GEOMETRY OF LIMIT SETS OF DISCRETE GROUPS ACTING ON REAL INFINITE DIMENSIONAL HYPERBOLIC SPACE

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1. Introduction

Dennis Sullivan, in his IHÉS *Seminar on Conformal and Hyperbolic Geometry* [40] that ran during the late 1970's and early '80s, indicated a possibility[1] of developing the theory of discrete groups acting by hyperbolic isometries on the open unit ball of a separable infinite dimensional real Hilbert space. Later in the early '90s, Misha Gromov lamented the paucity of results regarding such actions in his seminal lectures *Asymptotic Invariants of Infinite Groups* [19, 6A.III] where he encouraged their investigation in memorable terms: "The spaces like this [infinite dimensional symmetric spaces] ... look as cute and sexy to me as their finite dimensional siblings but they have been for years shamefully neglected by geometers and algebraists alike".

Gromov's lament had not fallen to deaf ears and the geometry and representation theory of infinite dimensional hyperbolic space $\mathbb{H}^\infty$ and its isometry group have been studied in the last decade by a handful of mathematicians. See, for example, the work by Burger-Iozzi-Monod [3], Delzant-Py [12], and Monod-Py [31]. However, infinite dimensional hyperbolic space has come into prominence most spectacularly through the recent resolution of a longstanding conjecture in

1991 *Mathematics Subject Classification.* Primary: 37F30, 37F35; Secondary: 30F40, 30F45, 58B20.

Key words and phrases. Fractal geometry, hyperbolic geometry, Hilbert manifold, Poincaré exponent, discrete group, Kleinian group, Fuchsian group, Hilbert space, indefinite metric, Poincaré series, critical exponent, Hausdorff dimension, geodesic dynamics.

The research was supported in part by the NSF Grant DMS 1001874.

[1]This was the earliest instance of such a proposal that we could find in the literature. It would be of interest to know whether such an idea may have been discussed prior to that.

algebraic geometry due to Enriques from the late nineteenth century. Cantat-Lamy [5] proved that the Cremona group (of birational transformations of the complex projective plane) has uncountably many non-isomorphic normal subgroups, i.e. it is not a simple group. Key to their enterprise is the fact, due to Manin [30], that the Cremona group admits a faithful isometric action on an infinite dimensional hyperbolic space called the Picard-Manin space.

We will be interested in subgroups of $\mathrm{Isom}(\mathbb{H}^\infty)$ whose natural actions are metrically proper, i.e. the orbit of an arbitrary point meets every bounded set in a set of finite cardinality. We call such groups *strongly discrete*. Now by a result of Gromov [6, Theorem 7.4.3] abstract groups that admit such actions correspond to those with the *Haagerup property*[2]. They include amenable groups, Coxeter groups and free groups, and are connected to various lines of investigation within geometric group theory, ergodic theory, representation theory and operator algebras, see [6]. For instance, it is an outstanding problem in geometric group theory to determine whether mapping class groups have the Haagerup property.

To make the connection with subgroups of $\mathrm{Isom}(\mathbb{H}^\infty)$ note that the boundary of infinite dimensional hyperbolic space is conformally equivalent to Hilbert space $\mathcal{H} := \partial\mathbb{H}^\infty \cup \{\infty\}$. As in finite dimensions, any isometry of $\mathcal{H}$ with respect to the Euclidean metric extends uniquely to an isometry of $\mathbb{H}^\infty$ which fixes ∞. Therefore there exists a correspondence between parabolic subgroups of the stabilizer $\mathrm{Stab}(\mathrm{Isom}(\mathbb{H}^\infty); \infty)$ and subgroups of $\mathrm{Isom}(\mathcal{H})$ whose orbits are unbounded. However, unlike in finite dimensions, such groups are not necessarily virtually nilpotent. Furthermore, even cyclic subgroups of $\mathrm{Isom}(\mathcal{H})$ are quite different from cyclic subgroups of $\mathrm{Isom}(\mathbb{R}^d)$ for $d \in \mathbb{N}$. Indeed, there is a well-known example of M. Edelstein [13] of a cyclic subgroup of $\mathrm{Isom}(\mathcal{H})$ whose orbits are unbounded but which is not strongly discrete.

This short note describes some of the first investigations regarding the Hasudorff geometry of limit sets of metrically proper isometric actions on real infinite dimensional hyperbolic space. Our goal is to present a generalization of the Bishop-Jones formula, equating the Poincaré exponent of the underlying group to the Hausdorff dimensions of the uniformly-radial and radial limit sets. To give a dynamical picture of what the Bishop-Jones relation is saying in terms of the geodesic flow on the underlying manifold, let us recall that radial limit points[3] correspond to geodesic excursions that return infinitely often to some bounded subset of the manifold, whereas uniformly radial directions correspond to geodesics that never leave a bounded region on the manifold. A priori, there seems to be no reason to believe that the Hausdorff dimensions of these sets are equal and their elegant result significantly generalized a large collection of previously known special cases, see for instance the work of Patterson [34], Sullivan [38] and Dani [8]. Our proof was inspired by Stratmann's presentation in [37]. Although the original proof and those of various generalizations thereafter (for instance [35, 7, 21]) crucially use the compactness of the sphere at infinity, our proof avoids such a dependence. We hope that it will shed some light on what aspects of this equality are "dimension-free" and follow from the presence of negative curvature. Finally, we indicate the robustness of strongly discrete convex-cobounded groups by showing such groups are finitely generated and of divergence type with finite Poincaré exponent. Further, these groups have compact limit sets, and the Hausdorff and packing measures on the limit sets are finite and positive and coincide with the conformal Patterson measure, up to a multiplicative constant.

The basic ideas behind these results were obtained by the authors during the summer of 2009 at the end of a productive conference, Dynamical Systems II, hosted at the University of North

[2]Such groups are also known as *a-T-menable* groups, since they are *morally* diametrically opposite to groups with Kazhdan's property (T).

[3]These were introduced in 1936 by Hedlund as *points of approximation*, where he proved that the geodesic flow on compact surfaces of constant negative curvature was topological mixing in [20].

Texas in Denton. These investigations have continued to develop and the reader is encouraged to follow-up this note with the work being done in collaboration with Lior Fishman and David S. Simmons in [17, 10, 11]. The more flexible concept of *partition structures* in these papers generalize the basic mass-redistribution principle that is used in this article. However, the ideas of the proof are more transparent in the setting of this paper and the authors are grateful for the gentle insistence of various colleagues to write such up.

2. Infinite dimensional models of hyperbolic geometry

We begin by defining Hilbert space, so let

$$\mathcal{H} \;=\; \ell_2(\mathbb{N}) \;:=\; \left\{ \mathbf{x} = (x_i)_1^\infty \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \sum_{i=1}^{\infty} {x_i}^2 < +\infty \right\}$$

and for $\mathbf{x} \in \mathcal{H}$, denote its norm by

$$\|\mathbf{x}\| := \left(\sum_{i=1}^{\infty} x_i^2 \right)^{1/2} .$$

A *Hilbert manifold* is a Banach manifold modelled on Hilbert space $(\mathcal{H}, \|\cdot\|)$ and an *infinite dimensional Riemannian manifold* is a Hilbert manifold equipped with a Riemannian structure, see [29]. Infinite dimensional real hyperbolic space is an infinite dimensional Riemannian manifold whose triangles are isometric to triangles in $\mathbb{H}^2$, the real hyperbolic plane. It follows that infinite dimensional hyperbolic space is a CAT(-1) space and therefore Gromov hyperbolic, see [2, 41].

Example 2.1. Hilbert space $\mathcal{H}$ can be considered by itself as an infinite dimensional Riemannian manifold, with at each point the standard inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{x}} := \sum_{i=1}^{\infty} u_i v_i .$$

Example 2.2. The space $\widehat{\mathcal{H}} := \mathcal{H} \cup \{\infty\}$ may be given the structure of a Hilbert manifold. The topology on $\widehat{\mathcal{H}}$ is defined as follows: a subset $U \subseteq \widehat{\mathcal{H}}$ is open if and only if $U \cap \mathcal{H}$ is open and if

$$\infty \in U \;\Rightarrow\; \mathcal{H} \setminus U \text{ is bounded.}$$

Warning 2.3. The topology on $\widehat{\mathcal{H}}$ is not a one-point compactification. Indeed, $\widehat{\mathcal{H}}$ with the topology defined above is not a compact space, since $\mathcal{H}$ is not locally compact.

2.1. The ball model and the upper half-space model.

There are several models of hyperbolic geometry, which are isometric as infinite dimensional Riemannian manifolds but which reflect different aspects of hyperbolic geometry. The models we will be interested in are the ball model ($\mathbb{B}$) and the upper half-space model ($\mathbb{H}$) and when we do not wish to specify a model we will write $\mathbb{H}$ (for "hyperbolic"). The ball model is the set

$$\mathbb{B} := \; \{ \mathbf{x} \in \mathcal{H} : \|\mathbf{x}\| < 1 \}$$

together with the Riemannian metric

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{x}, \mathbb{B}} \;:=\; \frac{4 \langle \mathbf{u}, \mathbf{v} \rangle}{(1 - \|\mathbf{x}\|^2)^2} \, .$$

The upper half-space model is the set

$$\mathbb{H} \;:=\; \{ \mathbf{x} \in \mathcal{H} : x_1 > 0 \}$$

together with the Riemannian metric

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{x},\mathbb{H}} := \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{x_1^2} .$$

Remark 2.4. In most references, the $(d+1)$-dimensional upper half-space model is defined to be the set $\{\mathbf{x} \in \mathbb{R}^{d+1} : x_{d+1} > 0\}$. When $d = \infty$, this does not make sense since there is no ∞th coordinate. Thus we have decided to use the first coordinate instead.

Note that the topological boundaries of $\mathbb{H}$ and $\mathbb{B}$ are also Hilbert manifolds (although it requires slightly more work to come up with coordinate charts):

$$\partial\mathbb{B} = \{\mathbf{x} \in \mathcal{H} : \|\mathbf{x}\| = 1\},$$
$$\partial\mathbb{H} = \{\mathbf{x} \in \mathcal{H} : x_1 = 0\} \cup \{\infty\}.$$

Further note that we have taken the boundary of $\mathbb{H}$ with respect to the Hilbert manifold $\widehat{\mathcal{H}}$ defined in Example 2.2. We shall always respect this convention. We note that the closures $\overline{\mathbb{B}}$ and $\overline{\mathbb{H}}$ are not Hilbert manifolds per se, but are Hilbert manifolds with boundary, see [29]. We will be content with considering them as topological subspaces of $\widehat{\mathcal{H}}$.

Finally, let us say a word about the geometric significance of $\mathbb{B}$ and $\mathbb{H}$. The ball model is best if you want to figure out what the hyperbolic world looks like if you are "at a point inside"; whereas the upper half-space model is best if you are "at a point on the boundary".

