VIRTUAL CENTERS OF HYPERBOLIC COMPONENTS IN THE TANGENT FAMILY; HAUSDORFF DIMENSION OUTLOOK

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Abstract. The tangent family $f_\lambda(z) = \lambda \tan z$ ($\lambda \in \mathbb{C} \setminus \{0\}, z \in \mathbb{C}$) is considered. It follows from [4] that the function ascribing to each parameter λ the Hausdorff dimension of the Julia set of f_λ is continuous at all hyperbolic parameters λ. Now, we prove that the hyperbolic dimension of the Julia set at each parameter λ_0 that is a virtual center of a hyperbolic component (in the sense of [2]) is equal to the limit of hyperbolic dimensions (which are also equal to ordinary Hausdorff dimensions) of the Julia sets at hyperbolic parameters λ canonically approaching λ_0 within this component. It is also shown that the Hausdorff dimension of the Julia set of f_{λ_0} is strictly larger than this limit.

1. Introduction

For every $\lambda \in \mathbb{C} \setminus \{0\}$ consider the meromorphic function $f_\lambda : \mathbb{C} \to \overline{\mathbb{C}}$ given by the formula

$$f_\lambda(z) = \lambda \tan(z).$$

Then for every integer $p \geq 1$ define the map g_p as follows.

$$g_p(\lambda) = f_\lambda^p(\lambda i).$$

The function g_p has a countable infinite set of poles and, if $p \geq 2$, a countable infinite set of essential singularities. A detailed description of the set of these singularities and its thorough analysis is provided in [2]. For every $r > 0$ put

$$\mathcal{A}^+(r) = \{z \in \mathbb{C} : \ \mathrm{Im} z > r\}$$

and for every $\alpha > 0$ set

$$S(r, \alpha) = \mathcal{A}^+(r) \cap \{z \in \mathbb{C} : \mathrm{Im} z \geq \alpha |\mathrm{Re} z|\}.$$

Thus the set $g_p^{-1}(S(r, \alpha))$ is an open subset of $\mathbb{C} \setminus \{0\}$ consisting of countably many connected components. It follows from [2] that if $r > 0$ is large enough, the boundary of each component V of $g_p^{-1}(S(r, \alpha))$ contains exactly one pole of g_p. Moreover, it follows from [2] and [3] that each pole λ of g_p belongs to the boundary of exactly one connected component of $g_p^{-1}(S(r, \alpha))$, which will be in the sequel denoted by

$$V(\lambda, r, \alpha).$$

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Obviously, if $r' > r$, then $V(\lambda, r', \alpha) = V(\lambda, r', \alpha) \cap g_p^{-1}(S(r.\alpha))$. For every $\lambda \in \mathbb{C} \setminus \{0\}$ let h_λ be the Hausdorff dimension of the Julia set $J(f_\lambda)$ and h_λ^* be the hyperbolic dimension of the set $J_r(f_\lambda)$ defined as the set of those points in $J(f_\lambda)$ whose ω-limit set contains a point lying out of the set $\{\infty, \lambda i, -\lambda i\}$. It follows from [4] that if λ is a hyperbolic parameter, meaning that f_λ has an attracting periodic cycle, then

$$h_\lambda = h_\lambda^*$$

(because h_λ-dimensional Hausdorff measure of the set of transitive points of f_λ is finite and positive). In addition on the space of hyperbolic parameters the function

$$\lambda \mapsto h_\lambda$$

is continuous. This can be deduced easily from uniqueness of h_λ-conformal measure, the fact which was proved in [4].

In this paper we prove the following theorem describing the behavior of dimension h_λ near the poles of the functions g_p.

Theorem A *Let $p \geq 1$ be an integer and let λ_0 be a pole of the function g_p. Then for every $\alpha > 0$ and every $r > 0$ large enough,*

$$\lim_{\lambda \to \lambda_0} h_\lambda = \lim_{\lambda \to \lambda_0} h_\lambda^* = h_{\lambda_0}^*,$$

where the limit is taken over the parameters λ in the connected component $V(\lambda_0, \alpha, r)$.

Remark that in the course of the proof of this theorem we reprove the fact known from [2] that the component $V(\lambda_0, r, \alpha)$ consists of hyperbolic parameters. The entire proof of Theorem A consists of several steps which are included in separate sections for the convenience of the reader.

Theorem A has actually two-fold meaning. First, 'continuity' of the hyperbolic dimension at poles of g_p, and secondly, discontinuity of the Hausdorff dimension of Julia sets at these poles; to see it, apart from Theorem A, make use of Skorulski's Theorem 5.6(iv) from [6] stating that $h_{\lambda_0}^* < 2$ ($\mathrm{HD}(J(f_{\lambda_0})) = 2$ since $J(f_{\lambda_0}) = 2$).

2. Preliminaries

For every $\lambda \in \mathbb{C} \setminus \{0\}$, we have that

$$f_\lambda(z) = \Big(\lambda i - \lambda i e^{2iz}\Big)\Big(1 - \frac{e^{2iz}}{1 + e^{2iz}}\Big).$$

Hence

$$
\begin{aligned}
f_\lambda(z)-\lambda i &= -\lambda i\frac{e^{2iz}}{1+e^{2iz}}-\lambda i e^{2iz}\Big(1-\frac{e^{2iz}}{1+e^{2iz}}\Big)\\
&= -\lambda i\Big(1+\frac{1}{1+e^{2iz}}\Big)e^{2iz}+\lambda i\frac{e^{4iz}}{1+e^{2iz}}.
\end{aligned}
\tag{2.1}
$$

