Skew product Smale endomorphisms over countable shifts of finite type

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Abstract. We introduce and study skew product Smale endomorphisms over finitely irreducible topological Markov shifts with countable alphabets. We prove that almost all conditional measures of equilibrium states of summable and locally Hölder continuous potentials are dimensionally exact, and that their dimension is equal to the ratio of the (global) entropy and the Lyapunov exponent. We also prove for them a formula of Bowen type for the Hausdorff dimension of all fibers. We develop a version of thermodynamic formalism for finitely irreducible two-sided topological Markov shifts with countable alphabets. We show that the exact dimensionality of conditional measures on fibers, implies the global exact dimensionality of the measure, when the projection measure in the base space is exact dimensional. We describe next the thermodynamic formalism for Smale skew products over countable-to-1 endomorphisms. We give several applications to equilibrium measures on natural extensions of endomorphisms. As applications we study equilibrium states and dimension for skew products over maps generated by graph directed Markov systems, in particular over expanding Markov-Rényi transformations; and we obtain global exact dimensionality of equilibrium states with respect to endomorphisms over the continued fractions transformation, and over parabolic maps, such as the Maneville-Pomeau maps. We prove next two results related to Diophantine approximation, which make the Doeblin-Lenstra Conjecture more general and more precise, for a larger class of measures than in the classical case. Also, we study the natural extensions of endomorphisms associated to graph-directed systems, and in particular to generalized Lüroth series expansions with countable partitions $\mathcal{I}$. In the end we prove exact dimensionality and find a computable formula for the (pointwise) dimension of equilibrium measures, for the induced maps of natural extensions $\mathcal{T}_\beta$ of beta-transformations, for arbitrary $\beta > 1$.

Contents

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1. Introduction

In this paper we explore the thermodynamic formalism for skew product Smale endomorphisms modeled on countable alphabet subshifts, and for conformal skew product Smale endomorphisms over countable-to-1 maps. This is very different than for finitely generated systems. Our notion of Smale space is different, although inspired by the respective notion from [28]. We first develop a fairly complete thermodynamic formalism of two-sided shift, building on the countable alphabet one-sided shift approach provided in [15] and [13]. Then for Smale endomorphisms that are fiberwise conformal, we prove exact dimensionality of equilibrium measures and find dimension formulas, and we prove a version of Bowen's formula giving a dynamical characterization of the Hausdorff dimension of every fiber. Exact dimensionality is an important property ([34], [1]), which implies that several dimensional quantities of the measure are equal (pointwise dimension, Hausdorff dimension, etc.)

We then pass to general skew products over countable-to-1 endomorphisms and we prove, under a condition of μ-injectivity for the coding of the base space, the exact dimensionality of conditional measures of equilibrium measures in the stable fibers, building on [16], [19]. Moreover in this case we show that, if the projection to the base of the equilibrium measure μ is exact dimensional, then the initial measure μ is exact dimensional globally. This will be applied to natural extensions of endomorphisms, which in certain cases can be viewed as skew products over the respective endomorphisms ([17]). Our results apply also to natural extensions of endomorphisms associated to iterated function systems.

We apply then our results to countable alphabet graph directed systems and iterated function systems (see [15]), and as particular cases to EMR (expanding Markov-Rényi) maps, such as the continued fraction (Gauss) map, and Manneville-Pomeau maps (which have parabolic points). We consider skew product endomorphisms over them.

We apply next our results to give a generalization of the Doeblin-Lenstra Conjecture in Diophantine Approximation (see [9], [5]), to the larger class of equilibrium measures on the natural extension of the Gauss map, and we prove and use the exact dimensionality of these measures. This will make the Doeblin-Lenstra Conjecture more precise, in obtaining statistical behaviour of the asymptotic frequencies of Diophantine approximants with regard to certain singular measures. We also prove exact dimensionality and find computable dimension formulas for equilibrium measures for the induced maps of natural extensions $\mathcal{T}_\beta$ of β-transformations, for arbitrary $\beta > 1$.

We now summarize the results of each Section. One of our objectives is to develop first the thermodynamic formalism on countably generated shifts. In order to do this, we recall in Section 2 the basic notions and results of thermodynamic formalism of one-sided subsifts of finite type modeled on a countable alphabet as developed in [15] and [13]. Passing on to two-sided shifts in Section 3, we provide a fairly complete thermodynamic formalism of locally Hölder continuous potentials with respect to dynamical systems generated by a two-sided subshift of finite type. This comprises topological pressure, variational principles, equilibrium and Gibbs states. It also includes the characterization of Gibbs states in terms of conditional measures; this has no counterpart in the context of one sided shifts.

We then define in Section 4 skew product Smale endomorphisms modeled on countable alphabet subshifts of finite type, and we specify several significant subclasses. We show that if a skew product Smale endomorphism is continuous and of compact type, then there is a bijection between invariant measures for the symbol dynamics, and those for the Smale endomorphism. Assuming the Smale endomorphism is Hölder continuous, we prove the existence and uniqueness of equilibrium states of locally Hölder continuous potentials.

Next, we study conformal Hölder continuous *Smale endomorphisms modeled on countable alphabet subshifts of finite type*, defined in Section 5. In Section 6 we prove two theorems for these Smale endomorphisms. First, in Theorem 6.2 we show that projections of a.e conditional measures of equilibrium states of summable locally Hölder continuous potentials are *dimensional exact*, and their (pointwise) *dimension* equals the ratio of the global entropy and the Lyapunov exponent. Then, we prove in Theorem 7.3 a version of Bowen's formula giving the Hausdorff dimension of each fiber essentially as the zero of pressure function of a geometric potential; we deal also with the case when the pressure function has no zero.

Another primary goal is studied in Section 8, where we consider general *skew product endomorphisms* $F : X \times Y \to X \times Y$, $F(x, y) = (f(x), g(x, y))$, over *countable-to-1 endomorphisms* $f : X \to X$ in the base X, where X is a general metric space (not only E_A^+), and $Y \subset \mathbb{R}^d$. The endomorphisms f are coded by using shift spaces with countably many symbols. We introduce a notion of *μ-injective coding*, and we prove in Theorem 8.4 a result about exact dimensionality and pointwise dimensions of conditional measures of equilibrium states in the fibers of the skew product; this will be used for several classes of applications. This is building on and extending a result about exact dimensionality of conditional measures in stable manifolds for endomorphisms from [16]. Then, in Theorem 8.7 we prove that, if the conditional measures of an equilibrium measures μ_ϕ on fibers are exact dimensional, and if the projection of μ_ϕ on the base space is also exact dimensional, then the measure μ_ϕ itself is *exact dimensional globally*.

We then study several classes of skew product Smale endomorphisms, in particular *natural extensions (inverse limits)* of certain endomorphisms (for eg [26], [29], [18]). In Section 9 we study skew products with the base map being given by *graph-directed Markov systems* (GDMS) of [15]. We prove in Theorem 9.4 the exact dimensionality of conditional measures for equilibrium measures. And moreover, in Theorem 9.6 we prove the *global exact dimensionality* of equilibrium measures of locally Holder continuous summable potentials for skew products. We then study certain classes of one–dimensional endomorphisms $f : I \to I$, including EMR (expanding Markov-Rényi) maps, and conformal skew product endomorphisms $F : I \times Y \to I \times Y$ over such maps f, in particular we consider the *continued fraction (Gauss) transformation* $f_1(x) = \{\frac{1}{x}\}, x \in (0,1]$, which is coded by a countable alphabet, and *parabolic systems*, such as the one given by the Manneville-Pomeau maps $f_2(x) = x + x^{1+\alpha} \mod 1,\ \ x \in [0,1]$, with $\alpha > 0$ (see [24], [12], [13], [14], [18], [21], [31], [32]). By using results of multifractal analysis, we prove in Corollary 9.14 that equilibrium measures are exact dimensional globally on $I \times Y$, with respect to the Smale endomorphism F over f_1. And in Corollary 9.20 we prove exact dimensionality of equilibrium measures for skew product endomorphisms over parabolic GDMS (such as the Maneville-Pomeau maps).

In Section 10 we apply the results obtained in previous Sections, to *Diophantine Approximation* of irrational numbers x. We generalize the *Doeblin-Lenstra Conjecture* (see [2], [9], [5]) about the approximation coefficients $\Theta_n(x)$ in continued fractions representation, *to equilibrium measures* of geometric potentials $-s\log|T'|,\ s > \frac{1}{2}$ (where T denotes here the Gauss map). We recall that if the continued fraction representation of an irrational number $x \in [0,1)$ is $x = \frac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3+\dots}}} = [a_1, a_2, \dots]$, with $a_i \geq 1$ integers for $i \geq 1$, and if $[a_1, \dots, a_n] = \frac{p_n(x)}{q_n(x)} \in \mathbb{Q}, n \geq 1$, then the approximation coefficients defined by

$$\Theta_n(x) := q_n(x)^2 \cdot \Big|x - \frac{p_n(x)}{q_n(x)}\Big|,\ n \geq 1,$$

are very important in diophantine approximation, and have been studied intensively (for eg [2], [9]). The original Doeblin-Lenstra Conjecture ([2], [5], [9]) gives information about the frequency of having consecutive $\Theta_k(x), \Theta_{k-1}(x)$ in some set, and involves the lift of the Gauss measure μ_G to the natural extension space $[0,1)^2$ of the continued fraction transformation; thus, it is valid for Lebesgue-a.e $x \in [0,1)$. In our case, by contrast, *we generalize* this to numbers x, from the complement of that set of zero Lebesgue measure. The natural extension $([0,1)^2, \mathcal{T})$ of the continued fraction transformation, is in fact a skew product endomorphism which falls into our class, so we can apply the results from previous Sections. Using the exact dimensionality of the lift $\hat{\mu}_s$ to the natural extension, of the equilibrium measure μ_s of the potential $-s\log|T'|, s > \frac{1}{2}$, *we also make the Doeblin-Lenstra Conjecture more precise*. Namely, for irrational x from a subset $\Lambda_s \subset [0,1)$ with $\mu_s(\Lambda_s) = 1$ and $\mathrm{HD}(\Lambda_s) > 0$ (but with $\mathrm{Leb}(\Lambda_s) = 0$), we estimate the asymptotic frequency of having the consecutive approximation coefficients $(\Theta_k(x), \Theta_{k-1}(x))$ r-close to arbitrary values (z, z'), for $1 \leq k \leq n$ and n large. This is contained in Theorem 10.1 and Theorem 10.2.

In the short Section 11 we consider countable-to-1 transformations S associated to generalized Lüroth series with countable interval partitions $\mathcal{I}$, and their natural extensions. Then, in Section 12 we study *β-transformations T_β, for arbitrary $\beta > 1$*. The natural extension (inverse limit) of such a transformations is denoted by $\mathcal{T}_\beta$, and in general it is defined on a stacked space which is complicated (see [6], [5]). However, by inducing to the square $Z_0 = [0,1)^2$, we prove exact dimensionality of the conditional measures on fibers $[0,1)$, of equilibrium states for the induced transformation $\mathcal{T}_{\beta,Z_0}$, and then also global exact dimensionality. This is contained in Theorems 12.2 and Theorem 12.4, in which we also give a *computable formula for the Lyapunov exponents*, and therefore a *computable formula for the pointwise dimension* of the measure.

Many authors studied various aspects in the thermodynamic formalism and dimension theory for endomorphisms, skew products, countable systems, for eg as a partial list, [1], [3], [4], [7], [8], [15], [16], [17], [19], [20], [21], [23], [24], [29], [30], [31], etc.

2. Thermodynamic Formalism on One-Sided Countable Shifts

In this section we collect some fundamental ergodic (thermodynamic formalism) results concerning one-sided symbolic dynamics. All of them can be found, with proofs, in [15], [13]. Let E be a countable set and let $A : E \times E \to \{0,1\}$ be a matrix. A finite or countable infinite tuple ω of elements of E is called A-admissible if and only if $A_{ab} = 1$ for any two consecutive elements a, b of E. The matrix A is said to be *finitely irreducible* provided that there exists a finite set F of finite A-admissible words such that for any two elements a, b of E there exists an element γ of F such that the word $a\gamma b$ is A-admissible. Throughout the entire paper the incidence matrix A is assumed to be finitely irreducible. Given $\beta > 0$ we define the metric d_β on $E^{\mathbb{N}}$ by setting

$$d_\beta\big((\omega_n)_0^{+\infty}, (\tau_n)_0^{+\infty}\big) = \exp\big(-\beta \max\{n \geq -1 : (0 \leq k \leq n) \Rightarrow \omega_k = \tau_k\}\big)$$

with the standard convention that $e^{-\infty} = 0$. Note that all the metrics d_β, $\beta > 0$, on $E^{\mathbb{N}}$ are Hölder continuously equivalent and they induce the product topology on $E^{\mathbb{N}}$. If a function $\psi : E^{\mathbb{N}} \to \mathbb{R}$ is Lipschitz continuous for the metric d_β on $E^{\mathbb{N}}$, then we will denote its Lipschitz constant by $L_\beta(\psi)$.

Now let us set

$$E_A^+ = \{(\omega_n)_0^{+\infty} : \forall_{n \in \mathbb{N}}\ A_{\omega_n \omega_{n+1}} = 1\}.$$

Obviously E_A^+ is a closed subset of $E^{\mathbb{N}}$ and we endow it with the topology and metrics d_β inherited from $E^{\mathbb{N}}$. The two-sided shift map $\sigma : E^{\mathbb{Z}} \to E^Z$ is defined by the formula

$$\sigma\big((\omega_n)_0^{+\infty}\big) = \big((\omega_{n+1})_{n=0}^{+\infty}\big).$$

Of course $\sigma(E_A^+) \subset E_A^+$ and $\sigma : E_A^+ \to E_A^+$ is a continuous mapping. For every finite word $\omega = \omega_0\omega_1 \ldots \omega_{n-1}$ put $|\omega| = n$, the length of ω and set

$$[\omega] = \{\tau \in E_A^+ : \forall_{(0 \leq j \leq n-1)} : \tau_j = \omega_j\}.$$

The set $[\omega]$ is called the cylinder generated by the finite word ω. Let $\psi : E_A^+ \to \mathbb{R}$ be a continuous function. The topological pressure $\mathrm{P}(\psi)$ is defined as follows.

$$\mathrm{P}(\psi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\big(\sup\big(S_n\psi|_{[\omega]}\big)\big)$$

and the limit exists, as the sequence $\log \sum_{|\omega|=n} \exp\big(\sup\big(S_n\psi|_{[\omega]}\big), n \in \mathbb{N}$, is sub-additive. The following theorem, a weaker version of the Variational Principle, was proved in [15].

Theorem 2.1. *If $\psi : E_A^+ \to \mathbb{R}$ is a continuous function and μ is a σ-invariant Borel probability measure on E_A^+ such that $\int \psi \, d\mu > -\infty$, then $\mathrm{h}_{\tilde{\mu}}(\sigma) + \int_{\mathrm{E}_\mathrm{A}^+} \psi \, \mathrm{d}\mu \le \mathrm{P}(\psi)$.*

We say that the function $\psi : E_A^+ \to \mathbb{R}$ is *summable* if and only if

$$\sum_{e\in E} \exp\big(\sup(\psi|_{[e]})\big) < +\infty.$$

A shift-invariant Borel probability measure μ on E_A^+ is called a Gibbs state of ψ provided that there are a constant $C \ge 1$ and $P \in \mathbb{R}$ such that

$$C^{-1} \le \frac{\mu([\omega])}{\exp(S_n\psi(\tau) - Pn)} \le C \tag{2.1}$$

for all $n \ge 1$, all admissible words ω of length n and all $\tau \in [\omega]$. It clearly follows from (2.1) that if ψ admits a Gibbs state, then $P = \mathrm{P}(\psi)$.

Definition 2.2. *A function $g : E_A^+ \to \mathbb{C}$ is said to be locally Hölder continuous if there exists $\beta > 0$, and a constant $C > 0$, such that for all $\omega, \omega' \in E_A^+$ with $\omega_0 = \omega_0'$, we have*

$$|g(\omega) - g(\omega')| \le C \exp\big(-\beta \mathit{max}\{n \ge 0, \mathit{with}\ \omega_k = \omega_k', 0 \le k \le n\}\big)$$

The following Remark is important for future applications, since it says that our results apply to unbounded locally Hölder continuous potentials.

Remark 2.3. *Notice that our local Hölder continuity condition is the same as what was called "Hölder continuity condition" in [13], [15]. However this is weaker than the usual definition of Hölder continuity on a symbolic space, since we do not say anything about sequences ω, ω' with $\omega_0 \ne \omega_0'$. Thus we allow also* ***unbounded*** *locally Hölder continuous potentials.*

The proofs of the following three results are in [15] (Theorems 2.1.7-2.1.8) and [13].

Theorem 2.4. *For every locally Hölder continuous summable potential $\psi : E_A^+ \to \mathbb{R}$ there exists a unique Gibbs state μ_ψ on E_A^+. The measure μ_ψ is ergodic.*

Theorem 2.5. *Suppose $\psi : E_A^+ \to \mathbb{R}$ is a locally Hölder continuous potential. Then, denoting by $\mathrm{P}_F(\psi)$ the topological pressure of $\psi|_{F_A^+}$ with respect to the shift map $\sigma : F_A^+ \to F_A^+$, we have*

$$\mathrm{P}(\psi) = \sup\{\mathrm{P}_F(\psi)\},$$

where the supremum is taken over all finite subsets F of E; equivalently over all finite subsets F of E such that the matrix $A|_{F\times F}$ is irreducible.

Theorem 2.6 (Variational Principle for One-Sided Shifts)**.** *Suppose that $\psi : E_A^+ \to \mathbb{R}$ is a locally Hölder continuous summable potential. Then*

$$\sup\left\{\mathrm{h}_\mu(\sigma) + \int_{\mathrm{E}_\mathrm{A}^+} \psi \mathrm{d}\mu,\ \mu\circ\sigma^{-1} = \mu \ \mathit{and} \int \psi \mathrm{d}\mu > -\infty\right\} = \mathrm{P}(\psi) = \mathrm{h}_{\mu_\psi}(\sigma) + \int_{E_A^+} \psi d\mu_\psi,$$

and μ_ψ is the only measure at which this supremum is attained.

Any measure that realizes the supremum in the above Variational Principle is called an equilibrium state for ψ. Then Theorem 2.6 can be reformulated as follows.

Theorem 2.7. *If $\psi : E_A^+ \to \mathbb{R}$ is a locally Hölder continuous summable potential, then the Gibbs state μ_ψ is a unique equilibrium state for ψ.*

We would also like to record the following straightforward consequence of irreducibility of the incidence matrix A.

Proposition 2.8. *A locally Hölder continuous potential $\psi : E_A^+ \to \mathbb{R}$ is summable if and only if $\mathrm{P}(\psi) < +\infty$.*

Definition 2.9. *Given two locally Hölder continuous functions $\gamma, \xi : E_A^+ \to \mathbb{R}$, define the sets*

$\Sigma(\gamma,\xi) = \{(q,t) \in \mathbb{R}^2 : q\gamma + t\xi$ *is summable*$\}$ *and* $\Sigma_{\mathbb{C}}(\gamma,\xi) = \{(q,t) \in \mathbb{C}^2 : (\mathrm{Re}\, q, \mathrm{Re}\, t) \in \Sigma(\gamma,\xi)\}$

Note that $\Sigma(\gamma,\xi)$ and $\Sigma_{\mathbb{C}}(\gamma,\xi)$ are open connected subsets respectively of $\mathbb{R}^2$ and $\mathbb{C}^2$. Invoking Theorem 2.4.6 from [15], Kato-Rellich Perturbation Theorem, Hartogs Theorem, and as the main ingredient, Theorem 2.6.8 from [15], plus at the end Theorem 2.3.3 from [15], we get the following.

Theorem 2.10. *If $\gamma, \xi : E_A^+ \to \mathbb{R}$ are locally Hölder continuous functions, then the function $\Sigma(\gamma,\xi) \ni (q,t) \mapsto \mathrm{P}(q\gamma + t\xi)$, is real-analytic.*

As the result complementary to this theorem, we immediately get from Proposition 2.6.13 and Proposition 2.6.14 in [15], the following.

Proposition 2.11. *If $\gamma, \xi : E_A^+ \to \mathbb{R}$ are locally Hölder continuous potentials, then for all $(q_0, t_0) \in \Sigma(\gamma,\xi)$,*

$$\frac{\partial}{\partial q}\Big|_{(q_0,t_0)} \mathrm{P}(q\gamma + t\xi) = \int \gamma \, d\mu_{q_0\gamma + t_0\xi}, \quad \frac{\partial}{\partial t}\Big|_{(q_0,t_0)} \mathrm{P}(q\gamma + t\xi) = \int \xi \, d\mu_{q_0\gamma + t_0\xi},$$

and

$$\frac{\partial^2}{\partial q \partial t}\Big|_{(q_0,t_0)} P(q\gamma + t\xi) = \sigma^2_{\mu_{q_0\gamma + t_0\xi}},$$

where $\mu_{q_0\gamma + t_0\xi}$ is the unique equilibrium state of the potential $q_0\gamma + t_0\xi$ and $\sigma^2_{\mu_{q_0\gamma+t_0\xi}}$ is the asymptotic covariance of the pair (γ,ξ) with respect to the measure $\mu_{q_0\gamma+t_0\xi}$ (see Proposition 2.6.14 in [15] for instance).

3. Thermodynamic Formalism on Two-Sided Countable Shifts

As in the previous section let E be a countable set and let $A : E \times E \to \{0,1\}$ be a finitely irreducible matrix. Given $\beta > 0$ we define the metric d_β on $E^{\mathbb{Z}}$ by setting

$$d_\beta\big((\omega_n)_{-\infty}^{+\infty}, (\tau_n)_{-\infty}^{+\infty}\big) = \exp\big(-\beta \max\{n \geq 0,\ \omega_k = \tau_k, \forall k \text{ with } |k| \leq n\}\big)$$

with the standard convention that $e^{-\infty} = 0$. Note that all the metrics d_β, $\beta > 0$, on $E^{\mathbb{Z}}$ are Hölder continuously equivalent and they induce the product topology on $E^{\mathbb{Z}}$. We set

$$E_A = \{(\omega_n)_{-\infty}^{+\infty} : \forall_{n \in \mathbb{Z}}\ A_{\omega_n \omega_{n+1}} = 1\}.$$

Obviously E_A is a closed subset of E^Z and we endow it with the topology and metrics d_β inherited from $E^{\mathbb{Z}}$. The two-sided shift map $\sigma : E^{\mathbb{Z}} \to E^Z$ is defined similarly as $\sigma\big((\omega_n)_{-\infty}^{+\infty}\big) = \big((\omega_{n+1})_{-\infty}^{+\infty}\big)$. Clearly $\sigma(E_A) = E_A$ and $\sigma : E_A \to E_A$ is a homeomorphism.

Definition 3.1. *A function $g : E_A \to \mathbb{C}$ is said to be locally Hölder continuous if there exists $\beta > 0$, and a constant $C > 0$, such that for all $\omega, \omega' \in E_A$ with $\omega_0 = \omega'_0$, we have*

$$|g(\omega) - g(\omega')| \leq C \exp\big(- \beta max\{n \geq 0, with\ \omega_k = \omega'_k, 0 \leq k \leq n\}\big)$$

For every $\omega \in E_A$ and all $-\infty \leq m \leq n \leq +\infty$, we set

$$\omega|_m^n = \omega_m \omega_{m+1} \dots \omega_n.$$

Let E_A^* be the set of all A-admissible finite words. For $\tau \in E^*$, $\tau = \tau_m \tau_{m+1} \dots \tau_n$, we set

$$[\tau]_m^n = \{\omega \in E_A : \omega|_m^n = \tau\}$$

and call $[\tau]_m^n$ the cylinder generated by τ from m to n. The family of cylinders from m to n will be denoted by C_m^n. If $m = 0$ we simply write $[\tau]$ for $[\tau]_m^n$.

Let $\psi : E_A \to \mathbb{R}$ be a continuous function. The topological pressure $\mathrm{P}(\psi)$ is defined as in the one-sided case by

$$\mathrm{P}(\psi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in C_0^{n-1}} \exp\big(\sup\big(S_n\psi|_{[\omega]}\big)\big), \tag{3.1}$$

and the limit exists due to the same subadditivity argument. Similarly we obtain:

Theorem 3.2. *If $\psi : E_A \to \mathbb{R}$ is a continuous function and μ is a σ-invariant Borel probability measure on E_A such that $\int \psi \, d\mu > -\infty$, then*

$$\mathrm{h}_\mu(\sigma) + \int_{\mathrm{E_A}} \psi \, \mathrm{d}\mu \leq \mathrm{P}(\psi).$$

A shift-invariant Borel probability measure μ on E_A is called a *Gibbs state* of ψ provided that there are a constant $C \geq 1$ and $P \in \mathbb{R}$ such that

$$C^{-1} \leq \frac{\mu([\omega|_0^{n-1}])}{\exp(S_n\psi(\omega) - Pn)} \leq C \tag{3.2}$$

for all $n \geq 1$ and all $\omega \in E_A$. It clearly follows from (3.2) that if ψ admits a Gibbs state, then $P = \mathrm{P}(\psi)$. Two functions ψ_1 and ψ_2 are called *cohomologous* in a class G of real-valued functions defined on E_A if and only if there exists $u \in G$ such that

$$\psi_2 - \psi_1 = u - u \circ \sigma.$$

Any function of the form $u - u \circ \sigma$ is called a *coboundary* in G. A function $\psi : E_A \to \mathbb{R}$ is called cohomologous to a constant, say $b \in \mathbb{R}$ provided that $\psi - b$ is a coboundary. Notice that any two functions on E_A, cohomologous in $C(E_A)$, the class of all real-valued bounded functions on B_A, have the same topological pressure and the same set of Gibbs measures.

A function $\psi : E_A \to \mathbb{R}$ is called *past-independent* if for every $\tau \in C_0^{+\infty}$ and for all $\omega, \rho \in [\tau]$, we have $\psi(\omega) = \psi(\tau)$. In order to apply the results from the previous section, we need the following.

Lemma 3.3. *Every locally Hölder continuous function $\psi : E_A \to \mathbb{R}$ is cohomologous to a past-independent locally Hölder continuous function $\psi^+ : E_A \to \mathbb{R}$ in the class H_B of bounded Hölder continuous functions.*

Proof. The proof is similar to the one in [3], Lemma 1.6. For every $e \in E$, fix an arbitrary $\overline{e} \in E_A(-\infty, -1)$ such that $A_{\overline{e}_{-1}e} = 1$. Then, for every $\omega \in E_A$ put $\overline{\omega} = \overline{e}\omega|_0^{+\infty}$, note that the mapping $\omega \mapsto \overline{\omega}$ is continuous and set

$$u(\omega) = \sum_{j=0}^{\infty}(\psi(\sigma^j(\omega)) - \psi(\sigma^j(\overline{\omega})).$$

We check first that u is well-defined and continuous. Fix $\beta > 0$ so that ψ is Lipschitz continuous with respect to the metric d_β. For every $j \geq 0$, $[\sigma^j(\omega)|_{-j}^{+\infty}] = [\sigma^j(\overline{\omega})|_{-j}^{+\infty}]$. Therefore $d_\beta(\sigma^j(\omega), \sigma^j(\overline{\omega})) \leq e^{-\beta j}$, and consequently

$$|\psi(\sigma^j(\omega)) - \psi(\sigma^j(\overline{\omega}))| \leq L_\beta(\psi)e^{-\beta j}. \tag{3.3}$$

Hence, $u : E_A \to \mathbb{R}$ is well-defined and continuous. If now $d_\beta(\omega, \tau) = e^{-\beta n}$, then $[\omega|_{-n}^{n}] = [\tau|_{-n}^{n}]$. Thus, for every $0 \leq j \leq n$, $|\psi(\sigma^j(\omega)) - \psi(\sigma^j(\tau))| \leq L_\beta(\psi)d_\beta(\sigma^j(\omega), \sigma^j(\tau)) \leq L_\beta(\psi)e^{-\beta(n-j)}$, and $|\psi(\sigma^j(\overline{\tau}))-\psi(\sigma^j(\overline{\omega}))| \leq L_\beta(\psi)d_\beta(\sigma^j(\overline{\tau}), \sigma^j(\overline{\omega})) \leq L_\beta(\psi)e^{-\beta(n-j)}$. Therefore, using also (3.3), we get

$$\begin{aligned}
|u(\omega) - u(\tau)| &= \Bigg| \sum_{j=0}^{[n/2]}(\psi(\sigma^j(\omega)) - \psi(\sigma^j(\tau))) + \sum_{j=0}^{[n/2]}(\psi(\sigma^j(\overline{\tau})) - \psi(\sigma^j(\overline{\omega}))) + \\
&\quad + \sum_{j>[n/2]}(\psi(\sigma^j(\omega)) - \psi(\sigma^j(\overline{\omega}))) + \sum_{j>[n/2]}(\psi(\sigma^j(\overline{\tau})) - \psi(\sigma^j(\tau)))\Bigg| \\
&\leq \sum_{j=0}^{[n/2]}|\psi(\sigma^j(\omega)) - \psi(\sigma^j(\tau))| + \sum_{j=0}^{[n/2]}|\psi(\sigma^j(\overline{\tau})) - \psi(\sigma^j(\overline{\omega}))| + \\
&\quad + \sum_{j>[n/2]}|\psi(\sigma^j(\omega)) - \psi(\sigma^j(\overline{\omega}))| + \sum_{j>[n/2]}|\psi(\sigma^j(\overline{\tau})) - \psi(\sigma^j(\tau))| \\
&\leq 2L_\beta(\psi)\sum_{j=0}^{[n/2]}e^{-\beta(n-j)} + 2L_\beta(\psi)\sum_{j>[n/2]}e^{-\beta j} \leq 2L_\beta(\psi)\left(\frac{e^{-\beta\frac{n}{2}}}{1-e^{-\beta}} + \frac{e^{-\beta\frac{n}{2}}}{1-e^{-\beta}}\right)
\end{aligned}$$

So, the function $u : E_A \to \mathbb{R}$ is Lipschitz continuous with respect to the metric $d_{\beta/2}$, and by (3.3) it is bounded. So, $u \in \mathrm{H}_{\beta/2}$. Hence $\psi^+ = \psi - u + u \circ \sigma$ is also Lipschitz continuous with respect to the metric $d_{\beta/2}$. We are therefore left to show that ψ^+ is past-independent. So, let $\omega|_0^{+\infty} = \tau_0^{+\infty}$. Then $\overline{\omega} = \overline{\tau}$ and

$$\begin{aligned}
\psi^+(\omega) &= \psi(\omega) - \left(\sum_{j=0}^{\infty}(\psi(\sigma^j(\omega)) - \psi(\sigma^j(\overline{\omega}))\right) + \sum_{j=0}^{\infty}(\psi(\sigma^{j+1}(\omega)) - \psi(\sigma^{j+1}(\overline{\omega})) \\
&= \psi(\overline{\omega}) = \psi(\overline{\tau}) = \psi^+(\tau).
\end{aligned}$$

□

In the setting of the above lemma, let $\overline{\psi}^+$ be the factorization of ψ^+ on E_A^+, i.e. $\psi^+ = \overline{\psi}^+ \circ \pi$. As an immediate consequence of this lemma we get the following:

Lemma 3.4. *If $\psi : E_A \to \mathbb{R}$ is a locally Hölder continuous potential, then $\mathrm{P}(\psi) = \mathrm{P}(\overline{\psi}^+)$, where, we remind, the former pressure is taken with respect to the two-sided shift $\sigma : E_A \to E_A$ while the latter one is taken with respect to the one-sided shift $\sigma : E_A^+ \to E_A^+$*

If F is a subset of E, we will also denote the 2-sided shift on the symbols from F by F_A^{+-}. Then, from this lemma and Theorem 2.5, we get the following:

Theorem 3.5. *Suppose that $\psi : E_A \to \mathbb{R}$ is a locally Hölder continuous potential. Then, denoting by $\mathrm{P}_F(\psi)$ the topological pressure of $\psi|_{F_A^{+-}}$ with respect to the shift map $\sigma : F_A^{+-} \to F_A^{+-}$, we have that*

$$\mathrm{P}(\psi) = \sup\{\mathrm{P}_F(\psi)\},$$

where the supremum is taken over all finite subsets F of E; equivalently over all finite subsets F of E such that the matrix $A|_{F\times F}$ is irreducible.