2.2. **Equivalence of models.** A $\mathcal{C}^\infty$ diffeomorphism $\Psi : X \to Y$ between infinite dimensional Riemannian manifolds is an *isomorphism* if $\langle \Psi'(x)[u], \Psi'(x)[v] \rangle_{\Psi(x),Y} = \langle u, v \rangle_{x,X}$ for all $x \in X$ and for all $u, v \in T_x X$. Note that every isomorphism of Riemannian manifolds $\Psi : X \to Y$ is also an *isometry*, i.e.

$$d_Y(\Psi(x_1), \Psi(x_2)) = d_X(x_1, x_2) \;\; \forall x_1, x_2 \in X. \tag{2.1}$$

The converse (known in finite dimensions as the Myers-Steenrod theorem) is also true, but much less trivial:

Theorem 2.5 (Theorem 7 of [18])**.** *Let X and Y be infinite dimensional Riemannian manifolds, and let $\Psi : X \to Y$ be a bijection. If Ψ satisfies* (2.1)*, then Ψ is an isomorphism (and in particular is $\mathcal{C}^\infty$).*

It may be shown by direct calculation (see [4]) that the map

$$e_{\mathbb{B},\mathbb{H}}(\mathbf{x}) = -\mathbf{e}_1 + 2\frac{\mathbf{x} + \mathbf{e}_1}{\|\mathbf{x} + \mathbf{e}_1\|^2}$$

is an isomorphism of Riemannian manifolds and in particular an isometry. Furthermore, the map $e_{\mathbb{B},\mathbb{H}}$ extends uniquely to a homeomorphism between $\overline{\mathbb{B}}$ and $\overline{\mathbb{H}}$.

2.3. **Comparison with the classical theory.** In contrast with the infinite dimensional setting of this article, we make a few brief remarks in this subsection about analogous considerations in finite dimensions. In particular about conformal maps, Möbius transformations and the notion of preserving orientation. To define conformal maps we first need the notion of a similarity.

Definition 2.6. Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. T is a *similarity* if it can be written as the product of a positive real number (called the *scaling constant* of T) and a linear isometry. An affine map $A : \mathcal{H} \to \mathcal{H}$ is a *similarity* if its linear part $\mathbf{x} \mapsto A(\mathbf{x}) - A(\mathbf{0})$ is a similarity. The group of similarities of $\mathcal{H}$ will be denoted $\mathrm{Sim}(\mathcal{H})$.

Definition 2.7. Let X and Y be infinite dimensional Riemannian manifolds, and let $f : X \to Y$ be a diffeomorphism. We say that f is *conformal* if for each $x \in X$, $f'(x) : T_xX \to T_{f(x)}Y$ is a similarity.

As in finite dimensions, the quintessential (non-linear) conformal map is the inversion with respect to a sphere[4]. The following theorem generalizes the classical result known as Liouville's theorem, which tells us that for $d \geq 3$, any conformal diffeomorphism betwen two subsets of $\mathbb{R}^d$ is the restriction of a Möbius transformation.

Theorem 2.8 (Liouville's theorem in Hilbert space)**.** *Let $U, V \subseteq \mathcal{H}$ be nonempty open connected sets and let $\phi : U \to V$ be a conformal diffeomorphism. Then one of the following two cases holds:*

(NL) *ϕ is the composition of an inversion and an affine similarity, or*

(L) *ϕ is an affine similarity.*

Note that in either case the map ϕ extends uniquely to a conformal map $\widehat{\phi} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$. As in the finite dimensional case, the map $\widehat{\phi}$ is called a *Möbius transformation* and we denote the class of such maps by $\mathrm{Mob}(\widehat{\mathcal{H}})$. The nonlinear (NL) case corresponds to the case when ∞ is not preserved, and the linear (L) case corresponds to when ∞ is preserved. Theorem 2.8 follows from the observation (see [23]) that R. Nevanlinna's proof of the finite dimensional Liouville's theorem [32] extends to infinite dimensions.

Remark 2.9. Notice that if, motivated by the finite dimensional theory, we restricted to the subclass of $\mathrm{Mob}(\widehat{\mathcal{H}})$ defined by

$$\mathrm{Mob}^*(\widehat{\mathcal{H}}) := \{\ g \in \mathrm{Mob}(\widehat{\mathcal{H}}) \mid g \textit{ is the composition of finitely many inversions}\},$$

then, unlike the finite dimensional case, we have

$$\mathrm{Mob}^*(\widehat{\mathcal{H}}) \subsetneq \mathrm{Mob}(\widehat{\mathcal{H}}).$$

In fact, the map g an be written as a finite composition of inversions if and only if $\mathrm{Fix}(g)$ has finite codimension. For example, the shift map on $\ell_2(\mathbb{Z})$ cannot be written as a finite composition of inversions. We give an indication of why this is true. Say $\mathrm{Fix}(g)$ has finite codimension, then one can find a finite dimensional subspace V such that the entire map is the Poincaré extension of its restriction to V. This reduces us to the finite dimensional statement [36] that every Möbius map is a composition of finitely many inversions which may then be re-extended. On the other hand if one computes the composition of two inversions it can be shown that $\mathrm{Fix}(g)$ has codimension 1 and so composing finitely many inversions only adds one finitely many times to the codimension.

Remark 2.10. One cannot make sense of *orientation-preserving* transformations in infinite dimensions as one cannot define a meaningful notion of orientation. If one wanted to define orientation-preserving via the kernel of a continuous homomorphism $\mathcal{O} : \mathscr{O}(\mathcal{H}) \to \mathbb{Z}_2$ one would easily fall into a trap. (Here $\mathbb{Z}_2$ is the group with two elements.) For example, any reflection in a hyperplane on $\ell_2(\mathbb{Z})$ would be orientation-preserving. For a construction of such a map, take the commutator of the shift map squared and the map that switches consecutive pairs of nonnegative coordinates, i.e. 0 and 1, 2 and 3, etc.

[4] Fix $\mathbf{p} \in \mathcal{H}$ and $\alpha > 0$, and let $S(\mathbf{p}, \alpha)$ denote the sphere around $\mathbf{p}$ of radius α. The *inversion with respect to the sphere* $S(\mathbf{p}, \alpha)$ is the map

$$\mathrm{i}_{\mathbf{p},\alpha} : \mathbf{x} \mapsto \alpha^2 \frac{\mathbf{x} - \mathbf{p}}{\|\mathbf{x} - \mathbf{p}\|^2} + \mathbf{p}.$$

We make the conventions that $\mathrm{i}_{\mathbf{p}} = \mathrm{i}_{\mathbf{p},1}$ and $\mathrm{i} = \mathrm{i}_{\mathbf{0}}$.

3. Classification of isometries

In [3] one may find the following classification of isometries of $\mathbb{H}$ based on results in [25] and [26]. Every isometry $g \in \mathrm{Isom}(\mathbb{H})$ is exactly one of the following three types: if it has bounded orbits then it is called *elliptic*; if its orbit is unbounded and it has one fixed point on the boundary then it is called *parabolic*, and if its orbit is unbounded and it fixes two points on the boundary it is called *hyperbolic*. We may conjugate each $g \in \mathrm{Isom}(\mathbb{H})$ to a "normal form" whose geometrical significance is clearer. The normal form will depend on the classification of g as elliptic, parabolic, or hyperbolic. We will not prove the remaining propositions in this section, but proofs may be found in [10]. Let us start with some

Notation 3.1. If G is a group acting on a space X, then for each $x \in X$ we will denote its stabilizer by

$$\mathrm{Stab}(G; x) := \{g \in G : g(x) = x\}.$$

For any Hilbert space $\mathcal{H}$, by $\mathscr{O}(\mathcal{H})$ we denote the group of linear isometries of $\mathcal{H}$. Let us write $\partial\mathbb{H} = \mathcal{E} \cup \{\infty\}$, where $\mathcal{E} := \{\mathbf{x} \in \mathcal{H} : x_1 = 0\}$. As in finite dimensions, for any $g \in \mathrm{Sim}(\mathcal{E})$ there exists a unique $\widehat{g} \in \mathrm{Sim}(\mathcal{H})$ so that $\widehat{g} \upharpoonright \mathcal{E} = g$ and so that $\widehat{g}(\mathbb{H}) = \mathbb{H}$. The map $\widehat{g}$ is called the *Poincaré extension* of g.

Proposition 3.2. *Fix $g \in \mathrm{Isom}(\mathbb{H})$.*

(i) *If g is elliptic, then g is conjugate to a map of the form $T \upharpoonright \mathbb{B}$ for some linear isometry $T \in \mathscr{O}(\mathcal{H})$.*

(ii) *If g is parabolic, then g is conjugate to a map of the form $\mathbf{x} \mapsto \widehat{T}[\mathbf{x}] + \mathbf{p} : \mathbb{H} \to \mathbb{H}$, where $T \in \mathscr{O}(\mathcal{E})$ and $\mathbf{p} \in \mathcal{E}$.*

(iii) *If g is hyperbolic, then g is conjugate to a map of the form $\mathbf{x} \mapsto \lambda\widehat{T}[\mathbf{x}] : \mathbb{H} \to \mathbb{H}$, where $0 < \lambda < 1$ and $T \in \mathscr{O}(\mathcal{H})$.*

In the first (*elliptic*) case the orbit[5] $(g^n(0))_1^\infty$ remains fixed forever, and in the third (*hyperbolic*) case it diverges to the boundary. In the latter, there is a pair of fixed points at infinity: one is attracting and the other repelling. In this case, every orbit is unbounded and the forward orbit approaches the attractive fixed point while the backward orbit approaches the repelling fixed point. Further, there exists a unique fixed geodesic connecting the two fixed points that is invariant under the action of g. On the other hand, things can get far more interesting in the second case when g is *parabolic*: then the orbit can oscillate, both accumulating at infinity and returning infinitely often to a bounded region. Note that this is forbidden in finite dimensions. We record this phenomena in the following

Proposition 3.3. *There exists a parabolic $g \in \mathrm{Stab}(\mathrm{Isom}(\mathbb{H}); \infty)$ whose orbit $(g^n(0))_1^\infty$ is unbounded but returns infinitely often to a bounded region and in fact accumulates at* 0.

We remark that this propostion is equivalent to a construction of M. Edelstein[6], who in Theorem 2.1 of [13] constructed a fixed-point-free isometry $g \in \mathrm{Isom}(\ell_2(\mathbb{C}))$ and a sequence $(n_k)_1^\infty$ so that $g^{n_k}(0) \underset{k}{\to} 0$.

4. Discrete groups of isometries

Let $X = \mathbb{H}$ and let G be a subgroup of the isometry group $\mathrm{Isom}(X)$. In finite dimensions, i.e. when $X = \mathbb{H}^n$ the following definitions are equivalent:

[5] We will write 0 instead of $\mathbf{0}$ and expect the reader to not interpret such as the numerical quantity zero.

[6] See [42] for a recent presentation.

(1) For every bounded $B \subseteq X$, $\#[g \in G : gB \cap B \neq \emptyset] < \infty$.
(2) For every $x \in X$, there exists an open set $U \ni x$ with

$$gU \cap U \neq \emptyset \Rightarrow gx = x.$$

(3) G is a discrete subset of $\mathrm{Isom}(X)$ w.r.t. compact-open topology.

