THE DYNAMICS AND GEOMETRY OF THE FATOU FUNCTIONS

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ABSTRACT. We deal with the Fatou functions $f_\lambda(z) = z + e^{-z} + \lambda$, $\mathrm{Re}\lambda \geq 1$. We consider the set $J_r(f_\lambda)$ consisting of those points of the Julia set of f_λ whose real parts do not escape to infinity under positive iterates of f_λ. Our ultimate result is that the function $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$ is real-analytic. In order to prove it we develop the thermodynamic formalism of potentials of the form $-t\log|F'_\lambda|$, where F_λ is the projection of f_λ to the infinite cylinder. It includes appropriately defined topological pressure, Perron-Frobenius operators, geometric and invariant generalized conformal measures (Gibbs states). We show that our Perron-Frobenius operators are quasicompact, that they embed into a family of operators depending holomorphically on an appropriate parameter and we obtain several other properties of these operators. We prove an appropriate version of Bowen's formula that the Hausdorff dimension of the set $J_r(f_\lambda)$ is equal to the unique zero of the pressure function. Since the formula for the topological pressure is independent of the set $J_r(f_\lambda)$, Bowen's formula also indicates that $J_r(f_\lambda)$ is the right set to deal with. What concerns geometry of the set $J_r(f_\lambda)$ we also prove that the $\mathrm{HD}(J_r(f_\lambda))$-dimensional Hausdorff measure of the set $J_r(F_\lambda)$ is positive and finite whereas its $\mathrm{HD}(J_r(f_\lambda))$-dimensional packing measure is locally infinite. This last property allows us to conclude that $\mathrm{HD}(J_r(f_\lambda)) < 2$. We also study in detail the properties of quasiconformal conjugations between the maps f_λ. As a byproduct of our main course of reasoning we prove stochastic properties of the dynamical system generated by F_λ and the invariant Gibbs states μ_t such as the Central Limit Theorem and the exponential decay of correlations.

1. INTRODUCTION

Given $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, let the transcendental entire function $f_\lambda : \mathbb{C} \to \mathbb{C}$ be defined by the formula

$$f_\lambda(z) = z + e^{-z} + \lambda. \tag{1.1}$$

Its derivative is this.

$$f'_\lambda(z) = 1 - e^{-z}. \tag{1.2}$$

For $\lambda = 1$, $f_1(z) = z + 1 + e^{-z}$ is the function considered by P. Fatou (see [14], Example 1). To his honor we will call all the functions f_λ Fatou functions. The equivalence relation $\sim$ on $\mathbb{C} \times \mathbb{C}$ is defined by the requirement that $w \sim z$ if and only if $w - z \in 2\pi i\mathbb{Z}$. The quotient

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spaces $\mathbb{C}/\sim$ is the infinite cylinder endowed with Riemann surface structure induced by the canonical quotient map

$$\Pi : \mathbb{C} \to Q.$$

Since the map $f_\lambda : \mathbb{C} \to \mathbb{C}$ respects the equivalence relation $\sim$, it induces a unique map

$$F_\lambda : Q \to Q$$

such that $F_\lambda \circ \Pi = \Pi \circ f_\lambda$. This map will be our main auxiliary, and also interesting itself, object of interest in this paper. As long as we are not interested in our considerations in the dependence on the parameter λ, we will skip the superscript λ and will simply write f and F for f_λ and F_λ respectively.

The aim of this paper is multi-fold. It is well known and very easy to check that all Fatou functions with $\mathrm{Re}\lambda \geq 1$ (in fact $\mathrm{Re}\lambda > 0$) have Baker's domain at infinity. Given the "asymptotic" similarity to the exponential function $z \mapsto e^{-z}$, we wanted to check whether Baker's domain prevails and destroys the fractal geometry of hyperbolic exponential functions discovered in [25] and [26], or, just the contrary, the similarities take over and the fractal geometry of the Julia sets of Fatou functions with $\mathrm{Re}\lambda \geq 1$ flourishes. Our idea to deal with this issue was to synthesize the approaches from [25] and [26] into one coherent entity. And indeed, this project has proved to work and the class of Fatou functions with $\mathrm{Re}\lambda \geq 1$ has turned out to posses the same kind of strikingly regular fractal geometry as the class of hyperbolic exponential functions. Undertaking such approach we have frequently repeated many parts of [25] and [26] very closely. As a result we have provided a uniform approach, including all the proofs for the sake of completeness and the convenience of the reader. Most often the technical details of our proofs were different than those in [25] and [26], especially in Section 3 and Section 9.

It follows from Bergweiler's result ([6]) that $J(F) = \Pi(J(f))$. In view of Lemma 2.7 the Julia set $J(f)$ is thin in the sense of [20]. Since all the critical points of the Fatou function f_λ are of the form $z_k = 2k\pi i$, $k \in \mathbb{Z}$, its all critical values of are of the form $2k\pi i + 1 + \lambda$. So, if $\mathrm{Re}\lambda \geq 1$, these points escape uniformly to ∞, and in consequence (invoking also Theorem 2.3), the Julia set $J(f)$ is at a positive distance from their forward trajectory. Combining this fact and thinness of $J(f)$, it follows from [24] that the Lebesgue measure of $J(f) = 0$. Let

$$I_\infty(F) = \{z \in Q : \lim_{n\to\infty} F^n(z) = \infty\}.$$

and let

$$I_\infty(f) := \{z \in \mathbb{C} : \mathrm{Re}(f^n(z)) \to -\infty\}.$$

$I_\infty(F)$ is the set of points escaping to infinity under forward iterates of F. Obviously

$$I_\infty(F) = \{z \in Q : \lim_{n\to\infty} \mathrm{Re}(F^n(z)) = -\infty\} = \Pi(I_\infty(f)).$$

Following closely the reasoning from [20] one can show that $\mathrm{HD}(I_\infty(F)) = \mathrm{HD}(I_\infty(f)) = \mathrm{HD}(J(F)) = \mathrm{HD}(J(f)) = 2$. These sets are dynamically rather boring and their fractal geometry has drastically different features (see [17]) than those characteristic for (classical) conformal expanding repellers. Therefore, following [25] and [26], we introduce the following sets.

$$J_r(F) = J(F) \setminus I_\infty(F) \quad \text{and} \quad J_r(f) = J(f) \setminus I_\infty(f).$$

Obviously $J_r(F) = \Pi(J_r(f))$. Our ultimate aim is to show that the function $\lambda \mapsto \mathrm{HD}(J(F_{\lambda,r}))$, $\mathrm{Re}\lambda > 1$, is real-analytic. In the course of the paper we also establish other interesting fractal and dynamical properties of the set $J_r(F)$. We mention them describing now in greater detail the content of our paper. Section 2 contains some basic properties of the Fatou functions. Section 3 consists of the proof of the fact that the bounded orbits of F have Hausdorff dimension strictly greater than 1. Since each bounded orbit is contained in $J_r(F)$, this implies that $\mathrm{HD}(J_r(F)) > 1$. In Section 4 we define appropriate in this context topological pressure of the potentials $-t\log|F'|$, $t \geq 0$, Perron-Frobenius operators with some more general potentials and generalized conformal (Gibbs) measures, whose existence is shown by proving tightness of an appropriate sequence of Borel probability measures on the Julia set. Using the existence of these measures we prove three basic properties of the Perron-Frobenius operators in Lemmas 4.6, 4.7 and 4.8. It is also shown that these generalized conformal measures give full measure to the set $J_r(F)$. We end this section with the proof of the uniqueness and ergodicity of conformal measures. In Section 5 we show that our Perron-Frobenius operators satisfy the assumptions of the Ionescu-Tulcea and Marinescu theorem. In particular the Perron-Frobenius operator acting on an appropriate Banach space is quasicompact and the full description of its spectral properties is provided in Theorem 5.4. The Section 6 is very short. It establishes the existence of F-invariant Borel probability measures μ_t equivalent to conformal measures m_t. This section also collects straightforward (by now) ergodic and stochastic consequences of the spectral properties of Perron-Frobenius operators, proven in the previous section, to the dynamical systems (F, μ_t). The Section 7 is devoted to the presentation of fractal properties of the set $J_r(F)$. First, it contains the appropriated version of Bowen's formula: Hausdorff dimension of the set $J_r(F)$ is equal to the unique zero of the pressure function. Since the formula for the topological pressure is independent of the set $J_r(f)$, Bowen's formula also indicates that, similarly as for exponential functions, this is the right set to deal with. The next result in this section concerns packing measures. It states that the h-dimensional ($h = \mathrm{HD}(J_r(F))$) packing measure of $J_r(F)$ is locally bounded at every point of $J_r(F)$. As an immediate consequence of this fact, we conclude that $\mathrm{HD}(J_r(F)) < 2 = \mathrm{HD}(J(F))$. Section 8 provides a sufficient condition (Corollary 8.7) for our Perron-Frobenius operators to depend holomorphically on the appropriate parameters. Section 9 establishes quasiconformal conjugations between maps F_λ, $\mathrm{Re}\lambda > 1$, proves that they form a holomorphic motion and shows that this motion has a "bounded speed" (Proposition 9.5). As its consequence the uniform Hölder continuity of these quasiconformal conjugations is established. These results are in particular used to prove in Section 10 the boundedness of $|\log\psi_z(\lambda)|$ (formula (10.5)) and its Hölder continuity. We remark that the function ψ plays the main role among all the auxiliary objects appearing in the proof of Proposition 10.2. In this Section 10, perhaps most technical part of our paper, we first prove continuity of the topological pressure with respect to the parameter λ and then we check that the conditions presented in Section 8 are satisfied. This means that, as our main technical argument, we carefully construct complex analytic extensions of our Perron-Frobenius operators so that the assumptions of Corollary 8.7 are satisfied. The section ends with, perhaps the main result of our paper, saying that the Hausdorff dimension function $\lambda \mapsto \mathrm{HD}(J_r(F_\lambda))$, $\mathrm{Re}\lambda > 1$, is real-analytic.

Ending this introduction we would like to mention that all the analysis we did in this paper for parameters λ with $\mathrm{Re}\lambda \geq 1$ could be, in view of (9.1) and the whole paragraph containing it, actually done assuming only that $\mathrm{Re}\lambda > 0$ (in particular a Baker's domain at infinity the exists). However, this would bring additional technical difficulties, and would make the whole paper much less readable. So, we have decided to work only with parameters λ for which $\mathrm{Re}\lambda \geq 1$.

2. Preliminaries

Put

$$P := \{z \in \mathbb{C} \colon 0 < \mathrm{Im}(z) < 2\pi\}.$$

Abusing a little bit notation, we will also frequently treat the strip P as a subset of the cylinder Q. Given $M \leq 0$, $D \subset \mathbb{C}$ and $E \subset Q$, we let

$$D_M = \{z \in \mathbb{C} \colon M \leq \mathrm{Re} \leq 0\} \ \text{ and } E_M = \{z \in Q : M \leq \mathrm{Re} \leq 0\}.$$

We also put $D_M^c = \mathbb{C} \setminus D_M$, $E_M^c = Q \setminus E_M$.- Let us prove the following.

Lemma 2.1. *The map $f : P \to \mathbb{C}$ is bijective.*

Proof. Put

$$P_- := \{z \in \mathbb{C} \colon 0 < \mathrm{Im}(z) < \pi\} \quad \text{and} \quad P^- := \{z \in \mathbb{C} \colon \pi < \mathrm{Im}(z) < 2\pi\}.$$

If $z \in P_-$ then $\sin(\mathrm{Im}(z)) > 0$ and therefore

$$\mathrm{Im}(f_\lambda(z)) = \mathrm{Im}(z) + \mathrm{Im}(\lambda) - e^{-\mathrm{Re}(z)} \sin(\mathrm{Im}(z)) < \mathrm{Im}(z) + \mathrm{Im}(\lambda) < \pi + \mathrm{Im}(\lambda).$$

Hence

$$f(P_-) \subset \{z \in \mathbb{C} \colon \mathrm{Im}(z) < \pi + \mathrm{Im}(\lambda)\}. \tag{2.1}$$

An analogous argument shows that

$$f_\lambda(P^-) \subset \{z \in \mathbb{C} \colon \mathrm{Im}(z) > \pi + \mathrm{Im}(\lambda)\}. \tag{2.2}$$

Now, if $z \in P_-$ then $\mathrm{Im}(f(z)) = -e^{-\mathrm{Re}(z)} \sin(\mathrm{Im}(z)) < 0$. Therefore, using also the fact that P_- is convex we conclude (see [22]) that the map $f|_{P_-}$ is injective. Analogously, if $z \in P^-$, then $\mathrm{Im}(f'(z)) > 0$, and the map $f|_{P^-}$ is also injective. Combining these two facts with (2.1) and (2.2), we see that the map f_λ restricted to the union $P_- \cup P^-$ is one-to-one. Since $f(\{z \in P : \mathrm{Im}(z) = \pi\}) = \{z \in P : \mathrm{Im}(z) = \pi + \mathrm{Im}(\lambda)\}$, and, since by a direct inspection, f is one-to-one on the set $\{z \in P : \mathrm{Im}(z) = \pi\}$, we finally conclude that $f|_P$ is injective. Since a direct calculation shows that $\partial(f(P)) \subset f(\partial P) \subset f(P)$, we conclude that $\partial f(P) = \emptyset$. Thus, $f(P) = \mathbb{C}$ and we are done. ■

Let $\mathrm{PC}(F)$ denote the postcritical set of F, i.e.

$$\mathrm{PC}(F) = \overline{\{F^n(\Pi(0)) : n \geq 0\}}.$$

Note that f (and F) has no finite asymptotic values. Let $f_*^{-1} : \mathbb{C} \to P$ be the (holomorphic) inverse map to the map $f : P \to \mathbb{C}$ proven to be bijective in Lemma 2.1. We shall show that the projected Fatou function F is expanding on its Julia set. We start with the following weaker result.

Lemma 2.2. *If $z \in J(F)$, then $\limsup_{n\to\infty} |(F^n)'(z)| = +\infty$.*

Proof. Let $\rho(z)|dz|$ be the Poincaré metric on $Q \setminus \mathrm{PC}(F)$ and let $||F'(z)||$ denote the norm of the derivative of F at a point $z \in Q \setminus \mathrm{PC}(F)$ taken with respect to the Poincaré metric ρ. Notice that $F^{-1}(Q \setminus \mathrm{PC}(F))$ is an open connected subset of $Q \setminus \mathrm{PC}(F)$ containing the Julia set $J(F)$ and the map $F : Q \setminus \mathrm{PC}(F) \to Q$ is a covering map. Hence, $F : Q \setminus \mathrm{PC}(F) \to Q$ is a local isometry with respect to the Poincaré metrics respectively on $Q \setminus \mathrm{PC}(F)$ and on Q. Since the norm of the derivative of the injection map from $F^{-1}(Q \setminus \mathrm{PC}(F))$ to $Q \setminus \mathrm{PC}(F)$ is at every point less than 1 when taken with respective Poincare metrics, we conclude that

$$||F'(z)|| > 1 \tag{2.3}$$

for all $z \in F^{-1}(Q \setminus \mathrm{PC}(F))$. Now fix $z \in J(F)$. If $\lim_{n\to\infty} F^n(z) = -\infty$, then it immediately follows from (1.2) that $\lim_{n\to\infty} |(F^n)'(z)| = +\infty$. So, suppose that $F^n(z)$ does not tend to $-\infty$. Then there exists a compact set $L \subset Q$ and an increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers such that $F^{n_j}(z) \in L$ for all $j \geq 1$. Since the function $w \mapsto ||F'(w)||$ is continuous, it follows from (2.3)

$$\eta := \inf\{||F'(z)|| : z \in L\} > 1. \tag{2.4}$$

Since the function $w \mapsto \rho(w)$ is continuous, we get

$$\gamma := \sup\{\rho(z) : z \in L\} < \infty. \tag{2.5}$$

Since for every $n \geq 1$, $|(F^n)'(z)| = \rho(z)||(F^n)'(z)||/\rho(F^n(z))$, combining (2.5), (2.4) and (2.3), we conclude that for every $j \geq 1$

$$||(F^{n_j})'(z)|| \geq \frac{\rho(z)}{\gamma}\eta^j.$$

The proof is complete. ■.

A direct calculation show that if $\mathrm{Re}(\lambda) \geq 1$ ($\mathrm{Re}(\lambda) > 1$), then if $\mathrm{Re}(z) > 0$ ($\mathrm{Re}(z) \geq 0$), then $\mathrm{Re}(f_\lambda(z)) > \mathrm{Re}(z)$. This proofs the following.

Theorem 2.3. *If* $\mathrm{Re}(\lambda) \geq 1$ *(*$\mathrm{Re}(\lambda) > 1$*), then the map* $f_\lambda : \mathbb{C} \to \mathbb{C}$ *has a Baker domain* D *at* ∞ *and* $\{z \in \mathbb{C} \colon \mathrm{Re}(z) > 0\}$ *(*$\{z \in \mathbb{C} \colon \mathrm{Re}(z) \geq 0\}$*) is contained in this domain. In particular* $J(f_\lambda) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) \leq 0\}$ *(*$\{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$*). Even more, if* $\mathrm{Re}(\lambda) > 1$*, then there exists* $\epsilon_\lambda > 0$ *such that* $J(f_\lambda) \subset \{z \in \mathbb{C} \colon \mathrm{Re}(z) < -\epsilon_\lambda\}$.

From now on λ is assumed to have the real part $\mathrm{Re}(\lambda) \geq 1$.

The following theorem provides a complete description of the structure of the Fatou set of the function $f : \mathbb{C} \to \mathbb{C}$.

Theorem 2.4. *The Fatou set of the Fatou function* $f : \mathbb{C} \to \mathbb{C}$ *consists exactly of the images of all backward iterates of the Baker's domain* D *at infinity containing the right half-plane.*

Proof. Fix w in the Fatou set of f and a ball B centered at w and contained in the connected component of the Fatou set that contains w. The map $H \circ \Pi : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a conformal semiconjugacy between f_λ and $G_\lambda : \mathbb{C} \to \mathbb{C}$ (see the paragraph containing (9.1) for the definitions of H and G_λ, and the relations between them). Since in addition G_λ has only two singular values: $z = 0$ is the asymptotic value, which is an attracting fixed point of G_λ and $z = 1$ is the only critical point and it is attracted to 0 under forward iterates of G_λ. In particular G_λ is in the class $\mathcal{S}$. Therefore the Fatou set of G_λ is formed by the basin of attraction to 0. Since $(H \circ \Pi) \circ f_\lambda^n(B) = G_\lambda^n \circ (H \circ \Pi)(B)$, $n \geq 0$, and since by Bergweiler's result ([6]), $\Pi(B)$ is contained in the Fatou set of G_λ, we conclude that for all $n \geq 0$ large enough $H \circ \Pi(f_\lambda^n(B))$ is contained in as small neighborhood of of 0 as one wishes. It therefore follows ($H(+\infty) = 0$)) that all points of $\Pi(f_\lambda^n(B))$ have uniformly arbitrarily large real parts. Obviously then the same is true for $f_\lambda^n(B)$, and consequently, $f_\lambda^n(B)$ is contained in D. We are done. ■

Utilizing this theorem and iterating backward the half-plane $\{z \in \mathbb{C} \colon \mathrm{Re} z > 0\}$, we see that the following is true.

Proposition 2.5. *The Julia set* $J(f)$ *is a Cantor bouquet in the sense of* [12].

Let us prove now the following stronger property of F', the one we were really after.

Proposition 2.6. *There are* $c > 0$ *and* $\kappa > 1$ *such that*

$$|(F^n)'(z)| \geq c\kappa^n$$

for all $z \in J(F)$ *and all* $n \geq 1$.

Proof. For every $k \geq 1$ let

$$A_k = \{z \in J(F) : |(F^k)'(z)| > 2\}.$$

In view of Lemma 2.2, $\bigcup_{k=1}^{\infty} A_k \supset J(F)$ and in view of (1.2) there exists $M < 0$ such that $Q_M^c \subset A_1$. Since by Lemma 2.3 $J(F) \setminus Q_M^c$ is a compact set and all the sets A_k, $k \geq 1$, are open, there exists $q \geq 1$ such that $J(F) \setminus Q_M^c \subset A_1 \cup A_2 \cup \ldots \cup A_q$. Thus

$$J(F) \subset A_1 \cup A_2 \cup \ldots \cup A_q.$$

Since $J(F)$ is a closed subset of Q without critical points, using (1.2), we deduce that

$$u := \min\{1, \inf\{|F'(z)| : z \in J(F)\}\} > 0.$$

Fix $z \in J(F)$ and $n \geq 1$. Using the finite cover $\{A_1, A_2, \ldots, A_q\}$ of $J(F)$, we can divide the sequence $z, F(z), \ldots, F^n(z)$ into blocks of length $\leq q$ such that the modulus of the derivative of the composition along each (possibly except for the last one) such a block is larger than 2. This gives that $|(F^n)'(z)| > 2^{E(n/q)} u^q$ and we are done. ■

Let us prove now the following simple but useful lemma.

