# GIBBS AND EQUILIBRIUM MEASURES FOR ELLIPTIC FUNCTIONS

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Abstract. Because of its double periodicity, each elliptic function canonically induces a holomorphic dynamical system on a punctured torus. We introduce on this torus a class of summable potentials. With each such potential associated is the corresponding transfer (Perron-Frobenius-Ruelle) operator. The existence and uniquenss of "Gibbs states" and equilibrium states of these potentials are proved. This is done by a careful analysis of the transfer operator which requires a good control of all inverse branches. As an application a version of Bowen's formula for expanding elliptic maps is obtained.

## 1. Introduction

We consider an arbitrary non-constant elliptic function $F : \mathbb{C} \to \hat{\mathbb{C}}$. This function is periodic with respect to a lattice $\Lambda$. Denote by $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ the canonical projection from $\mathbb{C}$ to the torus $\mathcal{T} = \mathbb{C}/\Lambda$. Now the map $F$ naturally projects down to a holomorphic map $f : \mathcal{T} \setminus \pi(F^{-1}(\infty)) \to \mathcal{T}$ by means of the semi-conjugacy $\pi$ so that $\pi \circ F = f \circ \pi$. This dynamical system $f$ is a natural object to study and is interesting itself. In addition, with its help we obtain valuable information about the dynamics and the geometry of the Julia set of the intial map $F : \mathbb{C} \to \hat{\mathbb{C}}$.

We introduce in the Section 3, on the torus $\mathcal{T}$, a class of summable potentials. With each such potential associated is the corresponding transfer operator, which is represented as a sum of an infinite series. The right natural choice of the class of our summable potentials $\varphi$ makes this series converge, and the represented by it transfer operator $\mathcal{L}_\varphi$ acts continuously on the Banach space

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of continuous functions on the torus $\mathcal{T}$. For every $x \in \mathcal{T}$, put

$$\begin{aligned} P(x,\varphi) &= \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{L}_\varphi^n 1(x) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \sum_{f^n(y)=x} \exp\left\{\varphi(y) + \varphi(f(y)) + ... + \varphi(f^{n-1}(y))\right\} . \end{aligned}$$

The main result of our paper is this.

**Theorem 1.1.** *Let $\varphi$ be an summable potential such that $\sup \varphi < \sup_{x\in\mathcal{T}} P(x,\varphi)$. Then:*

(1) *The limit*

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_\varphi^n 1(x)$$

*exists and is independend of $x \in \mathcal{T}$. It is called the topological pressure of the potential $\varphi$.*

(2) *There is a unique $\exp\{P(\varphi) - \varphi\}$–conformal measure $\nu$ on $\mathcal{T}$ and a unique Gibbs state $\mu$, i.e. a unique $f$-invariant measure that is equivalent with respect to $\nu$. Moreover, both measures are ergodic and supported on the conical limit set.*

(3) *The Radon-Nikodym derivative $h = d\,\mu/d\,\nu$ is continuous (and $\log h \in L^\infty$).*

We want to add that in the case the elliptic function $F$ is expanding the assumption $P(\varphi) > \sup(\varphi)$ is not needed. In this case all the potentials $-t\log|f'|$ are summable and Bowen's formula for the Hausdorff dimension of the Julia set of $f$ (or equivalently of $F$) holds.

The Theorem 1.1 is proven by a detailed analysis of the transfer operator and its decomposition into "bad" and "good" parts. This end requires a careful control of all inverse branches of the map $f$. In order to make the picture more complete, we show that the transfer operator is almost periodic and, consequently, the dynamical system $(f,\mu)$ is metrically exact.

We also show that the Gibbs states coming from Theorem 1.1 are the only equilibrium states for potentials $\varphi$ in the sense of classical variational principle.

## 2. Preliminaries on elliptic functions

An elliptic function is a meromorphic function $F : \mathbb{C} \to \hat{\mathbb{C}}$ which is doubly periodic: there is a lattice $\Lambda = < w_1, w_2 >$, $w_1, w_2 \in \mathbb{C}$ with $\Im\left(\frac{w_1}{w_2}\right) \neq 0$, such that $F(z+\omega) = F(z)$ for every $\omega \in \Lambda$. If $\mathcal{T} = \mathbb{C}/\Lambda$ is the quotient torus and $\pi : \mathbb{C} \to \mathcal{T}$ the canonical projection, then there is a induced map $f_0 : \mathcal{T} \to \hat{\mathbb{C}}$

(defined by $f_0 \circ \pi = F$) which is a finite branched covering map. Let $d$ be the number of critical points of $f_0$ counted with multiplicity.

If $\mathcal{R} = \{t_1 w_1 + t_2 w_2 \; ; \; 0 \le t_1, t_2 < 1\}$ is the basic fundamental parallelogram of $\Lambda$, then $F(\mathcal{R}) = F(\mathbb{C}) = \hat{\mathbb{C}}$. The set of poles is

$$\mathcal{P}_0 = F^{-1}(\infty) = \bigcup_{m,n \in \mathbb{Z}} \left(\mathcal{R} \cap F^{-1}(\infty) + m w_1 + n w_2\right) \, .$$

For every pole $b$ of $F$ let $q_b$ denote its multiplicity.

The main example is the Weierstrass elliptic function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \, .$$

As usual we denote by $\mathcal{F}_F$ the Fatou set which is the set of points $z \in \mathbb{C}$ such that all the iterates of $F$ are defined and form a normal family on a neighborhood of $z$. The Julia set $\mathcal{J}_F$ is the complement of $\mathcal{F}_F$ in $\hat{\mathbb{C}}$. The periodicity of $F$ is reflected in these sets:

$$\mathcal{J}_F + \omega = \mathcal{J}_F \quad and \quad \mathcal{F}_F + \omega = \mathcal{F}_F \quad for \;\; all \;\; w \in \Lambda \, .$$

Therefore, the natural way of studying the dynamics of the elliptic function $F$ is to consider its projection $f$ on the torus $\mathcal{T}$ which is given by semi-conjugation via the projection $\pi$:

$$\begin{array}{ccc} \mathbb{C} \setminus \mathcal{P}_0 & \xrightarrow{F} & \mathbb{C} \\ \big\downarrow \pi & & \big\downarrow \pi \\ \mathcal{T} \setminus \mathcal{P} & \xrightarrow{f} & \mathcal{T} \end{array} \tag{2.1}$$

where $\mathcal{P} = \pi(\mathcal{P}_0)$.

The conical set $\Lambda_c$ is the subset of the Julia set where the dynamics can be nicely rescaled. More precisely, $z \in \Lambda_c$ if there is $r > 0$ and an increasing sequence of integers $n_j \to \infty$ such that $f^{n_j} : U_j \to \mathbb{D}(f^{n_j}(z), r)$ is conformal with bounded distortion, where $U_j$ is the component of $f^{-n_j}(\mathbb{D}(f^{n_j}(z), r))$ that contains $z$.

## 3. Our class of potentials

The transfer operator of a potential $\varphi : \mathcal{T} \to \mathbb{R}$ is, for the moment formally, defined by

$$\mathcal{L}_\varphi \psi(x) = \sum_{y \in f^{-1}(x)} \psi(y) e^{\varphi(y)} \quad . \tag{3.1}$$

This operator is well defined as a positive, continuous and linear operator on the space of continuous functions $C(\mathcal{T})$ if the following condition is satisfied (it appeared in [Wal]): there is $K > 0$ such that

$$\mathcal{L}_\varphi 1(x) = \sum_{y \in f^{-1}(x)} e^{\varphi(y)} \leq K \quad for \ \ all \ \ x \in \mathcal{T} \quad . \tag{3.2}$$

Fix $x_0 \in \pi^{-1}(x)$. Then $y \in f^{-1}(x)$ if and only if there is $\omega \in \Lambda$ such that $f_0(y) = x_0 + \omega$. Therefore,

$$\mathcal{L}_\varphi 1(x) = \sum_{\omega \in \Lambda} \sum_{y \in f_0^{-1}(x_0 + \omega)} e^{\varphi(y)} \quad .$$

If $|\omega|$ is big, then $y \in f_0^{-1}(x_0 + \omega)$ is near a pole $b$ of $f_0 : \mathcal{T} \to \hat{\mathbb{C}}$, where we can write

$$x_0 + \omega = f_0(y) = \frac{G_b(y)}{(y-b)^{q_b}} \tag{3.3}$$

with $G_b$, a holomorphic function defined near $b$ such that $G_b(b) \neq 0$ and where $q_b$ is the multiplicity of the pole $b$. If we compare here with the series $\sum_{\omega \in \Lambda} |x_0 + \omega|^{-(2+\varepsilon_b)}$ convergent for every $\varepsilon_b > 0$, we get the following sufficient condition for the transfer operator to be continuous: there is a constant $C > 0$ such that, still for $y$ near the pole $b$,

$$\exp \varphi(y) \leq C|x_0 + \omega|^{-(2+\varepsilon_b)} = C \left( \frac{|y-b|^{q_b}}{|G_b(y)|} \right)^{2+\varepsilon_b} \quad .$$

It follows that there is a Hölder continuous function $H_b$ defined near the pole $b$ such that

$$\varphi(y) \leq H_b(y) + (2 + \varepsilon_b) q_b \log|y - b| \ \ near \ \ b \, .$$

Later on we will need equality here. So we are lead to the following class of potentials:

**Definition 3.1.** *[Class of Potentials] We will always assume that the potential $\varphi : \mathcal{T} \setminus \mathcal{P}_0 \to \mathbb{R}$ satisfies:*

- **C1:** *$\varphi$ is Hölder continuous on $\mathcal{T} \setminus V(\mathcal{P}_0)$ for any neighborhood $V(\mathcal{P}_0)$ of $\mathcal{P}_0$.*

**C2:** *For every pole $b \in \mathcal{P}_0$ there is $\varepsilon_p > 0$ and a Hölder continuous function $H_b$ such that*

$$\varphi(y) = H_b(y) + (2 + \varepsilon_b) q_b \log |y - b| \;\; near \;\; b \, ,$$

*where $q_b$ is the multiplicity of the pole $b$.*

*Such a potential will be called summable.*

We can now resume the above discussion by:

**Proposition 3.2.** *For every summable potential $\varphi$ the transfer operator $\mathcal{L}_\varphi$ is a well defined positive and continuous operator on the space of continuous functions on the torus $\mathcal{T}$. Since The Julia set $\mathcal{J}_f$ is $f$-invariant, the same is true for the transfer operator $\mathcal{L}_\varphi$ acting on the space $C(\mathcal{J}_f)$ of continuous functions on $\mathcal{J}_f$.*

## 4. Distortion and good inverse branches

We start with some definitions: $\|.\|$ denotes the sup–norm and $\|\varphi\|_E = \sup_{x \in E} |\varphi(x)|$, $E \subset \mathcal{T}$. We will denote as usual by $mod(A)$ the modulus of an annulus $A$. For a simply connected bounded domain $U$ we denote

$$Distortion(U) = R/r,$$

where $R = \inf\{R > 0 \; U \subset \mathbb{D}(z, R)\}$ and $r = \sup\{r > 0 \; U \supset \mathbb{D}(z, r)\}$.