Any one of them may be taken as a definition of a *discrete* group of isometries. However in infinite dimensions one must proceed more carefully. Notice that although (1) $\Rightarrow$ (2) even in infinite dimensions, there exist natural examples of groups that show us (2) $\not\Rightarrow$ (1).

Example 4.1. Consider $\mathbb{H}$ and let

$$V := \bigcup_{m \geq 2} \{(0, n_2, n_3, \ldots, n_m, 0, 0, \ldots) : n_i \in \mathbb{Z} \ \ \forall i = 2, \ldots, m\} \subseteq \mathcal{E}$$

Then v is a $\mathbb{Z}$-vector space and the group

$$G := \langle x \mapsto x + v : v \in V \rangle$$

is an example of one that satisfies (2) but not (1).

It may be somewhat harder to imagine the right infinite dimensional analogue(s) of (3), however [10] contains a detailed study. Let us start with the following definitions:

Definition 4.2. The group G is *strongly discrete* if (1) holds, i.e. for every bounded $B \subseteq X$,

$$\#[g \in G : gB \cap B \neq \emptyset] < \infty.$$

The group G is *weakly discrete* if (2) holds, i.e. for every $x \in X$, there exists an open set $U \ni x$ with

$$gU \cap U \neq \emptyset \Rightarrow gx = x.$$

Definition 4.3. An group G acts *properly discontinuously* if for every $x \in X$, there exists an open set $U \ni x$ with

$$gU \cap U \neq \emptyset \Rightarrow g = \mathrm{id}.$$

Equivalently, if there exists $r > 0$ such that

$$B(0, r) \ \cap \bigcup_{g \in G \setminus \{\mathrm{id}\}} g\left(B(0, r)\right) = \emptyset.$$

A group is *torsion-free* if every element of finite order is the identity. A group action is *free* if $\mathrm{Fix}(g) \neq \emptyset \Rightarrow g = \mathrm{id}$.

Unlike strong discreteness which turns out to be a rather robust notion in infinite dimensions, the notion of being properly discontinuous is far more fragile. We summarize the connections between our various notions in the following

Observation 4.4. Let (X, d) be a metric space and let $G < \mathrm{Isom}(X)$. Then:

1. Strongly discrete actions are weakly discrete.
2. Torsion-free strongly discrete actions are free.
3. Properly discontinuous actions are weakly discrete and free.
4. Torsion-free strongly discrete actions are properly discontinuous.
5. If X is a CAT(0) space, then free actions are torsion-free.

Remark 4.5. Strongly discrete torsion-free groups are always properly discontinuous. In the reverse direction if in finite dimensions, or when X is proper, properly discontinuous groups are strongly discrete via 3., since weakly discrete actions are strongly discrete in such a situation. On the other hand 5. tells that in a CAT(0) setting properly discontinuous groups are torsion-free.

We also remark that in a CAT(0) setting, the existence of torsion elements kills proper discontinuity. However for properly discontinuous groups the fixed points of elements $g \in G$ do not occur in the interiors of our models of hyperbolic Hilbert space. Furthermore, if $g \in G$ has three fixed points, then $g = \mathrm{id}$.

Proof. [Of Observation 4.4] Part 1: Fix $x \in X$. Then $\#[g : d(x, gx)] < \infty$ by strong discreteness. Therefore $\varepsilon := \min\{d(x, gx) : gx \neq x\} > 0$. So set $U := B(x, \varepsilon/2)$. Then we have that for every g with $gx \neq x$ we have $d(x, gx) \geq \varepsilon$ which implies $gU \cap U = \emptyset$. Therefore strongly discrete actions are weakly discrete. Part 2: By way of contradiction if G were not free then $gx = x$ for some $g \neq \mathrm{id}$. Then for every n we have $d(g^n x, x) = 0$. Now strong discreteness implies that $\#[g^n : n \in \mathbb{Z}] < \infty$ which in turn produces an N for which $g^N = \mathrm{id}$. This contradicts the assumption that G was torsion-free. Part 3 follows straight from the definitions and Part 4 follows from Parts 1., 2. and 3. Part 5: Suppose not, then $g^n = \mathrm{id}$ for some $g \neq \mathrm{id}$. Then $\#[H] < \infty$ where $H := \langle g \rangle$. Therefore the orbit $H(0)$ is bounded and since we are in a CAT(0) space Cartan's lemma produces an $x \in \mathrm{Fix}(H) = \mathrm{Fix}(g)$. But this provides a contradiction since we had assumed that G was free. □

We make the simplifying assumption that our groups are *without torsion-elements.*

Observation 4.6. A group $G < \mathrm{Isom}(\mathbb{B})$ is strongly discrete if and only if for every sequence $(g_n)_1^\infty$ of distinct elements of G,

$$\limsup_{n\to\infty} d_{\mathcal{H}}(g_n(0), \partial\mathbb{B}) = 0.$$

This may be observed from the following equivalences

$$\limsup_{n\to\infty} d_{\mathcal{H}}(g_n(0), \partial\mathbb{B}) = 0 \Leftrightarrow \liminf_{n\to\infty} d_{\mathcal{H}}(g_n(0), 0) = 1 \Leftrightarrow \liminf_{n\to\infty} d_{\mathbb{B}}(g_n(0), 0) = +\infty.$$

5. Poincaré series and the critical exponent

Definition 5.1. Fix $x \in \mathbb{H}$ and $s > 0$. The *Poincaré series* of the group G is defined by

$$\Sigma_{x,y}(s) = \sum_{g\in G} e^{-sd_{\mathbb{H}}(x,g(y))}. \tag{5.1}$$

With this notation, the *Poincaré exponent*[7] of G is given by

$$\delta_G = \inf\{s > 0 : \Sigma_{x,y}(s) < +\infty\}. \tag{5.2}$$

We reserve $\Sigma_s(G) := \Sigma_{0,0}(s) = \sum_{g\in G} e^{-sd_{\mathbb{H}}(0,g(0))}$.

It follows using the triangle inequality that the definition is independent of our choice in x and y.

Observation 5.2. In the finite-dimensional case the Poincaré exponent of a discrete group is always finite. However in infinite dimensions one may construct a strongly discrete Schottky group G such that $\delta_G = +\infty$. Construct a sequence of positive reals (a_n) with $a_n \to 0$ and $\sum_{n\geq 1} a_n^s = \infty$ for every s. Then with some care one may choose infinitely many generators $(g_n)_n$ such that for every n we have that $e^{-d(g_n(0),0)} \asymp a_n$ and so that each generator has a distinct coordinate axis as its hyperbolic axis.

[7] Also known as the *critical exponent.*

However, the following fact remains true even in infinite dimensions.

Observation 5.3. If G is a subgroup of $\mathrm{Isom}(\mathbb{H})$ and $\delta_G < +\infty$, then the group G is strongly discrete.

Proof. Suppose that G is not strongly discrete. Therefore that there exists some bounded set W such that $\#W \cap \overline{G(0)} = \infty$. Now the fact that W is bounded implies there exists some $B_{\mathbb{H}}(0,K) \supseteq W$ and therefore for every $g(0) \in W$, we have that

$$\sum_{g(0)\in W} e^{-td(0,g(0))} \geq \sum_{g(0)\in W} e^{-tK} = +\infty.$$

Since t was arbitrary we are done. $\square$

6. Limit sets

Let us define the limit sets of our group actions and move on to describe basic properties of such sets. When not specified, we assume that $G < \mathrm{Isom}(\mathbb{H})$ acts properly discontinuously[8] on $\mathbb{H}$.

Definition 6.1. The limit set of a group $G < \mathrm{Isom}(\mathbb{H})$ is defined to be

$$L(G) \ := \ \{\alpha \in \partial\mathbb{H} \mid \exists (g_n)_n \ \lim_{n\to\infty} g_n(x) = \alpha\}.$$

This definition is independent of the choice of $x \in \mathbb{H}$ and it is also clear from the definition that $L(G)$ is closed and G-invariant.

Definition 6.2. A group G is called *elementary* whenever $\#L(G) \in \{0,1,2\}$.

Theorem 6.3 (No Global Fixed Points)**.** *Any non-elementary weakly discrete group G has no global fixed points.*

Proof. Assume by way of contradiction that we have a global fixed point and then conjugate to send it to ∞. Then by Liouville's theorem g must be of the form

$$g(x) = \lambda_g T[x] + b.$$

The proof now splits into two cases, viz.

Case 1: [$\lambda_g = 1$ for every $g \in G$.]
Then each hyperplane $\{x : x_0 = \alpha\}$ is fixed by G for every $\alpha > 0$ and therefore $L(G) = \{\infty\}$. This leads to a contradiction since we assumed that G was non-elementary.

Case 2: [There exists a $g \in G$ with $\lambda_g < 1$.]
Without loss of generality we assume that $g(0) = 0$ since g is hyperbolic and so $l_{0,\infty}$ is the invariant axis. Then g is of the form

$$g(x) = g_1(x) = \lambda_1 T_1[x]$$

with $\lambda_1 < 1$. We claim that there exists a $g_2(x) := \lambda_2 T_2[x] + b$ with $\lambda_2 < 1$ and $b \neq 0$. Indeed, since G is non-elementary, there exists a third limit point say ξ that is neither 0 nor ∞. Now fix a point on the invariant axis at distance greater than one from the boundary $\partial\mathbb{H}$. Then there exists an element of the group that brings it arbitrarily close to ξ and this element is the one we were after. Note that since $\xi \notin \{0,\infty\}$ its translation vector is non-zero as claimed.
Now let us calculate the commutator of g_1 and g_2 and call it $g_3 := [g_2, g_1]$. Let

$$y = g_3(x) = g_2^{-1} \circ g_1^{-1} \circ g_2 \circ g_1(x).$$

[8]However in most cases this is more than we need and it suffices to assume that G is weakly discrete.

One may compute the following form

$$y = T_2^{-1} \circ T_1^{-1} \circ T_2 \circ T_1[x] + T_2^{-1} \circ T_1^{-1} \frac{I - \lambda_1 T_1}{\lambda_1 \lambda_2}[b]$$

which we rewrite as $g_3(x) = \hat{T}[x] + \hat{b}$. Note that $\hat{b} \neq 0$ [suppose not, then $T_2^{-1} \circ T_1^{-1}(I - \lambda_1 T_1)[b] = 0 \Leftrightarrow \lambda_1 T_1[b] = b$ which leads to a contradiction since $\lambda_1 < 1$ and $b \neq 0$]. Recall that $\uparrow = e_1$. Now consider

$$\begin{aligned} g_1^n g_3 g_1^{-n}(\uparrow) &= g_1^n g_3(\lambda_1^{-n} T_1^{-n} \uparrow) \\ &= g_1^n \big(\lambda_1^{-n} \hat{T}[T_1^{-n} \uparrow] + \hat{b}\big) \\ &= T_1^n \hat{T}[T_1^{-n} \uparrow] + \lambda_1^n T_1^n[\hat{b}] \\ &= \uparrow + \lambda_1^n T_1^n[\hat{b}]. \end{aligned}$$

Note that the last term goes to zero in norm as $n \to \infty$ since $\lambda_1 < 1$. But this contradicts the fact that G is weakly discrete.