Lemma 2.7. *There exists $\theta > 0$ such that*

$$J(f) \subset \bigcup_{n \in \mathbb{Z}} \{z \in \mathbb{C} \colon 2\pi n + \theta < \mathrm{Im}(z) < 2\pi(n+1) - \theta\}.$$

Proof. Since the function is $x \to \mathrm{Re}(f(x)) = x + e^{-x} + \mathrm{Re}(\lambda)$ is decreasing at semi-line $(-\infty, 0]$, the image of this semi-line under the map $\mathrm{Re} f$ is contained in $[1+\mathrm{Re}(\lambda), \infty) \subset (0, \infty)$. It therefore follows from Theorem 2.3 that the semi-line $(-\infty, 0]$ is contained in the Fatou set of (f_λ). Since the Fatou set is invariant under translation by $2\pi i$, it therefore suffices to show the existence of some $M < 0$ such that

$$J(f) \cap P_M^c \subset \{z \in \mathbb{C} \colon \frac{\pi}{3} < \mathrm{Im}(z) < 2\pi - \frac{\pi}{3} \quad \text{and} \quad \mathrm{Re}(z) \leq M\}.$$

And indeed, obviously there exists $M < 0$ so small that if $x \leq M$, then $x + \frac{1}{2}e^{-x} > 0$. Hence, if $z \in P_M^c$ and either $0 < \mathrm{Im}(z) < \frac{\pi}{3}$ or $2\pi - \frac{\pi}{3} < \mathrm{Im}(z) < 2\pi$, then $\mathrm{Re}(f_\lambda(z)) = \mathrm{Re}(z) + e^{-\mathrm{Re}(z)} \cos(\mathrm{Im}(z)) + \mathrm{Re}(\lambda) \geq \mathrm{Re}(z) + \frac{1}{2}e^{-\mathrm{Re}(z)} + 1$. Thus $f(z)$ is in the Fatou set of f, and consequently the point z is also in this set. We are done. ■

3. Bounded orbits

Set

$$J_{bd}(F) = \{z \in J(F) : \inf\{\mathrm{Re}(F^n(z)) : n \geq 0\} > -\infty\}.$$

Obviously $J_{bd}(F) \subset J_r(F)$. We shall prove the following.

Theorem 3.1. *We have that* $\mathrm{HD}(J_{bd}(F)) > 1$

Proof. In order to prove this theorem we shall estimate from below the Hausdorff dimension of the limit sets J_R of the iterated functions systems H_R whose construction we just begin to describe. Fix $R \geq 1$. Let

$$S_R = \{z \in P : -4R \leq \mathrm{Re}(z) \leq -R \quad \text{and} \quad 0 < \epsilon_R < \mathrm{Im}(z) \leq 2\pi - \epsilon_R\},$$

where ϵ_R will be defined later in the course of the proof. Fix an integer $k \geq 1$. Consider the inverse map $F_k^{-1} : S_R \to P$ given by the formula

$$F_k^{-1}(z) = f_*^{-1}(z + 2k\pi i).$$

We want to find $k \geq 1$ and $R \geq 1$ such that $F_k^{-1}(S_R) \subset S_R$. Notice that $e^{-F_k^{-1}(z)} + F_k^{-1}(z) + \lambda = z + 2k\pi i$ and therefore

$$|z + 2k\pi i| \leq |e^{-F_k^{-1}(z)}| + |F_k^{-1}(z)| + |\lambda| \leq 2e^{-\mathrm{Re}(F_k^{-1}(z))}$$

for all $k \geq 1$ sufficiently large. Also

$$|z + 2k\pi i| \geq \left|e^{-F_k^{-1}(z)}\right| - |F_k^{-1}(z)| - |\lambda| \geq \frac{1}{2}e^{-\mathrm{Re}(F_k^{-1}(z))}$$

for all $k \geq 1$ large enough. Therefore, if $k \in [e^{2R}, e^{3R}]$ and $R \geq 1$ is large enough, then

$$-4R \leq \mathrm{Re}(F_k^{-1}(z)) \leq -R.$$

Now, if $w = x + iy \in P \setminus S_R$, then

$$\begin{aligned}|\mathrm{Im} f(w)| &= |\mathrm{Im}(e^{-w}) + \mathrm{Im}(w) + \mathrm{Im}(\lambda)| \leq e^{-x}|\sin y| + |y| + |\mathrm{Im}(\lambda)| \\ &\leq e^{4R}\sin(\epsilon_R) + 2\pi + |\mathrm{Im}(\lambda)| \leq \frac{1}{2}e^R\end{aligned}$$

provided that $\epsilon_R > 0$ is sufficiently small. So, if $k \in [e^{2R}, e^{3R}]$ and $\epsilon_R > 0$ is small enough, we have that

$$F_k^{-1}(S_R) \subset S_R.$$

We have produced in this way the iterated function system H_R defined as

$$H_R := \{F_k^{-1} : S_R \to S_R, \quad e^{2R} \leq k \leq e^{3R}\}. \tag{3.1}$$

By the location of S_R and due to Koebe's distortion theorem, the iterated function system H_R satisfies all the requirements of [21], and consequently, all the results proven there apply to the system H_R. Let J_R be the limit set (see [21]) of the system H_R. Since $J_R \subset J_{bd}(F)$, it suffices to show that $\mathrm{HD}(J_R) > 1$. First note that, if $w = f(z) = z + e^{-z} + \lambda$, then

$$(f_*^{-1})'(w) = \frac{1}{f_*^{-1}(w) - w + \lambda + 1} \tag{3.2}$$

Let $z \in S_R$. Applying then (3.2), we get

$$\begin{aligned}|(F_k^{-1})'(z)| &= \frac{1}{|f_*^{-1}(z + 2k\pi i) - (z + 2k\pi i) + 1 + \lambda|} \geq \frac{1}{|z + 2k\pi i| + (4R + 2\pi i + 1 + |\lambda|)} \\ &\geq \frac{1}{9k + (4R + 2\pi + 1 + |\lambda|)} \geq \frac{1}{10k}\end{aligned} \tag{3.3}$$

if R is sufficiently large and $k \in [e^{2R}, e^{3R}]$. Fix $t > 0$ and let $\mathrm{P}_R(t)$ be the topological pressure of the iterated function system H_R evaluated at t (see [21] for its definition, basic properties and further references). Then by (3.3) we have for some $z \in S_R$ that

$$\begin{aligned} P_R(1) &= \log\left(\sum_{k=e^{2R}}^{e^{3R}} |(F_k^{-1})'(z)|\right) \geq \log\left(\sum_{k=e^{2R}}^{e^{3R}} \frac{1}{10k}\right) \\ &= -\log 10 + \left(\log(e^{3R}) - \log(e^{2R})\right) \\ &= -\log 10 + R > 0 \end{aligned}$$

if $R \geq 1$ is sufficiently large. It therefore follows from Theorem 3.15 of [21] that $\mathrm{HD}(J_R) > 1$. We are done. ■

4. Pressure, Perron-Frobenius Operators and Generalized Conformal Measures

For every $t \geq 0$ and every $z \in Q \setminus \mathrm{PC}(F)$ define the lower and upper topological pressure respectively by

$$\underline{\mathrm{P}}_z(t) = \liminf_{n\to\infty} \frac{1}{n} \log \sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-t} \text{ and } \overline{\mathrm{P}}_z(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-t}.$$

Since any two points in $J(F)$ belong to an open simply connected set disjoint from $\mathrm{PC}(F)$, it follows from Koebe's distortion theorem that $\underline{\mathrm{P}}_z(t)$ and $\overline{\mathrm{P}}_z(t)$ are independent of z and we denote their respective values by $\underline{\mathrm{P}}(t)$ and $\overline{\mathrm{P}}(t)$. It follows from (1.1) that for every $t \geq 0$ and every $z \in Q \setminus \mathrm{PC}(F)$

$$\begin{aligned} \mathrm{P}_z(1,t) :&= \sum_{x\in F^{-1}(z)} |F'(x)|^{-t} = \sum_{x\in F^{-1}(z)} |1 - e^{-x}|^{-t} = \sum_{k=-\infty}^{+\infty} |1 - e^{-z_k}|^{-t} \\ &= \sum_{k=-\infty}^{+\infty} |1 + z_k + \lambda - (\tilde{z} + 2\pi i k)|^{-t}, \end{aligned} \tag{4.1}$$

where $\tilde{z}$ is the only point in $\pi^{-1}(z) \cap P$ and $z_k = f_*^{-1}(\tilde{z} + 2\pi i k)$, $k \in \mathbb{Z}$ is the only point in P such that $f(z_k) = \tilde{z} + 2\pi i k$. This meaning of $\tilde{z}$ and z_k will be kept from now on throughout the entire paper. We will several times need the following.

Lemma 4.1. *If $t > 1$, then*

$$||\mathrm{P}(1,t)||_\infty := \sup\{\mathrm{P}_z(1,t) : z \in J(F)\} < +\infty.$$

Proof. Let

$$\mathbb{Z}_\lambda = \{k \in \mathbb{Z} : \pi|k| \geq |\mathrm{Im}\lambda| + 4\pi\}.$$

If $z \in J(F)$ and $k \in \mathbb{Z}_\lambda$, then

$$|1 + z_k + \lambda - (\tilde{z} + 2\pi ik)| \geq |\mathrm{Im}(1 + z_k + \lambda - (\tilde{z} + 2\pi ik))| \geq 2\pi|k| - |\mathrm{Im}\lambda| \geq \pi|k| \tag{4.2}$$

Fix now $T > 0$ so large that if $\mathrm{Re}z \leq -T$, then $\mathrm{Re}(z_k) \leq -1$ for all $k \in \mathbb{Z}$. So, for all such z and all $k \in \mathbb{Z}$

$$|1 - e^{-z_k}| \geq |e^{-z_k}| - 1 = e^{\mathrm{Re}(-z_k)} - 1 \geq e - 1 \geq 1. \tag{4.3}$$

Since $z_k \in J(F)$, it follows from Theorem 2.3 that the equality $1 - e^{-z_k} = 0$ never holds, and therefore

$$M = \inf\{|1 - e^{-z_k}| : z \in Q_{-t},\, k \in \mathbb{Z} \setminus \mathbb{Z}_\lambda\} > 0.$$

Combining this along with (4.3) and (4.2), we get for all $t > 1$ and all $z \in J(F)$ that

$$\begin{aligned} \mathrm{P}_z(1,t) &= \sum_{k \in \mathbb{Z}_\lambda} |1 + z_k + \lambda - (\tilde{z} + 2\pi ik)|^{-t} + \sum_{k \in \mathbb{Z} \setminus \mathbb{Z}_\lambda} |1 - e^{-z_k}|^{-t} \\ &\leq \sum_{k \in \mathbb{Z}_\lambda} (\pi|k|)^{-t} + \#(\mathbb{Z} \setminus \mathbb{Z}_\lambda) \max(1, M)^{-t} < +\infty. \end{aligned}$$

We are done. ■

Observe now that for every $n \geq 1$ and every $z \in J(F)$

$$\begin{aligned} \sum_{x \in F^{-n}(z)} |(F^n)'(x)|^{-t} &= \sum_{w \in F^{-(n-1)}(z)} \sum_{x \in F^{-1}(w)} |(F^{n-1})'(w)|^{-t} |F'(x)|^{-t} \\ &= \sum_{w \in F^{-(n-1)}(z)} |(F^{n-1})'(w)|^{-t} \sum_{x \in F^{-1}(w)} |F'(x)|^{-t} \\ &\leq ||\mathrm{P}(1,t)||_\infty \sum_{w \in F^{-(n-1)}(z)} |(F^{n-1})'(w)|^{-t}. \end{aligned}$$

Therefore, we obtain by induction that $\sum_{x \in F^{-n}(z)} |(F^n)'(x)|^{-t} \leq ||\mathrm{P}(1,t)||_\infty^n$ and consequently

$$\overline{\mathrm{P}}(t) = \overline{\mathrm{P}}_z(t) \leq \log ||\mathrm{P}(1,t)||_\infty \tag{4.4}$$

for all $t > 1$. We shall now establish the existence of conformal measures for the map $F : J(F) \to J(F)$. A Borel measure m_t is called (t, α_t)-conformal (with $t > 1$ and $\alpha_t \geq 0$) if for any Borel set $A \subset Q$ on which F is injective, we have

$$m_t(F(A)) = \int_A \alpha_t |F'|^t dm_t$$

As the first and most important step in the proof of the existence of these measures we shall prove the tightness of an appropriate sequence of "semi"-conformal measures. In order to define this sequence we apply the general method worked out in [11] (comp. also Chapter 10 of [23]). So, fix $n \geq 1$ and consider the set

$$K_n = \bigcap_{j \geq 0} F^{-j}\Big(Q_{-n}\Big)$$

Since $F : Q \to Q$ is continuous, K_n is a F-forward invariant compact subset of Q. The results from [11] and Chapter 10 of [23] establish for all $n \geq 1$ the existence of a Borel probability measure m_n supported on K_n and a non-decreasing sequence $\{\mathrm{P}_n(t)\}_{n=1}^{\infty}$ of "pressure-like" numbers with the following two properties. If $A \subset Q_{-n}$ is a Borel set such that $F|_A$ is 1-to-1, then

$$m_n(F(A)) \geq e^{\mathrm{P}_n(t)} \int_A |F'|^t dm_n.$$

If in addition, $A \cap \partial Q_{-n} = \emptyset$, the inequality sign in the above formula can be replaced by the equality sign. We are now ready to state and to prove the announced tightness.

Proposition 4.2. *The sequence $\{m_n\}_{n=1}^{\infty}$ is tight.*

Proof. Fix $\epsilon > 0$ and $M > 0$. We shall estimate the measure m_n of the following set

$$\begin{aligned}
m_n(\{z \in J(F) : \mathrm{Re}(F(z)) \leq -M\}) &\leq m_n(F^{-1}(\{z \in Q : \mathrm{Re}(z) \leq -M\})) \\
&= m_n(\bigcup_{k \in \mathbb{Z}} F_k^{-1}(\{z \in Q : \mathrm{Re}(z) \leq -M\}) \\
&= \sum_{k \in \mathbb{Z}} m_n(F_k^{-1}(Q^c_{-M})) \\
&\leq \sum_{k \in \mathbb{Z}} e^{-\mathrm{P}_n(t)} \sup_{z \in Q^c_{-M}} \{|(F_k^{-1})'(z)|^t\} m_n(Q^c_{-M}) \\
&\leq e^{-\mathrm{P}_n(t)} \sum_{k \in \mathbb{Z}} \sup_{z \in Q^c_{-M}} \{|(F_k^{-1})'(z)|^t\}
\end{aligned}$$

If $z \in Q^c_{-M}$, then

$$|(F_k^{-1})'(z)| = \frac{1}{|f_*^{-1}(z + 2k\pi i) - (z + 2k\pi i) + 1 + \lambda|}. \tag{4.5}$$

Recall that for every $z \in Q$ we have set

$$z_k := F_k^{-1}(z) = f_*^{-1}(z + 2k\pi i). \tag{4.6}$$

If $M > 0$ is large enough, then it easily follows from (1.1) that $-\mathrm{Re}(z_k)$ becomes as large as we wish uniformly with respect to $k \in \mathbb{Z}$. Since in addition $0 \leq \mathrm{Im}(z_k) \leq 2\pi$, we therefore get that $|e^{-z_k} + z_k + \lambda| \geq \frac{1}{2}|e^{-z_k}| = \frac{1}{2}e^{-\mathrm{Re}(z_k)}$. Hence $2|z + 2\pi ik| \geq e^{-\mathrm{Re}(z_k)}$ and consequently $-\mathrm{Re}(F_k^{-1}(z)) \leq \log 2 + \log|z + 2\pi ik|$. Thus

$$|\mathrm{Re}(z_k) - \mathrm{Re}(z)| \geq \mathrm{Re}(z_k) - \mathrm{Re}(z) \geq -\log 2 - \log|z + 2\pi ik| - \mathrm{Re}(z) \tag{4.7}$$

Consider $\hat{k}$, the largest $k \geq 0$ such that $\log|z + 2\pi ki| \leq -\frac{1}{3}\mathrm{Re}(z)$ for all $k \in \mathbb{Z}$ with $|k| \leq \hat{k}$. Then $\log|z + 2\pi i(\hat{k} + 1)| \geq -\frac{1}{3}\mathrm{Re}(z)$ or equivalently $|z + 2\pi i(\hat{k} + 1)| \geq e^{-\mathrm{Re}(z)/3}$. Since $|z + 2\pi i(\hat{k} + 1)| \leq 2\pi(\hat{k} + 1) + |z| \leq 2\pi(\hat{k} + 1) + \frac{1}{2}e^{-\mathrm{Re}(z)/3}$, assuming that M is large enough, we get $2\pi(\hat{k} + 1) \geq \frac{1}{2}e^{-\mathrm{Re}(z)/3}$ consequently

$$\hat{k} \geq e^{-\mathrm{Re}(z)/4} \geq e^{M/4} \tag{4.8}$$

again provided that $M > 0$ is sufficiently large. Now if $|k| \le \hat{k}$, then using (4.7) we get

$$|\mathrm{Re}(z_k) - \mathrm{Re}(z)| \ge -\log 2 - \frac{2}{3}\mathrm{Re}(z) \ge -\frac{1}{2}\mathrm{Re}(z) \ge M/2$$

if only $M > 0$ is large enough. Therefore

$$\begin{aligned}
|z_k - &(z + 2\pi ik) + \lambda + 1| \\
&\ge \frac{1}{2}\,|\mathrm{Re}(z_k - (z + 2\pi ik) + \lambda + 1)| + \frac{1}{2}\,|\mathrm{Im}(z_k - (z + 2\pi ik) + \lambda + 1)| \\
&\ge \frac{1}{2}\,|\mathrm{Re}(z_k) - \mathrm{Re}(z)| - \frac{1}{2}|\mathrm{Re}(\lambda + 1)| + \frac{1}{2}\,(2\pi|k| - 4\pi - \mathrm{Im}(\lambda + 1)) \\
&\ge \frac{M}{2} - \frac{1}{2}|\mathrm{Re}(\lambda + 1)| + \frac{1}{2}\,(2\pi|k| - 4\pi - \mathrm{Im}(\lambda + 1)) \ge \frac{M}{3} + \pi|k|
\end{aligned} \tag{4.9}$$

provided that M is large enough. If $|k| \ge \hat{k}$ then

$$|z_k - (z + 2\pi ik) + \lambda + 1| \ge |\mathrm{Im}(z_k - (z + 2\pi k) + \lambda + 1)| \ge 2\pi|k| - 4|\pi| - \mathrm{Im}(\lambda + 1) \ge \pi|k|$$

if only M is sufficiently large. Combining this, (4.9) and (4.8) we obtain

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} &e^{-\mathrm{P}_n(t)} \sup_{z\in Q^c_{-M}} \{|(F_k^{-1})'(z)|\}^t \\
&= e^{-\mathrm{P}_n(t)} \sum_{|k|\le\hat{k}} \sup_{z\in Q^c_{-M}} \{|(F_k^{-1})'(z)|\}^t + e^{-\mathrm{P}_n(t)} \sum_{|k|>\hat{k}} \sup_{z\in Q^c_{-M}} \{|(F_k^{-1})'(z)|\}^t \\
&\le e^{-\mathrm{P}_n(t)} \sum_{|k|\le\hat{k}} \left(\frac{M}{3} + \pi|k|\right)^{-t} + e^{-\mathrm{P}_n(t)} \sum_{|k|>\hat{k}} \frac{1}{(\pi|k|)^t} \\
&\le 2e^{-P_n(t)} \sum_{k=0}^{\infty} \left(\frac{M}{3} + \pi|k|\right)^{-t} + \frac{2}{\pi^t} e^{-\mathrm{P}_n(t)} \sum_{k\ge\hat{k}} k^{-t} \\
&\le C_t^{(1)} e^{-\mathrm{P}_n(t)} M^{1-t} + C_t^{(2)} e^{-P_n(t)} \hat{k}^{1-t} \\
&\le C_t e^{-\mathrm{P}_n(t)} \max\{M^{1-t},\, e^{\frac{M}{4}(1-t)}\} \\
&= C_t e^{-\mathrm{P}_n(t)} M^{1-t}
\end{aligned} \tag{4.10}$$

if M is large enough, where $C_t > 0$ is a constant depending only on t. Since

$$\sup\{|F'(z)| : z \in K_n\} < \infty,$$

$\mathrm{P}_n(t) > -\infty$ for all n large enough, say $n \ge n_0$. Since the sequence $\{\mathrm{P}_n(t)\}_{n=0}^{\infty}$ is non-decreasing, we get from (4.10) that for all $n \ge n_0$ and all $B \subset Q^c_{-M}$

$$m_n(F^{-1}(B)) \le C_t e^{-\mathrm{P}_{n_0}(t)} M^{1-t} m_n(B) \tag{4.11}$$

If $\mathrm{Re} z \le -M$, then $|e^{-z}| = e^{-\mathrm{Re} z} \ge e^M$. Hence, if in addition $\mathrm{Re}(f(z)) \ge -M$, then $|\mathrm{Im}(f(z))| \ge \sqrt{e^{2M} - M^2} \ge e^M/2$ provided that $M > 0$ is sufficiently large. Now, for every

$k \geq e^{2M}/2$, we have $|(F_k^{-1})'(z)| = |f_*^{-1}(z+2k\pi i) - (z+2k\pi i) + 1 + \lambda| \geq 2\pi k - 4\pi - |\lambda+1| \geq k$ if only M is large enough. Hence

$$m_n(\{z \in J(F) : \mathrm{Re}(z) \leq -M \text{ and } \mathrm{Re}(f(z)) > -M\}) \leq e^{-\mathrm{P}_{n_0}(t)} \sum_{k \geq e^{M}/2} k^{-t} \preceq e^{-\mathrm{P}_{n_0}(t)} e^{M(1-t)}. \tag{4.12}$$

Combining this and (4.11), we see that the sequence $\{m_n\}_{n \geq 1}$ is tight. ■

Recall that for every $n \geq n_0$, $\mathrm{P}_n(t) \in [\mathrm{P}_{n_0}(t), \overline{\mathrm{P}}(t)]$, and that the sequence $\{\mathrm{P}_n(t)\}_{n=1}^{\infty}$ is non-decreasing. Denote the exponent of its limit by α_t. It follows from Lemma 4.2 and Prokhorov's theorem that passing to a subsequence $\{n_k\}_{k=1}^{\infty}$, we may assume that the sequence $\{m_{n_k}\}_{k=1}^{\infty}$ converges weakly, say to a measure m_t on Q. Since there could be a problem with conformality of measures m_{n_k} only on sets $\{z \in J(F) : |\mathrm{Re} z| = -n_k\}$, since $n_k \nearrow +\infty$ when $k \nearrow +\infty$, and since $F : J(F) \to J(F)$ is an open map, which has no critical points, proceeding, with obvious modifications, as in [11] (comp. Chapter 10 of [23]), we obtain the following first basic result.

