Measure-theoretic degrees and topological pressure for non-expanding transformations

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Abstract

We consider invariant sets Λ of saddle type, for non-invertible smooth maps f, and equilibrium measures μ_ϕ associated to Hölder potentials ϕ on Λ. We define a notion of *measure-theoretic asymptotic degree of* $f|_\Lambda : \Lambda \to \Lambda$, with respect to the measure μ_ϕ on the fractal set Λ. In our case, the equilibrium measure μ_ϕ is the unique linear functional in $\mathcal{C}(\Lambda)^*$ tangent to the pressure function $P(\cdot) : \mathcal{C}(\Lambda) \to \mathbb{R}$ at ϕ. In particular, for the measure of maximal entropy μ_0 of $f|_\Lambda$, we obtain the *asymptotic degree* of $f|_\Lambda$, which represents the average rate of growth of the *number of n-preimages of x that remain* in Λ when $n \to \infty$; notice that, in general, Λ is not totally invariant for f. To this end, we will obtain first a formula for the Jacobians of the probability μ_ϕ, with respect to arbitrary iterates $f^m, m \geq 2$. We then show that a formula for the topological pressure $P(\phi)$ that holds in the expanding case, is no longer true on saddle sets. In the saddle case we find a *new formula for the pressure*, involving weighted sums on preimage sets. We also apply the asymptotic degrees, together with various pressure functionals, in order to obtain estimates for the Hausdorff dimension of stable slices through certain sets of full μ_ϕ-measure in the fractal Λ. In the end, we give also some concrete examples on saddle folded sets.

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1 Introduction and outline of main results.

We consider smooth maps $f : M \to M$ on a manifold M, which are hyperbolic and non-invertible on saddle locally maximal sets Λ, and associated equilibrium (Gibbs) measures μ_ϕ, of Hölder potentials ϕ on Λ. We investigate several notions related to them, like the Jacobian of such a measure, and a new, measure-theoretic notion of "degree" of $f|_\Lambda : \Lambda \to \Lambda$, in the case when the number of f-preimages that remain in Λ of an arbitrary point x, is not constant, when x varies in Λ. We will also look more closely at the pressure functional $P(\cdot)$, on the Banach space $\mathcal{C}(\Lambda)$ of continuous real-valued functions on Λ, when Λ is such a saddle non-invertible fractal set.

The hyperbolic non-expanding and non-invertible case is very different from the expanding case, and from the hyperbolic diffeomorphism case (for eg [5], [22], [8]). One difficulty is that branches of inverse iterates do not contract small balls on Λ, which means that the machinery from the expanding case cannot be used here; in fact inverse branches dilate on stable directions. Another difficulty is that, as the fractal set Λ is not necessarily totally invariant with respect to f, there may be sudden variations in the number of f-preimages in Λ of a point x, when x ranges in Λ; also, there exist in general many (possibly uncountably) local unstable manifolds through points in Λ.

We will obtain first a formula for the Jacobian of the equilibrium measure μ_ϕ with respect to an arbitrary iterate $f^m, m \geq 2$, in this saddle case (the Jacobian is in fact a Radon-Nikodym derivative). Using this, we obtain then a formula for the pressure $P(\phi)$ in terms of the preimage sets of μ_ϕ-almost all points x, and of the folding entropy of μ_ϕ. This formula is different from the one in the expanding case (for eg from [17]).

In general the map f is not constant-to-1 on the saddle fractal Λ. Thus, we want to determine a good notion of "degree" for the restriction $f|_\Lambda$. We find one such notion of asymptotic degree with respect to the measure μ_ϕ. If we consider in particular the measure of maximal entropy μ_0 on Λ, we obtain then the average logarithmic growth of the number of n-preimages that remain in Λ (when $n \to \infty$), which can be considered as the "degree" of f over Λ. By using the above notions of asymptotic degree with respect to μ_ϕ, we will obtain next estimates for the dimension of stable slices through certain explicitly constructed sets of full μ_ϕ-measure in Λ.

Hence, the asymptotic degrees, the formula for Jacobians of equilibrium states with respect to arbitrary iterates and the associated methods, are useful in obtaining:

a) the *rate of growth* of the number of n-preimages remaining in Λ, when $n \to \infty$;

b) a *formula for the pressure* $P(\phi)$ in the saddle non-invertible case, in terms of the n-preimages of x that remain in Λ, for μ_ϕ-a.e point x in Λ;

c) *estimates on the Hausdorff dimension* of certain slices through the fractal Λ.

The *Jacobian* of an invariant measure μ with respect to an endomorphism f of a Lebesgue space X (see Parry, [14]) describes locally the ratio between $\mu(f(A))$ and $\mu(A)$, given that an arbitrary point in X may have several f-preimages and that $\mu(f(A)) = \mu(f^{-1}(f(A)))$. Thus the Jacobian $J_f(\mu)$ is a Radon-Nikodym derivative between two absolutely continuous measures.

Here we are concerned with the case when f is a $\mathcal{C}^2$ endomorphism (i.e a non-invertible map) on a manifold M, having a compact invariant set $\Lambda \subset M$. We assume that the non-invertible map f is *hyperbolic* on Λ (see [22]). The map f is **not** assumed expanding on Λ, thus we do not have the machinery from the expanding case here. Hyperbolicity of f on Λ implies the existence of local stable manifolds of size r (for some small $r > 0$), which depend only on their base point and are denoted by $W^s_r(x), x \in \Lambda$. Hyperbolicity implies the existence of local unstable manifolds $W^u_r(\hat{x})$ which depend on whole past trajectories $\hat{x} \in \hat{\Lambda}$, where $\hat{\Lambda}$ is the inverse limit of the system $(\Lambda, f|_\Lambda)$. Through points $x \in \Lambda$ there may pass uncountably many local unstable manifolds of prehistories of x in $\hat{\Lambda}$, which is an important difference from the diffeomorphism case.

By *basic set* (or *locally maximal set* [6]), we mean a compact f-invariant set $\Lambda \subset M$, such that $\Lambda = \bigcap\limits_{n \in \mathbb{Z}} f^n(U)$ for a neighbourhood U of Λ, and such that f is topologically transitive on Λ. Such sets will also be referred to sometimes as *folded* fractals. The term basic set is not used in the sense of the Spectral Decomposition Theorem here. The fractal set Λ may *not* be totally f-invariant, so it may happen that some of the f-preimages of $x \in \Lambda$ do not remain in Λ. Examples of hyperbolic basic sets for smooth endomorphisms appeared for instance in [1], [5], [16], [24], [25], [8].

In our case, the *topological pressure* is a convex and Lipschitz continuous function $P : \mathcal{C}(\Lambda) \to \mathbb{R}$, on the Banach space $\mathcal{C}(\Lambda)$ of continuous real-valued functions on Λ (see for eg. [3], [6], [26], [27]). In our hyperbolic case, there exists a unique *equilibrium measure* (Gibbs state) of a Hölder continuous potential ϕ on Λ (see for eg [3], [6], [26]), and this measure will be denoted by μ_ϕ. The probability measure μ_ϕ is maximizing in the Variational Principle for the topological pressure, i.e we have:

$$P(\phi) = \sup\{h_\mu + \int_\Lambda \phi \, d\mu, \ \mu \text{ is } f - \text{invariant probability on } \Lambda\} = h_{\mu_\phi} + \int_\Lambda \phi \, d\mu_\phi$$

Then the positive linear functional $\mathcal{F}_\phi$ from the dual space $\mathcal{C}(\Lambda)^*$, represented by the equilibrium measure μ_ϕ by the Riesz Representation Theorem (for eg [18]), is in fact *tangent* to the convex pressure function P at ϕ; hence, for every continuous function $\psi \in \mathcal{C}(\Lambda)$ we have (for eg. [26], [27]),

$$\mu_\phi(\psi) + P(\phi) \le P(\phi + \psi)$$

In our case the entropy map $\mu \to h_\mu$ associated to $f|_\Lambda$, is upper semi-continuous (see [6], [26]). Hence by using properties of Legendre-Fenchel transforms and a form of Hahn-Banach Theorem, it follows that conversely, every linear functional $F \in \mathcal{C}(\Lambda)^*$ tangent to $P(\cdot)$ at ϕ, is in fact given by the equilibrium measure μ_ϕ (which is the only equilibrium measure of ϕ, in our hyperbolic case). Also, Walters showed that the pressure function $P(\cdot)$ has a unique tangent functional at ϕ if and only if $P(\cdot)$ is Gâteaux differentiable at ϕ (see [27]). It can be shown, moreover, that the equilibrium measure μ_ϕ is mixing on the fractal set Λ.

If μ is an f-invariant probability measure on Λ, then one can define the *folding entropy* $F_f(\mu)$, as the conditional entropy $H_\mu(\epsilon | f^{-1}\epsilon)$, where ϵ is the single point partition and $f^{-1}\epsilon$ is the fiber partition associated to f on Λ, see [21] (also [7]); we may denote it also by $F(\mu)$ if no confusion arises. Many ergodic properties of the measure-preserving transformation f on the probabilistic space (Λ, μ), can also be expressed in terms of the spectral properties of the associated Koopman operator $U_f : L^2(\mu) \to L^2(\mu), \ U_f(\chi) = \chi \circ f, \ \chi \in L^2(\mu)$ (see for eg. [26], [18]).

The **main results** of the paper are the following:

In **Theorem 1** we will prove a formula (and definition) for a *measure-theoretic asymptotic degree* with respect to the probability μ_ϕ. This degree involves only those n-preimages of x (i.e preimages with respect to f^n) which behave well with respect to μ_ϕ; the number of these well-behaved n-preimages of x is denoted by $d_n(x, \mu_\phi, \tau)$ (see Definition 3). Notice that the dynamics of f on Λ is basically the same as that of f^n on Λ; the iterate f^n invariates Λ and the measure μ_ϕ. So in a sense, one may take any iterate of f and study the preimages of points with respect to that iterate. The map f may not be constant-to-1 on Λ.