□

Corollary 6.4. *Any non-elementary weakly discrete group contains a rank two free subgroup generated by two hyperbolic isometries.*

Proof. Suppose not, then by Gromov's theorem [22, Theorem 1.2] there exists a finite subset of the boundary that is invariant under the group. Let H be a stabilizer of some point in that set. Then by the previous Theorem 6.3 the subgroup H is elementary. Now $G = \bigcup_{i=1}^n g_i H$, where $g_1, \ldots, g_n$ is a sequence of coset representatives of H in G. It follows that $L(G) = \bigcup_{i=1}^n g_i L(H)$ is finite and therefore that G is elementary which is absurd. □

Definition 6.5. A properly discontinuous group G is of *compact type* when $L(G)$ is compact.

Theorem 6.6. *For a properly discontinuous group G acting on $\mathbb{B}$, the following are equivalent:*

(1) *G is of compact type.*
(2) *Every infinite subset of $G(0)$ contains an accumulation point.*
(3) *Each sequence $(g_n(0))_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \|g_n(0)\| = 1$ has a converging subsequence, which necessarily accumulates at an element in $L(G)$.*

Proof. Notice that (2) $\Rightarrow$ (3) is immediate. Let's start by proving (3) $\Rightarrow$ (1). Suppose that $(\xi_n)_n \subseteq L(G)$. Then for every n, there exists a m_n such that $\|g_{m_n}(0) - \xi_n\| \leq 1/n$. Since we have a discontinuous action, orbits must accumulate on the boundary, i.e. $\|g_{m_n}(0)\| \to 1$ and $n \to \infty$. By hypothesis, there exists a subsequence $(n_k)_k$ and $\xi \in L(G)$ such that $g_{m_{n_k}}(0) \to \xi$ as $k \to \infty$. We now have that

$$\|\xi_{n_k} - \xi\| \leq \|g_{m_{n_k}}(0) - \xi_n\| + \|g_{m_{n_k}}(0) - \xi\|,$$

where the first term is bounded by $1/n_k$ and therefore it and the second term both vanish as $k \to \infty$.

Next we we show that (1) $\Rightarrow$ (2). Let $(g_n(0))_n$ be our infinite subset of $G(0)$. Then for every n, $g_n \in G$. Now pick two points $\alpha, \beta \in L(G)$ and let $[\alpha, \beta]$ denote the geodesic between them. Now fix some $z \in [\alpha, \beta]$ and pick some common subsequence such that $g_n(\alpha) \to \gamma$ and $g_n(\beta) \to \delta$ as $n \to \infty$. Now one may verify via calculation (or geometric intuition) that for every $\epsilon > 0$ there exists N_ϵ such that for every $n \geq N_\epsilon$, $g_n([\alpha, \beta]) \subseteq B_{\mathcal{H}}([\gamma, \delta], \epsilon)$. One may check that the Hausdorff distance between the $g_n([\alpha, \beta])$ is dependent on the distances $\|\gamma - g_n(\alpha)\|$ and $\|\delta - g_n(\beta)\|$ and vanishes as they decrease to zero. Therefore there exists a subsequence such that $g_n([\alpha, \beta]) \subseteq B_{\mathcal{H}}([\alpha, \beta], 1/n)$ for every n. It now follows that for every n there exists $w_n \in [\gamma, \delta]$

such that $d_{\mathcal{H}}(g_n(z), w_n) \leq 1/n$. Now there exists a subsequence $(w_n)_n$ (by compactness of $[\gamma, \delta]$) such that $w_n \to w \in [\gamma, \delta]$ as $n \to \infty$. Finally one can see that $g_n(z) \to w$ as $n \to \infty$ since

$$\|g_n(z) - w\| \leq \|g_n(z) - w_n\| + \|w_n - w\|.$$

□

Remark 6.7. This theorem may be extended to non-proper CAT(-1) spaces modulo the construction of geodesics between two boundary points, see [10].

7. Convex-cobounded groups of compact type

Fix $\alpha \in \partial\mathbb{K}$ and let S_α refer to the *hyperbolic ray* from 0, the origin, to α. As in finite dimensions the Busemann function may be extended to $\partial\mathbb{K} \times \mathbb{K} \times \mathbb{K}$ via

$$\lim_{z \to \alpha} \mathcal{B}_z(x, y) =: \mathcal{B}_\alpha(x, y),$$

where $\mathcal{B}_z(x, y) := d(x, z) - d(y, z)$.

Definition 7.1. Let us define the following distinguished subsets of $L(G)$:

- $\alpha \in L(G)$ is a *horospherical limit point* of G if there exists a sequence of orbit points $g(0)$ with
$$\mathcal{B}_\alpha(g(0), 0) \to -\infty.$$
The set of such points is called the *horospherical limit set* and is denoted by L_h or by $L_h(G)$.
- $\alpha \in L(G)$ is a *radial limit point* of G if there exists a constant $c = c(\alpha) > 0$ such that
$$S_\alpha \cap B(g(0), c) \neq \varnothing$$
for infinitely many $g(0)$ for every $g \in G$. The set of such points is called the *radial limit set* and is denoted by L_r or by $L_r(G)$.
- $\alpha \in L(G)$ is a *uniformly radial limit point* of G if there exists a positive constant $c = c(\alpha) > 0$ such that
$$S_\alpha \subseteq \cup_{g \in G} B(g(0), c).$$
The set of such points is called the *uniformly radial limit set* and is denoted by L_{ur} or by $L_{ur}(G)$.

In other words, we may write

$$L_{ur}(G) = \bigcup_{\sigma > 0} L_{ur,\sigma}(G)$$

and

$$L_r(G) = \bigcup_{\sigma > 0} L_{r,\sigma}(G)$$

where $\alpha \in L_{ur,\sigma}(G)$ when the geodesic from 0 to α, S_α, is covered by hyperbolic balls $B(g(0), \sigma)$ over the G-orbit of the origin and, similarly where $\alpha \in L_{r,\sigma}(G)$ when S_α intersects infinitely many hyperbolic balls $B(g_n(0), \sigma)$ for some subsequence of the G-orbit of the origin.
Notice that we have the following inclusions:

$$L_{ur} \subseteq L_r \subseteq L_h \subseteq L.$$

Definition 7.2. We define the projection map $\Pi : X \setminus \{0\} \to \partial X$ to be the unique map so that for all $x \in X \setminus \{0\}$, x is on the geodesic joining 0 and $\Pi(x)$. For $x \in X$ and $\sigma > 0$, it is useful to consider the set $\Pi(B(x, \sigma))$, which is called the "shadow" of the ball $B(x, \sigma)$, which we may also denote by $\mathrm{Shad}(x, \sigma)$. One may imagine shining a light from the point 0 onto the boundary.

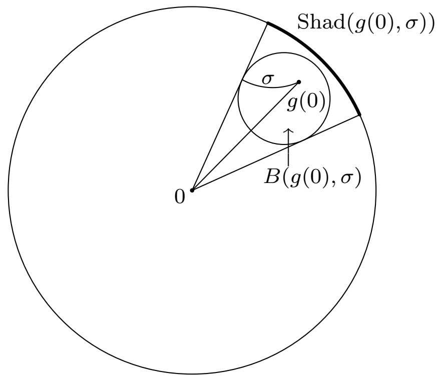


FIGURE 1. A useful estimate regarding the Euclidean diameter of shadows that will be used in the sequel is $\mathrm{diam}\big(\mathrm{Shad}(g(0),\sigma)\big) \asymp \sigma e^{-d(0,g(0))}$, see [37, Lemma 2.1].

Remark 7.3. In general there is nothing distinguished about the origin 0 and in fact we can similarly define Π_z and Shad_z when we are shining a light from $z \in \mathbb{H}$. Without the subscript we assume that the light is based at the origin. The reader is invited to verify that we would get the same sets $L_{ur}(G)$, $L_r(G)$, and $L_h(G)$ if in our constructions we replaced 0 by an arbitrary but frozen point $z \in \mathbb{H}$.

Definition 7.4. Given $\alpha, \beta \in \overline{\mathbb{B}}$, define $[\alpha,\beta]$ to be the unique geodesic joining α and β. Let

$$\mathcal{C} := \mathcal{C}\big(L(G)\big) = \bigcup_{\alpha,\beta\in L(G)} [\alpha,\beta]$$

A strongly discrete G is *convex-cobounded* if there exists $R>0$ such that

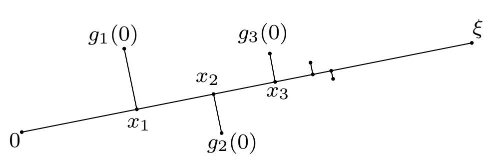
$$\mathcal{C} \subseteq G(B(0,R)).$$

Theorem 7.5. *Suppose G is a strongly discrete group of compact type. Then the following are equivalent:*

(A) *G is convex-cobounded.*
(B) $L(G) = L_{ur}(G)$
(C) $L(G = L_r(G)$
(D) $L(G = L_h(G)$

Proof. Since $(B) \Rightarrow (C) \Rightarrow (D)$, it is enough to prove $(A) \Rightarrow (B)$ and $(D) \Rightarrow (A)$.

[Proof of $(A) \Rightarrow (B)$] Fix some $\xi \in L(G)$. In view of Remark 7.3, we may assume without loss of generality that $[0,\xi] \subseteq \mathcal{C}$. Therefore, for every $x \in [0,\xi]$ there exists some $g \in G$ with $d(g(0),x) \le R$, with R coming from the definition of convex-coboundedness. Choose $x_n \in [0,\xi]$ with $d(0,x_n) = n$ and let $g_n \in G$ denote the corresponding elements that move the origin R-close to x_n.



Then notice that $d(g_n(0), g_{n+1}(0)) \leq 2R+1$ and so it is clear that $\xi \in L_{ur}(G)$.

[Proof of $(D) \Rightarrow (A)$] By way of contradiction suppose that G is not convex-cobounded. Define D to be the set

$$D := \{x : d(0,x) < d(0, g(x)) \quad \forall g \in G \text{ such that } g(0) \neq 0\}$$

and let $\Delta := D \cap \mathcal{C}$. Fix a point $x \in \mathcal{C} \setminus G(B(0,R))$ and note that the set of points in the orbit of 0 whose distance to x is less than the distance from x to 0 is finite, since G is strongly discrete. Pick the element $g(0) \in G(0)$ which is closest to x and then $g^{-1}(x) \in \Delta \setminus B(0,R)$[9]. Therefore $\Delta \subsetneq B(0,R)$ for every $R > 0$. Hence, we may construct a sequence $y_n \in G(x) \cap (\Delta \setminus B(0,n))$, $n \in \mathbb{N}$. Let ξ_n and η_n be points in $L(G)$ such that $y_n \in [\xi_n, \eta_n]$. Now since $\lim_{n\to\infty}(1 - \|y_n\|) = 0$, and the group G has a compact limit set, there exists a convergent subsequence $(y_{n_k})_{k=1}^{\infty}$ to some point $\xi \in L(G)$. Since $\lim_{n\to\infty} \min\{\|y_n - \xi_n\|, \|y_n - \eta_n\|\} = 0$, because of the symmetry of ξ_n and η_n, passing to yet another subsequence, we may assume without loss of generality that

$$\lim_{k\to\infty} \|y_{n_k} - \xi_{n_k}\| = 0$$

and consequently that

$$\lim_{k\to\infty} \xi_{n_k} = \lim_{k\to\infty} y_{n_k} = \xi.$$

We show that such a ξ must belong to $L(G) \setminus L_h(G)$, which will give a contradiction. For all $g \in G$ and for every $y_n \in \Delta$ we have that $d(0, y_n) \leq d(g(0), y_n)$, which in turn implies that $\mathcal{B}_\xi(g(0), 0) \geq 0$ and therefore $\xi \notin L_h(G)$. We are done with the proof. $\square$

8. The theorem of Bishop and Jones

Throughout this section let (X, ρ) be a complete metric space. We start with a version of mass-redistribution principle without any reference to hyperbolic geometry or groups.