Theorem 4.3. *For every $t > 1$ there exist $\alpha_t \geq 0$ and a (t, α_t)-conformal measure m_t for the map $F : J(F) \to J(F)$. In addition $m_t(J(F)) = 1$.*

Let $C_b = C_b(J(F))$ be the Banach space of all bounded continuous complex-valued functions on $J(F)$. Lemma 4.1 enables us to define for all $t \in \mathbb{C}$ with $\mathrm{Re} t > 1$ the Perron-Frobenius operator $\mathcal{L} = \mathcal{L}_t : C_b \to C_b$ by the formula

$$\begin{aligned}\mathcal{L}_t g(z) &= \sum_{x \in F^{-1}(z)} |F'(x)|^{-t} g(x) = \sum_{x \in F^{-1}(z)} |1 - e^{-x}|^{-t} g(x) = \sum_{k=-\infty}^{+\infty} |1 - e^{-z_k}|^{-t} g(z_k) \\ &= \sum_{k=-\infty}^{+\infty} |1 + z_k + \lambda - (\tilde{z} + 2\pi i k)|^{-t} g(z_k)\end{aligned} \tag{4.13}$$

It immediately follows from (4.13) that

$$|\mathcal{L}_t g(z)| \leq ||\mathcal{L}_t 1\!\!1||_\infty ||g||_\infty. \tag{4.14}$$

Applying and improving the formulas obtained in the proof of Lemma 4.1, we shall prove the following.

Lemma 4.4. *We have*

$$\lim_{z \to -\infty} \mathcal{L}_t 1\!\!1(z) = 0.$$

Proof. Consider $z \in J(F)$ and $\tilde{z} \in \Pi^{-1}(z) \cap P$. It follows from (1.1) that

$$\lim_{z \to -\infty} z_k = -\infty \tag{4.15}$$

uniformly with respect to $k \in \mathbb{Z}$. Hence, applying (1.1) again, we see that

$$\lim_{z \to -\infty} (\tilde{z} - z_k) = \infty \text{ or equivalently } \lim_{z \to -\infty} |\text{Re}(\tilde{z} - z_k)| = +\infty \tag{4.16}$$

uniformly with respect to $k \in \mathbb{Z}$. Fix $M > 0$. Then for all $k \in \mathbb{Z}$ and all $z \in J(F)_{T_1}^c$ with $-T_1 > 0$ large enough, we get

$$\begin{aligned} |1 + z_k + \lambda - (\tilde{z} + 2\pi ik)| &\geq \frac{1}{2}(|\text{Im}\Big(1 + z_k + \lambda - (\tilde{z} + 2\pi ik)\Big)| + |\text{Re}\Big(1 + z_k + \lambda - (\tilde{z} + 2\pi ik)\Big)|) \\ &\geq \frac{\pi}{2}|k| + \frac{1}{2}\Big(\text{Re}(\tilde{z} - z_k) - (1 + \text{Re}\lambda)\Big) \geq \frac{\pi}{2}|k| + M. \end{aligned} \tag{4.17}$$

Therefore, applying (4.13), we get for all $z \in J(F)_{T_1}^c$ that

$$\mathcal{L}_t(1\!\!1)(z) \leq \sum_{k \in \mathbb{Z}} \left(\frac{\pi}{2}|k| + M \right)^{-t}.$$

So, the proof is concluded by letting $M \nearrow +\infty$. ■

Notice also that $\mathcal{L}_t : C_b \to C_b$ is a bounded operator and its norm is equal to $||\text{P}(1,t)||_\infty$. Assume from now on throughout this section that $t \in (1, \infty)$ and consider the dual operator $\mathcal{L}_t^* : C_b^* \to C_b^*$ given by the formula $\mathcal{L}_t^*\mu(g) = \mu(\mathcal{L}_t g)$. A straightforward calculation (see Proposition 2.2 in [8] for example, where the finiteness of the partition can be replaced by its countability) shows the following.

Proposition 4.5. *For every $t > 1$, $\mathcal{L}_t^* m_t = \alpha_t m_t$.*

Let

$$\delta = \frac{1}{2} \min \left\{ \frac{1}{2}, \text{dist}\big(J(F), \text{PC}(F)\big) \right\} \tag{4.18}$$

Observe that for every $v \in J(F)$ and every $n \geq 1$ there exists a unique holomorphic inverse branch $F_v^{-n} : B(F^n(v), 2\delta) \to P$ of F^{-n} sending $F^n(v)$ to v. In particular $F_v^{-n}(J(F) \cap B(F^n(v), 2\delta)) \subset J(F)$. Fix now $t > 1$ and define

$$\hat{\mathcal{L}}_t = \alpha_t^{-1} \mathcal{L}_t.$$

Fix any two points w and z in $Q_x \setminus \text{PC}(F)$. There then exists the shortest smooth arc $\gamma_{w,z}$ joining w and z in $Q \setminus B\big(\text{PC}(F), 2\delta\big)$. The supremum of (Euclidean) lengths of arcs $\gamma_{w,z}$ taken over all pairs $w, z \in P_x$ is finite and consequently there exists a number $l_x \geq 1$ such that each such arc $\gamma_{w,z}$ can be covered by a chain of at most balls l_x balls of radius δ centered at points of $\gamma_{w,z}$. We may assume in addition that $U_{w,z}$, the union of these balls is a simply connected set. It then follows from Koebe's distortion theorem that there exists $K_x \geq 1$ such that if $F_*^{-n} : U_{w,z} \to Q$ is a holomorphic branch of F^{-n}, then

$$\frac{|(F_*^{-n})'(w)|}{|(F_*^{-n})'(z)|} \leq K_x$$

and consequently

$$K_x^{-t} \le \frac{\hat{\mathcal{L}}_t^n(\mathbb{1})(w)}{\hat{\mathcal{L}}_t^n(\mathbb{1})(z)} \le K_x^t. \tag{4.19}$$

In the remaining part of this section we develop the corresponding arguments from [26]. We start with the following.

Lemma 4.6. $\Theta = \sup_n\{||\hat{\mathcal{L}}_t^n(\mathbb{1})||_\infty\} < \infty$.

Proof. In view of Lemma 4.4 there exists $x \le 0$ so large in absolute value that for every $w \in Q_x^c$

$$\alpha_t^{-1}\mathcal{L}_t(\mathbb{1})(w) \le 1. \tag{4.20}$$

We shall prove by induction that for every $n \ge 0$

$$||\hat{\mathcal{L}}_t^n(\mathbb{1})||_\infty \le \frac{K_x^t}{m_t(Q_x)}.$$

And indeed, for $n = 0$ this estimate is immediate. So, suppose that it holds for some $n \ge 0$ and let $z^{n+1} \in Q$ be such a point that $\hat{\mathcal{L}}_t^{n+1}(\mathbb{1})(z^{n+1}) = ||\hat{\mathcal{L}}_t^{n+1}(\mathbb{1})||_\infty$ (such a point exists due to Lemma 4.4). If $z^{n+1} \in Q_x$, then using (4.19) and (4.13), we obtain

$$1 = \int \hat{\mathcal{L}}_t^{n+1}(\mathbb{1})dm \ge \int_{Q_x} \hat{\mathcal{L}}_t^{n+1}(\mathbb{1})dm \ge K_x^{-t}||\hat{\mathcal{L}}_t^{n+1}(\mathbb{1})||_\infty m(Q_x)$$

and consequently $||\hat{\mathcal{L}}_t^{n+1}(\mathbb{1})||_\infty \le K_x^t\Big(m(Q_x)\Big)^{-1}$. If $z^{n+1} \notin Q_x$, then it follows from (4.20) and the inductive assumption that

$$\begin{aligned} ||\hat{\mathcal{L}}_t^{n+1}(\mathbb{1})||_\infty &= \hat{\mathcal{L}}_t^{n+1}(\mathbb{1})(z^{n+1}) = \sum_{k=-\infty}^{+\infty} \hat{\mathcal{L}}_t^n(\mathbb{1})\Big(z_k^{n+1}\Big)\alpha_t^{-1}|1 - e^{-\tilde{z}^{n+1}}|^{-t} \\ &\le \sum_{k=-\infty}^{+\infty} \alpha_t^{-1}||\hat{\mathcal{L}}_t^n(\mathbb{1})||_\infty|1 - e^{-\tilde{z}^{n+1}}|^{-t} \le K_x^t\Big(m(Q_x)\Big)^{-1}\alpha_t^{-1}\sum_{k=-\infty}^{+\infty}|1 - e^{-\tilde{z}^{n+1}}|^{-t} \\ &= K_x^t\Big(m(Q_x)\Big)^{-1}\alpha_t^{-1}\hat{\mathcal{L}}_t(\mathbb{1})(z^{n+1}) \le K_x^t\Big(m(Q_x)\Big)^{-1}, \end{aligned}$$

where the meaning of the points z_k^{n+1} is provided by (4.6). We are done. ■

Lemma 4.7. *There exists $x_0 \le 0$ such that for every $x \ge x_0$*

$$\inf_{n\ge 0}\sup_{z\in Q_x}\{\hat{\mathcal{L}}_t^n(\mathbb{1})(z)\} \ge \frac{1}{4}.$$

Proof. Let Θ come from Lemma 4.6. Let $x_0 \le 0$ be so large in absolute value that $m(Q_{x_0}^c) \le 1/(4\Theta)$. Suppose for the contrary that $\mathcal{L}_0^n(1\!\!1)(z) < 1/4$ for some $n \ge 0$ and all $z \in Q_{x_0}$. Then

$$1 = \int \hat{\mathcal{L}}_t^n(1\!\!1)dm = \int_{Q_{x_0}} \hat{\mathcal{L}}_t^n(1\!\!1)dm + \int_{Q_{x_0}^c} \hat{\mathcal{L}}_t^n(1\!\!1)dm \le \frac{1}{4}m(Q_{x_0}) + \Theta m(Q_{x_0}^c) \le \frac{1}{4} + \Theta\frac{1}{4\Theta} = \frac{1}{2}.$$

This contradiction finishes the proof. ■

As an immediate consequence of this lemma and (4.19) we get the following.

Lemma 4.8. *For every $x \le x_0$ we have*

$$\inf_{n\ge 0}\inf_{z\in Q_x}\{\hat{\mathcal{L}}_t^n(1\!\!1)(z)\} \ge \frac{1}{4}\Big(\max\{K_x, K_{x_0}\}\Big)^{-t}.$$

We shall prove the following.

Proposition 4.9. *For every $t > 1$ we have $\underline{\mathrm{P}}(t) = \overline{\mathrm{P}}(t) = \log\alpha_t$.*

Proof. It follows from Lemma 4.6 that $\mathcal{L}_t^n(1\!\!1)(z) \le \Theta\alpha_t^n$ for every $z \in P$. Hence

$$\overline{\mathrm{P}}(t) = \overline{\mathrm{P}}_z(t) = \limsup_{n\to\infty}\frac{1}{n}\log\mathcal{L}^n(1\!\!1)(z) \le \log\alpha_t.$$

In view of Lemma 4.8, $\mathcal{L}_t^n(1\!\!1)(x_0) \ge \frac{1}{4}K_{x_0}^{-t}\alpha_t^n$ and therefore

$$\underline{\mathrm{P}}(t) = \underline{\mathrm{P}}_z(t) = \liminf_{n\to\infty}\frac{1}{n}\log\mathcal{L}_t^n(1\!\!1)(z) \ge \log\alpha_t.$$

We are done. ■

Denote the common value of $\underline{\mathrm{P}}(t)$ and $\overline{\mathrm{P}}(t)$ by $\mathrm{P}(t)$. Its basic properties are listed in the following.

Lemma 4.10. *The function $t \mapsto \mathrm{P}(t)$, $t \ge 0$, has the following properties.*

- *(a) There exists $t \in (0,1)$ such that $0 \le \mathrm{P}(t) < +\infty$.*
- *(b) $\mathrm{P}(t) < +\infty$ for all $t > 1$.*
- *(c) The function $\mathrm{P}(t)$ restricted to the interval $(1,+\infty)$ is convex, continuous and strictly decreasing.*
- *(d) $\lim_{t\to+\infty}\mathrm{P}(t) = -\infty$.*
- *(e) There exists exactly one $t > 1$ such that $\mathrm{P}(t) = 0$.*

Proof. The convexity of the function $t \mapsto \mathrm{P}(t)$, $t > 1$, follows immediately from Hölders inequality. Thus, this function is continuous. The facts that $\mathrm{P}(t)$, $t > 1$, is strictly decreasing and that $\lim_{t\to+\infty}\mathrm{P}(t) = -\infty$ follow from Proposition 2.6. Thus, the items (c) and (d) are proven. The item (b) follows from Lemma 4.1 and (4.4). In order to prove item (a), consider the iterated function system H_R introduced in the proof of Theorem 3.1 with $R > 0$ so large that $\mathrm{HD}(J_R) > 1$. Then by (b) $\mathrm{P}(\mathrm{HD}(J_R)) < +\infty$ and $\mathrm{P}(\mathrm{HD}(J_R)) \geq \mathrm{P}_R(\mathrm{HD}(J_R)) = 0$. So, item (a) is proved. Item (e) is an immediate consequence of items (a)-(d). We are done. ■

Recall

$$I_\infty(F) = \{z \in J(F) : \lim_{n\to\infty} F^n(z) = -\infty\},$$

i.e. $I_\infty(F)$ is the set of points escaping to infinity under forward iterates of F. Analogously define

$$I_\infty(f) = \{z \in J(f) : \lim_{n\to\infty} \mathrm{Re}(f^n(z)) = -\infty\}.$$

Denote

$$J_r(F) = J(F) \setminus I_\infty(F) \quad \text{and} \quad J_r(f) = J(f) \setminus I_\infty(f)$$

and notice that

$$I_\infty(f) = \Pi^{-1}(I_\infty(F)).$$

For $t > 1$ let m_t be the $(t, e^{\mathrm{P}(t)})$-conformal measure constructed in Theorem 4.3 (due to Proposition 4.9 $\alpha_t = e^{\mathrm{P}(t)}$). We shall prove the following.

Proposition 4.11. *For every $t > 1$ there exists $M > 0$ such that for m_t-a.e. x*

$$\limsup_{n\to\infty} \mathrm{Re}(F^n(x)) \geq -M.$$

In particular, $m_t(I_\infty(F)) = 0$ or equivalently $m_t(J_r(F)) = 1$.

Proof. Replacing in the formula (4.11) m_n by m_t (and consequently $\mathrm{P}_{n_0}(t)$ by $\mathrm{P}(t)$), we get for all $M > 0$ large enough and all $B \subset J_{-M}^c$ that

$$m_t(F^{-1}(B)) < C_t e^{-\mathrm{P}(t)} M^{1-t} m_t(B) \tag{4.21}$$

Hence, by a straightforward induction,

$$m_t\Big(F^{-1}(B) \cap \cdots \cap F^{-n}(B)\Big) \leq (C_t e^{-\mathrm{P}(t)} M^{1-t})^n m_t(B)$$

This implies that for all M large enough

$$m_t(\bigcap_{n=0}^{\infty} F^{-n}(Q_{-M}^c)) = 0$$

and consequently

$$m_t(\bigcup_{k=0}^{\infty} F^{-k}(\bigcap_{n=0}^{\infty} F^{-n}(Q_{-M}^c))) = 0.$$

The proof is finished. ■

Let us show now that the estimates used in Proposition 4.11 and Proposition 4.2 lead to the following.

Corollary 4.12.

$$m_t(Q^c_{-M}) \le Ce^{(1-t)M}$$

for some constant C and all $M \ge 0$ large enough.

Proof. It follows from (4.21) that

$$m_t\Big(Q^c_{-M} \cap F^{-1}Q^c_{-M}\Big) \le C_t e^{-\mathrm{P}(t)} M^{1-t} m_t(Q^c_{-M})$$

The formula (4.12) with m_n replaced by m_t reads that

$$m_t\Big(Q^c_{-M} \cap F^{-1}Q_{-M}\Big) \le \hat{C}_t e^{M(1-t)}$$

with an appropriate constant $\hat{C}_t > 0$. These two sets cover the whole set Q^c_{-M}. The first inequality says that (for all M sufficiently large) the first set covers less than, say, one half of the measure of Q^c_{-M}. Thus,

$$m_t(Q^c_{-M}) \le 2m\Big(Q^c_{-M} \cap F^{-1}Q_{-M}\Big) \le 2\hat{C}e^{(1-t)M}$$

and the proof is complete. ■

In order to prove Theorem 4.14 we will need the following generalization of Proposition 4.11.

Lemma 4.13. *If ν is a (t, β^j)-conformal measure for F^j with $t > 1$ and $\beta > 0$, then there exists $M > 0$ such that*

$$\limsup_{n\to\infty} \mathrm{Re}(F^{jn}(x)) \ge -M.$$

for ν-a.e. $z \in J(F)$. In particular, $\nu(I_\infty(F)) = 0$ or equivalently $\nu(J_r(F)) = 1$.

