geq \delta > \varepsilon$. Thus, the points z_g and z_h are (n, ε)-separated and, in particular $z_g \neq z_h$. This latter statement implies that the map $g \to z_g$, $g \in \hat{G}_n$ is bijective, and therefore $s(n, \varepsilon, X) \geq card(\hat{G}_n) = 2^n$. In consequence $h(G, G_1) \geq \log 2 > 0$. We are done. ∎

As an immediate consequence of this theorem and Theorem 3, we get the following.

Corollary 1. *If for $f_1, f_2 \in Homeo(S^1)$ there exists a closed interval $I \subset S^1$ such that*

$$f_1^{-1}(I),\ f_2^{-1}(I) \subset I$$

and

$$f_1^{-1}(I) \cap f_2^{-1}(I) = \emptyset$$

then, the semigroup generated by id_X, f_1, f_2 has positive entropy.

Theorem 6. *If (G, G_1) is a semigroup of Lipschitz transformations, acting on a compact Riemannian manifold, then the entropy $h(G, G_1)$ is finite.*

Proof. Denote by d the metric on a Riemannian compact manifold M and denote the dimension of this manifold by m. It follows from our assumptions that for any $g \in G_1$ there exists a positive L_g such that for any $x, y \in M$ we have

$$d(g(x), g(y)) \leq L_g d(x, y)$$

Let $L = \max\{L_g : g \in G_1\}$ and denote by A a maximal $(n, \varepsilon)-$ separated subset of M. Then, for any distinct $a_1, a_2 \in A$ we obtain

$$\varepsilon \leq d_n^{max}(a_1, a_2) \leq L^n d(a_1, a_2).$$

Thus $d(a_1, a_2) \geq \varepsilon L^{-n}$ which means that A is a $(0, \varepsilon L^{-n})-$separated subset of M. Hence

$$s(n, \varepsilon) \leq s(0, \varepsilon L^{-n}) \leq \frac{volM}{min_{x \in M} volB(x, 2^{-1}\varepsilon L^{-n})}.$$

For an m-dimensional manifold M we have that a ball $B(x, r)$ centered at a point $x \in M$ and a radius r, satisfies the inequality

$$volB(x, r) \geq cr^m$$

with some positive constant c independent of x and r. Thus,

$$\frac{volM}{min_{x \in M} volB(x, 2^{-1}\varepsilon L^{-n})} \leq \frac{volM}{c(2^{-1}\varepsilon L^{-n})^m},$$

and consequently

$$s(n, \varepsilon) \leq \frac{volM}{c(2^{-1}\varepsilon L^{-n})^m}.$$

Passing to the suitable limits we thus get that $h(G, G_1) \leq L^m$, which finishes the proof. ■

5 Zero entropy and final remarks

Theorem 7. *Let (G, G_1) be a finitely generated semigroup acting on a compact metric space X. Assume that the family $\{g : X \to X\}_{g \in G}$ is equicontinous. Then $h(G, G_1) = 0$.*

Proof. Denote by d the metric on the compact metric space X. Fix $\varepsilon > 0$. Since the semigroup G acts equicontinuously on X, there exists $\delta > 0$ such that if $x, y \in X$ and $d(x, y) < \delta$, then $d(g(x), g(y)) < \varepsilon$ for all $g \in G$. Conequently, if $A \subset X$ is δ-spanning (with respect to the metric d) subset of X, then A is $(n, \varepsilon)-$spanning. Hence $r(n, \varepsilon) \leq cardA < \infty$, (the latter inequality is true since X is compact) and therefore

$$h(G, G_1) \leq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log(r(n, \varepsilon)) = 0.$$

We are done. ■

As an immediate consequence of this theorem, we get the following.

Corollary 2. *If (G, G_1) is a finitely generated semigroup of isometries acting on a compact metric space X, then $h(G, G_1) = 0$.*

Problem 1. *It is well known that for a homeomorphism $f : X \to X$ of a compact metric space X the equality $h(f) = h(f^{-1})$ holds. Is this also true in the case of semigroups generated respectively by homeomorphisms $id_X, f_1, ..., f_k$ and $id_X, f_1^{-1}, ..., f_k^{-1}$ of the space X?*

Acknowledgement. We are grateful to the Mathematics Department of the University of Warwick, where the major part of our research has been conducted, for a warm hospitality and excellent working conditions.

References

[1] R. Adler,A. Konheim,M. McAndrew, *Topological entropy*, Trans. Amer.Math.Soc. **114** (1965), 303-319.

[2] A. Biś, *Entropies of a semigroup of maps*, to appear in Discrete and Continous Dynamical Systems.

[3] A. Biś and P. Walczak, *Entropies of hyperbolic groups and some foliated spaces*, in "Foliations – Geometry and Dynamics", Proc. of the Euroworkshop, Warsaw 2000, eds. P. Walczak *et al.*, World Sci. Publ., Singapore 2002, 197 – 211.

[4] R. Bowen, *Entropy for group endomorphisms and homogenous spaces*, Trans. Amer. Math. Soc. **153**, (1971)

[5] E. I. Dinaburg, *The relation between topological entropy and metric enntropy*, Soviet Math. Dokl. **11**, (1970), 13-16.

[6] S. Friedland, *Entropy of graphs, semigroups and groups*, in Ergodic theory of Z^d actions, eds. M. Policott and K. Schmidt, London Math. Soc, London, 1996, 319-343.

[7] E. Ghys, R. Langevin, P. Walczak, *Entropie geometrique des feuilletages*, Acta Math. **160**, (1988), 105-142

[8] M. Hurley, *On topological entropy of maps*, Ergodic Th. and Dynam. Sys. **15**, (1995), 557-568.

[9] R. Langevin, F. Przytycki, *Entropie de l'image inverse d'une application*, Bull. Soc. Math. France **120**, (1992), 237-250.

[10] R. Langevin, P. Walczak, *Entropie d'une dynamique*, C.R. Acad. Sci. Paris, Ser I **312**, (1991), 141-144.

[11] E. Mihailescu, M. Urbański, *Inverse topological pressure with applications to holomorphic dynamics of several complex variables*, preprint 2002.

[12] Z. Nitecki, *Topological entropy and the preimage structure of maps,* preprint 2003.

[13] Z. Nitecki, F. Przytycki, *Preimage entropy for mapping*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **9**, 1815-1843.

[14] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag 1982.

Andrzej Biś
E-mail: andbis@math.uni.lodz.pl
Faculty of Mathematics
University of Lodz
ul. Banacha 22
90-238 Lodz
Poland

Mariusz Urbański
urbanski@unt.edu
Department of Mathematics
University of North Texas
P.O. Box 311430
Denton, TX 76203-1430
USA