We say that the function $\psi : E_A \to \mathbb{R}$ is *summable* if and only if

$$\sum_{e\in E} \exp\bigl(\sup(\psi|_{[e]})\bigr) < +\infty.$$

As in the case of one-sided shift, we have the following.

Proposition 3.6. *A locally Hölder continuous potential $\psi : E_A \to \mathbb{R}$ is summable if and only if $\mathrm{P}(\psi) < +\infty$.*

Thus from Lemma 3.3 (note that the coboundary appearing there is bounded), we get the following.

Lemma 3.7. *Every locally Hölder continuous summable function $\psi : E_A \to \mathbb{R}$ is cohomologous to a past-independent locally Hölder continuous summable function $\psi^+ : E_A \to \mathbb{R}$ in the class H_B of all bounded Hölder continuous functions.*

Now, we are ready to prove the first main result of this section:

Theorem 3.8. *For every locally Hölder continuous summable potential $\psi : E_A \to \mathbb{R}$ there exists a unique Gibbs state μ_ψ on E_A, and the measure μ_ψ is ergodic.*

Proof. Let ψ^+ be the past-independent locally Hölder continuous summable potential ascribed to ψ according to Lemma 3.7. Treating ψ^+ as defined on the one-sided symbol space E_A^+, it follows from Theorem 2.4 that there exists a unique Borel probability shift-invariant measure μ_ψ^+ on E_A^+ for which the formula (3.2) is satisfied. In addition μ_ψ^+ is ergodic. Since the measure μ_ψ^+ is shift-invariant, we conclude that the formula

$$\mu_\psi([\omega|_m^n]) = \mu_\psi^+(\sigma^m([\omega|_m^n])) = \mu_\psi^+((([|_0^{n-m}]), \ |\omega| = n - m + 1,$$

gives rise to a Borel probability shift-invariant measure μ_ψ on E_A, for which (3.2) holds. Thus μ_ψ is a Gibbs state for ψ, and it is ergodic since μ_ψ^+ was ergodic. Also if μ is a Gibbs state for ψ, then it follows from its shift-invariance and (3.2) that, $\forall n \geq 0, \forall \omega \in E_A$,

$$C^{-1} \leq \frac{\mu([\omega|_{-n}^n])}{\exp\bigl(S_{2n+1}\psi(\sigma^{-n}(\omega)) - \mathrm{P}(\psi)n\bigr)} \leq C.$$

So any two Gibbs states of ψ are equivalent and, as μ_ψ is ergodic, uniqueness follows. $\square$

Remark 3.9. *Denote by* $\pi_1 : E_A \to E_A^+$, $\pi_1(\tau) = \tau|_0^\infty, \tau \in E_A$, *the canonical projection from* E_A *onto* E_A^+. *It follows from the above proof of Theorem 3.8 that if* $\psi : E_A^+ \to \mathbb{R}$ *is a locally Hölder continuous summable potential, then*

$$\mu_\psi = \mu_{\psi\circ\pi_1} \circ \pi_1^{-1}.$$

Let us now provide a variational characterization of Gibbs states.

Theorem 3.10 (Variational Principle for Two-Sided Shifts)**.** *Suppose that* $\psi : E_A \to \mathbb{R}$ *is a locally Hölder continuous summable potential. Then*

$$\sup\left\{\mathrm{h}_\mu(\sigma) + \int_{\mathrm{E_A}} \psi \mathrm{d}\mu : \mu\circ\sigma^{-1} = \mu \ \ \textit{and} \ \ \int\psi\mathrm{d}\mu > -\infty\right\} = \mathrm{P}(\psi) = \mathrm{h}_{\mu_\psi}(\sigma) + \int_{E_A} \psi d\mu_\psi,$$

and μ_ψ *is the only measure at which this supremum is attained.*

Proof. The claim of this theorem is equivalent to the same claim with ψ replaced by the past-independent locally Hölder continuous summable potential ψ^+ resulting from Lemma 3.7. Since the dynamical system (σ, E_A), is canonically isomorphic to the Rokhlin's natural extension of (σ, E_A^+), the mapping $\mu \mapsto \mu\circ\pi^{-1}$ establishes a bijection between M_σ^{+-} and M_σ^+, which preserves entropies. Since also $\mathrm{P}(\psi) = \mathrm{P}(\overline{\psi}^+)$ by Lemma 3.4, and since for every $\mu \in M_\sigma^{+-}$, we have

$$\int_{E_A^+} \overline{\psi}^+ d\mu\pi^{-1} = \int_{E_A} \overline{\psi}^+ \circ \pi d\mu = \int_{E_A} \psi^+ d\mu,$$

we are done because of Theorem 2.6, the Variational Principle for one-sided shifts. $\square$

Any measure that realizes the supremum value in the above Variational Principle is called an **equilibrium state** for ψ. With this terminology, Theorem 3.10 can be reformulated as follows.

Theorem 3.11. *If* $\psi : E_A \to \mathbb{R}$ *is a locally Hölder continuous summable potential, then the Gibbs state* μ_ψ *is a unique equilibrium state for* ψ.

We will need however more characterizations of Gibbs states. Let

$$\mathcal{P}_- = \{[\omega|_0^{+\infty}] : \omega \in E_A\} = \{[\omega] : \omega \in E_A^+\}.$$

Obviously $\mathcal{P}_-$ is a measurable partition of E_A and two elements $\alpha, \beta \in E_A$ belong to the same element of this partition if and only if $\alpha|_0^{+\infty} = \beta|_0^{+\infty}$. If μ is a Borel probability measure on E_A, we let

$$\{\overline{\mu}^\tau : \tau \in E_A\}$$

be a *canonical system of conditional measures* induced by partition $\mathcal{P}_-$ and measure μ (see Rokhlin [26]). Each $\overline{\mu}^\tau$ is a Borel probability measure on $[\tau|_0^{+\infty}]$ and we will frequently write with no confusion $\overline{\mu}^\omega$, $\omega \in E_A^+$, to denote the corresponding conditional measure on $[\omega]$. Recall the canonical projection

$$\pi_1 : E_A \to E_A^+, \ \pi_1(\tau) = \tau|_0^\infty, \tau \in E_A,$$

The system $\{\overline{\mu}^\omega : \omega \in E_A^+\}$ of conditional measures is entirely determined by the property that

$$\int_{E_A} g\, d\mu = \int_{E_A^+} \int_{[\omega]} g\, d\overline{\mu}^\omega\, d(\mu \circ \pi_1^{-1})(\omega)$$

for every measurable function $g \in L^1(\mu)$ ([26]). It is evident from this characterization that if we change such a system on a set of zero $\mu \circ \pi_1^{-1}$-measure, then we also obtain a system of conditional measures. The canonical system of conditional measures induced by μ is uniquely defined up to a set of zero $\mu \circ \pi_1^{-1}$-measure. We say that a collection

$$\{\overline{\mu}^\omega : \omega \in E_A^+\}$$

defines a *global system of conditional measures* of μ if this is indeed a system of conditional measures of μ and a measure $\overline{\mu}^\omega$ is defined for every $\omega \in E_A^+$, rather than only on a set of full $\mu \circ \pi_1^{-1}$-measure. The first characterization of Gibbs states is the following.

Theorem 3.12. *Suppose that $\psi : E_A \to \mathbb{R}$ is a locally Hölder continuous summable potential. Let μ be a Borel probability shift-invariant measure on E_A. Then $\mu = \mu_\psi$, the unique Gibbs state for ψ if and only if there exists $D \geq 1$ such that*

$$D^{-1} \leq \frac{\overline{\mu}^\omega([\tau\omega])}{\exp\big(S_n\psi(\rho) - \mathrm{P}(\psi)n\big)} \leq D \tag{3.4}$$

for every $n \geq 1$, $\mu \circ \pi_1^{-1}$-a.e. $\omega \in E_A^+$, $\overline{\mu}^\omega$-a.e. $\tau\omega \in E_A(-n, +\infty)$ with $A_{\tau_{-1}\omega_0} = 1$, and every $\rho \in \sigma^{-n}([\tau\omega|_{-n}^{+\infty}]) = [\tau\omega|_0^{+\infty}]$. Furthermore, there exists a global system of conditional measures of μ_ψ such that

$$D^{-1} \leq \frac{\overline{\mu}_\psi^\omega([\tau\omega])}{\exp\big(S_n\psi(\rho) - \mathrm{P}(\psi)n\big)} \leq D \tag{3.5}$$

for every $\omega \in E_A^+$, every $n \geq 1$, every $\tau \in E_A(-n, -1)$ with $A_{\tau_{-1}\omega_0} = 1$, and every $\rho \in \sigma^{-n}([\tau\omega|_{-n}^{+\infty}]) = [\tau\omega|_0^{+\infty}]$.

Proof. Suppose that (3.4) holds. Also, if $\eta = (\eta_0, \ldots, \eta_{n-1})$ is an arbitrary finite word, we denote by $\eta|_j^k$ the word with elements $\eta_0, \ldots, \eta_{n-1}$ on the positions $j, \ldots, k$ for any integers $j < k$ with $j - k = n - 1$. Then for every $\omega \in E_A^+$ (note that here, for every ω, although (3.4) is assumed to hold only for $\mu \circ \pi_1^{-1}$-a.e. $\omega \in E_A^+$), and every $n \geq 1$, we get

$$\begin{aligned}
\mu([\omega|_0^{n-1}]) = \mu(\sigma^n([\omega|_0^{n-1}])) = \mu([\omega|_0^{n-1}|_{-n}^{-1}]) &= \int_{E_A^+} \overline{\mu}^\tau\big([\omega|_0^{n-1}|_{-n}^{-1}\tau]\big) d\mu \circ \pi^{-1}(\tau) \\
&= \int_{E_A^+ : A_{\omega_{n-1}\tau_0} = 1} \overline{\mu}^\tau\big([\omega|_0^{n-1}|_{-n}^{-1}\tau]\big) d\mu \circ \pi^{-1}(\tau) \\
&\asymp \int_{E_A^+ : A_{\omega_{n-1}\tau_0} = 1} \exp\big(S_n\psi\big(\omega|_{-\infty}^{n-1}\tau\big) - \mathrm{P}(\psi)n\big) d\mu \circ \pi^{-1}(\tau) \\
&\asymp \int_{E_A^+ : A_{\omega_{n-1}\tau_0} = 1} \exp\big(S_n\psi(\omega) - \mathrm{P}(\psi)n\big) d\mu \circ \pi^{-1}(\tau) \\
&\asymp \exp\big(S_n\psi(\omega) - \mathrm{P}(\psi)n\big) \sum_{e \in E : A_{\omega_{n-1}e} = 1} \mu([e])
\end{aligned} \tag{3.6}$$

Consequently,

$$\mu([\omega|_0^{n-1}]) \preceq \exp\bigl(S_n\psi(\omega) - \mathrm{P}(\psi)n\bigr). \tag{3.7}$$

In order to prove the opposite inequality notice that because of finite irreducibility of the matrix A there exists a finite set $F \subset E$ such that for every $a \in E$ there exists $b \in F$ such that $A_{ab} = 1$. Since μ is a non-zero measure, there exists $c \in E$ such that $\mu([c]) > 0$. Invoking finite irreducibility of A again, we see that for every $e \in E$ there exists a finite word α such that $e\alpha c$ is A-admissible. Put $k = |e\alpha|$. It then follows from (3.6) that

$$\mu([e]) \geq \mu([e\alpha]) \succeq \exp\bigl(S_k\psi(\rho) - \mathrm{P}(\psi)k\bigr)\mu([c]) > 0$$

for every $\rho \in [e\alpha]$. Hence $T = \min\{\mu([e]) : e \in F\} > 0$. Continuing (3.6), we obtain:

$$\mu([\omega|_0^{n-1}]) \succeq T\exp\bigl(S_n\psi(\omega) - \mathrm{P}(\psi)n\bigr).$$

Combining this with (3.7) we see that μ is a Gibbs state for the potential ψ, and the first assertion of our theorem is established.

Now, in order to complete the proof, we need to define a global system of conditional measures of μ_ψ, so that (3.5) holds for every $\omega \in E_A^+$, $n \geq 1$, $\tau \in E_A(-n,-1)$ with $A_{\tau_{-1}\omega_0} = 1$, and every $\rho \in \sigma^{-n}([\tau\omega|_{-n}^{+\infty}]) = [\tau\omega|_0^{+\infty}]$. Indeed, let

$$L : \ell_\infty \to \mathbb{R}$$

be a Banach limit. Note that

$$\begin{aligned} \frac{\mu_\psi\bigl([\tau\omega|_{-n}^{k-1}]\bigr)}{\mu_\psi\bigl(\omega|_0^{k-1}]\bigr)} &= \frac{\mu_\psi\bigl([\tau\omega|_0^{n+k-1}]\bigr)}{\mu_\psi\bigl(\omega|_0^{k-1}]\bigr)} \asymp \frac{\exp\bigl(S_{n+k}\psi(\rho) - \mathrm{P}(\psi)(n+k)\bigr)}{\exp\bigl(S_k\psi(\sigma^n(\rho) - \mathrm{P}(\psi)k\bigr)} \\ &= \exp\bigl(S_n\psi(\rho) - \mathrm{P}(\psi)n\bigr) \asymp \mu_\psi([\tau]_0^{n-1}), \end{aligned} \tag{3.8}$$

with absolute comparability constants resulting from the Gibbs property of μ_ψ, belongs to ℓ_∞. In particular, the sequence

$$\left(\frac{\mu_\psi\bigl([\tau\omega|_{-n}^{k-1}]\bigr)}{\mu_\psi\bigl(\omega|_0^{k-1}]\bigr)}\right)_{k=1}^{\infty}$$

belongs to ℓ_∞. We can therefore define

$$\overline{\mu}_\psi^\omega\bigl([\tau\omega|_{-n}^{+\infty}]\bigr) := L\left(\left(\frac{\mu_\psi\bigl([\tau\omega|_{-n}^{k-1}]\bigr)}{\mu_\psi\bigl(\omega|_0^{k-1}]\bigr)}\right)_{k=1}^{\infty}\right)$$

Now for every $g : [\omega] \to \mathbb{R}$ and every linear combination $\sum_{j=1}^{s} a_j \mathbb{1}_{[\tau^{(j)}\omega|_{-n_j}^{+\infty}]}$, the sequence

$$\frac{\mu_\psi\left(\sum_{j=1}^{s} a_j \mathbb{1}_{[\tau^{(j)}\omega|_{-n_j}^{k-1}]}\right)}{\mu_\psi\bigl(\omega|_0^{k-1}]\bigr)} \asymp \mu_\psi\left(\sum_{j=1}^{s} a_j \mathbb{1}_{[\tau^{(j)}]_0^{n_j-1}}\right), \tag{3.9}$$

belongs to ℓ_∞, with the same comparability constants as above. We can then define

$$\overline{\mu}_\psi^\omega\left(\sum_{j=1}^s a_j \mathbb{1}_{[\tau^{(j)}\omega|_{-n_j}^{+\infty}]}\right) := L\left(\left(\frac{\mu_\psi\left(\sum_{j=1}^s a_j \mathbb{1}_{[\tau^{(j)}\omega|_{-n_j}^{k-1}]}\right)}{\mu_\psi\left(\omega|_0^{k-1}]\right)}\right)_{k=1}^\infty\right)$$

So, we have defined a function $\overline{\mu}_\psi^\omega$ from the vector space $\mathcal{V}$ of all linear combinations as above to $\mathbb{R}$. Since the Banach limit is a positive linear operator, so is the function $\overline{\mu}_\psi^\omega : \mathcal{V} \to \mathbb{R}$. Furthermore, because of the monotonicity of Banach limits and of (3.9), $\overline{\mu}_\psi^\omega(g_n) \searrow 0$ whenever $(g_n)_{n=1}^\infty$ is a monotone decreasing sequence of functions in $\mathcal{V}$ converging pointwise to 0. Therefore, Daniell-Stone Theorem gives a unique Borel probability measure on $[\omega]$, whose restriction to $\mathcal{V}$ coincides with $\overline{\mu}_\psi^\omega$. We keep the same symbol $\overline{\mu}_\psi^\omega$ for this extension. Now, it follows from Martingale's Theorem that for $\mu_\psi \circ \pi_1^{-1}$-a.e. $\omega \in E_A^+$ and every $\tau \in E_A(-n,-1)$ with $A_{\tau_{-1}\omega_0} = 1$ the limit

$$\lim_{k\to\infty} \frac{\mu_\psi\left([\tau\omega|_{-n}^{k-1}]\right)}{\mu_\psi\left(\omega|_0^{k-1}]\right)}$$

exists and coincides with the conditional measure of μ_ψ on $[\omega]$. By properties of Banach limits,

$$\frac{\mu_\psi\left([\tau\omega|_{-n}^{k-1}]\right)}{\mu_\psi\left(\omega|_0^{k-1}]\right)} = \lim_{k\to\infty} \frac{\mu_\psi\left([\tau\omega|_{-n}^{k-1}]\right)}{\mu_\psi\left(\omega|_0^{k-1}]\right)},$$

and we thus conclude that the collection

$$\{\overline{\mu}_\psi^\omega : \omega \in E_A^+\}$$

is indeed a global system of conditional measures of μ_ψ. Invoking now formula (3.8) (particularly its middle line), completes the proof.

□

Similarly, let

$$\mathcal{P}_+ = \{[\omega|_{-\infty}^{-1}] : \omega \in E_A\},$$

and given a Borel probability measure μ on E_A, let $\{\mu^{+\omega} : \omega \in E_A\}$ the corresponding canonical system of conditional measures. As in Theorem 3.12, we prove the following.

Theorem 3.13. *Suppose that $\psi : E_A \to \mathbb{R}$ is a locally Hölder continuous summable potential. Let μ be a Borel probability shift-invariant measure on E_A. Then $\mu = \mu_\psi$, the unique Gibbs state for ψ if and only if there exists $D \geq 1$ such that*

$$D^{-1} \leq \frac{\mu^{+\omega}([\omega\tau|_{-\infty}^{n-1}])}{\exp\big(S_n\psi(\rho) - \mathrm{P}(\psi)n\big)} \leq D \tag{3.10}$$

for every $\omega \in E_A(-\infty,-1)$, every $n \geq 1$, every $\tau \in E_A(0,n-1)$ with $A_{\omega_{-1}\tau_0} = 1$, and every $\rho \in [\omega\tau|_{-\infty}^{n-1}]$.

Noting that $\overline{(q\gamma + t\xi)}^+ = q\overline{\gamma}^+ + t\overline{\xi}^+$, as an immediate consequence of Lemma 3.4 and respectively Theorem 2.10 and Proposition 2.11, we get the following two results.

Theorem 3.14. *If $\gamma, \xi : E_A \to \mathbb{R}$ are locally Hölder continuous functions, then the function $\Sigma(\gamma,\xi) \ni (q,t) \mapsto \mathrm{P}(q\gamma + t\xi)$, is real-analytic.*

Proposition 3.15. *If $\gamma, \xi : E_A \to \mathbb{R}$ are locally Hölder continuous potentials, then for all $(q_0, t_0) \in \Sigma(\gamma, \xi)$,*

$$\frac{\partial}{\partial q}\Big|_{(q_0,t_0)} \mathrm{P}(q\gamma + t\xi) = \int \gamma \, d\mu_{q_0\gamma+t_0\xi}, \quad \frac{\partial}{\partial t}\Big|_{(q_0,t_0)} \mathrm{P}(q\gamma + t\xi) = \int \xi \, d\mu_{q_0\gamma+t_0\xi}, \text{ and,}$$

$$\frac{\partial^2}{\partial q \partial t}\Big|_{(q_0,t_0)} P(q\gamma + t\xi) = \sigma^2_{\mu_{q_0\gamma+t_0\xi}},$$

where $\mu_{q_0\gamma+t_0\xi}$ is the unique equilibrium state of the potential $q_0\gamma + t_0\xi$ and $\sigma^2_{\mu_{q_0\gamma+t_0\xi}}$ is the asymptotic covariance of the pair (γ, ξ) with respect to the measure $\mu_{q_0\gamma+t_0\xi}$.

4. Skew Product Smale Spaces of Countable Type

We keep the notation from the previous two sections.

Definition 4.1. *Let (Y, d) be a complete bounded metric space, and take for every $\omega \in E_A^+$ an arbitrary set $Y_\omega \subset Y$ and a continuous injective map $T_\omega : Y_\omega \to Y_{\sigma(\omega)}$. Define*

$$\hat{Y} := \bigcup_{\omega \in E_A^+} \{\omega\} \times Y_\omega \subset E_A^+ \times Y.$$

Define the map $T : \hat{Y} \to \hat{Y}$ by the formula

$$T(\omega, y) = (\sigma(\omega), T_\omega(y)).$$

The pair $(\hat{Y}, T : \hat{Y} \to \hat{Y})$ is called a skew product Smale endomorphism, if T is fiberwise uniformly contracting, i.e $\exists \lambda > 1$ so that $\forall \omega \in E_A^+$ and all $y_1, y_2 \in Y_\omega$,

$$d(T_\omega(y_2), T_\omega(y_1)) \le \lambda^{-1} d(y_2, y_1) \tag{4.1}$$

Note that for every $\tau \in E_A(-n, +\infty)$ the composition

$$T_\tau^n = T_{\tau|_{-1}^{+\infty}} \circ T_{\tau|_{-2}^{+\infty}} \circ \ldots \circ T_{\tau|_{-n}^{+\infty}} : Y_\tau \to Y_{\tau|_0^{+\infty}}$$

is well-defined. Then for every $\tau \in E_A$ we define the map $T_\tau^n := T^n_{\tau|_{-n}^{+\infty}} := T_{\tau|_{-1}^{+\infty}} \circ T_{\tau|_{-2}^{+\infty}} \circ \ldots \circ T_{\tau|_{-n}^{+\infty}} : Y_{\tau|_{-n}^{+\infty}} \to Y_{\tau|_0^{+\infty}}$. The sequence $\big(T_\tau^n\big(Y_{\tau|_{-n}^{+\infty}}\big)\big)_{n=0}^\infty$ consists of descending sets, and

$$\mathrm{diam}\big(T_\tau^n\big(Y_{\tau|_{-n}^{+\infty}}\big)\big) \le \lambda^{-n} \mathrm{diam}(Y). \tag{4.2}$$

The same is also true for the closures of these sets, so since (Y, d) is complete, we have that

$$\bigcap_{n=1}^\infty \overline{T_\tau^n\big(Y_{\tau|_{-n}^{+\infty}}\big)}$$

is a singleton. Denote its only element by $\hat{\pi}_2(\tau)$. So, we have defined the map

$$\hat{\pi}_2 : E_A \to Y,$$

and next define $\hat{\pi} : E_A \to E_A^+ \times Y$ by the formula

$$\hat{\pi}(\tau) = (\tau|_0^{+\infty}, \hat{\pi}_2(\tau)), \tag{4.3}$$

and the truncation to the elements of non-negative indices by

$$\pi_1 : E_A \to E_A^+, \ \pi_1(\tau) = \tau|_0^{+\infty}$$

In the notation for π_1 we drop the hat symbol, as π_1 is independent of the skew product on $\hat{Y}$. For all $\omega \in E_A^+$ we also define the $\hat{\pi}_2$-projection of the cylinder $[\omega] \subset E_A$, namely

$$J_\omega := \hat{\pi}_2([\omega]) \in Y,$$

and call these sets the *stable Smale fibers* of the system T (or simply *the fibers* of the Smale system). The global invariant set

$$J := \hat{\pi}(E_A) = \bigcup_{\omega \in E_A^+} \{\omega\} \times J_\omega \subset E_A^+ \times Y,$$

is called the **Smale space** induced by the Smale system T. This is different from the notion of Smale space of [28]. For each $\tau \in E_A$ we have $\hat{\pi}_2(\tau) \in \overline{Y}_{\tau|_0^{+\infty}}$; therefore $J_\omega \subset \overline{Y}_\omega$, for every $\omega \in E_A^+$. Since all the maps $T_\omega : Y_\omega \to Y_{\sigma(\omega)}$ are Lipschitz continuous with a Lipschitz constant λ^{-1}, all of them extend uniquely to continuous maps from $\overline{Y}_\omega$ to $\overline{Y}_{\sigma(\omega)}$ and these extensions are Lipschitz continuous with a Lipschitz constant λ^{-1}. We prove the following.

Proposition 4.2. *For every $\omega \in E_A^+$ we have that*

$$T_\omega(J_\omega) \subset J_{\sigma(\omega)}, \ \textit{and} \ \bigcup_{\substack{e \in E \\ A_{e\omega_0}=1}} T_{e\omega}(J_{e\omega}) = J_\omega, \ \textit{and} \tag{4.4}$$

$$T \circ \hat{\pi} = \hat{\pi} \circ \sigma, \tag{4.5}$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} E_A & \xrightarrow{\sigma} & E_A \\ \hat{\pi}\big\downarrow & & \big\downarrow\hat{\pi} \\ \hat{Y} & \xrightarrow{T} & \hat{Y}. \end{array}$$

Proof. First we prove formula (4.4), let $y \in J_\omega$. Then there exists $\tau \in E_A(-\infty, -1)$ such that $A_{\tau_{-1}\omega_0} = 1$ and $y = \hat{\pi}_2(\tau\omega)$. Then

$$\begin{aligned} \{T_\omega(y)\} &= T_\omega\Big(\bigcap_{n=1}^{+\infty} \overline{T_{\tau\omega}^n(Y_{\tau|_{-n}^{-1}\omega})}\Big) \subset \bigcap_{n=1}^{+\infty} T_\omega\big(\overline{T_{\tau\omega}^n(Y_{\tau|_{-n}^{-1}\omega})}\big) \subset \bigcap_{n=1}^{+\infty} \overline{T_\omega\big(T_{\tau\omega}^n(Y_{\tau|_{-n}^{-1}\omega})\big)} \\ &= \bigcap_{n=1}^{+\infty} \overline{T_{\tau\omega}^{n+1}(Y_{\tau|_{-n}^{-1}\omega}))} = \bigcap_{n=1}^{+\infty} \overline{T^{n+1}_{\tau|_{-\infty}^{-1}\omega_0(\sigma(\omega))}\big(Y_{\tau|_{-\infty}^{-1}\omega_0(\sigma(\omega))}\big)} \\ &= \big\{\hat{\pi}_2\big(\tau|_{-\infty}^{-1}\omega_0(\sigma(\omega))\big)\big\} \subset J_{\sigma(\omega)} \end{aligned} \tag{4.6}$$

Thus $T_\omega(J_\omega) \subset J_{\sigma(\omega)}$ and, as $\{T_\omega(y)\}$ and $\big\{\hat{\pi}_2\big(\tau|_{-\infty}^{-1}\omega_0(\sigma(\omega))\big)\big\}$, the respective sides of (4.6), are singletons, we therefore get

$$T_\omega \hat{\pi}_2(\tau\omega) = \hat{\pi}_2 \circ \sigma(\tau\omega), \tag{4.7}$$

meaning that (4.5) holds. The inclusion $\bigcup_{\substack{e\in E\\ A_{e\omega_0}=1}} T_{e\omega}(J_{e\omega}) \subset J_\omega$ holds because of (4.4). In order to prove the opposite one, let $z \in J_\omega$. Then $z = \hat{\pi}(\gamma\omega)$ with some $\gamma \in E_A(-\infty, -1)$, where $A_{\gamma_{-1}\omega_0} = 1$. Formula (4.7) then yields

$$z = \hat{\pi}_2 \circ \sigma\big(\gamma|_{-\infty}^{-2}\gamma_{-1}|_{-\infty}^{0}\omega\big) = T_{\gamma_{-1}\omega} \circ \hat{\pi}_2\big(\gamma|_{-\infty}^{-2}\gamma_{-1}|_{-\infty}^{0}\omega\big) \in T_{\gamma_{-1}\omega}\big(J_{\gamma_{-1}\omega}\big)$$

This means that $J_\omega \subset \bigcup_{\substack{e\in E\\ A_{e\omega_0}=1}} T_{e\omega}(J_{e\omega})$, and formula (4.4) is proved. $\square$

The proof of the following generalization of (4.4) is straightforward.

$$J_\omega = \bigcup_{\substack{\tau\in E_A^n\\ A_{\tau_n\omega_0}=1}} T_{\tau\omega}\big(J_{\tau\omega}\big)$$

for all $\omega \in E_A^+$, and $n > 0$. By formula (4.4) we have $T(J) \subset J$, so we may consider the dynamical system

$$T : J \to J$$

and we call it the *skew product Smale endomorphism* generated by the Smale system $T : \hat{Y} \to \hat{Y}$. By formula (4.4) we have the following.

Observation 4.3. *The map $T : J \to J$ is surjective.*

Let us now record another straightforward but important observation.

Observation 4.4. *If $T : \hat{Y} \to \hat{Y}$ is a skew product Smale system, then the following statements are equivalent:*

(a) *For every $\xi \in J$, the fiber $\hat{\pi}^{-1}(\xi) \subset E_A$ is compact.*

(b) *For every $y \in Y$, the fiber $\hat{\pi}_2^{-1}(y) \subset E_A$ is compact.*

(c) *For every $\xi = (\omega, y) \in J$, the set $\big\{e \in E : A_{e\omega_0} = 1 \text{ and } y \in T_{e\omega}(J_{e\omega})\big\}$ is finite.*

If either of these three above conditions is satisfied, we call the skew product Smale system $T : J \to J$ of *compact type.*

Remark 4.5. *In item (a) of Observation 4.4 one can replace J by $\hat{Y}$.*

Observation 4.6. *If for every $y \in Y$ the set*

$$\big\{e \in E : A_{e\omega_0} = 1 \text{ and } y \in T_{e\omega}(J_{e\omega})\big\}$$

is finite for every $\omega \in E_A^+$, then $T : J \to J$ is of compact type.

From now on we assume $T : \hat{Y} \to \hat{Y}$ to be a skew product Smale system of compact type.

If for every $\xi \in \hat{Y}$ (or in J), the fiber $\hat{\pi}^{-1}(\xi) \subset E_A$ is finite, we call the skew product Smale system T of *finite type.* Let us record an easy observation.

Observation 4.7. *If the skew product Smale system $T : \hat{Y} \to \hat{Y}$ is of finite type, then it is also of compact type.*

The Smale system $T : \hat{Y} \to \hat{Y}$ is called of *bijective type* if, for every $\xi \in J$ the fiber $\hat{\pi}^{-1}(\xi)$ is a singleton. Equivalently, the map $\hat{\pi} : E_A \to J$ is injective; then also $T : J \to J$ is bijective. A Smale skew product of bijective type is clearly of finite type, and thus of compact type.

Definition 4.8. *We call a Smale endomorphism continuous if the global map $T : J \to J$ is continuous with respect to the relative topology inherited from $E_A^+ \times Y$.*

Later in this section we will provide a construction giving rise to continuous Smale endomorphisms, in fact all of them will be Hölder continuous.

Lemma 4.9. *For every $n \geq 1$ and every $\tau \in E_A(-n, +\infty)$, we have*

$$\hat{\pi}_2\big([\tau]\big) = T_\tau^n\big(J_\tau\big). \tag{4.8}$$

Equivalently for every $\tau \in E_A$, we have that

$$\hat{\pi}_2\big([\tau|_{-n}^{+\infty}]\big) = T_\tau^n\big(J_{\tau|_{-n}^{+\infty}}\big). \tag{4.9}$$

Proof. Using formula (4.5) we get $T_\tau^n\big(J_{\tau|_{-n}^{+\infty}}\big) = T_\tau^n \circ \hat{\pi}_2\big([\tau|_{-n}^{+\infty}]|_0^{+\infty}\big) = \hat{\pi}_2 \circ \sigma^n\big([\tau|_{-n}^{+\infty}]|_0^{+\infty}\big) = \hat{\pi}_2\big([\tau|_{-n}^{+\infty}]\big)$. □

As an immediate consequence of (4.2), we get the following

Observation 4.10. *For every $\omega \in E_A$, the map*

$$[\omega]_0^{+\infty} \ni \tau \mapsto \hat{\pi}_2(\tau) \in J_{\omega|_0^\infty} \subset Y$$

is Lipschitz continuous if E_A is endowed with the metric $d_{\lambda^{-1}}$. In consequence, it is Hölder continuous with respect to any metric d_β, $\beta > 0$, on E_A.

Note that for every $\tau \in E_A^n$, $n \geq 1$

$$\hat{\pi}([\tau]) = \bigcup_{\omega \in [\tau]} \{\omega\} \times J_\omega.$$

Let $M(E_A)$ be the topological space of all Borel probability measures on E_A endowed with the topology of weak convergence, and $M_\sigma(E_A)$ be its closed subspace consisting of σ-invariant measures. Likewise, let $M(J)$ be the topological space of all Borel probability measures on J endowed with the topology of weak convergence, and let $M_T(J)$ be its closed subspace consisting of T-invariant measures. First we recall the following fact, which is well known in measure theory.