4.1. **Selecting good inverse branches.** Fix $m \geq 1$ an integer and $U \subset \mathcal{T}$ a topological disk that does not contain any critical value of $f^m$. In our applications we can always assume that

**a:** $U$ has a lift $U_0 \subset \mathbb{C}$, i.e. $\pi_{|U_0} : U_0 \to U$ conformal, with $U_0 \subset \mathbb{D}(0, r)$ for a fixed radius $r > 0$, and

**b:** the domain $U$ has a piecewise smooth boundary.

In this situation all the inverse branches

$$h_j^{(m)} : U \to U_j^{(m)} \quad ; \;\; j \in I_m \, ,$$

of $f^m$ are well defined. Before taking further inverse branches and in order to obtain the distortion control, we first have to replace the image domains $U_j^{(m)} = h_j^{(m)}(U)$ by bigger once as follows:

**Lemma 4.1.** *There are constants $K \geq 1$, $\kappa \in \mathbb{N}$, with $\kappa$ depending only on (the fixed) radius $r > 0$, and there are simply connected domains $V_j^{(m)}$, $j \in I_m$, such that for all $j \in I_m$ the following hold:*

(1) $\overline{U_j^{(m)}} \subset V_j^{(m)}$,

(2) *mod* $\left(V_j^{(m)} \setminus \overline{U_j^{(m)}}\right) \geq \frac{1}{K}$,

(3) *Distortion*$(U_j^{(m)}) \leq K$,

(4) *the family* $\{V_j^{(m)},\ j \in I_m\}$ *is of multiplicity at most* $\kappa$*, i.e. any point* $z \in \mathcal{T}$ *is in at most* $\kappa$ *sets* $V_j^{(m)}$.

*Proof.* Recall that $f = \pi \circ f_0$, where $f_0 : \mathcal{T} \setminus \mathcal{P} \to \mathbb{C}$ and $\pi : \mathbb{C} \to \mathcal{T}$ is the natural projection, and that $f_0$ (respectively $f$) do have at most $d$ critical points counted with multiplicities. Let $U_0$ be a lift of $U$ to $\mathbb{C}$ coming from item a) and suppose that $\overline{U_0} \subset V_0 = \mathbb{D}(0, r)$. For $\omega \subset \Lambda$, we write $U_\omega = U_0 + \omega$ and $V_\omega = V_0 + \omega$. Clearly, there is $\kappa \in \mathbb{N}$ (depending only on the fixed $r > 0$) such that the family $\{V_\omega,\ w \in \Lambda\}$ is of multiplicity at most $\kappa$.

A map $h_j^{(m)} : U \to U_j^{(m)}$, $j \in I_m$, is the composition of an inverse branch of $\pi$, say $\pi_{\omega_j}^{-1} : U \to U_{\omega_j}$, with one of the inverse branches $g_j^{-1}$ of $g = f_0 \circ f^{m-1}$.

If $V_{\omega_j}$ is without critical values of $g$, then $g_j^{-1}$ is well defined on $V_{\omega_j}$ and it suffices to put $V_j^{(m)} = g_j^{-1}(V_{\omega_j})$. A second case which is also easy to handle is when critical points of $g$ do belong to $V_{\omega_j}$ but not to $\overline{U_{\omega_j}}$. It suffices then to shrink $V_{\omega_j}$ in order to get a simply connected domain that still contains $\overline{U_{\omega_j}}$ but no critical value of $g$. Then we can proceed as before and define $V_j^{(m)} = g_j^{-1}(V_{\omega_j})$. Notice that such a new choice of $V_{\omega_j}$ is only necessary in finitely many cases, the map $g$ having only finitely many critical values.

Let us consider the remaining third case, namely if there is a critical value of $g$ in the boundary of $U_{\omega_j}$. This means that for some $k \in 0, ..., m-1$, the set $\overline{f^k(U_j^{(m)})}$ does contain a critical points of $f$. Choose $k$ to be minimal with this property. Then $f^k$ is without critical point in $\overline{U_j^{(m)}}$. We now can choose a simply connected domain $V$ such that $\overline{f^k(U_j^{(m)})} \subset V$ and such that the inverse of the map $f^k : U_j^{(m)} \to f^k(U_j^{(m)})$ does extend conromally to $V$. The image of $V$ under this inverse gives the set $V_j^{(m)}$ we look for (in case $k = 0$ one has $V_j^{(m)} = V$).

Remark that in this third case only finitely many sets $V$ are chosen. Indeed, this is due to the fact that the map $f$ has only finitely many critical points and since for every $k \in \{0, ..., m-1\}$, the multiplicity of the family $\{f^k(\overline{U}_j^{(m)});\ j \in I_m\}$ is bounded above by $\kappa$. This, together with Koebe's Distortion Theorem, immediatley prove the assertions (2) and (3). In order to obtain (4) one possibly has to shrink the domains $V$ such that

$$f_0 \circ f^{m-1-k}(V) \subset V_{\omega_j} \ .$$

The assertion follows then because the $V_\omega$ are of multiplicity bounded by $\kappa$. $\square$

**Lemma 4.2.** *In the previous setting, there exist, for every $n \geq m$, holomorphic inverse branches*

$$h_i^{(n)} : U \to U_i^{(n)} \subset \mathcal{T} \quad ; \quad i \in I_n \ ,$$

*of $f^n$ having the following properties:*

(1) *For any $i \in I_{n+1}$ there is $j \in I_n$ such that $f \circ h_i^{(n+1)} = h_j^{(n)}$.*
(2) *There is $K > 0$ such that, for all $n \geq m$ and $i \in I_n$,*

$$Distortion(U_i^{(n)}) \leq K \quad and$$

$$\frac{|(h_i^{(n)} \circ f^m)'(x)|}{|(h_i^{(n)} \circ f^m)'(x')|} \leq K \quad for\ all\ \ x, x' \in f^{n-m}(U_i^{(n)}) \ .$$

(3) *Fix $x \in U$ arbitrary. For $n > m$, let $H_n(x)$ be the set of $y \in f^{-n}(x)$ such that there exists $j \in I_{n-1}$ with $h_j^{(n-1)}(x) = f(y)$ but $h_j^{(n)}(x) \neq y$ for all $j \in I_n$. Then*

$$\sharp H_n(x) \leq \kappa d \quad for\ \ all\ \ n > m \ .$$

*Proof.* For $n = m$ everything follows from Lemma 4.1. The inductive step goes as follows:

Let $n \geq m$ and suppose that the inverse branches $h_i^{(n)} : U \to U_i^{(n)}$, $n \in I_n$, of $f^n$ are constructed such that every $h_i^{(n)}$ is of the form $\Psi_{i,j} \circ h_j^{(m)}$ with $\Psi : V_j^{(m)} \to V_i^{(n)}$, $\overline{U_i^{(n)}} \subset V_i^{(n)}$, and $f^{n-m} \circ \Psi_{i,j} = id$. Write then

$$f^{-1}(V_j^{(n)}) = \bigcup_i V_{i,j}$$

with $V_{i,j}$, the connected components of $f^{-1}(V_j^{(n)})$. Clearly the family built by all these sets $V_{i,j}$ is of multiplicity at most $\kappa$. Since $f$ has $d$ critical points, it follows that at most $\kappa d$ sets $V_{i,j}$ can contain critical points. Therefore, all but at most $\kappa d$ inverse branches $\Psi_{i,j} : V_j^{(n)} \to V_{i,j}$ of $f$ defined on the sets $V_{i,j}$, $j \in I_n$, do exist. The mappings $h_j^{(n+1)}$ we look for are relabelling of the $\Psi_{i,j} \circ h_j^{(n)}$. Remark that each a map $h_j^{(n+1)}$ again is of the form $\Psi \circ h_k^{(m)}$ with $\Psi : V_j^{(m)} \to V_i^{(n+1)}$. From Koebe's Theorem and the distortion control of the $U_k^{(m)}$ in Lemma 4.1 we finally get (2). The proof is complete. $\square$

Among these inverse branches, only those that shrink exponentially will be good for our applications. The others have to be controlled. That is the aim of the next Lemma where we use the previous notation again.

**Lemma 4.3.** *Let $0 < \lambda < 1$, $E_m = \emptyset$ and, for $n > m$, let $E_n$ be the set of all $j \in I_n$ such that* $\operatorname{diam}\left(U_j^{(n)}\right) > K\lambda^{\frac{n-m}{2}}$. *Then*

$$\sharp E_n \leq \lambda^{-(n-m)} \quad for\ all\ n \geq m.$$

*Proof.* The distortion control in the item (2) of Lemma 4.2 gives

$$l_2(U_j^{(n)}) \geq \frac{1}{K^2}\operatorname{diam}(U_j^{(n)})^2 \quad for\ all\ \ j \in I_n\ .$$

The domains $U_j^{(n)}$, $j \in I_n$, being disjoint

$$1 \geq l_2\left(\bigcup_{j\in E_n} U_j^{(n)}\right) = \sum_{j\in E_n} l_2(U_j^{(n)}) \geq \lambda^{n-m}\sharp E_n \quad .$$

Therefore $\sharp E_n \leq \lambda^{-(n-m)}$. □

The index set $J_n \subset I_n$ corresponding to the exponential shrinking branches according to the previous Lemma is defined inductively as follows: set $J_m = I_m$ (because in Lemma 4.3 the set $E_m = \emptyset$), suppose that $J_n \subset I_n$ is already defined for some $n \geq m$ and put then

$$J_{n+1} = \left\{j \in I_{n+1}\ ;\ f \circ h_j^{(n+1)} = h_i^{(n)} \ \ for\ some\ i \in J_n\right\} \setminus E_{n+1}\ .$$

Note that for any $j \in I_n$ there is $(j_n, j_{n-1}, ..., j_m)$ with $j = j_n$ and such that $f \circ h_{j_{k+1}}^{(k+1)} = h_{j_k}^{(k)}$, $k = m, ..., n-1$. Then $j \in I_n \setminus J_n$ equivalently means that there is some $k \in \{m+1, ..., n\}$ such that $j_{k-1} \in J_{k-1}$ but $j_k \in E_k$.

4.2. **Distortion estimation.** Along exponentially shrinking inverse branches, the variation of the function

$$S_n\varphi = \sum_{j=0}^{n-1} \varphi \circ f^j \quad , \quad n \geq 1\ , \tag{4.1}$$

can be controlled uniformly as follows.