Proof. Define inductively for all integers $k \ge 0$ respectively the functions ν_k whose domains coincide with all Borel sets $A \subset J(F)$ such that $F^k|_A$ is injective, as follows

$$\nu_0(A) = \nu(A) \text{ and } \nu_{k+1}(A) = \int_{F^k(A)} \beta^{-1} \left|\left(\left(F^k|_{F^k(A)}\right)^{-1}\right)'\right|^t d\nu_k.$$

The same computation as that leading to (4.10) and (4.11) gives us that if $k \ge 0$, $F^k|_B$ is injective and $B \subset J(F)^c_{-M}$, then

$$\nu_{k+1}(F^{-1}(B)) \le C\beta^{-1}M^{1-t}\nu_k(B).$$

Thus we get by induction that for every $B \subset J(F)^c_{-M}$ and every $n \ge 0$

$$\nu_n\Big(F^{-n}(B)\Big) \le (C\beta^{-1}M^{1-t})^n \nu(B).$$

Since, by our assumptions, $\nu_j = \nu$, we therefore get

$$\nu(F^{-j}(B)) \le (C\beta^{-1}M^{1-t})^j \nu(B).$$

Similarly as in the proof of Proposition 4.11 this implies that $\nu\left(\bigcap_{s=0}^{\infty} F^{-js}(Q_{-M}^c)\right) = 0$ and consequently

$$\nu(\bigcup_{k=0}^{\infty} F^{-jk}(\bigcap_{n=0}^{\infty} F^{-jn}(Q_{-M}^c))) = 0.$$

The proof is finished. ■

Theorem 4.14. *The $(t, e^{\mathrm{P}(t)})$-conformal measure $m = m_t$ is a unique (t, β)-conformal measure for F with $t > 1$. In addition it is ergodic with respect to each iterate of F.*

Proof. Fix $j \geq 1$. Suppose that ν is a (t, β^j)-conformal measure for F^j with some $t > 1$ and $\beta > 0$. In view of Lemma 4.13 we have $\nu(I_\infty(F)) = 0$. Given $N < 0$ let $J_{r,N}(F)$ be the subset of $J_r(F)$ defined as follows: $z \in J_{r,N}(F)$ if the trajectory of z under F^j has an accumulation point in $J(F)_N^c$. Obviously, $\bigcup_N J_{r,N}(F) = J_r(F)$ and by Proposition 4.11 and Lemma 4.13 there exists $M < 0$ such that $\nu(J_{r,M}(F)) = m(J_{r,M}(F)) = 1$. Fix $z \in J_{r,N}(F)$ and let us recall that $\delta \leq \mathrm{dist}(J(F), \mathrm{PC}(F))/2$. Then there exist $y \in J(F)$ such that $\mathrm{Re}(y) > N$ and an increasing sequence $\{n_k\}_{k=1}^\infty$ such that $y = \lim_{k\to\infty} F^{jn_k}(z)$. Considering for k large enough the sets $F_z^{-jn_k}(B(y,\delta))$ and $F_z^{-jn_k}(B(y,\delta/K))$, where $F_z^{-jn_k}$ is the holomorphic inverse branch of F^{n_k} defined on $B(y, 2\delta)$ and sending $F^{jn_k}(z)$ to z, using conformality of measures m and ν along with Koebe's distortion theorem we easily deduce that

$$B_N(\nu)^{-1}\beta^{-jn_k}|(F^{jn_k})'(z)|^{-t} \leq \nu\Big(B(z, c|(F^{jn_k})'(z)|^{-1})\Big) \leq B_N(\nu)\beta^{-jn_k}|(F^{jn_k})'(z)|^{-t} \tag{4.22}$$

for all $k \geq 1$ large enough, where $c \asymp 1$, $K \geq 1$ is the constant appearing in the Koebe's distortion theorem and ascribed to the scale $1/2$, $B_N(\nu)$ is some constant depending on ν and N. Let M be fixed as above. Fix now E, an arbitrary bounded Borel set contained in $J_r(F)$ and let $E' = E \cap J_{r,M}(F)$. Since m is regular, for every $x \in E'$ there exists a radius $r(x) > 0$ of the form from (4.22) (and the corresponding number $n(x) = n_k(x)$ for an appropriate k) such that

$$m(\bigcup_{x\in E'} B(x, r(x)) \setminus E') \leq \epsilon. \tag{4.23}$$

Now by the Besicovič theorem (see [G]) we can choose a countable subcover $\{B(x_i, r(x_i))\}_{i=1}^\infty$ with $r(x_i) \leq \epsilon$ and $jn(x_i) \geq \epsilon^{-1}$, from the cover $\{B(x, r(x))\}_{x\in E'}$ of E', of multiplicity bounded by some constant $C \geq 1$, independent of the cover. Therefore, assuming $e^{\mathrm{P}(t)} < \beta$

and using (4.22) along with (4.23), we obtain

$$\begin{aligned}
\nu(E) = \nu(E') &\le \sum_{i=1}^{\infty} \nu(B(x_i, r(x_i)))\beta^{-jn(x_i)} \le B_M(\nu) \sum_{i=1}^{\infty} r(x_i)^t \beta^{-jn(x_i)} \\
&\le B_M(\nu)B_M(m) \sum_{i=1}^{\infty} m(B(x_i, r(x_i)))\beta^{-jn(x_i)} e^{\mathrm{P}(t)jn(x_i)} \\
&\le B_M(\nu)B_M(m)Cm(\bigcup_{i=1}^{\infty} B(x_i, r(x_i)))\Big(e^{\mathrm{P}(t)}\beta^{-1}\Big)^{jn(x_i)} \\
&\le B_M(\nu)B_M(m)Cm(\bigcup_{i=1}^{\infty} B(x_i, r(x_i)))\Big(e^{\mathrm{P}(t)}\beta^{-1}\Big)^{\epsilon^{-1}} \\
&\le CB_M(\nu)B_M(m)\Big(e^{\mathrm{P}(t)}\beta^{-1}\Big)^{\epsilon^{-1}}(\epsilon + m(E')) \\
&= CB_M(\nu)B_M(m)\Big(e^{\mathrm{P}(t)}\beta^{-1}\Big)^{\epsilon^{-1}}(\epsilon + m(E)).
\end{aligned} \tag{4.24}$$

Hence letting $\epsilon \searrow 0$ we obtain $\nu(E) = 0$ and consequently $\nu(J(F)) = 0$ which is a contradiction. We obtain a similar contradiction assuming that $\beta < e^{\mathrm{P}(t)}$ and replacing in (4.24) the roles of m and ν. Thus $\beta = e^{\mathrm{P}(t)}$ and letting $\epsilon \searrow 0$ again, we obtain from (4.24) that $\nu(E) \le CB_M(\nu)B_M(m)m(E)$. Exchanging m and ν, we obtain $m(E) \le CB_M(\nu)B_M(m)\nu(E)$. These two conclusions along with the already mentioned fact that $m(J_r(F)) = \nu(J_r(F)) = 1$, imply that the measures m and ν are equivalent with Radon-Nikodym derivatives bounded away from zero and infinity.

Let us now prove that any $(t, e^{\mathrm{P}(t)})$-conformal measure ν is ergodic with respect to F^j. Indeed, suppose to the contrary that $F^{-j}(G) = G$ for some Borel set $G \subset J(F)$ with $0 < \nu(G) < 1$. But then the two conditional measures ν_G and $\nu_{J(F)\setminus G}$

$$\nu_G(B) = \frac{\nu(B \cap G)}{\nu(G)}, \quad \nu_{J(F)\setminus G}(B) = \frac{\nu(B \cap J(F) \setminus G)}{\nu(J(F) \setminus G)}$$

would be $(t, e^{j\mathrm{P}(t)})$-conformal for F^j and mutually singular. This contradiction finishes the proof. ■

5. Old and New Perron-Frobenius Operators and Their Finer Properties

Recall that $C_b = C_b(J(F))$ is the space of all bounded continuous complex valued functions defined on $J(F)$. Fix $\alpha \in (0, 1]$. Given $g \in C_b$ let

$$v_\alpha = \inf\{L \ge 0 : |g(y) - g(x)| \le L|y - x|^\alpha \ \text{ for all } \ x, y \in J(F) \ \text{ with } |y - x| \le \delta\},$$

be the α-variation of the function g, where $\delta > 0$ was defined in formula (4.18) and let

$$||g||_\alpha = v_\alpha(g) + ||g||_\infty.$$

Clearly the space

$$\mathrm{H}_\alpha = \mathrm{H}_\alpha(J(F)) = \{g \in J(F) : ||g||_\alpha < \infty\}$$

endowed with the norm $||\cdot||_\alpha$ is a Banach space densely contained in C_b with respect to the $||\cdot||_\infty$ norm.

Recall that for every $n \geq 1$ and every $v \in J(F)$,

$$F_v^{-n} : B(F^n(v), 2\delta) \to Q$$

was defined to be the holomorphic inverse branch of F^n defined on $B(F^n(v), 2\delta)$ and sending $F^n(v)$ to v. It follows from Proposition 2.6 and Koebe's distortion theorem that there exist constants $L > 0$ and $0 < \beta < 1$ such that for every $n \geq 0$, every $v \in J(F)$ and every $z \in B(F^n(v), \delta)$, we have

$$|(F_v^{-n})'(z)| \leq L\beta^n \tag{5.1}$$

We say that a continuous function $\phi : J(F) \to \mathbb{C}$ is dynamically Hölder with an exponent $\alpha > 0$ if there exists $c_\phi > 0$ such that

$$|\phi_n(F_v^{-n}(y)) - \phi_n(F_v^{-n}(x))| \leq c_\phi |\phi_n(F_v^{-n}(x))||y-x|^\alpha \tag{5.2}$$

for all $n \geq 1$, all $x, y \in J(F)$ with $|x-y| \leq \delta$ and all $v \in F^{-n}(x)$, where

$$\phi_n(z) = \phi(z)\phi(F(z)) \dots \phi(F^{n-1}(z)).$$

We say that a continuous function $\phi : J(F) \to \mathbb{C}$ is summable if

$$\sup_{z \in J(F)} \left\{ \sum_{v \in F^{-1}(z)} ||\phi \circ F_v^{-1}||_\infty \right\} < \infty.$$

If the continuous function ϕ is summable then the formula

$$\mathcal{L}_\phi g(z) = \sum_{x \in F^{-1}(z)} \phi(x) g(x) \tag{5.3}$$

defines a bounded operator $\mathcal{L}_\phi : C_b \to C_b$ called the Perron-Frobenius operator associated with the potential ϕ. We shall prove the following.

Lemma 5.1. *If $\phi : J(F) \to \mathbb{C}$ is a summable dynamically Hölder potential with an exponent $\alpha > 0$ then $\mathcal{L}_\phi(\mathrm{H}_\alpha) \subset \mathrm{H}_\alpha$. If, in addition, $\phi(J(F)) \subset [0, \infty)$ and $\sup_{n\geq 1}\{||\mathcal{L}_\phi^n(\mathbb{1})||_\infty\} < \infty$, then there exists a constant $c_1 > 0$ such that*

$$||\mathcal{L}_\phi^n g||_\alpha \leq \frac{1}{2}||g||_\alpha + c_1||g||_\infty$$

for all $n \geq 1$ large enough and every $g \in \mathrm{H}_\alpha$.

Proof. Fix $n \geq 1$, $g \in \mathrm{H}_\alpha$ and $x, y \in J(F)$ with $|y-x| \leq \delta$. Put $V_n = F^{-1}(x)$. Then we have

$$\begin{aligned}
&\left|\mathcal{L}_\phi^n g(y) - \mathcal{L}_\phi^n g(x)\right| = \\
&= \left|\sum_{v\in V_n} \phi_n(F_v^{-n}(y))g(F_v^{-n}(y)) - \sum_{v\in V_n} \phi_n(F_v^{-n}(x))g(F_v^{-n}(x))\right| \\
&= \Big|\sum_{v\in V_n} \phi_n(F_v^{-n}(x))(g(F_v^{-n}(y)) - g(F_v^{-n}(x))) + \sum_{v\in V_n} g(F_v^{-n}(y))(\phi_n(F_v^{-n}(y)) - \phi_n(F_v^{-n}(x)))\Big| \\
&\leq \sum_{v\in V_n} \left|g(F_v^{-n}(y))\right| \left|\phi_n(F_v^{-n}(y)) - \phi_n(F_v^{-n}(x))\right| + \\
&+ \sum_{v\in V_n} |\phi_n(F_v^{-n}(x))||g(F_v^{-n}(y)) - g(F_v^{-n}(x))| \\
&\leq \sum_{v\in V_n} ||g||_\infty \sum_{v\in V_n} |\phi_n(F_v^{-n}(x))| \cdot |x-y|^\alpha + \sum_{v\in V_n} |\phi_n(F_v^{-n}(x))| v_\alpha(g) |F_v^{-n}(y) - F_v^{-n}(x)|^\alpha \\
&\leq c_\phi ||g||_\infty \mathcal{L}_{|\phi|}^n(1\!\!1)(x)|y-x|^\alpha + v_\alpha(g)(L\beta^n)^\alpha |y-x|^\alpha \sum_{v\in V_n} |\phi_n(F_v^{-n}(x))| \\
&\leq ||\mathcal{L}_{|\phi|}^n(1\!\!1)||(c_\phi||g||_\infty + L^\alpha\beta^{\alpha n} v_\alpha(g))|y-x|^\alpha.
\end{aligned}$$

This shows that

$$v_\alpha(\mathcal{L}_\phi^n g) \leq \mathcal{L}_{|\phi|}^n(1\!\!1)(c_\phi||g||_\infty + L^\alpha\beta^{\alpha n}||g||_\alpha) < \infty \tag{5.4}$$

and, in particular, $\mathcal{L}_\phi^n(g) \in \mathrm{H}_\alpha$. The inclusion $\mathcal{L}_\phi(H_\alpha) \subset H_\alpha$ is proved. Suppose now that $\phi(J(F)) \subset [0,\infty)$ and $\Theta_\phi = \sup_{n\geq 1}\{||\mathcal{L}_\phi^n(1\!\!1)||_\infty\}$ is finite. It then follows from (5.4) that

$$||\mathcal{L}_\phi^n g||_\alpha \leq \Theta_\phi L^\alpha\beta^{\alpha n}||g||_\alpha + c_\phi\Theta_\phi||g||_\infty + ||\mathcal{L}_\phi^n g||_\infty \leq \Theta_\phi L^\alpha\beta^{\alpha n}||g||_\alpha + \Theta_\phi(c_\phi+1)||g||_\infty.$$

The proof is thus finished by taking $n \geq 1$ so large that $\Theta_\phi L^\alpha\beta^{\alpha n} \leq \frac{1}{2}$. ■

We say that a summable dynamically Hölder potential $\phi : J(F) \to (0,\infty)$ satisfies condition (*) if

$$\Theta_\phi = \sup_{n\geq 1}\{||\mathcal{L}_\phi^n(1\!\!1)||_\infty\} < \infty$$

and we say that ϕ is rapidly decreasing if

$$\lim_{\mathrm{Re} z\to -\infty} \mathcal{L}_\phi(1\!\!1)(z) = 0.$$

In order to apply the theorem of Ionescu-Tulcea and Marinescu we also need the following.

Lemma 5.2. *Suppose that $\phi : J(F) \to (0,\infty)$ is a rapidly decreasing summable dynamically Hölder potential satisfying condition (*). If B is a bounded subset of H_α (with the $||\cdot||_\alpha$ norm), then $\mathcal{L}_\phi(B)$ is a pre-compact subset of C_b (with the $||\cdot||_\infty$ norm).*

Proof. Fix an arbitrary sequence $\{g_n\}_{n=1}^\infty \subset B$. Since, by Lemma 5.1, the family $\mathcal{L}_\phi(B)$ is equicontinuous and, since the operator $\mathcal{L}_\phi$ is bounded, this family is bounded, it follows from Ascoli's theorem that we can choose from $\{\mathcal{L}_\phi(g_n)\}_{n=1}^\infty$ an infinite subsequence $\{\mathcal{L}_\phi(g_{n_j})\}_{j=1}^\infty$ converging uniformly on compact subsets of $J(F)$ to a function $\psi \in C_b$. Fix now $\epsilon > 0$. Since B is a bounded subset of C_b, it follows from (4.14) that there exists $T < 0$ such that $|\mathcal{L}_\phi g(z)| \le \epsilon/2$ for all $g \in B$ and all $z \in J(F)_T^c$. Hence

$$|\psi(z)| \le \epsilon/2 \tag{5.5}$$

for all $z \in J_T^c$. Thus $|\mathcal{L}_\phi(g_{n_j})(z) - \psi(z)| \le \epsilon$ for all $j \ge 1$ and all $z \in J_T^c$. In addition, there exists $p \ge 1$ such that $|\mathcal{L}_\phi(g_{n_j})(z) - \psi(z)| \le \epsilon$ for every $j \ge p$ and every $z \in J_T$. Therefore $|\mathcal{L}_\phi(g_{n_j})(z) - \psi(z)| \le \epsilon$ for all $j \ge p$ and all $z \in J(F)$. This means that $||\mathcal{L}_\phi(g_{n_j}) - \psi||_\infty \le \epsilon$ for all $j \ge p$. Letting $\epsilon \searrow 0$ we conclude from this and from (5.5) that $\mathcal{L}_\phi(g_{n_j})$ converges uniformly on $J(F)$ to $\psi \in C_b$. We are done. ■

Combining now Lemma 5.1 and Lemma 5.2, we see that the assumptions of Theorem 1.5 in [16] are satisfied with Banach spaces $\mathrm{H}_\alpha \subset C_b$ and the bounded operator $\mathcal{L}_\phi : C_b \to C_b$ preserves H_α. It gives us the following.

Theorem 5.3. *If the assumptions of Lemma 5.2 are satisfied then there exist finite numbers $\gamma_1, \dots, \gamma_p \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$, finitely many bounded finitely dimensional operators $Q_1, \dots, Q_p : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ and an operator $S : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ such that*

$$\mathcal{L}_\phi^n = \sum_{i=1}^p \gamma_i^n Q_i + S^n$$

for all $n \ge 1$,

$$Q_i^2 = Q_i,\ Q_i \circ Q_j = 0, (i \ne j),\ Q_i \circ S = S \circ Q_i = 0$$

and

$$||S^n||_\alpha \le C\xi^n$$

for some constant $C > 0$, some constant $\xi \in (0,1)$ and all $n \ge 1$. In particular all numbers $\gamma_1, \dots, \gamma_p$ are isolated eigenvalues of the operator $\mathcal{L}_\phi : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ and this operator is quasi-compact.

Since for all $t \in \mathbb{C}$ with $\mathrm{Re}t \ge 0$, all $n \ge 1$, all $x, y \in J(F)$ with $|y - x| \le \delta$, all $v \in F^{-n}(x)$ and some constant $M_t > 0$,

$$\Big| |(F_v^{-n})'(y)|^t - |(F_v^{-n})'(x)|^t \Big| \le M_t |(F_v^{-n})'(x)|^{\mathrm{Re}t} |y - x|,$$

it follows from Lemma 4.1, Lemma 4.4 and Lemma 4.6 that if t is real and $\mathrm{Re}t > 1$, then $\phi_t(z) = e^{-\mathrm{P}(t)} |F'(z)|^{-t}$ is a rapidly decreasing summable dynamically Hölder potential satisfying condition (*) which means that all the assumptions of Theorem 5.3 are satisfied. Note that $\mathcal{L}_{\phi_t} = \hat{\mathcal{L}}_t$. Using heavily Theorem 5.3 we shall prove the following

Theorem 5.4. *If $t > 1$ then we have the following.*

(a) *The number 1 is a simple isolated eigenvalue of the operator $\hat{\mathcal{L}}_t : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$.*

(b) *The eigenspace of the eigenvalue 1 is generated by nowhere vanishing function $\psi_t \in \mathrm{H}_\alpha$ such that $\int \psi_t dm_t = 1$ and $\lim_{\mathrm{Re}z \to -\infty} \psi_t(z) = 0$.*

(c) *The number 1 is the only eigenvalue of modulus 1.*

(d) *With $S : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ as in Theorem 5.3, we have*

$$\hat{\mathcal{L}}_t = Q_1 + S,$$

where $Q_1 : \mathrm{H}_\alpha \to \mathbb{C}\psi_t$ is a projector on the eigenspace $\mathbb{C}\psi_t$, $Q_1 \circ S = S \circ Q_1 = 0$ and

$$||S^n||_\alpha \le C\xi^n$$

for some constant $C > 0$, some constant $\xi \in (0, 1)$ and all $n \ge 1$.