Lemma 4.11. *Let W and Z be Polish spaces. Let μ be a Borel probability measure on Z, let $\hat{\mu}$ be its completion, and denote by $\hat{\mathcal{B}}_\mu$ the complete σ-algebra of all $\hat{\mu}$-measurable subsets of Z. Let $f : W \to Z$ be a Borel measurable surjection and let $g : W \to \overline{\mathbb{R}}$ be a Borel measurable function. Define the functions $g_*, g^* : Z \to \overline{\mathbb{R}}$ respectively by*

$$g_*(z) := \inf\{g(w) : w \in f^{-1}(z)\} \text{ and } g^*(z) := \sup\{g(w) : w \in f^{-1}(z)\}.$$

Then these two functions are measurable with respect to the σ-algebra $\hat{\mathcal{B}}_\mu$. If in addition the map $f : W \to Z$ is locally 1-to-1, then both g_* and $g^* : Z \to \overline{\mathbb{R}}$ are Borel measurable.

We now prove the following:

Theorem 4.12. *If $T : J \to J$ is a continuous skew product Smales endomorphism of compact type, then the map*

$$M_\sigma(E_A) \ni \mu \longmapsto \mu \circ \hat{\pi}^{-1} \in M_T(J)$$

is surjective.

Proof. Fix $\mu \in M_T(J)$. Let $\mathcal{B}_b(E_A)$ and $\mathcal{B}_b(J)$ be the vector spaces of all bounded Borel measurable real-valued functions defined respectively on E_A and on J. Let $\mathcal{B}_b^+(E_A)$ and $\mathcal{B}_b^+(J)$ be the respective convex cones consisting of non-negative functions. Define also:

$$\hat{\mathcal{B}}_b(E_A) := \{g \circ \hat{\pi} : g \in \mathcal{B}_b(J)\}.$$

Clearly $\hat{\mathcal{B}}_b(E_A)$ is a vector subspace of $\mathcal{B}_b(E_A)$ and, as $\hat{\pi} : E_A \to J$ is a surjection, for each $h \in \hat{\mathcal{B}}_b(E_A)$ there exists a unique $g \in \mathcal{B}_b(J)$ such that $h = g \circ \hat{\pi}$. Thus, treating, via integration, μ as a linear functional from $\mathcal{B}_b(J)$ to $\mathbb{R}$, the formula

$$\hat{\mathcal{B}}_b(E_A) \ni g \circ \hat{\pi} \longmapsto \hat{\mu}(g \circ \hat{\pi}) := \mu(g) \in \mathbb{R},$$

defines a positive linear functional from $\hat{\mathcal{B}}_b(E_A)$ to $\mathbb{R}$. Since, by Lemma 4.11 applied to the map f being equal to $\hat{\pi} : E_A \to J$, for every $h \in \mathcal{B}_b(E_A)$, the function $h_* \circ \hat{\pi} : E_A \to \mathbb{R}$ belongs to $\hat{\mathcal{B}}_b(E_A)$. Since $h - h_* \circ \hat{\pi} \geq 0$, thus $h - h_* \circ \hat{\pi} \in \mathcal{B}_b^+(E_A)$, the Riesz Extension Theorem applies to produce a positive linear functional $\mu^* : \mathcal{B}_b(E_A) \to \mathbb{R}$ such that:

$$\mu^*(h) = \hat{\mu}(h)$$

for every $h \in \hat{\mathcal{B}}_b(E_A)$. But μ^* restricted to the vector space $C_b(E_A)$ of all bounded continuous real-valued functions on E_A, remains linear and positive. We prove the following.

Claim 1^0: If $(g_n)_{n=1}^\infty$ is a monotone decreasing sequence of non-negative functions in $C_b(E_A)$ converging pointwise in E_A to the function identically equal to zero, then $\lim_{n\to\infty} \mu^*(g_n)$ exists and is equal to zero.

Proof. Clearly, $(g_n^*)_{n=1}^\infty$ is a monotone decreasing sequence of non-negative bounded functions that, by Lemma 4.11, all belong to $\mathcal{B}(J)$, thus to $\mathcal{B}_b^+(J)$. Fix $y \in J$. Since our map $T : J \to J$ is of compact type, the set $\hat{\pi}^{-1}(y) \subset E_A$ is compact. Therefore Dini's Theorem applies to let us conclude that the sequence $\left(g_n|_{\hat{\pi}^{-1}(y)}\right)_{n=1}^\infty$ converges uniformly to zero. Since all these functions are non-negative, this just means that the sequence $(g_n^*)_{n=1}^\infty$ converges to zero. In conclusion $(g_n^*)_{n=1}^\infty$ is a monotone decreasing sequence of functions in $\mathcal{B}_b^+(J)$ converging pointwise to zero. Therefore, as also $g_n \leq g_n^* \circ \hat{\pi}$, we get

$$0 \leq \varlimsup_{n\to\infty} \mu^*(g_n) \leq \varlimsup_{n\to\infty} \mu^*(g_n^* \circ \hat{\pi}) = \varlimsup_{n\to\infty} \hat{\mu}(g_n^* \circ \hat{\pi}) = \varlimsup_{n\to\infty} \mu(g_n^*) = 0.$$

So, the limit $\lim_{n\to\infty} \mu^*(g_n)$ exists and is equal to zero. The proof of Claim 1^0 is complete. $\square$

Having Claim 1^0, Daniell-Stone Representation Theorem applies to tell us that μ^* extends uniquely from $C_b(E_A)$ to an element of $M(E_A)$. We denote it by the same symbol μ^*. Now we shall prove the following.

Claim 2^0: For every $\varepsilon > 0$ there exists K_ε, a compact subset of E_A such that $\hat{\pi}^{-1}(\hat{\pi}(K_\varepsilon)) = K_\varepsilon$ and $\mu(\hat{\pi}(K_\varepsilon)) \geq 1 - \frac{\varepsilon}{2}$.

Proof. Fix any $k \in \mathbb{Z}$ and denote by $p_k : E^{+-} \to E$ is the canonical projection on the kth coordinate, i.e.

$$p_k\big((\gamma_n)_{n=-\infty}^{+\infty}\big) = \gamma_k.$$

Fix $\varepsilon > 0$. In the sequel we will assume without loss of generality that $E = \{1, 2, \dots\}$. Since the map $T : J \to J$ is of compact type, each set $\hat{\pi}^{-1}(y) \subset E_A$, $y \in J$, is compact, and consequently, the function $p_k^* : J \to \overline{\mathbb{R}}$, defined in Lemma 4.11, takes values in $\mathbb{R}$. So we have the function $p_k^* : J \to E$ which, by Lemma 4.11, is Borel measurable; thus $p_k^* \circ \hat{\pi} : E_A \to \mathbb{N}$ is also Borel measurable. Hence, there exists $n_k \geq 1$ such that

$$\mu\big((p_k^*)^{-1}([n_k + 1, +\infty))\big) < 2^{-|k|-4}\varepsilon. \tag{4.10}$$

Since the measure μ is inner (closed sets) regular, by Lusin's Theorem Borel measurability of the function $p_k^* : J \to \mathbb{N}$ yields the existence of closed subsets $J_k \subset J$ such that $\mu(J_k) \geq 1 - \varepsilon 2^{-|k|-4}$ and the restriction $p_k^*|_{J_k} : J_k \to \mathbb{N}$ is continuous. Define

$$J_\infty := \bigcap_{k \in \mathbb{Z}} J_k.$$

Then J_∞ is a closed subset of J,

$$\mu(J_\infty) \geq 1 - \frac{\varepsilon}{4}, \tag{4.11}$$

and each map $p_k^*|_{J_\infty} : J_\infty \to \mathbb{N}$ is continuous. Define

$$K_\varepsilon := \bigcap_{k \in \mathbb{Z}} \big(p_k^*|_{J_\infty} \circ \hat{\pi}|_{\hat{\pi}^{-1}(J_\infty)}\big)^{-1}([1, n_k]).$$

By the definition of the maps p_k^* we have that

$$\hat{\pi}^{-1}(\hat{\pi}(K_\varepsilon)) = K_\varepsilon \tag{4.12}$$

and

$$\hat{\pi}(K_\varepsilon) = J_\infty \cap \bigcap_{k \in \mathbb{Z}} (p_k^*)^{-1}([1, n_k]).$$

Therefore, employing (4.11) and (4.10), we get

$$\mu(J \setminus \hat{\pi}(K_\varepsilon)) \leq \mu(J \setminus J_\infty) + \sum_{k \in \mathbb{Z}} \mu\big((p_k^*)^{-1}([n_k + 1, +\infty))\big) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \tag{4.13}$$

Since all the maps $p_k^*|_{J_\infty}$, $k \in \mathbb{Z}$ are continuous, K_ε is a closed subset of E_A. But $K_\varepsilon \subset \prod_{k \in \mathbb{Z}} [1, n_k]$ and since this Cartesian product is compact, we can conclude that K_ε is compact. Along with (4.12) and (4.13) this completes the proof of Claim 2^0.

□

Using the T-invariance of μ and Urysohn's Approximation Method, we prove the following:

Claim 3^0: If $\varepsilon > 0$ and $K_\varepsilon \subset E_A$ is the compact set produced in Claim 2^0, then

$$\mu^* \circ \sigma^{-j}(K_\varepsilon) \geq 1 - \varepsilon$$

for all integers $j \geq 0$.

Proof. Fix $\varepsilon > 0$ arbitrary. Fix an integer $j \geq 0$. Since the measure $\mu^* \circ \sigma^{-j} \circ \hat{\pi}^{-1}$ is outer regular, and $\hat{\pi}(K_\varepsilon)$ is a compact set, there exists an open set $U \subset J$ such that

$$\hat{\pi}(K_\varepsilon) \subset U \quad \text{and} \quad \mu^* \circ \sigma^{-j} \circ \hat{\pi}^{-1}\big(U \setminus \hat{\pi}(K_\varepsilon)\big) \leq \varepsilon/2.$$

Now, Urysohn's Lemma produces a continuous function $u : J \to [0,1]$ such that $u|_{\hat{\pi}(K_\varepsilon)} \equiv 1$ and $u(E_A \setminus U) \subset \{0\}$. Then, by our construction of μ^* and by Claim 2^0, we get

$$\begin{aligned}
\mu^* \circ \sigma^{-j}(K_\varepsilon) &= \mu^* \circ \sigma^{-j} \circ \hat{\pi}^{-1}\big(\hat{\pi}(K_\varepsilon)\big) \geq \mu^* \circ \sigma^{-j} \circ \hat{\pi}^{-1}(U) - \frac{\varepsilon}{2} \\
&= \mu^*(1\!\!1_U \circ \hat{\pi} \circ \sigma^j) - \frac{\varepsilon}{2} = \mu^*(1\!\!1_U \circ T^j) - \frac{\varepsilon}{2} \geq \mu^*(u \circ T^j \circ \hat{\pi}) - \frac{\varepsilon}{2} \\
&= \hat{\mu}\big((u \circ T^j) \circ \hat{\pi}\big) - \frac{\varepsilon}{2} = \mu(u \circ T^j) - \frac{\varepsilon}{2} = \mu(u) - \frac{\varepsilon}{2} \geq \mu(\hat{\pi}(K_\varepsilon)) - \frac{\varepsilon}{2} \geq 1 - \varepsilon.
\end{aligned}$$

□

Now, for every $n \geq 1$ set

$$\mu_n^* := \frac{1}{n}\sum_{j=0}^{n-1} \mu^* \circ \sigma^{-j}.$$

It directly follows from Claim 3^0 that

$$\mu_n^*(K_\varepsilon) \geq 1 - \varepsilon$$

for every $\varepsilon > 0$ and all $n \geq 1$. Also, since, by Claim 2^0, each set K_ε is compact, the sequence of measures $(\mu_n^*)_{n=1}^\infty$ is tight with respect to the weak topology on $M_\sigma(E_A)$. There thus exists $(n_k)_{k=1}^\infty$, an increasing sequence of positive integers such that $(\mu_{n_k}^*)_{k=1}^\infty$ converges weakly, and denote its limit by $\nu \in M(E_A)$. A standard argument shows that $\nu \in M_\sigma(E_A)$. By the definitions of $\hat{\mu}$ and μ^*, we get for every $g \in C_b(E_A)$, and every $n \geq 1$, that

$$\begin{aligned}
\mu_n^* \circ \hat{\pi}^{-1}(g) = \mu_n^*(g \circ \hat{\pi}) &= \frac{1}{n}\sum_{j=0}^{n-1} \mu^* \circ \sigma^{-j}(g \circ \hat{\pi}) = \frac{1}{n}\sum_{j=0}^{n-1} \mu^*(g \circ \hat{\pi} \circ \sigma^j) \\
&= \frac{1}{n}\sum_{j=0}^{n-1} \mu^*\big((g \circ T^j) \circ \hat{\pi}\big) = \frac{1}{n}\sum_{j=0}^{n-1} \hat{\mu}\big((g \circ T^j) \circ \hat{\pi}\big) \\
&= \frac{1}{n}\sum_{j=0}^{n-1} \mu(g \circ T^j) = \frac{1}{n}\sum_{j=0}^{n-1} \mu(g) = \mu(g).
\end{aligned}$$

Therefore $\mu_n^* \circ \hat{\pi}^{-1} = \mu$ for every $n \geq 1$, hence,

$$\nu \circ \hat{\pi}^{-1} = \big(\lim_{k\to\infty} \mu_{n_k}^*\big) \circ \hat{\pi}^{-1} = \lim_{k\to\infty}\big(\mu_{n_k}^* \circ \hat{\pi}^{-1}\big) = \mu.$$

□

Let us now record the following straightforward but important observation.

Observation 4.13. *If T is a Smale endomorphism and $\mu \in M_\sigma(E_A)$, then*

$$\mathrm{h}_{\mu\circ\hat{\pi}^{-1}}(\mathrm{T}) = \mathrm{h}_\mu(\sigma).$$

Proof. It is standard, we include it for completeness. We have the well-known inequalities:

$$\mathrm{h}_{\mu\circ\hat{\pi}^{-1}}(\mathrm{T}) \le \mathrm{h}_{\mu}(\sigma) \ \text{ and } \ \mathrm{h}_{\mu\circ\hat{\pi}^{-1}\circ\pi_1^{-1}}(\sigma) \le \mathrm{h}_{\mu\circ\hat{\pi}^{-1}}(\mathrm{T}). \tag{4.14}$$

But $\pi_1 : E_A \to E_A^+$, $\pi_1(\tau) = \tau|_0^\infty$ is the canonical projection from E_A to E_A^+. So, $\mu \in M_\sigma(E_A)$ is the Rokhlin's natural extension of the measure $\mu\circ\hat{\pi}^{-1}\circ\pi_1^{-1} \in M_\sigma(E_A^+)$. Hence, $\mathrm{h}_{\mu\circ\hat{\pi}^{-1}\circ\pi_1^{-1}}(\sigma) = \mathrm{h}_\mu(\sigma)$. Along with (4.14) this implies that $\mathrm{h}_{\mu\circ\hat{\pi}^{-1}}(\mathrm{T}) = \mathrm{h}_\mu(\sigma)$. □

Now, we define the topological pressure of continuous real-valued functions on J with respect to the dynamical system $T : J \to J$. Since the space J is **not compact**, there is no canonical candidate for such definition. We choose the definition which will turn out to behave well on the theoretical level (variational principle), and serves well for practical purposes (Bowen's formula). For every finite admissible word $\omega \in E_A^{+*}$ let

$$[\omega]_T =: \hat{\pi}_2([\omega]) \subset J.$$

If $\psi : J \to \mathbb{R}$ is a continuous function, we define

$$\mathrm{P}(\psi) = \mathrm{P}_T(\psi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\bigl(\sup(S_n\psi|_{[\omega]_T}\bigr), \ \text{where,}$$

$$S_n\psi = \sum_{j=0}^{n-1} \psi\circ T^j, \quad n \ge 1$$

The limit above exists, since the sequence $\log\sum_{\omega\in C^{n-1}}\exp\bigl(\sup(S_n\psi|_{[\omega]_T}\bigr), n \in \mathbb{N}$, is subadditive. We call $\mathrm{P}_T(\psi)$ the *topological pressure* of the potential $\psi : J \to \mathbb{R}$ with respect to the dynamical system $T : J \to J$. As an immediate consequence of this definition and Definition 3.1, we get the following:

Observation 4.14. *If $\psi : J \to \mathbb{R}$ is a continuous function, then*

$$\mathrm{P}_T(\psi) = \mathrm{P}_\sigma(\psi\circ\hat{\pi}).$$

The following theorem follows immediately from Theorem 3.10, Observation 4.14, Theorem 4.12, and Observation 4.13.

Theorem 4.15. *If $\psi : J \to \mathbb{R}$ is a continuous function, and $\mu \in M_T(J)$ is such that $\psi \in L^1(J,\mu)$ and $\int \psi\, d\mu > -\infty$, then $\mathrm{h}_\mu(\mathrm{T}) + \int_{\mathrm{J}} \psi\, \mathrm{d}\mu \le \mathrm{P}_{\mathrm{T}}(\psi)$.*

We adopt the following two definitions.

Definition 4.16. *The measure $\mu \in M_T(J)$ is called an equilibrium state of the continuous potential $\psi : J \to \mathbb{R}$, if $\int \psi\, d\mu > -\infty$ and*

$$\mathrm{h}_\mu(\mathrm{T}) + \int_{\mathrm{J}} \psi\, \mathrm{d}\mu = \mathrm{P}_{\mathrm{T}}(\psi).$$

Definition 4.17. *The potential $\psi : J \to \mathbb{R}$ is called summable if*

$$\sum_{e\in E} \exp\bigl(\sup(\psi|_{[e]_T})\bigr) < +\infty.$$

As an immediate observation we get the following.

Observation 4.18. *A potential $\psi : J \to \mathbb{R}$ is summable if and only if $\psi \circ \hat{\pi} : E_A \to \mathbb{R}$ is summable.*

Definition 4.19. *We call a continuous skew product Smale endomorphism $T : \hat{Y} \to \hat{Y}$ Hölder, if the projection $\hat{\pi} : E_A \to J$ is Hölder continuous.*

We shall now first establish an important property of Hölder skew product Smale endomorphisms of compact type, and then will describe a fairly general construction of such endomorphisms.

Theorem 4.20. *If $T : \hat{Y} \to \hat{Y}$ is Hölder skew product Smale endomorphism of compact type and $\psi : J \to \mathbb{R}$ is a locally Hölder summable potential, then ψ admits a unique equilibrium state, denoted by μ_ψ. In addition*

$$\mu_\psi = \mu_{\psi\circ\hat{\pi}} \circ \hat{\pi}^{-1},$$

where $\mu_{\psi\circ\hat{\pi}}$ is the unique equilibrium state of $\psi \circ \hat{\pi} : E_A \to \mathbb{R}$ with respect to $\sigma : E_A \to E_A$.

Proof. By our hypotheses, $\psi \circ \hat{\pi} : E_A \to \mathbb{R}$ is a summable locally Hölder continuous potential. It therefore has a unique equilibrium state $\mu_{\psi\circ\hat{\pi}}$ by Theorem 2.7. By Observation 4.14 and Observation 4.4 we then have that

$$\mathrm{h_T}\big(\mu_{\psi\circ\hat{\pi}} \circ \hat{\pi}^{-1}\big) + \int_{\mathrm{J}} \psi \, \mathrm{d}\big(\mu_{\psi\circ\hat{\pi}} \circ \hat{\pi}^{-1}\big) = \mathrm{h}_\sigma(\mu_{\psi\circ\hat{\pi}}) + \int_{\mathrm{E_A}} \psi \circ \hat{\pi} \, \mathrm{d}(\mu_{\psi\circ\hat{\pi}}) = \mathrm{P}_\sigma(\psi \circ \hat{\pi}). = \mathrm{P_T}(\psi)$$

Thus, in order to complete the proof we are only left to show that if μ is an equilibrium state of the potential ψ, then $\mu = \mu_{\psi\circ\hat{\pi}} \circ \hat{\pi}^{-1}$. So, assume that μ is such equilibrium. It then follows from Theorem 4.12 that $\mu = \nu \circ \hat{\pi}^{-1}$ for some $\nu \in M_\sigma(E_A)$. But then by Observation 4.14, we get

$$\mathrm{h}_\nu(\sigma) + \int_{\mathrm{E_A}} \psi \circ \hat{\pi} \, \mathrm{d}\nu \geq \mathrm{h}_{\nu\circ\hat{\pi}^{-1}}(\mathrm{T}) + \int_{\mathrm{J}} \psi \, \mathrm{d}\big(\nu \circ \hat{\pi}^{-1}\big) = \mathrm{h}_\mu(\mathrm{T}) + \int_{\mathrm{J}} \psi \, \mathrm{d}\mu = \mathrm{P_T}(\psi) = \mathrm{P}_\sigma(\psi \circ \hat{\pi}).$$

Hence, ν is an equilibrium state of the potential $\psi \circ \hat{\pi} : E_A \to \mathbb{R}$ and the dynamical system $\sigma : E_A \to E_A$. Thus from Theorem 2.7, $\nu = \mu_{\psi\circ\hat{\pi}}$.

□

Now we provide the promised construction of Hölder Smale skew product endomorphisms. We start with (Y, d), a complete bounded metric space, and we assume that for every $\omega \in E_A^+$ there is given a continuous closed injective map $T_\omega : Y \to Y$ satisfying the following two conditions:

$$d(T_\omega(y_2), T_\omega(y_1)) \leq \lambda^{-1} d(y_2, y_1), \tag{4.15}$$

for all $y_1, y_2 \in Y$ and some $\lambda > 1$ independent of ω.

$$d_\infty(T_\beta, T_\alpha) := \sup\big\{d(T_\beta(\xi), T_\alpha(\xi)) : \xi \in Y\big\} \leq C d_\kappa(\beta, \alpha) \tag{4.16}$$

with some constants $C \in (0, +\infty)$, $\kappa > 0$, and all $\alpha, \beta \in E_A^+$.

Then let

$$\hat{Y} = E_A^+ \times Y,$$

and call $T : \hat{Y} \to \hat{Y}$ a skew product Smale system of global character. One may assume without loss of generality that

$$\kappa \le \frac{1}{2}\log\lambda. \tag{4.17}$$

We shall prove the following.

Theorem 4.21. *Each skew product Smale system of global character is Hölder.*

Proof. Let $T : E_A^+ \times Y \to E_A^+ \times Y$ be such skew product Smale system of global character. We first show that $T : E_A^+ \times Y \to E_A^+ \times Y$ is continuous. Clearly it is enough to show that $p_2 \circ T : E_A^+ \times Y \to Y$ is continuous, where p_2 denotes here the canonical projection onto the second coordinate. Indeed, for all $\alpha, \beta \in E_\omega^+$ and all $z, w \in Y$, we have

$$\begin{aligned}
d(p_2 \circ T(\alpha, z), p_2 \circ T(\beta, w)) &= d(T_\alpha(z), T_\beta(w)) \le d(T_\alpha(z), T_\beta(z)) + d(T_\beta(z), T_\beta(w)) \\
&\le d_\infty(T_\alpha, T_\beta) + \lambda^{-1} d(z, w) \\
&\le C d_\kappa(\alpha, \beta) + \lambda^{-1} d(z, w),
\end{aligned}$$

and continuity of the map $p_2 \circ T : E_A^+ \times Y \to Y$ is proved. So the continuity of $T : E_A^+ \times Y \to E_A^+ \times Y$ is proved, and thus $T : J \to J$ is continuous too. We now show that $T : J \to J$ is Hölder. So, fix an integer $n \ge 0$, two words $\alpha, \beta \in E_A$ and $\xi \in Y$. We then have

$$\begin{aligned}
d\big(T_\alpha^{n+1}(\xi), T_\beta^{n+1}(\xi)\big) &= d\Big(T_\alpha^n\big(T_{\alpha|_{-(n+1)}^{+\infty}}(\xi)\big), T_\beta^n\big(T_{\beta|_{-(n+1)}^{+\infty}}(\xi)\big)\Big) \\
&\le d\Big(T_\alpha^n\big(T_{\alpha|_{-(n+1)}^{+\infty}}(\xi)\big), T_\alpha^n\big(T_{\beta|_{-(n+1)}^{+\infty}}(\xi)\big)\Big) + d\Big(T_\alpha^n\big(T_{\beta|_{-(n+1)}^{+\infty}}(\xi)\big), T_\beta^n\big(T_{\beta|_{-(n+1)}^{+\infty}}(\xi)\big)\Big) \\
&\le \lambda^{-n} d\big(T_{\alpha|_{-(n+1)}^{+\infty}}(\xi)\big), T_{\beta|_{-(n+1)}^{+\infty}}(\xi)\big) + d_\infty\big(T_\alpha^n, T_\beta^n\big) \\
&\le \lambda^{-n} C d_\kappa\big(\alpha|_{-(n+1)}^{+\infty}, \beta|_{-(n+1)}^{+\infty}\big) + d_\infty\big(T_\alpha^n, T_\beta^n\big).
\end{aligned} \tag{4.18}$$

Let $p \ge -1$ be uniquely determined by the property that

$$d_\kappa(\alpha, \beta) = e^{-\kappa p}. \tag{4.19}$$

Consider two cases. First assume that

$$d_\kappa(\alpha, \beta) \ge e^{-\kappa n}. \tag{4.20}$$

Then using also (4.17), we get

$$\lambda^{-n} d_\kappa\big(\alpha|_{-(n+1)}^{+\infty}, \beta|_{-(n+1)}^{+\infty}\big) \le e^{-2\kappa n} \le e^{-\kappa n} d_\kappa(\alpha, \beta). \tag{4.21}$$

So, assume that

$$d_\kappa(\alpha, \beta) < e^{-\kappa n}. \tag{4.22}$$

Then $n < p$, so $n + 1 \le p$, whence

$$\begin{aligned}
d_\kappa\big(\alpha|_{-(n+1)}^{+\infty}, \beta|_{-(n+1)}^{+\infty}\big) &= \exp\big(-\kappa((n+1) + 1 + p)\big) = e^{-\kappa(n+2)} e^{-\kappa p} \\
&= e^{-\kappa(n+2)} d_\kappa(\alpha, \beta) \le e^{-\kappa n} d_\kappa(\alpha, \beta).
\end{aligned}$$

Hence, $\lambda^{-n} d_\kappa\big(\alpha|_{-(n+1)}^{+\infty}, \beta|_{-(n+1)}^{+\infty}\big) \le e^{-\kappa n} d_\kappa(\alpha,\beta)$. Inserting this and (4.21) to (4.18), in either case, yields

$$d\big(T_\alpha^{n+1}(\xi), T_\beta^{n+1}(\xi)\big) \le d_\infty\big(T_\alpha^n, T_\beta^n\big) + Ce^{-\kappa n} d_\kappa(\alpha,\beta).$$

Hence, taking the supremum over all $\xi \in Y$, we get

$$d_\infty\big(T_\alpha^{n+1}, T_\beta^{n+1}\big) \le d_\infty\big(T_\alpha^n, T_\beta^n\big) + Ce^{-\kappa n} d_\kappa(\alpha,\beta).$$

This in turn gives by immediate induction that

$$d_\infty\big(T_\alpha^n, T_\beta^n\big) \le Cd_\kappa(\alpha,\beta)\sum_{j=0}^{n-1} e^{-\kappa n} \le Cd_\kappa(\alpha,\beta)\sum_{j=0}^{\infty} e^{-\kappa n} = C(1-e^{-\kappa})^{-1} d_\kappa(\alpha,\beta) \tag{4.23}$$

for all $\alpha, \beta \in E_A$ and all integers $n \ge 0$. Recall that the integer $p \ge -1$ is determined by (4.19). Assume that $p \ge 0$. Then using (4.23), (4.22), and (4.2), we get

$$\begin{aligned} d(\hat\pi_2(\alpha), (\hat\pi_2(\alpha)) &\le \operatorname{diam}\big(T_\alpha^p(Y)\big) + \operatorname{diam}\big(T_\beta^p(Y)\big) + d_\infty\big(T_\alpha^p, T_\beta^p\big) \\ &\le \lambda^{-p}\operatorname{diam}(Y) + \lambda^{-p}\operatorname{diam}(Y) + C(1-e^{-\kappa})^{-1} d_\kappa(\alpha,\beta) \\ &\le 2\operatorname{diam}(Y) d_\kappa^{\frac{\log\lambda}{\kappa}}(\alpha,\beta) + C(1-e^{-\kappa})^{-1} d_\kappa(\alpha,\beta). \end{aligned}$$

Since also d is a bounded metric and since $d_\kappa(\alpha,\beta) = e^\kappa$ if $p=-1$, we now conclude that $\hat\pi_2 : E_A \to Y$ is Hölder continuous. Hence $\hat\pi : E_A \to Y$ is also Hölder continuous.

□

As an immediate consequence of Theorem 3.14 and Proposition 3.15 along with Observation 4.14 and Theorem 4.20, we get the following two last results of this section. We mention that with somewhat different setting and methods, such results were proved by Sarig [30]. Define now the set $\Sigma(\gamma,\xi)$ similarly to Definition 2.9.

Theorem 4.22. *Suppose that $T : \hat Y \to \hat Y$ a Hölder skew product Smale system, for example a Smale systems of global character. If $\gamma, \xi : J \to \mathbb{R}$ are locally Hölder continuous functions, then the function $\Sigma(\gamma,\xi) \ni (q,t) \mapsto \mathrm{P}_T(q\gamma + t\xi)$ is real-analytic.*

Proposition 4.23. *Suppose that $T : \hat Y \to \hat Y$ a Hölder skew product Smale system, for axample a Smale systems of global character. If $\gamma, \xi : J \to \mathbb{R}$ are locally Hölder continuous potentials, then for all $(q_0, t_0) \in \Sigma(\gamma,\xi)$,*

$$\frac{\partial}{\partial q}\Big|_{(q_0,t_0)} \mathrm{P}_T(q\gamma + t\xi) = \int \gamma\, d\mu_{q_0\gamma + t_0\xi}, \quad \frac{\partial}{\partial t}\Big|_{(q_0,t_0)} \mathrm{P}_T(q\gamma + t\xi) = \int \xi\, d\mu_{q_0\gamma + t_0\xi},$$

and

$$\frac{\partial^2}{\partial q \partial t}\Big|_{(q_0,t_0)} P_T(q\gamma + t\xi) = \sigma^2_{\mu_{q_0\gamma + t_0\xi}},$$

where $\mu_{q_0\gamma + t_0\xi}$ is the unique equilibrium state of the potential $q_0\gamma + t_0\xi$ and $\sigma^2_{\mu_{q_0\gamma+t_0\xi}}$ is the asymptotic covariance of the pair (γ,ξ) with respect to the measure $\mu_{q_0\gamma+t_0\xi}$ (see Proposition 2.6.14 in [15] for instance).

5. Conformal Skew Product Smale Endomorphisms of Countable Type

In this section we keep the setting of skew product Smale endomorphisms. However we assume more about the spaces Y_ω, $\omega \in E_A^+$, and the fiber maps $T_\omega : Y_\omega \to Y_{\sigma(\omega)}$, namely:

(a) Y_ω is a closed bounded subset of $\mathbb{R}^d$, with some $d \geq 1$ such that $\overline{\mathrm{Int}(Y_\omega)} = Y_\omega$.

(b) Each map $T_\omega : Y_\omega \to Y_{\sigma(\omega)}$ extends to a C^1 conformal embedding from Y_ω^* to $Y_{\sigma(\omega)}^*$, where Y_ω^* is a bounded connected open subset of $\mathbb{R}^d$ containing Y_ω. We keep the same symbol T_ω to denote this extension and we assume that the maps $T_\omega : Y_\omega^* \to Y_{\sigma(\omega)}^*$ enjoy the following properties:

(c) Formula (4.1) holds for all $y_1, y_2 \in Y_\omega^*$, perhaps with some smaller constant $\lambda > 1$.

(d) (Bounded Distortion Property 1) There exist constants $\alpha > 0$ and $H > 0$ such that for all $y, z \in Y_\omega^*$ we have that

$$\big| \log |T_\omega'(y)| - \log |T_\omega'(z)| \big| \leq H||y - z||^\alpha.$$

(e) The function $E_A \ni \tau \longmapsto \log |T_\tau'(\hat{\pi}_2(\omega))| \in \mathbb{R}$ is Hölder continuous.