Definition 8.1. Given two sets $C, D \subseteq X$ and some $\kappa > 0$ we say that

$$C \subseteq_\kappa D \Leftrightarrow B(C, \kappa \mathrm{diam}(C)) \subseteq D,$$

and read it as the *κ-thickening of C is contained in D*. We denote $B(C, \kappa\mathrm{diam}(C)) := C_\kappa$ to be the κ-thickening of C.

We prove the following mass-redistribution result.

Proposition 8.2 (Mass-Redistribution)**.** *Let (X, ρ) be a complete metric space and fix $t \geq 0$ and $\kappa \in (0,1)$. For every $n \geq 1$ let E_n be a finite set. Set $E_i^j := E_i \times \ldots \times E_j$ for $0 \leq i \leq j$ and to avoid clutter we write E^n for E_1^n. Suppose that $\Sigma \subseteq E^* := \bigcup_{n \geq 1} E^n$, has the property that*

$$\widehat{\Sigma} := \{\omega|_{|\omega|-1} \in E^{|\omega|-1} : \omega \in \Sigma\} \subseteq \Sigma \cup \{\varnothing\}.$$

We denote $\Sigma^n := \Sigma \cap E^n$. Suppose further that for every $\omega \in \Sigma$, there exists a closed subset $A(\omega) \subseteq X$ with the following properties:

(a) *For every $\omega \in \Sigma$, $A(\omega) \subseteq_\kappa A(\omega|_{|\omega|-1})$;*
(b) *For every $\omega \in \Sigma$, $\mathrm{diam}\big(A(\omega|_{|\omega|-1})\big) \leq \kappa^{-1}\mathrm{diam}(A(\omega))$;*
(c) *For $\omega, \tau \in E^n \cap \Sigma$, $\omega \neq \tau$, $A(\omega) \cap A(\tau) = \emptyset$;*

[9] The existence of such a g for every x proves that D is a *fundamental domain* based at 0, see [36, Chapter 6]. Since we do not require this notion in the sequel, we refrain from defining it and listing its properties.

(d) *For every* $\omega \in \Sigma$

$$\sum_{\substack{e \in E_{|\omega|+1} \\ \omega e \in \Sigma}} \operatorname{diam}^t\big(A(\omega e)\big) \geq \operatorname{diam}^t\big(A(\omega)\big);$$

and

(e) $\lim_{n\to\infty} \max\{\operatorname{diam}(A(\omega)) : \omega \in \Sigma^n\} = 0$.

Then it follows that

$$\operatorname{HD}\left(\bigcap_{n=1}^{\infty} \bigcup_{\omega \in \Sigma^n} A(\omega)\right) \geq t.$$

Remark 8.3. We make a few small remarks, before starting the proof of the Proposition.

- The sets E_i may be thought of as index sets or as alphabets if we think in terms of symbolic dynamics, or IFSs. Note that if we take all of them to be equal to the same set E then E^n is simply the n-fold product of the set E.
- It may help to explain what each of the conditions are saying
  1. Condition (a) says that the κ-thickenings are decreasing, while
  2. Condition (b) tells us that they do not decrease too fast.
  3. Condition (d) is the appropriate redistribution of mass that leads to the appropriate measure in the limit
  4. Condition (c) is a natural disjointness condition which is necessary when building a measure.

  We note that we may prove the same proposition by specifying rates, i.e. different κ's for (a) and (b).
- In general metric spaces $\operatorname{diam}(A_\kappa) \leq (1+2\kappa)\operatorname{diam}(A)$. Whereas for Banach spaces, it turns out that we have equality. Therefore when in Hilbert spaces for instance, we have that (a) implies condition (e).

Proof. Decreasing Σ if necessary we may assume without loss of generality that $A(\omega) \neq \emptyset$ for every $\omega \in \Sigma$. For every $n \geq 1$, let

$$J_n := \bigcup_{\omega \in E^n \cap \Sigma} A(\omega) \quad \text{and} \quad J := \bigcap_{n=1}^{\infty} J_n.$$

Note that for every $\omega \in \Sigma$, we have that $J \cap A(\omega) \neq \emptyset$ and so we fix a point $x_\omega \in J \cap A(\omega)$. Now define inductively the following sequence of Borel probability measures $(\mu_n)_{n\geq 0}$ on J as follows. Let μ_0 be an arbitrary Borel probability measure on J. For the inductive step, suppose that $n \geq 0$ and $\mu_0, \mu_1, \ldots, \mu_n$ (each a Borel probability measure on J) have already been defined. Then set

$$\mu_{n+1} := \sum_{\omega \in E^{n+1} \cap \Sigma} \frac{\operatorname{diam}^t(A(\omega))}{\sum_{e \in E: \omega|_n e \in \Sigma} \operatorname{diam}^t(A(\omega|_n e))} \mu_n(A(\omega|_n))\delta_{x_\omega}. \tag{8.1}$$

Now since μ_n is a probability measure and by (c) all the sets $A(\tau), \tau \in E^n \cap \Sigma$ are pairwise disjoint, a straightforward direct computation shows that μ_{n+1} is a Borel probability measure. Now since $x_\omega \in J$ for every $\omega \in \Sigma$, we immediately get from (8.1) that

$$\mu_n(J) = 1 \text{ for every } n \geq 0. \tag{8.2}$$

Now suppose that $\tau \in E^n \cap \Sigma$. In view of (8.1) and (c) we then get

$$
\begin{aligned}
\mu_{n+1}(A(\tau)) &= \sum_{\omega\in E^{n+1}\cap\Sigma} \frac{\operatorname{diam}^t(A(\omega))}{\sum_{e\in E:\omega|_n e\in\Sigma}\operatorname{diam}^t(A(\omega|_n e))}\mu_n(A(\omega|_n))\delta_{x_\omega}(A(\tau)) \\
&= \sum_{a\in E:\tau a\in\Sigma} \frac{\operatorname{diam}^t(A(\tau a))}{\sum_{e\in E:\tau e\in\Sigma}\operatorname{diam}^t(A(\tau e))}\mu_n(A(\tau)) \\
&= \mu_n(A(\tau)) \sum_{a\in E:\tau a\in\Sigma} \frac{\operatorname{diam}^t(A(\tau a))}{\sum_{e\in E:\tau e\in\Sigma}\operatorname{diam}^t(A(\tau e))} \qquad (8.3)\\
&= \mu_n(A(\tau))\frac{\sum_{a\in E:\tau a\in\Sigma}\operatorname{diam}^t(A(\tau a))}{\sum_{e\in E:\tau e\in\Sigma}\operatorname{diam}^t(A(\tau e))} \\
&= \mu_n(A(\tau)).
\end{aligned}
$$

Because of (c) and since $x_\omega \in J \cap A(\omega)$ for every $\omega \in \Sigma$, we get that for every $n \geq 0$ and for all $0 \leq i \leq j$ that

$$
\mu_{n+j}(A(\tau)) = \mu_{n+j}\left(\bigcup_{\omega\in E_{n+1}^{n+i}:\tau\omega\in\Sigma} A(\tau\omega)\right) = \sum_{\omega\in E_{n+1}^{n+i}:\tau\omega\in\Sigma} \mu_{n+j}(A(\tau\omega)). \tag{8.4}
$$

Having this formula and treating (8.3) as the inductive step, we make the following

Observation 8.4. $\mu_{n+k}(A(\tau)) = \mu_n(A(\tau))$ for every $n, k \geq 0$ and all $\tau \in E^n \cap \Sigma$.

Now let $c := \left[\sum_{e\in E_1}\operatorname{diam}^t(A(e))\right]^{-1} < \infty$. Next we prove the following.

Observation 8.5. For every $n \geq 1$ and for every $\omega \in E^n \cap \Sigma$, we have that $\mu_n(A(\omega)) \leq c \cdot \operatorname{diam}^t(A(\omega))$.

Proof. We prove this by induction on $n \geq 1$. If $e \in E \cap \Sigma$, then it follows from (8.1) that

$$
\begin{aligned}
\mu_1(A(e)) &= \left[\sum_{a\in E_1}\operatorname{diam}^t(A(a))\right]^{-1}\operatorname{diam}^t(A(e)) \cdot \mu_0(X) \\
&= c \cdot \operatorname{diam}^t(A(e)).
\end{aligned}
$$

So suppose that the claim holds for some $n \geq 1$. For every $\omega \in E^{n+1} \cap \Sigma$, we then have from (8.3) and (d) that

$$
\begin{aligned}
\mu_{n+1}(A(\omega)) &= \left[\sum_{e\in E_{n+1}:\omega|_n e\in\Sigma}\operatorname{diam}^t(A(\omega|_n e))\right]^{-1}\operatorname{diam}^t(A(\omega)) \cdot \mu_n(A(\omega|_n)) \\
&\leq \left[\operatorname{diam}^t(A(\omega|_n))\right]^{-1}\operatorname{diam}^t(A(\omega)) \cdot \mu_n(A(\omega|_n)) \\
&\leq c \cdot \operatorname{diam}^t(A(\omega)).
\end{aligned}
$$

We are done. ∎

Now the set $J = \bigcap_{n\geq1}\bigcup_{\omega\in E^n\cap\Sigma} A(\omega) =: \bigcap_{n\geq1} J_n$ is closed as it is the intersection of closed sets J_n. Recall that (e) gives us that

$$
\lim_{n\to\infty} \max\{\operatorname{diam}(A(\omega)) : \omega \in E^n \cap \Sigma\} = 0,
$$

and now because of the finiteness of the index sets E_n, the set J is totally bounded. Thus J is compact since X is complete. Therefore by the Banach-Alaoglu theorem, the sequence $(\mu_n)_1^\infty$ of

Borel probability measures on J contains a weakly convergent subsequence. Denote its weak limit by μ. Since $A(\omega) \cap J$ is a clopen subset of J (with respect to the topology relative to J) for every $\omega \in \Sigma$, it follows from Observation 8.5 that for every $\omega \in \Sigma$,

$$\mu(A(\omega)) \leq c \cdot \operatorname{diam}^t(A(\omega)). \tag{8.5}$$

Note that we have not yet used conditions (a) and (b) and we now do so in estimating from above the measures of balls centered at the points of J.

