RIGIDITY OF LIMIT SETS FOR NONPLANAR GEOMETRICALLY FINITE KLEINIAN GROUPS OF THE SECOND KIND

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Abstract. We consider the relation between geometrically finite groups and their limit sets in both finite-dimensional and infinite-dimensional hyperbolic space. We prove three main results. Our first main result is that if G_1 and G_2 are finite-dimensional geometrically finite nonplanar groups of the second kind whose limit sets are equal, then G_1 and G_2 are commensurable, and in fact the index of the subgroup $G_1 \cap G_2$ in $\langle G_1, G_2 \rangle$ is finite. Our second and third main results are as follows: our first main result does not generalize to infinite dimensions, while a weaker rigidity theorem of Susskind and Swarup ('92) does generalize to infinite dimensions. Susskind and Swarup's theorem differs from ours in that it assumes from the outset that G_1 is a subgroup of G_2.

1. Introduction

Fix $2 \leq d \leq \infty$, let $\mathbb{H}^d$ denote d-dimensional hyperbolic space, and let $\mathrm{Isom}(\mathbb{H}^d)$ denote the isometry group of $\mathbb{H}^d$. In this paper we consider the following rigidity question: If $G_1, G_2 \leq \mathrm{Isom}(\mathbb{H}^d)$ are discrete groups whose limit sets $\Lambda(G_1), \Lambda(G_2)$ are equal, are G_1 and G_2 commensurable? In general the answer is no; additional hypotheses are needed. The following result is due to P. Susskind and G. A. Swarup:

Theorem 1.1 ([6, Theorem 1]; cf. [4, Theorem 3] for the case $d = 2$)**.** *Fix $2 \leq d < \infty$, and let $G_1, G_2 \leq \mathrm{Isom}(\mathbb{H}^d)$ be discrete groups whose limit sets are equal. If G_1 is nonelementary and geometrically finite and is a subgroup of G_2, then G_1 and G_2 are commensurable.*

The requirement here that $G_1 \leq G_2$ is quite a strong hypothesis, and the theorem is certainly false without it. To see this, note that if $G_1, G_2 \leq \mathrm{Isom}(\mathbb{H}^d)$ are lattices, then $\Lambda(G_1) = \partial\mathbb{H}^d = \Lambda(G_2)$, but it is quite possible that $G_1 \cap G_2 = \{\mathrm{id}\}$. In this paper we will prove a rigidity theorem similar to Theorem 1.1, but avoiding the hypothesis that $G_1 \leq G_2$, and in fact avoiding any hypothesis relating G_1 and G_2 other than the equality $\Lambda(G_1) = \Lambda(G_2)$. Based on the example above, we will need to rule out the case $\Lambda(G_1) = \Lambda(G_2) = \partial\mathbb{H}^d$, so we will assume that G_1 and G_2 are of the second kind. We will also assume that G_1 and G_2 are *nonplanar*, i.e. that their common limit set Λ is not contained in the closure of any proper totally geodesic subspace of $\mathbb{H}^d$. Our first main result, whose hypotheses are all necessary (cf. Remark 3.2 below), is as follows:

Theorem 1.2. *Fix $2 \leq d < \infty$, and let $G_1, G_2 \leq \mathrm{Isom}(\mathbb{H}^d)$ be two geometrically finite nonplanar groups of the second kind whose limit sets are equal. Then G_1 and G_2 are commensurable; in fact,*

$$[\langle G_1, G_2 \rangle : G_1 \cap G_2] < \infty.$$

In infinite dimensions, the situation is different: Theorem 1.1 generalizes to infinite dimensions (with "discrete" becoming "strongly discrete", see below), but Theorem 1.2 fails in infinite dimensions. We prove these results as Theorems 4.1 and 4.3, and they constitute our second and third main results. These results illustrates the fact that Theorem 1.2 is significantly more powerful than Theorem 1.1.

In Section 2, we define the terms used in our theorems, keeping an eye on the infinite-dimensional case. In Section 3, we prove Theorem 1.2, and in Section 4 we prove the assertions made above regarding the infinite dimensional analogues of Theorems 1.1 and 1.2.