(f) (Open Set Condition) For every $\omega \in E_A^+$ and for all $a, b \in E$ with $A_{a\omega_0} = A_{b\omega_0} = 1$ and $a \neq b$, we have

$$T_{a\omega}(\mathrm{Int}(Y_{a\omega})) \cap T_{b\omega}(\mathrm{Int}(Y_{b\omega})) = \emptyset.$$

(g) (Boundary Condition) There exists a measurable function $\delta : E_A^+ \to (0, \infty)$ so that

$$J_\omega \cap \big(Y_\omega \setminus \overline{B}(Y_\omega^c, \delta(\omega)\big) \neq \emptyset$$

for all $\omega \in E_A^+$. We mention that Open Set Condition plus the Boundary Condition give what is called the Strong Open Set Condition.

Any skew product Smale endomorphism satisfying conditions (a)–(g) will be called in the sequel a *conformal skew product Smale endomorphism*.

Remark 5.1. *The Bounded Distortion Property 1, i.e (d), is always automatically satisfied if $d \geq 2$. If $d = 2$, this is so because of Koebe's Distortion Theorem and because each conformal map in $\mathbb{C}$ is either holomorphic or antiholomorphic. If $d \geq 3$ this follows from Liouville's Representation Theorem asserting that each conformal map in $\mathbb{R}^d$, $d \geq 3$, is either a Möbius transformation or similarity, see [15] for details.*

A standard straightforward calculation based on (c), (d), and (e), yields in fact the following.

(BDP2) (Bounded Distortion Property 2) Perhaps with a larger constant H than in (d), we have that

$$\Big| \log \big| \big(T_\tau^n\big)'(y) \big| - \log \big| \big(T_\tau^n\big)'(z) \big| \Big| \leq H||y - z||^\alpha.$$

for all $\tau \in E_A$, $y, z \in Y^*_{\tau|_{-n}^{+\infty}}$, and all $n > 0$.

An immediate consequence of (BDP2) is the following version.

(BDP3) (Bounded Distortion Property 3) For all $\tau \in E_A$, all $n \geq 0$, and all $y, z \in Y^*_{\tau|_{-n}^{+\infty}}$,

$$K^{-1} \leq \frac{\left|\left(T_\tau^n\right)'(y)\right|}{\left|\left(T_\tau^n\right)'(z)\right|} \leq K,$$

where $K = \exp\bigl(H\mathrm{diam}^\alpha(Y)\bigr)$.

Recall also that we say that a conformal skew product Smale endomorphism is *Hölder*, if the condition of Hölder continuity for $\hat{\pi} : E_A \to J$ is satisfied, see Definition 4.19.

Remark 5.2. *Note that condition (e) is satisfied for instance if $T : \hat{Y} \to \hat{Y}$ is of global character (then by Theorem 4.21, it is Hölder) and if in addition*

$$||T'_\alpha - T'_\beta||_\infty \leq C d_\kappa(\alpha, \beta) \tag{5.1}$$

for all $\alpha, \beta \in E_A^+$. Actually if the conformal endomorphism $T : \hat{Y} \to \hat{Y}$ is of global character, then (5.1) also automatically follows in all dimensions $d \geq 2$. For $d = 2$ this is just Cauchy's Formula for holomorphic functions, and for $d \geq 3$ it would follow from the Liouville's Representation Theorem, although in this case the proof is not straightforward.

As an immediate consequence of the Open Set Condition (f) we get the following:

Lemma 5.3. *Let $T : \hat{Y} \to \hat{Y}$ a conformal skew product Smale endomorphism. If $n \geq 1$, $\alpha, \beta \in E_A(-n, \infty)$, $\alpha|_0^{+\infty} = \beta|_0^{+\infty}$, and $\alpha|_{-n}^{-1} \neq \beta|_{-n}^{-1}$, then*

$$T_\alpha^n(\mathrm{Int}(Y_\alpha)) \cap T_\beta^n(\mathrm{Int}(Y_\beta)) = \emptyset, \text{ and}$$

$$T_\alpha^n(\mathrm{Int}(Y_\alpha)) \cap T_\beta^n(Y_\beta) = \emptyset = T_\alpha^n(Y_\alpha) \cap T_\beta^n(\mathrm{Int}(Y_\beta)).$$

Now as a consequence of the Open Set Condition in fibers, we obtain:

Lemma 5.4. *Let $T : \hat{Y} \to \hat{Y}$ be a conformal skew product Smale endomorphism. If $n \geq 1$ and $\tau \in E_A(-n, \infty)$, then*

$$\hat{\pi}_2^{-1}\bigl(T_\tau^n(\mathrm{Int}(Y_\tau))\bigr) \subset [\tau].$$

Proof. let $\gamma \in \hat{\pi}_2^{-1}\bigl(T_\tau^n(\mathrm{Int}(Y_\tau))\bigr)$. This means that $\gamma|_0^{+\infty} = \tau|_0^{+\infty}$ and $\hat{\pi}_2(\gamma) \in T_\tau^n(\mathrm{Int}(Y_\tau)) \subset Y_{\tau|_0^{+\infty}}$. On the other hand, $\hat{\pi}_2(\gamma) \in T^n_{\gamma|_{-n}^{+\infty}}\bigl(Y_{\gamma|_{-n}^{+\infty}}\bigr)$. It therefore follows from the second formula of Lemma 5.3 that $\gamma|_{-n}^0 = \tau$. So, $\gamma \in [\tau]$ and the proof is complete. $\square$

We will also use the following condition:

(h) (Uniform Geometry Condition) $\exists(R > 0)\ \forall(\omega \in E_A^+)\ \exists(\xi_\omega \in Y_\omega)$

$$B(\xi_\omega, R) \subset Y_\omega.$$

The primary significance of the Uniform Geometry Condition (h) lies in:

Lemma 5.5. *If $T : \hat{Y} \to \hat{Y}$ is a Hölder conformal skew product Smale endomorphism satisfying Uniform Geometry Condition (h), then for every $\gamma \geq 1$, $\exists\ \Gamma_\gamma > 0$ such that:*

If $\mathcal{F} \subset E_A^*(-\infty, -1)$ is a collection of mutually incomparable (finite) words, so that $A_{\tau_{-1}\omega_0} = 1$ for some $\omega \in E_A^*$ and all $\tau \in \mathcal{F}$, and so that for some $\xi \in Y_\omega$,

$$T_{\tau\omega}^{|\tau|}(Y_{\tau\omega}) \cap B(\xi, r) \neq \emptyset \ \text{ with } \gamma^{-1} r \leq \mathrm{diam}\bigl(T_{\tau\omega}^{|\tau|}(Y_{\tau\omega})\bigr) \leq \gamma r,$$

then the cardinality of $\mathcal{F}$ is bounded above by Γ_γ.

Proof. From the definition of conformal Smale skew product endomorphisms of Hölder type, the family $\{T^{|\tau|}_{\tau\omega}(\mathrm{Int}(Y_{\omega\tau})) : \tau \in \mathcal{F}\}$ consists of mutually disjoint subsets of Y_ω.
Also, from the Uniform Geometry condition (as without it, one may not guarantee the existence of balls of radii comparable to r inside $T^{|\tau|}_{\tau\omega}(\mathrm{Int}(Y_{\tau\omega}))$), we obtain:

$$\begin{aligned}T^{|\tau|}_{\tau\omega}(\mathrm{Int}(Y_{\tau\omega})) &\supset T^{|\tau|}_{\tau\omega}(B(\xi_{\tau\omega}, R)) \supset B\big(T^{|\tau|}_{\tau\omega}(\xi_{\tau\omega}, K^{-1}R\big|\big(T^{|\tau|}_{\tau\omega}\big)'(\xi_{\tau\omega})\big|\big)\\ &\supset B\big(T_{\tau\omega}(\xi_{\tau\omega}), K^{-2}R\gamma^{-1}r\big)\end{aligned}$$

Moreover, $T^{|\tau|}_{\tau\omega}(\mathrm{Int}(Y_{\omega\tau})) \subset B(\xi, (1+\gamma)r)$. Therefore the conclusion of the Lemma follows. ∎

6. Volume Lemmas

We are in the setting of Section 5. Thus let $T : \hat{Y} \to \hat{Y}$ a conformal skew product Smale endomorphism, i.e one satisfying conditions (a)–(g) of Section 5. Condition (h), the Uniform Geometry Condition is not required in the current section, it will be used in the next one.

First of all, we recall the definition of *exact dimensional measure*, from Young [34]:

Definition 6.1. *Let μ be a Borel probability measure on a metric space X. We say that μ is exact dimensional if there exists a value d_μ so that, for μ-a.e. $x \in X$,*

$$\varliminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \varlimsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = d_\mu$$

Exact dimensionality of a measure is important, since it implies that all dimensions of that measure are equal (pointwise dimension, Hausdorff dimension, box dimension); it was studied in various dynamical settings (see for eg [34], [1], etc.)

Now if μ is a Borel probability σ-invariant measure on E_A, then by $\chi_\mu(\sigma)$ we denote its *Lyapunov exponent*, defined by the formula

$$\chi_\mu(\sigma) := -\int_{E_A} \log\left|T'_{\tau|_0^{+\infty}}\big(\hat{\pi}_2(\tau)\big)\right| d\mu(\tau) = -\int_{E_A^+}\int_{[\omega]} \log\left|T'_\omega\big(\hat{\pi}_2(\tau)\big)\right| d\overline{\mu}^\omega(\tau)\, dm(\omega),$$

where $m = \mu\circ\pi_1^{-1} = \pi_{1*}\mu$ is the canonical projection of μ onto E_A^+. We shall prove:

Theorem 6.2. *Let $T : \hat{Y} \to \hat{Y}$ be a Hölder conformal skew product Smale endomorphism, and let $\psi : E_A \to \mathbb{R}$ be a locally Hölder continuous summable potential. Then for the projection $= \overline{\mu}^\omega_\psi \circ \hat{\pi}_2^{-1}$, of the conditional measure onto the fiber J_ω, we have that*

$$\mathrm{HD}\big(\overline{\mu}^\omega_\psi\circ\hat{\pi}_2^{-1}\big) = \frac{\mathrm{h}_{\mu_\psi}(\sigma)}{\chi_{\mu_\psi}(\sigma)} = \frac{\mathrm{P}_\sigma(\psi) - \int\psi\, d\mu_\psi}{\chi_{\mu_\psi}(\sigma)}$$

for m_ψ-a.e $\omega \in E_A^+$, where $m_\psi = \mu_\psi\circ\pi_1^{-1}$. Moreover for m_ψ-a.e $\omega\in E_A^+$ the measure $\overline{\mu}^\omega_\psi\circ\hat{\pi}_2^{-1}$ is exact dimensional, and its pointwise dimension is given by:

$$\lim_{r\to 0}\frac{\log \overline{\mu}^\omega_\psi\circ\hat{\pi}_2^{-1}(B,r))}{\log r} = \frac{\mathrm{h}_{\mu_\psi}(\sigma)}{\chi_{\mu_\psi}(\sigma)}, \tag{6.1}$$

for m_ψ-a.e. $\omega \in E_A^+$ and $\overline{\mu}_\psi^\omega \circ \hat{\pi}_2^{-1}$-a.e. $z \in J_\omega$ (and equivalently for $\mu_\psi \circ \hat{\pi}^{-1}$-a.e. $(\omega, z) \in J$).

Proof. According to Definition 6.1, we only need to show that formula (6.1) holds a.e. Since the measure μ_ψ is ergodic, Birkhoff's Ergodic Theorem applied to the map $\sigma^{-1} : E_A \to E_A$ produces a measurable set $E_{A,\psi} \subset E_A$ such that $\mu_\psi(E_{A,\psi}) = 1$, and for every $\tau \in E_{A,\psi}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \left(T_\tau^n\right)'(\hat{\pi}_2(\sigma^{-n}(\tau)))\right| = -\chi_{\mu_\psi}(\sigma) \tag{6.2}$$

and

$$\lim_{n\to\infty} \frac{1}{n} S_n\psi(\sigma^{-n}(\tau)) = \int_{E_A} \psi \, d\mu_\psi \tag{6.3}$$

For arbitrary $\omega \in E_A^+$ denote now

$$\nu_\omega := \overline{\mu}_\psi^\omega \circ \hat{\pi}_2^{-1},$$

which is a Borel probability measure on J_ω. Fix $\tau \in E_{A,\psi}$. Fix also a radius $r \in \big(0, \operatorname{diam}\big(Y_{p_2(\tau)}\big)/2\big)$. Let $z = \hat{\pi}_2(\tau)$, and take the least integer $n = n(z,r) \geq 0$ so that

$$T_\tau^n\big(Y_{\tau|_{-n}^{+\infty}}\big) \subset B(z,r). \tag{6.4}$$

If $r > 0$ is small enough (depending on τ), then $n \geq 1$ and $T_\tau^{n-1}\big(Y_{\tau|_{-(n-1)}^{+\infty}}\big) \not\subset B(z,r)$. Since $z \in T_\tau^{n-1}\big(Y_{\tau|_{-(n-1)}^{+\infty}}\big)$, this implies that

$$\operatorname{diam}\left(T_\tau^{n-1}\big(Y_{\tau|_{-(n-1)}^{+\infty}}\big)\right) \geq r. \tag{6.5}$$

Write $\omega := \tau|_0^{+\infty}$. It follows from (6.4), Lemma 4.9, and Theorem 3.12 that

$$\begin{aligned} \nu_\omega(B(z,r)) &\geq \nu_\omega\big(\hat{\pi}_2\big([\tau|_{-n}^{+\infty}]\big)\big) = \overline{\mu}_\psi^\omega \circ \hat{\pi}_2^{-1}\big(\hat{\pi}_2\big([\tau|_{-n}^{+\infty}]\big)\big) \geq \overline{\mu}_\psi^\omega\big([\tau|_{-n}^{+\infty}]\big) \\ &\geq D^{-1} \exp\big(S_n\psi(\sigma^{-n}(\tau)) - \mathrm{P}_\sigma(\psi)n\big). \end{aligned} \tag{6.6}$$

By taking logarithms and using (6.5), this gives that

$$\frac{\log \nu_\omega(B(z,r))}{\log r} \leq \frac{-\log D + S_n\psi(\sigma^{-n}(\tau)) - \mathrm{P}_\sigma(\psi)n}{\log\left(\operatorname{diam}\left(T_\tau^{n-1}\big(Y_{\tau|_{-(n-1)}^{+\infty}}\big)\right)\right)}$$

So applying (BDP3), we get that

$$\begin{aligned} \frac{\log \nu_\omega(B(z,r))}{\log r} &\leq \frac{-\log D + S_n\psi(\sigma^{-n}(\tau)) - \mathrm{P}_\sigma(\psi)n}{\log K + \log\left(\operatorname{diam}\left(Y_{\tau|_{-(n-1)}^{+\infty}}\right)\right) + \log\left|\left(T_\tau^{n-1}\right)'\left(\hat{\pi}_2\big(\sigma^{-n}(\tau)\big)\right)\right|} \\ &\leq \frac{-\log D + S_n\psi(\sigma^{-n}(\tau)) - \mathrm{P}_\sigma(\psi)n}{\log K + \log(\operatorname{diam}(Y)) + \log\left|\left(T_\tau^{n-1}\right)'\left(\hat{\pi}_2\big(\sigma^{-n}(\tau)\big)\right)\right|} \\ &= \frac{-\frac{\log D}{n} + \frac{1}{n}S_n\psi(\sigma^{-n}(\tau)) - \mathrm{P}_\sigma(\psi)}{\frac{\log K}{n} + \frac{\log(\operatorname{diam}(Y))}{n} + \frac{1}{n}\log\left|\left(T_\tau^{n-1}\right)'\left(\hat{\pi}_2\big(\sigma^{-n}(\tau)\big)\right)\right|}, \end{aligned}$$

and by virtue of (6.2) and (6.3) this yields

$$\varlimsup_{r\to 0} \frac{\log \nu_\omega(B(z,r))}{\log r} \leq \frac{\lim_{n\to\infty} \frac{1}{n}S_n\psi(\sigma^{-n}(\tau)) - \mathrm{P}_\sigma(\psi)}{\lim_{n\to\infty} \frac{1}{n}\log\left|\left(T_\tau^{n-1}\right)'\left(\hat{\pi}_2\big(\sigma^{-n}(\tau)\big)\right)\right|} = \frac{\mathrm{P}_\sigma(\psi) - \int \psi \, d\mu_\psi}{\chi_{\mu_\psi}(\sigma)}. \tag{6.7}$$

In order to establish the opposite inequality note that the set $\hat{\pi}_2^{-1}\big(J_\omega \setminus \overline{B}(Y_\omega^c, \delta(\omega))\big)$ is open in $[\omega] \subset E_A$, is not empty by (g), and therefore

$$\overline{\mu}_\psi^\omega\big(\hat{\pi}_2^{-1}\big(J_\omega \setminus \overline{B}(Y_\omega^c, \delta(\omega))\big)\big) > 0$$

for every $\omega \in E_A^+$. In consequence $\mu_\psi(Z) > 0$, where

$$Z := \bigcup_{\omega \in E_A^+} \hat{\pi}_2^{-1}\big(J_\omega \setminus \overline{B}(Y_\omega^c, \delta(\omega))\big)$$

Also, since $\delta : E_A^+ \to (0, +\infty)$ is measurable, there exists $R > 0$ so that $\mu_\psi(Z_R) > 0$, where

$$Z_R := \bigcup_{\omega \in E_A^+} \hat{\pi}_2^{-1}\big(J_\omega \setminus \overline{B}(Y_\omega^c, R)\big)$$

Consider now the set of integers

$$N(\tau) := \{k \geq 0 : \sigma^{-k}(\tau) \in Z_R\}.$$

Represent the set $N(\tau)$ as a strictly increasing sequence $(k_n(\tau))_{n=1}^\infty$. By Birkhoff's Ergodic Theorem, there is a measurable set $\tilde{E}_{A,\psi} \subset E_{A,\psi}$ with $\mu_\psi(\tilde{E}_{A,\psi}) = 1$ and for every $\tau' \in \tilde{E}_{A,\psi}$,

$$\lim_{n\to\infty} \frac{Card\{0 \leq i \leq n,\ \sigma^{-i}(\tau') \in Z_R\}}{n} = \mu_\psi(Z_R)$$

Now we put $k_n(\tau) \geq n$, instead of n above, and notice that $Card\{0 \leq i \leq k_n(\tau),\ \sigma^{-i}(\tau') \in Z_R\} = n$. Therefore as $\mu_\psi(Z_R) > 0$, we obtain for every $\tau \in \tilde{E}_{A,\psi}$ and any n large, that:

$$\lim_{n\to\infty} \frac{k_n(\tau)}{n} = \frac{1}{\mu_\psi(Z_R)}$$

Hence for every $\tau \in \tilde{E}_{A,\psi}$, we have

$$\lim_{n\to\infty} \frac{k_{n+1}(\tau)}{k_n(\tau)} = 1 \tag{6.8}$$

Fix $\tau \in \tilde{E}_{A,\psi}$. Keep $\omega = \tau|_0^{+\infty}$ and consider the largest $n = n(\tau, r) \geq 1$ such that with $k_j := k_j(\tau)$, $j \geq 1$, we obtain

$$K^{-1}\left|\left(T_\tau^{k_n}\right)'\left(\hat{\pi}_2(\sigma^{-k_n}(\tau))\right)\right| R \geq r. \tag{6.9}$$

Then

$$K^{-1}\left|\left(T_\tau^{k_{n+1}}\right)'\left(\hat{\pi}_2(\sigma^{-k_{n+1}}(\tau))\right)\right| R < r. \tag{6.10}$$

It follows from (6.9) and (BDP3) that

$$B(z, r) \subset T_\tau^{k_n}\big(B(\hat{\pi}_2(\sigma^{-k_n}(\tau)), R)\big) \subset T_\tau^{k_n}\left(\mathrm{Int}\left(Y_{\tau|_{-k_n}^{+\infty}}\right)\right).$$

Hence, invoking also Lemma 5.4 and Theorem 3.12, we infer that

$$\nu_\omega(B(z, r)) \leq \overline{\mu}_\psi^\omega\big([\tau|_{-k_n}^{+\infty}]\big) \leq D \exp\left(S_{k_n}\psi(\sigma^{-k_n}(\tau)) - \mathrm{P}_\sigma(\psi) k_n\right).$$

By taking logarithms and using (6.10), this gives that

$$\frac{\log \nu_\omega(B(z,r))}{\log r} \geq \frac{\log D + S_{k_n}\psi(\sigma^{-k_n}(\tau)) - \mathrm{P}_\sigma(\psi)k_n}{-\log K + \log\left|\left(T_\tau^{k_{n+1}}\right)'\left(\hat{\pi}_2\left(\sigma^{-k_{n+1}}(\tau)\right)\right)\right|}$$
$$= \frac{\frac{\log D}{k_n} + \frac{1}{k_n}S_{k_n}\psi(\sigma^{-k_n}(\tau)) - \mathrm{P}_\sigma(\psi)}{\frac{-\log K}{k_n} + \frac{1}{k_n}\log\left|\left(T_\tau^{k_{n+1}}\right)'\left(\hat{\pi}_2\left(\sigma^{-k_{n+1}}(\tau)\right)\right)\right|}.$$

By virtue of (6.2), (6.8) and (6.3), this yields

$$\varliminf_{r\to 0}\frac{\log \nu_\omega(B(z,r))}{\log r} \geq \frac{\lim_{n\to\infty}\frac{1}{k_n}S_{k_n}\psi(\sigma^{-k_n}(\tau)) - \mathrm{P}_\sigma(\psi)}{\lim_{n\to\infty}\frac{1}{k_n}\log\left|\left(T_\tau^{k_{n+1}}\right)'\left(\hat{\pi}_2\left(\sigma^{-k_{n+1}}(\tau)\right)\right)\right|} = \frac{\mathrm{P}_\sigma(\psi) - \int\psi\,d\mu_\psi}{\chi_{\mu_\psi}(\sigma)}.$$

Along with (6.7) and since $\mathrm{P}_\sigma(\psi) - \int\psi\,d\mu_\psi = \mathrm{h}_{\mu_\psi}(\sigma)$. this gives that formula (6.1) holds for all $\tau \in \tilde{E}_{A,\psi}$, and the proof of Theorem 6.2 is complete.

□

If μ is now a Borel probability T-invariant measure on the fibered limit set J, then by $\chi_\mu(T)$ we denote its *Lyapunov exponent*, which is defined by the formula

$$\chi_\mu(T) := -\int_J \log\left|T_\omega'(z)\right| d\mu(\omega,z) = -\int_{E_A^+}\int_{J_\omega}\log\left|T_\omega'(z)\right| d\overline{\mu}^\omega(z)\,dm(\omega),$$

where $m = \mu\circ\pi_1^{-1}$ is the canonical projection of μ onto E_A^+, and $\left(\overline{\mu}^\omega\right)_{\omega\in E_A^+}$ is the canonical system of conditional measures of μ with respect to the measurable partition $\{\{\omega\}\times J_\omega\}_{\omega\in E_A^+}$. Now we prove the following.

Corollary 6.3. *Let $T : \hat{Y} \to \hat{Y}$ be a Hölder conformal Smale endomorphism of compact type. Let $\psi : J \to \mathbb{R}$ be a locally Hölder continuous summable potential. Then*

$$\mathrm{HD}\left(\overline{\mu}_\psi^\omega\right) = \frac{\mathrm{h}_{\mu_\psi}(\mathrm{T})}{\chi_{\mu_\psi}(T)} = \frac{\mathrm{P}_T(\psi) - \int\psi\,d\mu_\psi}{\chi_{\mu_\psi}(T)}$$

for m_ψ-a.e. $\omega \in E_A^+$, where $m_\psi = \mu_\psi\circ\pi_1^{-1}$. Moreover, for m_ψ-a.e. $\omega\in E_A^+$ the measure $\overline{\mu}_\psi^\omega$ is exact dimensional and

$$\lim_{r\to 0}\frac{\log\overline{\mu}_\psi^\omega(B(z,r))}{\log r} = \frac{\mathrm{h}_{\mu_\psi}(\mathrm{T})}{\chi_{\mu_\psi}(T)} \tag{6.11}$$

for m_ψ-a.e. $\omega\in E_A^+$ and $\overline{\mu}_\psi^\omega$-a.e. $z\in J_\omega$ (equivalently for μ_ψ-a.e. $(\omega,z)\in J$).

Proof. Let $\hat{\psi} := \psi\circ\hat{\pi} : E_A \to \mathbb{R}$. By Theorem 4.20 $\mu_\psi = \mu_{\hat{\psi}}\circ\hat{\pi}^{-1}$ is the unique equilibrium state of the potential ψ and the shift map $\sigma : E_A \to E_A$. By Observation 4.14, $\mathrm{P}_T(\psi) = \mathrm{P}_\sigma(\hat{\psi})$, and by Observation 4.13, $\mathrm{h}_{\mu_\psi}(\mathrm{T}) = \mathrm{h}_{\mu_{\hat{\psi}}}(\sigma)$. Since in addition $\chi_{\mu_\psi}(T) = \chi_{\mu_{\hat{\psi}}}(\sigma)$, the proof of our corollary follows immediately from Theorem 6.2 applied to the potential $\hat{\psi} : E_A \to \mathbb{R}$. □

7. Bowen Type Formula

We keep the setting of Sections 5 and Section 6, so $T : \hat{Y} \to \hat{Y}$ is a conformal skew product Smale endomorphism, i.e. one satisfying conditions (a)–(g) of Section 5. Furthermore we do however emphasize that in the present section Condition (h), i.e. the Uniform geometry Condition, is now needed and it is assumed.

For every $t \geq 0$ let $\psi_t : J \to \mathbb{R}$ be the function function given by the following formula.

$$\psi_t(\omega, y) = -t \log |T'_\omega(y)|.$$

Define $\mathcal{F}(T)$ to be the set of parameters $t \geq 0$ for which the potential ψ_t is summable, i.e.

$$\sum_{e \in E} \exp\big(\sup\big(\psi_t|_{[e]_T}\big)\big) < +\infty.$$

This means that

$$\sum_{e \in E} \sup \big\{ ||T_{e\tau}||_\infty^t : \tau \in E_A(1, +\infty),\ A_{e\tau_1} = 1 \big\} < +\infty.$$

For every $t \geq 0$ we abbreviate

$$\mathrm{P}(t) := \mathrm{P}_T(\psi_t),$$

and call $\mathrm{P}(t)$ the topological pressure of the parameter t. From Proposition 3.6, we have

$$\mathcal{F}(T) = \{t \geq 0 : \mathrm{P}(t) < +\infty\}.$$

We record the following basic properties of the pressure function $[0, +\infty) \ni t \mapsto \mathrm{P}(t)$.

Proposition 7.1. *The pressure function* $t \mapsto \mathrm{P}(t),\ t \in [0, \infty)$ *has the following properties:*

(a) P *is monotone decreasing*

(b) $\mathrm{P}|_{\mathcal{F}(T)}$ *is strictly decreasing.*

(c) $\mathrm{P}|_{\mathcal{F}(T)}$ *is convex, real-analytic, and Lipschitz continuous.*

Proof. All these statements except real analyticity follow easily from definitions, plus, due to Lemma 3.4 and Observation 4.14, from their one-sided shift counterparts. The real analyticity assertion is an immediate consequence of Theorem 4.22.

$\square$

Now we can define two significant numbers associated with the conformal skew product Smale endomorphism T:

$$\theta_T := \inf \big\{ t \geq 0 : \mathrm{P}(t) < +\infty \big\} \text{ and } B_T := \inf \big\{ t \geq 0 : \mathrm{P}(t) \leq 0 \big\}.$$

B_T is called the *Bowen's parameter* of the system T. Clearly $\theta_T \leq B_T$.
The main result of this section is the following.

Theorem 7.2. *If* $T : \hat{Y} \to \hat{Y}$ *is a Hölder conformal skew product Smale endomorphism satisfying the Uniform Geometry Condition (h), then for every* $\omega \in E_A^+$,

$$\mathrm{HD}(J_\omega) = B_T.$$

We first shall prove this theorem under the assumption that the alphabet E is finite. In this case we will actually prove more. Recall that if (Z,ρ) is a separable metric space, then a finite Borel measure ν on Z is called *Ahlfors regular* (or *geometric*) if and only if

$$C^{-1}r^h \le \nu(B(z,r)) \le Cr^h,$$

for all $r>0$, with some independent constants $h\ge 0$, $C\in(0,+\infty)$. It is well known and easy to prove that there is at most one h with such property and all Ahlfors regular measures on Z are mutually equivalent, with bounded Radon-Nikodym derivatives. Moreover

$$h=\mathrm{HD}(Z)=\mathrm{PD}(Z)=\mathrm{BD}(Z),$$

the two latter dimensions being, respectively the packing and box-counting dimensions of Z. In addition, the h-dimensional Hausdorff measure H_h, and the h-dimensional packing measure P_h on Z, are Ahlfors regular, equivalent to each other and equivalent to ν. Now, if the alphabet E is finite, then the Smale endomorphism $T:\hat Y\to\hat Y$ is of compact type, and in particular, for every $t\ge 0$ there exists μ_t, a unique equilibrium state for the potential $\psi_t:J\to\mathbb{R}$. Since $0\le \mathrm{P}(0)<+\infty$ it follows from Proposition 7.1 that

$$\mathrm{P}(B_T)=0.$$

We shall prove first the case of a finite alphabet E.

Theorem 7.3. *If $T:\hat Y\to\hat Y$ is a Hölder conformal skew product Smale endomorphism satisfying the Uniform Geometry Condition (h) and the alphabet E is finite, then $\overline{\mu}^\omega_{B_T}$ is an Ahlfors regular measure on J_ω, for every $\omega\in E_A^+$. In particular, for every $\omega\in E_A^+$,*

$$\mathrm{HD}(J_\omega)=B_T$$

Proof. Put $h:=B_T$. Fix $\omega\in E_A^+$ and $z=\hat\pi_2(\tau)\in J_\omega$ arbitrary. Let $n=n(z,r)$ be given by (6.4), and let us denote

$$\nu_\omega:=\overline{\mu}^\omega_h\circ\hat\pi_2^{-1}.$$

Formula (6.6) gives, for $\psi=\psi_h$,

$$\nu_\omega(B(z,r))\ge D^{-1}\exp\big(S_n\psi(\sigma^{-n}(\tau))\big)=D^{-1}\big|\big(T^n_\tau\big)'(\hat\pi_2(\sigma^{-n}(\tau)))\big|^h. \tag{7.1}$$

Now, since the set E_A is compact (as E is finite) and since the function $E_A\ni\tau\mapsto |T'_\tau(\hat\pi_2(\tau))|\in(0,+\infty)$ is continuous, in fact Hölder continuous, we conclude that there exists a constant $M\in(0,+\infty0$ such that

$$M^{-1}\le\inf\big\{|T'_\tau(\hat\pi_2(\tau))|:\tau\in E_A\big\}\le\sup\big\{|T'_\tau(\hat\pi_2(\tau))|:\tau\in E_A\big\}\le M. \tag{7.2}$$

Having this an inserting (6.5) to (7.1), we get

$$\nu_\omega(B(z,r))\ge (DM^h)^{-1}r^h. \tag{7.3}$$

In order to prove an inequality in the opposite direction, define:

$$\mathcal{F}(z,r):=\Big\{\tau\in E_A^*(-\infty,-1):T^{|\tau|}_{\tau\omega}(Y_{\tau\omega})\cap B(z,r/2)\ne\emptyset,$$
$$\mathrm{diam}\big(T^{|\tau|}_{\tau\omega}(Y_{\tau\omega})\big)\le r/2 \text{ and } \mathrm{diam}\Big(T^{|\tau|}_{\tau|^{-1}_{-(|\tau|-1)}\omega}\big(Y_{\tau|^{-1}_{-(|\tau|-1)}\omega}\big)\Big)>r/2\Big\}.$$

By its very definition $\mathcal{F}(z,r)$ consists of mutually incomparable elements of $E_A^*(-\infty,-1)$, so using (7.2) along with (BDP3), we get for every $\tau\in\mathcal{F}(z,r)$, with $n:=|\tau|$, that

$$\begin{aligned}
\operatorname{diam}\big(T_{\tau\omega}^n(Y_{\tau\omega})\big) &= \operatorname{diam}\Big(T^{n-1}_{\tau|_{-(n-1)}^{-1}\omega}\big(T_{\tau\omega}(Y_{\tau\omega})\big)\Big) \ge K^{-1}\Big\|\Big(T^{n-1}_{\tau|_{-(n-1)}^{-1}\omega}\Big)'\Big\|_\infty \operatorname{diam}\big(T_{\tau\omega}(Y_{\tau\omega})\big)\\
&\ge K^{-2}\Big\|\Big(T^{n-1}_{\tau|_{-(n-1)}^{-1}\omega}\Big)'\Big\|_\infty \|T'_{\tau\omega}\|_\infty \operatorname{diam}(Y_{\tau\omega}) \ge 2K^{-2}M^{-1}R\Big\|\Big(T^{n-1}_{\tau|_{-(n-1)}^{-1}\omega}\Big)'\Big\|_\infty\\
&\ge 2K^{-3}M^{-1}R\operatorname{diam}^{-1}(Y)\operatorname{diam}\Big(T^{n-1}_{\tau|_{-(n-1)}^{-1}\omega}\big(T_{\tau\omega}\big(Y_{\tau|_{-(n-1)}^{-1}\omega}\big)\big)\Big)\\
&\ge K^{-3}M^{-1}R\operatorname{diam}^{-1}(Y)r.
\end{aligned}$$

Thus Lemma 5.5 applies with the radius equal to $r/2$, since $\#\mathcal{F}(z,r)\le\Gamma_\gamma$, where

$$\gamma:=\max\{1,2K^3MR^{-1}\operatorname{diam}(Y)\}.$$

Since also

$$\hat\pi_2^{-1}(B(z,r))\subset\bigcup_{\tau\in\mathcal{F}(z,r)}[\tau\omega],$$

we therefore get

$$\begin{aligned}
\nu_\omega(B(z,r)) &\le \overline{\mu}_h^\omega\circ\hat\pi_2^{-1}\left(\bigcup_{\tau\in\mathcal{F}(z,r)}[\tau\omega]\right)\le\sum_{\tau\in\mathcal{F}(z,r)}\overline{\mu}_h^\omega\circ\hat\pi_2^{-1}([\tau\omega])\\
&\le\sum_{\tau\in\mathcal{F}(z,r)}\big\|\big(T_{\tau\omega}^{|\tau|}\big)'\big\|_\infty^h\le K^h\sum_{\tau\in\mathcal{F}(z,r)}\operatorname{diam}^h\big(T_{\tau\omega}^{|\tau|}(Y_{\tau\omega})\big)\le(2K)^h\#_\Gamma r^h
\end{aligned}$$

Hence, along with (7.3), this shows that ν_ω is Ahlfors regular with exponent $h=B_T$.