It therefore follows that if $\mathrm{Im}z>0$ is large enough, then

$$\frac{3}{2}|\lambda|e^{-2\mathrm{Im}z}\le |f_\lambda(z)-\lambda i| \tag{2.2}$$

and

$$|f_\lambda(z)-\lambda i|\le 3|\lambda|e^{-2\mathrm{Im}z}. \tag{2.3}$$

A straightforward calculation shows that

$$f_\lambda'(z)=\frac{4\lambda e^{2iz}}{(1+e^{2iz})^2}. \tag{2.4}$$

We assume that $\lambda_0, r>0$ and $\alpha>0$ are given by the hypothesis of Theorem A. Presenting our reasoning we keep λ_0 and α fixed but we let $r>0$ to be appropriately large. Put

$$V_r := V(\lambda_0,\alpha,r)$$

and

$$U_r := iV_r.$$

Since $f_{\lambda_0}^{p-1}(\lambda_0 i)$ is a pole of f_{λ_0}, there exists $n\in\mathbb{Z}$ such that $f_{\lambda_0}^{p-1}(\lambda_0 i)=s_n=(n+\frac{1}{2})\pi$. So taking $r>0$ large enough, there exists $R>0$ so small that the function

$$F: V_r\times U_r\times B(s_n,R)\to\overline{\mathbb{C}}$$

given by the formula

$$F(\lambda,z,w)=f_\lambda^{p-1}(z)-w$$

does not take the value ∞ (so F is $\mathbb{C}$-valued). Since $F(\lambda_0,i\lambda_0,s_n)=0$ and

$$\frac{\partial F}{\partial z}(\lambda_0,i\lambda_0,s_n)=(f_{\lambda_0}^{p-1})'(\lambda_0 i)\neq 0,$$

it follows from the Implicit Function Theorem that for all $r>0$ large enough there exists a unique holomorphic function $\zeta: V_r\times B(s_n,R_1)\to U_r$ (R_1 is small enough) such that $\zeta(\lambda_0,s_n)=i\lambda_0$ and $F(\lambda,\zeta(\lambda,w),w)=0$. For every $\lambda\in V_r$ define the function $f_{\lambda,*}^{-(p-1)}: B(s_n,R_1)\to U_r$ by the formula

$$f_{\lambda,*}^{-(p-1)}(w)=\zeta(\lambda,w).$$

Then analyticity of ζ yields that $f_{\lambda,*}^{-(p-1)}: B(s_n,R_1)\to U_r$ is analytic and

$$f_\lambda^{p-1}(f_{\lambda,*}^{-(p-1)}(w))=F(\lambda,f_{\lambda,*}^{-(p-1)}(w),w)+w=F(\lambda,\zeta(\lambda,w),w)+w=w.$$

So $f_{\lambda,*}^{-(p-1)}$ is an analytic inverse branch of f_λ^{p-1}. Since $f_\lambda^p(i\lambda) = g_p(\lambda) \neq \infty$ for all $\lambda \in V_r \setminus \{\lambda_0\}$, we have that $f_\lambda^{p-1}(i\lambda)$ is not a pole of f_λ, in particular $f_\lambda^{p-1}(i\lambda) \neq s_n$. Thus

$$v_\lambda := f_{\lambda,*}^{-(p-1)}(s_n) \neq i\lambda, \quad \lambda \in V_r \setminus \{\lambda_0\}. \tag{2.5}$$

Since $v_\lambda = \zeta(\lambda, s_n)$, we see that the function $\lambda \mapsto v_\lambda$, $\lambda \in V_r$, is analytic and

$$\lim_{\lambda \to \lambda_0} v_\lambda = \lim_{\lambda \to \lambda_0} \zeta(\lambda, s_n) = \zeta(\lambda_0, s_n) = i\lambda_0. \tag{2.6}$$

Since for every $\lambda \in V_r \setminus \{\lambda_0\}$ the ball $B(v_\lambda, |v_\lambda - i\lambda|)$ contains no singular values of f_λ, we see that for every $\xi \in f_{\lambda,\xi}^{-1}(v_\lambda)$ there exists a unique holomorphic inverse branch $f_{\lambda,\xi}^{-1}$ defined on $B(v_\lambda, |v_\lambda - i\lambda|)$ of f_λ sending v_λ to ξ. Let $v(\lambda) := v_\lambda$. For every $\lambda \in V_r$ let

$$\eta_\lambda = \lambda i - v(\lambda).$$

Since $\lim_{\lambda \to \lambda_0} \lambda i = i\lambda_0$ it follows from (2.6) that

$$\lim_{\lambda \to \lambda_0} |\eta_\lambda| = 0. \tag{2.7}$$

For every $M > 0$ large enough put

$$D_M^+ = \sup\{|(f_\lambda^{p-1})'(z)| : \ \lambda \in V_M, z \in U_M\} \quad \text{and} \quad D_M^- = \inf\{|(f_\lambda^{p-1})'(z)| : \ \lambda \in V_M, z \in U_M\}.$$

Define

$$\kappa = \frac{\alpha}{\sqrt{1+\alpha^2}}.$$

A straightforward calculation shows that if $z \in S(r, \alpha)$ then

$$\mathrm{Im} z \geq \kappa |z|. \tag{2.8}$$

By $K \geq 1$ we denote throughout the entire paper the constant ascribed to the scale $1/2$ in Koebe's Distortion Theorem. For every $\lambda \in V_r$ put

$$\mathcal{G}(\lambda) = \left\{ z \in \overline{\mathbb{C}} \colon \mathrm{Im} z > -\frac{1}{2}\log|\eta_\lambda| + 5 + \frac{1}{2}\log|\lambda_0| - \frac{1}{2}\log\kappa + \frac{1}{2}\log K + \log D_M^+ - \log D_M^- \right\}.$$

3. The size of the Julia set

We shall prove first the following.