**Lemma 4.4.** *Let $0 < \lambda < 1$, $m \geq 1$ and $U$ be a topological disk in $\mathcal{T}$ that does not contain any critical value of $f^m$. Then there is $A > 0$ (depending on $\lambda, m$ and the Hölder constants of $\varphi$ but not on $U$) such that for all $x, x' \in U$ and all $j \in J_n$, $n \geq m$, we have*

$$\left|S_n\varphi\left(h_j^{(n)}(x)\right) - S_n\varphi\left(h_j^{(n)}(x')\right)\right| \leq A \quad .$$

*Proof.* Looking at the structure of the inverse images $f^{-1}(z)$ (see the paragraph following formula (3.1)) one can choose open neighborhoods $V_1 = \bigcup_{b\in\mathcal{P}_0} D(b,r) \supset V_2$ of the poles $\mathcal{P}_0$ such that, whenever $f_I^{-N}$ is an inverse branch of $f^N$ defined on some domain $\Omega \subset \mathcal{T}$, then

$$f_I^{-N}(\Omega) \subset \mathcal{T}\setminus V_2 \quad \mathit{or} \quad f_I^{-N}(\Omega) \subset D(b,r) \;\text{ for some pole }\; b \in \mathcal{P}_0 \;.$$

Let $c, \alpha$ be Hölder constants that are common to $\varphi_{|\mathcal{T}\setminus V_2}$ and to the $H_b$ functions on $D(b,r)$, $b\in\mathcal{P}_0$ (cf. condition (C2) of the definition of the class of potentials). These constants are independent of $U$.

Call $x_n = h_j^{(n)}(x)$ and $x_{n-i} = f^i(x_n)$, $0\le i\le n$ and define analogously points $x_i'$. Then the definition of the sets $J_n$ yields

$$|x_i - x_i'| \le K\lambda^{\frac{i-m}{2}} = K^*\lambda^{\frac{i}{2}} \;. \tag{4.2}$$

Consider first the case when $x_i, x_i'$ are in one of the discs $D(b,r)$, $b\in\mathcal{P}_0$. Then

$$\begin{aligned} |\varphi(x_i)-\varphi(x_i')| &\le |H_b(x_i)-H_b(x_i')| + (2+\varepsilon_b)q_b\left|\log(|x_i-b|) - \log(|x_i'-b|)\right| \\ &\le c|x_i-x_i'|^\alpha + (2+\varepsilon_b)\log\left(\tfrac{|x_i-b|}{|x_i'-b|}\right)^{q_b} \;. \end{aligned}$$

Now, with an appropriate $\omega\in\Lambda$

$$\left(\frac{|x_i-b|}{|x_i'-b|}\right)^{q_b} = \frac{|G_b(x_i)|}{|G_b(x_i')|}\frac{|x_{i-1}'+\omega|}{|x_{i-1}+\omega|},$$

where we may suppose that $G_b$ is holomorphic and $G_b\neq 0$ on $D(b,2r)$, cf. (3.3). Clearly

$$\log\left(\frac{|G_b(x_i)|}{|G_b(x_i')|}\right) \le \log\left(1+\left|\frac{|G_b(x_i)|-|G_b(x_i')|}{|G_b(x_i')|}\right|\right) \le c_b|x_i-x_i'|$$

and

$$\log\left(\frac{|x_{i-1}'+\omega|}{|x_{i-1}+\omega|}\right) \le \frac{|x_{i-1}'-x_{i-1}|}{|x_{i-1}+\omega|} \le |x_{i-1}'-x_{i-1}| \;.$$

Altogether we have, in this case, the estimation:

$$\begin{aligned} |\varphi(x_i)-\varphi(x_i')| &\le c|x_i-x_i'|^\alpha + (2+\varepsilon_b)\left(c_b|x_i-x_i'| + |x_{i-1}-x_{i-1}'|\right) \\ &\le C_b\lambda^{\frac{\alpha}{2}i} \;. \end{aligned}$$

In the other case, namely $x_i, x_i' \in \mathcal{T}\setminus V_2$, one gets

$$|\varphi(x_i)-\varphi(x_i')| \le c|x_i-x_i'|^\alpha \le cK^{*\alpha}\lambda^{\frac{\alpha}{2}i} \;.$$

To conclude this proof one just has to add up these estimations for all $0 < i \le n$. $\square$

Later on we will need an asymptotically sharper version of Lemma 4.4.

**Lemma 4.4'.** *Let $0 < \lambda < 1$, $m \geq 1$ and $U$ be a topological disk in $\mathcal{T}$ that does not contain any critical value of $f^m$. Then, for every $\varepsilon > 0$, there is $\delta > 0$ such that*

$$\left| S_n\varphi\left(h_j^{(n)}(x)\right) - S_n\varphi\left(h_j^{(n)}(x')\right)\right| \leq \varepsilon$$

*for all $j \in J_n$, $n \geq 1$ and for all $x, x' \in U$ with $\sigma_U(x, x') < \delta$ where*

$$\sigma_U(x, x') = \inf\{|\gamma|;\ \gamma \text{ path in } U \text{ joining } x \text{ and } x'\}$$

*is the internal chord arc distance in $U$.*

*Proof.* We will prove the following claim from which the statement follows. Indeed, it suffices then to inject this new estimation in the proof of Lemma 4.4.

**Claim 4.5.** *For every $\varepsilon > 0$ there is $\delta > 0$ such that, whenever $x, x' \in U$ with $\sigma_U(x, x') < \delta$,*

$$|h_j^{(n)}(x) - h_j^{(n)}(x')| \leq \varepsilon \operatorname{diam}(U_j^{(n)})$$

*for all $j \in J_n$ and $n \geq m$.*

From the construction of the inverse branches and Lemma 4.1 we see that each branch $h_j^{(n)}$ is the composition of two mappings, the first one $g_1$ being an inverse branch of some $f^k$ defined on $U$ and the second one $g_2$, an inverse branch of $f^{n-k}$ this time defined on $U' = f^{n-k}(U_j^{(n)})$. As is explained in the proof of Lemma 4.1, the map $g_2$ is defined on a larger domain $V$ such that $mod(V \setminus \overline{U'}) \geq 1/K$ which means that Koebe's Distortion Theorem applies to all these maps $g_2$. Moreover, all the possible maps $g_1$ are taken from a finite set of conformal maps defined on the domain $U$. Since the boundaries of $U$ and $g_1(U)$ are locally connected, the maps $g_1$ are uniformly continuous with respect to the internal chordal metrics on $U$ and $g_1(U)$. Now taking compositions $g_2 \circ g_1$, the claim thus follows. □

## 5. Conformal measures and Pressure

As we have seen in the previous section, if $\varphi$ is a potential from our class then the transfer operator $\mathcal{L}_\varphi$ is well defined as a continuous operator of the space of continuous functions on $\mathcal{T}$. It follows that the map

$$\mu \mapsto \frac{\mathcal{L}_\varphi^* \mu}{\int \mathcal{L}_\varphi 1\, d\mu}$$

is also continuous on the space of probability measures $\mathcal{M}(\mathcal{T})$. The Schauder-Tychonoff fixed point theorem applies and gives a measure $\nu \in \mathcal{M}(\mathcal{T})$ such that

$$\mathcal{L}_\varphi^* \nu = \rho\nu \quad with \quad \rho = \int \mathcal{L}_\varphi 1 \, d\nu \ .$$

The first equality means that the Radon-Nikodym derivative is given by the formula

$$\frac{d\,\nu \circ f}{d\,\nu} = \rho e^{-\varphi} \ .$$

Such a measure is called $\rho e^{-\varphi}$– *conformal.*

Denker and Urbański [DU1] gave an explicit construction of conformal measures from which precise information on the number $\rho$ follows. Indeed, conformal measures constructions in general involve Poincaré series

$$\Sigma(\alpha, x) = \sum_{n=1}^{\infty} e^{-n\alpha} \mathcal{L}_\varphi^n 1(x) = \sum_{n=1}^{\infty} \sum_{y \in f^{-n}(x)} \exp\Big(S_n\varphi(x) - n\alpha\Big) \ .$$

For such a series there is a transition parameter:

$$P(x, \varphi) = \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{L}_\varphi^n 1(x) \ .$$

It signifies that $\Sigma(\alpha, x)$ converges for $\alpha > P(x,\varphi)$ and diverges if $\alpha < P(x,\varphi)$. Usually $P(x,\varphi)$ is also called the topological pressure of $\varphi$ at $x$. Notice that $P(x,\varphi)$ is finite which directly follows from the inequality (3.2). Now, if one applies the Denker-Urbański method to our situation, then one obtains ([DU1]):

**Proposition 5.1.** *Let $x \in \mathcal{T}$. There then exists a $\rho e^{-\varphi}$–conformal measure $\nu$ with $\log(\rho) = P(x,\varphi)$. Moreover, this measure $\nu$ is without atoms provided that $P(x,\varphi) > \sup\varphi$ .*

## 6. Existence of Gibbs states

6.1. **Decomposition of the transfer operator.** In this section we adapt the arguments of [DU2]. They are based on the hypothesis

$$\sup\varphi < \sup_{x \in \mathcal{T}} P(x,\varphi) = \sup_{x \in \mathcal{T}} \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{L}_\varphi^n 1(x) \tag{6.1}$$

to establish the existence of an Gibbs state.

**Remark 6.1.** *Notice that if the map $F$ is expanding (see Appendix) then all the inverse branches are good and all the results of the following sections are true without the hypothesis (6.1)*

Fix a point $x_0 \in \mathcal{T}$ for which

$$\log(\sigma) := \sup \varphi - P(x_0, \varphi) < 0$$

and let $\nu$ be a $\rho e^{-\varphi}$–conformal measure with $\rho = \exp(P(x_0, \varphi))$ (which exists and is without atoms because of the Denker-Urbański construction).