□

Proof of Theorem 7.2: Fix $t>B_T$ arbitrary. Then $\mathrm{P}(t)<0$. It therefore follows from the definition of topological pressure and of the potential ψ_t that for every integer $n\ge1$ large enough and for every $\omega\in E_A^+$, we have that

$$\sum_{\substack{\tau\in E_A^*(-n,-1)\\ A_{\tau_{-1}\omega_0}=1}}\big\|\big(T_{\tau\omega}^n\big)'\big\|_\infty^t\le\exp\left(\frac12\mathrm{P}(t)n\right).$$

Therefore by (BDP2),

$$\sum_{\substack{\tau\in E_A^*(-n,-1)\\ A_{\tau_{-1}\omega_0}=1}}\operatorname{diam}^t\big(T_{\tau\omega}^n(Y_{\tau\omega})\big)\le K^t\exp\left(\frac12\mathrm{P}(t)n\right). \tag{7.4}$$

Since $\mathrm{P}(t)<0$, since the family $\big\{T_{\tau\omega}^n(Y_{\tau\omega}):\tau\in E_A^*(-n,-1),\,A_{\tau_{-1}\omega_0}=1\big\}$ is a cover of J_ω and since the diameters of this cover converge to zero $\big(\operatorname{diam}\big(T_{\tau\omega}^n(Y_{\tau\omega})\big)\le\lambda^{-n}\operatorname{diam}(Y)\big)$, it follows from (7.4), by letting $n\to\infty$, that $\mathrm{H}_t(J_\omega)=0$. Therefore $\mathrm{HD}(J_\omega)\le t$, and, by arbitrariness of $t>B_T$,

$$\mathrm{HD}(J_\omega)\le B_T. \tag{7.5}$$

In order to prove the opposite inequality fix $0 \leq t < B_T$. Then $\mathrm{P}(t) > 0$ and it thus follows from Theorem 3.5 that $\mathrm{P}_F(t) > 0$ for some finite set $F \subset E$ such that the matrix $A|_{F\times F}$ is irreducible. It then further follows from Theorem 7.3 that $\mathrm{HD}(J_\omega(F)) > t$ for all $\omega \in E_A^+$. Since $J_\omega(F) \subset J_\omega$, this yields $\mathrm{HD}(J_\omega) \geq t$. Thus, by arbitrariness of $t < B_T$, we get that $\mathrm{HD}(J_\omega) \geq B_T$. Along with (7.5) this completes the proof of Theorem 7.2

□

8. General Skew Products over Countable-to-1 Endomorphisms. Beyond the Symbol Space

We want to enlarge the class of endomorphisms of skew products for which we can prove exact dimensionality of conditional measures on fibers. For general thermodynamic formalism of endomorphisms related to our approach, one can see [28], [16], [18], [17], [20], etc. Also our result below on exact dimensionality of conditional measures in fibers extends a result on exact dimensionality of conditional measures on stable manifolds of hyperbolic endomorphisms (see [16]). We will investigate fibered systems which are at most countable–to–1 in the base, and have fibered maps satisfying conditions (a)–(g) in Section 5. We want to apply the results obtained in the previous sections to endomorphisms of skew products over countable-to-1 endomorphisms. This includes systems generated by conformal iterated function systems, and several important classes such as β-transformations ($\beta > 1$), generalized Lüroth series, EMR-maps, the continued fraction transformation, and Manneville-Pomeau maps.

First, let us prove a general result about skew products whose base transformations are modeled by 1-sided shifts on a countable alphabet. Assume that we have a skew product $F : X \times Y \to X \times Y$, where X and Y are complete bounded metric spaces, $Y \subset \mathbb{R}^d$ for some $d \geq 1$, and

$$F(x, y) = (f(x), g(x, y)),$$

where the map

$$Y \ni y \longmapsto g(x, y)$$

is injective and continuous for every $y \in Y$; we denote the map $Y \ni y \mapsto g(x, y)$ also by $g_x(y)$. Assume that the base map $f : X \to X$ is at most countable-to-1, and that the dynamics of f is modeled by a 1-sided Markov shift on a countable alphabet E with the matrix A finitely irreducible, i.e there exists a surjective Hölder continuous map, called *coding*,

$$p : E_A^+ \to X \quad \text{such that} \quad p \circ \sigma = f \circ p$$

We assume that conditions (a)–(g) from Section 5 are satisfied for the maps $T_\omega : Y_\omega \to Y_{\sigma\omega}$, $\omega \in E_A^+$. Then we call $F : X \times Y \to X \times Y$ a *generalized conformal skew product Smale endomorphism.*

Given the skew product F as above, we can also form a skew product endomorphism in the following way: define for every $\omega \in E_A^+$, the fiber map $\hat{F}_\omega : Y \to Y$ by

$$\hat{F}_\omega(y) = g(p(\omega), y).$$

The system $(\hat{Y}, \hat{F})$ is called *the symbolic lift* of F. If $\hat{Y} = E_A^+ \times Y$, we obtain a conformal skew product Smale endomorphism $\hat{F} : \hat{Y} \to \hat{Y}$ given by

$$\hat{F}(\omega, y) = (\sigma(\omega), \hat{F}_\omega(y)), \tag{8.1}$$

and the following diagram commutes,

$$\begin{array}{ccc} E_A^+ \times Y & \xrightarrow{\hat{F}} & E_A^+ \times Y \\ p \times \mathrm{Id} \big\downarrow & & \big\downarrow p \times \mathrm{Id} \\ X \times Y & \xrightarrow{F} & X \times Y. \end{array}$$

As in the beginning of Section 4, we study the structure of fibers $J_\omega,\ \omega \in E_A^+$ and later of the sets J_x, $x \in X$. From definition, $J_\omega = \hat{\pi}_2([\omega])$ and it is the set of points of type

$$\bigcap_{n \geq 1} \overline{\hat{F}_{\tau_{-1}\omega} \circ \hat{F}_{\tau_{-2}\tau_{-1}}\omega \circ \ldots \circ \hat{F}_{\tau_{-n}\ldots\tau_{-1}\omega}(Y)}.$$

Let us call *n-prehistory* of the point x with respect to the system (f, X), any finite sequence of points in X:

$$(x, x_{-1}, x_{-2}, \ldots, x_{-n}) \in X^{n+1},$$

where

$$f(x_{-1}) = x,\ f(x_{-2}) = x_{-1}, \ldots, f(x_{-n}) = x_{-n+1}.$$

Call a *complete prehistory* (or simply a *prehistory*) of x with respect to the system (f, X), any infinite sequence of consecutive preimages in X, i.e.

$$\hat{x} = (x, x_{-1}, x_{-2}, \ldots),$$

where

$$f(x_{-i}) = x_{-i+1}$$

forall integers $i \geq -1$. The space of complete prehistories is denoted by $\hat{X}$ and is called the *natural extension* (or *inverse limit*) of the system (f, X). We have a bijection $\hat{f} : \hat{X} \to \hat{X}$,

$$\hat{f}(\hat{x}) = (f(x), x, x_{-1}, \ldots).$$

In this paper, we use the terms inverse limit and natural extension interchangeably, without having necessarily a fixed invariant measure defined on the space X.
We consider on $\hat{X}$ the canonical metric, which induces the topology equivalent to the one inhereted from the product (Tichonov) topology on $X^{\mathbb{N}}$. With respect to this topology $\hat{f}$ becomes a homeomorphism. For more on the dynamics of endomorphisms and their inverse limits, one can see [29], [18], [20], [17].

In the above notation, we have $f(p(\tau_{-1}\omega)) = p(\omega) = x$, and for all the prehistories of x, $\hat{x} = (x, x_{-1}, x_{-2}, \ldots) \in \hat{X}$, consider the points of type

$$\bigcap_{n \geq 1} \overline{g_{x_{-1}} \circ g_{x_{-2}} \circ \ldots \circ g_{x_{-n}}(Y)},$$

The set of such points is denoted by J_x.

Notice that, if $\hat{\eta} = (\eta_0, \eta_1, \ldots)$ is another sequence in E_A^+ such that $p(\hat{\eta}) = x$, then for any η_{-1} so that $\eta_{-1}\hat{\eta} \in E_A^+$, we have $p(\eta_{-1}\hat{\eta}) = x'_{-1}$ where x'_{-1} is some 1-preimage (i.e preimage of order 1) of x. Hence from the definitions and the discussion above, we see that

$$J_x = \bigcup_{\omega \in E_A^+, p(\omega)=x} J_\omega \tag{8.2}$$

Let us denote the respective fibered limit sets for T and F by:

$$J = \bigcup_{\omega \in E_A^+} \{\omega\} \times J_\omega \subset E_A^+ \times Y \quad \text{and} \quad J(X) := \bigcup_{x \in X} \{x\} \times J_x \subset X \times Y \tag{8.3}$$

Then $\hat{F}(J) = J$ and $F(J(X)) = J(X))$. In addition, with the Hölder continuous projection $p_J : J \to J(X)$ defined by the formula

$$p_J(\omega, y) = (p(\omega), y),$$

i.e $p_J = (p \times \mathrm{Id})|_J$, the following diagram commutes.

$$\begin{array}{ccc} J & \xrightarrow{\hat{F}} & J \\ p_J \big\downarrow & & \big\downarrow p_J \\ J(X) & \xrightarrow{F} & J(X). \end{array}$$

In the sequel, $\hat{\pi}_2 : E_A \to Y$ and $\hat{\pi} : E_A \to E_A^+ \times Y$ are the maps defined in Section 4, and,

$$\hat{\pi}(\tau) = (\tau|_0^\infty, \hat{\pi}_2(\tau)).$$

Now, it will be important to know if enough points $x \in X$ have unique coding sequences in E_A^+.

Definition 8.1. *Let $F : X \times Y \to X \times Y$ be a generalized conformal skew product Smale endomorphism. Let μ be a Borel probability measure X. We then say that the coding $p : E_A^+ \to X$ is μ-injective, if there exists a μ-measurable set $G \subset X$ with $\mu(G) = 1$ such that for every point $x \in G$, the set $p^{-1}(x)$ is a singleton in E_A^+.*
Denote such a set G by G_μ and for $x \in G_\mu$ the only element of $p^{-1}(x)$ by $\omega(x)$.

Proposition 8.2. *If the coding $p : E_A^+ \to X$ is μ-injective, then for every $x \in G_\mu$, we have*

$$J_x = J_{\omega(x)}.$$

Proof. Take $x \in G_\mu$, and let $x_{-1} \in X$ be an f-preimage of x, i.e $f(x_{-1}) = x$. Since $p : E_A^+ \to X$ is surjective, there exists $\eta \in E_A^+$ such that $p(\eta) = x_{-1}$. But this implies that

$$f(x_{-1}) = f \circ p(\eta) = p \circ \sigma(\eta) = x.$$

Then, from the uniqueness of the coding sequence for x, it follows that $\sigma(\eta) = \omega(x)$, whence $x_{-1} = p(\omega_{-1}\omega(x))$, for some $\omega_{-1} \in E$. Since

$$J_x = \bigcap_{n \geq 1} \overline{g_{x_{-1}} \circ g_{x_{-2}} \circ \ldots \circ g_{x_{-n}}(Y)},$$

it follows that $J_x = J_{\omega(x)}$. $\square$

In the sequel we work only with μ-injective codings, and the measure μ will be clear from the context. Also given a metric space X with a coding $p : E_A^+ \to X$, and a potential $\phi : X \to \mathbb{R}$, we say that ϕ is *locally Hölder continuous* if $\phi \circ p$ is locally Hölder continuous.

Now consider a potential $\phi : J(X) \to \mathbb{R}$ such that the potential

$$\widehat{\phi} := \phi \circ p_J \circ \hat{\pi} : E_A \to \mathbb{R}$$

is locally Hölder continuous and summable. For example, $\widehat{\phi}$ is locally Hölder continuous if $\phi : J(X) \to \mathbb{R}$ is itself locally Hölder continuous. This case will be quite frequent in certain of our examples given later, when we will have locally Hölder continuous potentials ϕ on a set in $\mathbb{R}^2$ containing $J(X)$; however we will need and we will deal with the the above less restrictive case as well.

Define now

$$\mu_\phi := \mu_{\widehat{\phi}} \circ (p_J \circ \hat{\pi})^{-1}, \tag{8.4}$$

and call it the equilibrium measure of ϕ on $J(X)$ with respect to the skew product F.

Now, let us consider the partition ξ' of $J(X)$ into the fiber sets $\{x\} \times J_x$, $x \in X$, and the conditional measures μ_ϕ^x associated to μ_ϕ with respect to the measurable partition ξ' (see [26]). Recall that for each $\omega \in E_A^+$, we have $\hat{\pi}_2([\omega]) = J_\omega$.

Denote by $p_1 : X \times Y \to X$ the canonical projection onto the first coordinate, i.e.

$$p_1(x, y) = x.$$

Theorem 8.3. *Let $F : X \times Y \to X \times Y$ be a generalized conformal skew product Smale endomorphism. Let $\phi : J(X) \to \mathbb{R}$ be a potential such that $\widehat{\phi} = \phi \circ p_J \circ \hat{\pi} : E_A \to \mathbb{R}$ is a locally Hölder continuous summable potential on E_A. Assume that the coding $p : E_A^+ \to X$ is $\mu_\phi \circ p_1^{-1}$–injective, and denote the corresponding set $G_{\mu_\phi} \subset X$ by G_ϕ. Then:*

1. *$J_x = J_{\omega(x)}$ for every $x \in G_\phi$.*
2. *With $\bar{\mu}_\psi^\omega$, $\omega \in E_A^+$ the conditional measures of $\mu_{\widehat{\phi}}$, we have for $\mu_\phi \circ p_1^{-1}$–a.e. $x \in G_\phi$,*

$$\mu_\phi^x = \bar{\mu}_{\widehat{\phi}}^{\omega(x)} \circ (p_J \circ \hat{\pi})^{-1},$$

or equivalently, if μ_ϕ^x and $\bar{\mu}_{\widehat{\phi}}^{\omega(x)}$ are viewed as measures on J_x and E_A^-,

$$\mu_\phi^x = \bar{\mu}_{\widehat{\phi}}^{\omega(x)} \circ \hat{\pi}_2^{-1}$$

Proof. Part (1) is just a copy of Proposition 8.2. We thus deal with part (2) only. By the definition of canonical conditional measures, we have for every μ_ϕ–integrable function $H : J(X) \to \mathbb{R}$ that

$$\int_{J(X)} H d\mu_\phi = \int_{E_A} H \circ p_J \circ \hat{\pi} d\mu_{\widehat{\phi}} = \int_{E_A^+} \int_{[\omega]} H \circ p_J \circ \hat{\pi}\, d\bar{\mu}_{\widehat{\phi}}^\omega \, d\mu_{\widehat{\phi}} \circ \pi_1^{-1}(\omega) \tag{8.5}$$

and

$$\int_{J(X)} H d\mu_\phi = \int_X \int_{\{x\} \times J_x} H \, d\mu_\phi^x \, d\mu_\phi \circ p_1^{-1}(x) \tag{8.6}$$

But from the definitions of various projections:

(8.7) $\mu_\phi \circ p_1^{-1} = \mu_{\widehat{\phi}} \circ (p_J \circ \hat{\pi})^{-1} \circ p_1^{-1} = \mu_{\widehat{\phi}} \circ (p_1 \circ p_J \circ \hat{\pi})^{-1} = \mu_{\widehat{\phi}} \circ (p \circ \pi_1)^{-1} = \mu_{\widehat{\phi}} \circ \pi_1^{-1} \circ p^{-1}$.

Therefore, remembering also that $\mu_\phi \circ p_1^{-1}(G_\phi) = 1$, we get that

(8.8)

$$\begin{aligned}\int_{E_A^+} \int_{[\omega]} H \circ p_J \circ \hat{\pi}\, d\bar{\mu}_{\widehat{\phi}}^{\omega}\, d\mu_{\widehat{\phi}} \circ \pi_1^{-1}(\omega) &= \int_{E_A^+} \int_{\{p(\omega)\} \times J_{p(\omega)}} H d\bar{\mu}_{\widehat{\phi}}^{\omega} \circ (p_J \circ \hat{\pi})^{-1}\, d\mu_{\widehat{\phi}} \circ \pi_1^{-1}(\omega) \\ &= \int_{G_\phi} \int_{\{x\} \times J_x} H d\bar{\mu}_{\widehat{\phi}}^{\omega(x)} \circ (p_J \circ \hat{\pi})^{-1}\, d\mu_{\widehat{\phi}} \circ \pi_1^{-1} \circ p^{-1}(x) \\ &= \int_{G_\phi} \int_{\{x\} \times J_x} H d\bar{\mu}_{\widehat{\phi}}^{\omega(x)} \circ (p_J \circ \hat{\pi})^{-1}\, d\mu_\phi \circ p_1^{-1}(x).\end{aligned}$$

Hence this, together with (8.5) and (8.6), gives

$$\int_{G_\phi} \int_{\{x\} \times J_x} H\, d\mu_\phi^x\, d\mu_\phi \circ p_1^{-1}(x) = \int_{G_\phi} \int_{\{x\} \times J_x} H d\bar{\mu}_{\widehat{\phi}}^{\omega(x)} \circ (p_J \circ \hat{\pi})^{-1}\, d\mu_\phi \circ p_1^{-1}(x).$$

Thus, the uniqueness of the system of Rokhlin's canonical conditional measures yields

$$\mu_\phi^x = \bar{\mu}_{\widehat{\phi}}^{\omega(x)} \circ (p_J \circ \hat{\pi})^{-1}$$

for $\mu_\phi \circ p_1^{-1}$–a.e. $x \in G_\phi$. This means that the first part of (2) is established. Next, note that $p_J \circ \hat{\pi} = (p \circ \pi_1) \times \hat{\pi}_2$ and thus $p_J \circ \hat{\pi}|_{[\omega(x)]} = \{x\} \times \hat{\pi}_2|_{[\omega(x)]}$.

□

As in the previous Section, define a *Lyapunov exponent* for an F-invariant measure μ on the fibered limit set $J(X) = \bigcup_{x \in X} \{x\} \times J_x$, by:

$$\chi_\mu(F) = -\int_{J(X)} \log |g_x'(y)|\, d\mu(x, y).$$

In conclusion, as an immediate consequence of Theorem 8.3, Theorem 6.2, and definition (8.4), we obtain the following result for skew product endomorphisms over countable-to-1 maps $f : X \to X$:

Theorem 8.4. *Let $F : X \times Y \to X \times Y$ a generalized conformal skew product Smale endomorphism. Let $\phi : J(X) \to \mathbb{R}$ be a potential such that*

$$\psi := \phi \circ p_J \circ \hat{\pi} : E_A \to \mathbb{R}$$

is locally Hölder continuous summable. Assume the coding $p : E_A^+ \to X$ is $\mu_\phi \circ p_1^{-1}$–injective.

Then, for $\mu_\phi \circ p_1^{-1}$-a.e $x \in X$, the conditional measure μ_ϕ^x is exact dimensional on J_x, and moreover

$$\lim_{r \to 0} \frac{\log \mu_\phi^x(B(y, r))}{\log r} = \frac{\mathrm{h}_{\mu_\phi}(\mathrm{F})}{\chi_{\mu_\phi}(F)} = \mathrm{HD}(\mu_\phi^x),$$

for μ_ϕ^x-a.e $y \in J_x$; hence, equivalently, for μ_ϕ-a.e $(x, y) \in J(X)$.

In turn, as an immediate consequence of this theorem, we get the following.

Corollary 8.5. *Let $F : X \times Y \to X \times Y$ a generalized conformal skew product Smale endomorphism. Let $\phi : J(X) \to \mathbb{R}$ be a locally Hölder continuous potential such that*

$$\sum_{e \in E} \exp(\sup(\phi|_{\pi([e]) \times Y})) < \infty.$$

Assume that the coding $p : E_A^+ \to X$ is $\mu_\phi \circ p_1^{-1}$–injective.

Then, for $\mu_\phi \circ p_1^{-1}$-a.e $x \in X$, the conditional measure μ_ϕ^x is exact dimensional on J_x, and moreover

$$\lim_{r \to 0} \frac{\log \mu_\phi^x(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_\phi}(\mathrm{F})}{\chi_{\mu_\phi}(F)} = \mathrm{HD}(\mu_\phi^x),$$

for μ_ϕ^x-a.e $y \in J_x$; hence, equivalently, for μ_ϕ-a.e $(x, y) \in J(X)$.

Remark 8.6. *Note that if in the settings of this section, in particular in Theorem 8.3, Theorem 8.4, and Corollary 8.5, the "symbol" map $\hat{F} : J \to J$ is assumed to be of compact type, then*

$$\mu_\phi = \mu_{\phi \circ p_J} \circ p_J^{-1},$$

where $\mu_{\phi \circ p_J}$ is the unique equilibrium state (in the sense of variational principle) of the potential $\phi \circ p_J : J \to \mathbb{R}$, produced in Theorem 4.20.

By using Theorem 8.4, we will be able to prove exact dimensionality of conditional measures of equilibrium states on fibers, for many types of skew products. In the forthcoming sections we will describe such (large) classes of applications.

First, let us prove a general result about exact dimensionality of measures on whole fibered limit sets $J(X)$. We want to prove that, if the conditional measures on fibers are exact dimensional, with the same value of the dimension regardless of fiber, and if the projection on the first coordinate is also exact dimensional, then the original measure μ is exact dimensional with its dimension equal to the sum of the above dimensions.

Theorem 8.7. *Let $F : X \times Y \to X \times Y$ a generalized conformal skew product Smale endomorphism. Assume that $X \subset \mathbb{R}^d$ with some integer $d \geq 1$. Let μ be a Borel probability F–invariant measure on $J(X)$, and $(\mu^x)_{x \in X}$ be the Rokhlin's canonical sytem of conditional measures of μ, with respect to the partition $\big(\{x\} \times J_x\big)_{x \in X}$. Assume that:*

a) There exists $\alpha > 0$ such that for $\mu \circ p_1^{-1}$-a.e $x \in X$ the conditional measure μ^x is exact dimensional and $\mathrm{HD}(\mu_x) = \alpha$,

b) The measure $\mu \circ p_1^{-1}$ is exact dimensional on X.

Then the measure μ is exact dimensional on $J(X)$, and for μ-a.e $(x, y) \in J(X)$,

$$\mathrm{HD}(\mu) = \lim_{r \to 0} \frac{\log \mu(B((x,y),r))}{\log r} = \alpha + \mathrm{HD}(\mu \circ p_1^{-1}).$$

Proof. Denote the canonical projection to first coordinate by $p_1 : X \times Y \to X$. Let then

$$\nu := \mu \circ p_1^{-1}.$$

Denote the Hausdorff dimension $\mathrm{HD}(\nu)$ by γ. From the exact dimensionality of the conditional measures of μ, we know that for ν–a.e $x \in X$ and for μ^x–a.e $y \in Y$,

$$\lim_{r\to 0} \frac{\log \mu^x(B(y,r))}{\log r} = \alpha.$$

Then for any $\varepsilon \in (0,\alpha)$ and any integer $n \geq 1$, consider the following Borel set in $X \times Y$:

$$A(n,\varepsilon) := \left\{ z = (x,y) \in X \times Y : \ \alpha - \varepsilon < \frac{\log \mu^x(B(y,r))}{\log r} < \alpha + \varepsilon \ \text{ for all } \ r \in (0, 1/n) \right\}.$$

From definition it is clear that $A(n,\varepsilon) \subset A(n+1,\varepsilon)$ for all $n \geq 1$. Moreover, setting

$$X_Y' := \bigcap_{\varepsilon>0} \bigcup_{n=1}^{\infty} A(n,\varepsilon),$$

it follows from the exact dimensionality of almost all the conditional measures of μ and from the equality of their pointwise dimensions, that

$$\mu(X_Y') = 1.$$

For $\varepsilon > 0$ and $n \geq 1$, consider also the following Borel subset of X:

$$D(n,\varepsilon) := \left\{ x \in X : \ \gamma - \varepsilon < \frac{\log \nu(B(x,r))}{\log r} < \gamma + \varepsilon \ \text{ for all } \ r \in (0,1/n) \right\}.$$

We know that $D(n,\varepsilon) \subset D(n+1,\varepsilon)$ for all $n \geq 1$, and from the exact dimensionality of ν, we obtain that for every $\varepsilon > 0$, we have

$$\nu\left(\bigcup_{n=1}^{\infty} D(n,\varepsilon) \right) = 1.$$

For $\varepsilon > 0$ and an integer $n \geq 1$, let us denote now

$$E(n,\varepsilon) := A(n,\varepsilon) \cap p_1^{-1}(D(n,\varepsilon)).$$

Clearly from above, we have that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(E(n,\varepsilon)) = 1. \tag{8.9}$$

From the definition of conditional measures and the definition of $A(n,\varepsilon)$ and $D(n,\varepsilon)$, we have that, for any $z \in E(n,\varepsilon)$, $x = \pi_1(z)$ and any $n \geq 1, \varepsilon > 0, 0 < r < 1/n$,

$$\begin{aligned} \mu(E(n,\varepsilon) \cap B(z,r)) &= \int_{D(n,\varepsilon)\cap B(x,r)} \mu^y\big(B(z,r) \cap (\{y\} \times Y) \cap A(n,\varepsilon)\big)\, d\nu(y) \\ &\leq \int_{D(n,\varepsilon)\cap B(x,r)} r^{\alpha-\varepsilon}\, d\nu(y) = r^{\alpha-\varepsilon}\nu(D(n,\varepsilon)\cap B(x,r)) \\ &\leq r^{\alpha+\gamma-2\varepsilon}. \end{aligned} \tag{8.10}$$

Since $\mu(E(n,\varepsilon)) > 0$ for all $n \geq 1$ large enough, it follows from Borel Density Lemma - Lebesgue Density Theorem that, for μ-a.e $z \in E(n,\varepsilon)$, we have that

$$\lim_{r\to 0} \frac{\mu(B(z,r) \cap E(n,\varepsilon))}{\mu(B(z,r))} = 1.$$

Thus for any $\theta > 1$ arbitrary, there exists a subset $E(n, \varepsilon, \theta)$ of $E(n, \varepsilon)$, such that

$$\mu(E(n, \varepsilon, \theta)) = \mu(E(n, \varepsilon)),$$

and for every $z \in E(n, \varepsilon, \theta)$ there exists $r(z, \theta) > 0$ so that for any $0 < r < \inf\{r(z, \theta), 1/n\}$, we have from 8.10:

$$\mu(B(z, r)) \leq \theta \mu(E(n, \varepsilon) \cap B(z, r)) \leq \theta \cdot r^{\alpha+\gamma-2\varepsilon}$$

Thus for $z \in E(n, \varepsilon, \theta)$, we obtain

$$\lim_{r\to 0} \frac{\log \mu(B(z, r))}{\log r} \geq \alpha + \gamma - 2\varepsilon.$$

Now, since $\mu(E(n, \varepsilon, \theta)) = \mu(E(n, \varepsilon))$, it follows from (8.9) that $\mu(\bigcup\limits_n E(n, \varepsilon, \theta)) = 1$. Hence

$$\mu\left(\bigcap_{\varepsilon>0}\bigcap_{\theta>1}\bigcup_{n=1}^{\infty} E(n, \varepsilon, \theta)\right) = 1,$$

and for points $z \in \bigcap\limits_{\varepsilon>0}\bigcap\limits_{\theta>1}\bigcup\limits_{n} E(n, \varepsilon, \theta)$, we have

$$\lim_{r\to 0} \frac{\log \mu(B(z, r))}{\log r} \geq \alpha + \gamma$$

Conversely, from the exact dimensionality of ν and of the conditional measures of μ, and with $x = \pi_1(z)$, we have that for $r \in (0, 1/n)$,

$$\begin{aligned} \mu(B(z, r) \cap E(n, \varepsilon)) &= \int_{D(n,\varepsilon)\cap B(x,r)} \mu^y\big(B(z, r) \cap A(n, \varepsilon) \cap \{y\} \times Y\big)\, d\nu(y) \\ &\geq r^{\alpha+\gamma+2\varepsilon} \end{aligned} \tag{8.11}$$

Thus we have that $\mu(B(z, r)) \geq \mu(B(z, r) \cap E(n, \varepsilon)) \geq r^{\alpha+\gamma+2\varepsilon}$, for $z \in E(n, \varepsilon)$ and $r \in (0, 1/n)$. Making use of (8.9) we deduce that μ is exact dimensional, and for μ-a.e $z \in X \times Y$ we have:

$$\lim_{r\to 0} \frac{\log \mu(B(z, r))}{\log r} = \alpha + \gamma$$

□

By using the results of this Section, we will be able to prove in the next section exact dimensionality of conditional measures on fibers for equilibrium measures with respect to Smale endomorphisms on skew products over various base maps. In the next Sections we will apply the results obtained above to skew products over systems that are modeled by shifts with countable alphabets. We will study several classes for which we will say more and be more specific, such as Lüroth series (GLS) and their natural extensions, beta transformations and their natural extensions, expanding Markov Rényi maps, the Gauss map, Manneville-Pomeau maps (which have parabolic points), etc.

9. Skew Products with the Base Maps Being Graph-Directed Markov Systems

In this section we consider systems for which the base map is induced by a countable alphabet conformal graph directed Markov system (GDMS) as in [15]. Our main goal here is to prove that equilibrium measures for skew products over the base maps are exact dimensional. We will use among other results Theorem 8.7 and a result from [21]. A *directed multigraph* consists of:

- A finite set V of vertices,
- A countable (either finite or infinite) set E of directed edges,
- A map $A : E \times E \to \{0,1\}$ called an *incidence matrix* on (V,E),
- Two functions $i,t : E \to V$, such that $A_{ab} = 1$ implies $t(b) = i(a)$.