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2. Definitions of terms

Fix $2 \leq d \leq \infty$, and let $\mathbb{H}^d$ denote d-dimensional real hyperbolic space; see [3, §2] for background regarding the case $d = \infty$. We will use [3] as our standard reference regarding Kleinian groups, for the reason that it explicitly considers the infinite-dimensional case. A group $G \leq \mathrm{Isom}(\mathbb{H}^d)$ is called *(strongly) discrete* if

$$\#\{g \in G : d(\mathbf{0}, g(\mathbf{0})) \leq R\} < \infty \;\; \forall R > 0.$$

The adverb "strongly" is used in infinite dimensions since in that case there are other, weaker, notions of discreteness; cf. [3, §5]. The group G is called *nonplanar* if it preserves neither any proper closed totally geodesic subspace of $\mathbb{H}^d$ nor any point on $\partial\mathbb{H}^d$. This property was called *acting irreducibly* in [3, §7.6].

The *limit set* of G is the set

$$\Lambda(G) := \{\xi \in \partial\mathbb{H}^d : \exists (g_n)_1^\infty \text{ in } G \;\; g_n(\mathbf{0}) \underset{n}{\to} \xi\}.$$

G is called *nonelementary* if its limit set contains at least three points, in which case its limit set must contain uncountably many points [3, Proposition 10.5.4]. Recall that a set $A \subseteq \mathbb{H}^d$ is said to be *convex* if the geodesic segment connecting any two points of A is contained in A, and that when G is nonelementary, the *convex hull* of the limit set is the smallest convex subset of $\mathbb{H}^d$ whose closure contains Λ. We denote the convex hull of the limit set by $\mathcal{C}(G)$.

A strongly discrete group $G \leq \mathrm{Isom}(\mathbb{H}^d)$ is called *geometrically finite* if there exists a disjoint G-invariant collection of horoballs $\mathscr{H}$ and a radius $\sigma > 0$ such that

$$\mathcal{C}(G) \subseteq G(B(\mathbf{0}, \sigma)) \cup \bigcup_{H \in \mathscr{H}} H.$$

This definition appears in the form presented here in [3, Definition 12.4.1], and in a similar form in [1, Definition (GF1)]. G is called *convex-cobounded* if the collection $\mathscr{H}$ is empty, i.e. if

$$\mathcal{C}(G) \subseteq G(B(\mathbf{0}, \sigma)).$$

Finally, G is of *compact type* if its limit set is compact. It was shown in [3] that every geometrically finite group is of compact type.

If a sequence $(x_n)_1^\infty$ in $\mathbb{H}^d$ converges to a point $\xi \in \partial\mathbb{H}^d$, then as usual we call the convergence *radial* if there is a cone with vertex ξ which contains the sequence $(x_n)_1^\infty$. By [3, Proposition 7.1.1], the convergence is radial if and only if the numerical sequence $(\langle o|\xi\rangle_{x_n})_1^\infty$ is bounded. Here $\langle\cdot|\cdot\rangle$ denotes the Gromov product:

$$\langle y|\xi\rangle_z = \lim_{x\to\xi} \frac{1}{2}[d(z,y) + d(z,x) - d(y,x)].$$

Given $\xi \in \Lambda(G)$, we denote by $\mathcal{B}_\xi$ the *Busemann function* based at ξ, i.e.

$$\mathcal{B}_\xi(y,z) = \lim_{x\to\xi}[d(x,y) - d(x,z)].$$

In the sequel we will find the following results useful:

Proposition 2.1 (Minimality of limit sets, [3, Proposition 7.4.1])**.** *Fix $G \leq \mathrm{Isom}(\mathbb{H}^d)$. Any closed G-invariant subset of ∂X which contains at least two points contains $\Lambda(G)$.*

Proposition 2.2 ([3, Proposition 7.6.3])**.** *Let G be a nonelementary subgroup of $\mathrm{Isom}(\mathbb{H}^d)$. Then the following are equivalent:*

(A) *G is nonplanar.*
(B) *There does not exist a nonempty closed totally geodesic subspace $V \subsetneq \mathbb{H}$ whose closure contains $\Lambda(G)$.*

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2, and then show that none of its hypotheses can be dropped. To do so we will need the following theorem:

Theorem 3.1 ([5, Theorem 2])**.** *Fix $2 \leq d < \infty$, and suppose that $G \leq \mathrm{Isom}(\mathbb{H}^d)$ is nonplanar and is not dense in $\mathrm{Isom}(\mathbb{H}^d)$. Then G is discrete.*