Theorem 1 (Measure-theoretic asymptotic degree for equilibrium states). *Let $f : M \to M$ be a $\mathcal{C}^2$ non-invertible map and Λ a basic set for f so that f is hyperbolic on Λ and does not have critical points in Λ. Let also ϕ be a Hölder continuous potential on Λ and μ_ϕ be the equilibrium measure associated to ϕ. Then we have the following formula:*

$$\lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_\Lambda \log d_n(x, \mu_\phi, \tau) \ d\mu_\phi(x) = F_f(\mu_\phi)$$

In **Corollary 1** we will use the formula proved in Theorem 1 in order to calculate the average value with respect to μ_0, of the **logarithmic growth of the number of n-preimages** of x in Λ. As Λ is not necessarily totally invariant, the measurable (but possibly discontinuous) function

$$d_n(x) := Card(f^{-n}(f^n(x)) \cap \Lambda), \ x \in \Lambda,$$

may be non-constant on Λ; see the examples in [8]. It is natural to study the average value of $\log d_n(\cdot)$.

Corollary 1 (Average rate of growth of the number of n-preimages $d_n(\cdot)$, when $n \to \infty$). *In the setting of Theorem 1, denote by μ_0 the unique measure of maximal entropy for f on Λ. If $d_n(x)$ denotes the cardinality of $f^{-n}(f^n x) \cap \Lambda$ for $n \geq 1$, then we have:*

$$\lim_{n \to \infty} \frac{1}{n} \log d_n(x) = F_f(\mu_0), \ \mu_0 - a.a \ x \in \Lambda, \quad and \quad \lim_{n \to \infty} \frac{1}{n} \int_\Lambda \log d_n(x) \ d\mu_0(x) = F_f(\mu_0)$$

Corollary 1 allows us to make the following:

Definition 1. In the setting of Theorem 1, define the **asymptotic logarithmic degree of $f|_\Lambda$** (with respect to the measure of maximal entropy μ_0) by: $a_l(f, \Lambda) := \lim_n \frac{1}{n} \int_\Lambda \log d_n(x) d\mu_0(x)$. The **asymptotic degree** of $f|_\Lambda$ is then defined as the number

$$d_\infty(f, \Lambda) := e^{a_l(f, \Lambda)}$$

Similarly we define the **asymptotic degree with respect to the measure μ_ϕ** on Λ, as

$$d_\infty(f, \mu_\phi) := \exp\big(\lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_\Lambda \log d_n(x, \mu_\phi, \tau) \ d\mu_\phi(x)\big)$$

In particular if $f|_\Lambda$ is d-to-1, then $d_\infty(f, \Lambda) = d$, and $F(\mu_0) = \log d$.

To prove Theorem 1 we will need Proposition 1 which gives a formula for the **Jacobian** of an equilibrium measure μ_ϕ, with respect to an *arbitrary iterate* f^n; the estimates do not depend on n.

Proposition 1 (Jacobians of equilibrium measures with respect to iterates of endomorphisms). *Let f be a $\mathcal{C}^2$ hyperbolic endomorphism on a folded basic set Λ, which has no critical points in Λ; let also ϕ be a Hölder continuous potential on Λ and let μ_ϕ the unique equilibrium measure of ϕ on Λ. Then there exists a comparability constant $C > 0$ independent of $m \geq 2$ and of $x \in \Lambda$, such that for $\mu_\phi - a.e \ x \in \Lambda$, the Jacobian of μ_ϕ with respect to the iterate f^m satisfies:*

$$C^{-1} \cdot \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m \phi(\zeta)}}{e^{S_m \phi(x)}} \leq \ J_{f^m}(\mu_\phi)(x) \leq \ C \cdot \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m \phi(\zeta)}}{e^{S_m \phi(x)}} \tag{1}$$

Recall now (see [17]) that in the expanding case we have the following formula for pressure:

Theorem (Relation between preimage sets and pressure in the expanding case, [17])**.** *Let $f : X \to X$ a topologically transitive open distance expanding map, then for every Hölder continuous potential $\phi : X \to \mathbb{R}$ and every $x \in X$ we have the equality*

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x)} e^{S_n\phi(y)}$$

In our *saddle* set setting we obtain however the following different **formula for the pressure**:

Theorem 2 (Relation between preimage sets and pressure in the saddle non-invertible case)**.** *In the setting of Proposition 1 and for an arbitrary Hölder continuous potential ϕ on Λ, we have for μ_ϕ-a.e $x \in \Lambda$,*

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{y \in f^{-n}(f^n(x))\cap\Lambda} e^{S_n\phi(y)} - \log d_\infty(f, \mu_\phi) + h_{\mu_\phi}$$

The difference in the formula above for the saddle case, is due to the negative Lyapunov exponents. The Remark after the proof of Theorem 2 shows that, in general $F_f(\mu_\phi) \neq h_{\mu_\phi}$. Once we have a formula for the pressure on a saddle set for a non-invertible map, we can obtain the measure-theoretic entropy h_μ for **any** f-invariant measure μ on Λ, by a reverse Variational Principle (see [26]), using the fact that: $h_\mu = \inf\left\{P(\psi) - \int_\Lambda \psi \, d\mu, \ \psi \text{ Hölder continuous on } \Lambda\right\}$, as the entropy map is upper semi-continuous in our case.

Another application will be in the next Corollary, where we compute the μ_ϕ-measure of an **arbitrary ball** centered on Λ; for a map $f : X \to X$ on a metric space X, we denote by $B_n(x, \varepsilon) := \{y \in X, d(f^i y, f^i x) < \varepsilon, i = 0, \ldots, n-1\}, x \in X, \varepsilon > 0$, an arbitrary dynamical (Bowen) ball.

Corollary 2. *In the same setting as in Proposition 1, assuming f is conformal on both stable and unstable local manifolds, there is $C > 0$ such that the μ_ϕ-measure of an arbitrary ball is given by:*

$$\frac{1}{C} \int_{B_n(z,\varepsilon)} \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x))\cap\Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}} \, d\mu_\phi(x) \le \mu_\phi(B(f^m z, \rho) \le C \int_{B_n(z,\varepsilon)} \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x))\cap\Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}} \, d\mu_\phi(x),$$

where ε is fixed and m, n are the largest integers s.t $\varepsilon |Df_s^m(z)| \ge \rho$ and $\varepsilon |Df_u^{n-m}(f^m z)|^{-1} \ge \rho$, for any $z \in \Lambda, \rho > 0$.

We also apply the asymptotic degrees in order to obtain **estimates** for the Hausdorff dimension of various **slices** through Λ. We recall that from Definition 1, that $\log d_\infty(f, \mu_\phi) = F(\mu_\phi)$; in particular $\log d_\infty(f, \Lambda) = F(\mu_0)$, where μ_0 is the measure of maximal entropy of $f|_\Lambda$. If $\Phi^s(x) := \log |Df_s(x)|, \ x \in \Lambda$, then for any fixed number $\gamma \le h_{top}(f|_\Lambda)$, we have that the function

$$t \mapsto P(t\Phi^s - \gamma),$$

is strictly decreasing and convex, it has a value larger or equal than 0 when $t = 0$, and converges to $-\infty$ when $t \to \infty$. Hence this pressure function has a unique zero (called also a solution of the

Bowen-type equation, [2]), which will prove important in dimension estimates. In the next result, given the basic saddle set Λ for the map f, we denote by E^s_x the stable tangent space at $x \in \Lambda$; hence $Df|_{E^s_x}$ is a linear contraction.

Theorem 3 (Dimension estimates for certain stable slices). *In the setting of Theorem 1, assume that f is conformal on local stable manifolds over the saddle basic set Λ, and that μ_ϕ is the equilibrium measure of a Hölder continuous potential ϕ on Λ; denote $\Phi^s(y) := \log |Df|_{E^s(y)}|,\ y \in \Lambda$. Then there exists a Borel set $\mathcal{K}(\mu_\phi) \subset \Lambda$ such that $\mu_\phi(\mathcal{K}(\mu_\phi)) = 1$, and for every $x \in \Lambda$ we have:*

$$HD\big(W^s_r(x) \cap \mathcal{K}(\mu_\phi)\big) \leq t^s_{d_\infty(f,\mu_\phi)},$$

where $t^s_{d_\infty(f,\mu_\phi)}$ is the unique zero of the pressure function $t \to P(t\Phi^s - \log d_\infty(f, \mu_\phi))$.

We remark that the set $\mathcal{K}(\mu_\phi)$ is constructed explicitly in the proof of Theorem 3 above; and that it is not contained necessarily in the generic set of the stable potential Φ^s. In fact, we obtain a whole class of sets of type $\mathcal{K}(\mu_\phi)$, according to various conditions.

In Section 3 we will give also several examples of hyperbolic basic fractal sets, and apply the results above to the equilibrium measures on them. Such examples may be obtained for example from parametrized families with transversality conditions; from solenoids with self-intersections; or from perturbations of some known hyperbolic endomorphisms.