Proof. Let us show that 1 is an eigenvalue of $\hat{\mathcal{L}}_t$ and let us identify the eigenfunction claimed in part (b). And indeed, in view of Lemma 5.1, $||\hat{\mathcal{L}}_t^n(1\!\!1)||_\alpha \le C_1$ for some constant $C_1 > 0$ and all $n \ge 0$. Thus,

$$\left\|\frac{1}{n}\sum_{j=1}^{n} \hat{\mathcal{L}}_t^j(1\!\!1)\right\|_\alpha = \left\|\hat{\mathcal{L}}_t\left(\frac{1}{n}\sum_{j=0}^{n-1} \hat{\mathcal{L}}_t^j(1\!\!1)\right)\right\|_\alpha \le C_1$$

for every $n \ge 1$. Therefore, it follows from Lemma 5.2 that there exists a strictly increasing sequence of positive integers $\{n_k\}_{k\ge1}$ such that the sequence $\left\{\frac{1}{n_k}\sum_{j=1}^{n_k} \hat{\mathcal{L}}_t^j(1\!\!1)\right\}_{k\ge1}$ converges in the Banach space C_b to a function $\psi_t : J(F) \to I\!\!R$. Obviously, $||\psi_t||_\alpha \le C_1$ and, in particular $\psi_t \in \mathrm{H}_\alpha$. Since m_t is a fixed point of the operator conjugate to $\hat{\mathcal{L}}_t$, $\int \hat{\mathcal{L}}_t^j(1\!\!1)d\,m_t = 1$ for every $j \ge 0$. Consequently, $\int \frac{1}{n}\sum_{j=0}^{n-1} \hat{\mathcal{L}}_t^j(1\!\!1)d\,m_t = 1$ for every $n \ge 1$. So, applying Lebesgue's dominated convergence theorem along with Lemma 4.6, we conclude that $\int \psi_t d\,m_t = 1$. It immediately follows from Lemma 4.8 that $\psi_t > 0$ throughout $J(F)$. Since $\psi_t = \hat{\mathcal{L}}_t\psi_t$, it follows from Lemma 4.4 that $\lim_{z\to-\infty} \psi_t(z) = 0$. Thus, in order to complete the proof of the items (a), (b), (c) (that 1 is an isolated eigenvalue of $\hat{\mathcal{L}}_t : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ follows from Theorem 5.3) it suffices to show that if $\beta \in S^1$ is an eigenvalue of $\hat{\mathcal{L}}_t : \mathrm{H}_\alpha \to \mathrm{H}_\alpha$ and ρ is its eigenfunction, then $\beta = 1$ and $\rho \in \mathbb{C}\psi_t$. But this can be done in exactly the same way as in the proof of Theorem 35(ii) in [9] using ergodicity of each iterate of F proven in Theorem 4.14 (comp. also Theorem 6.1). The item (d) is now an immediate consequence of Theorem 5.3 and items (a), (b) and (c). ■

6. Invariant Measures

The following theorem immediately follows from Theorem 5.4, Proposition 4.11 and Theorem 4.14.

Theorem 6.1. *If $t > 1$, then the measure $\mu = \mu_t = \psi_t m_t$ is F-invariant, ergodic with respect to each iterate of F and equivalent to the measure m_t. In particular $\mu(J_r(F)) = 1$.*

Due to Theorem 5.4 the F-invariant measure μ has much finer stochastic properties than ergodicity of all iterates of F. Here these follow.

Theorem 6.2. *The dynamical system (F, μ_t) is metrically exact i.e., its Rokhlin natural extension is a K-system.*

The proof of this fact is the same as the proof of Corollary 37 in [9]. The next two theorems are standard consequences of Theorem 5.4 (see [7] and [23] for example). Let g_1 and g_2 be real square-μ integrable functions on $J_r(F)$. For every positive integer n the n-th correlation of the pair g_1, g_2, is the number

$$C_n(g_1, g_2) := \int g_1 \cdot (g_2 \circ F^n)\, d\mu - \int g_1\, d\mu \int g_2\, d\mu.$$

provided the above integrals exist. Notice that due to the F-invariance of μ we can also write

$$C_n(g_1, g_2) = \int (g_1 - Eg_1)\Big((g_2 - Eg_2) \circ F^n\Big)\, d\mu,$$

where we write $Eg = \int g\, d\mu$. We have the following.

Theorem 6.3. *There exists $C \geq 1$ and $\rho < 1$ such that for all $g_1 \in \mathrm{H}_\alpha(P), g_2 \in L^1(\mu_t)$*

$$C_n(g_1, g_2) \leq C\rho^n \|g_1 - Eg_1\|_\alpha \|g_2 - Eg_2\|_{L^1}.$$

Let $g : J_r(F) \to R$ be a square-integrable function. The limit

$$\sigma^2(g) = \lim_{n\to\infty} \frac{1}{n} \int \Big(\sum_{j=0}^{n-1} g \circ F^j - nEg\Big)^2 d\mu_t$$

is called asymptotic variance or dispersion, provided it exists.

Theorem 6.4. *If $g \in \mathrm{H}_\alpha(\mathrm{P})$, $\alpha \in (0,1)$, then $\sigma^2(g)$ exists and, if $\sigma^2(g) > 0$, then the sequence of random variables $\{g \circ F^n\}_{n=0}^\infty$ with respect to the probability measure μ_t satisfies the Central Limit Theorem, i.e.*

$$\mu\left(\left\{x \in J_r(F) : \frac{\sum_{j=0}^{n-1} g \circ F^j - nEg}{\sqrt{n}} < r\right\}\right) \to \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{r} e^{-t^2/2\sigma^2}\, dt.$$

7. Bowen's Formula, Hausdorff and Packing Measures

For every $n \geq 1$ let

$$A_n = \{z \in J(F) : -n-1 \leq \text{Re}z \leq -n\}$$

We shall prove the following

Lemma 7.1. *If $t > 1$ then, $\int \log |F'| d\mu_t < +\infty$.*

Proof. Since $\psi_t : J(F) \to (0, +\infty)$ is bounded (even more $\lim_{z\to-\infty} \psi_t(z) = 0$), applying (1.2) and Corollary 4.12, we obtain

$$\begin{aligned}\int \log |F'| d\mu_t \preceq \int \log |F'| dm_t &= \sum_{n=1}^{\infty} \int_{A_n} \log |F'| dm_t \\ &\leq \sum_{n=1}^{\infty} m_t\Big(Q_{-n}^c\Big) \log(1+e^n) \\ &\leq \sum_{n=1}^{\infty} C \exp\Big((1-t)n\Big)(1+n) < +\infty.\end{aligned}$$

We are done. ■

We shall prove now an analog of the celebrated Bowen's formula. Since the definition of the pressure function $\mathrm{P}(t)$ has apriori nothing to do with the set $J_r(F)$, this theorem in particular indicates that $J_r(F)$ is the right object to deal with. From now on throughout the whole paper we put

$$h = \text{HD}(J_r(F)).$$

Theorem 7.2. $h = \text{HD}(J_r(F))$ *is the unique zero of the pressure function $t \mapsto \mathrm{P}(t)$, $t > 1$.*

Proof. Let $\eta > 1$ be the unique number t (produced in Lemma 4.10) such that $\mathrm{P}(t) = 0$. Given $k \geq 1$ let

$$X_k = \{z \in J_r(F) : \limsup_{n\to\infty} \text{Re}(F^n(z)) > -k\}.$$

Choose an arbitrary point $z \in J(F)$. Fix $t > \eta$. Take $n \geq 1$ so large that

$$\frac{1}{j} \log \sum_{x\in F^{-j}(z)} |(F^j)'(x)|^{-t} \leq \frac{1}{2}\mathrm{P}(t) \quad \text{(note that } \mathrm{P}(t) < 0 \text{ by Lemma 4.10).}$$

for all $j \geq n$. Cover Q_{-k} by finitely many open balls $B(z_1, \delta), B(z_2, \delta), \dots, B(z_l, \delta)$. Since

$$X_k \subset \bigcup_{j=n}^{\infty} \bigcup_{i=1}^{l} \bigcup_{x\in F^{-j}(z_i)} F_x^{-j}(Q_{-k}),$$

we conclude that

$$\mathrm{H}^t(X_k) \le \lim_{n\to\infty} \sum_{j=n}^{\infty}\sum_{i=1}^{l}\sum_{x\in F^{-j}(z_i)} K^t|(F^j)'(x)|^{-t}(2\delta)^t \le l(2\delta K)^t \lim_{n\to\infty}\sum_{j=n}^{\infty}\exp(\frac{1}{2}\mathrm{P}(t)j) = 0$$

since $\mathrm{P}(t)/2 < 0$. Hence $\mathrm{HD}(X_k) \le t$. Since $J_r(F) = \bigcup_{k=0}^{\infty} X_k$, this implies that $\mathrm{HD}(J_r(F)) \le t$. Letting now $t \searrow \eta$, we conclude that $\mathrm{HD}(J_r(F)) \le \eta$. In order to prove the opposite inequality fix $\epsilon > 0$. Since $\mu_\eta(J_r(F)) = 1$ and since μ_η is ergodic F-invariant, it follows from Birkhoff's ergodic theorem and Jegorov's theorem that there exist a Borel set $Y \subset J_r(F)$ and the integer $k \ge 1$ such that $\mu(Y) \ge \frac{1}{2}$ and for every $x \in Y$ and every $n \ge k$

$$\left|\frac{1}{n}\log|(F^n)'(x)| - \chi\right| < \epsilon, \tag{7.1}$$

where $\chi = \int \log|F'|d\mu_\eta$ is finite due to Lemma 7.1. Put $\nu = m_{\eta|Y}$. Given $x \in Y$ and $0 < r < \delta$, let $n \ge 0$ be the largest integer such that

$$B(x,r) \subset F_x^{-n}(B(F^n(x),\delta)). \tag{7.2}$$

Then $B(x,r)$ is not contained in $F_x^{-(n+1)}(B(F^{n+1}(x),\delta))$ and applying $\frac{1}{4}$-Koebe's distortion theorem, we get

$$r \ge 4^{-1}\delta|(F^{n+1})'(x)|^{-1}. \tag{7.3}$$

Taking $r > 0$ sufficiently small, we may assume that $n \ge k$. Combining now (7.2) along with $\mathrm{P}(\eta) = 0$, Koebe's Distortion Theorem and (7.3), we obtain

$$\begin{aligned} m_\eta(B(x,r)) &\le m_\eta(F_x^{-n}(B(F^n(x),\delta))) \asymp |(F_x^{-n})'(F^n(x))|^\eta m_\eta(B(F^n(x),\delta)) \\ &\preceq |(F_x^{-n})'(F^n(x))|^\eta \preceq r^\eta \frac{|(F^{n+1})'(x)|^\eta}{|(F^n)'(x)|^\eta} \end{aligned}$$

Employing now (7.1), we thus get

$$m_\eta(B(x,r)) \le r^\eta e^{(\chi+\epsilon)(n+1)} e^{-(\chi-\epsilon)n} \asymp r^\eta e^{2\epsilon n} \tag{7.4}$$

Now, it follows from (7.2), Koebe's distortion theorem and (7.1) that $r \le K\delta|(F^n)'(x)|^{-1} \le K\delta e^{-(\chi-\epsilon)n}$. Thus $e^{(\chi-\epsilon)n} \preceq r^{-1}$ and consequently $e^{2\epsilon n} \preceq r^{-\frac{2\epsilon}{\chi-\epsilon}}$. This and (7.4) imply that $\nu(B(x,r)) \le m_\eta(x,r)) \preceq r^{\eta-\frac{2\epsilon}{\chi-\epsilon}}$. Consequently $\mathrm{HD}(J_r(F)) \ge \mathrm{HD}(\nu) \ge \eta - \frac{2\epsilon}{\chi-\epsilon}$ and letting $\epsilon \to 0$ we finally obtain $\mathrm{HD}(J_r(F)) \ge \eta$. We are done. ■

Remark 7.3. *We have already used this fact in the proof of Theorem 7.2 but we would like to emphasize it separately that due to Theorem 4.14 m_h is a unique t-conformal ($(t,1)$-conformal) measure for F with $t > 1$. From now onwards the measure m_h will be simply denoted by m.*

Let H^h and P^h be be respectively the h-dimensional Hausdorff and packing measures (see [10], comp. [23] for example, for its definition and some basic properties). The results of this section provide in a sense a complete description of the geometrical structure of the sets $J_r(F)$

and $J_r(f)$ and also they exhibit the geometrical meaning of the h-conformal measure m. The short proof of the first result improves on the argument from the proof of Proposition 4.9 in [25].

Proposition 7.4. *We have* $\mathrm{P}^h(J_r(F)) = \infty$. *In fact* $\mathrm{P}^h(G) = \infty$ *for every open nonempty subset of* $J_r(F)$.

Proof. Since $m(J_r(F) \cap Q_M^c)) > 0$ for every $M \in I\!R$, it follows from Birkhoff's ergodic theorem, Theorem 6.1 and Theorem 4.14 that there exists a set $E \subset J_r(F)$ such that $m(E) = 1$ and $\limsup_{n\to\infty} \mathrm{Re}F^n(z) = -\infty$ for every $z \in E$. Fix $z \in E$. The above formula means that there exists an unbounded increasing sequence $\{n_k\}_{k=1}^\infty$, depending on z, such that

$$\lim_{k\to\infty} \mathrm{Re}(F^{n_k}(z)) = -\infty. \tag{7.5}$$

Since $B(F^{n_k}(z), 2\delta) \cap \mathrm{PC}(F) = \emptyset$, for every $k \ge 1$ there exists a unique analytic inverse branch $F_z^{-n} : B(F^n(z), 1) \to C\!\!\!\!I$ of F^{n_k} mapping $F^n(z)$ to z. In virtue of $\frac14$- Koebe's distortion theorem we have

$$B(z, 4^{-1}\delta|(F^{n_k})'|(z)|^{-1}) \subset F_z^{-n_k}(B(F^{n_k}(z), \delta)),$$

Applying the standard version of Koebe's distortion theorem and conformality of the measure m, we obtain

$$\begin{aligned} m(B(z, 4^{-1}\delta|(F^{n_k})'(z)|^{-1}) &\le K^h|(F^{n_k})'(z)|^{-h} m(B(F^{n_k}(z), \delta)) \\ &\le (4K)^h(4^{-1}|(F^{n_k})'(z)|^{-1})^h m(Q^c_{\mathrm{Re}F^{n_k}(z)+\delta}) \end{aligned}$$

Since by (7.5), $\lim_{k\to\infty} m(Q^c_{\mathrm{Re}F^{n_k}(z)+\delta}) = 0$, we see that

$$\liminf_{r\to 0} \frac{m(B(z,r))}{r^h} = 0.$$

Since $m(G \cap J_r(F)) > 0$ for every non-empty open subset of $J_r(F)$, this implies (see an appropriate Converse Frostman's Type Theorem in [10] or [23]) that $\mathrm{P}^h(G) = \infty$. We are therefore done. ■

Since the 2-dimensional packing measure on $C\!\!\!\!I$ is proportional to the 2-dimensional Lebesgue measure and this latter one is not locally infinite, we immediately get from Proposition 7.4 the following.

Corollary 7.5. *It holds* $h = \mathrm{HD}(J_r(F)) < 2$.

Theorem 7.6. $0 < \mathrm{H}^h(J_r(F)) < \infty$.

Proof. It follows from (4.22) applied with the measure m that the h-dimensional Hausdorff measure $\mathrm{H}^h(J_{r,M}(F))$ is finite, where M is given by Lemma 4.11. Since for every $n \ge M$, $m(J_{r,n}(F) \setminus J_{r,M}(F)) = 0$ (for the definiotion of the set $J_{r,k}$ see the beginning of the proof of Theorem 4.14, we deduce in the similar way (using again (4.22)) that $\mathrm{H}^h(J_{r,n}(F)\setminus J_{r,M}(F))=0$ for all $n\ge M$. Since $\bigcup_{n\ge M}J_{r,n}(F)=J_r(F)$, we thus conclude that $\mathrm{H}^h(J_r(F))=\mathrm{H}^h(J_{r,M}(F))<\infty$. We are therefore to show that $\mathrm{H}^h(J_r(F))>0$. The proof follows closely the proof of Theorem 4.10 in [25]. Since $m(J_r(F))=1$, it suffices to demonstrate that for every $z\in J_r(F)$ and all $r>0$ sufficiently small (depending on z)

$$m(B(z,r))\le Cr^h$$

for some constant $0\le C<\infty$ independent of z and r. And indeed, put

$$\theta=\min\{\pi,\operatorname{dist}(J(f),\Pi^{-1}(\mathrm{PC}(F)))\}.$$

Fix $z\in J_r(F)$, $0<r\le\theta(32|f'(z)|)^{-1}$. Since $F:J(F)\to J(F)$ is an expanding map, there exists a largest $n\ge 1$ such that

$$r|(f^n)'(z)|\le\frac{\theta}{32}. \tag{7.6}$$

Thus

$$r|(f^{n+1})'(z)|>\frac{\theta}{32}. \tag{7.7}$$

It follows from the definition of θ that the holomorphic inverse branch $f_z^{-n}:B(f^n(z),\theta)\to\mathbb{C}$ of f^n sending $f^n(z)$ to z, is well-defined. Since $f|_{B(f^n(z),\theta)}$ is 1-to-1 and since, by Koebe's $\frac14$-Theorem, $f(B(f^n(z),\theta))\supset B\Big(f^{n+1}(z),\frac14\theta|f'(f^n(z))|\Big)$, we conclude that the holomorphic inverse branch $f_z^{-(n+1)}:B\Big(f^{n+1}(z),\frac14\theta|f'(f^n(z))|\Big)\to\mathbb{C}$ of f^{n+1} mapping $f^{n+1}(z)$ to z, is well-defined. Since

$$4r|(f^{n+1})'(z)|=4r|(f^n)'(z)|\cdot|f'(f^n(z))|=\theta\Big(\frac{32}{\theta}r|(f^n)'(z)|\Big)\cdot\frac18|f'(f^n(z))|$$

and since, by (7.6), $\frac{32}{\theta}r|(f^n)'(z)|\le1$, we conclude that $4r|(f^{n+1})'(z)|\le\frac18\theta|f'(f^n(z))|$. Applying Koebe's $\frac14$-Theorem again, we see that

$$f_z^{-(n+1)}\Big(B\Big(f^{n+1}(z),4r|(f^{n+1})'(z)|\Big)\Big)\supset B\Big(z,|(f^{n+1})'(z)|^{-1}r|(f^{n+1})'(z)|\Big)=B(z,r).$$

Since the ball $B\Big(f^{n+1}(z),4r|(f^{n+1})'(z)|\Big)$ intersects at most $\frac{1}{2\pi}4r|(f^{n+1})'(z)|+1\le r|(f^{n+1})'(z)|$ horizontal strips of the form $2\pi ik+P$, $k\in\mathbb{Z}$, using Koebe's Distortion Theorem, h-conformality of the measure m and, at the end, (7.7), we get

$$\begin{aligned}
r^{-h}(m(B(z,r))&\le r^{-h}K^h|(f^{n+1})'(z)|^{-h}(r|(f^{n+1})'(z)|)m\Big(\Pi\Big(B\Big(f^{n+1}(z),4r|(f^{n+1})'(z)|\Big)\Big)\Big)\\
&\le r^{-h}K^h|(f^{n+1})'(z)|^{-h}(r|(f^{n+1})'(z)|)\\
&=K^h(r|(f^{n+1})'(z)|)^{1-h}\le K^h\left(\frac{32}{\theta}\right)^{h-1}.
\end{aligned}$$

We are done by applying an appropriate Converse Frostman's Type Theorem in [10] or [23]). ■

8. Analyticity of Perron-Frobenius operators

Let us start this section with the following.