Lemma 3.1. *For all $r > 0$ large enough and all $\lambda \in V_r$*

$$f_\lambda^{p+1}(\mathcal{G}(\lambda)) \subset \mathcal{G}(\lambda).$$

Proof. Taking $r>0$ sufficiently large so that $|\eta_\lambda|>0$ is so small as we wish, it follows from (2.3) that

$$\begin{aligned}
f_\lambda(\mathcal{G}(\lambda)) &\subset B\left(i\lambda, 3|\lambda|\exp\left(-2\left(-\frac{1}{2}\log|\eta_\lambda|+5+\frac{1}{2}\log|\lambda_0|+\frac{1}{2}\log\left(\frac{K}{\kappa}\right)+\log\left(\frac{D_M^+}{D_M^-}\right)\right)\right)\right)\\
&= B\Big(i\lambda, 3e^{-10}K^{-1}\left(D_M^+/D_M^-\right)^{-2}\kappa|\lambda||\lambda_0|^{-1}|\eta_\lambda|\Big)\\
&\subset B\Big(i\lambda, e^{-8}K^{-1}\left(D_M^+/D_M^-\right)^{-2}\kappa|\lambda||\lambda_0|^{-1}|\eta_\lambda|\Big) \qquad (3.1)\\
&\subset B\Big(i\lambda, e^{-7}K^{-1}\left(D_M^+/D_M^-\right)^{-2}\kappa|\eta_\lambda|\Big).
\end{aligned}$$

Assume that $r\ge M$ to be so large that

$$\bigcup_{\lambda\in V_r}\mathcal{G}(\lambda)\subset\mathcal{A}^+(M).$$

Applying the Mean Value Inequality, it follows from (3.1) that

$$f_\lambda^p(\mathcal{G}(\lambda))\subset B\Big(f_\lambda^{p-1}(i\lambda), e^{-7}K^{-1}(D_M^+)^{-1}(D_M^-)^2\kappa|\eta_\lambda|\Big). \qquad (3.2)$$

Applying the Mean Value Inequality for the same function f_λ^{p-1} we also get that

$$|f_\lambda^{p-1}(i\lambda)-s_n|=|f_\lambda^{p-1}(i\lambda)-f_\lambda^{p-1}(v_\lambda)|\le D_M^+|i\lambda-v_\lambda|=D_M^+|\eta_\lambda|. \qquad (3.3)$$

Applying in turn the Mean Value Inequality to the inverse branch $f_{\lambda,*}^{-(p-1)}: B(s_n,R_1)\to U_r$ and using (2.5), we get for every $\lambda\in V_r$,

$$\begin{aligned}
|\eta_\lambda| = |i\lambda-v(\lambda)| &= |f_{\lambda,*}^{-(p-1)}(f_\lambda^{p-1}(i\lambda))-f_{\lambda,*}^{-(p-1)}(s_n)|\\
&\le \sup\{|(f_{\lambda,*}^{-(p-1)})'(z)| : z\in f_\lambda^{p-1}(U_M)\}|f_\lambda^{p-1}(i\lambda)-s_n|\\
&\le (D_M^-)^{-1}|f_\lambda^{p-1}(i\lambda)-s_n|
\end{aligned}$$

or equivalently

$$|f_\lambda^{p-1}(i\lambda)-s_n|\ge D_M^-|\eta_\lambda|. \qquad (3.4)$$

Since $\lim_{M\to+\infty}\frac{D_M^+}{D_M^-}=1$, we therefore see that with $M>0$ large enough,

$$s_n\notin B(f_\lambda^{p-1}(i\lambda), e^{-1}D_M^+|\eta_\lambda|).$$

Combining this, (3.4) and the fact that s_n is a simple pole of the tangent function, we see that with $r\ge M$ large enough (so that $f_\lambda^{p-1}(i\lambda)$ is as close to s_n as we wish), the map f_λ restricted to the ball $B(f_\lambda^{p-1}(i\lambda), 2D_M^+e^{-5}|\eta_\lambda|)$ is univalent. Since, s_n is a simple pole of the the tangent function $f_1(z)=\tan(z)$, with residuum equal to 1, there exists $R_2>0$ such that

$$\frac{1}{2}|\lambda||z-s_n|^{-2}\le|f_\lambda'(z)|\le 2|\lambda||z-s_n|^{-2} \qquad (3.5)$$

and

$$|f_\lambda(z)|\ge\frac{1}{2}|\lambda||z-s_n|^{-1} \qquad (3.6)$$

for all $z \in B(s_n, R_2)$. Since $\lim_{\lambda\to\lambda_0} f_\lambda^{p-1}(i\lambda) = s_n$ and since $\lim_{\lambda\to\lambda_0} |\eta_\lambda| = 0$, taking $r \geq M$ large enough, we have that $B(f_\lambda^{p-1}(i\lambda), 2D_M^+ e^{-5}|\eta_l|) \subset B(s_n, R_2)$. Combining this, (3.5) with $z = f_\lambda^{p-1}(i\lambda)$, Koebe's Distortion Theorem and (3.4) we conclude that

$$\begin{aligned}
f_\lambda\Big(B(f_\lambda^{p-1}(i\lambda), e^{-7}K^{-1}(D_M^+)^{-1}(D_M^-)^2\kappa|\eta_\lambda|)\Big) &\subset \\
&\subset B\Big(f_\lambda^p(i\lambda), K|f_\lambda'(f_\lambda^{p-1}(i\lambda))|e^{-7}K^{-1}(D_M^+)^{-1}(D_M^-)^2\kappa|\eta_\lambda|\Big) \\
&\subset B\Big(f_\lambda^p(i\lambda), 2|\lambda||f_\lambda^{p-1}(i\lambda) - s_n|^{-2}e^{-7}(D_M^+)^{-1}(D_M^-)^2\kappa|\eta_\lambda|\Big) \\
&\subset B\Big(f_\lambda^p(i\lambda), 2|\lambda|e^{-7}(D_M^+)^{-1}\kappa|\eta_\lambda|^{-1}\Big) \\
&\subset B\Big(f_\lambda^p(i\lambda), (4D_M^+)^{-1}\kappa|\lambda||\eta_\lambda|^{-1}\Big)
\end{aligned}$$

and using (3.2), we get that

$$f_\lambda^{p+1}(\mathcal{G}(\lambda)) \subset B\Big(f_\lambda^p(i\lambda), (4D_M^+)^{-1}\kappa|\lambda||\eta_\lambda|^{-1}\Big). \tag{3.7}$$