Fix $\sigma < \lambda < 1$, denote

$$\alpha = \frac{\mu d}{1-\sigma} + \frac{1}{\lambda - \sigma}$$

and fix $m \geq 1$ such that $\alpha \sigma^m < 1$. Choose then a topological disk $U \subset \mathcal{T}$ as in Section 4. So, in particular, $U$ does not contain any critical value of $f^m$ and this disk can be chosen to be dense in $\mathcal{T}$ with $U \cap \mathcal{J}_f \neq \emptyset$ (which implies $\nu(U) > 0$). Lemmas 4.2 and 4.3 on good inverse branches apply on $U$ and allow us to decompose the normalized transfer operator

$$\mathcal{N}_\varphi = \rho^{-1} \mathcal{L}_\varphi = e^{-c} \mathcal{L}_\varphi$$

into

$$\mathcal{N}_\varphi^n = e^{-nc} \mathcal{L}_\varphi^n = \mathcal{G}_\varphi^n + \mathcal{A}_\varphi^n + \mathcal{B}_\varphi^n \quad , \quad n \geq m \ , \tag{6.2}$$

where

$$\begin{aligned} \mathcal{G}_\varphi^n \psi(x) &= \sum_{j \in J_n} \psi\Big(h_j^{(n)}(x)\Big) \exp\Big\{S_n\varphi(h_j^{(n)}(x)) - nc\Big\} \ , \\ \mathcal{B}_\varphi^n \psi(x) &= \sum_{j \in I_n \setminus J_n} \psi\Big(h_j^{(n)}(x)\Big) \exp\Big\{S_\varphi(h_j^{(n)}(x)) - nc\Big\} \end{aligned}$$

and

$$\mathcal{A}_\varphi^n = \mathcal{N}_\varphi^n - \mathcal{G}_\varphi^n - \mathcal{B}_\varphi^n \ .$$

6.2. **Behavior of the good part.** Let us first make the following observations on the good part of the operator:

**Lemma 6.2.** *There is $c_1 \geq 1$ such that*

$$\mathcal{G}_\varphi^n 1(x) \leq c_1 \mathcal{G}_\varphi^n 1(x') \quad \textit{for all } x, x' \in U \textit{ and } n \geq m \ . \tag{6.3}$$

*Furthermore,*

$$\mathcal{G}_\varphi^n 1(x) \leq c_1 \quad \textit{for all } x \in U \textit{ and } n \geq m \ . \tag{6.4}$$

*Proof.* Assertion (6.3) immediately follows from Lemma 4.4. The second assertion follows from the first one and from the inequality

$$\int_U \mathcal{G}_\varphi^n 1 \, d\nu \leq \int \mathcal{N}_\varphi^n 1 \, d\nu = \int 1 \, d\nu = 1 \ ,$$

since this implies the existence of a point $x_0 \in U$ for which $\mathcal{G}_\varphi^n 1(x_0) \leq \frac{1}{\nu(U)}$. $\square$

6.3. **Estimations for the bad parts.** We first handle the part corresponding to the preimages that cannot be reached by inverse branches.

**Lemma 6.3.** *For every $n \geq 1$,*

$$\|\mathcal{A}_\varphi^n 1\|_U \leq \mu d \sigma^{m+1} \sum_{k=0}^{n-m-1} \sigma^k \|\mathcal{N}_\varphi^{n-m-1-k} 1\| \, .$$

*Proof.* With the notations from Lemma 4.2,

$$\begin{aligned} \mathcal{A}_\varphi^n(\psi)(x) &= \mathcal{N}_\varphi^n(\psi)(x) - \mathcal{G}_\varphi^n(\psi)(x) - \mathcal{B}_\varphi^n(\psi)(x) \\ &= \sum_{k=m+1}^{n} \sum_{y \in H_k(x)} \exp\{S_k\varphi(y) - kc\} \mathcal{N}_\varphi^{n-k}(\psi)(y) \ . \end{aligned}$$

Therefore, for any $n \geq m$ and $x \in U$,

$$\mathcal{A}_\varphi^n 1(x) \leq \sum_{k=m+1}^{n} \sharp H_k(x) \sigma^k \|\mathcal{N}_\varphi^{n-k} 1\| \leq \mu d \sigma^{m+1} \sum_{k=0}^{n-m-1} \sigma^k \|\mathcal{N}_\varphi^{n-m-1-k} 1\|$$

and we are done. □

The corresponding statement for the remaining part is:

**Lemma 6.4.** *For every $n > m$,*

$$\|\mathcal{B}_\varphi^n 1\|_U \leq \frac{\sigma^{m+1}}{\lambda} \sum_{k=0}^{n-m-1} \left(\frac{\sigma}{\lambda}\right)^k \|\mathcal{N}_\varphi^{n-m-1-k} 1\| \, .$$

*Proof.* The estimation goes as follows. Let $n > m$ and $x \in U$. Then, using Lemma 4.3, we obtain.

$$\begin{aligned} \mathcal{B}_\varphi^n 1(x) &= \sum_{k=m+1}^{n} \sum_{\substack{(j = j_n, \ldots, j_m) \\ j_{k-1} \in J_{k-1} \textit{ and } j_k \in E_k}} \exp\{S_n\varphi(h_j^{(n)}(x)) - nc\} \\ &\leq \sum_{k=m+1}^{n} \sum_{\substack{i \in E_k\ , \\ y = h_i^{(k)}(x)}} \exp\{S_k\varphi(y) - kc\} \mathcal{N}_\varphi^{n-k}(y) \\ &\leq \sum_{k=m+1}^{n} \sharp E_k \sigma^k \|\mathcal{N}_\varphi^{n-k}\| \leq \lambda^m \sum_{k=m+1}^{n} \left(\frac{\sigma}{\lambda}\right)^k \|\mathcal{N}_\varphi^{n-k}\| \ . \end{aligned}$$

□

6.4. **The normalized operator is uniformly bounded.** .

Recall that $\sigma < \lambda < 1$ and that $m \geq 1$ has been chosen such that $\alpha\sigma^m < 1$ where $\alpha = \frac{\mu d}{1-\sigma} + \frac{1}{\lambda - \sigma}$. We further define

$$c_2 = \max\left\{ \frac{c_1}{1-\alpha\sigma^{m+1}} \; , \; \|\mathcal{N}_\varphi^k 1\| \; ; \; 0 \leq k \leq m \right\},$$

where $c_1$ is the constant from Lemma 6.2.

**Proposition 6.5.** *$\|\mathcal{N}_\varphi^n 1\| \leq c_2$ for every $n \geq 0$.*

*Proof.* We proceed again by induction. Let $n > m$ and suppose $\|\mathcal{N}_\varphi^k 1\| \leq c_2$ for $k = 0, 1, ..., n-1$. Lemmas 6.3 and 6.4 give, for every $x \in U$,

$$\mathcal{A}_\varphi^n 1(x) + \mathcal{B}_\varphi^n 1(x) \leq c_2 \left\{ \mu d \frac{\sigma^{m+1}}{1-\sigma} + \frac{1}{\lambda - \sigma}\sigma^{m+1} \right\} = \alpha\sigma^{m+1} c_2 \; .$$

Therefore

$$\mathcal{N}_\varphi^n(x) = \mathcal{G}_\varphi^n 1(x) + \mathcal{A}_\varphi^n 1(x) + \mathcal{B}_\varphi^n 1(x) \leq c_1 + \alpha\sigma^{m+1} c_2 \leq c_2$$

for every $x \in U$, and the proposition follows by density of $U$ in $\mathcal{T}$ and continuity of $\mathcal{N}_\varphi^n 1$. $\square$

**Remark 6.6.** *Once found this upper bound $c_2$ for the operators $\mathcal{N}_\varphi^n$, we may, in posteriori, suppose in the sequel that $m \geq 1$ has been chosen so big that $\alpha\sigma^{m+1}c_2$ is arbitrarily small. This means that*

$$\|\mathcal{N}_\varphi^n 1 - \mathcal{G}_\varphi^n 1\|_U = \|\mathcal{A}_\varphi^n 1 + \mathcal{B}_\varphi^n 1\|_U \leq \alpha\sigma^{m+1} c_2$$

*is arbitrarily small.*

**Proposition 6.7.** *There is a constant $c_3 > 0$ such that*

$$\mathcal{N}_\varphi^n 1(x) \geq c_3 \quad for \;\; all \;\; n \geq 1 \;\; and \;\; x \in \mathcal{T} \; .$$

*Proof.* We may suppose that

$$\|\mathcal{A}_\varphi^n 1 + \mathcal{B}_\varphi^n 1\|_U \leq \frac{1}{4} \quad for \;\; all \;\; n > m \; . \tag{6.5}$$

Lemma 6.2 says that $c_1 \mathcal{G}_\varphi^n 1(x) \geq \mathcal{G}_\varphi^n(x')$ for all $n > m$ and all $x, x' \in U$. On the other hand, $\int \mathcal{N}_\varphi^n 1(x)\, d\nu(x) = 1$ and so $\mathcal{N}_\varphi^n 1(x) \geq 1$ for some $x \in \mathcal{T}$. Since $\overline{U} = \mathcal{T}$ and $\mathcal{N}_\varphi^n 1$ is continuous, there is $x' \in U$ such that $\mathcal{N}_\varphi^n 1(x') \geq 1/2$. Therefore,

$$\begin{aligned} \mathcal{N}_\varphi^n 1(x) &\geq \mathcal{G}_\varphi^n 1(x) \geq \frac{1}{c_1} \mathcal{G}_\varphi^n 1(x') \\ &= \frac{1}{c_1}\Big( \mathcal{N}_\varphi^n 1(x') - \big( \mathcal{A}_\varphi^n 1(x') + \mathcal{B}_\varphi^n 1(x') \big) \Big) \geq \frac{1}{4c_1} \end{aligned}$$

for all $x \in U$ and, again by density and continuity, also for all $x \in \mathcal{J}_f$. The constant we look for is

$$c_3 = \min\left\{\frac{1}{4c_1}\ ,\ \inf_{x\in\mathcal{J}_f} \mathcal{N}_\varphi^k 1(x)\ ;\ k = 0, ..., m\right\} .$$

□

6.5. **Existence of the Gibbs state.**

**Theorem 6.8.** *Let $\varphi$ be a potential from our class, $x_0 \in \mathcal{T}$ such that $\log(\rho) = P(x_0, \varphi) > \sup(\varphi)$ and $\nu$ a $\rho e^\varphi$–conformal measure. Then there exists a $f$–invariant measure $\mu$ which is absolutely continuous with respect to $\nu$. Moreover, the density function $h = d\mu/d\nu$ satisfies $c_3 \le h(x) \le c_2$ for every $x \in \mathcal{J}_f$.*

*Proof.* We have to construct a normalized fixed point $h$ of $\mathcal{N}_\varphi$. Consider first

$$\tilde{h}(x) = \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{N}_\varphi^k 1(x) \quad , \quad x \in \mathcal{J}_f .$$

Clearly, if $h_n = \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathcal{N}_\varphi^k 1$, then

$$\mathcal{N}_\varphi(h_n) = h_n + \frac{1}{n}\left(\mathcal{N}_\varphi^{n+1} 1 - \mathcal{N}_\varphi 1\right) .$$

Fix $x \in \mathcal{J}_f$ and choose $n_j \to \infty$ such that $h_{n_j}(x) \to \tilde{h}(x)$. Then $\mathcal{N}_\varphi(h_{n_j})(x) \to \tilde{h}(x)$.