2 Main results and proofs.

In the sequel, let a smooth (say $\mathcal{C}^2$) non-invertible map $f : M \to M$ defined on a compact Riemannian manifold, and let Λ be a fixed basic set of f, such that f is hyperbolic on the compact Λ. In general, the fractal set Λ is not totally invariant, i.e we do not always have $f^{-1}(\Lambda) = \Lambda$. As said before, hyperbolicity is understood here in the sense of *endomorphisms* (i.e non-invertible maps), i.e there exists a continuous splitting of the tangent bundle into stable and unstable directions, over the inverse limit $\hat\Lambda$ consisting of sequences of consecutive preimages, $\hat\Lambda = \{\hat x = (x, x_{-1}, x_{-2}, \dots,)$ with $x_{-i} \in \Lambda, f(x_{-i}) = x_{-i+1}, i \geq 1\}$. For any $\hat x \in \hat\Lambda$ we have a stable space E^s_x and an unstable space $E^u_{\hat x}$. There is a small $r > 0$ and local stable/unstable manifolds, $W^s_r(x)$ and $W^u_r(\hat x)$, for any $\hat x \in \hat\Lambda$. Denote also

$$Df_s(x) := Df|_{E^s_x},\ x \in \Lambda \text{ and } Df_u(\hat x) := Df|_{E^u_{\hat x}},\ \hat x \in \hat\Lambda \tag{2}$$

The endomorphism f is assumed to have stable directions too, so it is non-expanding. More about hyperbolicity for endomorphisms can be found for example in [22], [10], etc. When the map is not invertible, there appear significantly different phenomena and different techniques than in the case of diffeomorphisms (as for example in [1], [20], [25], [8]).

We will use in the sequel the notion of *Jacobian of an invariant measure* introduced by Parry in [14]. Let $f : M \to M$ be a continuous endomorphism on the manifold M and μ an f-invariant probability on M. Assume also that f is *essentially countable-to-one*, i.e that the canonical measures (mod 0) of μ with respect to the partition into fibers $f^{-1}(\epsilon)$, are purely atomic (see [14]); in other words, modulo μ the fibers $f^{-1}(x)$ are countable, f is measurable and positively non-singular with respect to μ, i.e $\mu(A) = 0$ implies $\mu(f(A)) = 0$. Then, as shown by Rohlin ([19], [14]), there exists a measurable partition $\xi = (A_0, A_1, \dots)$ so that f is injective on each A_i, and the push-forward measure $((f|_{A_i})^{-1})_*\mu$ is absolutely continuous on A_i with respect to μ. The respective Radon-Nykodim derivative, will be called the **Jacobian** of μ with respect to f:

$$J_f(\mu)(x) = \frac{d\mu \circ (f|_{A_i})}{d\mu}(x), \ \mu - \text{a.e on } A_i, i \geq 0$$

Notice that from the f-invariance of μ, we have $J_f(\mu)(x) \geq 1, \mu - \text{a.e } x \in M$. Consider now in general $f : M \to M$ a $\mathcal{C}^1$ endomorphism and μ an f-invariant probability on the manifold M; then the *folding entropy* $F_f(\mu)$ of μ is the conditional entropy: $F_f(\mu) := H_\mu(\epsilon|f^{-1}\epsilon)$, where ϵ is the partition into single points. From [19], we can use the measurable single point partition ϵ in order to desintegrate μ into a canonical family of conditional measures μ_x on the finite fiber $f^{-1}(x)$ for μ-a.e x. Hence the entropy of the conditional measure of μ restricted to $f^{-1}(x)$ is $H(\mu_x) = -\Sigma_{y \in f^{-1}(x)} \mu_x(y) \log \mu_x(y)$. From [14] we have also $J_f(\mu)(x) = \frac{1}{\mu_{f(x)}(x)}, \ \mu - \text{a.e } x$, hence

$$F_f(\mu) = \int \log J_f(\mu)(x) d\mu(x) \tag{3}$$

Definition 2. Given two positive functions $Q_1(n, x), Q_2(n, x)$, we will say that they are **comparable** if there exists a positive constant C so that $\frac{1}{C} \leq \frac{Q_1(n,x)}{Q_2(n,x)} \leq C$ for all n, x.

Recall that, given a continuous function $f : X \to X$ on a compact metric space X, the *topological pressure* $P(\phi)$ of a continuous real-valued function $\phi \in \mathcal{C}(X)$, is defined by

$$P(\phi) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \inf\{\sum_{z \in F} e^{S_n\phi(z)}, \ F \subset X \text{ such that } \bigcup_{z \in F} B_n(z, \varepsilon) = X\},$$

with $B_n(z, \varepsilon) = \{y \in X, d(f^i z, f^i y) < \varepsilon, 0 \leq i \leq n-1\}$ and $S_n\phi(z) = \sum\limits_{0 \leq i \leq n-1} \phi(f^i z), \ z \in X, n \geq 1$.

If $f : X \to X$ is a homeomorphism on X having the specification property, then the equilibrium measure μ_ϕ of the Hölder potential $\phi \in \mathcal{C}(X)$, is defined as the unique measure which maximizes in the Variational Principle for topological pressure (see for eg [6], [26]), namely:

$$P(\phi) = \sup\{h_\mu + \int \phi d\mu, \ \mu \text{ probability measure on } \ X\}$$

It was shown (see for eg [3], [6]), that the probability measure μ_ϕ is ergodic and satisfies the estimates $A_\varepsilon e^{S_n\phi(x) - nP(\phi)} \leq \mu_\phi(B_n(x, \varepsilon)) \leq B_\varepsilon e^{S_n\phi(x) - nP(\phi)}$, where $B_n(x, \varepsilon) := \{y \in X, d(f^i y, f^i x) < \varepsilon, i = 0, \dots, n-1\}$, $P(\phi)$ denotes the topological pressure of ϕ with respect to f, and where the positive constants $A_\varepsilon, B_\varepsilon$ are independent of x, n.

The general homeomorphism framework above allows us to apply this result to equilibrium measures on the inverse limit $\hat{\Lambda}$. If $\pi : \hat{\Lambda} \to \Lambda, \pi(\hat{x}) := x, \hat{x} \in \hat{\Lambda}$ is the *canonical projection* and if ϕ is a Hölder potential on Λ, then μ_ϕ is the unique equilibrium measure for ϕ on Λ if and only if $\mu_\phi = \pi_* \mu_{\phi\circ\pi}$, where $\mu_{\phi\circ\pi}$ is the unique equilibrium measure of $\phi \circ \pi$ on the compact metric space $\hat{\Lambda}$; here the homeomorphism $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ is the shift map defined by $\hat{f}(x, x_{-1}, x_{-2}, \ldots) = (f(x), x, x_{-1}, \ldots)$. So for the non-invertible map f and the measure μ_ϕ on Λ, we obtain the same estimates as above:

$$A_\varepsilon e^{S_n\phi(x)-nP(\phi)} \le \mu_\phi(B_n(x,\varepsilon)) \le B_\varepsilon e^{S_n\phi(x)-nP(\phi)},$$

with positive constants $A_\varepsilon, B_\varepsilon$ independent of n, x, where the consecutive sum $S_n\phi$ is defined as $S_n\phi(x) := \phi(x) + \ldots + \phi(f^{n-1}(x))$, for $x \in \Lambda,\ n \in \mathbb{N}$. In particular, if $\phi \equiv 0$, we obtain the measure of maximal entropy μ_0.

Let us give now the proof of the formula for the asymptotic logarithmic degree with respect to μ_ϕ, on the set Λ; this degree takes into consideration those n-preimages which behave well with respect to μ_ϕ. We assume for the moment that Proposition 1 is known; its proof is independent of Theorem 1 and will be given later in the paper. First, for an f-invariant probability μ on Λ, $\tau > 0$ small, $n \in \mathbb{N}$ and $x \in \Lambda$ let us define the finite set:

$$G_n(x,\mu,\tau) := \{y \in f^{-n}(f^n x) \cap \Lambda, \text{ s.t } |\frac{S_n\phi(y)}{n} - \int \phi d\mu| < \tau\}, \tag{4}$$

Definition 3. In the above setting, denote by $d_n(x,\mu,\tau) := \text{Card } G_n(x,\mu,\tau), x \in \Lambda, n > 0, \tau > 0$. The function $d_n(\cdot,\mu,\tau)$ is measurable, nonnegative and finite on Λ.

Proof of Theorem 1. First let us recall formula (3) for an arbitrary f-invariant measure μ, $F_f(\mu) = \int_\Lambda \log J_f(\mu)(x) d\mu(x)$. From the Chain Rule for Jacobians, we have $J_{f^n}(\mu)(x) = J_f(\mu)(x) \ldots J_f(\mu)(f^{n-1}(x))$, for μ-a.e $x \in \Lambda$, for any $n \ge 1$. On the other hand, since μ is f-invariant, we know that $\int \log J_f(\mu)(x) d\mu(x) = \int \log J_f(\mu)(f(x)) d\mu(x) = \int \log J_f(\mu)(f^k x) d\mu(x)$, for all $k \ge 1$. These facts imply that for any $n \ge 1$,

$$F_f(\mu) = \frac{1}{n} \int \log J_{f^n}(\mu)(x) d\mu(x) \tag{5}$$

As we saw above, since f is hyperbolic on Λ, then any Hölder continuous potential ϕ on Λ has a unique equilibrium measure μ_ϕ on Λ. Therefore from Proposition 1, since the constant C is independent of n we obtain that:

$$F_f(\mu_\phi) = \lim_{n\to\infty} \frac{1}{n} \int_\Lambda \log \frac{\sum\limits_{y \in f^{-n}(f^n(x))\cap\Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) \tag{6}$$

Now since Λ is compact, each point $x \in \Lambda$ has only finitely many f-preimages in Λ, i.e there exists a positive integer d s.t $\text{Card}(f^{-1}x) \le d, x \in \Lambda$. Since μ_ϕ is an ergodic measure (as it is an equilibrium state) and from Birkhoff Ergodic Theorem we obtain that

$$\mu_\phi\left(x \in \Lambda, |\frac{S_n\phi(x)}{n} - \int \phi d\mu| > \tau/2\right) \underset{n\to\infty}{\to} 0,$$

for any small $\tau > 0$. Thus for any $\eta > 0$ there exists a large integer $n(\eta)$ such that for $n \geq n(\eta)$,

$$\mu_\phi(x \in \Lambda, |\frac{S_n\phi(x)}{n} - \int \phi d\mu| > \tau/2) < \eta \tag{7}$$

Let us now take a point $x \in \Lambda$ with $|\frac{S_n\phi(x)}{n} - \int \phi d\mu| < \tau$. From Definition 3 we have

$$\frac{e^{n(\int \phi d\mu_\phi - \tau)} d_n(x, \mu_\phi, \tau) + r_n(x, \mu_\phi, \tau)}{e^{n(\int \phi d\mu + \tau)}} \leq \frac{\sum\limits_{y \in f^{-n}(f^n x) \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} \leq \frac{e^{n(\int \phi d\mu_\phi + \tau)} d_n(x, \mu_\phi, \tau) + r_n(x, \mu_\phi, \tau)}{e^{n(\int \phi d\mu_\phi - \tau)}}, \tag{8}$$

where $r_n(x, \mu_\phi, \tau)$ is the remainder $\sum\limits_{y \in f^{-n} f^n(x) \setminus G_n(x, \mu_\phi, \tau)} e^{S_n\phi(y)}$. In order to simplify notation, we will also denote $r_n(x, \mu_\phi, \tau)$ by r_n when no confusion can arise.