Combining now (3.3) and (3.6), we get that

$$|f_\lambda^p(i\lambda)| \geq \frac{1}{2}|\lambda||f_\lambda^{p-1}(i\lambda) - s_n|^{-1} \geq \frac{1}{2}(D_M^+)^{-1}|\lambda||\eta_\lambda|^{-1}.$$

Now assume that $\lambda \in V_r$. Then, using (2.8), we obtain

$$\mathrm{Im}(f_\lambda^p(i\lambda)) \geq \kappa|f_\lambda^p(i\lambda)| \geq (2D_M^+)^{-1}\kappa|\lambda||\eta_\lambda|^{-1}.$$

It therefore follows from (3.7) that every $z \in f_\lambda^{p+1}(\mathcal{G}(\lambda))$

$$\begin{aligned}
\mathrm{Im}(z) &\geq (2D_M^+)^{-1}\kappa|\lambda||\eta_\lambda|^{-1} - (4D_M^+)^{-1}\kappa|\lambda||\eta_\lambda|^{-1} \\
&= (4D_M^+)^{-1}\kappa|\lambda||\eta_\lambda|^{-1} \\
&\geq (8D_M^+)^{-1}\kappa|\lambda||\eta_\lambda|^{-1}.
\end{aligned}$$

So, looking at the definition of $\mathcal{G}(\lambda)$, we see that if $r \geq M$ is large enough (so that $|\eta_\lambda|$ is as small as we wish), then $f_\lambda^{p+1}(\mathcal{G}(\lambda)) \subset \mathcal{G}(\lambda)$, and we are done. $\square$

For every $\lambda \in V_r$ let

$$\Delta(\lambda) = \frac{1}{2}\Big(-\log|\eta_\lambda| + 10 + \log|\lambda_0| - \log\kappa + \log K + 2\log D_M^+ - 2\log D_M^-\Big).$$

Since both the Fatou set and Julia set of f_λ are symmetric with respect to the real axis, and since $\mathcal{G}(\lambda)$ is contained in the Fatou set of f_λ by Lemma 3.1, we get the following.

Lemma 3.2. *For all $r > 0$ large enough and all $\lambda \in V_r$, we have*

$$J(f_\lambda) \subset \{z : \ |\mathrm{Im} z| \leq \Delta(\lambda)\}.$$

4. Estimates of conformal measure

Let $\sim$ be a relation on $\mathbb{C}$ determined by the requirement that $w \sim z$ if and only if $w - z \in \mathbb{Z}$. Obviously $\sim$ is an equivalence relation on $\mathbb{C}$. Let $Q = \mathbb{C}/\sim$ be the corresponding quotient space, an infinite cylinder, and let $\pi : \mathbb{C} \to Q$ be the canonical projection map, f_λ invariant with respect to the lattice $\pi\mathbb{Z}$, it projects down to the map $F_\lambda : Q \to Q \cup \{\infty\}$ such that the following diagram commutes,

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f_\lambda"] \arrow[d, "\Pi"'] & \mathbb{C} \arrow[d, "\Pi"] \\ Q \arrow[r, "F_\lambda"] & Q \end{tikzcd} \tag{4.1}$$

i.e. $\Pi \circ f_\lambda = F_\lambda \circ \pi$. Let $J(F_\lambda) = \Pi(J(f_\lambda))$. It easily follows from lemma 3.1 that if $\lambda \in V_r$, then the sets $\mathcal{A}_r(\lambda)$ and its reflection with respect to the real axis have both attracting periodic points of f_λ of period $p+1$. The asymptotic values $\pm i\lambda$ must therefore belong to their basis of immediate attraction. Since in addition the map f_λ has no critical points, we conclude that the map f_λ satisfies condition (*) from [4]. Theorem 2, p. 621 from [4] than states that h_λ-dimensional Hausdorff measure (defined by means of spherical metric) of the Julia set $J(f_\lambda)$ is positive and finite, where $h_\lambda = \mathrm{HD}(J(f_\lambda)) = \mathrm{HD}(J(F_\lambda))$ is the Hausdorff dimension of the Julia set $J(f_\lambda)$. Invoking now Lemma 3.2 we conclude that the h_λ-dimensional Hausdorff measure (defined by means of Euclidean metric) $\mathrm{H}^h(J(F_\lambda))$ of $J(F_\lambda)$ is positive and finite. Then $m_\lambda = \mathrm{H}^h_{|J(F_\lambda)}/\mathrm{H}^h(J(F_\lambda))$ is a Borel probability measure on $J(F_\lambda)$, and it is obviously h_λ-conformal in the sense that

$$m_\lambda(F_\lambda(A)) = \int_A |F_\lambda'|^{h_\lambda} dm_\lambda$$

whenever $A \subset J(F_\lambda)$ is a Borel set and $F_{\lambda|A}$ is one-to-one.

Assume that $\lambda_0 \in \mathbb{C} \setminus \{0\}$ is given as in the hypothesis of Theorem A. Let $M > 0$ be the constant determined in the previous section. For every integer $k \geq M$ put

$$X_k = \{z \in Q : k \leq \mathrm{Im} z \leq k+1\}.$$

Our first result is this.