Let $\varepsilon > 0$ and $j \ge j_0$ such that $\mathcal{N}_\varphi(h_{n_j})(x) - \tilde{h}(x) < \varepsilon$. The series

$$\sum_{y\in f^{-1}(x)} e^{\varphi(y)-c} = \mathcal{N}_\varphi 1(x)$$

being convergent and $c_3 \le h_{n_j}, \tilde{h} \le c_2$, for all $j$, there are $y_1, ..., y_N \in f^{-1}(x)$ such that

$$\left|\mathcal{N}_\varphi(\tilde{h})(x) - \sum_{k=1}^{N} \tilde{h}(y_k)e^{\varphi(y_k)-c}\right| < \varepsilon .$$

On the other hand,

$$\varepsilon > \mathcal{N}_\varphi(h_{n_j})(x) - \tilde{h}(x) > \sum_{k=1}^{N} h_{n_j}(y_k)e^{\varphi(y_k)-c} - \tilde{h}(x) .$$

Let $j_1 \ge j_0$ such that for all $j \ge j_1$ and $k = 1, ..., N$

$$h_{n_j}(y_k)e^{\varphi(y_k)-c} \ge \tilde{h}(y_k)e^{\varphi(y_k)-c} - \varepsilon/N .$$

It follows that

$$2\varepsilon > \sum_{k=1}^{N} \tilde h(y_k) e^{\varphi(y_k)-c} - \tilde h(x) \geq \mathcal{N}_\varphi(\tilde h)(x) - \tilde h(x) - \varepsilon \; .$$

Therefore $\tilde h(x) \geq \mathcal{N}_\varphi(\tilde h)(x)$ for all $x \in \mathcal{J}_f$. Equality follows from $\int \mathcal{N}_\varphi(\tilde h)\, d\nu = \int \tilde h \, d\nu$. Put now

$$h = \tilde h \Big/ \int \tilde h \, d\nu \; .$$

Then $d\mu = h\, d\nu$ defines an $f$–invariant probability measure having all the required properties. □

6.6. **Pressure.**

**Proposition 6.9.** *Let $\varphi$ be a potential of our class such that* $\sup(\varphi) < \|P(.,\varphi)\|$. *Then $x \mapsto P(x,\varphi)$ is constant on $\mathcal{T}$. The common value*

$$P(\varphi) = P(x,\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_\varphi^n 1(x) \quad \textit{for all} \;\; x \in \mathcal{J}_f$$

*will be called the topological pressure of $\varphi$. If $m \in \mathcal{M}(\mathcal{T})$ is any $te^{-\varphi}$–conformal measure, then* $\log(t) = P(\varphi)$.

*Proof.* Let $x_0$ be a point such that $P(x_0,\varphi) > \sup(\varphi)$. Then we know from Proposition 6.5 and 6.7 that

$$c_3 \leq \rho^{-n} \mathcal{L}_\varphi^n 1(x) \leq c_2 \quad \textit{for all} \;\; n \geq 1 \;\; \textit{and} \;\; x \in \mathcal{T} \tag{6.6}$$

where $\rho = \exp(P(x_0,\varphi))$. Therefore $x \mapsto P(x,\varphi)$ is constant on $\mathcal{T}$ equal to say $P(\varphi)$.

Consider now $m$ any $te^{\varphi}$–conformal measure. Then $\mathcal{L}_\varphi^* m = tm$. Iterating and integrating this equation gives

$$\log(t) = \frac{1}{n} \log \int \mathcal{L}_\varphi^n 1 \, dm \quad \textit{for all} \;\; n \geq 1 \; .$$

Applying (6.6) gives $t = \rho = e^{P(\varphi)}$. □

## 7. Uniqueness and ergodicity of Gibbs states

Right now we know that for any $\rho e^{-\varphi}$–conformal measure the factor of conformality $\rho = e^{P(\varphi)}$.

**Theorem 7.1.** *Let $\varphi$ be a potential from our class. Then there exists a unique probability measure $\nu$ that is $e^{P(\varphi)-\varphi}$–conformal. Moreover, this measure $\nu$ is ergodic and supported on the conical set: $\nu(\Lambda_c) = 1$.*

**Remark 7.2.** *Any invariant measure $\mu$ that is absolutely continuous with respect to the unique $e^{P(\varphi)-\varphi}$–conformal measure $\nu$ is also ergodic. Therefore there is only a unique ergodic invariant measure which has this property. This is the measure $d\mu = h\,d\nu$, that has been obtained in Theorem 6.8, and it will be called in the sequel the Gibbs state of $\varphi$.*

*Proof.* We have to show that a $e^{P(\varphi)-\varphi}$–conformal measure $\nu$ has non-zero mass on the conical set. Ergodicity and uniqueness follow then by known arguments (see [DMNU] and [McM1]).

We take again the notation of section 5.1 and may suppose that the constants have been chosen such that

$$\|\mathcal{A}_\varphi^n 1 + \mathcal{B}_\varphi^n 1\|_U \le \frac{1}{2}\frac{c_3}{c_2} \quad , \; for \;\; all \;\; n > m \, ,$$

with $c_2, c_3$ the bounds of the density function $h$ in Theorem 6.8 (cf. Remark 6.6).

For $i \in J_n$, call $U_i^{(n)} = h_i^{(n)}(U)$. Then

$$\nu(U_i^{(n)}) = \int_U \exp\left(S_n\varphi(h_i^{(n)}(x)) - nc\right) d\nu(x) \quad ,$$

where $c = P(\varphi)$. Therefore,

$$\begin{aligned} \sum_{i\in J_n} \nu(U_i^{(n)}) &= \int_U \sum_{i\in J_n} \exp\left(S_n\varphi(h_i^{(n)}(x)) - nc\right) d\nu(x) \\ &= \int_U \mathcal{G}_\varphi n 1(x)\, d\nu(x) \\ &= \int_U \mathcal{N}_\varphi^n 1(x)\, d\nu(x) - \int_U \left(\mathcal{A}_\varphi^n 1(x) + \mathcal{B}_\varphi^n 1(x)\right) d\nu(x) \, . \end{aligned}$$

But

$$\begin{aligned} \int_U \mathcal{N}_\varphi^n 1\, d\nu = \int \mathcal{N}_\varphi^n 1_{f^{-n}(U)}\, d\nu = \nu(f^{-n}(U)) \ge \frac{1}{c_2}\mu(f^{-n}(U)) \ge \frac{c_3}{c_2}\nu(U) \\ = \frac{1}{c_2}\mu(U) = \frac{c_3}{c_2} \end{aligned}$$

because of the invariance of $\mu$ and the fact that $d\mu = h\,d\nu$ with $c_3 \le h \le c_2$ (Theorem 6.8). In conclusion

$$\sum_{i\in J_n} \nu(U_i^{(n)}) \ge \frac{c_3}{2c_2} \, . \tag{7.1}$$

The points that are in $\bigcup_{i\in J_n} U_i^{(n)}$ for infinitely many $n>m$ are conical. Hence,

$$\mathcal{E}=\bigcap_{k>m}\mathcal{E}_k=\bigcap_{k>m}\bigcup_{n\geq k}\Big(\bigcup_{i\in J_n}U_i^{(n)}\Big)\subset\Lambda_c\ .$$

Since $\mathcal{E}_k$ is a decreasing sequence of sets with

$$\nu(\mathcal{E}_k)\geq\nu\Big(\bigcup_{i\in J_k}U_i^{(k)}\Big)\geq\frac{c_3}{2c_2}\ ,$$

for all $k\geq m$, we get

$$\nu(\Lambda_c)\geq\nu(\mathcal{E})\geq\frac{c_3}{2c_2}>0\ .$$

Since the measure $\nu$ is ergodic and since the conical set $\Lambda_c$ is $f$–invariant, we get that $\nu(\Lambda_c)=1$. □

## 8. Almost periodicity of the transfer operator

**Theorem 8.1.** *For any $\Phi\in C(\mathcal{T})$, the Banach space of continuous functions on the torus $\mathcal{T}$, the family $\{\mathcal{N}_\varphi^n\Phi\}_n$ is equicontinuous.*

In particular we see that the sequence of functions $h_n=\frac{1}{n}\sum_{k=1}^n\mathcal{N}_\varphi^k 1$, $n\geq 1$ forms an equicontinuous family. Arzéla-Ascoli's Theorem applies and gives this.

**Corollary 8.2.** *The Radon-Nikodym derivative $h$ of the Gibbs state $\mu$ with respect to the $e^{P(\varphi)-\varphi}$–conformal measure $\nu$ is continuous.*

Theorem 8.1 means that the normalized transfer operator $\mathcal{N}_\varphi$ is almost periodic. This leads to the following spectral properties (see [DU2] for details):

**Corollary 8.3.** *The space of complex valued continuous functions $C(\mathcal{J}_f)$ decomposes into a direct sum $C(\mathcal{J}_f)=C(\mathcal{J}_f)_u+C(\mathcal{J}_f)_0$ with*

$$C(\mathcal{J}_f)_u=\mathbb{C}h$$

*is the closure of the linear span of the unitary eigenvectors of $\mathcal{N}_\varphi$ and*

$$C(\mathcal{J}_f)_0=\left\{\Phi\ ;\ \int\Phi\,d\nu=0\right\}\ .$$

*Moreover, if $\Phi=\Phi_u+\Phi_0$ with $\Phi_u\in C(\mathcal{J}_f)_u$ and $\Phi_0\in C(\mathcal{J}_f)_0$, then $\Phi_u=(\int\Phi\,d\nu)h$.*

As an immediate consequence of Theorem 8.1 and of Corollary 8.3, we get the following (see [DU2]). Denote by $\mathcal{B}$ the $\sigma$–algebra of Borel sets on $\mathcal{J}_f$.

**Corollary 8.4.** *The dynamical system $(f,\mu)$ is metrical exact, the intersection $\bigcap_{n=0}^\infty f^{-n}(\mathcal{B})$ is the trivial $\sigma$–algebra consiting only of sets of measure zero and one and, consequently, its Rokhlin natural extension is a $K$–automorphism.*

*Proof of Theorem 8.1:* Let $0 < \varepsilon < 1$. We use again the decomposition (6.2) of the normalized transfer operator $\mathcal{N}_\varphi^n = \mathcal{G}_\varphi^n + \mathcal{A}_\varphi^n + \mathcal{B}_\varphi^n$, $n \geq m$, on a topological disk $U$ that is dense in $\mathcal{T}$ and has the properties mentioned sooner. Because of Remark 6.6, this can be done such that $\|\mathcal{A}_\varphi^n 1 + \mathcal{B}_\varphi^n 1\|_U \leq \frac{\varepsilon}{4\|\Phi\|}$, for all $n \geq m$.