Proof of Theorem 1.2. Fix $2 \leq d < \infty$, and let $G_1, G_2 \leq \mathrm{Isom}(\mathbb{H}^d)$ be two geometrically finite nonplanar groups of the second kind whose limit sets are equal. Let Λ denote the common limit set of G_1 and G_2, let $G_+ = \langle G_1, G_2\rangle$, and let $G_- = G_1 \cap G_2$. Since Λ is a G_+-invariant closed subset of $\partial\mathbb{H}^d$ which contains at least two points, it follows from Proposition 2.1 that $\Lambda = \Lambda(G_+)$. In particular $\Lambda(G_+) \neq \partial\mathbb{H}^d$, which implies that G_+ is not dense in $\mathrm{Isom}(\mathbb{H}^d)$. On the other hand G_+ is nonplanar since it contains a nonplanar subgroup. Thus by Theorem 3.1, G_+ is discrete. Applying Theorem 1.1, we see that both G_1 and G_2 are commensurable with G_+. Thus G_1 and G_2 are commensurable, and in particular

$$[G_+ : G_-] \leq [G_+ : G_1] \cdot [G_+ : G_2] < \infty,$$

which completes the proof. $\square$

Remark 3.2. All three hypotheses of Theorem 1.2 are necessary.

1. The necessity of G_1 (and by symmetry G_2) being geometrically finite can be seen by letting G_2 be a Schottky group generated by two loxodromic isometries $g, h \in \mathrm{Isom}(\mathbb{H}^2)$ and then letting

$$G_1 := \langle g^{-n} h g^n : n \in \mathbb{N}\rangle.$$

Clearly G_1 and G_2 are not commensurable. On the other hand, G_1 is a normal subgroup of G_2 and so its limit set is preserved by G_2; thus by the minimality of limit sets we have $\Lambda(G_1) = \Lambda(G_2)$. Another example based on Jørgensen fibrations is given at the end of [6].

2. The necessity of G_1 (or equivalently, G_2) being nonplanar can be seen as follows: Let G_1 be a Schottky group generated by two loxodromic isometries $g, h \in \mathrm{Isom}(\mathbb{H}^4)$ such that
   (i) the axes of g and h are coplanar,
   (ii) the plane P generated by their axes is preserved by G_1, and
   (iii) h commutes with every rotation of $\mathbb{H}^4$ that fixes every point of P.

   Let j be an irrational rotation that fixes every point of P, and let
$$G_2 = \langle g, hj \rangle.$$
   Then for all $n \neq 0$, we have $j^n \notin G_2$ and $(hj)^n = h^n j^n \in G_2$ and thus $h^n \notin G_2$. It follows that G_1 and G_2 are not commensurable. On the other hand, $G_1|P = G_2|P$, which implies that $\Lambda(G_1) = \Lambda(G_2)$.
3. The necessity of G_1 (or equivalently, G_2) being of the second kind can be seen quite easily, as it suffices to consider any two lattices in $\mathrm{Isom}(\mathbb{H}^d)$ which have no common element.

4. Infinite dimensions

In this section we demonstrate our second and third main results, namely that while Theorem 1.1 can be generalized to infinite dimensions, Theorem 1.2 cannot. We remark that our counterexample to an infinite-dimensional version of Theorem 1.2 is also a counterexample to an infinite-dimensional version of Theorem 3.1, since the proof of Theorem 1.2 does not use finite-dimensionality in any way except for the use of Theorem 3.1.

Theorem 4.1. *Fix $2 \leq d \leq \infty$, and let $G_1, G_2 \leq \mathrm{Isom}(\mathbb{H}^d)$ be strongly discrete groups whose limit sets are equal. If G_1 is nonelementary and geometrically finite and is a subgroup of G_2, then G_1 and G_2 are commensurable.*

Note that the finite-dimensional case of this theorem also provides another proof of Theorem 1.1.

Proof of Theorem 4.1. Let Λ denote the common limit set of G_1 and G_2, and let $\mathcal{C}$ denote the convex hull of Λ. Fix $o \in \mathcal{C}$ and let $T \subseteq G_2$ be a transversal[1] of G_2/G_1 with the following minimality property: for all $g \in T$ and for all $h \in G_1$,

$$d(o, g(o)) \leq d(o, h^{-1}g(o)) = d(h(o), g(o)). \tag{4.1}$$

Here d denotes the hyperbolic metric on $\mathbb{H}^d$. Equivalently, (4.1) says that $g(o)$ is in the closed Dirichlet domain $\mathcal{D}$ centered at o for the group G_1 (cf. [3, Definition 12.1.4]).

By contradiction we suppose that $[G_2 : G_1] = \#(T) = \infty$. Since G_1 is geometrically finite, it is of compact type [3, Theorem 12.4.4], and thus G_2 is also of compact type. On the other hand, G_2 is strongly discrete, so by [3, Proposition 7.7.2], there exists a sequence

[1] I.e. a set for which each left coset gG_1 of G_1 intersects T exactly once.