Given n large, let us consider now a partition $(A_i^n)_{1 \leq i \leq K}$ of Λ (modulo μ_ϕ) so that for each $0 \leq i \leq K$, there exists a point $z_i \in A_i^n$ so that for any n-preimage $\xi_{ij} \in f^{-n}(z_i) \cap \Lambda, 1 \leq j \leq d_{n,i}$, we have $A_i^n \subset f^n(B_n(\xi_{ij}, \varepsilon)), 1 \leq j \leq d_{n,i}, 1 \leq i \leq K$. For the above partition, let us denote by A_{ij}^n the part of the n-preimage of A_i^n which belongs to the Bowen ball $B_n(\xi_{ij}, \varepsilon)$, i.e

$$A_{ij}^n := f^{-n}(A_i^n) \cap B_n(\xi_{ij}, \varepsilon), 1 \leq j \leq d_{n,i}, 1 \leq i \leq K$$

Since A_i^n were chosen disjoint, also the pieces of their preimages, A_{ij}^n, i, j, are mutually disjoint.

We will decompose the integral in (6) over the sets A_{ij}^n. Notice that if $y, z \in A_{ij}^n$, then since ϕ is Hölder continuous and $A_{ij}^n \subset B_n(\xi_{ij}, \varepsilon)$, it follows that we have

$$|S_n\phi(y) - S_n\phi(z)| \leq C(\varepsilon), \tag{9}$$

where $C(\varepsilon)$ is a positive function with $C(\varepsilon) \underset{\varepsilon \to 0}{\to} 0$. So we will obtain:

$$\int_\Lambda \log \frac{\sum\limits_{y \in f^{-n} f^n x \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) = \sum_{0 \leq j \leq d_i, 0 \leq i \leq K} \int_{A_{ij}^n} \log \frac{\sum\limits_{y \in f^{-n} f^n x \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) \tag{10}$$

Let us now denote by $R_n(i, \mu_\phi, \tau)$ the set of preimages ξ_{ij} with $\xi_{ij} \notin G_n(\xi_{ik_0}, \mu_\phi, \tau)$, for some n-preimage ξ_{ik_0}; in fact, as can be seen from (4), it does not matter which preimage ξ_{ik_0} we choose in the set $f^{-n}(f^n\xi_{ik_0})$. Then, denote simply by $R_{n,i}$ the set of indices $j, 1 \leq j \leq d_{n,i}$ with $\xi_{ij} \in R_n(i, \mu_\phi, \tau)$ for every $1 \leq i \leq K$. Now in the decomposition from (10) we notice that the integral over those sets A_{ij}^n with $j \in R_{n,i}$ will not matter significantly. Indeed as $\text{Card}(f^{-1}x \cap \Lambda) \leq d, x \in \Lambda$ and as $-M \leq \phi(x) \leq M, x \in \Lambda$ we have,

$$1 \leq \frac{\sum\limits_{y \in f^{-n} f^n x \cap \Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} \leq d^n e^{2nM}$$

Now recall that $A_{ij}^n \subset B_n(\xi_{ij}, \varepsilon)$ and the sets A_{ij}^n, i, j are mutually disjoint (w. r. t μ_ϕ). Hence by using inequalities (7) and (9) and the fact that $\xi_{ij} \notin G_n(\xi_{ik_0}, \mu_\phi, \tau)$ when $j \in R_{n,i}$, we obtain:

$$\sum_{0\le i\le K, j\in R_{n,i}} \frac{1}{n}\int_{A^n_{ij}} \log \frac{\sum\limits_{y\in f^{-n}f^n x\cap\Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) \le \frac{1}{n}\log(d^n e^{2nM})\cdot\eta = \eta(\log d + 2M) \tag{11}$$

But by using the comparison between different parts of the n-preimage of a small set from the proof of Proposition 1 (see (18)), we deduce that the last term of formula (10) is comparable to

$$\sum_{i,j} \mu_\phi(A^n_{ij}) \log \frac{d_n(z_i,\mu_\phi,\tau)\mu_\phi(A^n_{ij}) + \tilde r_n(z_i,\mu_\phi,\tau)}{\mu_\phi(A^n_{ij})}, \tag{12}$$

where $\tilde r_n(z_i,\mu,\tau) := \sum\limits_{\xi_{ij}\in f^{-n}(z_i)\cap\Lambda,\ \xi_{ij}\notin G_n(\xi_{ik_0},\mu_\phi,\tau)} \mu_\phi(A^n_{ij})$. Hence from (18), (11) and (12) we obtain:

$$\begin{aligned}
&\frac{1}{n}\sum_{i,j\notin R_{n,i}} \mu_\phi(A^n_{ij})\log d_n(z_i,\mu_\phi,\tau) + \frac{1}{n}\sum_{i,j\notin R_{n,i}} \mu_\phi(A^n_{ij})\log(1+\frac{\tilde r_n(z_i,\mu_\phi,\tau)}{d_n(z_i,\mu_\phi,\tau)\mu_\phi(A^n_{ij})}) - \delta(\tau) - \eta C' \le \\
&\le \int_\Lambda \frac{1}{n}\log\frac{\sum\limits_{y\in f^{-n}f^n x\cap\Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) \le \\
&\le \frac{1}{n}\sum_{i,j\notin R_{n,i}} \mu_\phi(A^n_{ij})\log d_n(z_i,\mu_\phi,\tau) + \frac{1}{n}\sum_{i,j\notin R_{n,i}} \mu_\phi(A^n_{ij})\log(1+\frac{\tilde r_n(z_i,\mu_\phi,\tau)}{d_n(z_i,\mu_\phi,\tau)\mu_\phi(A^n_{ij})}) + \delta(\tau) + \eta C',
\end{aligned} \tag{13}$$

with $C' = \log d + 2M$ being the constant found in (11), and where the positive constant $\delta(\tau)$ comes from the uniformly bounded variation of $\frac{1}{n}S_n\phi(x)$ when x is in A^n_{ij} and when $1\le i\le K, j\notin R_{n,i}$ vary; clearly we have $\delta(\tau) \underset{\tau\to 0}{\to} 0$.

Now we know that in general $\log(1+x)\le x$, for $x>0$. Thus $\log(1+\frac{\tilde r_n(z_i,\mu_\phi,\tau)}{d_n(z_i,\mu_\phi,\tau)\mu_\phi(A^n_{ij})}) \le \frac{\tilde r_n(z_i,\mu_\phi,\tau)}{d_n(z_i,\mu_\phi,\tau)\mu_\phi(A^n_{ij})}, i, j$ and hence in (13) we have, for n large enough that:

$$\begin{aligned}
&\sum_{i,j\notin R_{n,i}} \mu_\phi(A^n_{ij})\log(1+\frac{\tilde r_n(z_i,\mu_\phi,\tau)}{d_n(z_i,\mu_\phi,\tau)\mu_\phi(A^n_{ij})}) \le \sum_{i,j\notin R_{n,i}} \mu_\phi(A^n_{ij})\frac{\tilde r_n(z_i,\mu_\phi,\tau)}{d_n(z_i,\mu_\phi,\tau)\mu_\phi(A^n_{ij})} = \\
&= \sum_{1\le i\le K} \tilde r_n(z_i,\mu_\phi,\tau) \le \eta,
\end{aligned} \tag{14}$$

where we used that by definition, there are $d_n(z_i,\mu_\phi,\tau)$ indices j in $\{1,\dots,d_{n,i}\}\setminus R_{n,i}$ for any $1\le i\le K$. Therefore from the last displayed inequality and (13) we obtain that, for $n\ge n(\eta)$,

$$\left| \frac{1}{n}\int_\Lambda \log\frac{\sum\limits_{y\in f^{-n}f^n x\cap\Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) - \frac{1}{n}\int_\Lambda \log d_n(z,\mu_\phi,\tau)d\mu_\phi(z)\right| \le \delta(\tau)+\eta, \tag{15}$$

where $\delta(\tau)\underset{\tau\to 0}{\to} 0$. Then by taking $n\to\infty$ and $\tau\to 0$, we will obtain the conclusion of the Theorem from (6) and (15), namely that $F_f(\mu_\phi) = \lim\limits_{\tau\to 0}\lim\limits_{n\to\infty}\frac{1}{n}\int_\Lambda \log d_n(x,\mu_\phi,\tau)d\mu_\phi(x)$.