Let $\varepsilon' > 0$ and choose then $\delta > 0$ according to Lemma 4.4'. Let $x, x' \in U$ with $\sigma_U(x, x') < \delta$ and let $i \in I_n$. We denote $e_i^{(n)}(x) = \exp\Big(S_n\varphi(h_i^{(n)}(x)) - nc\Big)$. It follows from Lemma 4.4' that

$$\begin{aligned}
\sum_{i \in J_n} |e_i^{(n)}(x) - e_i^{(n)}(x')| &= \sum_{i \in I_n} \Big( e_i^{(n)}(x') \Big| 1 - \exp\Big(S_n\varphi(h_i^{(n)}(x)) - S_n\varphi(h_i^{(n)}(x'))\Big)\Big| \Big) \\
&\leq 2\varepsilon' \sum_{i \in J_n} e_i^{(n)}(x') = 2\varepsilon \mathcal{G}_\varphi^n 1(x') \\
&\leq 2\varepsilon' c_1 \quad for \ \ all \ \ n \geq m
\end{aligned}$$

by Lemma 6.2. Therefore,

$$\Big|\mathcal{G}_\varphi^n \Phi(x) - \mathcal{G}_\varphi^n \Phi(x')\Big| = \left| \sum_{i \in I_n} \Phi(h_i^{(n)}(x)) e_i^{(n)}(x) - \Phi(h_i^{(n)}(x')) e_i^{(n)}(x') \right|$$

$$\begin{aligned}
&\leq \|\Phi\| \sum_{i \in I_n} |e_i^{(n)}(x) - e_i^{(n)}(x')| + \sum_{i \in I_n} e_i^{(n)}(x') \Big| \Phi(h_i^{(n)}(x)) - \Phi(h_i^{(n)}(x')) \Big| \\
&\leq \|\Phi\| 2\varepsilon' c_1 + c_1 \sup_{i \in I_n} \Big| \Phi(h_i^{(n)}(x)) - \Phi(h_i^{(n)}(x')) \Big| \ .
\end{aligned}$$

Due to (uniform) continuity of $\Phi$, this expression is arbitrarily small, say less then $\frac{\varepsilon}{2}$, provided $\sigma_U(x, x')$ is sufficiently small. Using the decomposition,

$$\begin{aligned}
\Big|\mathcal{N}_\varphi^n \Phi(x) - \mathcal{N}_\varphi^n \Phi(x')\Big| &\leq \Big|(\mathcal{A}_\varphi^n + \mathcal{B}_\varphi^n)\Phi(x) - (\mathcal{A}_\varphi^n + \mathcal{B}_\varphi^n)\Phi(x')\Big| + \Big|\mathcal{G}_\varphi^n \Phi(x) - \mathcal{G}_\varphi^n \Phi(x')\Big| \\
&\leq 2\|\mathcal{A}_\varphi^n + \mathcal{B}_\varphi^n\|_U \|\Phi\| + \frac{\varepsilon}{2} \leq \varepsilon
\end{aligned}$$

for any $n \geq m$ and any $x, x' \in U$ with $\sigma_U(x, x') < \delta$.

The general case easily follows by continuity, by density of $U$ in $\mathcal{T}$ and from accessibility of the points in the piecewise smooth boundary of $U$. $\square$

## 9. Bowen's formula for expanding elliptic functions

In the setting of expanding rational functions it is well known that the Hausdorff dimension of the Julia set is the only zero of the pressure function. As an application of our investigations we here extend this result to expanding elliptic functions. This section also builds a bridge between the present paper and the articles [KU1, KU2] written by Kotus and Urbański.

In what follows we consider $F:\mathbb{C}\to\hat{\mathbb{C}}$ elliptic and expanding, i.e. there are $c>0$ and $\lambda>1$ such that

$$|(F^n)'(z)| \geq c\lambda^n \quad \text{for all } z\in\mathcal{J}_F \text{ and all } n\geq 1 \tag{9.1}$$

(see Appendix for a equivalent topological characterization of expanding mappings). Denote again by $f$ the projection of $F$ to the underlying torus.

9.1. **Summable potentials and study of the pressure function.** Let

$$\varphi_t(z) = -t\,\log|f'(z)|,\ z\in\mathcal{J}_f\ ,$$

and let $P(t)=P(\varphi_t)$ be the corresponding pressure. Consider also

$$q=\max\{q_b\,;\,b\in F^{-1}(\infty)\},$$

the maximal multiplicity of $F$ at poles and let $\theta=\frac{2q}{q+1}$. A straight forward calculation gives the following.

**Lemma 9.1.** *For every $t>\theta$, $\varphi_t$ is a summable potential .*

The lower bound for $t$ here turns out to be optimal because of [KU1]:

**Lemma 9.2.** *The limit $\lim_{t\downarrow\theta}P(t)=\infty$.*

*Proof.* Recall that $P(t)=\lim_{n\to\infty}\frac{1}{n}\log\sum_{f^n(y)=x}|(f^n)'(y)|^{-t}$ where $x\in\mathcal{T}$ can be chosen arbitrarily. The divergence of this series $\sum_{f^n(y)=x}|(f^n)'(y)|^{-\theta}$ follows from the proof of Theorem 1 in [KU1]. Indeed, it has been shown there that there exists a conformal iterated function system $S=\{\Phi_j\}$ with generators $\Phi_j$ being convenable chosen inverse branches of $F^2$ defined on some disk $B\subset\mathbb{C}$ and having the property

$$\Psi(\theta,x)=\sum_j|\Phi_j'(x)|^\theta=\infty \quad \text{for all } x\in B\,.$$

Therefore, again with $x\in B$,

$$\frac{1}{2n}\log\sum_{F^{2n}(y)=x}|(F^{2n})'(y)|^{-\theta}\geq\frac{1}{2n}\log\sum_{|\omega|=n}|\Phi_\omega'(x)|^\theta=\infty\ ,$$

where $\Phi_\omega=\Phi_{\omega_1}\circ\ldots\circ\Phi_{\omega_n}$, from which the Lemma follows. □

At this point we can formulate the following Proposition who's proof now is standard.

**Proposition 9.3.** *The pressure function $P:[\theta,\infty)\to\overline{\mathbb{R}}$ is continuous, convex, strictly decreasing with $P(t)<0$ for sufficiently big $t>\theta$ (in fact $\lim_{t\to\infty}P(t)=-\infty$). Consequently there is a unique zero $\delta$ of this function.*

9.2. **Hausdorff dimension of the Julia set.** We now are ready to show Bowen's formula in this setting:

**Theorem 9.4.** *The Hausdorff dimension of the Julia set of a expanding elliptic function coincides with the only zero of the pressure function.*

*Proof.* For every $t > \theta$ there is a unique $e^{P(t)}|f'|^t$–conformal measure $\nu_t$ for $f$. In particular, for $t = \delta$, the unique zero of $t \mapsto P(t)$, Theorem 1.1 asserts that there is a unique (classical) $|f'|^\delta$–conformal measure $\nu_\delta$ (usually simply called $\delta$–conformal measure). Clearly this measure lifts to a $\Lambda$–periodic $\delta$–conformal measure of $F$, and there is only one such measure up to a multiplicative constant.

On the other hand, it has been shown in [KU2, Theorem 4.1] that the packing measure $\Pi^h$, with $h = \mathrm{HD}(\mathcal{J}_F)$, is such a measure. It follows that $\Pi^h = \nu_\delta$ up to a multiplicative constant and that $h = \delta$, proving the Theorem. □

An immediate consequence of Proposition 9.3 and Theorem 9.4 is

**Corollary 9.5.** *If $F$ is a expanding elliptic function, then*

$$HD(\mathcal{J}_F) > \theta = \frac{2q}{q+1} .$$

This is only an alternative point of view of the main Theorem of [KU1] where this last statement has been proven for all elliptic functions.

## 10. Variational principle and equilibrium states

Given a summable potential $\varphi : \mathcal{T} \to \mathbb{R}$ denote by $M_\varphi$ the space of all Borel probability $f$-invariant measures on $J(f)$ for which $\int \varphi d\mu > -\infty$. Since $\varphi$ is bounded above, this equivalently means that $\int |\varphi| d\mu < \infty$, i.e. the function $\varphi$ is integrable. We shall prove in this section two main results. The first one, the appropriate form of the variational principle is this.

**Theorem 10.1.** *We have that*

$$P(\varphi) = \sup\{h_\mu + \int \varphi d\mu : \mu \in M_\varphi\}.$$

Following the classical definition of equilibrium states, a measure $\mu \in M_\varphi$ is called an equilibrium state of the potential $\varphi$ if and only if $h_\mu + \int \varphi d\mu = P(\varphi)$. Our second main theorem is this.

**Theorem 10.2.** *The Gibbs state $\mu_\varphi$ of the summable potential $\varphi$ is a unique equilibrium state for $\varphi$.*

The proof of this theorm will follow as an outcome of several auxiliary results, some of them interesting themselves. If a Borel probability measure $\mu$ on $J(f)$ is $f$-invariant and the function $\log_+|f'|$ is integrable with respect to the measure $\mu$, then the integral $\int \log|f'|d\mu$ is well defined, although its value can be equal to $-\infty$, is denoted by $\chi_\mu$ and is called the Lyapunov characteristic exponent. We start with the following little observation.

**Lemma 10.3.** *If $\mu \in M_\varphi$, then the function $\log_+|f'|$ is integrable with respect to the measure $\mu$.*

*Proof.* From conditions (C1) and (C2) of Definition 3.1 and the behaviour of the derivative $f'$ near poles $\mathcal{P}_0$, it follows that there exists a constant $C > 0$ so big that $\log_+|f'| \le C(|\varphi| + 1)$ and we are done. □

We will also need the following.

**Lemma 10.4.** *If $\mu \in M_\varphi$, then the family of functions $\{\log_+|f' \circ f^j|\}_{j=0}^\infty$ is uniformly integrable with respect to the measure $\mu$. Precisely, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_A \log_+|f' \circ f^j|d\mu \le \varepsilon$ for every Borel set such that $\mu(A) \le \delta$.*

*Proof.* Since, by Lemma 10.3, the function $\log_+|f'|$ is integrable , there exists an open neighbourhood $B$ of $\mathcal{P}_0$ such that $\int_B \log_+|f'|d\mu \le \varepsilon/2$. Then $M = ||\log_+|f'|||_{J(f)\setminus B} < \infty$. Choose $\delta = \frac{\varepsilon}{2M}$. For every $j \ge 0$ and every Borel set $A$ with $\mu(A) \le \delta$, we have

$$\begin{aligned}
\int_A \log_+|f' \circ f^j|d\mu &= \int_{A\cap f^{-j}(B)} \log_+|f' \circ f^j|d\mu + \int_{A\setminus f^{-j}(B)} \log_+|f' \circ f^j|d\mu \\
&\le \int_{f^{-j}(B)} \log_+|f' \circ f^j|d\mu + \int_{A\setminus f^{-j}(B)} Md\mu \\
&= \int_B \log_+|f'|d\mu + M\mu(A \setminus f^{-j}(B)) \\
&\le \frac{\varepsilon}{2} + M\mu(A) \\
&\le \frac{\varepsilon}{2} + M\frac{\varepsilon}{2M} = \varepsilon.
\end{aligned}$$