Lemma 4.1. *There is a constant* $C > 0$ (independent of λ) such that if $\lambda \in V_r^+$, $h_\lambda \geq 1$, and $M \leq k \leq -\frac{1}{2}\log|\eta_\lambda| + \frac{1}{2}\log(|\lambda_0|/16)$, then

$$m_\lambda(X_k) \leq Ce^{2k(1-h_\lambda)}.$$

Proof. Fix $z \in \Pi^{-1}(X_k)$. By (2.3) we get that

$$|f_\lambda(z) - \lambda i| \leq 3|\lambda|e^{-2\mathrm{Im} z} \leq 3|\lambda|e^{-2k}. \tag{4.2}$$

Our upper bound on k equivalently means that

$$|\eta_\lambda| \le \frac{1}{16}|\lambda_0|e^{-2k}. \tag{4.3}$$

So, using the Mean Value Inequality, (3.3), (4.2) and (4.3) we obtain

$$\begin{aligned} |f_\lambda^p(z) - s_n| &\le |f_\lambda^{p-1}(f_\lambda(z)) - f_\lambda^{p-1}(i\lambda)| + |f_\lambda^{p-1}(i\lambda) - s_n| \\ &\le D_M^+|f_\lambda(z) - i\lambda| + D_M^+|\eta_\lambda| \\ &\le 3D_M^+|\lambda|e^{-2k} + \frac{1}{16}D_M^+|\lambda_0|e^{-2k} \\ &\le 5D_M^+|\lambda_0|e^{-2k}. \end{aligned} \tag{4.4}$$

Now, fix in turn $\xi \in f_\lambda^p(\Pi^{-1}(X_k))$. Then $f_{\lambda,*}^{-(p-1)}(\xi) \in f_\lambda(\Pi^{-1}(X_k))$. Recall also that $f_{\lambda,*}^{-(p-1)}(s_n) = v_\lambda$. Now, using the Mean Value Inequality, (2.2) and (4.3), we get that

$$\begin{aligned} D_M^+|\xi - s_n| &\ge |f_{\lambda,*}^{-(p-1)}(\xi) - f_{\lambda,*}^{-(p-1)}(s_n)| \\ &\ge |f_{\lambda,*}^{-(p-1)}(\xi) - i\lambda| - |f_{\lambda,*}^{-(p-1)}(s_n) - i\lambda| \\ &\ge \frac{3}{2}|\lambda|e^{-2(k+1)} - |\eta_\lambda| \\ &\ge \frac{1}{8}|\lambda_0|e^{-2k} - |\eta_\lambda| \\ &\ge \frac{1}{16}|\lambda_0|e^{-2k} \end{aligned}$$

So

$$|\xi - s_n| \ge \frac{1}{16D_M^+}|\lambda_0|e^{-2k}. \tag{4.5}$$

Now, we shall estimate from above the width of the set $f_\lambda^{p+1}(\Pi^{-1}(X_k))$. By (4.5) (3.5) and (4.4), we get for all $z_1, z_2 \in \Pi^{-1}(X_k)$ that

$$\begin{aligned} |f_\lambda^{(p+1)}(z_2) - f_\lambda^{(p+1)}(z_1)| &= |f_\lambda(f_\lambda^p(z_2)) - f_\lambda(f_\lambda^p(z_1))| \\ &\le 2|\lambda|\Big(16D_M^+|\lambda_0|^{-1}\Big)^2 e^{4k}\Big(\pi/2\Big)|f_\lambda^p(z_2) - f_\lambda^p(z_1)| \\ &\le 2^9\pi|\lambda_0|^{-1}\Big(D_M^+\Big)^2 e^{4k}\Big(|f_\lambda^p(z_2) - s_n| + |f_\lambda^p(z_1) - s_n|\Big) \\ &\le 2^{13}\pi|\lambda_0|^{-1}\Big(D_M^+\Big)^3 e^{2k}. \end{aligned}$$

Hence

$$\sup \mathrm{Re}\Big(f^{p+1}(\Pi^{-1}(X_k))\Big) - \inf \mathrm{Re}\Big(f^{p+1}(\Pi^{-1}(X_k))\Big) \le 2^{13}\pi|\lambda_0|^{-1}\Big(D_M^+\Big)^3 e^{2k}. \tag{4.6}$$

Now, take arbitrary point $z \in X_k$. Choose then any point $\tilde{z} \in \Pi^{-1}(z)$. We get by (3.4), (3.5) and (4.4) that

$$\begin{aligned} |(F_\lambda^{p+1})'(z)| &= |(f_\lambda^{p+1})'(\tilde{z})| = |f_\lambda'(f_\lambda^p(z))||(f_\lambda^{p-1})'(f_\lambda(z))||f_\lambda'(z)| \\ &\geq \frac{1}{2}|\lambda||f_\lambda^p(z) - s_n|^{-2} D_M^- 2|\lambda| e^{-2(k+1)} \\ &\geq e^{-2} D_M^- |\lambda|^2 (5D_M^+)^{-2} |\lambda_0|^{-2} e^{4k} e^{-2k} \\ &\geq (2|\lambda_0|)^{-1} D_M^- (5eD_M^+)^{-2} e^{2k}. \end{aligned} \tag{4.7}$$

Since the map f_λ^{p+1} restricted to the set

$$\left\{ z \in \mathbb{C} \colon -\frac{\pi}{2} \leq \mathrm{Re}(z) \leq -\frac{\pi}{2} \quad \text{and} \quad k \leq \mathrm{Im}(z) < k+1 \right\}$$

is one-to-one, it follows from (4.6) that the map $F_{|X_k}^{p+1}$ is at most $\left(2^{13}(D_M^+)^3|\lambda_0|^{-1}e^{2k}\right)$-to-one. Therefore, applying (4.7) and using h_λ-conformality of the measure m_λ, we get that

$$\begin{aligned} 1 \geq m_\lambda(F_\lambda^{p+1}(X_k)) &\geq 2^{-13}(D_M^+)^{-3}|\lambda_0|e^{-2k} \int_{X_k} |(F_\lambda^{p+1})'(z)|^{h_\lambda} dm_\lambda(z) \\ &\geq 2^{-13}(D_M^+)^{-3}|\lambda_0|e^{-2k}\left((2|\lambda_0|)^{-1}D_M^-(5eD_M^+)^{-2}\right)^{h_\lambda} e^{2kh_\lambda} m(X_k) \\ &\geq C^{-1} e^{2k(h_\lambda - 1)} m(X_k), \end{aligned}$$

where the existence of a constant $C > 0$ (independent of λ) follows from the middle line and the observation that $h_\lambda \in [1, 2]$. We are done. $\square$

Now, for each $\lambda \in V_r$, $R > 0$ and integer $j \geq 1$, let

$$P_\lambda = \left\{ z \in \mathbb{C} \colon \mathrm{Re}(w_0(\lambda)) - \frac{\pi}{2} \leq \mathrm{Re} z < \mathrm{Re}(w_0(\lambda)) + \frac{\pi}{2} \right\}$$

and

$$A_j(\lambda, R) = A(w_0(\lambda), (j+1)^{-1}R, j^{-1}R) \cap P_\lambda$$

be the annulus centered at $w_0(\lambda)$ with inner radius $(j+1)^{-1}R$ and outer radius $j^{-1}R$ intersected with the vertical strip P_λ. We shall prove the following.