Let $z \in J = \cap_{n \geq 1} J_n$. Since for every $n \geq 1$, the sets $E_n \cap \Sigma$ are finite, it follows from König's Lemma that there exists $\omega \in E^{\mathbb{N}}$ such that $\omega|_n \in \Sigma$ for every $n \geq 1$ and $\{z\} = \bigcap_{n=1}^{\infty} A(\omega|_n)$. Because of (c) this $\omega \in E^{\mathbb{N}}$ is unique.

Now fix a radius $r \in (0, \kappa \min\{\operatorname{diam}(A(\omega)) : \omega \in E^2 \cap \Sigma\})$. Then there exists a largest $n = n(\omega, r) \geq 2$ such that

$$r \leq \kappa \operatorname{diam}(A(\omega|_n)). \tag{8.6}$$

Since $z \in A(\omega|_n)$ it follows from (a) that

$$B(z, r) \subseteq B\big(z, \kappa \operatorname{diam}(A(\omega|_n))\big) \subseteq B\big(A(\omega|_n), \kappa \operatorname{diam}(A(\omega|_n)))\big) \subseteq A(\omega|_{n-1}).$$

Now (8.5) implies that

$$\mu(B(z, r)) \leq c \cdot \operatorname{diam}^t(A(\omega|_{n-1})). \tag{8.7}$$

By the definition of n, we have that $\kappa \operatorname{diam}(A(\omega|_{n+1})) < r$ and then applying (b) twice we get that

$$\operatorname{diam}(A(\omega|_{n-1})) \leq \kappa^{-1} \operatorname{diam}(A(\omega|_n)) \leq \kappa^{-2} \operatorname{diam}(A(\omega|_{n+1})) \leq \kappa^{-3} r.$$

Inserting this into (8.7) we finally get

$$\mu(B(z, r)) \leq c \kappa^{-3t} r^t. \tag{8.8}$$

Note that $\mu(J) = 1$ (since its a probability measure on J) and thus by a direct application of the [14, Proposition 2.1 (Mass distribution principle)], we have that $HD(J) \geq t$. □

Remark 8.6. When (X, ρ) is a finite dimensional Euclidean space, condition (a) can be replaced by the requirement that all the sets $A(\omega)$ for $\omega \in \Sigma$ are uniformly undistorted balls and then a standard volume argument [28, Lemma 4.2.6] would work to get (8.8). Our Proposition 8.2 requires no extra structure on X and condition (a) will be proved to be satisfied in the course of our proof of the Bishop-Jones theorem for Hilbert spaces. We should note however that condition (a) is somewhat strong and for example it fails in the standard construction of C, the middle-third Cantor set if $X = [0, 1]$. Note however if $X = C$, then (a) is satisfied. Therefore one must take some care in the choice of X.

Notation 8.7. For $\xi \in \partial\mathbb{B}$, we denote $\widehat{B}(\xi, r)$ to be the union of all geodesics (in $\mathbb{B}$) with both endpoints in the ball in $\partial\mathbb{B}$ centered at ξ and with spherical radius equal to r. In particular $\Pi[\widehat{B}(\xi, r)]$ is equal to this ball.

Next we shall prove the following

Lemma 8.8. *If G is a non-elementary strongly discrete group acting on $\mathbb{B}$, then for every $t < \delta_G$ there exist two distinct points $\xi_1, \xi_2 \in L(G)$ such that for all $r > 0$,*

$$\sum_{\gamma(0) \in \widehat{B}(\xi_i, r)} e^{-t d(\gamma(0), 0)} = +\infty \quad \textit{for} \quad i = 1, 2.$$

*We call such points ξ_i, t-*divergent points *of $L(G)$.*

Proof. We assume that our group is non-elementary and so there exist at least two distinct hyperbolic elements, by Corollary 6.4. We prove the existence of one such point (as claimed in the Lemma) and the same argument will provide another point distinct from the first; as we have that these hyperbolic elements have distinct pairs of fixed points.

Suppose by way of contradiction, that there exist no t-divergent points in the limit set. Now pick a hyperbolic element g with axis l_g whose attracting and repelling endpoints respectively are ξ_g^+ and ξ_g^- on $L(G)$. Let's look at ξ_g^- and refer to it as simply ξ. Then there exists an r_ξ such that the sum over the G-orbit of 0 within $A := \widehat{B}(\xi, r_\xi)$ is finite, i.e.

$$\sum_{\gamma(0)\in A} e^{-td(\gamma(0),0)} < +\infty \ .$$

Then we have that

$$\sum_{\gamma(0)\in \mathbb{B}\setminus A} e^{-td(\gamma(0),0)} = +\infty \ .$$

Now for an arbitrary $\varepsilon > 0$, there exists $n \geq 0$ large enough such that $g^n(\mathbb{B} \setminus A) \subseteq \widehat{B}(\xi_g^+, \varepsilon)$. It is enough to show that for such n, we have that

$$\sum_{\gamma(0)\in \mathbb{B}\setminus A} e^{-td(g^n\gamma(0),0)} = +\infty \ ,$$

since then we would have shown ξ_g^+ to be t-divergent and thus derived a contradiction. Notice that

$$\begin{aligned}
\sum_{\gamma(0)\in \mathbb{B}\setminus A} e^{-td(g^n\gamma(0),0)} &\geq \sum_{\gamma(0)\in \mathbb{B}\setminus A} e^{-td(0,g^n(0))} e^{-td(g^n(0),g^n\gamma(0))} \\
&\qquad\qquad \text{[by the triangle inequality]} \\
&= e^{-td(0,g^n(0))} \sum_{\gamma(0)\in \mathbb{B}\setminus A} e^{-td(g^n(0),g^n\gamma(0))} \\
&= e^{-td(0,g^n(0))} \sum_{\gamma(0)\in \mathbb{B}\setminus A} e^{-td(0,\gamma(0))} \\
&\qquad\qquad \big[\text{since } g \text{ is an isometry}\big] \\
&= +\infty \ .
\end{aligned}$$

We are done. ∎

Definition 8.9. Fix $\tau > 0$. For every integer $n \geq 0$ let

$$A_n(\tau) := \{z \in G(0) : \tau n \leq d(z, 0) \leq \tau(n + 1)\}.$$

The set $A_n(\tau)$ is called the *hyperbolic* (n, G)*-annulus* centered at 0 and of width τ.

Lemma 8.10. *Let G be a strongly discrete group. Fix $\tau > 0$ and $0 \leq s < t < \delta_G$. Let $\xi \in L(G)$ be a t-divergent point. Then for every $M > 0$ and for every $r > 0$ there exists $(n_j(\xi))_{j=1}^{\infty}$ an increasing sequence of positive integers such that*

$$\sum_{g(0)\in \widehat{B}(\xi,r)\cap A_{n_j(\xi)}(\tau)} e^{-sd(g(0),0)} \geq M \quad \textit{for every} \quad j \geq 1.$$

Proof. Suppose by way of contradiction that there exist $M, r > 0$ and an integer $q \geq 0$ such that

$$\sum_{z \in \widehat{B}(\xi,r) \cap A_n(\tau)} e^{-sd(z,0)} < M \quad \text{for every} \quad n \geq q+1.$$

Take $r_* \in (0, r]$ so small that $A_q(\tau) \cap \widehat{B}(\xi, r_*) = \emptyset$. Then $A_n(\tau) \cap \widehat{B}(\xi, r_*) = \emptyset$ for all $n = 0, 1, 2, \ldots, q$ and thus we get that

$$\sum_{z \in \widehat{B}(\xi,r_*) \cap A_n(\tau)} e^{-sd(z,0)} = 0 \leq M \quad \text{if} \quad n \leq q$$

and that

$$\sum_{z \in \widehat{B}(\xi,r_*) \cap A_n(\tau)} e^{-sd(z,0)} \leq \sum_{z \in \widehat{B}(\xi,r) \cap A_n(\tau)} e^{-sd(z,0)} \leq M \quad \text{if} \quad n \geq q+1.$$

Therefore we have that $\sum_{z \in \widehat{B}(\xi,r_*) \cap A_n(\tau)} e^{-sd(z,0)} \leq M$ for all $n \geq 0$. Hence

$$\begin{aligned}
\sum_{z \in G(0) \cap \widehat{B}(\xi,r_*)} e^{-td(z,0)} &= \sum_{n=0}^{\infty} \sum_{z \in \widehat{B}(\xi,r_*) \cap A_n(\tau)} e^{-(t-s)d(z,0)} e^{-sd(z,0)} \\
&\leq \sum_{n=0}^{\infty} \sum_{z \in \widehat{B}(\xi,r_*) \cap A_n(\tau)} e^{-(t-s)\tau n} e^{-sd(z,0)} \\
&= \sum_{n=0}^{\infty} e^{-(t-s)\tau n} \sum_{z \in \widehat{B}(\xi,r_*) \cap A_n(\tau)} e^{-sd(z,0)} \\
&\leq M \sum_{n=0}^{\infty} e^{-(t-s)\tau n} \\
&< +\infty.
\end{aligned}$$

The last inequality follows since s was chosen strictly smaller that t and so we have a geometric series that converges. But then this contradicts the hypothesis that ξ is a t-divergent point and finishes the proof. □

We sometimes, for emphasis, will use the shorthand B_e, d_e and similarly B_h, d_h to distinguish between the Euclidean and hyperbolic settings respectively. In general the absence of subscripts refers to the hyperbolic setting, though diam without a subscript will refer to Euclidean diameter.

Lemma 8.11. *There exists $\alpha > 0$ such that for all $\sigma > \log 2$ we have*

$$\Pi[B_h(z, \alpha\sigma)] \subseteq B_e\left(\Pi[z], \frac{1}{8} \mathrm{diam}_e \Pi[B_h(z, \sigma)]\right),$$

for every z with $d(z, 0) > \sigma$.

Proof. The proof immediately follows from the estimate

$$\mathrm{diam}_e\big(\Pi[B(z,\sigma)]\big) = \mathrm{diam}_e\big(\,\mathrm{Shad}(z,\sigma)\big) \asymp \sigma e^{-d(z,g(z))}$$

described in Figure 7 above and [37, Lemma 2.1]. □

For every $\sigma > 0$, let $r_\sigma > 0$ be chosen so small that

$$r_\sigma < \pi e^{-\alpha\sigma}. \tag{8.9}$$

From this point on, for the remainder of the section, fix an arbitrary $\tau > 0$. The main ingredient, forming the inductive step in our proof of the Bishop-Jones Theorem, is the following lemma, whose proof is illustrated on the Figure 2 and is provided after formulation of the lemma.