We are done. □

**Theorem 10.5.** *(Ruelle's Inequality) If $\mu \in M_\varphi$ is ergodic, then $h_\mu(f) \le 2\max\{0, \chi_\mu\}$.*

*Proof.* Inspecting the proof of Theorem 9.1.1 from [PU], we see that in our context we get that

$$h_\mu \leq 2\int \frac{1}{n}\log\left(c(|(f^n)'(x)+2)|\right)d\mu(x) \tag{10.1}$$

for all $n \geq 1$ with some universal constant $c \geq 1$ depending only on the geometry of the torus $\mathcal{T}$. Now fix $\varepsilon > 0$. Choose $n_1 \geq 1$ so large that

$$\frac{2}{n_1}\log c \leq \frac{\varepsilon}{4} \quad \text{and} \quad \frac{1}{n_1}(2\log 2+\log 3) \leq \frac{\varepsilon}{8}.$$

Take $\delta > 0$ corresponding to the number $\varepsilon/8$ according to Lemma 10.4. By Lemma 10.3 the Lyapunov exponent $\chi_\mu = \int \log|f'|d\mu$ is well defined, and therefore, in view of Birkhoff's Ergodic Theorem, there exist a Borel set $A \subset J(f)$ and an integer $n_2 \geq n_1$ such that $\mu(A) \leq \delta$ and $|(f^k)'(x)| \leq \exp\big(\chi_\mu+\varepsilon)k\big)$ for all $x \in J(f)\setminus A$ and all $k \geq n_2$. Now fix an arbitrary $n \geq n_2$ and put

$$H_n = \{x \in J(f) : |(f^n)'(x)| \leq 1\}.$$

If $x \notin H_n$, then $|(f^n)'(x)|+2 \leq 2|(f^n)'(x)|$. For all $n \geq n_2$ we then have

$$\begin{aligned}
&\int \frac{1}{n}\log\big(|(f^n)'(x)|+2)\big)d\mu(x) \\
&\quad= \int_{H_n}\log\big(|(f^n)'(x)+2)|\big)d\mu(x) + \int_{A\setminus H_n}\log\big(|(f^n)'(x)|+2)\big)d\mu(x)+ \\
&\qquad+\int_{J(f)\setminus(A\cup H_n}\log\big(|(f^n)'(x)|+2)\big)d\mu(x) \\
&\quad\leq \frac{1}{n}\log 3\mu(H_n) + \int_{A\setminus H_n}\log\big(2|(f^n)'(x)\big)d\mu(x)+ \\
&\qquad+\int_{J(f)\setminus(A\cup H_n}\log\big(2|(f^n)'(x)|\big)d\mu(x) \\
&\quad\leq \frac{\log 3}{n}+\frac{\log 2}{n}\mu(A\setminus H_n) + \int_{A\setminus H_n}\log(|(f^n)'(x)|)d\mu(x)+ \\
&\qquad+\frac{\log 2}{n}\mu\big(J(f)\setminus(A\cup H_n)\big)+\max\{0,\chi_\mu+\varepsilon\}\mu\big(J(f)\setminus(A\cup H_n)\big) \\
&\quad\leq \frac{1}{n}(2\log 2+\log 3) + \int_A\sum_{j=0}^{n-1}\log_+|f'\circ f^j|d\mu + \max\{0,\chi_\mu+\varepsilon\} \\
&\quad\leq \frac{\varepsilon}{4}+\max\{0,\chi_\mu+\varepsilon\}.
\end{aligned}$$

Thus, using (10.1) and the choice of $n_1$, we get that $h_\mu(f) \leq \varepsilon+\max\{0,\chi_\mu+\varepsilon\}$. So, letting $\varepsilon \searrow 0$ finishes the proof. $\square$

**Lemma 10.6.** *If $\varphi : \mathcal{T} : \mathbb{R}$ is a summable potential, then $\mu_\varphi \in M_\varphi$.*

*Proof.* In view of conditions (C1) and (C2) from Definition 3.1 and of Theorem 1.1, it suffices to prove that

$$\int_{B(b,R)} -\log|z-b| d\nu_\varphi(z) < +\infty$$

for every pole $b$ on the torus $\mathcal{T}$ and some, sufficiently small, $R > 0$. Indeed, fix $b \in \mathcal{T}$ and $R > 0$ so small that

$$|f_0(z)| \asymp |z-b|^{-q_b} \quad \text{and} \quad |f'(z)| \asymp |z-b|^{-q_b-1}$$

for all $z \in B(0, 2R)$. Given $w \in \mathbb{C}$ and $0 \le r_1 \le r_2$ let

$$A(w, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z-w| \le r_2\}$$

be the corresponding annulus. Using then properties (C1) and (C2) along with Theorem 1.1, we see that for every $k \ge 0$ we have this.

$$\begin{aligned}
1 &\succeq \nu_\varphi\big(f\big(A(b, Re^{-(k+1)}, Re^{-k})\big)\big) \\
&\succeq \frac{1}{q_b}\frac{e^{P(\varphi)}}{l_2\big(f\big(A(b, Re^{-(k+1)}, Re^{-k})\big)\big)} \exp\big(k(2+\varepsilon_b)q_b\big)\nu_\varphi\big(A(b, Re^{-(k+1)}, Re^{-k})\big) \\
&\succeq \frac{\exp\big(kq_b(2+\varepsilon_b)\big)}{l_2\big(A(0, Re^{q_bk}, Re^{q_b(k+1)})\big)}\nu_\varphi\big(A(b, Re^{-(k+1)}, Re^{-k})\big) \\
&\asymp \exp(\varepsilon_b q_b k)\nu_\varphi\big(A(b, Re^{-(k+1)}, Re^{-k})\big).
\end{aligned}$$

Hence $\nu_\varphi\big(A(b, Re^{-(k+1)}, Re^{-k})\big) \preceq \exp(-\varepsilon_b q_b k)$, and therefore

$$\begin{aligned}
\int_{B(b,R)} -\log|z-b| d\nu_\varphi(z) &= \sum_{k=0}^{\infty} \int_{A(b, Re^{-(k+1)}, Re^{-k})} -\log|z-b| d\nu_\varphi(z) \\
&= O(1) + \sum_{k=0}^{\infty} k\exp(-\varepsilon_b q_b k) < +\infty.
\end{aligned}$$

We are done. $\square$

**Lemma 10.7.** *If $\mu \in M_\varphi$ is ergodic and $\chi_\mu > 0$, then there exists a countable generator for $\mu$ that has finite entropy.*

*Proof.* Since $\mu$ is ergodic and $\chi_\mu > 0$, an appropriate version of Pesin's theory (see [PU], Section 9.2) can be developed to give that for $\mu$-a.e. $z$ (say $z \in Y_1$ with $\mu(Y_1) = 1$) there exist $\delta \in (0, 1]$ and $C > 0$ such that for every integer $n$ in some set $N(z) \subset \{1, 2, 3, \ldots\}$ with density $> 1/2$ and every $0 \le j \le n$, there exists a unique holomorphic inverse branch $f^{n-j}_{f^j(z)}: B(f^n(z), 2\delta) \to \mathcal{T}$ of $f^{n-j}$ such that $f^{n-j}_{f^j(z)}\big(f^n(z)\big)= f^j(z)$ and

$$|(f_z^{-n})'(w)| \le C\exp\Big(-\frac{\chi_\mu}{2}n\Big) \tag{10.2}$$

for all $w \in B(f^n(z), \delta)$. Let $K \ge 1$ be the constant coming from Koebe's Distortion Theorem associated with the scale 1/2. Since the set $\mathcal{P}$ is finite, there exists $\beta > 0$ so small that for every $z \in J(f) \setminus \mathcal{P}$, the map $f : \mathcal{T} \setminus \mathcal{P} \to \mathcal{T}$ restricted to the ball $B\big(z, (8K + 1)\beta\mathrm{dist}(z, \mathcal{P})\big)$ is injective. It was established in the proof of Lemma 10.6 that the logarithm of the function $\rho(z) = \min\{\delta, \beta\mathrm{dist}(z, \mathcal{P})\big)$ is integrable. There thus exists (see Lemma 9.3.2 in [PU]) a countable partition $\alpha$ by Borel sets such that $\mathrm{H}_\mu(\alpha) < +\infty$ and

$$\alpha(z) \subset B(z, \rho(z)) \tag{10.3}$$

for $\mu$-a.e. $z \in J(f)$, say $z \in Y_2 \subset Y_1$ with $\mu(Y_2) = 1$ and $f(Y_2) \subset Y_2$. Fix $x \in Y_2$. We shall show that

$$\alpha_n(x) \subset f_x^{-n}(B(f^n(x), \delta)) \tag{10.4}$$

for all $n \ge 0$, where $\alpha_n(x)$ is the uniqe atom of the refined partition $\alpha \vee f^{-1}(\alpha) \vee f^{-2}(\alpha) \vee \ldots \vee f^{-n}(\alpha)$ containing $x$. In order to achieve this we shall show by induction with respect to $k = 0, 1, \ldots, n$ that

$$\alpha_k(f^{n-k}(x)) \subset f^{-k}_{f^{n-k}(x)}(B(f^n(x), \delta)). \tag{10.5}$$

Indeed, for $k = 0$, this formula follows immediately from (10.3), the definition of the function $\rho$, and the inclusion $f(Y_2) \subset Y_2$. Suppose now that (10.5) holds for some $0 \le k \le n - 1$, and consider two cases.

Case $1^0$:

$$\mathrm{diam}\Big(f^{-(k+1)}_{f^{n-(k+1)}(x)}(B(f^n(x), \delta))\Big) \le 8K\rho\big(f^{n-(k+1)}(x)\big).$$

Then, using (10.3) and the inclusion $f^{n-(k+1)}(Y_2) \subset Y_2$, we get that

$$\begin{aligned}\alpha_k(f^{n-(k+1)}(x)) \cup f^{-(k+1)}_{f^{n-(k+1)}(x)}(B(f^n(x), \delta)) \subset& \\ \subset B\big(f^{n-(k+1)}(x), \rho\big(f^{n-(k+1)}(x)\big)&+8K\rho\big(f^{n-(k+1)}(x)\big)\big) \\ \subset B\big(f^{n-(k+1)}(x), (8K+1)\rho&\big(f^{n-(k+1)}(x)\big).\end{aligned}$$

Case $2^0$:

$$\mathrm{diam}\Big(f^{-(k+1)}_{f^{n-(k+1)}(x)}(B(f^n(x), \delta))\Big) \ge 8K\rho\big(f^{n-(k+1)}(x)\big).$$

Then, applying Koebe's $\frac{1}{4}$-Distortion Theorem, the standard version of Koebe's Distortion Theorem and, at the end, (10.3) we get that

$$\begin{aligned} f^{-(k+1)}_{f^{n-(k+1)}(x)}(B(f^n(x),\delta)) &\supset B\big(f^{n-(k+1)}(x),(8K)^{-1}\text{diam}\left(f^{-(k+1)}_{f^{n-(k+1)}(x)}(B(f^n(x),\delta))\right)\\ &\supset B\big(f^{n-(k+1)}(x),\rho\big(f^{n-(k+1)}(x)\big)\supset \alpha(f^{n-(k+1)}(x)). \end{aligned}$$