□

We give now the proof of the auxilliary Proposition 1, which is independent of Theorem 1.

Proof of Proposition 1.

We know from definition that the Jacobian $J_{f^m}(\mu_\phi)$ is the Radon-Nikodym derivative of $\mu_\phi \circ f^m$ with respect to μ_ϕ on sets of injectivity for f^m. In order to estimate the Jacobian of μ_ϕ with respect to f^m, we have to compare the measure μ_ϕ on different components of the preimage set $f^{-m}(B)$, for a small Borel set B, where $m \geq 1$ is fixed.

Let us consider two subsets E_1, E_2 of Λ so that $f^m(E_1) = f^m(E_2) \subset B$ and E_1, E_2 belong to two disjoint balls $B_m(y_1, \varepsilon)$, respectively $B_m(y_2, \varepsilon)$. This happens if $\mathrm{diam}(B)$ is small enough, since f has no critical points in Λ and thus there exists a positive distance ε_0 between any two different preimages from $f^{-1}(y)$ for $y \in \Lambda$. As in [6] since the Borelian sets with boundaries of measure zero form a sufficient collection, we can assume that each of the sets E_1, E_2 have boundaries of μ_ϕ-measure zero. Recall that $f^m(E_1) = f^m(E_2)$. As in [6], μ_ϕ is the weak limit of the sequence of measures: $\tilde{\mu}_n := \frac{1}{P(f,\phi,n)} \cdot \sum\limits_{x \in \mathrm{Fix}(f^n) \cap \Lambda} e^{S_n \phi(x)} \delta_x$, where $P(f, \phi, n) := \sum\limits_{x \in \mathrm{Fix}(f^n) \cap \Lambda} e^{S_n\phi(x)}, n \geq 1$ (see for eg [18] for weak convergence of measures). Thus we have

$$\tilde{\mu}_n(E_1) = \frac{1}{P(f, \phi, n)} \cdot \sum_{x \in \mathrm{Fix}(f^n) \cap E_1} e^{S_n \phi(x)}, n \geq 1 \tag{16}$$

Consider a periodic point $x \in \mathrm{Fix}(f^n) \cap E_1$; it follows that $f^m(x) \in f^m(E_1)$, so there exists a point $y \in E_2$ such that $f^m(y) = f^m(x)$. However the point y is not necessarily periodic. We will use now the Specification Property ([6], [3]) on hyperbolic locally maximal sets. If $\varepsilon > 0$ is fixed, there exists a constant $M_\varepsilon > 0$ such that for all $n > M_\varepsilon$, there is a point $z \in \mathrm{Fix}(f^n) \cap \Lambda$ which ε-shadows the $(n - M_\varepsilon)$-orbit of y. In particular $z \in B_m(y_2, 2\varepsilon)$, since $E_2 \subset B_m(y_2, \varepsilon)$.

Let now $V \subset B_m(y_2, \varepsilon)$ be an arbitrary neighbourhood of the set E_2. Take two points $x, x' \in \mathrm{Fix}(f^n) \cap E_1$ and assume the same periodic point $z \in V \cap \mathrm{Fix}(f^n)$ corresponds to both of them through the previous shadowing procedure. Thus the $(n - M_\varepsilon - m)$-orbit of $f^m(z)$ ε-shadows the $(n - M_\varepsilon - m)$-orbit of $f^m(x)$ and also the $(n - M_\varepsilon - m)$-orbit of $f^m(x')$. So the $(n - M_\varepsilon - m)$-orbit of $f^m(x)$ 2ε-shadows the $(n - M_\varepsilon - m)$-orbit of $f^m(x')$. But we took $x, x' \in E_1 \subset B_m(y_1, \varepsilon)$, so $x' \in B_m(x, 2\varepsilon)$ and hence from above, $x' \in B_{n - M_\varepsilon}(x, 2\varepsilon)$.

We partition now the set $B_{n-M_\varepsilon}(x, 2\varepsilon)$ into smaller Bowen balls of type $B_n(\zeta, 2\varepsilon)$, and let us denote their number by N_ε. In each of these $(n, 2\varepsilon)$-Bowen balls we may have at most one fixed point for f^n. Then if $d(f^i\xi, f^i\zeta) < 2\varepsilon, i = 0, \ldots, n-1$ and if ε is small enough, we can apply the Inverse Function Theorem at each step, and thus there exists only one fixed point for f^n in $B_n(\zeta, 2\varepsilon)$.

So there exist at most N_ε periodic points in Λ from $\mathrm{Fix}(f^n) \cap E_1$ having the same point $z \in V \cap \mathrm{Fix}(f^n)$ associated to them by the above procedure. Notice also that if $x, x' \in \mathrm{Fix}(f^n) \cap E_1$ have the same point $z \in V$ attached to them, then $x' \in B_{n-M_\varepsilon}(x, 2\varepsilon)$ and then, from the Hölder continuity of ϕ it follows $|S_n\phi(x) - S_n\phi(x')| \leq \hat{C}_\varepsilon$, for some positive constant $\hat{C}_\varepsilon$ depending on ϕ (but independent of n, m, x). This can be used then in the estimate for $\tilde{\mu}_n(E_1)$ from (16). Notice also that if $z \in B_{n-M_\varepsilon}(y, \varepsilon)$, then $f^m(z) \in B_{n-M_\varepsilon-m}(f^m(x), \varepsilon)$. Thus from the Hölder continuity of ϕ and since $x \in E_1 \subset B_m(y_1, \varepsilon)$, it follows that there exists a positive constant $\hat{C}'_\varepsilon$ satisfying: $|S_n\phi(z) - S_n\phi(x)| \le |S_m\phi(y_1) - S_m\phi(y_2)| + \hat{C}'_\varepsilon$, for $n > n(\varepsilon, m)$. Then using also (16), and since there are at most N_ε points $x \in \mathrm{Fix}(f^n) \cap E_1$ having the same $z \in V \cap \mathrm{Fix}(f^n) \cap \Lambda$ corresponding to them, we obtain that there exists a constant $C_\varepsilon > 0$ s.t:

$$\tilde{\mu}_n(E_1) \le C_\varepsilon \tilde{\mu}_n(V) \cdot \frac{e^{S_m\phi(y_1)}}{e^{S_m\phi(y_2)}}, \tag{17}$$

where we recall that $E_1 \subset B_m(y_1, \varepsilon), E_2 \subset B_m(y_2, \varepsilon)$ and $f^m(E_1) = f^m(E_2)$. But $\partial E_1, \partial E_2$ were assumed of μ_ϕ-measure zero, hence: $\mu_\phi(E_1) \le C_\varepsilon \mu_\phi(V) \cdot \frac{e^{S_m\phi(y_1)}}{e^{S_m\phi(y_2)}}$. But V was chosen arbitrarily as a neighbourhood of E_2, and by applying the same procedure for E_1 we obtain:

$$\frac{1}{C}\mu_\phi(E_2)\frac{e^{S_m\phi(y_1)}}{e^{S_m\phi(y_2)}} \le \mu_\phi(E_1) \le C\mu_\phi(E_2)\frac{e^{S_m\phi(y_1)}}{e^{S_m\phi(y_2)}}, \tag{18}$$

where $C > 0$ does not depend on m, E_1, E_2.

Now, the Jacobian $J_{f^m}(\mu_\phi)$ is the Radon-Nikodym derivative of $\mu_\phi \circ f^m$ with respect to μ_ϕ on sets of injectivity for f^m, hence

$$\mu_\phi(f^m(D)) = \int_D J_{f^m}(\mu_\phi)(x) d\mu_\phi(x),$$

for any Borelian set D on which f^m is injective. Hence from the Lebesgue Density Theorem, we have that, by putting $D = B(x, r)$ for small $r > 0$, we obtain:

$$J_{f^m}(\mu_\phi)(x) = \lim_{r \to 0} \frac{\mu_\phi(f^m(B(x,r)))}{\mu_\phi(B(x,r))}, \tag{19}$$

for μ_ϕ-a.e $x \in \Lambda$. On the other hand from the invariance of μ_ϕ, we have for any Borel set D that:

$$\mu_\phi(f^m(D)) = \mu_\phi(f^{-m}(f^m D)) \tag{20}$$

Thus if D is a small ball around x, one has to consider the m-preimages y of x, belonging to Λ. If $\zeta \in B_m(y, \varepsilon)$ then, from the Hölder continuity of ϕ we have that $|S_m\phi(\zeta) - S_m\phi(y)| \le \hat{C}_\varepsilon$, where the constant $\hat{C}_\varepsilon$ does not depend on $m > 0, y \in \Lambda$. So in the comparison inequlities of (18), we can take instead of y_1, y_2, the respective m-preimages of x belonging to Λ.

Therefore from (19), the invariance in (20), and the comparison between different pieces of the m-preimage from (18), it follows that the Jacobian of μ_ϕ with respect to f^m satisfies:

$$J_{f^m}(\mu_\phi)(x) \approx \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}}, \quad \mu_\phi - \text{a.e } x \in \Lambda,$$

where the comparability constant $C > 0$ is independent of $m > 1, x \in \Lambda$.

$\square$

Let us recall now that in the expanding case we have a formula relating $P(\phi)$ to the preimage sets of $f^n, n \ge 1$ (given in Section 1, see [17]); however the proof for that result does not work in the saddle case.

We give then, in our saddle case, the proof of the formula for $P(\phi)$ in terms of the folding entropy and the preimage sets, announced in Theorem 2 in Section 1:

Proof of Theorem 2.