Lemma 4.2. *There exists a constant $C_1 > 0$ (independent of λ) such that for every $\lambda \in V_r$ with $h_\lambda \geq 1$, every $R > 0$, and every integer $j \geq 1$,*

$$m_\lambda(\Pi(A_j(\lambda, r)) \leq C_1 \left(\min\left\{\frac{\pi}{2}, R\right\}\right)^{-1} R^{2h_\lambda} |\eta_\lambda|^{h_\lambda - 1} j^{1-2h_\lambda}.$$

Proof. Since it is easy to calculate that

$$\mathrm{Im}(w_0(\lambda)) = \mathrm{Im}\Big(\frac{1}{2i}\log_0\Big(\frac{i\eta_\lambda}{2\lambda - i\eta_\lambda}\Big)\Big) = -\frac{1}{2}\log\left|\frac{i\eta_\lambda}{2\lambda - i\eta_\lambda}\right| = -\frac{1}{2}\log|\eta_\lambda| + \frac{1}{2}\log|2\lambda - i\eta_\lambda|,$$

where $\log_0\Big(\frac{i\eta_\lambda}{2\lambda - i\eta_\lambda}\Big)$ is a logarithm of $\frac{i\eta_\lambda}{2\lambda - i\eta_\lambda}$, we get that

$$-\frac{1}{2}\log|\eta_\lambda| + \frac{1}{2}\log|\lambda_0| \le \mathrm{Im}(w_0(\lambda)) \le -\frac{1}{2}\log|\eta_\lambda| + \frac{1}{2}\log|3\lambda_0|. \tag{4.8}$$

It therefore follows from (2.4) that every $z \in B(w_0(\lambda), R)$

$$|f_\lambda'(z)| \le 8|\lambda_0|\exp(\log|\eta_\lambda| - \log|\lambda_0|) = 8|\eta_\lambda|.$$

Thus, applying the Mean Value Inequality, we obtain for every $j \ge 1$, the following

$$f_\lambda(B(w_0(\lambda), Rj^{-1})) \subset B(v(\lambda), 8|\eta_\lambda|Rj^{-1}). \tag{4.9}$$

It follows from (4.8) and (2.4) that for every $z \in B(w_0(\lambda), R)$,

$$|f_\lambda'(z)| \ge 2|\lambda_0|\exp(\log|\eta_\lambda| - \log(3|\lambda_0|) = \frac{2}{3}|\eta_\lambda|. \tag{4.10}$$

Applying $\frac{1}{4}$-Koebe's Distortion Theorem, we therefore get that

$$\begin{aligned} f_\lambda\Big(B(w_0(\lambda), R(j+1)^{-1}) \cap P_\lambda\Big) &\supset f_\lambda\Big(B\Big(w_0(\lambda), \min\Big\{\pi/2, R(j+1)^{-1}\Big\}\Big)\Big) \\ &\supset B\left(v(\lambda), \frac{1}{6}|\eta_\lambda|\min\Big\{\pi/2, R(j+1)^{-1}\Big\}\right) \\ &\supset B\left(v(\lambda), \frac{1}{6}|\eta_\lambda|\min\{\pi/2, R\}(j+1)^{-1}\right). \end{aligned}$$

Since the map f_λ restricted to the set $B(w_0(\lambda), R) \cap P_\lambda$ is one-to-one, we thus conclude (using also (4.9) that

$$f_\lambda(A_j(\lambda, R)) \subset A\left(v(\lambda), \frac{1}{6}\min\{\pi/2, R\}|\eta_\lambda|(j+1)^{-1}, 8R|\eta_\lambda|j^{-1}\right). \tag{4.11}$$

Since $f_\lambda^{p-1}(v_\lambda) = s_n$ and since $(f_\lambda^{p-1})'(i\lambda_0) \ne 0$, it follows from (2.6) that there exists a universal radius $T > 0$ such that the map f_λ^{p-1} (restricted to to the ball $B(v_\lambda, T)$ is univalent and

$$\frac{1}{2}|(f_{\lambda_0}^{p-1})'(i\lambda_0)| \le |(f_\lambda^{p-1})'(z)| \le 2|(f_{\lambda_0}^{p-1})'(i\lambda_0)| \tag{4.12}$$

for all $r > 0$ large enough (so that λ is as close to λ_0 as one wishes) and for all $z \in B(v_\lambda, T)$. So applying the Mean Value Theorem, $\frac{1}{4}$-Koebe Distortion Theorem and (4.11) yields that for all $\lambda \in V_r$ and all $j \ge 1$ that

$$f_\lambda^p(A_j(\lambda, R)) \subset A\left(s_n, \frac{1}{48}|(f_{\lambda_0}^{p-1})'(i\lambda_0)|\min\{\pi/2, R\}|\eta_\lambda|(j+1)^{-1}, 16|(f_{\lambda_0}^{p-1})'(i\lambda_0)|R|\eta_\lambda|j^{-1}\right) \tag{4.13}$$

Hence, if $z \in A_j(\lambda, R)$ then using (3.5) we get that

$$|f_\lambda'(f_\lambda^p(z))| \ge 2^{-10}|\lambda_0||(f_{\lambda_0}^{p-1})'(i\lambda_0)|^{-2}R^{-2}|\eta_\lambda|^{-2}j^2.$$