Lemma 8.1. *Supppose that $\{\phi_\sigma : J(F) \to \mathbb{C}\}_{\sigma\in G}$ is a family of continuous summable potentials , where G is an open connected subset of $\mathbb{C}$. If for every $z \in J(F)$ the function $\sigma \mapsto \phi_\sigma(z), \sigma \in G$, is holomorphic and the map $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(\mathrm{H}_\alpha)$ is continuous on G, then the map $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(\mathrm{H}_\alpha)$ is holomorphic on G.*

Proof. Let $\gamma \subset G$ be a simple closed curve. Fix $g \in H_\alpha$ and $z \in J(F)$. Let $W \subset G$ be a bounded open set such that $\gamma \subset W \subset \bar{W} \subset G$. Since for each $x \in F^{-1}(z)$ the function $\sigma \mapsto g(x)\phi_\sigma(x)$ is holomorphic on G and since for each $\sigma \in W$

$$\left|\sum_{x\in F^{-1}(z)} g(x)\phi_\sigma(x)\right| \le ||\mathcal{L}_{\phi_\sigma} g||_\infty \le ||\mathcal{L}_{\phi_\sigma} g||_\alpha \le ||g||_\alpha \sup\{||\mathcal{L}_{\phi_\theta}||_\alpha : \theta \in \bar{W}\} < \infty$$

by compactness of $\bar{W}$ and continuity of the mapping $\sigma \mapsto \mathcal{L}_{\phi_\sigma}$, we conclude that the function

$$\sigma \mapsto \mathcal{L}_{\phi_\sigma} g(z) = \sum_{x\in F^{-1}(z)} \phi_\sigma(x)g(x) \in \mathbb{C},\ \sigma \in W,$$

is holomorphic. Hence, by Cauchy's theorem $\int_\gamma \mathcal{L}_{\phi_\sigma} g(z)d\sigma = 0$. Since the function $\sigma \mapsto \mathcal{L}_{\phi_\sigma} g \in H_\alpha$ is continuous, the integral $\int_\gamma \mathcal{L}_{\phi_\sigma} g d\sigma$ exists and for every $z \in J(F)$, we have $\int_\gamma \mathcal{L}_{\phi_\sigma} g d\sigma(z) = \int_\gamma \mathcal{L}_{\phi_\sigma} g(z)d\sigma = 0$. Hence, $\int_\gamma \mathcal{L}_{\phi_\sigma} g d\sigma = 0$. Now, since the function $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(H_\alpha)$ is continuous, the integral $\int_\gamma \mathcal{L}_{\phi_\sigma} d\sigma$ exists and for every $g \in H_\alpha$, $\int_\gamma \mathcal{L}_{\phi_\sigma} d\sigma(g) = \int_\gamma \mathcal{L}_{\phi_\sigma} g d\sigma = 0$. Thus, $\int_\gamma \mathcal{L}_{\phi_\sigma} d\sigma = 0$ and in view of Morera's theorem, the function $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(H_\alpha)$ is holomorphic in G. The proof is complete. ■

In order to prove the main result of this section we need the following auxiliary definitions and few elementary lemmas. Given $w \in J(F)$ we define $\mathrm{H}_{\alpha,w}$ to be the set of all bounded functions $g : B(w,\delta) \to \mathbb{C}$ such that there exists a constant $C \ge 0$ such that if $x, y \in B(w,\delta)$ and $|y - x| \le \delta$, then $|g(y) - g(x)| \le C|y - x|^\alpha$. The α-variation $v_\alpha(g)$ is defined to be the least C with this property. $\mathrm{H}_{\alpha,w}$ endowed with the norm $||g||_\alpha = v_\alpha(g) + ||g||_\infty$ is a Banach space.

Lemma 8.2. *If $v \in J(F)$ and $\phi \in \mathrm{H}_\alpha$ then the operator $A_{v,\phi} : \mathrm{H}_\alpha \to \mathrm{H}_{\alpha,F(v)}$ given by the formula*

$$A_{v,\phi} g(z) = \phi(F_v^{-1}(z))g(F_v^{-1}(z)),\ z \in B(F(v),\delta)$$

is continuous, and

$$||A_{v,\phi}||_\alpha \le (2 + (L\beta)^\alpha)||\phi \circ F_v^{-1}||_\alpha.$$

Proof. For every $g \in \mathrm{H}_\alpha$ and $z \in B(F(v),\delta)$ we have

$$|A_{v,\phi} g(z)| = |\phi(F_v^{-1}(z))| \cdot |g(F_v^{-1}(z))| \le ||\phi \circ F_v^{-1}||_\alpha \cdot ||g||_\alpha \tag{8.1}$$

If, in addition, $w \in B(F(v), \delta)$ and $|w - z| \le \delta$, then similarly as in the proof of Lemma 5.1, we get

$$\begin{aligned}
&|A_{v,\phi}g(w) - A_{v,\phi}g(z)| \le \\
&\qquad |g(F_v^{-1}(w))||\phi \circ F_v^{-1}(w) - \phi \circ F_v^{-1}(z)| + |\phi \circ F_v^{-1}(z)||g(F_v^{-1}(w)) - g(F_v^{-1}(z))| \\
&\qquad \le ||g||_\infty ||\phi \circ F_v^{-1}||_\alpha |w - z|^\alpha + v_\alpha(g)||\phi \circ F_v^{-1}||_\infty L^\alpha \beta^\alpha |w - z|^\alpha \\
&\qquad \le ||g||_\alpha (1 + (L\beta)^\alpha)||\phi \circ F_v^{-1}||_\alpha |w - z|^\alpha
\end{aligned}$$

Hence, $v_\alpha(A_{v,\phi}g) \le (1 + (L\beta)^\alpha)||\phi \circ F_v^{-1}||_\alpha ||g||_\alpha$ and combining this with (8.1), we obtain $||A_{v,\phi}g||_\alpha \le (2 + (L\beta)^\alpha)||\phi \circ F_v^{-1}||_\alpha ||g||_\alpha$. Consequently, $A_{v,\phi}(\mathrm{H}_\alpha) \subset \mathrm{H}_{\alpha,F(v)}$, the operator $A_{v,\phi} : \mathrm{H}_\alpha \to \mathrm{H}_{\alpha,F(v)}$ is continuous, and $||A_{v,\phi}||_\alpha \le (2 + (L\beta)^\alpha)||\phi \circ F_v^{-1}||_\alpha$. The proof is complete. ■

Lemma 8.3. *If $\phi : J(F) \to \mathbb{C}$ is dynamically Hölder then for every $v \in J(F)$,*

$$||\phi \circ F_v^{-1}||_\alpha \le (c_\phi + 1)||\phi \circ F_v^{-1}||_\infty.$$

Proof. It follows from (5.2) that for all $x, y \in B(F(v), \delta)$ with $|x - y| \le \delta$ we have

$$|\phi \circ F_v^{-1}(y) - \phi \circ F_v^{-1}(x)| \le c_\phi |\phi(F_v^{-1}(x))| \cdot |y - x|^\alpha \le c_\phi ||\phi \circ F_v^{-1}||_\infty |y - x|^\alpha$$

and, therefore, $v_\alpha(\phi \circ F_v^{-1}) \le c_\phi ||\phi \circ F_v^{-1}||_\infty$. Thus, $||\phi \circ F_v^{-1}||_\alpha \le (c_\phi + 1) \cdot ||\phi \circ F_v^{-1}||_\infty$. We are done. ■

A straightforward calculation proves the following.

Lemma 8.4. *If $\phi \in \mathrm{H}_\alpha$, then for every $n \ge 1$ and every $v \in J(F)$*

$$||\phi \circ F_v^{-n}||_\alpha \le (1 + L^\alpha \beta^{\alpha n})||\phi||_\alpha,$$

where $g \mapsto g \circ F_v^{-n} : B(F^n(v), \delta) \to \mathbb{C}$ is an operator from H_α to $\mathrm{H}_{\alpha,F^n(v)}$.

Lemma 8.5. *If $\rho : X \to \mathrm{H}_\alpha$ is a continuous mapping defined on a metric space X , then for every $v \in J(F)$ the function $x \mapsto A_{v,\rho(x)} \in L(\mathrm{H}_\alpha, \mathrm{H}_{\alpha,F^n(v)})$, $x \in X$, is continuous.*

Proof. Fix $x_0 \in X$, $\varepsilon > 0$ and take $\theta > 0$ so small that for every $x \in B(x_0, \theta)$ and every $v \in J(F)$, $||\rho(x) - \rho(x_0)||_\alpha \le (2 + (L\beta)^\alpha)^{-2}\varepsilon$. Then applying Lemma 8.2 and Lemma 8.4 we see that for every $x \in B(x_0, \theta)$ and every $v \in V_1$, we have

$$\begin{aligned}
||A_{v,\rho(x)} - A_{v,\rho(x_0)}||_\alpha = ||A_{v,\rho(x)-\rho(x_0)}||_\alpha &\le (2 + (L\beta)^\alpha)||(\rho(x) - \rho(x_0)) \circ F_v^{-1}||_\alpha \\
&\le (2 + (L\beta)^\alpha)(1 + (L\beta)^\alpha)||\rho(x) - \rho(x_0)||_\alpha \le \varepsilon
\end{aligned}$$

The proof is complete. ■

Denote the class of Hölder continuous summable functions on $J(F)$ by H_α^s. Now we are in position to prove the main result of this section.

Theorem 8.6. *Suppose that G is an open connected subset of the complex plane $\mathbb{C}$ and that $\phi_\sigma : J(F) \to \mathbb{C}$, $\sigma \in G$, is a family of mappings such that the following assumptions are satisfied.*

(a) *For every $\sigma \in G$, ϕ_σ is in H_α^s.*
(b) *For every $\sigma \in G$ the function ϕ_σ is dynamically Hölder.*
(c) *The function $\sigma \mapsto \phi_\sigma \in \mathrm{H}_\alpha$ ($\sigma \in G$) is continuous.*
(d) *The family $\{c_{\phi_\sigma}\}_{\sigma\in G}$ is bounded.*
(e) *The function $\sigma \mapsto \phi_\sigma(z) \in \mathbb{C}$, $\sigma \in G$, is holomorphic for every $z \in J(F)$.*
(f) $\forall(\sigma_2 \in G)\,\exists(r > 0)\,\exists(\sigma_1 \in G)$

$$\sup\left\{\left|\frac{\phi_\sigma}{\phi_{\sigma_1}}\right| : \sigma \in \overline{B(\sigma_2, r)}\right\} < \infty.$$

Then the function $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(\mathrm{H}_\alpha)$, $\sigma \in G$, is holomorphic.

Proof. In view of Lemma 8.1 it suffices to demonstrate that the function $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(\mathrm{H}_\alpha)$, $\sigma \in G$, is continuous. First notice that in view of Lemma 8.2, Lema 8.3 and the assumption (d), we have for every $v \in J(F)$ and every $\sigma \in \overline{B(\sigma_2, r)}$ that

$$||A_{v,\phi_\sigma}||_\alpha \le (2 + (L\beta)^\alpha)||\phi_\sigma \circ F_v^{-1}||_\alpha \le M||\phi_\sigma \circ F_v^{-1}||_\infty,$$

where $M = (2+(L\beta)^\alpha)\sup\{c_{\phi_\sigma}, \sigma \in G\} < \infty$. We can continue the above estimate as follows.

$$\begin{aligned}
||A_{v,\phi_\sigma}||_\alpha \le M||\phi_\sigma \circ F_v^{-1}||_\infty &= M\left|\left|\phi_{\sigma_1} \circ F_v^{-1} \cdot \frac{\phi_\sigma \circ F_v^{-1}}{\phi_{\sigma_1} \circ F_v^{-1}}\right|\right|_\infty \\
&\le M||\phi_{\sigma_1} \circ F_v^{-1}||_\infty \left|\left|\frac{\phi_\sigma \circ F_v^{-1}}{\phi_{\sigma_1} \circ F_v^{-1}}\right|\right|_\infty \qquad (8.2) \\
&\le M\left|\left|\frac{\phi_\sigma}{\phi_{\sigma_1}}\right|\right|_\infty ||\phi_{\sigma_1} \circ F_v^{-1}||_\infty \le MT||\phi_{\sigma_1} \circ F_v^{-1}||_\infty,
\end{aligned}$$

where T is the supremum taken from the assumption (f). For every $z \in J(F)$ define the operator $\mathcal{L}_{\phi_\sigma,z} : \mathrm{H}_\alpha \mapsto \mathrm{H}_{\alpha,z}$ by the formula

$$\mathcal{L}_{\phi_\sigma,z} = \sum_{v\in F^{-1}(z)} \phi \circ F_v^{-1} \cdot g \circ F_v^{-1} = \sum_{v\in F^{-1}(z)} A_{v,\phi_\sigma}. \qquad (8.3)$$

Notice that

$$\mathcal{L}_{\phi_\sigma,z}(g) = \mathcal{L}_{\phi_\sigma}(g)|_{B(z,\delta)} \qquad (8.4)$$

for every $g \in \mathrm{H}_\alpha$. Fix now $\epsilon > 0$ and two elements $\sigma, \tau \in B(\sigma_2, r)$. Then there exist $g_\epsilon \in B_{\mathrm{H}_\alpha}(0,1)$ and two points $x, y \in J(F)$ such that

$$
\begin{aligned}
||\mathcal{L}_{\phi_\sigma} - \mathcal{L}_{\phi_\tau}||_\alpha &= \sup\{||\mathcal{L}_{\phi_\sigma}(g) - \mathcal{L}_{\phi_\tau}(g)||_\alpha : g \in B_{\mathrm{H}_\alpha}(0,1)\} \\
&\le ||\mathcal{L}_{\phi_\sigma}(g_\epsilon) - \mathcal{L}_{\phi_\tau}(g_\epsilon)||_\alpha + \frac{\epsilon}{5} \\
&= v_\alpha\Big(\mathcal{L}_{\phi_\sigma}(g_\epsilon) - \mathcal{L}_{\phi_\tau}(g_\epsilon)\Big) + \frac{\epsilon}{5} + ||\mathcal{L}_{\phi_\sigma}(g_\epsilon) - \mathcal{L}_{\phi_\tau}(g_\epsilon)||_\infty \\
&\le v_\alpha\Big(\mathcal{L}_{\phi_\sigma,x}(g_\epsilon) - \mathcal{L}_{\phi_\tau,x}(g_\epsilon)\Big) + \frac{\epsilon}{5} + ||\mathcal{L}_{\phi_\sigma,y}(g_\epsilon) - \mathcal{L}_{\phi_\tau,y}(g_\epsilon)||_\infty + \frac{\epsilon}{5} + \frac{\epsilon}{5} \\
&\le ||\mathcal{L}_{\phi_\sigma,x}(g_\epsilon) - \mathcal{L}_{\phi_\tau,x}(g_\epsilon)||_\alpha + ||\mathcal{L}_{\phi_\sigma,y}(g_\epsilon) - \mathcal{L}_{\phi_\tau,y}(g_\epsilon)||_\alpha + \frac{3\epsilon}{5} \qquad (8.5) \\
&\le 2||\mathcal{L}_{\phi_\sigma,w}(g_\epsilon) - \mathcal{L}_{\phi_\tau,w}(g_\epsilon)||_\alpha + \frac{3\epsilon}{5} \\
&\le 2||\mathcal{L}_{\phi_\sigma,w} - \mathcal{L}_{\phi_\tau,w}||_\alpha + \frac{3\epsilon}{5},
\end{aligned}
$$

where w is either x or y depending upon which number $||\mathcal{L}_{\phi_\sigma,x}(g_\epsilon) - \mathcal{L}_{\phi_\tau,x}(g_\epsilon)||_\alpha$ or $||\mathcal{L}_{\phi_\sigma,y}(g_\epsilon) - \mathcal{L}_{\phi_\tau,y}(g_\epsilon)||_\alpha$ is larger. Since ϕ_{σ_1} is a summable function (see (a)), there exists a finite set $V \subset F^{-1}(w)$ such that

$$\sum_{v \in F^{-1}(w)\setminus V} ||\phi_{\sigma_1} \circ F_v^{-1}||_\infty \le \frac{\epsilon}{10MT}. \tag{8.6}$$

Since, by (c), the function $\xi \mapsto \phi_\xi \in \mathrm{H}_\alpha$ is continuous, it follows from Lemma 8.5 that the function $\xi \mapsto A_{v,\phi_\xi} \in L(\mathrm{H}_\alpha, \mathrm{H}_{\alpha,F(v)})$ is continuous. Consequently there exists $\eta \in (0, r)$ such that

$$||A_{v,\phi_\sigma} - A_{v,\phi_\tau}||_\alpha \le \frac{\epsilon}{10\#V} \tag{8.7}$$

for all $\sigma, \tau \in B(\sigma_2, \eta)$ and all $v \in V$. Combining now (8.5), (8.4), (8.3), (8.7) and (8.6), we get

$$||\mathcal{L}_{\phi_\sigma} - \mathcal{L}_{\phi_\tau}||_\alpha \le \frac{3\epsilon}{5} + 2\sum_{v \in V} ||A_{v,\phi_\sigma} - A_{v,\phi_\tau}||_\alpha + 2\sum_{v \in F^{-1}(w)\setminus V} \Big(||A_{v,\phi_\sigma}||_\alpha + ||A_{v,\phi_\tau}||_\alpha\Big) \le \frac{3\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} = \epsilon.$$

We are done. ■

Due to Hartogs' theorem, as an immediate consequence of Theorem 8.6 we obtain the following.

Corollary 8.7. *Suppose that G is an open connected subset of $\mathbb{C}^n$, $n \ge 1$, and that $\phi_\sigma : J(F) \to \mathbb{C}$, $\sigma \in G$, is a family of mappings such that the following assumptions are satisfied.*

(a) *For every $\sigma \in G$, ϕ_σ is in H_α^s.*
(b) *For every $\sigma \in G$ the function ϕ_σ is dynamically Hölder.*
(c) *The function $\sigma \mapsto \phi_\sigma \in \mathrm{H}_\alpha$ $(\sigma \in G)$ is continuous.*
(d) *The family $\{c_{\phi_\sigma}\}_{\sigma \in G}$ is bounded.*
(e) *The function $\sigma \mapsto \phi_\sigma(z) \in \mathbb{C}$, $\sigma \in G$, is holomorphic for every $z \in J(F)$.*

(f) $\forall(\sigma_2 \in G)\,\exists(r > 0)\,\exists(\sigma_1 \in G)$

$$\sup\left\{\left|\frac{\phi_\sigma}{\phi_{\sigma_1}}\right| : \sigma \in \overline{B(\sigma_2, r)}\right\} < \infty.$$

Then the function $\sigma \mapsto \mathcal{L}_{\phi_\sigma} \in L(\mathrm{H}_\alpha)$, $\sigma \in G$, *is holomorphic.*

9. Quasiconformal conjugacies in the family $\{f_\lambda\}$

We will need in the sequel the following simple result.

Lemma 9.1. *If* $\mathrm{Re}\lambda_0 > 1$, $\lim_{n\to\infty} \lambda_n = \lambda_0$, *if* $z_n \in J(F_{\lambda_n})$ *for all* $n \geq 1$, *and if* $\lim_{n\to\infty} z_n = z$ *for some* $z \in \mathbb{C}$, *then* $z \in J(F_{\lambda_0})$.

Proof. Suppose on the contrary that z is in the Fatou set of F_{λ_0}. Then there exists $k \geq 1$ so large that $\mathrm{Re}\Big(F_{\lambda_0}^k(z)\Big) > 1$. By continuity, $\mathrm{Re}\Big(F_{\lambda_n}^k(z)\Big) > 1$ for all $n \geq 1$ large enough. Hence $F_{\lambda_n}^k(z)$, and consequently also z, belong to the Fatou set of F_{λ_n} for all $n \geq 1$ large enough. This contradition finishes the proof. ■

The two covering maps $\Pi : \mathbb{C} \to Q = \mathbb{C}/\sim$ and the map $z \mapsto e^{-z} \in \mathbb{C}\setminus\{0\}$, $z \in \mathbb{C}$, induce a conformal homeomorphism $H : Q \to \mathbb{C}\setminus\{0\}$ which extends to a conformal homeomorphism, denoted by the same symbol H, from $Q \cup \{+\infty\}$ to $\mathbb{C}$, sending $+\infty$ to 0. Each map $G_\lambda = H \circ F_\lambda \circ H^{-1} : \mathbb{C} \to \mathbb{C}$ is given by the formula

$$G_\lambda(z) = e^{-\lambda} z e^{-z}. \tag{9.1}$$

Assuming that $\mathrm{Re}\lambda > 0$, the map G_λ has exactly one attracting fixed point, namely $z = 0$, and its multiplier is equal to $e^{-\lambda}$. G_λ has only one singularity $z = 1$, which its critical point. This critical point belongs to the Fatou set of G_λ and is attracted to 0 under forward iterates of G_λ. In particular G_λ is in the class $\mathcal{S}$. Obviously, with the terminology of [13], $G_\lambda \in M = M_{G_1}$, even more $\{G_\lambda\}_{\{\mathrm{Re}\lambda>0\}} \subset \Sigma$. It was shown in the proof of Theorem 10 in [13] that each element of Σ is structurally stable and the conjugating maps are quasiconformal. In particular, we get the following.

Theorem 9.2. *Fix* $\lambda_0 \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda_0) > 0$. *Then for every* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) > 0$, *there exists a quasiconformal homeomorphism conjugating* G_λ *and* G_{λ_0} *(i.e.* $G_\lambda \circ H_\lambda = H_\lambda \circ G_{\lambda_0}$*). These conjugating homeomorphisms can be chosen so that the following properties are satisfied.*

(a) *For every* $z \in \mathbb{C}$, *the map* $\lambda \mapsto H_\lambda(z)$ *is holomorphic.*
(b) *The mapping* $(\lambda, z) \to H_\lambda(z)$ *is continuous.*
(c) *The dilation of the maps* H_λ *converges to* 1 *when* $\lambda \to \lambda_0$.

Since each map F_λ is conjugated with G_λ by the same conformal homeomorphism $H : Q \to \mathbb{C} \setminus \{0\}$, as an immediate consequence of Theorem 9.2, we get the following.