First recall that ϕ is a Hölder continuous function on the hyperbolic basic set Λ, so its unique equilibrium measure μ_ϕ is ergodic. From the properties of the Jacobian, we know that it satisfies the Chain Rule, i.e $J_{f\circ g}(\mu_\phi)(x) = J_f(\mu_\phi)(g(x)) \cdot J_g(\mu_\phi)(x)$ for μ_ϕ-a.e $x \in \Lambda$. Hence μ_ϕ-a.e we have, $\log J_{f^m}(\mu_\phi)(x) = \log J_f(\mu_\phi)(x) + \ldots + \log J_f(\mu_\phi)(f^{m-1}(x))$. This means that we can apply Birkhoff Ergodic Theorem and obtain that

$$\frac{\log J_{f^m}(\mu_\phi)}{m} \underset{m\to\infty}{\to} \int_\Lambda \log J_f(\mu_\phi) d\mu_\phi = F_f(\mu_\phi)$$

We apply now Proposition 1 to get μ_ϕ-a.e the convergence

$$\frac{\log \sum\limits_{y\in f^{-m}(f^m x)\cap\Lambda} e^{S_m\phi(y)} - \log e^{S_m\phi(x)}}{m} \underset{m\to\infty}{\to} F_f(\mu_\phi) \tag{21}$$

But again from Birkhoff Ergodic Theorem, $\frac{S_m\phi(x)}{m} \to \int \phi d\mu_\phi$ for μ_ϕ-a.e $x \in \Lambda$. Thus from (21) and the definition of equilibrium measure $P(\phi) = \int \phi \, d\mu_\phi + h_{\mu_\phi}$, we obtain the required formula:

$$\frac{\log \sum\limits_{y\in f^{-m}(f^m x)\cap\Lambda} e^{S_m\phi(y)}}{m} \underset{m\to\infty}{\to} F_f(\mu_\phi) + P(\phi) - h_{\mu_\phi}$$

□

Remark 1. In general $F_f(\mu_\phi) \neq h_{\mu_\phi}$. Indeed consider the inverse SRB measure μ^- introduced on a d-to-1 hyperbolic repeller Λ in [9]. Then this measure μ^- satisfies

$$h_{\mu^-}(f) = \log d - \int_\Lambda \sum_{i,\lambda_i(\mu^-,x)<0} \lambda_i(\mu^-, x) m_i(\mu^-, x) d\mu^-(x),$$

where $\lambda_i(\mu^-, x)$ are the Lyapunov exponents of μ^- at x and $m_i(\mu^-, x)$ their respective multiplicities. So if there are negative Lyapunov exponents on Λ, as for the hyperbolic repellers introduced in [9] and explained below in Example 3), then $F_f(\mu^-) = \log d$, whereas $h_{\mu^-} > \log d$.

□

In the case of the measure of maximal entropy μ_0 of $f|_\Lambda$, we have that $\phi \equiv 0$, so $\frac{\log \sum_{y\in f^{-m}(f^m x)\cap\Lambda} e^{S_m\phi(y)}}{m} = \frac{\log d_m(x)}{m}$ and $P(0) = h_{\mu_0}$. We obtain then, from Theorem 1 and the proof of Theorem 2, also the proof for **Corollary 1**, thus giving the asymptotic degree of $f|_\Lambda$ in terms of the folding entropy of the measure of maximal entropy μ_0 on Λ.

Corollary 1 expresses the average value of the **logarithmic growth of the preimage counting function of** $f^n|_\Lambda$; as Λ is not necessarily totally invariant, $d_n(\cdot)$ may be non-constant and it is difficult to obtain the number of preimages of x in Λ; see the examples from [8] which are not constant-to-one on their respective basic sets, and the effect of preimages on the hyperbolic dynamics and on stable dimension (for eg [11], [12]).

A useful consequence of Proposition 1 is Corollary 2, which gives the measure μ_ϕ of **an arbitrary** ball in Λ. It can be proved by writing an arbitrary ball $B(y, \rho)$ as a certain iterate $f^m(B_n(z, \varepsilon))$ of some Bowen ball, in such a way that the iterate $f^m(B_n(z, \varepsilon)$ has roughly the same sides in the stable and unstable directions; here $y = f^m(z)$ and $\rho > 0$ is arbitrary.

Asymptotic degrees are useful also in order to obtain **estimates** for the Hausdorff dimension of various **slices** through the fractal Λ. In [12] we obtained that, if the number of f-preimages of any point $x \in \Lambda$ is at least d, then the *stable dimension* $\delta^s(x) := HD(W^s_r(x) \cap \Lambda)$ is less or equal than the unique zero t^s_d of the pressure $t \to P(t\Phi^s - \log d)$. However in general, the number of preimages of points varies discontinuously in Λ. Since the Hausdorff dimension of stable slices is not changed by taking iterates f^m, we will use the asymptotic degrees to obtain estimates for the dimension of certain stable slices.

Proof of Theorem 3.

We want to prove an upper estimate for the dimension of a certain slice through Λ, by using the asymptotic degree with respect to the equilibrium measure μ_ϕ of a Hölder continuous function ϕ on Λ. Let us denote by $G(\mu_\phi) := \{y \in \Lambda,\ \frac{1}{n} \log J_{f^n}(\mu_\phi)(y) \underset{n\to\infty}{\to} F(\mu_\phi)\}$. As we showed above, $\mu_\phi(G(\mu_\phi)) = 1$. Let us recall also from Proposition 1 that there exists a positive constant C independent of n, such that for μ_ϕ-almost all $y \in \Lambda$ we have:

$$J_{f^n}(\mu_\phi)(y) \geq C \cdot \frac{\sum\limits_{z \in f^{-n}(f^n y) \cap \Lambda} e^{S_n\phi(z)}}{e^{S_n\phi(y)}} \geq C' \cdot d_n(y, \mu_\phi, \tau), \tag{22}$$

with $C' \in (0, 1)$ also a constant independent of n. For $y \in G(\mu_\phi)$, we have $\lim\limits_{n\to\infty} \frac{1}{n} \log J_{f^n}(\mu_\phi)(y) = F(\mu_\phi)$. On the other hand, from Theorem 1 and from (22), it follows that:

$$\lim_{\tau\to 0} \lim_{n\to\infty} \int_\Lambda \frac{\log d_n(y, \mu_\phi, \tau)}{n} d\mu_\phi(y) = F(\mu_\phi), \text{ and also,}$$

$$\int_\Lambda \frac{\log d_n(y, \mu_\phi, \tau)}{n} d\mu_\phi(y) \leq \int_\Lambda \frac{\log \frac{J_{f^n}(\mu_\phi)(y)}{C'}}{n}\, d\mu_\phi(y)$$

From the fact that $\frac{\log J_{f^n}(\mu_\phi)(y)}{n} \underset{n\to\infty}{\to} F(\mu_\phi)$ for μ_ϕ-a.e $y \in \Lambda$, it follows that, for any positive integer N and $\delta > 0$, there exists a Borel set

$$\Theta_N(\mu_\phi, \delta) := \{y \in \Lambda,\ e^{n(F(\mu_\phi)-\delta)} \leq J_{f^n}(\mu_\phi)(y) \leq e^{n(F(\mu_\phi)+\delta)},\ n \geq N\},$$

and that $\mu_\phi(\Theta_N(\mu_\phi, \delta)) \geq 1 - \rho(\delta, N)$, where $\rho(\delta, N) \underset{N\to\infty}{\to} 0$ for any fixed $\delta > 0$. Moreover, notice that we have in general $d_n(y) \geq d_n(y, \mu_\phi, \tau),\ y \in \Lambda, \tau > 0$, and that $d_n(y, \mu_\phi, \tau) \geq d_n(y, \mu_\phi, \tau')$

if $\tau > \tau' > 0$. Now for fixed small $\delta > 0, \tau > 0$, and n integer, let $K_n(\mu_\phi, \delta)$ be the set of points $y \in \Theta_n(\mu_\phi, \delta) \subset \Lambda$ so that

$$d_n(y, \mu_\phi, \tau) \geq \frac{1}{2} \cdot e^{n(F(\mu_\phi)-\delta)}$$

Let us assume that $\mu_\phi(K_n(\mu_\phi, \delta))$ does not converge to 1 when $n \to \infty$. Then, there exists $\alpha > 0$ and a subsequence $(m_n)_n$ of integers, such that for every n, we have $\mu_\phi(K_{m_n}(\mu_\phi, \delta)) < 1 - \alpha$. So $\mu_\phi(\Lambda \setminus K_{m_n}(\mu_\phi, \delta)) > \alpha, \ n > 0$. But for every point $z \in \Lambda \setminus K_{m_n}(\mu_\phi, \delta)$, we have

$$d_{m_n}(z, \mu_\phi, \tau) < \frac{1}{2} e^{m_n(F(\mu_\phi)-\delta)}, \text{ hence } \frac{\log d_{m_n}(z, \mu_\phi, \tau)}{m_n} < F(\mu_\phi) - \delta - \frac{\log 2}{m_n}$$

Therefore, using also (22) we would obtain the following estimate:

$$\int_\Lambda \frac{\log d_{m_n}(z, \mu_\phi, \tau)}{m_n} d\mu_\phi(z) \leq \big(F(\mu_\phi) - \delta\big)\mu_\phi(\Lambda \setminus K_{m_n}(\mu_\phi, \delta)) - \frac{\log C' + \log 2}{m_n} + \int_{K_{m_n}(\mu_\phi,\delta)} \frac{\log J_{f^{m_n}}(\mu_\phi)}{m_n} d\mu_\phi \tag{23}$$