Consequently using (4.10) and (4.12) we get that for every $\zeta \in \Pi(A_j,(\lambda,R))$ $\zeta = \Pi(z)$, $z \in A_j(\lambda,R)$ that

$$\begin{aligned} |(F_\lambda^{p+1})'(\zeta)| =&|(f_\lambda^{p+1})'(\zeta)| \\ =&|f_\lambda'(z)||(f_\lambda^{p-1})'(f(z))||f_\lambda'(f_\lambda^p(z))| \\ \geq& 3^{-1}2^{-10}|\lambda_0||(f_{\lambda_0}^{p-1})'(i\lambda_0)|^{-1}R^{-2}|\eta_\lambda|^{-1}j^2. \end{aligned} \tag{4.14}$$

Looking at (4.13) and (3.6) we see that

$$f_\lambda^{p+1}(A_j(\lambda,R)) \subset B(0, 96|\lambda_0||(f_{\lambda_0}^{p-1})'(i\lambda_0)|^{-1}(\min\{\pi/2,R\})^{-1}|\eta_\lambda|^{-1}(j+1).$$

Since f_λ^{p+1} restricted to $A_j(\lambda,R)$ is one-to-one, we therefore conclude that the map F_λ^{p+1} restricted to $\Pi(A_j(\lambda,R))$ is at most $96\pi^{-1}|\lambda_0||(f_\lambda^{p-1})'(i\lambda_0)|^{-1}(\min\{\frac{\pi}{2},R\})^{-1}|\eta_\lambda|^{-1}j$-to-one. Hence, using this fact, (4.14) and conformality of the measure m_λ, we get that

$$\begin{aligned} 1 &\geq m_\lambda(F_\lambda^{p+1}(A_j(\lambda,R))) \\ &\geq \Big(96\pi^{-1}|\lambda_0||(f_\lambda^{p-1})'(i\lambda_0)|(\min\{\pi/2,R\})^{-1}|\eta_\lambda|^{-1}j\Big)^{-1}\int_{A_j(\lambda,R)}|(F_\lambda^{p+1})'|^{h_\lambda}dm_\lambda \\ &\geq \frac{\pi}{96}|(f_{\lambda_0}^{p-1})'(i\lambda_0)|^{-1}\min\{\pi/2,R\}|\eta_\lambda|j^{-1}\Big(3^{-1}2^{-10}|\lambda_0|(f_{\lambda_0}^{p-1})'(i\lambda_0)|^{-1}R^{-2}|\eta_\lambda|^{-1}j^2\Big)^{h_\lambda}\cdot \\ &\quad \cdot m_\lambda(A_j(\lambda,R)). \end{aligned}$$

Thus

$$m_\lambda(A_j(\lambda,R)) \leq C_1(\min\{\pi/2,R\})^{-1}R^{2h_\lambda}|\eta_\lambda|^{h_\lambda-1}j^{1-2h_\lambda},$$

where $C_1 > 0$ is a universal constant, since $h_\lambda \in [1,2]$. We are done. ☐

Since hyperbolic sets are stable under perturbations and their Hausdorff dimension is determined by Bowen's formula, we get that

$$\varliminf_{\lambda\to\lambda_0} h_\lambda^* \geq h_{\lambda_0}^*. \tag{4.15}$$

Since B. Skorulski proved in Theorem 5.6 of [6] that $h_{\lambda_0}^* > 1$, we therefore conclude that there is $s > 1$ such that

$$h_\lambda = h_\lambda^* \geq s > 1$$

for all $r > 0$ large enough and all $\lambda \in V(\lambda_0,\alpha,r)$. Writing $C_2 = C_1\sum_{j=1}^\infty j^{1-2s}$, we obtaining the following immediate consequence of Lemma 4.2.

Corollary 4.3. *For all $r > 0$ large enough, all $\lambda \in V_r$ and all $R \geq \frac{\pi}{2}$*

$$m_\lambda(\Pi(B(w(\lambda_0),R))) \leq \frac{C_2\pi}{2}R^4|\eta_\lambda|^{s-1}.$$

5. Tightness of conformal measures

We shall prove in this section the following.

Lemma 5.1. *With the assumptions of Theorem A, the family*

$$\{m_\lambda\}_{\lambda\in V(\lambda_0,\alpha,r)}$$

is tight, where m_λ is the h_λ-conformal measure for $F_\lambda : Q\to Q$ introduced in the beginning of Section 4.

Proof. Since

$$\lim_{t\to+\infty} D_t^+ = \lim_{t\to+\infty} D_t^- = |(f_{\lambda_0}^{p-1})'(i\lambda_0)|,$$

we have that

$$\frac{1}{2}|(f_{\lambda_0}^{p-1})'(i\lambda_0)| \le D_M^- \le D_M^+ \le 2|(f_{\lambda_0}^{p-1})'(i\lambda_0)|$$

for all $M\ge T$, with some fixed $T>0$ large enough. It therefore follows from Lemma 3.2 that for all $\lambda\in V(\lambda_0,\alpha,r)$

$$J(f_\lambda)\subset\{z\in \mathbb{C}\colon |\mathrm{Im}z|\le \Delta^*(\lambda)\} \tag{5.1}$$

where

$$\begin{aligned}\Delta^*(\lambda) =&\frac{1}{2}\Big(-\log|\eta_\lambda| + 10 + \log|\lambda_0| - \log\kappa + \log K\\ &+ 2\log(2|(f_{\lambda_0}^{p-1})'(i\lambda_0)|) - 2\log((1/2)|(f_{\lambda_0}^{p-1})'(i\lambda_0)|) \quad .\\ =&\frac{1}{2}\Big(-\log|\eta_\lambda| + 10 + 4\log 2 + \log|\lambda_0| - \log\kappa + \log K\Big)\end{aligned}$$