Theorem 9.3. *Fix $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_0) > 0$. Then for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$, there exists a quasiconformal homeomorphism conjugating F_λ and F_{λ_0} (i.e. $F_\lambda \circ h_\lambda = h_\lambda \circ F_{\lambda_0}$). These conjugating homeomorphisms can be chosen so that the following properties are satisfied.*

(a) *For every $z \in Q$, the map $\lambda \mapsto h_\lambda(z)$ is holomorphic.*
(b) *The mapping $(\lambda, z) \to h_\lambda(z)$ is continuous.*
(c) *The dilation of the maps h_λ converges to 1 when $\lambda \to \lambda_0$.*

We will need the following improvment of Proposition 2.6.

Lemma 9.4. *If $\mathrm{Re}\lambda_0 > 1$, then there exist constants $r > 0$, $c > 0$ and $\kappa > 1$ such that*

$$|(F_\lambda^n)'(z)| \geq c\kappa^n$$

for all $\lambda \in B(\lambda_0, r)$, all $z \in J(F_\lambda)$ and all $n \geq 1$.

Proof. For every λ with $\mathrm{Re}\lambda > 1$ and every $k \geq 1$ let

$$A_k(\lambda) = \{z \in \mathbb{C} \colon |(F_\lambda^k)'(z)| > 2\}.$$

In view of (1.2) there exists $M < 0$ so small that

$$Q_M^c \subset A_1(\lambda) \tag{9.2}$$

for all λ with $\mathrm{Re}\lambda > 1$. It was shown in the proof of Proposition 2.6 that there exists $q \geq 1$ such that

$$J(F_{\lambda_0}) \subset A_1(\lambda_0) \cup A_2(\lambda_0) \cup \ldots \cup A_q(\lambda_0). \tag{9.3}$$

We shall show that

$$J(F_\lambda) \subset A_1(\lambda) \cup A_2(\lambda) \cup \ldots \cup A_q(\lambda) \tag{9.4}$$

for all $\lambda \in \mathbb{C}$ sufficiently close to λ_0. Indeed, suppose on the contrary that there exist a sequence $\{\lambda_n\}_{n=1}^\infty$ converging to λ_0 and a sequence $\{z_n\}_{n=1}^\infty$ ($z_n \in J(F_{\lambda_n})$) such that

$$|(F_{\lambda_n}^j)'(z_n)| \leq 2 \tag{9.5}$$

for all $n \geq 1$ and all $j = 1, 2, \ldots, q$. By (9.2) $z_n \in Q_M$ for all $n \geq 1$. Since in addition $\mathrm{Re}z_n \leq 0$ for all $n \geq 1$, passing to a subsequence, we may assume without loss of generality that the sequence $\{z_n\}_{n=1}^\infty$ converges to a point $z \in \mathbb{C}$. It therefore follows from Lemma 9.1 that

$$z \in J(F_{\lambda_0}). \tag{9.6}$$

Since due to Hartogs's theorem, each map $(\lambda, z) \mapsto F_\lambda^n(z)$, $n \geq 1$, is an analytic function from $\mathbb{C}^2 \to \mathbb{C}$, using (9.5) we obtain

$$|(F_\lambda^j)'(z)| = \lim_{n\to\infty} |(F_{\lambda_n}^j)'(z_n)| \leq 2$$

for all $j = 1, 2, \dots, q$. Since $z \in J(F_{\lambda_0})$ (see (9.6)), these inequalities (for $j = 1, 2, \dots, q$) contradict (9.3), and the proof of formula (9.4) is finished. Now the proof of our lemma can be concluded in exactly the same way as the proof of Proposition 2.6 was concluded. ■

For every $z \in Q$ we will write $h_\lambda'(z)$ to denote the derivative of the holomorphic function $\lambda \mapsto h_\lambda(z)$. Also, in order to use a more convenient notation in the proof of the next proposition, we will freaquently write $F(\lambda, z)$ for $F_\lambda(z)$ and more generally $F^n(\lambda, z)$ for $F_\lambda^n(z)$. We shall now prove the following.

Proposition 9.5. *For every $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_0) > 1$ there exists $r > 0$ such that*

$$T := \sup\{|h_\lambda'(z)| : \lambda \in B(\lambda_0, r), z \in J(F_{\lambda_0})\} < \infty.$$

Proof. Consider a periodic point $z \in J(F_{\lambda_0})$ of F_{λ_0}. Fix a period $n \geq 1$ of z. This means that

$$F^n(\lambda, h_\lambda(z)) = h_\lambda(z)$$

for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 1$. Differentiating this equation with respect to the (first) variable λ, we get

$$D_1F^n(\lambda, h_\lambda(z)) + D_2F^n(\lambda, h_\lambda(z))h_\lambda'(z) = h_\lambda'(z)$$

or equivalently

$$h_\lambda'(z) = \frac{D_1F^n(\lambda, h_\lambda(z))}{1 - D_2F^n(\lambda, h_\lambda(z))}.$$

It follows from Proposition 2.6 that if $n \geq 1$, kept to be a period of z, is large (depending on λ) enough, then

$$|h_\lambda'(z)| \leq 2\frac{|D_1F^n(\lambda, h_\lambda(z))|}{|D_2F^n(\lambda, h_\lambda(z))|}. \tag{9.7}$$

So, all we need to do is to get a satisfactory upper bound for the right-hand side of this inequality. We have for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 1$, all $w \in \mathbb{C}$ and all $n \geq 1$, that

$$\begin{aligned}
D_1F^n(\lambda, w) &= D_1\Big(F(\lambda, F^{n-1}(\lambda, w))\Big) \\
&= D_1F(\lambda, F^{n-1}(\lambda, w)) + D_2F(\lambda, F^{n-1}(\lambda, w))D_1F^{n-1}(\lambda, w) \\
&= 1 + D_2F(\lambda, F^{n-1}(\lambda, w))D_1F^{n-1}(\lambda, w).
\end{aligned}$$

Therefore, by induction

$$D_1F^n(\lambda, w) = 1 + \sum_{j=0}^{n-1} D_2F^{n-j}(\lambda, F^j(\lambda, w)).$$

Hence, it follows from (9.7) and Lemma 9.4 hat

$$|h'_\lambda(z)| \le 2\sum_{j=0}^{n} |(F_\lambda^j)'(h_\lambda(z))|^{-1} \le 2\sum_{j=0}^{\infty} |(F_\lambda^j)'(h_\lambda(z))|^{-1} \le 2c^{-1}\sum_{j=0}^{\infty}\kappa^{-j} = \frac{2\kappa}{c(\kappa-1)} < \infty \tag{9.8}$$

for all $\lambda \in \mathbb{C}$ sufficiently close to λ_0, say for $\lambda \in B(\lambda_0, r)$. Now fix an arbitrary point $w \in J(F_{\lambda_0})$. Since periodic points of F are dense in $J(F_{\lambda_0})$, there exists a sequence $\{w_n\}_{n=1}^{\infty} \subset J(F_{\lambda_0})$ consisting of periodic points of F_{λ_0} such that $\lim_{n\to\infty} w_n = w$. It then follows from condition (b) of Theorem 9.3 that the sequence $\{\lambda \mapsto h_\lambda(w_n)\}_{n=1}^{\infty}$ converges to the function $\lambda \mapsto h_\lambda(w)$ uniformly on all compact subsets of $\{z \in \mathbb{C} : \mathrm{Re} z > 1\}$. Since the item (a) of Theorem 9.3 says that all the functions $\lambda \mapsto h_\lambda(\xi)$, $\xi \in J(F_{\lambda_0})$, are analytic, it follows from (9.8) that

$$|h'_\lambda(w)| = \lim_{n\to\infty} |h'_\lambda(w_n)| \le \frac{2\kappa}{c(\kappa-1)}.$$

We are done. ■

Given $K, \alpha > 0$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 1$ we say that a map $h : \mathbb{C} \to \mathbb{C}$ is (K, α, λ)-Hölder continuous if

$$|h(x) - h(y)| \le K|x-y|^\alpha$$

for all $z \in J(F_\lambda)$ and all $x, y \in B(z, 2^{-10})$.

Proposition 9.6. *For every $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}\lambda_0 > 1$ there exists $\hat{K}_{\lambda_0} \ge 1$ such that if $\lambda \in \mathbb{C}$ is sufficiently close to λ_0, then the conjugating homeomorphism $h_\lambda : \mathbb{C} \to \mathbb{C}$ is $(\hat{K}_{\lambda_0}, 1/q_\lambda, \lambda_0)$-Hölder continuous, where q_λ is the quasiconformality constant of h_λ.*

Proof. In view of Theorem 9.3, we may assume $r > 0$ produced in Proposition 9.5 to be so small that $q_\lambda < 2$ for all $\lambda \in B(\lambda_0, r)$. Fix $x \in J(F_{\lambda_0})$. Let $G = h_\lambda(B(x,1)))$. Since $h_\lambda : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, G is an open connected simply connected set. Let $R : B(0,1) \to G$ be its conformal representation such that $R(0) = h_\lambda(x)$. In view of Proposition 2.5, there exists a connected subset of the Julia set $J(F_{\lambda_0})$ joining x and the boundary $\partial B(x,1)$ of $B(x,1)$. Since $J(F_\lambda) = h_\lambda\big(J(F_{\lambda_0})\big)$, there thus exists a connected subset of $J(F_\lambda)$ joining $h_\lambda(x)$ and ∂G. Consequently, there exists $z \in \partial B(0,1/2)$ such that $R(z) \in J(F_\lambda)$. Hence, $R(B(0,1/2))$ does not contain any ball centered at $h_\lambda(x)$ and with radius $> |R(z) - h_\lambda(x)|$. But writing $R(z) = h_\lambda(w)$, where $w \in B(x,1) \cap J(F_{\lambda_0})$ as $R(z) \in G \cap J(F_\lambda)$, it follows from Proposition 9.5

that

$$
\begin{aligned}
|R(z)-h_\lambda(x)| &= |h_\lambda(w)-h_\lambda(x)| \le |h_\lambda(w)-w|+|w-x|+|x-h_\lambda(x)| \\
&= |h_\lambda(w)-h_{\lambda_0}(w)|+1+|h_{\lambda_0}(x)-h_\lambda(x)| \\
&= \left|\int_{\lambda_0}^{\lambda} h_\gamma'(w)d\gamma\right|+1+\left|\int_{\lambda_0}^{\lambda} h_\gamma'(x)d\gamma\right| \\
&\le \int_{\lambda_0}^{\lambda} |h_\gamma'(w)||d\gamma|+1+\int_{\lambda_0}^{\lambda} |h_\gamma'(x)||d\gamma| \\
&\le 1+2T|\lambda-\lambda_0| \le 1+2rT.
\end{aligned}
$$

Therefore $R(B(0,1/2))$ does not contain the ball $B(h_\lambda(x),2(1+2rT))$. So, Koebe's $\frac{1}{4}$-distortion theorem implies that

$$|R'(0)| \le 16(1+2rT). \tag{9.9}$$

The map $g=R^{-1}\circ h_\lambda : B(x,1)\to B(0,1)$ is a quasiconformal homeomorphism between two disks of radius 1. Hence, by Mori's theorem, $|g(z_1)-g(z_2)| \le 16|z_1-z_2|^{1/q_\lambda}$ for all $z_,z_2 \in B(x,1)$. In particular, for every $z\in B(x,1)$,

$$|g(z)| = |g(z)-g(x)| \le 16|z-x|^{1/q_\lambda}.$$

This implies that if $|z-x|\le 2^{-10}$, then $|g(z)|\le 16\left(2^{-10}\right)^{1/q_\lambda} \le 16\left(2^{-10}\right)^{1/2}=1/2$. Therefore, if $|z_1-x|,|z_2-x|\le 2^{-10}$, then using (9.9), we obtain

$$
\begin{aligned}
|h_\lambda(z_2)-h_\lambda(z_1)| &= |R(g(z_2))-R(g(z_2))| \le K|R'(0)||g(z_2)-g(z_1)| \\
&\le 16K|R'(0)|\cdot|z_2-z_1|^{1/q_\lambda} \le 16K(1+2rT)|z_2-z_1|^{1/q_\lambda}
\end{aligned}
$$

where K is the Koebe distortion constant corresponding to the scale $1/2$. We are done. ■

10. Real Analyticity of the Hausdorff Dimension

In this section we prove Theorem 10.3, our main result in this paper. We will need the following continuity result.

Lemma 10.1. *The function* $(t,\lambda)\mapsto \mathrm{P}_\lambda(t)$, $(t,\lambda)\in(1,+\infty)\times\{\lambda\in\mathbb{C} : \mathrm{Re}\lambda>1\}$, *is continuous.*

Proof. A standard application of Hölder's inequality shows that for every $\lambda\in\mathbb{C}$ with $\mathrm{Re}\lambda>1$, the function $t\mapsto \mathrm{P}_\lambda(t)$, $t\in(1,\infty)$, is continuous. There thus exists $\xi_1\in(0,t_0-1)$ such that if $t\in(t_0-\xi_1,t_0+\xi_1)$, then

$$|\mathrm{P}_{\lambda_0}(t)-\mathrm{P}_{\lambda_0}(t_0)| < \epsilon/2. \tag{10.1}$$

Now fix now $\gamma>1$ so small that $(t_0+\xi_1)\log\gamma<\epsilon/3$. Let $0<r_1\le r$ be so small that

$$M_8=\sup\{\sup\{\mathrm{Re}(J(F_\lambda)\} :\ \lambda\in B(\lambda_0,r_1)\}<0.$$

Then there exists $0 < r_2 \le r_1$ so small that if $\mathrm{Re} z, \mathrm{Re} w \le M_8$ and $|z-w| \le r_2$ then

$$\gamma^{-1} < \frac{|1-e^{-w}|}{|1-e^{-z}|} < \gamma.$$

In view of Proposition 9.5 there exists $\eta \in (0, \min\{\xi_1, r_2\})$ so small that if $|\lambda - \lambda_0| < \xi$ then $|h_\lambda(z) - z| < r_2$ away from a small fixed neighbourhood of the attracting periodic orbit. Now fix $z \in J(F_{\lambda_0})$. Take $n \ge 1$ and $x \in F_{\lambda_0}^{-n}(z)$. Since the homeomorphism h_λ described in Section 8 conjugates F_λ and F_{λ_0}, $h_\lambda(F_{\lambda_0}^{-n}(z)) = F_\lambda^{-n}(z)$. Also, for every $0 \le i \le n$ and every $x \in F_{\lambda_0}^{-n}(z)$, $h_\lambda(f_{\lambda_0}^i(x)) = f^i(h_\lambda(x))$ and, therefore $|f_{\lambda_0}^i(x) - f_\lambda^i(h_\lambda(x))| \le r_2$. Hence,

$$\begin{aligned}\frac{|(F_\lambda^n)'(h_\lambda(x))|}{|(F_{\lambda_0}^n)'(x)|} &= \frac{|(f_\lambda^n)'(h_\lambda(x))|}{|(f_{\lambda_0}^n)'(x)|} = \frac{|\prod_{i=0}^{n-1} f_\lambda'(f_\lambda^i(h_\lambda(x)))|}{|\prod_{i=0}^{n-1} f_{\lambda_0}'(f_{\lambda_0}^i(x))|}\\ &= \prod_{i=1}^{n} \frac{|1-\exp(-f_\lambda^i(h_\lambda(x)))|}{|1-\exp(-f_{\lambda_0}^i(x)))|} \in (\gamma^{-n}, \gamma^n).\end{aligned}$$

Since $h_\lambda : F_{\lambda_0}^{-n}(z) \to F_\lambda^{-n}(h_\lambda(z))$ is a bijection, we therefore conclude that

$$\frac{\sum_{x \in F_\lambda^{-n}(h_\lambda(z))} |(F_\lambda^n)'(x)|^{-t}}{\sum_{x \in F_{\lambda_0}^{-n}(z)} |(F_{\lambda_0}^n)'(x)|^{-t}} \in (\gamma^{-tn}, \gamma^{tn})$$

and from this,

$$\frac{1}{n}\log\left(\sum_{x \in F_\lambda^{-n}(h_\lambda(z))} |(F_\lambda^n)'(x)|^{-t}\right) - \frac{1}{n}\log\left(\sum_{x \in F_{\lambda_0}^{-n}(z)} |(F_{\lambda_0}^n)'(x)|^{-t}\right) \in (-t\log\gamma, t\log\gamma).$$

So, $\mathrm{P}_\lambda(t) - \mathrm{P}_{\lambda_0}(t) \in (-t\log\gamma, t\log\gamma)$ for all $\lambda \in B(\lambda_0, \xi)$ and consequently $|\mathrm{P}_\lambda(t) - \mathrm{P}_{\lambda_0}(t)| < \epsilon/2$ for all $(t, \lambda) \in (t_0 - \xi, t_0 + \xi) \times B(\lambda_0, \xi)$. Combining this with (10.1), we see that $|\mathrm{P}_\lambda(t) - \mathrm{P}_{\lambda_0}(t_0)| < \epsilon$ for all $(t, \lambda) \in (t_0 - \xi, t_0 + \xi) \times B(\lambda_0, \xi)$. The continuity of the function $(t, \lambda) \mapsto \mathrm{P}_\lambda(t)$, $\lambda \in (1, +\infty) \times \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 1\}$ is established. ■

Fix now $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}\lambda_0 > 1$ and $t_0 \in (1, \infty)$. Since by Proposition 9.6, $h_\lambda : J(F_{\lambda_0}) \to J(F_\lambda)$ is Hölder continuous with the Hölder exponent $1/q_\lambda$ depending on λ and since q_λ converges to 1 as $\lambda \to \lambda_0$, we get that for every $r > 0$ sufficiently small that

$$\alpha = \inf\left\{\frac{1}{q_\lambda} : \lambda \in B(\lambda_0, r)\right\} > 0. \tag{10.2}$$

For every $\lambda \in B(\lambda_0, r)$ and every $t > 1$ let $\mathcal{L}^0_{\lambda,t} : \mathrm{H}_\alpha(J(F_{\lambda_0})) \to \mathrm{H}_\alpha(J(F_{\lambda_0}))$ be the operator induced by the weight function $|F_\lambda' \circ h_\lambda|^{-t} : J(F_{\lambda_0}) \to \mathbb{R}$, i.e.

$$\mathcal{L}^0_{\lambda,t} g(z) = \sum_{x \in F_{\lambda_0}^{-1}(z)} |F_\lambda'(h_\lambda(x))|^{-t} g(x).$$

Our aim is to use Corollary 8.7. However, the potential $|F_\lambda' \circ h_\lambda|^{-t}$ does not depend on $(\lambda, t) \in \mathbb{C}^2$ in a holomorphic way. For this reason, we have to embed λ into $\mathbb{C}^2$ and t into $\mathbb{C}$.

We embed the complex plane $\mathbb{C}$ into $\mathbb{C}^2$ by the formula $x+iy \mapsto (x,y) \in \mathbb{C}^2$. So, $\lambda \in \mathbb{C} = \mathbb{R}^2$ may be treated as an element of $\mathbb{C}^2$. Fix

$$f = f_{\lambda_0} \text{ and } F = F_{\lambda_0}.$$

The technical result of this section is provided by the following.