But we know that $\lim\limits_{n\to\infty} \frac{1}{n} \log J_{f^n}(\mu_\phi)(y) = F(\mu_\phi)$, for μ_ϕ-a.e $y \in \Lambda$. From Proposition 1 and the fact that the number of f-preimages of $x \in \Lambda$ is bounded, it also follows that the functions $\frac{\log J_{f^n}(\mu_\phi)}{n}$ are bounded above by some constant $C > 0$ independent of n. Thus from Lebesgue dominated convergence theorem, one obtains the convergence towards 0 of the sequence $(\rho_n)_n$, defined by:

$$\begin{aligned}\rho_n &:= \int_{K_{m_n}(\mu_\phi,\delta)} \big(\frac{\log J_{f^{m_n}}(\mu_\phi)(x)}{m_n} - F(\mu_\phi)\big) \ d\mu_\phi(x) = \\ &= \int_\Lambda \big(\frac{\log J_{f^{m_n}}(\mu_\phi)(x)}{m_n} - F(\mu_\phi)\big) \cdot \chi_{K_{m_n}(\mu(\phi),\delta)}(x) \ d\mu_\phi(x) \underset{n\to\infty}{\to} 0\end{aligned} \tag{24}$$

But then in (23), we can replace the term $\int_{K_{m_n}(\mu_\phi,\delta)} \frac{\log J_{f^{m_n}}(\mu_\phi)}{m_n} d\mu_\phi$ by $\int_{K_{m_n}(\mu_\phi,\delta)} \big(\frac{\log J_{f^{m_n}}(\mu_\phi)(x)}{m_n} - F(\mu_\phi)\big) \ d\mu_\phi(x) + F(\mu_\phi)\mu_\phi(K_{m_n}(\mu_\phi, \delta)) = \rho_n + F(\mu_\phi) \cdot \mu_\phi(K_{m_n}(\mu_\phi, \delta))$. Hence in the right-hand side of (23), we have the following sum:

$$\begin{aligned}&\big(F(\mu_\phi) - \delta\big)\mu_\phi(\Lambda \setminus K_{m_n}(\mu_\phi, \delta)) - \frac{\log C' + \log 2}{m_n} + \rho_n + F(\mu_\phi)\mu_\phi(K_{m_n}(\mu_\phi, \delta)) = \\ &= F(\mu_\phi) + \rho_n - \frac{\log C' + \log 2}{m_n} - \delta \cdot \mu_\phi(\Lambda \setminus K_{m_n}(\mu_\phi, \delta))\end{aligned}$$

In the last displayed equality, we know that $\rho_n \underset{n\to\infty}{\to} 0$, that $\frac{\log C' + \log 2}{m_n} \underset{n\to\infty}{\to} 0$ and that $\mu_\phi(\Lambda \setminus K_{m_n}(\mu_\phi, \delta)) > \alpha > 0$ for all n. Thus, from (23) we obtain

$$\lim_{n\to\infty} \int_\Lambda \frac{\log d_{m_n}(z, \mu_\phi, \tau)}{m_n} d\mu_\phi(z) \leq F(\mu_\phi) - \alpha\delta$$

Then, since $d_n(y, \mu_\phi, \tau') \leq d_n(y, \mu_\phi, \tau)$ for $0 < \tau' < \tau$, we obtain also $\lim\limits_{\tau'\to 0} \lim\limits_{n\to\infty} \int_\Lambda \frac{\log d_{m_n}(z, \mu_\phi, \tau')}{m_n} d\mu_\phi(z) \leq F(\mu_\phi) - \alpha\delta$; but this gives a contradiction with Theorem 1, since the last limit must be equal to $F(\mu_\phi)$. Hence there exists an integer N_0 such that for any $n > N_0$, the Borel set $K_n(\mu_\phi, \delta) \subset \Lambda$ has the property that if $y \in K_n(\mu_\phi, \delta)$, then:

$$d_n(y) \geq \ d_n(y, \mu_\phi, \tau) \geq \frac{1}{2} \cdot e^{n(F(\mu_\phi)-\delta)}, \tag{25}$$

and for every $n \geq N_0$ there exists some $\chi_n(\delta) > 0$, with $\chi_n(\delta) \underset{n\to\infty}{\searrow} 0$, such that:

$$\mu_\phi(K_n(\mu_\phi, \delta)) > 1 - \chi_n(\delta).$$

Next, let us notice that from the f-invariance of the probability μ_ϕ, we have that:

$$\mu_\phi(f^n K_n(\mu_\phi, \delta)) \geq \mu_\phi(K_n(\mu_\phi, \delta)) \geq 1 - \chi_n(\delta), \ \forall n \geq N_0 \tag{26}$$

Now, for any integer $n \geq N_0$, consider a strictly increasing sequence of integers $\big(p_i(n)\big)_{i\geq 1}$, such that $p_1(n) = n$, and such that the following condition is satisfied:

$$\sum_{i\geq 1} \chi_{p_i(n)}(\delta) < 3\chi_n(\delta) \tag{27}$$

For a sequence $(p_i(n))_i$ of integers as above, let us define the following Borel subsets of Λ:

$$\mathcal{K}_n(\mu_\phi, \delta) := \bigcap_{i\geq 1} f^{p_i(n)} K_{p_i(n)}(\mu_\phi, \delta), \ \text{and} \ \ \mathcal{K}(\mu_\phi, \delta) := \bigcup_{n\geq 1} \mathcal{K}_n(\mu_\phi, \delta)$$

We will cover now the set $\mathcal{K}_n(\mu_\phi, \delta)$ with open sets, in order to estimate its Hausdorff dimension. First of all, notice that if $z \in K_n(\mu_\phi, \delta)$, then any other point z' from $f^{-n}(f^n z)$ belongs also to $K_n(\mu_\phi, \delta)$. Notice also that if $\varepsilon < \varepsilon_0$, then the set $f^{-n}(y)$ is (n, ε)-separated for any point $y \in \Lambda$, due to $\mathcal{C}_f \cap \Lambda = \emptyset$. In addition, for any $y \in \mathcal{K}_n(\mu_\phi, \delta)$ and for any integer $i \geq 1$, we know that:

$$d^*_{p_i(n)}(y) \geq \frac{1}{C'} \cdot e^{p_i(n)\cdot(F(\mu_\phi)-\delta)}, \tag{28}$$

where in general $d^*_n(y) := Card\{f^{-n}(y) \cap \Lambda\}$ is the number of n-preimages in Λ of y, $n \geq 1$. Take next an arbitrary number $t > \tilde{t}^s_{F(\mu_\phi)-\delta}$, where $\tilde{t}^s_{F(\mu_\phi)-\delta}$ is the unique zero of the pressure function:

$$t \mapsto P(t\Phi^s - F(\mu_\phi) + \delta)$$

It is clear that such a zero exists and is unique, since $F(\mu_\phi) \leq h_{top}(f|_\Lambda)$; from notations, we have $\tilde{t}^s_\gamma = t^s_{e^\gamma}$, $\forall \gamma$, (where we recall that, for $\chi > 0$, t^s_χ denotes the unique zero of the function $t \to P(t\Phi^s - \log \chi)$). Therefore, $P(t\Phi^s - F(\mu_\phi) + \delta) < 0$.

Since for all $i \geq 1$, $f^{p_i(n)}(K_{p_i(n)}(\mu_\phi, \delta))$ covers $\mathcal{K}_n(\mu_\phi, \delta)$, it follows that we can find a cover $\mathcal{V}^{(n,i)}$ of $W \cap \mathcal{K}_n(\mu_\phi, \delta)$, with sets of type $f^{p_i(n)}(B_{p_i(n)}(\xi, \varepsilon))$, where ξ ranges in a $(p_i(n), \varepsilon)$-spanning set $F_{p_i(n)}$ of Λ, and $n, i \geq 1$. Clearly from the conformality of f on local stable manifolds, it follows that the diameter of any ball in W of type $W \cap f^{p_i(n)}(B_{p_i(n)}(\xi, \varepsilon))$, converges to 0 when $i \to \infty$ (and n is fixed).

Now, we will use (28) and the procedure of succesive elimination of covers from the proof of Theorem 1.2 of [12], plus the fact that $t > \tilde{t}^s_{F(\mu_\phi)-\delta}$. Hence it follows that for any $n, i \geq 1$, we can extract from the above very rich cover $\mathcal{V}^{(n,i)}$, some finite "optimal" open subcover $\mathcal{U}^{(n,i)} = \big(U^{(n,i)}_j\big)_{j\in I(n,i)}$ of the set $W \cap \mathcal{K}_n(\mu_\phi, \delta)$, such that we have:

$$\sum_{j\in I(n,i)} \text{diam}(U^{(n,i)}_j)^t < \frac{1}{2}$$

Hence, as the diameters of the sets in $\mathcal{U}^{(n,i)}$ decrease to 0 when $i \to \infty$, we conclude that $HD(W \cap \mathcal{K}_n(\mu_\phi, \delta)) \le t$, and since t was chosen arbitrary larger than $\tilde{t}^s_{F(\mu_\phi)-\delta}$, we obtain $HD(W \cap \mathcal{K}_n(\mu_\phi, \delta)) \le \tilde{t}^s_{F(\mu_\phi)-\delta}$. Therefore using the fact that $\mathcal{K}(\mu_\phi, \delta) = \underset{n}{\cup} \mathcal{K}_n(\mu_\phi, \delta)$, and also the properties of the Hausdorff dimension of a countable union, it follows that

$$HD(W \cap \mathcal{K}(\mu_\phi, \delta)) \le \tilde{t}^s_{F(\mu_\phi)-\delta} \tag{29}$$

Moreover, from (26) and (27), we obtain that

$$\mu_\phi(\mathcal{K}_n(\mu_\phi, \delta)) \ge 1 - \sum_{i \ge 1} (1 - \mu_\phi(K_{p_i(n)}(\mu_\phi, \delta))) \ge 1 - \sum_{i \ge 1} \chi_{p_i(n)}(\delta) \ge 1 - 3\chi_n(\delta)$$