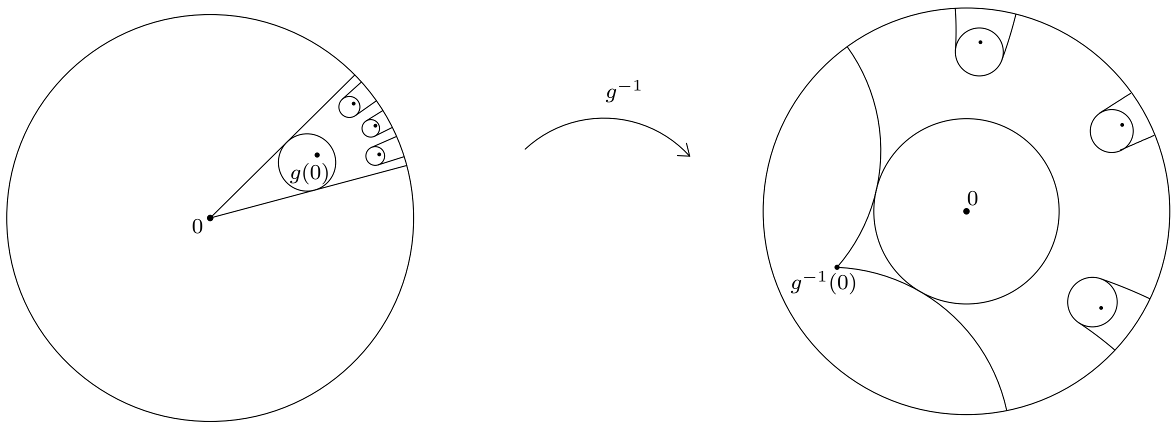


FIGURE 2. The strategy for the proof of Lemma 8.12 is to construct a collection of "children" of the point $g(0)$. We "pull back" the entire picture via g^{-1}. In the pulled-back picture, with the help of the Light Cone Lemma (cf. [37, Lemma 2.3]) we obtain the existence of many points $x \in G(0)$ such that $\mathrm{Shad}_{g^{-1}(0)}(x, \sigma) \subseteq \mathrm{Shad}_{g^{-1}(0)}(0, \sigma)$. These children can then be pushed forward via g to get children of $g(0)$.

Lemma 8.12. *Fix $0 < s < t < \delta_G$. Let $\xi_1, \xi_2 \in L(G)$ be two t-divergent points (see Lemma 8.8 for their existence). Then there exist $\sigma > 0$ and positive integers $l_1, l_2 \geq 1$ such that the following holds:*
For every $g \in G$ with $d(g(0), 0) > \sigma$, there exists a set $\Gamma(g)$ contained in one of the sets

$$\Gamma_i := \{h \in G : h(0) \in \widehat{B}(\xi_i, r_\sigma) \cap A_{l_i(\xi_i)}(\tau)\} \quad \textit{for } i = 1, 2;$$

such that the following hold:-

(a) *The family $\{\Pi[B(gh(0), \sigma)] : h \in \Gamma(g)\}$ consists of mutually disjoint balls.*
(b) *For every $h \in \Gamma(g)$,*

$$\Pi[gh(0)] \in B_e\left(\Pi[g(0)], \frac{1}{8}\mathrm{diam}_e \Pi[B(g(0), \sigma)]\right).$$

(c) *There exists a constant $\beta_\sigma \in (0, 1)$ depending only on l_1, l_2 and σ (in particular independent of g) such that for every $h \in \Gamma(g)$,*

$$\frac{\beta_\sigma}{4}\mathrm{diam}\Big(\Pi[B(g(0), \sigma)]\Big) \leq \mathrm{diam}\Big(\Pi[B(gh(0), \sigma)]\Big) \leq \frac{1}{4}\mathrm{diam}\Big(\Pi[B(g(0), \sigma)]\Big).$$

(d) *The following inequality holds*

$$\sum_{h \in \Gamma(g)} \mathrm{diam}^s\Big(\Pi[B(gh(0), \sigma)]\Big) \geq \mathrm{diam}^s\Big(\Pi[B(g(0), \sigma)]\Big).$$

Proof. Take $\sigma > \log 2$ and as large so that

$$d_e(\xi_1, \xi_2) > 6\pi e^{-\alpha\sigma}.$$

Now by the choice of r_σ, see (8.9), we have that

$$\inf\{d_e(x,y) : x \in \widehat{B}(\xi_1, r_\sigma), y \in \widehat{B}(\xi_2, r_\sigma)\} > 4\pi e^{-\alpha\sigma}.$$

Since $\alpha\sigma > \log 2$, it then follows from the Light Cone Lemma (cf. [37, Lemma 2.3]) that at least one of the balls $\widehat{B}(\xi_1, r_\sigma)$ or $\widehat{B}(\xi_2, r_\sigma)$ is contained in $g^{-1}\big(\Pi[B(g(0), \alpha\sigma)]\big)$. Assume without loss of generality that $\widehat{B}(\xi_1, r_\sigma) \subseteq g^{-1}\big(\Pi[B(g(0), \alpha\sigma)]\big)$. Consequently, if we fix two integers $l_1, l_2 \geq 1$ that we will specify later in the course of the proof, then

$$\Pi[h(0)] \in g^{-1}\big(\Pi[B(g(0), \alpha\sigma)]\big)$$

for every $h \in \Gamma_1$. Therefore $g(\Pi[h(0)]) \in \Pi[B(g(0), \alpha\sigma)]$ and g maps the geodesic from 0 to $\Pi[h(0)]$ with $h(0)$ on it to the geodesic from $g(0)$ to $g\Pi[h(0)]$ with $gh(0)$ on it. Since the light cone generated by 0 and $B(g(0), \alpha\sigma)$ is convex and contains both the points $g(0)$ and $g\Pi[h(0)]$ it also contains the point $gh(0)$ and thus

$$\Pi[gh(0)] \in \Pi[B(g(0), \alpha\sigma)] \subseteq B_e\left(\Pi[gh(0)], \frac{1}{8}\mathrm{diam}\Pi[B(g(0), \sigma)]\right)$$

where the inclusion follows from Lemma 8.11. Note that condition (b) of our Lemma has been established.

Assume now that $l_1 > \sigma/\tau$. Then $d(h(0), 0) > \sigma$ for all $h \in \Gamma_1$ and we can apply the Geometric Distortion Lemma (cf. [37, Lemma 2.2]) to get

$$d(g(0), 0) + d(h(0), 0) - 2\sigma \leq d(gh(0), 0) \leq d(g(0), 0) + d(h(0), 0) \tag{8.10}$$

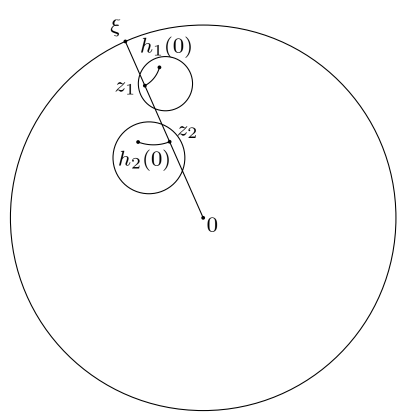
for every $h \in \Gamma_1$. Note that we also needed $g(\Pi[h(0)]) \in \Pi[B(g(0), \sigma)]$ to apply the Geometric Distortion Lemma, but we already have that $g(\Pi[h(0)]) \in \Pi[B(g(0), \alpha\sigma)] \subseteq \Pi[B(g(0), \sigma)]$. Now since $h(0) \in A_{l_1}(\tau)$, we have that $l_1\tau \leq d(h(0), 0) < l_1\tau + \tau$ for every $h \in \Gamma_1$, we then get

$$d(g(0), 0) + l_1\tau - 2\sigma \leq d(gh(0), 0) \leq d(g(0), 0) + l_1\tau + \tau \tag{8.11}$$

for every $h \in \Gamma_1$. Now define $\Gamma(g)$ to be a maximal (in the sense of inclusion) subset of Γ_1 such that

$$d(h_1(0), h_2(0)) > \tau + 6\sigma \tag{8.12}$$

for all $h_1, h_2 \in \Gamma_1$ with $h_1 \neq h_2$. Let us now prove item (a). Suppose by way of contradiction that, $\Pi[B(gh_1(0), \sigma)] \cap \Pi[B(gh_2(0), \sigma)] \neq \emptyset$ for some $h_1, h_2 \in \Gamma_1$ with $h_1 \neq h_2$. Let $\xi \in \Pi[B(gh_1(0), \sigma)] \cap \Pi[B(gh_2(0), \sigma)]$ and for each $i \in \{1, 2\}$ let $z_i \in B(gh_i(0), \sigma)$ be chosen on the ray $[0, \xi]$.



Then the following estimates are true using (8.11):

$$d(g(0),0)+l_1\tau-3\sigma<d(z_1,0)<d(g(0),0)+l_1\tau+\tau+\sigma$$

and

$$d(g(0),0)+l_1\tau-3\sigma<d(z_2,0)<d(g(0),0)+l_1\tau+\tau+\sigma.$$

It then follows that

$$\begin{aligned} d(z_1,z_2) &= |d(0,z_1)-d(0,z_2)| \\ &< [d(g(0),0)+l_1\tau+\tau+\sigma]-[d(g(0),0)+l_1\tau-3\sigma] \\ &= \tau+4\sigma. \end{aligned}$$

Therefore we have that

$$\begin{aligned} d(h_1(0),h_2(0)) &= d(gh_1(0),gh_2(0)) \\ &\le d(gh_1(0),z_1)+d(z_1,z_2)+d(z_2,gh_2(0)) \\ &< \sigma+[\tau+4\sigma]+\sigma \\ &= \tau+6\sigma \end{aligned}$$

which contradicts (8.12) and finishes the proof of item (a). It follows directly from the left-hand side of (8.11) and [37, Lemma 2.1] (see Figure 7) that with $l_1 \ge 1$ chosen large enough the right-hand side of (c) holds. Similarly the left-hand side of (c) follows from the right-hand side of (8.11) and [37, Lemma 2.1].