Invoking the definition of the function $\rho$, we see that in any case the function $f$ restricted to the union $\alpha_k(f^{n-(k+1)}(x))\cup f^{-(k+1)}_{f^{n-(k+1)}(x)}(B(f^n(x),\delta))$ is 1-to-1. Hence, using (10.5), we obtain that

$$\begin{aligned} \alpha_{k+1}\big(f^{n-(k+1)}(x)\big) &= \alpha(f^{n-(k+1)}(x))\cap f^{-1}\big(\alpha_k(f^{n-k}(x))\big)\\ &\subset \alpha(f^{n-(k+1)}(x))\cap f^{-1}\big(f^{-k}_{f^{n-k}(x)}(B(f^n(x),\delta))\big)\\ &\subset f^{-(k+1)}_{f^{n-(k+1)}(x)}(B(f^n(x),\delta)). \end{aligned}$$

Thus, the inductive proof of (10.5) is complete, and taking $k=n$ we obtain (10.4). Taking two distinct points $w,z\in Y_2$, we see from (10.4) and (10.2) that for all $n\in N(w)\cap N(z)$ (which is an infinite set) large enough, we have $\alpha_n(z)\subset B(z,|w-z|/2)$ and $\alpha_n(w)\subset B(z,|w-z|/2)$. In particular $\alpha_n(z)\cap\alpha_n(w)=\emptyset$ and we are done. $\square$

Combining this lemma along with formulas (8.10) and (10.5) from [Pa], we get the following.

**Lemma 10.8.** *If $\mu\in M_\varphi$ is ergodic and $\chi_\mu>0$, then*

$$h_\mu(f)=\int\log J_\mu d\mu,$$

*where $J_\mu$ is the Jacobian of $f$ with respect to the measure $\mu$. Note that $J_\mu$ is finite out of a set of measure zero.*

Now we can easily deduce the following.

**Lemma 10.9.** *If $\varphi:\mathcal{T}\to\mathbb{R}$ is a summable potential then, $P(\varphi)=h_{\mu_\varphi}+\int\varphi d\mu_\varphi$.*

*Proof.* It follows from Theorem 1.1 (2) and (3) that $J_{\mu_\varphi}=\frac{h\circ f}{h}\exp\big(P(\varphi)-\varphi\big)$. Hence

$$\int\log J_{\mu_\varphi}d\mu_\varphi=\int(P(\varphi)-\varphi)d\mu_\varphi=P(\varphi)-\int\varphi d\mu_\varphi>P(\varphi)-\sup(\varphi)>0. \tag{10.6}$$

Since $h_{\mu_\varphi}\ge\int\log J_{\mu_\varphi}d\mu_\varphi$ regardless whether a generating partition with finite entropy exists or not, we thus get that $h_{\mu_\varphi}>0$. Since by Lemma 10.6 $\mu_\varphi\in M_\varphi$, it follows from Ruelle's inequality (Theorem 10.5) that $\chi_{\mu_\varphi}>0$. So, since by

Theorem 1.1(2), the measure is ergodic, using Lemma 10.8 and (10.6), we obtain that $h_{\mu_\varphi} = P(\varphi) - \int \varphi d\mu_\varphi$. We are done. □

The next crucial step towards proving the varational principle and towards identifying the equilibrium states of $\varphi$ is given by the following.

**Lemma 10.10.** *If $\mu \in M_\varphi$, then $h_\mu + \int \varphi d\mu \leq P(\varphi)$ and if $\mu$ is an ergodic equilibrium state for $\varphi$, then*

$$J_\mu^{-1} = \frac{h}{h \circ f} \exp(\varphi - P(\varphi))$$

*$\mu$-a.e.*

*Proof.* Suppose that $\mu \in M_\varphi$ is ergodic. Let $\mathcal{L}_\mu : L^1(\mu) \to L^1(\mu)$ be the transfer operator associated to the measure $\mu$. The operator $\mathcal{L}_\mu$ is determined by the formula

$$\mathcal{L}_\mu(g)(x) = \sum_{y \in f^{-1}(x)} J_\mu^{-1}(y) g(y).$$

Using Theorem 1.1(3) (which implies that $\mathcal{N}_\varphi(h) = h$) and the $f$-invariance of $\mu$, we can write

$$\begin{aligned}
1 &= \int 1 d\mu = \int \frac{\mathcal{N}_\varphi(h)}{h} d\mu \\
&= \int \mathcal{L}_\mu \left( \frac{h \cdot \exp(\varphi - P(\varphi))}{J_\mu^{-1} \cdot h \circ f} \right) d\mu \\
&= \int \frac{h \cdot \exp(\varphi - P(\varphi))}{J_\mu^{-1} \cdot h \circ f} d\mu \geq 1 + \int \log \left( \frac{h \cdot \exp(\varphi - P(\varphi))}{J_\mu^{-1} \cdot h \circ f} \right) d\mu \qquad (10.7) \\
&= 1 + \int \log h d\mu - \int \log h \circ f d\mu + \int (\varphi - P(\varphi)) d\mu + \int \log J_\mu d\mu \\
&= 1 + \int \varphi d\mu - P(\varphi) + \int \log J_\mu d\mu.
\end{aligned}$$

If now $h_\mu = 0$, then $\int \log J_\mu d\mu = 0 = h_\mu$. If $h_\mu > 0$, then it follows from Ruelle's inequality (Theorem 10.5) that $\chi_\mu > 0$. So, $h_\mu = \int \log J_\mu d\mu$ in view of Lemma 10.8. Thus, (10.7) can be continued to give

$$1 + \int \varphi d\mu - P(\varphi) + \int \log J_\mu d\mu = 1 + \int \varphi d\mu - P(\varphi) + h_\mu.$$

Hence, $P(\varphi) \geq h_\mu + \int \varphi d\mu$ and equality holds if and only if $\frac{h \cdot \exp(\varphi - P(\varphi))}{J_\mu^{-1} \cdot h \circ f} = 1$ $\mu$-a.e. So, we are done in the ergodic case. In general, inequality $P(\varphi) \geq h_\mu + \int \varphi d\mu$ follows from the ergodic case and the Ergodic Decomposition Theorem. We are done. □

Our last lemma in the sequence is this.

**Lemma 10.11.** *If $\mu \in M_\varphi$ is an ergodic equilibrium state for the summable potential $\varphi$, then $\mu = \mu_\varphi$.*

*Proof.* In view of Lemma 10.10 we may assume that

$$J_\mu^{-1} = \frac{h}{h \circ f} \exp(\varphi - P(\varphi)) \tag{10.8}$$

everywhere throughout the set $J(f)$. Let $Y_1$ be the set established to exist in the proof of Lemma 10.7. Fix $z \in Y_1$ and take an arbitrary $z \in N(z)$. Pesin 's theory gives in fact more than (10.2), namely that

$$\left| \left( f_{f^j(z)}^{n-j} \right)' (w) \right| \le C \exp\left( -\frac{\chi_\mu}{2}(n-j) \right) \tag{10.9}$$

for all $0 \le j \le n$ and all $w \in B(f^n(z), \delta)$, where $f_{f^j(z)}^{n-j} : B(f^n(z), \delta) \to \mathbb{C}$ is the unique holomorphic inverse branch of $f^{n-j}$, defined on $B(f^n(z), \delta)$ and sending $f^n(z)$ to $f^j(z)$. A slight obvious modification of of the proof of Lemma 4.4 using (10.9), gives that

$$|S_n\varphi\big(f_z^{-n}(w)\big) - S_n\varphi(z)| \le B \tag{10.10}$$

with some constant $B > 0$, all $n \in N(z)$ and all $w \in B(f^n(z), \delta)$. It follows from Koebe's $\frac{1}{4}$-Distortion Theorem that $B(z, 8^{-1}|(f^n)'(z)|\delta) \subset f_z^{-n}\big(B(f^n(z), \delta)\big)$. Therefore, using (10.10) along with (10.8), we get that

(10.11)
$$\begin{aligned} \mu\big(B(z, 8^{-1}|(f^n)'(z)|\delta)\big) &\le e^B ||h||_\infty ||1/h||_\infty \exp\big(S_n\varphi(z) - P(\varphi)n\big)\mu\big(B(f^n(z), \delta)\big) \\ &\le e^B ||h||_\infty ||1/h||_\infty \exp\big(S_n\varphi(z) - P(\varphi)n\big). \end{aligned}$$

It follows from Koebe's Distortion Theorem that

$$B(z, 8^{-1}|(f^n)'(z)|\delta) \supset f_z^{-n}\big(B(f^n(z), (8K)^{-1}\delta)\big),$$

where $K \ge 1$ is the constant coming from Koebe's Distortion Theorem corresponding to the scale 1/2. Therefore, using Theorem 1.1(2) and (3) (implying that $J_{\mu_\varphi}^{-1} = \frac{h}{h\circ f} \exp(\varphi - P(\varphi))$ along with (10.10), we get that

$$\begin{aligned} \mu_\varphi\big(B(z, 8^{-1}|(f^n)'(z)|\delta)\big) &\ge \\ &\ge e^{-B}(||h||_\infty ||1/h||_\infty)^{-1} \exp\big(S_n\varphi(z) - P(\varphi)n\big)\mu_\varphi\big(B(f^n(z), (8K)^{-1}\delta)\big) \\ &\ge Me^{-B}(||h||_\infty ||1/h||_\infty)^{-1} \exp\big(S_n\varphi(z) - P(\varphi)n\big), \end{aligned}$$

where $M = \inf\{\mu_\varphi(B(\xi, (8K)^{-1}\delta)) : \xi \in J(f)\}$ is positive since $\mathrm{supp}(\mu_\varphi) = J(f)$. Combining this formula and (10.11), we get that

$$\mu\big(B(z, 8^{-1}|(f^n)'(z)|\delta)\big) \le M^{-1}e^{2B}(||h||_\infty ||1/h||_\infty)^2 \mu_\varphi\big(B(z, 8^{-1}|(f^n)'(z)|\delta)\big).$$

Since, by (10.9), $\lim_{n\in N(z)} 8^{-1}|(f^n)'(z)|\delta = 0$ and since $\mu(Y_1) = 1$, a strightforward argument, using Besicowic covering theorem, gives that $\mu$ is absolutely continuous with respect to $\mu_\varphi$. Since both measures $\mu$ and $\mu_\varphi$ are ergodic, we thus get that $\mu = \mu_\varphi$. □

Since, by the Ergodic Decomposition Theorem, the uniqueness of equilibrium states is equivalent to the uniqueness of ergodic equilibrium states, Theorema 10.1 and 10.2 follow now from the first part of Lemma 10.10, from Lemma 10.9 and Lemma 10.11.

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