Hence as $\chi_n(\delta) \underset{n\to\infty}{\to} 0$, it implies that $\mu_\phi(\mathcal{K}(\mu_\phi, \delta)) = \mu_\phi(\underset{n}{\cup} \mathcal{K}_n(\mu_\phi, \delta)) = 1$. Let us define now the following Borel subset of Λ,

$$\mathcal{K}(\mu_\phi) := \underset{\delta > 0}{\cap} \mathcal{K}(\mu_\phi, \delta)$$

Then, we obtain from above that $\mu_\phi(\mathcal{K}(\mu_\phi)) = 1$. Now we let $\delta \to 0$, and employ (29), the continuity of the pressure function, and the definition of $d_\infty(f, \mu_\phi)$, in order to obtain the required bound:

$$HD(W \cap \mathcal{K}(\mu_\phi)) \le \tilde{t}^s_{F(\mu_\phi)} = t^s_{d_\infty(f,\mu_\phi)}$$

□

3 Some examples of folded saddle-type systems.

1) Let us consider an iterated function system in the unit interval I, $g : I_1 \cup \ldots \cup I_p \to I$ for some $p \ge 2$, such that g is $\mathcal{C}^2$-smooth, and injective and expanding each I_j to I, i.e $g(I_j) = I = [0,1]$, $1 \le j \le p$. We define the compact space

$$X = \{x \in I_1 \cup \ldots \cup I_p,\ g^n(x) \in I_1 \cup \ldots \cup I_p, n \ge 0\}$$

Consider now parameters $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p$, with $||\lambda|| < \eta$ for some small $\eta > 0$, i.e $\lambda \in B_p(0, \eta) \subset \mathbb{R}^p$. Consider also the Lipschitz continuous functions $\phi_1, \ldots, \phi_p$ defined on $X^* := X \times [0,1] \times B_p(0,\eta)$, and assume that $\phi_1(x, \cdot, \cdot), \ldots, \phi_p(x, \cdot, \cdot)$ are $\mathcal{C}^2$ differentiable functions of (y, λ), with derivatives in (y, λ) depending Lipschitz continuously on (x, y, λ), and that there exist constants $\alpha, \alpha' > 0$ with $0 < \alpha' < |\frac{\partial}{\partial y}\phi_i| < \frac{1}{4}$ on X^*, for all $i = 1, \ldots, p$ and $|\frac{\partial}{\partial \lambda_j}\phi_i| < \alpha$ on X^*, for all $i, j = 1, \ldots, p$. If $\phi_i \le \beta$ on X^*, for $i = 1, \ldots, p$, then we assume also that $\eta + \beta < 1$.

We define now the parametrized maps $F_\lambda : X \times [0,1] \to X \times (0,1)$ by the formula

$$F_\lambda(x, y) = (g(x), \lambda_i + \phi_i(x, y, \lambda)),$$

if $x \in X_i, i = 1, \ldots, p$. From the conditions on $\phi_1, \ldots, \phi_p$, we see that F_λ is well defined and it is a hyperbolic fiberwise conformal skew product endomorphism. We see that $0 < \alpha' < |(\phi_x^\lambda)'| < \frac{1}{4}, x \in X, \lambda \in B_d(0, \eta)$, so the conditions from the definition of a uniformly transversal family in [13] are satisfied for the parametrized family $(F_\lambda)_\lambda$, as shown in Theorem 3.3 of [13]. For $x \in X \cap I_i$, let us denote by:

$$\Psi_i(x, y, \lambda) := \lambda_i + \Phi_i(x, y, \lambda), 1 \le i \le p$$

If $z \in X$, then define the contraction $\Psi^n_{z,\lambda} := \Psi_{x,\lambda} \circ \Psi_{g^{n-1}(z),\lambda} \ldots \circ \Psi_{z,\lambda}$, where $\Psi_{z,\lambda}(\cdot) = \Psi_i(z, \cdot, \lambda)$, for $z \in X \cap I_i$, $1 \le i \le p$ and $\lambda \in B_p(0, \eta)$. We define then the following fibered invariant fractal,

$$\Lambda_\lambda := \bigcup_{x\in X} \bigcap_{n\ge 1} \bigcup_{z\in g^{-n}(x)} \Psi^n_{z,\lambda}(I)$$

From [13], it follows that the stable dimension over the saddle set Λ_λ of F_λ, is given by a Bowen type equation on the natural extension $\hat{\Lambda}_\lambda$, for Lebesgue-almost all parameters λ. However, F_λ is not a homeomorphism on Λ_λ in general.

We can also estimate the asymptotic degree of the measure of maximal entropy $\mu_{0,\lambda}$ for F_λ on Λ_λ. The topological entropy $h_{top}(F_\lambda|_{\Lambda_\lambda}) = \log p$, since Bowen balls in Λ_λ are determined only by the dilation of g in the base. Also from our assumptions above, we see that the negative Lyapunov exponent $\tilde{\lambda}_1(\mu_{0,\lambda})$ is larger than $\log \alpha'$. We now use an inequality due to Ruelle (see [21] and [7]):

$$h_{\mu_{0,\lambda}} \le F(\mu_{0,\lambda}) - \sum_{\tilde{\lambda}_i<0} m_i \tilde{\lambda}_i(\mu_{0,\lambda}),$$

where m_i is the multiplicity of the Lyapunov exponent $\tilde{\lambda}_i(\mu_{0,\lambda})$. Consequently, since in our case we have only one negative Lyapunov exponent $\tilde{\lambda}_1(\mu_{0,\lambda})$, and since $h_{top}(F_\lambda|_{\Lambda_\lambda}) = \log p$, we obtain thus an estimate on the average rate of growth of the number of n-preimages remaining in the basic set Λ_λ. Namely, from Definition 1 and Theorem 1,

$$d_\infty(F_\lambda, \Lambda_\lambda) = \exp(F(\mu_{0,\lambda})) \ge p \cdot \alpha'$$

Hence $d_\infty(F_\lambda, \Lambda_\lambda) > 1$, if $p\alpha' > 1$. However, F_λ may not be constant-to-1 on the fractal Λ_λ. Using Proposition 1 and Theorem 2, we can infer also the Jacobian of μ_ϕ with respect to an arbitrary iterate of F_λ, and the pressure $P(\phi)$ for any Hölder continuous potential ϕ on Λ_λ.

2) Examples of *hyperbolic attractors* for endomorphisms can be obtained from *solenoids with self-intersections*, by the method of Bothe ([1]). We consider $f : \mathbb{D}^2 \times S^1 \to \mathbb{D}^2 \times S^1$ given by:

$$f(x, y, t) := (\lambda_1(t) \cdot x + z_1(t), \lambda_2(t) \cdot y + z_2(t), \psi(t)),$$

where $\psi(\cdot)$, $0 < \lambda_i(t) < 1$ and $z_i(\cdot), i = 1, 2$ are $\mathcal{C}^1$ functions, and where $\psi'(t) > 1$. We obtain then the hyperbolic saddle-type fractal attractor,

$$\Lambda = \bigcap_{j\ge 0} f^j(\mathbb{D}^2 \times S^1)$$

For certain choices of ϕ, λ_i, z_i, the map f is non-invertible and not constant-to-1 on Λ. Then, for an arbitrary Hölder potential ϕ on Λ, one obtains the measure μ_ϕ associated to ϕ. The negative Lyapunov exponents are given by the average values with respect to μ_ϕ, of $\log \lambda_i, i = 1, 2$ and the positive exponent is given by the average value of $\log |\psi'|$.

We can then estimate as before the asymptotic degree of $f|_\Lambda$ as $d_\infty(f, \Lambda)$, using Corollary 1; and more generally, we obtain the asymptotic degree $d_\infty(f, \mu_\phi)$, with respect to the equilibrium measure μ_ϕ of ϕ. From Proposition 1, we obtain also the Jacobian $J_{f^n}(\mu_\phi)$ of μ_ϕ, with respect to an arbitrary iterate $f^n, n \geq 1$. In the case when $\lambda_1 = \lambda_2$, we can also estimate the μ_ϕ-measure of an arbitrary ball in Λ, by applying Corollary 2.

3) Another class of examples are given by hyperbolic toral (linear) endomorphisms $f_A : \mathbb{T}^m \to \mathbb{T}^m$ and their *perturbations*; they are Anosov endomorphisms. Notice that a small $\mathcal{C}^2$ perturbation g of f_A, is not necessarily conjugated to f_A if f_A is not invertible ([16]). Also notice that f_A is $|\det(A)|$-to-1 on $\mathbb{T}^m$, and that the same is true also for g.

However, given an equilibrium measure μ_ϕ of a Hölder continuous potential ϕ for g, not necessarily all the g-preimages are well behaved with respect to μ_ϕ. Then by using Theorem 1 and Proposition 1, we obtain the Jacobians $J_{f^n}(\mu_\phi)$ of μ_ϕ with respect to iterates f^n, and the asymptotic logarithmic degree with respect to μ_ϕ. Moreover, by applying Corollary 2 to the smooth perturbation g of a hyperbolic toral endomorphism $f_A : \mathbb{T}^2 \to \mathbb{T}^2$, we can obtain the μ_ϕ-measure of any ball in $\mathbb{T}^2$.

In particular, since $h_{top}(g) = h_{top}(f_A) = \log \lambda_2$, it follows that the measure of maximal entropy $\mu_{0,g}$ of g, is given on any ball by:

$$\mu_{0,g}(B(y', \rho)) \approx |\det(A)|^m e^{-nh_{top}(g)} \approx |\det(A)|^m \lambda_2^{-n},$$

where λ_2 is the eigenvalue of A bigger than 1, and where $y' = g^m(y) \in \mathbb{T}^2, \rho > 0$, with the integers m, n being given by Corollary 2.

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