Now suppose that in addition, we have a collection of nonempty compact metric spaces $\{X_v\}_{v\in V}$ and a number $\lambda \in (0,1)$, and that for every $e \in E$, we have a one-to-one contraction $\phi_e : X_{t(e)} \to X_{i(e)}$ with Lipschitz constant $\leq \lambda$. Then the collection

$$\mathcal{S} = \{\phi_e : X_{t(e)} \to X_{i(e)}\}_{e\in E}$$

is called a *graph directed Markov system* (or *GDMS*). We now describe the limit set of the system $\mathcal{S}$. For every $\omega \in E_A^+$, the sets $\{\phi_{\omega|_n}\left(X_{t(\omega_n)}\right)\}_{n\geq 1}$ form a descending sequence of nonempty compact sets and therefore $\bigcap_{n\geq 1} \phi_{\omega|_n}\left(X_{t(\omega_n)}\right) \neq \emptyset$. Since for every $n \geq 1$,

$$\mathrm{diam}\left(\phi_{\omega|_n}\left(X_{t(\omega_n)}\right)\right) \leq \lambda^n \mathrm{diam}\left(X_{t(\omega_n)}\right) \leq \lambda^n \max\{\mathrm{diam}(X_v) : v \in V\},$$

we conclude that the intersection $\bigcap_{n\in\mathbb{N}} \phi_{\omega|_n}\left(X_{t(\omega_n)}\right)$ is a singleton, and we denote its only element by $\pi(\omega)$. In this way we have defined a map

$$\pi : E_A^+ \to \coprod_{v\in V} X_v,$$

where $X := \coprod_{v\in V} X_v$ is the disjoint union of the compact sets X_v ($v \in V$). The map π is called the *coding map*, and the set

$$J = J_{\mathcal{S}} = \pi(E_A^+)$$

is called the *limit set* of the GDMS $\mathcal{S}$. The sets $J_v = \pi(\{\omega \in E_A^+ : i(\omega_1) = v\})$, for $v \in V$, are called the *local limit sets* of $\mathcal{S}$.

We call the GDMS $\mathcal{S}$ *finite* if the alphabet E is finite. Furthermore, we call $\mathcal{S}$ *maximal* if for all $a,b \in E$, we have $A_{ab} = 1$ if and only if $t(b) = i(a)$. In [15] a maximal GDMS was called a *graph directed system* (abbr. GDS). Finally, we call a maximal GDMS $\mathcal{S}$ an *iterated function system* (or *IFS*) if V, the set of vertices of $\mathcal{S}$, is a singleton. Equivalently, a GDMS is an IFS if and only if the set of vertices of $\mathcal{S}$ is a singleton and all entries of the incidence matrix A are equal to 1.

Definition 9.1. *We call the GDMS $\mathcal{S}$ and its incidence matrix A* finitely (symbolically) irreducible *if there exists a finite set $\Lambda \subset E_A^*$ such that for all $a,b \in E$ there exists a word $\omega \in \Lambda$ such that the concatenation $a\omega b$ is in E_A^*. $\mathcal{S}$ and A are called* finitely primitive *if* *the set Λ may be chosen to consist of words all having the same length. Note that all IFSs are finitely primitive.*

Intending to pass to geometry, we call a GDMS *conformal* if for some $d \in \mathbb{N}$, the following conditions are satisfied:

(a) For every vertex $v \in V$, X_v is a compact connected subset of $\mathbb{R}^d$, and $X_v = \overline{\mathrm{Int}(X_v)}$.

(b) There exists a family of open connected sets $W_v \subset X_v$ $(v \in V)$ such that for every $e \in E$, the map ϕ_e extends to a C^1 conformal diffeomorphism from $W_{t(e)}$ into $W_{i(e)}$ with Lipschitz constant $\leq \lambda$.

(c) (Bounded Distortion Property BDP) There are two constants $L \geq 1$ and $\alpha > 0$ such that for every $e \in E$ and every pair of points $x, y \in X_{t(e)}$,

$$\left| \frac{|\phi_e'(y)|}{|\phi_e'(x)|} - 1 \right| \leq L\|y - x\|^\alpha,$$

where $|\phi_\omega'(x)|$ denotes the scaling of the derivative, which is a linear similarity map.

(d) (Open Set Condition OSC) For all $a, b \in E$, if $a \neq b$, then

$$\phi_a(\mathrm{Int}(X_a)) \cap \phi_b(\mathrm{Int}(X_b)) = \emptyset.$$

(e) (Boundary Condition) There exists $e \in E$ such that

$$J_{\mathcal{S}} \cap \mathrm{Int} X_e \neq \emptyset.$$

If the Open Set Condition and the Boundary Condition are both satisfied, then we say that the Strong Open Set Condition (SOSC) is satisfied.

Remark 9.2. *By Koebe's Distortion Theorem condition (c) is automaticallly satisfied if $d = 2$ and if $d \geq 3$ it is a, not too hard, consequence of Liouville's Representation Theorem. See [15] for details.*

We define the GDMS map $f = f_{\mathcal{S}} : J_{\mathcal{S}} \to J_{\mathcal{S}}$, associated to the system $\mathcal{S}$, by

$$f(\phi_e(x)) = x \tag{9.1}$$

if $x \in \mathrm{Int}(X_{t(e)})$ (then e is uniquely determined), and $f(z)$ to be some given preassigned point ξ of $J_{\mathcal{S}}$, if $z \notin \bigcup_{e \in E} \phi_e\big(\mathrm{Int}(X_{t(e)})\big)$.

A special class of conformal GDMSs is provided by one dimensional-systems. Precisely, if X is compact interval in $\mathbb{R}$, then the GDMSs, more precisely the derived maps $f_{\mathcal{S}} : J_{\mathcal{S}} \to J_{\mathcal{S}}$, associated to them, are sometimes called *expanding Markov-Rényi) maps (EMR maps)*; see [24]. A sufficient condition for (BDP), i.e. (c) is that

$$\sup_{e \in E} \sup_{x,y,z \in X} \left\{ \frac{|\phi_e''(x)|}{|\phi_e'(y)| \cdot |\phi_e'(z)|} \right\} < \infty.$$

It is known as the *Rényi condition.*

Let us now consider a general GDMS map $f : J_{\mathcal{S}} \to J_{\mathcal{S}}$, and a skew product $F : J_{\mathcal{S}} \times Y \to J_{\mathcal{S}} \times Y$, where $Y \subset \mathbb{R}^d$ is a bounded open set, with

$$F(x, y) = (f(x), g(x, y)).$$

Recall from Section 8, that the symbolic lift of F is $\hat{F} : E_A^+ \times Y \to E_A^+ \times Y$, given by

$$\hat{F}(\omega, y) = (\sigma(\omega), g(\pi(\omega), y)).$$

The map $p : E_A^+ \to X$ is now equal to the map $\pi_{\mathcal{S}} : E_A^+ \to X$. So, the map $p \times \mathrm{Id} : E_A^+ \times Y \to J_{\mathcal{S}} \times Y$ is given by the formula

$$(p \times \mathrm{Id})(\omega, y) := (\pi_{\mathcal{S}}(\omega), y).$$

Using the notation of Section 8, we denote its restriction to the set $J = \bigcup_{\omega \in E_A^+} \{\omega\} \times J_\omega$ by p_J. If the symbolic lift $\hat{F}$ is a Hölder conformal skew product Smale endomorphism, then we say by extension that F is a *Hölder conformal skew product endomorphism over* f. Recall also from (8.3) that the fibered limit set of F is

$$J(J_{\mathcal{S}}) = \bigcup_{x \in J_{\mathcal{S}}} \{x\} \times J_x.$$

The first result, easy but crucial for us is the following.

Lemma 9.3. *Let $f : J_{\mathcal{S}} \to J_{\mathcal{S}}$ be a finitely irreducible conformal GDMS map, let $Y \subset \mathbb{R}^d$ be an open bounded set, and let $F : J_{\mathcal{S}} \times Y \to J_{\mathcal{S}} \times Y$ be a Hölder conformal skew product endomorphism over f. If ν is a Borel probability shift–invariant ergodic measure on E_A^+ with full topological support, then the coding $p = \pi_{\mathcal{S}} : E_A^+ \to J_{\mathcal{S}}$ is $\nu \circ \pi_{\mathcal{S}}^{-1}$–injective.*

Proof. Since $\phi_e(\mathrm{Int} X_{t(e)}) \subset \mathrm{Int} X_{t(e)}$, we have that $\sigma^{-1}\big(\pi_{\mathcal{S}}^{-1}(\mathrm{Int} X)\big) \subset \pi_{\mathcal{S}}^{-1}(\mathrm{Int} X)$, where

$$\mathrm{Int} X := \bigcup_{e \in E} \mathrm{Int} X_{t(e)}$$

Since the Borel probability measure ν is shift–invariant and ergodic, it thus follows that $\nu\big(\pi_{\mathcal{S}}^{-1}(\mathrm{Int} X)\big) \in \{0, 1\}$. But since $\mathrm{supp}(\nu) = E_A^+$, it thus follows from the Strong Open Set Condition (e) that $\nu\big(\pi_{\mathcal{S}}^{-1}(\mathrm{Int} X)\big) > 0$. Hence,

$$\nu\big(\pi_{\mathcal{S}}^{-1}(\mathrm{Int} X)\big) = 1$$

Invoking shift–invariance of the measure ν again, we thus conclude that

$$\nu\left(\bigcap_{n=0}^{\infty} \sigma^{-n}(\pi_{\mathcal{S}}^{-1}(\mathrm{Int} X))\right) = 1$$

Denote the set in parentheses by $\mathrm{Int}_\infty(\mathcal{S})$. Then, by the Open Set Condition (d), the map $\pi_{\mathcal{S}}|_{\mathrm{Int}_\infty(\mathcal{S})}$ is one–to–one and $\pi_{\mathcal{S}}^{-1}\big(\pi_{\mathcal{S}}(\mathrm{Int}_\infty(\mathcal{S}))\big) = \mathrm{Int}_\infty(\mathcal{S})$. Thus,

$$\nu \circ \pi_{\mathcal{S}}^{-1}\big(\pi_{\mathcal{S}}(\mathrm{Int}_\infty(\mathcal{S}))\big) = \nu\big(\mathrm{Int}_\infty(\mathcal{S})\big) = 1,$$

and for every point $x \in \pi_{\mathcal{S}}(\mathrm{Int}_\infty(\mathcal{S}))$, the set $\pi_{\mathcal{S}}^{-1}(x)$ is a singleton.

□

The following result follows then directly from Theorem 8.4 and Lemma 9.3.

Theorem 9.4. *Let $f : J_{\mathcal{S}} \to J_{\mathcal{S}}$ be a finitely irreducible conformal GDMS map, let $Y \subset \mathbb{R}^d$ be an open bounded set, and let $F : J_{\mathcal{S}} \times Y \to J_{\mathcal{S}} \times Y$ be a Hölder conformal skew product endomorphism over f. Let $\phi : J(J_{\mathcal{S}}) \to \mathbb{R}$ be a potential such that*

$$\widehat{\phi} = \phi \circ p_J \circ \hat{\pi} : E_A \to \mathbb{R}$$

is a locally Hölder continuous summable potential on E_A. Then, for $\mu_\phi \circ p_1^{-1}$-a.e $x \in X$, the conditional measure μ_ϕ^x is exact dimensional on J_x, and moreover

$$\lim_{r \to 0} \frac{\log \mu_\phi^x(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_\phi}(\mathrm{F})}{\chi_{\mu_\phi}(F)} = \mathrm{HD}(\mu_\phi^x),$$

for μ_ϕ^x-a.e $y \in J_x$; hence, equivalently, for μ_ϕ-a.e $(x,y) \in J(J_{\mathcal{S}})$.

Proof. One only needs to notice that $\mathrm{supp}(\mu_{\widehat{\phi}} \circ \pi^{-1}) = E_A^+$ since $\mu_{\widehat{\phi}}$ is the equilibrium state of the locally Hölder continuous summable potential $\widehat{\phi}$ on E_A. Indeed one uses formula (8.7) to apply Lemma 9.3 with the measure $\nu := \mu_{\widehat{\phi}} \circ \pi^{-1}$, to conlude that the coding $p = \pi_{\mathcal{S}} : E_A^+ \to X$ is $\mu_\phi \circ p_1^{-1}$–injective. Hence Theorem 8.4 applies to end the proof. □

Now consider the following situation. Let $\mathcal{S}$, f, F, and Y be as above. Let $\theta : J_{\mathcal{S}} \to \mathbb{R}$ be an arbitrary potential such that $\theta \circ \pi_{\mathcal{S}} : E_A^+ \to \mathbb{R}$ is a locally Hölder continuous summable potential. Let

$$\phi_\theta := \theta \circ p_1 : J(J_{\mathcal{S}}) \to \mathbb{R}. \tag{9.2}$$

then we have the following.

Lemma 9.5. *The potential*

$$\widehat{\phi_\theta} = \phi_\theta \circ p_J \circ \hat{\pi} : E_A \to \mathbb{R}$$

is locally Hölder continuous and summable.

Proof. Since $\widehat{\phi_\theta} = (\theta \circ \pi_{\mathcal{S}}) \circ \pi_1$, it follows that $\widehat{\phi_\theta}$ is locally Hölder continuous as a composition of two locally Hölder continuous functions. From the definition of summability, the function $\widehat{\phi_\theta} : E_A \to \mathbb{R}$ is summable, since in its composition, the function $\theta \circ \pi_{\mathcal{S}} : E_A^+ \to \mathbb{R}$ is summable. □

In this setting, as a result related to Theorem 9.4, we get the following.

Theorem 9.6. *Let $\mathcal{S}$ be a finitely irreducible conformal GDMS. Let $f : J_{\mathcal{S}} \to J_{\mathcal{S}}$ be the corresponding GDMS map. Let $Y \subset \mathbb{R}^d$ be an open bounded set, and let $F : J_{\mathcal{S}} \times Y \to J_{\mathcal{S}} \times Y$ be a Hölder conformal skew product endomorphism over f. Let also $\theta : J_{\mathcal{S}} \to \mathbb{R}$ an arbitrary potential such that $\theta \circ \pi_{\mathcal{S}} : E_A^+ \to \mathbb{R}$ is a locally Hölder continuous summable. Then,*

a) *For $\mu_{\theta \circ \pi_{\mathcal{S}}} \circ \pi_{\mathcal{S}}^{-1}$–a.e. $x \in J_{\mathcal{S}}$, the conditional measure $\mu_{\phi_\theta}^x$ is exact dimensional on J_x, and*

$$\lim_{r \to 0} \frac{\log \mu_{\phi_\theta}^x(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_{\phi_\theta}}(F)}{\chi_{\mu_{\phi_\theta}}(F)}$$

for $\mu_{\phi_\theta}^x$–a.e. $y \in J_x$; hence, equivalently for μ_{ϕ_θ}–a.e $(x,y) \in J(J_{\mathcal{S}})$.

b) *The equilibrium state μ_{ϕ_θ} of $\phi_\theta : J(J_{\mathcal{S}}) \to \mathbb{R}$ for F, is exact dimensional on $J(J_{\mathcal{S}})$ and*
$$\mathrm{HD}\big(\mu_{\phi_\theta}\big) = \frac{\mathrm{h}_{\mu_{\phi_\theta}}(F)}{\chi_{\mu_{\phi_\theta}}(F)} + \mathrm{HD}\big(\mu_{\theta\circ\pi_{\mathcal{S}}} \circ \pi_{\mathcal{S}}^{-1}\big) = \frac{\mathrm{h}_{\mu_{\phi_\theta}}(F)}{\chi_{\mu_{\phi_\theta}}(F)} + \frac{\mathrm{h}_{\mu_{\theta\circ\pi_{\mathcal{S}}}}}{\chi_{\mu_{\theta\circ\pi_{\mathcal{S}}}}}.$$

Proof. It follows from (8.4) that
$$\mu_{\phi_\theta} \circ p_1^{-1} = \mu_{\phi_\theta\circ p_J\circ\hat{\pi}} \circ (p_J\circ\hat{\pi})^{-1} \circ p_1^{-1} = \mu_{\theta\circ p_1\circ p_J\circ\hat{\pi}} \circ (p_1\circ p_J\circ\hat{\pi})^{-1}.$$
Since $\pi_{\mathcal{S}} \circ \pi_1 = p_1 \circ p_J \circ \hat{\pi}$, we get from Remark 3.9 that
$$\mu_{\theta\circ\pi_{\mathcal{S}}} \circ \pi_{\mathcal{S}}^{-1} = \mu_{\theta\circ\pi_{\mathcal{S}}\circ\pi_1} \circ \pi_1^{-1} \circ \pi_{\mathcal{S}}^{-1} = \mu_{\theta\circ p_1\circ p_J\circ\hat{\pi}} \circ (p_1\circ p_J\circ\hat{\pi})^{-1}.$$
Therefore,
$$\mu_{\phi_\theta} \circ p_1^{-1} = \mu_{\theta\circ\pi_{\mathcal{S}}} \circ \pi_{\mathcal{S}}^{-1}.$$
Hence, a) follows now directly from Theorem 9.4, while b) follows from a) and Theorem 8.7 since exact dimensionality of the measure $\mu_{\theta\circ\pi_{\mathcal{S}}} \circ \pi_{\mathcal{S}}^{-1}$ has been proved in [21]. Indeed in [21] we proved, as a particular case of the random case, the exact dimensionality for all projections of ergodic invariant measures on limit sets of countable conformal IFS with arbitrary overlaps; and this result extends easily to GDMS. □

Remark 9.7. *Writing $\tilde{\phi}_\theta := \phi_\theta \circ p_J : J \to \mathbb{R}$, and defining*
$$\mu_{\tilde{\phi}_\theta} := \mu_{\tilde{\phi}_\theta\circ\hat{\pi}} \circ \hat{\pi}^{-1}$$
we will have
$$\mu_{\phi_\theta} = \mu_{\widehat{\phi}_\theta} \circ (p_J\circ\hat{\pi})^{-1} = \mu_{\tilde{\phi}_\theta\circ\hat{\pi}} \circ \hat{\pi}^{-1} \circ p_J^{-1} = \mu_{\tilde{\phi}_\theta} \circ p_J^{-1}.$$
Note also that if the symbolic lift map $\hat{F} : J \to J$ is assumed to be of compact type, then $\mu_{\tilde{\phi}_\theta}$ is the unique equilibrium state (in the usual sense of variational principle) of the potential $\tilde{\phi}_\theta : J \to \mathbb{R}$, produced in Theorem 4.20.

An immediate consequence of Theorem 9.4 is:

Corollary 9.8. *Let $f : J_{\mathcal{S}} \to J_{\mathcal{S}}$ be a finitely irreducible conformal GDMS map, let $Y \subset \mathbb{R}^d$ be an open bounded set, and let $F : J_{\mathcal{S}} \times Y \to J_{\mathcal{S}} \times Y$ be a Hölder conformal skew product endomorphism over f. Let $\phi : J(J_{\mathcal{S}}) \to \mathbb{R}$ be a locally Hölder continuous potential such that $\widehat{\phi} = \phi \circ p_J \circ \hat{\pi} : E_A \to \mathbb{R}$ is summable. Then, for $\mu_\phi \circ p_1^{-1}$-a.e $x \in X$, the conditional measure μ_ϕ^x is exact dimensional on J_x, and moreover*
$$\lim_{r\to 0} \frac{\log \mu_\phi^x(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_\phi}(\mathrm{F})}{\chi_{\mu_\phi}(F)} = \mathrm{HD}(\mu_\phi^x),$$
for μ_ϕ^x-a.e $y \in J_x$; hence, equivalently, for μ_ϕ-a.e $(x,y) \in J(J_{\mathcal{S}})$.

A Corollary of Theorem 9.6, which will be applied to EMR maps (in the sense of [24]), is then the following:

Corollary 9.9. *Let $\mathcal{S}$ be a finitely irreducible conformal GDMS. Let $f : J_{\mathcal{S}} \to J_{\mathcal{S}}$ be the corresponding GDMS map. Let $Y \subset \mathbb{R}^d$ be an open bounded set, and let $F : J_{\mathcal{S}}\times Y \to J_{\mathcal{S}}\times Y$ be a Hölder conformal skew product endomorphism over f. Let $\theta : J_{\mathcal{S}} \to \mathbb{R}$ be an arbitrary locally Hölder continuous potential such that $\theta \circ \pi_{\mathcal{S}} : E_A^+ \to \mathbb{R}$ is summable potential. Then,*

(a) *For $\mu_{\theta\circ\pi_{\mathcal{S}}}\circ\pi_{\mathcal{S}}^{-1}$–a.e. $x\in J_{\mathcal{S}}$, the conditional measure μ_ϕ^x is exact dimensional on J_x; in fact*

$$\lim_{r\to 0}\frac{\log\mu_{\phi_\theta}^x(B(y,r))}{\log r}=\frac{\mathrm{h}_{\mu_{\phi_\theta}}(F)}{\chi_{\mu_{\phi_\theta}}(F)}$$

for $\mu_{\phi_\theta}^x$–a.e. $y\in J_x$; hence, equivalently for μ_{ϕ_θ}–a.e $(x,y)\in J(J_{\mathcal{S}})$.

(b) *The equilibrium state μ_{ϕ_θ} of $\phi_\theta: J(J_{\mathcal{S}})\to\mathbb{R}$ for F, is exact dimensional on $J(J_{\mathcal{S}})$ and*

$$\mathrm{HD}\big(\mu_{\phi_\theta}\big)=\frac{\mathrm{h}_{\mu_{\phi_\theta}}(F)}{\chi_{\mu_{\phi_\theta}}(F)}+\mathrm{HD}\big(\mu_{\theta\circ\pi_{\mathcal{S}}}\circ\pi_{\mathcal{S}}^{-1}\big)=\frac{\mathrm{h}_{\mu_{\phi_\theta}}(F)}{\chi_{\mu_{\phi_\theta}}(F)}+\frac{\mathrm{h}_{\mu_{\theta\circ\pi_{\mathcal{S}}}}}{\chi_{\mu_{\theta\circ\pi_{\mathcal{S}}}}}.$$

Remark 9.10. *As mentioned above, it folows from the last Corollary that our results hold if the derived map $f_{\mathcal{S}}: I\to I$ associated to the GDMS $\mathcal{S}$, is an expanding Markov-Rényi (EMR) map, in the sense of* [24].

Now, consider further an arbitrary conformal GDMS

$$\mathcal{S}=\{\phi_e: X_{t(e)}\to X_{i(e)}\}_{e\in E}$$

Let $\theta: J_{\mathcal{S}}\to\mathbb{R}$ be a potential such that $\theta\circ\pi_{\mathcal{S}}: E_A^+\to\mathbb{R}$ is locally Hölder continuous and summable. Of particular importance are then the potentials $\theta_{q,t}: J_{\mathcal{S}}\to\mathbb{R}$, $t,q\in\mathbb{R}$,

$$\theta_{q,t}(\phi_e(x)):=t\log|\phi_e'(x)|+q(\theta(\phi_e(x))-\mathrm{P}(\theta)) \tag{9.3}$$

Then

$$\theta_{q,t}\circ\pi_{\mathcal{S}}(\omega)=t\log\big|\phi_{\omega_0}'(\pi_{\mathcal{S}}(\sigma(\omega)))\big|+q\big(\theta\circ\pi_{\mathcal{S}}(\omega)-\mathrm{P}(\phi)\big)$$

In terms of the GDMS map $f_{\mathcal{S}}: J_{\mathcal{S}}\to J_{\mathcal{S}}$ associated to the system $\mathcal{S}$, and defined by (9.1), we have

$$\theta_{q,t}(x):=-t\log|f'(x)|+q(\theta(x)-\mathrm{P}(\theta)). \tag{9.4}$$

Because of the Bounded Distortion Property (BDP), i.e condition (c) of the definition of conformal GDMSs, the first summand in the above formula is Hölder continuous and the second one is Hölder continuous by its very definition. Thus, we obtain the following.

Lemma 9.11. *For all $q,t\in\mathbb{R}$ the potential $\theta_{q,t}\circ\pi_{\mathcal{S}}: E_A^+\to\mathbb{R}$ is locally Hölder continuous.*

The problem of for which parameters $q,t\in\mathbb{R}$ the potentials $\theta_{q,t}\circ\pi_{\mathcal{S}}: E_A^+\to\mathbb{R}$ are summable is more delicate and has been treated in detail in [8]. Here we only want to state the following obvious.

Observation 9.12. *There exists a non-negative number (or $-\infty$), denoted in* [11] *and* [15] *by $\theta_{\mathcal{S}}$, such that the potential $\zeta_t:=\theta_{0,t}\circ\pi_{\mathcal{S}}: E_A^+\to\mathbb{R}$ is summable for every $t>\theta_{\mathcal{S}}$ and not summable for any $t\le\theta_{\mathcal{S}}$. The explicit formula for the potential $\zeta_{\mathcal{S},t}$ is*

$$\zeta_{\mathcal{S},t}(\omega)=t\log\big|\phi_{\omega_0}'(\pi_{\mathcal{S}}(\sigma(\omega)))\big|;$$

We also know that by Lemma 9.11 that this potential is locally Hölder continuous. In addtion, we record that

$$\theta_{0,t}(x)=-t\log|f'(x)|.$$

The importance of the geometric potentials $\theta_{q,t}\circ\pi_{\mathcal{S}}$, and in particular the issue of their summability, is primarily due to the fact that these are suitable for a description of geometry of the limit set $J_{\mathcal{S}}$. Firstly, if $q=0$, then the parameter $t\geq 0$ for which $\mathrm{P}(\theta_{0,t})=0$ (if it exists) coincides with the Hausdorff dimension $\mathrm{HD}(J_{\mathcal{S}})$ of the limit set $J_{\mathcal{S}}$, and secondly, these potentials play an indispensible role in multifractal analysis of the equilibrium state $\mu_\theta := \mu_{\theta\circ\pi_{\mathcal{S}}}\circ\pi_{\mathcal{S}}^{-1}$; for example as in [24], [8], [15], [27].

We denote by $\Sigma(\mathcal{S},\theta)$ the set of those pairs $(t,q)\in\mathbb{R}^2$ for which the potential $\theta_{q,t}\circ\pi_{\mathcal{S}}: E_A^+\to\mathbb{R}$ is summable and by $\Sigma_0(\mathcal{S},\theta)$ the set of those $q\in\mathbb{R}$ for which there exits (necessarily at most one) a real number $T(q)$ such that $(q,T(q))\in\Sigma(\mathcal{S},\theta)$ and moreover

$$\mathrm{P}\big(\theta_{q,T(q)}\circ\pi_{\mathcal{S}}\big)=0.$$

We now are in the setting of Lemma 9.5 and Theorem 9.6. For $q\in\Sigma_0(\mathcal{S},\theta)$ abbreviate

$$\psi_q := \phi_{\theta_{q,T(q)}} = \theta_{q,T(q)}\circ p_1 : J(J_{\mathcal{S}})\to\mathbb{R}.$$

As an immediate consequence of Theorem 9.6, we get the folowing.

Corollary 9.13. *With $\theta_{q,t}$ defined in (9.3) and with notation following it, we have the following. If $q\in\Sigma_0(\mathcal{S},\theta)$, then*

(a) *For $\mu_{\theta_{q,T(q)}\circ\pi_{\mathcal{S}}}\circ\pi_{\mathcal{S}}^{-1}$–a.e. $x\in J_{\mathcal{S}}$, the conditional measure $\mu^x_{\psi_q}$ is exact dimensional on J_x; in fact*

$$\lim_{r\to 0}\frac{\log\mu^x_{\psi_q}(B(y,r))}{\log r}=\frac{\mathrm{h}_{\mu_{\psi_q}}(F)}{\chi_{\mu_{\psi_q}}(F)}$$

for $\mu^x_{\psi_q}$–a.e. $y\in J_x$; hence, equivalently for μ_{ψ_q}–a.e $(x,y)\in J(J_{\mathcal{S}})$.

(b) *The equilibrium state μ_{ψ_q} of $\psi_q : J(J_{\mathcal{S}})\to\mathbb{R}$ for F, is exact dimensional on $J(J_{\mathcal{S}})$ and*

$$\mathrm{HD}\big(\mu_{\psi_q}\big)=\frac{\mathrm{h}_{\mu_{\psi_q}}(F)}{\chi_{\mu_{\psi_q}}(F)}+\mathrm{HD}\big(\mu_{\theta_{q,T(q)}\circ\pi_{\mathcal{S}}}\circ\pi_{\mathcal{S}}^{-1}\big)=\frac{\mathrm{h}_{\mu_{\phi_\theta}}(F)}{\chi_{\mu_{\phi_\theta}}(F)}+\frac{\mathrm{h}_{\mu_{\theta_{q,T(q)}\circ\pi_{\mathcal{S}}}}}{\chi_{\mu_{\theta_{q,T(q)}\circ\pi_{\mathcal{S}}}}}.$$

To be more specific, we now study two important subclasses of examples, based on the Gauss map, and parabolic iterated function systems (see [9], [33], [12], [14], [15]).

For every integer $n\geq 1$, let $g_n:[0,1]\to[0,1]$ be given by the formula

$$g_n(x)=\frac{1}{n+x}. \tag{9.5}$$

The collection of maps $\mathcal{G}:=\{g_n\}_{n=1}^\infty$ on $[0,1]$, forms a conformal iterated system; its associated map $G:=f_{\mathcal{G}}:[0,1]\to(0,1]$ is called the Gauss map, and is defined by

$$G(x)=\frac{1}{x}-n \quad \text{if} \quad x\in\Big(\frac{1}{n+1},\frac{1}{n}\Big], \tag{9.6}$$

and $G(x)=0$ otherwise. It is an EMR map. From Theorem 9.6, we get the following:

Corollary 9.14. *Let $G : I \to I$ be the Gauss map. Let $Y \subset \mathbb{R}^d$ be an open bounded set, and let $F : I \times Y \to I \times Y$ be a Hölder conformal skew product endomorphism over G. Let $\theta : I \to \mathbb{R}$ be an arbitrary potential such that $\theta \circ \pi_{\mathcal{G}} : \mathbb{N}^+ \to I$ is a locally Hölder continuous summable potential. Then,*

(a) *For $\mu_{\theta\circ\pi_{\mathcal{G}}} \circ \pi_{\mathcal{G}}^{-1}$–a.e. $x \in I$, the conditional measure $\mu^x_{\phi_\theta}$ is exact dimensional on J_x; and*
$$\lim_{r\to 0} \frac{\log \mu^x_{\phi_\theta}(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_{\phi_\theta}}(F)}{\chi_{\mu_{\phi_\theta}}(F)}$$
for $\mu^x_{\phi_\theta}$–a.e. $y \in J_x$; hence, equivalently for μ_{ϕ_θ}–a.e $(x,y) \in J(I)$.

(b) *The equilibrium state μ_{ϕ_θ} of $\phi_\theta : J(I) \to \mathbb{R}$ for F, is exact dimensional on $J(I)$ and*
$$\mathrm{HD}\big(\mu_{\phi_\theta}\big) = \frac{\mathrm{h}_{\mu_{\phi_\theta}}(F)}{\chi_{\mu_{\phi_\theta}}(F)} + \mathrm{HD}\big(\mu_{\theta\circ\pi_{\mathcal{G}}} \circ \pi_{\mathcal{G}}^{-1}\big) = \frac{\mathrm{h}_{\mu_{\phi_\theta}}(F)}{\chi_{\mu_{\phi_\theta}}(F)} + \frac{\mathrm{h}_{\mu_{\theta\circ\pi_{\mathcal{G}}}}}{\chi_{\mu_{\theta\circ\pi_{\mathcal{G}}}}}.$$

As a matter of fact the unit interval $I = [0,1]$ in the above corollary can be replaced by the limit set $J_{\mathcal{G}}$ of the Gauss iterated system $\mathcal{G}$, which is the set of irrational numbers in I. We can take in the last Corollary, the potential θ to be a geometric potential of type $\theta_{q,t}$. Also, one can choose an arbitrary subset E of $\mathbb{N}$ and perform the above construction for an arbitrary subsystem $\mathcal{G}_E = \{g_n : [0,1] \to [[0,1]\}_{n\in E}$.

Since
$$g_n'(x) = -\frac{1}{(n+x)^2}, \text{ so that } \ |g_n'(x)| \asymp n^{-2},$$
we obtain the following well–known:

Observation 9.15. *For the Gauss system we have*
$$\theta_{\mathcal{G}} = 1/2$$
and the potential $\zeta_{\mathcal{G},1/2}$ is not summable.

9.1. **Skew Products with Conformal Parabolic GDMSs in the Base.** Now we pass to the second large class of examples. As said, this class is built on parabolic iterated function systems.