$(g_n)_1^\infty$ in T so that $g_n(o) \to \xi \in \Lambda$. But G_1 is geometrically finite, so by [3, Theorem 12.4.4] we have that ξ is either a radial limit point or a bounded parabolic point of G_1.

If ξ is a radial limit point of G_1, then ξ is also a horospherical limit point of G_1, so there exists $h \in G_1$ such that $\mathcal{B}_\xi(o, h(o)) > 0$. But (4.1) gives

$$\mathcal{B}_\xi(o, h(o)) = \lim_{n\to\infty}[d(o, g_n(o)) - d(h(o), g_n(o))] \le 0,$$

a contradiction.

If ξ is a bounded parabolic point of G_1, then ξ is a parabolic point of G_2, so by [3, Remark 12.3.8], ξ is not a radial limit point of G_2. We will show that the sequence $(g_n(o))_1^\infty$ tends radially to ξ, a contradiction.

Given distinct points $p, q \in \mathbb{H}^d \cup \partial\mathbb{H}^d$, let $[p, q]$ denote the geodesic segment or ray connecting p and q. Now, $\mathcal{C}$ is cobounded in the *quasiconvex core* $\mathcal{C}_o = \bigcup_{g_1, g_2 \in G_1}[g_1(o), g_2(o)]$ [3, Proposition 7.5.3], which is in turn cobounded in the set $A = \bigcup_{g\in G_1}[g(o), \xi]$ by the thin triangles condition [3, Proposition 4.3.1(ii)]. Thus, there exists $\sigma > 0$ such that $\mathcal{C} \subseteq A^{(\sigma)}$, where $A^{(\sigma)}$ denotes the σ-thickening of A. On the other hand, since ξ is a bounded parabolic point of G_1, there exists a ξ-bounded set $S \subseteq \mathbb{H}^d$ such that $G_1(o) \subseteq H_1(S)$, where H_1 is the stabilizer of ξ in G_1. Thus, if we let

$$R = \bigcup_{x\in S}[x, \xi],$$

then $\mathcal{C} \subseteq \bigcup_{h\in H_1} h(R^{(\sigma)})$.

Claim 4.2. *The function*

$$f(y) = \min(\langle o|\xi\rangle_y, \langle y|\xi\rangle_o)$$

is bounded on $R^{(\sigma)}$.

Proof. Fix $y \in R$, say $y \in [x, \xi]$ for some $x \in S$. Since S is ξ-bounded, [3, Proposiiton 4.3.1(i)] implies that $d(o, [x, \xi])$ is bounded independent of x. Let $z \in [x, \xi]$ be the point closest to o. Then either $\langle y|\xi\rangle_z = 0$ or $\langle z|\xi\rangle_y = 0$, depending on whether z or y is closer to ξ. It follows that $f(y) \le d(o, z)$ is bounded independent of y. This shows that f is bounded on R; since f is uniformly continuous, it is also bounded on $R^{(\sigma)}$. ◁

Fix $n \in \mathbb{N}$. Since $x_n := g_n(o) \in T \subseteq \mathcal{C} \cap \mathcal{D}$, there exists $h_n \in H_1$ such that $x_n \in h_n(R^{(\sigma)})$. Since $x_n \in \mathcal{D}$, we have $d(o, x_n) \le d(o, h_n^{-1}(x_n))$ and thus $f(x_n) \le f(h_n^{-1}(x_n))$. Thus, the function f is bounded on the sequence $(x_n)_1^\infty$. Since $x_n \to \xi$, we must have $\langle x_n|\xi\rangle_o \to \infty$ (cf. [3, Observation 3.4.20]); thus the sequence $(\langle o|\xi\rangle_{x_n})_1^\infty$ is bounded. As remarked earlier, this is equivalent to the fact that $x_n \to \xi$ radially, which is a contradiction as observed earlier. □

Theorem 4.3. *There exist $G_1, G_2 \le \mathrm{Isom}(\mathbb{H}^\infty)$ convex-cobounded nonplanar groups of the second kind whose limit sets are equal satisfying $G_1 \cap G_2 = \{\mathrm{id}\}$. In particular, G_1 and G_2 are not commensurable.*

In the proof of Theorem 4.3, we will make use of the following:

Theorem 4.4 ([2, Theorem 1.1])**.** *Let T be a tree and let $V \subseteq T$ denote its set of vertices, and suppose that $\#(V) = \#(\mathbb{N})$. Then for every $\lambda > 1$, there is an embedding $\Psi_\lambda : V \to \mathbb{H}^\infty$ and a representation $\pi_\lambda : \mathrm{Isom}(T) \to \mathrm{Isom}(\mathbb{H}^\infty)$ such that*