We have only left to show item (d) and this where the final specification of l_1 will be made. First for every $h\in\Gamma(g)$, define

$$Q_g(h):=\{f\in\Gamma(g): d(f(0),h(0))\le\tau+6\sigma\}.$$

By maximality of $\Gamma(g)$, see (8.12), we have that

$$\bigcup_{h\in\Gamma(g)} Q_g(h)=\Gamma_1.$$

Now since our group is strongly discrete, the number of elements in $Q_g(h)$ is bounded above by some constant, $C_1=C_1(\tau,\sigma)$, depending only $(\tau+6\sigma)$ and of course on G. Also because of (8.11) and in [37, Lemma 2.1], there exists another constant, $C_2=C_2(\tau,\sigma)$, such that for every $g\in G$ with $d(g(0),0)>\sigma$ and all $h_1,h_2\in\Gamma_1$

$$C_2^{-1}\le\frac{\operatorname{diam}\big(\Pi[B(gh_2(0),\sigma)]\big)}{\operatorname{diam}\big(\Pi[B(gh_1(0),\sigma)]\big)}\le C_2.$$

Hence,

$$\begin{aligned} \sum_{h\in\Gamma_1}\operatorname{diam}^s(\Pi[B(gh(0),\sigma)]) &\le \sum_{h\in\Gamma(g)}\sum_{f\in Q_g(h)}\operatorname{diam}^s(\Pi[B(gf(0),\sigma)]) \\ &\le \sum_{h\in\Gamma(g)} C_2(\tau,\sigma)^s\sum_{f\in Q_g(h)}\operatorname{diam}^s(\Pi[B(gh(0),\sigma)]) \\ &= C_2(\tau,\sigma)^s\sum_{h\in\Gamma(g)}[\#Q_g(h)]\operatorname{diam}^s(\Pi[B(gh(0),\sigma)]) \\ &\le C_2(\tau,\sigma)^sC_1(\tau,\sigma)\sum_{h\in\Gamma(g)}\operatorname{diam}^s(\Pi[B(gh(0),\sigma)]). \end{aligned} \tag{8.13}$$

Now take $M = C_\sigma^s C_2(\tau,\sigma)^s C_1(\tau,\sigma)$, where $C_\sigma > 0$ comes from [37, Lemma 2.1], and choose $l_1 \geq 1$ to be one of the numbers $(n_j(\xi_1))_{j=1}^\infty$ appearing in Lemma 8.10 that is as large as required above. Now by Lemma 8.10, [37, Lemma 2.1] and the right-hand-side of (8.10), we get that

$$\begin{aligned}
\sum_{h\in\Gamma_1} \mathrm{diam}^s(\Pi[B(gh(0),\sigma)]) &\geq C_\sigma^{-s} \sum_{h\in\Gamma_1} e^{-sd(gh(0),0)} \\
&\geq C_\sigma^{-s} \sum_{h\in\Gamma_1} e^{-sd(g(0),0)} e^{-sd(h(0),0)} \\
&= C_\sigma^{-s} e^{-sd(g(0),0)} \sum_{h\in\Gamma_1} e^{-sd(h(0),0)} \\
&\geq M C_\sigma^{-s} e^{-sd(g(0),0)} \\
&\geq M C_\sigma^{-2s} \mathrm{diam}^s(\Pi[B(g(0),\sigma)]) \\
&\geq C_2(\tau,\sigma)^s C_1(\tau,\sigma) \mathrm{diam}^s(\Pi[B(g(0),\sigma)]).
\end{aligned}$$

Now inserting this into (8.13), we finally get

$$\sum_{h\in\Gamma(g)} \mathrm{diam}^s(\Pi[B(gh(0),\sigma)]) \geq \mathrm{diam}^s(\Pi[B(g(0),\sigma)])$$

which establishes (d) and finished the proof of our Lemma. □

We are now in a position to prove the main result of this section, viz. the extension of the Bishop-Jones theorem to the infinite dimensional case.

Theorem 8.13. *If G is a strongly discrete group acting isometrically on $\mathbb{B}$, then*

$$\mathrm{HD}(L_{ur}(G)) = \mathrm{HD}(L_r(G)) = \delta_G.$$

Proof. As $L_{ur}(G) \subseteq L_r(G)$, we have that

$$\mathrm{HD}(L_{ur}(G)) \leq \mathrm{HD}(L_r(G)). \tag{8.14}$$

We shall first show that

$$\mathrm{HD}(L_r(G)) \leq \delta_G.$$

If $\delta_G = +\infty$, then we are done and so let's assume that $\delta_G < +\infty$. Fix an arbitrary $s > \delta_G$. Write G as $(g_n)_{n=1}^\infty$. Fix $\sigma > 0$ and let

$$L_{r,\sigma}(G) := \bigcap_{n\geq 1} \bigcup_{k\geq n} \Pi\big[B(g_k(0),\sigma)\big].$$

Since $\sum_{n\geq 1} e^{-sd(g_n(0),0)} < +\infty$, we get that

$$\lim_{n\to\infty} \sum_{k\geq n} \mathrm{diam}^s(\Pi\big[B(g_k(0),\sigma)\big]) \asymp_\sigma \lim_{n\to\infty} \sum_{k\geq n} e^{-sd(g_k(0),0)} = 0.$$

Thus $\mathrm{HD}(L_{r,\sigma}(G)) \leq s$ and consequently that $\mathrm{HD}(L_{r,\sigma}(G)) \leq \delta_G$. Now $L_r(G) = \bigcup_{n\geq 1} L_{r,n}(G)$ and therefore by the σ-stability of Hausdorff dimension, we get

$$\mathrm{HD}(L_r(G)) = \sup_{n\geq 1}\{\mathrm{HD}(L_{r,n}(G))\} \leq \delta_G.$$

Along with (8.14) this gives $\mathrm{HD}(L_{ur}(G)) \leq \mathrm{HD}(L_r(G)) \leq \delta_G$, and we are left to show that

$$\mathrm{HD}(L_{ur}(G)) \geq \delta_G. \tag{8.15}$$

By means of Lemma 8.12 we will perform a construction to which Proposition 8.2 will apply. In the setting of Lemma 8.12, for every $n \geq 1$ let

$$E_n := E := \Gamma_1 \cup \Gamma_2.$$

We define the set $\Sigma \subseteq E^*$ and the sets $A(\omega)$ for $\omega \in \Sigma$ by induction with respect to word length in Σ. For the base of your recursion, we take $E \cap \Sigma := E := \Gamma_1 \cup \Gamma_2$ and $A(h) := \Pi[B(h(0), \sigma)]$ for all $h \in E$. For the inductive step, suppose that the set $E^n \cap \Sigma$ has been defined and that all the sets $A(f)$, $f \in E^n \cap \Sigma$ have been defined as well. To define $E^{n+1} \cap \Sigma$ consider all the elements $g = f_1, \ldots, f_n \in E^n \cap \Sigma$ and declare that $f = f_1, \ldots, f_{n+1} \in E^{n+1} \cap \Sigma$ if $f_{n+1} \in E$ and $f_{n+1} \in \Gamma(f_1 \circ \ldots \circ f_n) = \Gamma(g)$. Then put $A(f) = \Pi[B(f_1 \circ \ldots \circ f_{n+1}(0), \sigma)]$. Verifying now the hypotheses of Proposition 8.2, we see that $\widehat{\Sigma} \subseteq \Sigma$ directly by construction. Properties (c), (d), (b) and (a) of Proposition 8.2 follow respectively from property a) of Lemma 8.12; property (d) of Lemma 8.12; the left-hand side of property (c) of Lemma 8.12 and finally from both property (b) and the right-hand-side of property (c) of Lemma 8.12 with $\kappa = 1/4$. Therefore all the properties of Proposition 8.2 have been verified and as a result of applying it we get that

$$\mathrm{HD}\left(\bigcap_{n \geq 1} \bigcup_{\omega \in E^n \cap \Sigma} A(\omega) \right) \geq s.$$

It now follows from (c) of Lemma 8.12, or more directly from (8.11), that

$$\bigcap_{n \geq 1} \bigcup_{\omega \in E^n \cap \Sigma} A(\omega) \subseteq L_{ur}(G)$$

and so we have that $\mathrm{HD}(L_{ur}(G)) \geq s$. Since was arbitrarily smaller than δ_G, we therefore get that $\mathrm{HD}(L_{ur}(G)) \geq \delta_G$. Thus (8.15) has been established and we are done. □

9. Convex-cobounded groups revisited

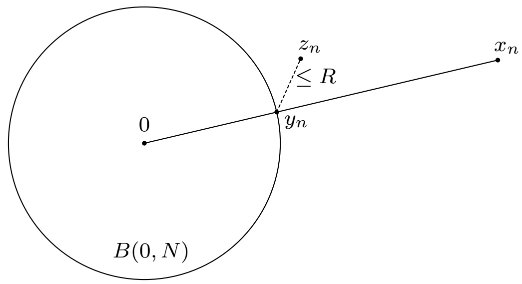
With the proof of the main theorem behind us, if only to whet the reader's appetite, we conclude with a proof sketch of the following theorem and encourage her to look up the papers [17, 10, 11].

Theorem 9.1. *Let $G < \mathrm{Isom}(\mathbb{H})$ be strongly discrete and convex-cobounded. Then G is finitely generated, has finite Poincaré exponent $\delta < \infty$, is of divergence type, and has a compact limit set. The δ-dimensional Hausdorff and packing measures on $L(G)$ are finite and positive and coincide up to a multiplicative constant with the δ-conformal Patterson measure which is Ahlfors δ-regular.*

Proof (sketch). The proof that the group G is finitely generated and then showing that the orbit map is a quasi-isometry follows from the Milnor-Schwarz Lemma [2, Proposition I.8.19], once we notice that $L(G)$ being compact implies that $\mathcal{C}$ is also compact, since we are in a Hilbert space. Say G has d generators, and let $|g|$ denote the word-length of $g \in G$, i.e. the length of the shortest word that g may be expressed in terms of the d generators. Then

$$\Sigma_s(G) = \sum_{g \in G} e^{-sd(0,g(0))} \leq \sum_{g \in G} e^{-s(\varepsilon|g|-K)} = e^{sK} \sum_{g \in G} e^{-s\varepsilon|g|} \leq e^{sK} \sum_{g \in F_d} e^{-s\varepsilon|g|} ,$$

where ε, K are the quasi-isometry constants and F_d denotes the free group on d generators which surjects onto G. By taking $s\varepsilon$ to be sufficiently large we can force convergence of the final sum above. Thus $\delta < \infty$. Next let us prove that G is of compact type. Consider a sequence $x_n \in G(0)$ with $d(0, x_n) \to \infty$. Choose $y_n \in \partial B(0, N) \cap [0, x_n]$ as in the figure below.



Since our group is convex-cobounded there exists a sequence $z_n \in G(0)$ with $d(0, z_n) \leq N + R$. Since the action is strongly discrete there are only finitely many such z_ns for every n and therefore we may extract a constant subsequence $(n_k)_k$ with $z_{n_k} = z$. Thus $d(y_{n_k}, y_{n_l}) \leq 2R$ and so

$$\langle x_{n_k} | x_{n_l} \rangle_0 \geq \langle y_{n_k} | y_{n_l} \rangle_0 \geq \frac{1}{2}[N + N - 2R] = N - R .$$

Here $\langle x|y \rangle_z := \frac{1}{2}[d(x,z) + d(y,z) - d(x,y)]$ denotes the Gromov product, see [41, Definition 2.7]. Now since N are arbitrary, we may extract a diagonal sequence such that for every $k, l \in \mathbb{N}$

$$\langle x_{n_k} | x_{n_l} \rangle_0 \geq \min\{k, l\} - R \underset{k,l}{\longrightarrow} \infty .$$

Thus $(x_{n_k})_k$ is a Gromov sequence [41, Section 5] whose distances from the origin become arbitrarily large and thus we have convergence to a limit point on the boundary. Thus $L(G)$ is compact by Theorem 6.6.

Note that the compactness of the limit set allows the usual Patterson-Sullivan machinery via weak limits to go through, see [38, Theorem 1]. The Ahlfors regularity of the Patterson–Sullivan measure (and thus its equivalence with Hausdorff and packing measures) follows from a well-known argument using Sullivan's shadow lemma; see [38, Proposition 3] and [27, Section 8]. Finally, since Theorem 7.5 shows that the Patterson–Sullivan measure is supported on $L_r(G)$, the easy direction of the well-known Ahlfors-Thurston-Tukia argument (viz. μ is s-conformal implies that $\Sigma_s(G) = \infty$, see [33, Theorem 8.2.2 and 8.2.3]) shows that the G is of divergence type. This completes the proof. $\square$

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