Assume again that we are given a directed multigraph (V, E, i, t) (E countable, V finite), an incidence matrix $A : E \times E \to \{0,1\}$, and two functions $i, t : E \to V$ such that $A_{ab} = 1$ implies $t(b) = i(a)$. Also, we have nonempty compact metric spaces $\{X_v\}_{v\in V}$. Suppose further that we have a collection of conformal maps $\phi_e : X_{t(e)} \to X_{i(e)}$, $e \in E$, satisfying the following conditions (which are more general than the above in that we don't necessarily assume that the maps are uniform contractions):

(1) (Open Set Condition) $\phi_a(\mathrm{Int}(X)) \cap \phi_b(\mathrm{Int}(X)) = \emptyset$ for all $a, b \in E$ with $a \neq b$.

(2) $|\phi_e'(x)| < 1$ everywhere except for finitely many pairs (e, x_e), $e \in E$, for which x_i is the unique fixed point of ϕ_e and $|\phi_e'(x_e)| = 1$. Such pairs and indices i will be called parabolic and the set of parabolic indices will be denoted by Ω. All other indices will be called hyperbolic. We assume that $A_{ee} = 1$ for all $e \in \Omega$.

(3) $\forall n \geq 1\ \forall \omega = (\omega_1\omega_2...\omega_n) \in E_A^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then ϕ_ω extends conformally to an open connected set $W_{t(\omega_n)} \subset \mathbb{R}^d$ and maps $W_{t(\omega_n)}$ into $W_{i(\omega_n)}$.

(4) If $e \in E$ is a parabolic index, then

$$\bigcap_{n\geq 0} \phi_{e^n}(X) = \{x_e\}$$

and the diameters of the sets $\phi_{e^n}(X)$ converge to 0.

(5) (Bounded Distortion Property) $\exists K \geq 1\ \forall n \geq 1\ \forall \omega \in E_A^n\ \forall x, y \in W_{t(\omega_n)}$, if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then

$$\frac{|\phi'_\omega(y)|}{|\phi'_\omega(x)|} \leq K.$$

(6) $\exists \kappa < 1\ \forall n \geq 1\ \forall \omega \in E_A^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then $\|\phi'_\omega\| \leq \kappa$.

(7) (Cone Condition) There exist $\alpha, l > 0$ such that for every $x \in \partial X \subset \mathbb{R}^d$ there exists an open cone $\mathrm{Con}(x, \alpha, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure α, and altitude l.

(8) There exists a constant $L \geq 1$ such that for every $e \in E$ and every $x, y \in V$,

$$\left|\frac{|\phi'_e(y)|}{|\phi'_e(x)|} - 1\right| \leq L\|y - x\|^\alpha,$$

We call such a system of maps $\mathcal{S} = \{\phi_e : e \in E\}$ a *subparabolic conformal graph directed Markov system.*

Definition 9.16. *If $\Omega \neq \emptyset$, we call the system $\mathcal{S} = \{\phi_i : i \in E\}$ parabolic.*

As declared in (2) the elements of the set $E \setminus \Omega$ are called hyperbolic. We extend this name to all the words appearing in (5) and (6). It follows from (3) that for every hyperbolic word ω,

$$\phi_\omega(W_{t(\omega)}) \subset W_{t(\omega)}.$$

Note that our conditions ensure that $\phi'_e(x) \neq 0$ for all $e \in E$ and all $x \in X_{t(i)}$. It was proved (though only for IFSs but the case of GDMSs can be treated completely similarly) in [12] (comp. [15]) that

$$\lim_{n\to\infty} \sup_{\omega \in E_A^n} \left\{\mathrm{diam}(\phi_\omega(X_{t(\omega)}))\right\} = 0. \tag{9.7}$$

This implies then:

Corollary 9.17. *The map $\pi : E_A^+ \to X := \bigoplus_{v\in V} X_v$,*

$$\pi(\omega) := \bigcap_{n\geq 0} \phi_{\omega|_n}(X),$$

is well defined, i.e. this intersection is always a singleton, and the map π is uniformly continuous.

As for hyperbolic (attracting) systems the limit set $J = J_{\mathcal{S}}$ of $\mathcal{S} = \{\phi_e\}_{e\in e}$ is

$$J_{\mathcal{S}} := \pi(E_A^+)$$

and it enjoys the following self-reproducing property: $J = \bigcup_{e\in E} \phi_e(J)$. We now want to associate to the parabolic system $\mathcal{S}$ a canonical hyperbolic system $\mathcal{S}^*$; this will be done by using the jump transform ([31]). We will then be able to apply the ideas from the previous section to $\mathcal{S}^*$. The set of edges is:

$$E_* := \big\{i^n j : n \geq 1,\ i \in \Omega,\ i \neq j \in E,\ A_{ij} = 1\big\} \cup (E \setminus \Omega) \subset E_A^*.$$

We set

$$V_* = t(E_*) \cup i(E_*)$$

and keep the functions t and i on E_* as the restrictions of t and i from E_A^*. The incidence matrix $A^* : E_* \times E_* \to \{0,1\}$ is defined in the natural (the only reasonable) way by declaring that $A^*_{ab} = 1$ if and only if $ab \in E_A^*$. Finally

$$\mathcal{S}^* = \{\phi_e : X_{t(e)} \to X_{t(e)} | \, e \in E^*\}.$$

It immediately follows from our assumptions (see [12] and [15] for more details) that the following result is true.

Theorem 9.18. *The system $\mathcal{S}^*$ is a hyperbolic (contracting) conformal GDMS and the limit sets $J_{\mathcal{S}}$ and $J_{\mathcal{S}^*}$ differ only by a countable set. If the system $\mathcal{S}$ is finitely irreducible, then so is the system $\mathcal{S}^*$.*

The most important advantage of $\mathcal{S}^*$ is that it is a an attracting conformal GDMS. On the other hand, the price we pay by replacing the non-uniform contractions in $\mathcal{S}$ with the uniform contractions in $\mathcal{S}^*$ is that even if the alphabet E is finite, the alphabet E_* of $\mathcal{S}^*$ is always infinite. Thus we will be able to apply our results on infinite Smale systems. We have the following quantitative behavior around parabolic points.

Proposition 9.19. *Let $\mathcal{S}$ be a conformal parabolic GDMS. Then there exists a constant $C \in (0, +\infty)$ and for every $i \in \Omega$ there exists some constant $\beta_i \in (0, +\infty)$ such that for all $n \geq 1$ and for all $z \in X_i := \bigcup_{j\in I\setminus\{i\}} \phi_j(X)$,*

$$C^{-1} n^{-\frac{\beta_i+1}{\beta_i}} \leq |\phi'_{i^n}(z)| \leq C n^{-\frac{\beta_i+1}{\beta_i}}.$$

Furthermore, if $d = 2$ then all constants β_i are integers ≥ 1 and if $d \geq 3$ then all constants β_i are equal to 1.

From Theorem 9.6 we obtain:

Corollary 9.20. *Let $\mathcal{S}$ be an irreducible conformal parabolic GDMS. Let $\mathcal{S}^*$ be the corresponding atracting conformal GDMS produced in Theorem 9.18. Furthermore, let $f : J_{\mathcal{S}^*} \to J_{\mathcal{S}^*}$ be the corresponding GDMS map. Let $Y \subset \mathbb{R}^d$ be an open bounded set, and let $F : J_{\mathcal{S}^*} \times Y \to J_{\mathcal{S}^*} \times Y$ be a Hölder conformal skew product endomorphism over f. Let $\theta : J_{\mathcal{S}^*} \to \mathbb{R}$ be an arbitrary potential such that $\theta \circ \pi_{\mathcal{S}^*} : E_{A^*}^+ \to \mathbb{R}$ is a locally Hölder continuous summable potential. Then,*

(a) *For $\mu_{\theta\circ\pi_{\mathcal{S}}} \circ \pi_{\mathcal{S}^*}^{-1}$–a.e. $x \in J_{\mathcal{S}^*}$, the conditional measure $\mu_{\phi_\theta}^x$ is exact dimensional on J_x; in fact,*

$$\lim_{r\to 0} \frac{\log \mu_{\phi_\theta}^x(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_{\phi_\theta}}(F)}{\chi_{\mu_{\phi_\theta}}(F)}$$

for $\mu_{\phi_\theta}^x$–a.e. $y \in J_x$; hence, equivalently for μ_{ϕ_θ}–a.e $(x,y) \in J(J_{\mathcal{S}^*})$.

(b) *The equilibrium state μ_{ϕ_θ} of $\phi_\theta : J(J_{\mathcal{S}}^*) \to \mathbb{R}$ for F, is exact dimensional on $J(J_{\mathcal{S}}^*)$ and*

$$\mathrm{HD}\big(\mu_{\phi_\theta}\big) = \frac{\mathrm{h}_{\mu_{\phi_\theta}}(F)}{\chi_{\mu_{\phi_\theta}}(F)} + \mathrm{HD}\big(\mu_{\theta\circ\pi_{\mathcal{S}^*}} \circ \pi_{\mathcal{S}^*}^{-1}\big) = \frac{\mathrm{h}_{\mu_{\phi_\theta}}(F)}{\chi_{\mu_{\phi_\theta}}(F)} + \frac{\mathrm{h}_{\mu_{\theta\circ\pi_{\mathcal{S}^*}}}}{\chi_{\mu_{\theta\circ\pi_{\mathcal{S}^*}}}}.$$

As an immediate consequence of Corollary 9.13, just replacing each $\mathcal{S}$ by $\mathcal{S}^*$, we get the folowing.

Corollary 9.21. *With $\theta_{q,t}$ defined in (9.3) and with notation following it, we have the following. If $q \in \Sigma_0(\mathcal{S}^*, \theta)$, then we obtain for the potential $\psi_q = \theta_{q,T(q)} \circ p_1 : J(J_{\mathcal{S}^*}) \to \mathbb{R}$ the same conclusions (a), (b) as in Corollary 9.20.*

We would like now to investigate one important concrete example. We call this system $\mathcal{I}$. It is formed by two inverse maps of the two continuous pieces of the Manneville-Pomeau map $f : [0,1] \to [0,1]$ defined by:

$$f(x) = x + x^{1+\alpha} \pmod 1,$$

where $\alpha > 0$ is a an aribrary fixed positive number. Of course the GDMS map resulting from $\mathcal{I}$ is just, the above defined, map f.

As an immediate consequence of Corollary 9.20, we obtain the following.

Corollary 9.22. *Let $Y \subset \mathbb{R}^d$ be an open bounded set, and let $F : J_{\mathcal{I}} \times Y \to J_{\mathcal{I}} \times Y$ be a Hölder conformal skew product endomorphism over the Manneville-Pomeau map f.*

Let $\theta : J_{\mathcal{I}} \to \mathbb{R}$ be a potential such that $\theta \circ \pi_{\mathcal{I}} : \mathbb{N}^+ \to \mathbb{R}$ is a locally Hölder continuous summable potential. Then, the conclusions of Corollary 9.20 will hold in this case.

Also for the geometric potentials $\theta_{q,T(q)}$ we obtain:

Corollary 9.23. *With $\theta_{q,t}$ defined in (9.3), then for $q \in \Sigma_0(\mathcal{I}, \theta)$, we obtain the same conclusions (a), (b) as in Corollary 9.22 for the potential $\theta_{q,T(q)}$.*

Remark 9.24. *In the construction of the attracting conformal GDMS of Theorem 9.18 we built on Schweiger's jump transformation from [31]. We could instead use inducing on each set X_v, $v \in V \setminus \Omega$, i.e. considering the system generated by the maps ϕ_ω where $i(\omega_1) = t\big(\omega|_{|\omega|}\big) = v$ and $i(\omega_k) \neq v$ for all $k = 2, 3, \dots, |\omega| - 1$. The "jump" construction of Theorem 9.18 seems to be somewhat better as it usually leads to a smaller system.*

9.2. **Natural Extensions of Graph Directed Markov Systems.** Finally in this section, we want to define and explore the systems we refer to which as natural extensions of graph directed Markov systems. Precisely, let $\mathcal{S} = \{\phi_e : X_{t(e)} \to X_{i(e)}\}_{e\in E}$ be a conformal finitely irreducible graph directed Markov system and let $f = f_{\mathcal{S}} : J_{\mathcal{S}} \to J_{\mathcal{S}}$ be the GDMS map associated to the system $\mathcal{S}$ and given by formula (9.1). Fix an arbitrarily chosen point $\xi \in J_{\mathcal{S}}$. We then define the skew product map

$$\tilde{f} : J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}} \to J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}}, \text{ by}$$

$$\tilde{f}(\phi_e(x), y) := (x, \phi_e(y)) = \big(f(\phi_e(x)), \phi_e(y)\big), \tag{9.8}$$

if $x \in \mathrm{Int}(X_{t(e)})$ (then e is uniquely determined), and

$$\tilde{f}(z, y) := (\xi, \xi) \tag{9.9}$$

in the case when $z \notin \bigcup_{e\in E} \phi_e\big(\mathrm{Int}(X_{t(e)})\big)$. We call it the *natural extension* (or *inverse limit*) of the map f.

From Theorem 9.4, we obtain the following:

Corollary 9.25. *Let $\mathcal{S}$ be a conformal finitely irreducible graph directed Markov system and let $f = f_{\mathcal{S}} : J_{\mathcal{S}} \to J_{\mathcal{S}}$ be the corresponding GDMS map. Furhermore, let $\tilde{f} : J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}} \to J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}}$ be the natural extension of the map f, as defined above. Let $\phi : J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}} \to \mathbb{R}$ be such a potential that*

$$\widehat{\phi} = \phi \circ p_J \circ \hat{\pi} : E_A \to \mathbb{R}$$

is locally Hölder continuous and summable. Then, for $\mu_\phi \circ p_1^{-1}$–a.e $x \in J_{\mathcal{S}}$, the conditional measure μ_ϕ^x is exact dimensional on $\overline{J}_{\mathcal{S}}$, and moreover,

$$\lim_{r\to 0} \frac{\log \mu_\phi^x(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_\phi}(\tilde{f})}{\chi_{\mu_\phi}(\tilde{f})},$$

for μ_ϕ^x–a.e $y \in \overline{J}_{\mathcal{S}}$; hence, equivalently, for μ_ϕ-a.e $(x,y) \in J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}}$.

As a consequence of Theorem 9.6 we obtain the following.

Corollary 9.26. *Let $\mathcal{S}$ be a conformal finitely irreducible graph directed Markov system and let $f = f_{\mathcal{S}} : J_{\mathcal{S}} \to J_{\mathcal{S}}$ be the corresponding GDMS map. Furhermore, let $\tilde{f} : J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}} \to J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}}$ be the natural extension of the map f. Let $\theta : J_{\mathcal{S}} \to \mathbb{R}$ be an arbitrary potential such that $\theta \circ \pi_{\mathcal{S}} : E_A^+ \to J_{\mathcal{S}}$ is a locally Hölder continuous summable potential. Then,*

(a) *For $\mu_{\theta\circ\pi_{\mathcal{S}}} \circ \pi_{\mathcal{S}}^{-1}$–a.e. $x \in J_{\mathcal{S}}$, the conditional measure $\mu_{\phi_\theta}^x$ is exact dimensional on J_x; in fact*

$$\lim_{r\to 0} \frac{\log \mu_{\phi_\theta}^x(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_{\phi_\theta}}(\tilde{f})}{\chi_{\mu_{\phi_\theta}}(\tilde{f})}$$

for $\mu_{\phi_\theta}^x$–a.e. $y \in J_x$; hence, equivalently for μ_{ϕ_θ}–a.e $(x,y) \in J(J_{\mathcal{S}})$.

(b) *The equilibrium state* μ_{ϕ_θ} *of* $\phi_\theta : J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}} \to \mathbb{R}$ *for* $\tilde{f}$, *is exact dimensional on* $J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}}$ *and*

$$\mathrm{HD}\big(\mu_{\phi_\theta}\big) = \frac{\mathrm{h}_{\mu_{\phi_\theta}}(\tilde{f})}{\chi_{\mu_{\phi_\theta}}(\tilde{f})} + \mathrm{HD}\big(\mu_{\theta\circ\pi_{\mathcal{S}}} \circ \pi_{\mathcal{S}}^{-1}\big) = \frac{\mathrm{h}_{\mu_{\phi_\theta}}(\tilde{f})}{\chi_{\mu_{\phi_\theta}}(\tilde{f})} + \frac{\mathrm{h}_{\mu_{\theta\circ\pi_{\mathcal{S}}}}}{\chi_{\mu_{\theta\circ\pi_{\mathcal{S}}}}}.$$

Thus, in the important case of the Gauss map and the continued fraction system, we have the exact dimensionality of the conditional measures on the fibers of the natural extension, and also the global exact dimensionality of equilibrium measures:

Corollary 9.27. *Let* $G : I \to I$ *be the Gauss map and let* $\tilde{G} : J_{\mathcal{G}} \times \overline{J}_{\mathcal{G}} \to J_{\mathcal{G}} \times \overline{J}_{\mathcal{G}}$ *be the natural extension of the map* G. *Let* $\theta : J_G \to \mathbb{R}$ *be an arbitrary potential such that* $\theta \circ \pi_{\mathcal{G}} : E_A^+ \to \mathbb{R}$ *is a locally Hölder continuous summable potential. Then, the conclusion of Corollary 9.26 hold.*

10. Diophantine Approximants and the Generalized Doeblin-Lenstra Conjecture

We want to apply the results about skew products to certain properties of Diophantine approximants, making the conjecture of Doeblin and Lenstra more general and precise. Let $\mathcal{G}$ be the Gauss system introduced by formula (9.5). Then the corresponding coding map $\pi_{\mathcal{G}} : \mathbb{N}^{\mathbb{N}} \to [0,1]$ is a bijection between $\mathbb{N}^{\mathbb{N}}$ and $J_{\mathcal{G}}$ which in this case is the set of irrational numbers in $[0,1]$.

$$\pi_{\mathcal{G}}(\omega) = [\omega_1, \omega_2, \ldots] = \cfrac{1}{\omega_1 + \cfrac{1}{\omega_2 + \cfrac{1}{\cdots + \cfrac{1}{\omega_n + \cdots}}}},$$

Recall also that the associated continued fraction (Gauss) transformation is given by:

$$G(x) = \frac{1}{x} - n \quad \text{if} \quad x \in \Big(\frac{1}{n+1}, \frac{1}{n}\Big], \tag{10.1}$$

and $G(x) = 0$ otherwise. If we truncate the representation $[\omega_1, \omega_2, \ldots]$ at an integer $n \geq 1$, then we obtain a rational number p_n/q_n, called the n-th convergent of $x := \pi_{\mathcal{G}}(\omega)$, where $p_n, q_n \geq 1$ are relatively prime integers, and

$$\frac{p_n}{q_n} = [\omega_1, \ldots, \omega_n]$$

If needed, we shall also denote p_n and q_n respectively by $p_n(\omega)$ and $q_n(\omega)$ or also by $p_n(x)$ and $q_n(x)$, in order to indicate their dependence on ω and x. We also sometimes write $\omega_n(x)$ for ω_n. Then, as in [2], [9], [5], one can introduce the numbers,

$$\Theta_n := \left| x - \frac{p_n}{q_n} \right| \cdot q_n^2, \quad n \geq 1$$

This number Θ_n also depends on ω or (equivalently) on x and will be also denoted by $\Theta_n(\omega)$ or $\Theta_n(x)$. Denote:

$$T_n = T_n(\omega) := \pi_{\mathcal{G}}(\sigma^n(\omega)) = [\omega_{n+1}, \omega_{n+2}, \ldots], \quad n \geq 1,$$

and by

$$V_n = V_n(\omega) := [\omega_n, \dots, \omega_1], \quad n \geq 1$$

We will also denote them respectively by $T_n(x)$ and $V_n(x)$. We see that $T_n(x)$ represents the future of x while the number $V_n(x)$ represents the past of x. It follows directly from the definitions that for every integer $n \geq 1$, we have that

$$V_n = \frac{q_{n-1}}{q_n}, \quad \Theta_{n-1} = \frac{V_n}{1+T_nV_n}, \quad \text{and} \quad \Theta_n = \frac{T_n}{1+T_nV_n} \tag{10.2}$$

Recall that we have started with the Gauss iterated function system $\mathcal{G}$, we then associated to it in formula (9.6) (or (10.1)) the corresponding map $G = f_{\mathcal{G}} : [0,1] \to (0,1]$; next we consider the natural extension $\tilde{G} = \tilde{f}_{\mathcal{G}} : J_{\mathcal{G}} \times [0,1] \to J_{\mathcal{G}} \times [0,1]$ of Subsection 9.2. It follows from (9.8) that $\tilde{G}$ is explicitely given by the formula

$$\tilde{G}(x,y) = \left(T_1(x), \frac{1}{\omega_1(x)+y}\right)$$

It follows from this that

$$\tilde{G}(x,0) = \left(T_1(x), \frac{1}{\omega_1(x)}\right) \quad \text{and} \quad \tilde{G}^2(x,0) = \left(T_2(x), \frac{1}{\omega_2(x)+\frac{1}{\omega_1(x)}}\right)$$

By induction, we obtain for every $n \geq 1$, that

$$\tilde{G}^n(x,0) = (T_n(x), [\omega_n(x), \dots, \omega_1(x)]) = (T_n(x), V_n(x))$$

The approximation coefficients Θ_n were the object of an important Conjecture originally stated by Doeblin and reformulated in the 80's by Lenstra (see [9], [5]), namely that for Lebesgue–a.e $x \in [0,1)$ the frequency of appearances of $\Theta_n(x)$ in the interval $[0,t]$, $t \in [0,1]$, is given by the function $F : [0,1] \to [0,1]$ given by

$$F(t) = \begin{cases} \frac{t}{\log 2} & \text{if } t \in [0,1/2], \\ \frac{1}{\log 2}(1-t+\log 2t) & \text{if } t \in [1/2,1]. \end{cases}$$

More precisely, the **Doeblin-Lenstra Conjecture** says that for Lebesgue–a.e. $x \in [0,1]$ and all $t \in [0,1]$,

$$\lim_{n\to\infty} \frac{\#\{1 \leq k \leq n : \Theta_k(x) \leq t\}}{n} = F(t),$$

i.e. the above limit exists and is equal to equal to the above function $F(t)$. This conjecture was solved by Bosma, Jager and Wiedijk [2] in the 1980's. In the proof, they needed fundamentally the **natural extension** $(J_{\mathcal{G}} \times [0,1], \tilde{G}, \tilde{\mu}_G)$, of the continued fraction Gauss dynamical system $\tilde{G}$ with the classical Gauss measure μ_G defined by

$$\mu_G(A) = \frac{1}{\log 2}\int_A \frac{1}{1+x}\,dx$$

for every Borel set $A \subset \mathbb{R}$. Indeed in the expression of Θ_n we have both the future T_n, as well as the past V_n, thus the natural extensions is the right construction in this case.

Let us now apply our results on skew products for the natural extension $\tilde{G}$ to the lifts of certain invariant measures, in fact some equilibrium states. Precisely, we recall (see Observation 9.12) the potentials

$$\theta_s := \theta_{0,s}(x) = -s \log |T'(x)|, \ x \in [0,1).$$

If $s > 1/2$, then (see Observation 9.12 and Observation 9.15), the corresponding potential $\zeta_{\mathcal{G},s} := \theta_s \circ \pi_{\mathcal{S}} : E_A^+ \to \mathbb{R}$ is locally Hölder continuous and summable. We let

$$\mu_s := \mu_{\zeta_{\mathcal{G},s}} \circ \pi_{\mathcal{G}}^{-1}$$

denote the equilibrium states of the potential θ_s. We also recall that the explicit formula for the potential $\zeta_{\mathcal{G},s}$ is

$$\zeta_{\mathcal{G},s}(\omega) = s \log \left| g'_{\omega_0}(\pi_{\mathcal{G}}(\sigma(\omega))) \right| = 2s \log\big(\omega_0 + \pi_{\mathcal{G}}(\sigma(\omega))\big) = 2s \log\big(\omega_0 + [\omega_1, \omega_2, \ldots]\big).$$

Let us now introduce the potential $\tau_s : J_{\mathcal{G}} \times [0,1] \to \mathbb{R}$ being by definition equal to

$$\tau_s = \theta_s \circ p_1,$$

as in formula (9.2). In other words, $\tau_s(x,y) = \theta_s(x)$. And consider the measure

$$\hat{\mu}_s := \mu_{\tau_s}$$

to be the equilibrium measure of τ_s with respect to the dynamical system $\tilde{\mathcal{G}}$ on $J_{\mathcal{G}} \times [0,1]$; more precisely, $\hat{\mu}_s$ is the measure $\mu_{\phi_{\theta_s}}$ of Corollary 9.27 applied with $\theta = \theta_s$. In particular,

$$\mathrm{h}_{\hat{\mu}_s}(\tilde{\mathrm{G}}) = \mathrm{h}_{\mu_s}(\mathrm{G}) \quad \text{and} \quad \chi_{\hat{\mu}_s} = \chi_{\mu_s}. \tag{10.3}$$

We know from this corollary that the measure $\hat{\mu}_s$ is exact dimensional on $J_{\mathcal{G}} \times [0,1]$ and we have a formula for its Hausdorff dimension.

Our purpose now is to describe asymptotic frequencies with which the approximation coefficients $\Theta_n(x)$ of $x \in [0,1]$ become close to certain given arbitrary values, when x is Lebesgue non-generic (i.e x belongs to a set of Lebesgue measure zero). In fact, these x's will be generic for equilibrium measures μ_s, which except for $s = 1$ are singular with respect to the Lebesgue measure.

First let us prove the following result about the asymptotic frequency of appearance of $(T_n(x), V_n(x))$ in all squares of $\mathbb{R}^2$, with respect to the measure $\hat{\mu}_s$:

Theorem 10.1. *If $s > 1/2$, then for μ_s-a.e $x \in [0,1]$ and for all four real numbers $a < b$ and $c < d$, we have that*

$$\lim_{n\to\infty} \frac{\#\big\{k \in \{0,1,\ldots,n-1\} : (T_k(x), V_k(x)) \in (a,b) \times (c,d)\big\}}{n} = \hat{\mu}_s\big((a,b) \times (c,d)\big)$$

Proof. Denote $A = (a,b) \times (c,d)$, and for every $\varepsilon > 0$ let

$$A(\varepsilon) := (a,b) \times (c-\varepsilon, d+\varepsilon) \quad \text{and} \quad A(-\varepsilon) = (a,b) \times (c+\varepsilon, d-\varepsilon)$$

Then

$$A(-\varepsilon) \subset A \subset A(\varepsilon)$$

Let $x \in [0,1] \setminus \mathbb{Q}$. Then $x = [\omega_1(x), \omega_2(x), \ldots]$. Hence, there exists an integer $n_\varepsilon \geq 1$ such that for every $y \in [0,1]$ and every integer $n \geq n_\varepsilon$, we have that

$$\big|[\omega_n(x), \omega_{n-1}(x), \ldots, \omega_1(x) + y] - [\omega_n(x), \omega_{n-1}(x), \ldots, \omega_1(x)]\big| < \varepsilon$$

Thus, if $\tilde{G}^n(x,y) = (T^n(x), [\omega_n(x), \omega_{n-1}(x), \dots, \omega_1(x) + y] \in A(-\varepsilon)$, then

$$(T_n(x), V_n(x)) \in A,$$

and if $(T_n(x), V_n(x)) \in A$, then

$$\tilde{G}^n(x,y) \in A(\varepsilon)$$

Therefore for every $x \in [0,1] \setminus \mathbb{Q}$ and every $y \in [0,1]$, we obtain that

(10.4)

$$\begin{aligned}\varliminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1\!\!1_{A(-\varepsilon)}(\tilde{G}^k(x,y)) &\le \varliminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1\!\!1_{A}(\tilde{G}^k(x,0)) \le \\ &\le \varlimsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1\!\!1_{A}(\tilde{G}^k(x,0)) \le \varlimsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1\!\!1_{A(\varepsilon)}(\tilde{G}^k(x,y)).\end{aligned}$$

Since the equilibrium measure $\hat{\mu}_s$ is ergodic on $[0,1]^2$ with respect to the map $\tilde{G}$ and since $\hat{\mu}_s$ projects on $\mu_s := \mu_{\theta_s}$ the equilibrium state of the potential θ_s, it follows from Birkhoff's Ergodic Theorem that for μ_s-a.e $x \in [0,1]$ there exist $y_1 \in [0,1]$ and $y_2 \in [0,1]$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1\!\!1_{A(-\varepsilon)}(\tilde{G}^k(x,y)) = \hat{\mu}_s(A(-\varepsilon)), \text{ and } \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1\!\!1_{A(\varepsilon)}(\tilde{G}^k(x,y)) = \hat{\mu}_s(A(\varepsilon))$$

Along with (10.4) these yield

$$\hat{\mu}_s(A(-\varepsilon)) \le \varliminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1\!\!1_{A}(\tilde{G}^k(x,0)) \le \varlimsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1\!\!1_{A}(\tilde{G}^k(x,0)) \le \hat{\mu}_s(A(\varepsilon)) \tag{10.5}$$

Noting that $\hat{\mu}_s$ does not charge the boundary of A and letting in the above inequality with $\varepsilon > 0$ to 0 over a (countable) sequence, we obtain that μ_s-a.e $x \in [0,1]$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1\!\!1_{A}\big(T_k(x), V_k(x)\big) = \hat{\mu}_s(A)$$

□

We prove now that for $x \in \Lambda_s$ (recall that Λ_s has zero Lebesgue measure, but μ_s-measure equal to 1), the approximation coefficients $\Theta_n(x), \Theta_{n-1}(x)$ behave *very erratically*. The following Theorem says that for irrational numbers $x \in \Lambda_s$, the behaviour of the consecutive numbers $\Theta_k(x), \Theta_{k-1}(x)$ is chaotic, and we can estimate the asymptotic frequency that $\Theta_k(x)$ is r-close to some z, while $\Theta_{k-1}(x)$ is r-close to some z'. This *asymptotic frequency* is comparable to $r^{\delta(\hat{\mu}_s)}$, *regardless* of the point $x \in \Lambda_s$, or the numbers z, z' chosen.

Theorem 10.2. *For every $s > 1/2$, there exists a Borel set $\Lambda_s \subset [0,1] \setminus \mathbb{Q}$ with $\mu_s(\Lambda_s) = 1$, so that:*

(1) $\mathrm{HD}(\Lambda_s) = \mathrm{h}_{\mu_s}(\mathrm{G})/\chi_{\mu_s}$.
(2) *For every $x \in \Lambda_s$ we have that:*

$$\lim_{n\to\infty} \frac{1}{n} \log\left| x - \frac{p_n(x)}{q_n(x)} \right| = \chi_{\mu_s}$$

(3) *For every* $x \in \Lambda_s$ *and* $\hat{\mu}_s$*-a.e* $(z, z') \in [0,1)^2$, *we have that*

$$\lim_{r\to 0} \log \lim_{n\to\infty} \frac{\#\left\{0 \le k \le n-1 : \big(\Theta_k(x), \Theta_{k-1}(x)\big) \in B\left(\frac{z}{1+zz'}, r\right) \times B\left(\frac{z'}{1+zz'}, r\right)\right\}}{n} = \mathrm{HD}(\hat{\mu}_s)$$

(4)

$$\mathrm{HD}(\hat{\mu}_s) = \frac{\mathrm{h}_{\mu_s}(\mathrm{G})}{\chi_{\mu_s}} + \frac{\mathrm{h}_{\hat{\mu}_s}(\tilde{\mathrm{G}})}{2\int_{[0,1)^2} \log(\omega_1(x) + y)\, d\hat{\mu}_s(x,y)}$$

In addition, we have:

(5) $\{\chi_{\mu_s}\}_{s>1/2} = [\chi_{\mu_{1/2}}, +\infty)$.

Proof. Formula (4) is a immediate consequence of both Corollary 9.27, applied to the potential $\theta_s : I \to \mathbb{R}$, and formula (10.3). By [15] the function $(1/2, +\infty) \ni s \longmapsto \chi_{\mu_s}$ is strictly increasing (note for example that $\chi_{\mu_s} = -\mathrm{P}'(\sigma, \zeta_{\mathcal{G},s}) > 0$), and that $\lim_{s\to+\infty} \chi_{\mu_s} = \infty$. Hence (5) follows. As $\mathrm{HD}(\mu_s) = \mathrm{h}_{\mu_s}(\mathrm{G})/\chi_{\mu_s}$, there exists a Borel set $\Lambda_s^* \subset I \setminus Q$ such that: $\mu_s(\Lambda_s^*) = 1$, $\mathrm{HD}(\Lambda_s^*) = \mathrm{h}_{\mu_s}(\mathrm{G})/\chi_{\mu_s}$, and each Borel subset of Λ_s^* with full measure μ_s has Hausdorff dimension equal to $\mathrm{h}_{\mu_s}(\mathrm{G})/\chi_{\mu_s}$.