(i) Ψ_λ *is* π_λ*-equivariant and extends equivariantly to a map* $\Psi_\lambda : \partial T \to \partial\mathbb{H}^\infty$,
(ii) *for all* $x, y \in V$,

$$\lambda^{d(x,y)} = \cosh d(\Psi_\lambda(x), \Psi_\lambda(y)), \text{ and}$$

(iii) *the set* $\Psi_\lambda(V)$ *is cobounded in the convex hull of the set* $\Lambda := \Psi_\lambda(\partial T)$.

Proof of Theorem 4.3. Let $\mathbb{F}_2$ be the free group on two elements, and let T be the right Cayley graph of $\mathbb{F}_2$. Fix any $\lambda > 1$, and apply the previous theorem to get Ψ_λ, π_λ, and Λ. Without loss of generality, we can suppose that there is no closed totally geodesic subspace of $\mathbb{H}^\infty$ containing Λ; otherwise, replace $\mathbb{H}^\infty$ by the smallest such subspace.

Lemma 4.5. *If* $\Gamma \leq \mathrm{Isom}(T)$ *acts sharply transitively on* V*, then* $G := \pi_\lambda(\Gamma)$ *is strongly discrete and convex-cobounded; moreover,* $\Lambda(G) = \Lambda$.

Proof. The equation $\Lambda(G) = \Lambda$ follows from the π_λ-equivariance of Ψ_λ together with the fact that $\Lambda(\Gamma) = \partial T$. Strong discreteness follows from (ii) of Theorem 4.4, and convex-coboundedness follows from (iii). ◁

Let $\Phi : \mathbb{F}_2 \to \mathrm{Isom}(T)$ be the natural left action of $\mathbb{F}_2$ on its right Cayley graph, and let $\Gamma_1 = \Phi(\mathbb{F}_2) \leq \mathrm{Isom}(T)$.

Lemma 4.6. *There exists* $\gamma \in \mathrm{Isom}(T)$ *such that* $\Gamma_1 \cap \gamma^{-1}\Gamma_1\gamma = \{\mathrm{id}\}$.

Proof. Write $\mathbb{F}_2 = \langle a, b\rangle$, and define $\gamma : \mathbb{F}_2 \to \mathbb{F}_2$ by the formula

$$\gamma(a^{n_1}b^{n_2}\cdots a^{n_{k-1}}b^{n_k}) = \begin{cases} a^{n_1}b^{n_2}\cdots a^{n_{k-1}}b^{n_k} & \text{if } n_1 \neq 0 \\ b^{-n_2}\cdots a^{-n_{k-1}}b^{-n_k} & \text{if } n_1 = 0 \end{cases}.$$

(The convention here is that $n_i \neq 0$ for $i = 2, \ldots, k-1$.) It can be verified directly that γ preserves edges in the Cayley graph, so γ extends uniquely to $\gamma \in \mathrm{Isom}(T)$. By contradiction, suppose there exist $x_1, x_2 \in \mathbb{F}_2 \setminus \{e\}$ with $\Phi_{x_1} = \gamma^{-1}\Phi_{x_2}\gamma$. Then $\gamma\Phi_{x_1} = \Phi_{x_2}\gamma$; evaluating at e gives $x_2 = \gamma(x_1)$. Write $x = x_1$; we have

$$\gamma(xy) = \gamma(x)\gamma(y) \quad \forall y \in \mathbb{F}_2. \tag{4.2}$$

Write $x = a^{n_1}b^{n_2}\cdots a^{n_{k-1}}b^{n_k}$. If $n_1 \neq 0$, then

$$\gamma(xb) = \gamma(x)b \neq \gamma(x)b^{-1} = \gamma(x)\gamma(b),$$

and if $n_1 = 0$, then

$$\gamma(xa) = \gamma(x)a^{-1} \neq \gamma(x)a = \gamma(x)\gamma(a).$$

Either equation contradicts (4.2). ◁

Let $\Gamma_2 = \gamma^{-1}\Gamma_1\gamma$. By Lemma 4.5, $G_1 = \pi_\lambda(\Gamma_1)$ and $G_2 = \pi_\lambda(\Gamma_2)$ are strongly discrete and convex-cobounded, and $\Lambda(G_1) = \Lambda = \Lambda(G_2)$. On the other hand, $G_1 \cap G_2 = \{\mathrm{id}\}$. □

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