Our first goal is to show if $r>0$ is large enough, then

$$\lim_{M\to+\infty}\sup\{m_\lambda(\{z\in Q:\mathrm{Im}z\ge M\}) : \quad \lambda\in V(\lambda_0,\alpha,r)\} = 0. \tag{5.2}$$

Indeed, let

$$R=\max\{10+4\log 2+\log K-\log\kappa, \log 30\}.$$

It then follows from (5.1) and (4.8) that

$$J(F_\lambda)\cap\{z\in Q:\ \ \mathrm{Im}z\ge M\}\subset\bigcup_{k=M}^{q_\lambda} X_k\cup B(w(\lambda_0),R)),$$

where

$$q_\lambda = -\frac{1}{2}\log|\eta_\lambda| + \frac{1}{2}(\log|\lambda_0|/16).$$

Hence, applying Lemma 4.1 and Corollary 4.3, we obtain with $r > 0$ large enough for every $\lambda \in V_r$, that

$$\begin{aligned} m_\lambda(J(F_\lambda) \cap \{z \in Q : \mathrm{Im} z \geq M\}) &\leq \sum_{k=M}^{q_\lambda} Ce^{2k(1-s)} + \frac{\pi C_2}{2} R^4 |\eta_\lambda|^{s-1} \\ &\leq \frac{Ce^{2M(1-s)}}{1-e^{2(1-s)}} + \frac{\pi C_2}{2} R^4 |\eta_\lambda|^{s-1}. \end{aligned}$$

Since $1 - s < 0$ and since $\lim_{\lambda \to \lambda_0} |\eta_\lambda| = 0$, formula (5.2) follows. Let $S : \mathbb{C} \to \mathbb{C}$ be given by the formula $S(z) = -z$. This map obviously projects down to the map $\hat{S} : Q \to Q$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{S} & \mathbb{C} \\ \Pi \big\downarrow & & \big\downarrow \Pi \\ Q & \xrightarrow{\hat{S}} & Q \end{array} \tag{5.3}$$

Let $m'_\lambda = m_\lambda \circ \hat{S}^{-1}$. Looking at this diagram, we see that $\hat{S} \circ \hat{S} = id$ (in particular $\hat{S}$ is invertible and $\hat{S}^{-1} = \hat{S}$) and $\hat{S}^{-1} \circ F_\lambda = F_\lambda \circ \hat{S}^{-1}$. Since $|(\hat{S})'(z)| = 1$ for all $z \in Q$, it also follows from (5.3) that $|F'_\lambda(S^{-1}(z))| = |F'_\lambda(z)|$. Consequently

$$\begin{aligned} \frac{dm'_\lambda \circ F_\lambda}{dm'_\lambda}(z) &= \frac{dm_\lambda \circ \hat{S}^{-1} \circ F_\lambda}{dm_\lambda \circ \hat{S}^{-1}}(z) \\ &= \frac{dm_\lambda \circ F_\lambda \circ \hat{S}^{-1}}{dm_\lambda \circ \hat{S}^{-1}}(z) \\ &= \frac{dm_\lambda \circ F_\lambda}{dm_\lambda}(\hat{S}^{-1}(z)) \\ &= |F'_\lambda(z)|^{h_\lambda}(\hat{S}^{-1}(z)) \\ &= |F'_\lambda(z)|^{h_\lambda} \end{aligned}$$

This means that m'_λ is an h_λ-conformal measure for F_λ. The uniqueness (see[4]) of such conformal measure implies that $m'_\lambda = m_\lambda$ or equivalently $m_\lambda \circ \hat{S}^{-1} = m_\lambda$. Hence

$$m_\lambda(\{z \in Q : \mathrm{Im} z \leq -M\}) = m_\lambda((\hat{S})^{-1}(\{z \in Q : \mathrm{Im} z \leq -M\})).$$

Combining this and (5.2), we see that

$$\lim_{M \to +\infty} \sup\{m_\lambda(\{z \in Q : |\mathrm{Im} z| \geq M\}) : \quad \lambda \in V(\lambda_0, \alpha, r)\} = 0.$$

Since the set $\{z \in Q : |\mathrm{Im} z| \leq M\}$ is compact, the tightness of the family of measures $\{m_\lambda\}_{\lambda \in V(\lambda_0, \alpha, r)}$ is established. $\square$

6. Conclusion of the proof of Theorem A

Take $\{\lambda_n\}_{n=0}^{\infty}$, any sequence of parameters from $V(\lambda_0, \alpha, r)$ such that $\lim_{n\to\infty} \lambda_n = \lambda_0$. In order to conclude the proof it suffices to show that

$$\lim_{n\to\infty} h_{\lambda_n} = h_{\lambda_0^*}.$$

Since $h_\lambda \in [s, 2]$, $s > 1$, we may assume without loss of generality that the sequence $\{h_{\lambda_n}\}_{n=1}^{\infty}$ converges, say, to $t \in [s, 2]$. Combining Lemma 5.1 and Prokhorov theorem (see [1], Theorem 5.1, Section 5, p. 59) , we conclude that there is a subsequence of the sequence $\{m_{\lambda_n}\}_{n=1}^{\infty}$ converges weakly to a Borel probability measure m on Q. We may assume without loss of generality that the sequence $\{m_{\lambda_n}\}_{n=1}^{\infty}$ converges itself to the measure m. It is by now standard and not too difficult (see [7] for details in a compatible situation) to show that m is a t-conformal measure for F_{λ_0}. Since, in addition $J(F_{\lambda_0}) = Q$ and since $t \geq s > 1$, it follows from Theorem 5.6 from [6] that $t = h^*_{\lambda_0}$, and we are done. $\square$

Proposition 6.1. *With the assumption of Theorem A, we have that*

$$\lim_{\lambda\to\lambda_0} m_\lambda = m_{\lambda_0},$$

where m_{λ_0} is the weak limit of $\{m_\lambda\}_{\lambda\in V(\lambda_0,r,\alpha)}$, and where m_λ is the only h_λ-conformal measure of F_λ on $J(F_{\lambda_0})$.

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