Define now Λ_s to be the set of all points x in Λ_s^* for which (3) holds and for which the assertion of Theorem 10.1 holds too. From this theorem, together with a result of [24], it follows that the set Λ_s satisfies the conditions (1), (2), and (3) above. Thus we have to prove only condition (4). So fix $x \in \Lambda_s$ and let $z, z' \in [0,1)$ be arbitrary. Because of formulas (10.2), there exists a constant $C \ge 1$ such that for all radii $r > 0$, we have that:

a) If for some integer $k \ge 1$, $(T_k(x), V_k(x)) \in B(z, r) \times B(z', r)$, then

$$(\Theta_k(x), \Theta_{k-1}(x)) \in B\left(\frac{z}{1+zz'}, Cr\right) \times B\left(\frac{z'}{1+zz'}, Cr\right)$$

b) If $(\Theta_k(x), \Theta_{k-1}(x)) \in B\left(\frac{z}{1+zz'}, r\right) \times B\left(\frac{z'}{1+zz'}, r\right)$, then

$$(T_k, V_k) \in B(z, Cr) \times B(z', Cr).$$

Therefore, it follows from Theorem 10.1 that:

(10.6)

$$\hat{\mu}_s\big(B(z, C^{-1}r) \times B(z', C^{-1}r)\big) \le$$

$$\le \lim_{n\to\infty} \frac{\#\left\{0 \le k \le n-1 : \big(\Theta_k(x), \Theta_{k-1}(x)\big) \in B\left(\frac{z}{1+zz'}, r\right) \times B\left(\frac{z'}{1+zz'}, r\right)\right\}}{n} \le$$

$$\le \hat{\mu}_s\big(B(z, Cr) \times B(z', Cr)\big)$$

Since by Corollary 9.27 the measure $\hat{\mu}_s$ is dimensional exact, we have that

$$\lim_{r\to 0} \frac{\log \hat{\mu}_s\big(B(z, C^{-1}r) \times B(z', C^{-1}r)\big)}{\log r} = \mathrm{HD}(\hat{\mu}_s) = \lim_{r\to 0} \frac{\log \hat{\mu}_s\big(B(z, Cr) \times B(z', Cr)\big)}{\log r}$$

Along with (10.6) this finishes the proof of formula (3), and the proof of Theorem 10.2.

□

11. Generalized Lüroth Systems and Their Inverse Limits

We include this short section since it is needed for the full treatment of β-transformations, for arbitrary $\beta > 1$, and of their natural extensions. It just collects the results of the previous sections in the case of a special subclass of maps.

Definition 11.1. *We call an iterated function system $\mathcal{S} = \{\phi_e : I \to I\}_{e\in E}$ a Lüroth system if the maps $\phi_e : I \to I$, $e \in E$, are of the form $x \longmapsto ax + b$, and* $\text{Leb}\left(\bigcup_{e\in E}\phi_e(I)\right) = 1$.

Let $f = f_{\mathcal{S}} : J_{\mathcal{S}} \to J_{\mathcal{S}}$ be the map associated to the system $\mathcal{S}$ and given by formula (9.1):

$$f(\phi_e(x)) = x \tag{11.1}$$

Writing

$$\phi_e(x) = a_e x + b_e, \quad e \in E,$$

with $a_e \in (0, 1)$, we rewrite (11.1) in the following more explicit form

$$f(x) = \begin{cases} a_e^{-1}x - a_e^{-1}b_e & \text{if } \ x \in \text{Int}(\phi_e(I)) \\ 0 & \text{if } \ x \notin \bigcup_{e\in E}\text{Int}(\phi_e(I)). \end{cases} \tag{11.2}$$

In particular, we took the given preassigned point ξ involved in (9.1) to be 0. In the sequel we will need however mainly only the definition of f on $\bigcup_{e\in E}\text{Int}(\phi_e(I))$.

The *natural extension* $\tilde{f} : J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}} \to J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}}$ of f is given by formula (9.8). In more explicit terms, we have:

$$\tilde{f}(x,y) = \begin{cases} (f(x), a_e y + b_e), \ x \in \text{Int}(\phi_e(I)) \\ (0,0), \ x \notin \bigcup_{e\in E}\text{Int}(\phi_e(I)) \end{cases} = \begin{cases} \left(a_e^{-1}x - a_e^{-1}b_e, a_e y + b_e\right), \ x \in \text{Int}(\phi_e(I)) \\ (0,0), \ x \notin \bigcup_{e\in E}\text{Int}(\phi_e(I)) \end{cases} \tag{11.3}$$

As consequences of Corollary 9.25, we have:

Corollary 11.2. *Let $\mathcal{S}$ be a Lüroth system system and let $f = f_{\mathcal{S}} : J_{\mathcal{S}} \to J_{\mathcal{S}}$ be the corresponding GDMS map. Furhermore, let $\tilde{f} : J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}} \to J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}}$ be the natural extension of the map f, as defined above. Let $\phi : J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}} \to \mathbb{R}$ be such a potential that*

$$\widehat{\phi} = \phi \circ p_J \circ \hat{\pi} : E_A \to \mathbb{R}$$

is locally Hölder continuous and summable. Then, for $\mu_\phi \circ p_1^{-1}$–a.e $x \in J_{\mathcal{S}}$, the conditional measure μ_ϕ^x is exact dimensional on $\overline{J}_{\mathcal{S}}$, and moreover,

$$\lim_{r\to 0}\frac{\log \mu_\phi^x(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_\phi}(\tilde{f})}{\chi_{\mu_\phi}(\tilde{f})},$$

for μ_ϕ^x–a.e $y \in \overline{J}_{\mathcal{S}}$; hence, equivalently, for μ_ϕ-a.e $(x,y) \in J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}}$.

In particular, the above conclusions hold if $\phi : J_{\mathcal{S}} \times \overline{J}_{\mathcal{S}} \to \mathbb{R}$ is a locally Hölder continuous potential with $\widehat{\phi} = \phi \circ p_J \circ \hat{\pi} : E_A \to \mathbb{R}$ summable. Similarly the above conclusions hold if $\theta : J_{\mathcal{S}} \to \mathbb{R}$ is so that $\theta \circ \pi_{\mathcal{S}} : E_A^+ \to J_{\mathcal{S}}$ is a locally Hölder continuous summable potential.

12. Thermodynamic Formalism for Inverse Limits of β-Maps, $\beta > 1$

For a real number $\beta > 1$, the β-transformation is $T_\beta : [0,1) \to [0,1)$, defined by

$$T_\beta(x) = \beta x \ (\text{mod } 1),$$

and the β-expansion of x is:

$$x = \sum_{k=1}^{\infty} \frac{d_k}{\beta^k},$$

where $d_k = d_k(x) = [\beta T_\beta^{k-1}(x)],\ k \geq 1$, where in general $[a]$ denotes the integer part of a real number a. The *digits* $d_k,\ k \geq 1$ are chosen from the finite set $\{0, 1, \ldots, [\beta]\}$. Not all infinite series of the form $\sum\limits_{k\geq 1} \frac{d_k}{\beta^k}$, with $d_k \in \{0, 1, \ldots, [\beta]\}$ are however β-expansions of some number. Thus we say that a sequence of digits $(d_1, d_2, \ldots)$ is *admissible* if there exists some $x \in [0,1)$ whose β-expansion is $x = \sum\limits_{k\geq 1} \frac{d_k}{\beta^k}$. The map T_β does not necessarily preserve the Lebesgue measure λ, however it has a unique probability measure $\nu_\beta = h_\beta d\lambda$, which is equivalent to λ and T_β-invariant. Its density h_β has an explicit form (see [22], [5]).

Consider now the inverse limit of the system $([0,1), T_\beta)$. First, let us take $\beta = \frac{\sqrt{5}+1}{2}$, as a simpler example. Define the following skew product map

$$\mathcal{T}_\beta(x,y) := \left(T_\beta(x), \frac{y + [\beta x]}{\beta} \right),$$

on a subset $Z \subset [0,1)^2$, where the horizontal $[0,1)$ is considered as the future-axis and the vertical $[0,1)$ is considered as the past-axis. The inverse limit of T_β must encapsulate both the forward iterates of T_β, and the backward trajectories of T_β. For $\beta = \frac{1+\sqrt{5}}{2}$, take

$$Z = [0, 1/\beta) \times [0,1) \bigcup [1/\beta, 1) \times [0, 1/\beta).$$

Then the map $\mathcal{T}_\beta : Z \to Z$ is well defined and bijective, and it is the inverse limit of T_β. For $\beta = \frac{\sqrt{5}+1}{2}$, the set of admissible sequences forms a subshift of finite type $E_2^+(11)$, defined as the set of sequences in E_2^+ which do not contain the forbidden word 11. In this case there is a Hölder continuous coding map $\pi : E_2^+(11) \to [0,1)$, $\pi\big((d_1, d_2, \ldots)\big) = \sum\limits_{n\geq 1} \frac{d_n}{\beta^n}$. We can then take skew product endomorphisms over $T_{\frac{1+\sqrt{5}}{2}}$, and obtain:

Theorem 12.1. *Let a skew product endomorphism $F : [0,1) \times Y \to [0,1) \times Y,\ F(x,y) = \big(\frac{1+\sqrt{5}}{2} x \ (mod\ 1), g(x,y)\big)$, where Y is an open bounded set in $\mathbb{R}^d$ and $g(x,\cdot) : Y \to Y$ is conformal for any $x \in [0,1)$. Assume also that for any $x_{-1}, x'_{-1} \in [0,1)$ with $T_{\frac{1+\sqrt{5}}{2}} x_{-1} = T_{\frac{1+\sqrt{5}}{2}} x'_{-1}$, we have $g(x_{-1}, Y) \cap g_{x'_{-1}}(Y) = \emptyset$, and that the fiber maps g_x satisfy conditions a) -g) of Section 5. Then, for any locally Hölder continuous $\phi : [0,1) \times Y \to \mathbb{R}$, its equilibrium measure μ_ϕ has exact dimensional conditional measures μ_ϕ^x, for $\mu_\phi \circ \pi_1^{-1}$–a.e $x \in [0,1)$.*

Proof. This case is simpler because the coding is with a subshift of finite type on finitely many symbols. Moreover we see that the coding $\pi : E_2^+(11) \to [0,1)$ over $T_{\frac{1+\sqrt{5}}{2}}$ is injective outside a countable set, and the associated symbolic skew product over $E_2^+(11)$,

$\hat{F} : E_2^+(11) \times Y \to E_2^+(11) \times Y$ satisfies conditions (a)–(g) in Section 5. Therefore if ϕ is locally Hölder we have also the summability condition, since we work with finitely many symbols, thus from Theorem 8.4 we obtain the conclusion.

□

In general, if $f : X \to X$ is a measurable map on a Borel space $(X, \mathcal{B})$ and μ is an f-invariant measure on $(X, \mathcal{B})$, let $A \subset X$ be a measurable subset with $\mu(A) > 0$. In our case, if we work with equilibrium measures, it will be enough to take A such that $int(A) \neq \emptyset$. By Poincaré Recurrence Theorem it follows that μ-a.e $x \in A$ is A-recurrent, i.e it returns infinitely often to A. Define then $n(x) := \inf\{m \geq 1, T^m(x) \in A\}$, to be the *first return time* of x to A. This permits to define the *induced map* on A,

$$T_A : A \to A, \ T_A(x) = T^{n(x)}(x), \ x \in A \tag{12.1}$$

It is easy to prove that if $\mu_A(B) := \frac{\mu(B)}{\mu(A)}$, for $B \subset A$, then the measure μ_A is T_A-invariant.

It was proved (see for eg [5]) that the induced transformation of the natural extension $\mathcal{T}_\beta$ onto a certain subset, is the natural extension of a GLS system. Recall that natural extensions are viewed only as dynamical systems, without measures. When $\beta = \frac{\sqrt{5}+1}{2}$, take the partition $\mathcal{I} = \{[0, \frac{1}{\beta}), [\frac{1}{\beta}, 1)\}$ and the associated GLS($\mathcal{I}$)-transformation,

$$S(x) = \begin{cases} T_\beta(x), \ x \in [0, \frac{1}{\beta}) \\ \\ T_\beta^2(x), \ x \in [\frac{1}{\beta}, 1) \end{cases}$$

So if $\beta = \frac{\sqrt{5}+1}{2}$, let $W = [0,1) \times [0, \frac{1}{\beta})$, and $\mathcal{T}_{\beta,W} : W \to W$, the induced transformation of $\mathcal{T}_\beta$ on W. If $(x, y) \in [0, \frac{1}{\beta}) \times [0, 1)$, we have $\mathcal{T}_\beta(x, y) \in [0, 1) \times [0, \frac{1}{\beta})$, so for such (x, y), we get $n(x, y) = 1$. If $(x, y) \in [\frac{1}{\beta}, 1) \times [0, \frac{1}{\beta})$, then $\mathcal{T}_\beta(x, y) \in [0, \frac{1}{\beta}) \times [\frac{1}{\beta}, 1) \notin W$, but $\mathcal{T}_\beta^2(x, y) \in W$, so $n(x, y) = 2$. Hence the induced map $\mathcal{T}_{\beta,W}$ of the natural extension $\mathcal{T}_\beta$ on W is:

$$\mathcal{T}_{\beta,W}(x, y) = \begin{cases} (\beta x, \frac{y}{\beta}), \ (x, y) \in [0, \frac{1}{\beta}) \times [0, \frac{1}{\beta}) \\ \\ (\beta(\beta x - 1), \frac{y+1}{\beta^2}) = (\beta^2 x - \beta, \frac{y+1}{\beta^2}), \ (x, y) \in [\frac{1}{\beta}, 1) \times [0, \frac{1}{\beta}) \end{cases} \tag{12.2}$$

Then from (11.3), the inverse limit of S is the map $\mathcal{S} : [0, 1)^2 \to [0, 1)^2$ given by:

$$\mathcal{S}(x, y) = \begin{cases} (\beta x, \frac{y}{\beta}), \ (x, y) \in [0, \frac{1}{\beta}) \times [0, 1) \\ \\ (\beta^2 x - \beta, \frac{1}{\beta} + y\frac{\beta-1}{\beta}) = (\beta^2 x - \beta, \frac{y+\beta}{\beta^2}), \ (x, y) \in [\frac{1}{\beta}, 1) \times [0, 1) \end{cases} \tag{12.3}$$

If $\Psi : [0, 1)^2 \to W, \Psi(x, y) = (x, \frac{y}{\beta})$, then Ψ is an isomorphism between $([0, 1)^2, \mathcal{S})$ and $([0, 1) \times [0, \frac{1}{\beta}), \mathcal{T}_{\beta,W})$. Hence, $\mathcal{T}_{\beta,W}$ is equal (mod Ψ) with the natural extension $\mathcal{S}$ of GLS($\mathcal{I}$).

For *general* $\beta > 1$, *not all sequences in* $E_{[\beta]}^+$ *are admissible*, i.e not all sequences of digits $(d_1, d_2, \dots)$ determine a point $x \in [0, 1)$ that has β-expansion $x = \sum\limits_{n \geq 1} \frac{d_n}{\beta^n}$. This is an important obstacle, since we cannot code T_β with subshifts of finite type. For general $\beta > 1$, one needs a more complicated GLS with partition $\mathcal{I}$ with countably many subintervals

$I_n, n \in \mathcal{D}$, and then to induce the natural extension of T_β to a subset, in order to obtain the natural extension of the GLS($\mathcal{I}$)-map. We will apply next Corollary 11.2 for equilibrium states on the natural extension of the GLS($\mathcal{I}$) expansion. The construction of the inverse limit of T_β can be found in [5], [6], we recall it for completion. Let the following rectangles

$$Z_0 = [0,1)^2,\ Z_i = [0, T_\beta^i 1) \times [0, \frac{1}{\beta^i}), i \geq 1$$

Consider the space Z of the natural extension as being obtained by placing the rectangle Z_{i+1} on top of Z_i, for all $i \geq 0$. The index i indicates at what height we are in this stack. If 1 has a finite β-expansion of length n, then only n such rectangles Z_i are stacked. Assume 1 has an infinite β-expansion; the finite case follows as for $\beta = \frac{1+\sqrt{5}}{2}$. Let the β-expansion of 1 be $1 = .b_1 b_2 \ldots,$ of x be $x = .d_1 d_2 \ldots$, and of y be $y = .0 \ldots 0 c_{i+1} c_{i+2} \ldots$ with 0 repeated i times. If $(x,y) \in Z_i$, then $d_1 \leq b_{i+1}$. Define $\mathcal{T}_\beta : Z \to Z,\ \mathcal{T}_\beta(x,y) = (T_\beta(x), \tilde{y}(x))$, with

$$\tilde{y}(x) = \begin{cases} \frac{b_1}{\beta} + \ldots + \frac{b_i}{\beta^i} + \frac{d_1}{\beta^{i+1}} + \frac{y}{\beta} = .b_1 \ldots b_i d_1 c_{i+1} c_{i+2} \ldots, \text{ if } d_1 < b_{i+1}, \\ \\ \frac{y}{\beta}, \text{ if } d_1 = b_{i+1} \end{cases} \tag{12.4}$$

If $(x,y) \in Z_i$ then $d_1 \leq b_{i+1}$, so

$$\mathcal{T}_\beta(x,y) \in \begin{cases} Z_0, \text{ if } d_1 < b_{i+1}, \\ \\ Z_{i+1}, \text{ if } d_1 = b_{i+1} \end{cases} \tag{12.5}$$

For $(x,y) \in Z_0$, if $d_1 < b_1$ then $\mathcal{T}_\beta(x,y) \in Z_0$; and if $d_i = b_i, 1 \leq i \leq n-1$ and $d_n < b_n$, then $\mathcal{T}_\beta^i(x,y) \in Z_i, i \leq n-1$ and $\mathcal{T}_\beta^n(x,y) \in Z_0$. Hence the induced map of $\mathcal{T}_\beta$ on $Z_0 = [0,1)^2$ is

$$\mathcal{T}_{\beta,Z_0}(x,y) = \begin{cases} \mathcal{T}_\beta(x,y), \text{ if } d_1 < b_1, \\ \\ \mathcal{T}_\beta^n(x,y), \text{ if } d_i = b_i, 1 \leq i \leq n-1, \text{ and } d_n < b_n \end{cases}$$

Partition Z_0 into subsets $Z_0^k := \{(x,y) \in Z_0,\ \inf\{n \geq 1, \mathcal{T}_\beta^n(x,y) \in Z_0\} = k\}$, and thus

$$\mathcal{T}_{\beta,Z_0}(x,y) = \begin{cases} \big(T_\beta(x), \frac{1}{\beta}(y + d_1)\big),\ (x,y) \in Z_0^1, \text{ if } d_1 < b_1, \\ \\ \big(T_\beta^k(x), \frac{b_1}{\beta} + \ldots + \frac{b_{k-1}}{\beta^{k-1}} + \frac{d_k}{\beta^k} + \frac{y}{\beta^k}\big),\ (x,y) \in Z_0^k, k \geq 2 \end{cases}$$

For any $n \geq 1$, if $b_0 := 0$, there exist unique integers $k = k(n) \geq 0$ and $1 \leq i \leq b_{k+1}$, so that $n = b_0 + b_1 + \ldots + b_k + (i-1)$. Define a partition $\mathcal{I} = \{I_n,\ n \geq 1\}$ of $[0,1)$, by

$$I_n := \left[b_0 + \frac{b_1}{\beta} + \ldots + \frac{b_k}{\beta^k} + \frac{i-1}{\beta^{k+1}},\ b_0 + \frac{b_1}{\beta} + \ldots + \frac{b_k}{\beta^k} + \frac{i}{\beta^{k+1}}\right) \tag{12.6}$$

From the definition of $\mathcal{T}_{\beta,Z_0}$ and of I_n we see that for $(x,y) \in I_n \times [0,1)$, we have:

$$\mathcal{T}_{\beta,Z_0}(x,y) = \mathcal{T}_\beta^{k+1}(x,y) = \left(T_\beta^{k+1} x,\ b_0 + \frac{b_1}{\beta} + \ldots \frac{b_k}{\beta^k} + \frac{i-1}{\beta^{k+1}} + \frac{y}{\beta^{k+1}}\right)$$

If we take the transformation S of GLS($\mathcal{I}$) and its natural extension $\mathcal{S}$, then (11.3) applies. If $x \in I_n$, then $s_1(x) = \beta^{k+1}$ and $h_1(x)/s_1(x) = b_0 + \frac{b_1}{\beta} + \ldots + \frac{b_k}{\beta^k} + \frac{i-1}{\beta^{k+1}}$. So by (11.3) we obtain equality between $\mathcal{S}$ and the induced map on Z_0 of the natural extension of T_β,

$$\mathcal{S} = \mathcal{T}_{\beta, Z_0} \tag{12.7}$$

We can now apply (12.7) to equilibrium states of locally Hölder continuous potentials, for the induced map of the natural extension $\mathcal{T}_\beta$, in order to prove the *exact dimensionality* of their conditional measures on fibers. By (12.7) and Corollary 11.2, we obtain the following result, for the induced map of the natural extension of the β-transformation:

Theorem 12.2. *Let $\beta > 1$ arbitrary and the β-transformation $T_\beta : [0,1) \to [0,1), T_\beta(x) = \beta x (mod\ 1)$. Let $\phi : [0,1)^2 \to \mathbb{R}$ a locally Hölder continuous map with*

$$\sum_{n \geq 1} \exp(\sup \phi|_{I_n \times [0,1)}) < \infty,$$

where $I_n, n \geq 1$ are given by (12.6), and let μ_ϕ be its equilibrium state with respect to the induced map $\mathcal{T}_{\beta,[0,1)^2}$ of the natural extension $\mathcal{T}_\beta$ on $[0,1)^2$. Denote by $\mathcal{S}$ the natural extension of the GLS($\mathcal{I}$) transformation, where $\mathcal{I}$ is the partition of $[0,1)$ given by $(I_n)_{n\geq 1}$. Then for $\mu_\phi \circ \pi_1^{-1}$-a.e $x \in [0,1)$, the conditional measure μ_ϕ^x of μ_ϕ is exact dimensional on $[0,1)$, and for μ_ϕ^x-a.e $y \in [0,1)$, its Hausdorff and pointwise dimension are both equal to

$$\mathrm{HD}(\mu_\phi^x) = \lim_{r \to 0} \frac{\mu_\phi^x(B(y,r))}{\log r} = \frac{\mathrm{h}_{\mu_\phi}(\mathcal{S})}{\chi_{\mu_\phi}(\mathcal{S})}.$$

Due to the particular expression of the induced map on $[0,1)^2$ of the natural extension $\mathcal{T}_\beta$ of the beta-transformation T_η, as a natural extension of a GLS-transformation, we can say more about the *Lyapunov exponent* $\chi_{\mu_\phi}(\mathcal{S})$. This Lyapunov exponent is then used in the formula above for the pointwise dimension of μ_ϕ^x:

Corollary 12.3. *In the setting of Theorem 12.2 with $\beta > 1$ arbitrary, write the β-expansion of 1 as $1 = .b_1 b_2 \ldots$. For arbitrary integer $n \geq 1$, define the integers $k = k(n) \geq 0$ and $1 \leq i = i(n) \leq b_{k+1}$ so that $n = b_1 + \ldots + b_k + i - 1$. Then,*

(a) *with the intervals $I_n, n \geq 1$ given by (12.6), we obtain the Lyapunov exponent as,*

$$\chi_{\mu_\phi}(\mathcal{S}) = \log \beta \cdot \sum_{n \geq 1} \big(k(n) + 1\big) \cdot \mu_\phi\big(I_n \times [0,1)\big).$$

Hence for $\mu_\phi \circ \pi_1^{-1}$-a.e $x \in [0,1)$, we have

$$\mathrm{HD}(\mu_\phi^x) = \frac{h_{\mu_\phi}(\mathcal{S})}{\log \beta \sum\limits_{n \geq 1} (k(n)+1) \cdot \mu_\phi\big(I_n \times [0,1)\big)}.$$

(b) *When $\beta = \frac{1+\sqrt{5}}{2}$, we obtain that the Lyapunov exponent of μ_ϕ as*

$$\chi_{\mu_\phi}(\mathcal{S}) = \log \frac{1+\sqrt{5}}{2} \cdot \Big(1 + \mu_\phi\big([\frac{1}{\beta}, 1) \times [0,1)\big)\Big)$$

Hence $\mu_\phi \circ \pi_1^{-1}$-a.e $x \in [0,1)$, we have

$$\mathrm{HD}(\mu_\phi^x) = \frac{h_{\mu_\phi}(\mathcal{S})}{\log \frac{1+\sqrt{5}}{2} \Big(1 + \mu_\phi\big([\frac{1}{\beta}, 1) \times [0,1)\big)\Big)}.$$

Proof. To prove (a) we apply (12.6), and (12.7). Let us write $\mathcal{S}(x,y) = (S(x), g_x(y))$, $(x,y) \in [0,1) \times [0,1)$, where S is the GLS($\mathcal{I}$)-transformation and $\mathcal{S}$ is its natural extension (see 11.3). The derivative of the fiber map g_x is constant and equal to L_n, for $x \in I_n$, where L_n is the length of the interval I_n and thus equal to $\frac{1}{\beta^{k+1}}$. Finally, for the Hausdorff (and pointwise) dimension of conditional measures we apply Theorem 12.2.

For (b) we apply (12.3) to get $g_x'(y) = \frac{1}{\beta}$ for $(x,y) \in [0, \frac{1}{\beta}) \times [0,1)$, and $g_x'(y) = \frac{1}{\beta^2}$, $(x,y) \in [\frac{1}{\beta}, 1) \times [0,1)$, with $\beta = \frac{1+\sqrt{5}}{2}$. Then use that $\mu_\phi([0,\frac{1}{\beta}) \times [0,1)) + \mu_\phi([\frac{1}{\beta},1) \times [0,1)) = 1$. □

Now from (12.7), we know that the induced map $\mathcal{T}_{\beta,[0,1)^2}$ is equal to the inverse limit $\mathcal{S}$ of the GLS transformation S associated to the countable partition $\mathcal{I}$, given by (12.6). According to (11.3), and if f is given by (11.2), then $\mathcal{S}(x,y)$ and $\mathcal{T}_{\beta,[0,1)^2}$ can be written as

$$\mathcal{S}(x,y) = \mathcal{T}_{\beta,[0,1)^2}(x,y) = (f(x), \frac{h_1}{s_1} + \frac{y}{s_1}),\ (x,y) \in [0,1)^2 \tag{12.8}$$

We will now use the explicit form of the induced map $\mathcal{T}_{\beta,W}$ of the natural extension $\mathcal{T}_\beta$ of T_β, $\beta > 1$, and Theorem 8.7, to prove that **any** $\mathcal{T}_{\beta,W}$-invariant equilibrium measure μ_ϕ is **exact dimensional globally** on $[0,1) \times [0,1)$, and to compute its pointwise (and Hausdorff) dimension. We use below the notation from (12.8) and Corollary 12.3; also denote by $\pi_1 : [0,1)^2 \to [0,1)$ the projection on first coordinate.

Theorem 12.4. *Let an arbitrary $\beta > 1$, $T_\beta(x) = \beta x \ (mod\ 1), x \in [0,1)$, $\mathcal{T}_\beta$ be the natural extension of T_β, and $\mathcal{T}_{\beta,[0,1)^2}$ the induced map of $\mathcal{T}_\beta$ on $[0,1)^2$. Let ϕ a locally Hölder continuous potential on $[0,1)^2$ satisfying $\sum\limits_{n\geq 1} \exp(\sup \phi|_{I_n \times [0,1)}) < \infty$, and μ_ϕ its equilibrium measure with respect to $\mathcal{T}_{\beta,[0,1)^2}$, and let $\nu := \pi_{1*}\mu_\phi$ be the projection of μ_ϕ.*
Then, μ_ϕ is exact dimensional on $[0,1)^2$, and its pointwise (and Hausdorff) dimension is

$$\mathrm{HD}(\mu_\phi) = \frac{2h_\nu(f)}{\log\beta \cdot \sum\limits_{n\geq 1} \big(k(n)+1\big)\mu_\phi\big(I_n \times [0,1)\big)}.$$

Proof. We recall from (12.8) that for $\beta > 1$ arbitrary, $\mathcal{T}_{\beta,[0,1)^2} = \mathcal{S}$. If ϕ is locally Hölder continuous and summable on $[0,1)^2$, and if μ_ϕ is its equilibrium measure with respect to $\mathcal{T}_{\beta,[0,1)^2}$, and thus also with respect to $\mathcal{S}$, then its projection $\nu = \pi_{1*}\mu_\phi$ is f-invariant and ergodic on $[0,1)$. Now, if $\mathcal{I}$ denotes the countable partition $(I_n)_{n\geq 1}$ from (12.6), then from the previous Section we obtain the coding $\pi : \Sigma_{\mathcal{I}}^+ \to [0,1)$ between $(\Sigma_{\mathcal{I}}^+, \sigma)$ and $([0,1), f)$. Moreover, we also have that ν gives zero measure to points, and thus ν is the projection of an ergodic measure $\tilde{\nu}$ on $\Sigma_{\mathcal{I}}^+$. The interval $[0,1)$ can be viewed as the limit set of the canonical iterated function system associated to the countable partition $\mathcal{I}$ of $[0,1)$, where the contractions are the inverses of the branches of f on the intervals $I_n, n \geq 1$.

Consider now the random system given, in the notation of [21] by: a parameter space $\Lambda = \{\lambda\}$ with a homeomorphism $\theta : \Lambda \to \Lambda, \theta(\lambda) = \lambda$, which preserves the Dirac measure $m = \delta_\lambda$, and to the shift space $(\Sigma_{\mathcal{I}}^+, \sigma)$ with the ergodic σ-invariant measure $\tilde{\nu}$. Then, the measure $\tilde{\nu}$ is in fact the only conditional measure of the product measure $\delta_\lambda \times \tilde{\nu}$ on $\Lambda \times \Sigma_{\mathcal{I}}^+$. On the other hand, since our potential ϕ is summable, it follows from our results in Section

4 and from the fact that $\big([0,1)^2, \mathcal{T}_{\beta,[0,1)^2}\big)$ is coded by a Smale space of countable type, that the entropy $h_{\mu_\phi}(\mathcal{T}_{\beta,[0,1)^2})$ is finite; therefore, since $\nu = \pi_{1*}\mu_\phi$, we obtain also $h_{\tilde\nu}(f) < \infty$. But then, from Remark 3.4 of [21], $H_{\delta_\lambda\times\tilde\nu}(\pi^{-1}_{\Sigma_I^+}(\xi)|\pi_\Lambda^{-1}(\epsilon_\Lambda)) < \infty$. So from Theorem 3.13 in [21] and the discussion above, it follows that ν is exact dimensional on $[0,1)$, and that,

$$\mathrm{HD}(\nu) = \frac{h_\nu(f)}{\chi_\nu(f)}$$

On the other hand, from Theorem 12.2, it follows that the conditional measures of μ_ϕ are exact dimensional on fibers (which are all equal to $[0,1)$ in this case).
Now, if the f-invariant measure ν is exact dimensional on $[0,1)$, and if the conditional measures of μ_ϕ are exact dimensional on fibers, we apply Theorem 8.7 to obtain that μ_ϕ is exact dimensional globally on $[0,1)^2$. Moreover from Theorem 8.7, we have that $HD(\mu_\phi)$ is the sum of $HD(\nu)$ and of the dimension of conditional measures. The last step is then to obtain the expression of the Lyapunov exponent $\chi_\nu(f)$. In our case, from (11.2) it follows that $|f'(x)| \equiv L_n^{-1}$, if $x \in I_n$, where L_n denotes the length of I_n, $n \geq 1$; and from (12.6), $L_n = \frac{1}{\beta^{k(n)+1}}$. Also $\nu(A) = \mu_\phi(A \times [0,1))$ for any Borel set $A \subset [0,1)$. Therefore,

$$\chi_\nu(f) = \log\beta \cdot \sum_{n\geq 1} \big(k(n)+1\big)\mu_\phi\big(I_n \times [0,1)\big)$$

In addition, remark that from Shannon-McMillan-Breiman Theorem ([10]), $h_{\mu_\phi}(\mathcal{S})$ is equal to $h_\nu(f)$, since $\mathcal{S}$ contracts in the second coordinate. In conclusion, from the above discussion and Corollary 12.3, the pointwise dimension of μ_ϕ is the one from the statement.

□

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