Proposition 10.2. *Fix λ_0 with $\mathrm{Re}\lambda_0 > 1$ and $t_0 > 1$. There then exist $R > 0$ and a holomorphic function*

$$L : \mathbb{D}_{\mathbb{C}^3}((\lambda_0, t_0), R) \to L(\mathrm{H}_\alpha(J(F(\lambda_0))))$$

(λ_0 is treated here as elements of $\mathbb{C}^2$, t_0 as an element of $\mathbb{C}$ and α comes from (10.2) with r replaced by R) such that for every $(\lambda, t) \in B(\lambda_0, R) \times B(t_0, R) \subset \mathbb{C} \times \mathbb{R}$

$$L(\lambda, t) = \mathcal{L}^0_{\lambda,t}. \tag{10.3}$$

Proof. For every $\lambda \in \mathbb{C}$ sufficiently close to λ_0, say $\lambda \in U$, let $\theta_\lambda = F'_\lambda \circ h_\lambda$ and for every $z \in J(F)$ let

$$\phi_{(\lambda,t)}(z) = |\theta_\lambda(z)|^{-t} \tag{10.4}$$

and

$$\psi_z(\lambda) = \frac{\theta_\lambda(z)}{\theta_{\lambda_0}(z)}, \ (\lambda, z) \in U \times J(F).$$

We claim that there exists $r > 0$ such that for every $z \in J(F)$ the holomorphic function $\log \psi_z : B(\lambda_0, r) \to \mathbb{C}$ is well defined and there exists a universal constant (independent of $z \in J(F)$ in particular) $M_1 > 0$ such that

$$|\log \psi_z(\lambda)| \le M_1 \tag{10.5}$$

for all $\lambda \in B(\lambda_0, r)$, where the branch $\log \psi_z(\lambda)$ is determined by the requirement that $\log \psi_z(\lambda_0) = 0$. Indeed, since

$$\lim_{z \to -\infty} \left(\frac{e^{-z}}{1 - e^{-z}} \right) = -1,$$

we see that

$$M := \sup\left\{ \left| \frac{e^{-z}}{1 - e^{-z}} \right| : \mathrm{Re} z < -\epsilon_{\lambda_0} \right\} < \infty, \tag{10.6}$$

where ϵ_{λ_0} comes from Theorem 2.3. Taking $r > 0$ suficiently small, it follows from Proposition 9.5 that

$$|h_\lambda(z) - z| = |h_\lambda(z) - h_{\lambda_0}(z)| = \left| \int_{\lambda_0}^{\lambda} h'_\gamma(z) d\lambda \right| \le \int_{\lambda_0}^{\lambda} |h'_\gamma(z)||d\gamma| \le T|\lambda - \lambda_0| \le Tr. \tag{10.7}$$

Observe that

$$|1 - e^{-w}| \le E|w| \tag{10.8}$$

for all $w \in B(0,Tr)$ and some $E>0$. Now, for every $z \in J(F)$, we have

$$\psi_z(\lambda)=\frac{F'_\lambda(h_\lambda(z))}{F'_{\lambda_0}(z)}=\frac{1-e^{-h_\lambda(z)}}{1-e^{-z}}=\frac{1-e^{-z}+e^{-z}-e^{-h_\lambda(z)}}{1-e^{-z}}=1+\frac{e^{-z}-e^{-h_\lambda(z)}}{1-e^{-z}}.$$

Hence, using (10.6), (10.8) and (10.7), we obtain

$$|\psi_z(\lambda)-1|=\left|\frac{e^{-z}-e^{-h_\lambda(z)}}{1-e^{-z}}\right|=\left|\frac{e^{-z}}{1-e^{-z}}\right||1-e^{z-h_\lambda(z)}|\le ME|h_\lambda(z)-z|\le METr<1/2,$$

where the last inequality was written assuming that $r>0$ is small enough. So, the required branch of $\log\psi_z$ is proven to exist and formula (10.5) is established . Fix $\xi_1,\xi_2 \in J(F)$. Since, by Theorem 2.3, the segment $[\xi_1,\xi_2]$ joining the points ξ_1 and ξ_2 lies entirely in $\{z\in Q:\mathrm{Re}z<-\epsilon_{\lambda_0}\}$, it follows from (10.6) that

$$|\log\Big(e^{-\xi_2}-1\Big)-\log\Big(e^{-\xi_1}-1\Big)|=\left|\int_{\xi_1}^{\xi_2}\frac{e^{-z}}{e^{-z}-1}dz\right|\le\int_{\xi_1}^{\xi_2}\left|\frac{e^{-z}}{e^{-z}-1}\right||dz|\le M|\xi_2-\xi_1|. \tag{10.9}$$

Fix now $z_1,z_2\in J(F)$ with $|z_2-z_1|\le\delta$. Applying Proposition 9.6, (10.2) and (10.9), we get

$$\begin{aligned}
&|\log\psi_{z_2}(\lambda)-\log\psi_{z_1}(\lambda)|=\\
&\quad=\Big|\log\Big(F'_\lambda\circ h_\lambda(z_2)\Big)-\log\Big(F'_{\lambda_0}\circ h_{\lambda_0}(z_2)\Big)-\log\Big(F'_\lambda\circ h_\lambda(z_1)\Big)+\log\Big(F'_{\lambda_0}\circ h_{\lambda_0}(z_1)\Big)\Big|\\
&\quad=\Big|\log\Big(1-e^{-h_\lambda(z_2)}\Big)-\log\Big(1-e^{-z_2}\Big)-\log\Big(1-e^{-h_\lambda(z_1)}\Big)+\log\Big(1-e^{-z_1}\Big)\Big|\\
&\quad\le\Big|\log\Big(1-e^{-h_\lambda(z_2)}\Big)-\log\Big(1-e^{-h_\lambda(z_1)}\Big)\Big|+\Big|\log\Big(1-e^{-z_2}\Big)-\log\Big(1-e^{-z_1}\Big)\Big|\\
&\quad\le M|h_\lambda(z_2)-h_\lambda(z_1)|+M|z_2-z_1|\\
&\quad\le M\hat{K}_{\lambda_0}|z_1-z_2|^\alpha
\end{aligned}$$

Hence for every $\lambda\in B(\lambda_0,r)$ the function $z\mapsto\log\psi_z(\lambda)$, $z\in J(F)$, belongs to H_α and its Hölder constant is bounded from above by $M\hat{K}_{\lambda_0}$. Since the function $\log\psi_z:B(\lambda_0,r)\to\mathbb{C}$ is holomorphic, it is uniquely represented as a power series

$$\log\psi_z(\lambda)=\sum_{n=1}^\infty a_n(z)(\lambda-\lambda_0)^n.$$

By Cauchy's inequalities,

$$|a_n(z)|\le\frac{M_1}{r^n} \tag{10.10}$$

for all $n\ge0$. For every $\lambda=x+iy\in B(\lambda_0,r)\subset\mathbb{C}$, we have

$$\begin{aligned}
\mathrm{Re}\log\psi_z(\lambda)&=\mathrm{Re}\left(\sum_{n=1}^\infty a_n(z)\Big((x-\mathrm{Re}(\lambda_0))+(y-\mathrm{Im}(\lambda_0))i\Big)^n\right)\\
&=\sum_{p,q=0}^\infty c_{p,q}(z)\Big(x-\mathrm{Re}(\lambda_0)\Big)^p\Big(y-\mathrm{Im}(\lambda_0)\Big)^q,
\end{aligned} \tag{10.11}$$

where $c_{p,q}(z) = a_{p+q}(z)\binom{p+q}{q}i^q$. Due to (10.10)

$$|c_{p,q}(z)| \le |a_{p+q}(z)| \cdot 2^{p+q} \le M_1 2^{p+q} r^{-(p+q)} \tag{10.12}$$

Hence, $\mathrm{Re}\log\psi_z$ extends by the same power series expansion

$$\sum_{p,q=0}^{\infty} c_{p,q}(z)\Big(x - \mathrm{Re}(\lambda_0)\Big)^p\Big(y - \mathrm{Im}(\lambda_0)\Big)^q$$

to a holomorphic function on the polydisk $\mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/4)$. We denote this extension by the same symbol $\mathrm{Re}\log\psi_z$ and we have

$$|\mathrm{Re}\log\psi_z(\lambda)| \le \sum_{p,q=0}^{\infty} M_1 2^{-(p+q)} = 4M_1 \tag{10.13}$$

on $\mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/4)$. So, for every $t \in B_{\mathbb{C}}(t_0, \rho)$, where $\rho = t_0 - 1$, the formula

$$\zeta_{(\lambda,t)}(z) = -\Big(t\mathrm{Re}\log\psi_z(\lambda) + t\log|\theta_{\lambda_0}(z)|\Big) \tag{10.14}$$

extends $-t\log|\theta_\lambda(z)|$ on the polydisk $\mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/4) \times B_{\mathbb{C}}(t_0, \rho)$. Now, due to (10.13), for every $(\lambda, t) \in B_{\mathbb{C}^2}(\lambda_0, r/4) \times B_{\mathbb{C}}(t_0, \rho)$ and every $z \in J(F)$, we have

$$\begin{aligned}|e^{\zeta_{\lambda,t}(z)}| &= \exp\left(\mathrm{Re}(-t\mathrm{Re}\log\psi_z(\lambda) - t\log|\theta_{\lambda_0}(z)|)\right) \\ &= \exp(\mathrm{Re}(-t\mathrm{Re}\log\psi_z(\lambda)))|\theta_{\lambda_0}(z)|^{-\mathrm{Re}(t)} \le \exp(|t||\mathrm{Re}\log\psi_z(\lambda)|)|\theta_{\lambda_0}(z)|^{-\mathrm{Re}(t)} \\ &\le e^{4M_1|t|}|\theta_{\lambda_0}(z)|^{-\mathrm{Re}(t)}\end{aligned} \tag{10.15}$$

Since the function $|\theta_{\lambda_0}|^{-\mathrm{Re}t}$ is summable, it therefore follows that each function

$$\phi_{(\lambda,t)} = e^{\zeta_{(\lambda,t)}},\ (\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/4) \times B_{\mathbb{C}}(t_0, \rho),$$

is summable. In order to prove the proposition, we shall check that the family of functions $\phi_{(\lambda,t)}$ satisfies all assumptions of Corollary 8.7. One part of the assumption (a) of Corollary 8.7 (summability) is already proven. Of course putting $L(\lambda, t) = \mathcal{L}^0_{\lambda,t}$, $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/4) \times B_{\mathbb{C}}(t_0, \rho)$, the condition (10.3) is satisfied. Obviously, the function $(\lambda, t) \mapsto \phi_{(\lambda,t)}(z)$, $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/4) \times B_{\mathbb{C}}(t_0, \rho)$, is holomorphic for every $z \in J(F)$ and the assumption (e) of Corollary 8.7 is established.

Now we shall show that for every $\lambda \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/4)$ the function $z \mapsto \mathrm{Re}\log\psi_z(\lambda)$, $z \in J(F)$, is in H_α. Since we have already proved that for every $\lambda \in B(\lambda_0, r)$ the function $z \mapsto \log\psi_z(\lambda)$, $z \in J(F)$, is in H_α and its Hölder constant is bounded from above by $2C$, using Cauchy's inequalities, we conclude that

$$|a_n(z) - a_n(w)| \le M\hat{K}_{\lambda_0}\left(\frac{4}{r}\right)^n |z - w|^\alpha$$

for all $z, w \in J(F)$ with $|z - w| \le \delta$. Therefore,

$$|c_{p,q}(z) - c_{p,q}(w)| \le 2C \cdot 2^{p+q}\left(\frac{4}{r}\right)^{p+q} |z - w|^\alpha = M\hat{K}_{\lambda_0}\left(\frac{8}{r}\right)^{p+q} |z - w|^\alpha \tag{10.16}$$

Hence,

$$|\mathrm{Re}\log\psi_z(\lambda)-\mathrm{Re}\log\psi_w(\lambda)|\le M\hat{K}_{\lambda_0}\sum_{p,q=0}^{\infty}\left(\frac{8}{r}\right)^{p+q}\left(\frac{r}{16}\right)^p\left(\frac{r}{16}\right)^q|z-w|^\alpha=4M\hat{K}_{\lambda_0}|z-w|^\alpha \tag{10.17}$$

for every $\lambda\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16))$ and all $z,w\in J(F)$ with $|z-w|\le\delta$. Hence, using (10.13), we see that the function $z\mapsto\mathrm{Re}\log\psi_z(\lambda)$, $z\in J(F)$, is in H_α for every $\lambda\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/4)$. Since $\theta_{\lambda_0}(z)=F'_{\lambda_0}(z)=1-e^{-z}$, we get that $\log|\theta_{\lambda_0}(z)|=\log|1-e^{-z}|$. Utilizing (10.17), (10.14) and (10.9), we conclude that

$$\begin{aligned}|\zeta_{(\lambda,t)}(z)-\zeta_{(\lambda,t)}(w)|&=|t|\left(|\mathrm{Re}\log\psi_z(\lambda)-\mathrm{Re}\log\psi_w(\lambda)|+\log|\theta_{\lambda_0}(z)|-\log|\theta_{\lambda_0}(w)||\right)\\&\le(t_0+\rho)\Big(4M\hat{K}_{\lambda_0}|z-w|^\alpha+|\log|\theta_{\lambda_0}(z)|-\log|\theta_{\lambda_0}(w)||\Big)\\&=(t_0+\rho)\Big(4M\hat{K}_{\lambda_0}|z-w|^\alpha+\Big|\log|e^{-z}-1|-\log|e^{-w}-1|\Big|\Big)\\&=(t_0+\rho)\Big(4M\hat{K}_{\lambda_0}|z-w|^\alpha+\Big|\mathrm{Re}\Big(\log(e^{-z}-1)-\log(e^{-w}-1)\Big)\Big|\Big)\\&\le(t_0+\rho)\Big(4M\hat{K}_{\lambda_0}|z-w|^\alpha+\Big|\log(e^{-z}-1)-\log(e^{-w}-1)\Big|\Big)\\&\le(t_0+\rho)\Big(4M\hat{K}_{\lambda_0}|z-w|^\alpha+M|z-w|\Big)\\&\le C(t_0+\rho)|z-w|^\alpha\end{aligned} \tag{10.18}$$

for every $(\lambda,t)\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)\times B_{\mathbb{C}}(t_0,\rho)$, where $C=M(4\hat{K}_{\lambda_0}+1)$. We shall now check the second part of the assumption (a) of Corollary 8.7 that $\phi_{(\lambda,t)}=e^{\zeta_{(\lambda,t)}}\in\mathrm{H}_\alpha$ for all $(\lambda,t)\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)\times B_{\mathbb{C}}(t_0,\rho)$. Indeed, first observe that due to (10.15) there exists a constant $M_2>0$ such that

$$|\phi(\lambda,t)(z)|=|e^{\zeta_{(\lambda,t)}(z)}|\le M_2 \tag{10.19}$$

for all $(\lambda,t)\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)\times B_{\mathbb{C}}(t_0,\rho)$ and all $z\in J(F)$. Obviously, there exists a constant $M_3>0$ such that $|e^\eta-1|<M_3|\eta|$ for all $\eta\in\mathbb{C}$ with $|\eta|\le C\delta^\alpha$. Applying (10.18) and (10.19), we obtain

$$\begin{aligned}|\phi_{(\lambda,t)}(z)-\phi_{(\lambda,t)}(w)|&=|e^{\zeta_{(\lambda,t)}(w)}|\cdot|e^{\zeta_{(\lambda,t)}(z)-\zeta_{(\lambda,t)}(w)}-1|\le M_3M_2|\zeta_{(\lambda,t)}(z)-\zeta_{(\lambda,t)}(w)|\\&\le CM_2M_3(|t_0|+\rho)|z-w|^\alpha\end{aligned}$$

for all $(\lambda,t)\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)\times B_{\mathbb{C}}(t_0,\rho)$ and all $z,w\in J(F)$ with $|z-w|\le\delta$. In particular, $\phi_{(\lambda,t)}\in\mathrm{H}_\alpha$ and assumption (a) of Corollary 8.7 is verified.

We shall now check the assumptions (b) and (d) of Corollary 8.7, i.e. that all the functions $\phi_{(\lambda,t)}=e^{\zeta_{(\lambda,t)}}$, $(\lambda,t)\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)\times B_{\mathbb{C}}(t_0,\rho)$, are dynamically Hölder (with the exponent α) with uniformly bounded constants c_{ϕ_α}. So, fix $\lambda\in\mathbb{D}_{\mathbb{C}^2}(\lambda_0,r/16)$, $n\ge1$, $v\in F^{-n}(x)$ and

$x, y \in J(F)$ with $|x-y| \le \delta$. Applying (5.1) and (10.18), we obtain

$$\begin{aligned}\Big|\sum_{i=0}^{n-1}\zeta_{(\lambda,t)}(F^i(F_v^{-n}(y))) - \sum_{i=0}^{n-1}\zeta_{(\lambda,t)}(F^i(F_v^{-n}(x)))\Big| \\ &\le \sum_{i=0}^{n-1}\Big|\zeta_{(\lambda,t)}(F^i(F_v^{-n}(y))) - \zeta_{(\lambda,t)}(F^i(F_v^{-n}(x)))\Big| \\ &\le \sum_{i=0}^{n-1} C(|t_0|+\rho)|F^i(F_v^{-n}(y)) - F^i(F_v^{-n}(x))|^\alpha \\ &\le CL^\alpha(|t_0|+\rho)\sum_{i=0}^{\infty}\beta^{(n-i)\alpha}|y-x|^\alpha \\ &\le \frac{CL^\alpha}{1-\beta^\alpha}(|t_0|+\rho)|y-x|^\alpha\end{aligned}$$

Therefore, putting $M_4 = \sup\left\{\left|\frac{e^z-1}{z}\right| : |z| \le \frac{CL^\alpha}{1-\beta^\alpha}(|t_0|+\rho)\right\} < \infty$, we get

$$\begin{aligned}|\phi_{(\lambda,t),n}(F_v^{-n}(y)) - \phi_{(\lambda,t),n}(F_v^{-n}(x))| \le \\ &= |\phi_{(\lambda,t),n}(F_v^{-n}(x))| \cdot \Big|\frac{\phi_{(\lambda,t),n}(F_v^{-n}(y))}{\phi_{(\lambda,t),n}(F_v^{-n}(x))} - 1\Big| \\ &= |\phi_{(\lambda,t),n}(F_v^{-n}(x))|\left|\exp\left(\sum_{i=0}^{n-1}\zeta_{(\lambda,t)}(F^i(F_v^{-n}(y))) - \sum_{i=0}^{n-1}\zeta_{(\lambda,t)}(F^i(F_v^{-n}(x)))\right) - 1\right| \\ &\le \frac{CM_4L^\alpha}{1-\beta^\alpha}(|t_0|+\rho)|\phi_{(\lambda,t),n}(F_v^{-n}(x))| \cdot |y-x|^\alpha.\end{aligned}$$

and the assumptions (b) and (d) of Corollary 8.7 have been verified. We shall now check the assumption (c) of Corollary 8.7 that the function

$$(\lambda, t) \mapsto \phi_{(\lambda,t)} \in \mathrm{H}_\alpha$$

is continuous in some neighbourhood of (λ_0, t_0) in $\mathbb{C}^3$. Since

$$\phi_{(\lambda,t)}(z) = e^{-t\mathrm{Re}\log\psi_z(\lambda)} \cdot |\theta_{\lambda_0}(z)|^{-t},$$

it is enough to show that both maps $z \mapsto e^{-t\mathrm{Re}\log\psi_z(\lambda)}$ and $z \mapsto |\theta_{\lambda_0}(z)|^{-t}$ are in H_α and that both maps

$$(\lambda, t) \mapsto e^{-t\mathrm{Re}\log\psi_{(\cdot)}(\lambda)} \in \mathrm{H}_\alpha$$

and

$$(\lambda, t) \mapsto |\theta_{\lambda_0}(\cdot)|^{-t} \in \mathrm{H}_\alpha$$

are continuous. First recall that the function $z \mapsto \mathrm{Re}\log\psi_z(\lambda)$, $z \in J(F)$, is in H_α and consequently the function $z \mapsto t\mathrm{Re}\log\psi_z(\lambda)$, $z \in J(F)$, is in H_α for every $t \in \mathbb{R}$. Our most direct aim now is to show first that the mapping $(\lambda, t) \mapsto t\mathrm{Re}\log\psi_{(\cdot)}(\lambda) \in \mathrm{H}_\alpha$ is continuous on a polydisk $\mathbb{D}_{\mathbb{C}^3}((\lambda_0, t_0), R)$ with sufficiently small $R > 0$. This function is obviously continuous with respect to the variable t on the polydisk $\mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times B_{\mathbb{C}}(t_0, \rho)$. It is therefore sufficient to prove the Lipschitz continuity of the functions $\lambda \mapsto -t\mathrm{Re}\log\psi_{(\cdot)}(\lambda) \in \mathrm{H}_\alpha$ with Lipschitz constants independent of t. In order to do it, fix $\lambda = (\lambda_x, \lambda_y), \lambda' = (\lambda'_x, \lambda'_y) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$ In view of (10.11) we have for all $z \in J(F)$ that

$$
\begin{aligned}
&|\mathrm{Re}\log\psi_z(\lambda') - \mathrm{Re}\log\psi_z(\lambda)| = \\
&\qquad = \sum_{p,q=0}^{\infty} c_{p,q}(z)\left((\lambda'_x - \mathrm{Re}\lambda_0)^p(\lambda'_y - \mathrm{Im}\lambda_0)^q - (\lambda_x - \mathrm{Re}\lambda_0)^p(\lambda_y - \mathrm{Im}\lambda_0)^q\right)
\end{aligned} \tag{10.20}
$$

Put $a_x = \lambda'_x - \mathrm{Re}\lambda_0$, $a_y = \lambda'_y - \mathrm{Im}\lambda_0$, $b_x = \lambda_x - \mathrm{Re}\lambda_0$ and $b_y = \lambda_y - \mathrm{Im}\lambda_0$. We then have

$$
\begin{aligned}
|a_x^p a_y^q - b_x^p b_y^q| &= |a_x^p(a_y^q - b_y^q) + b_y^q(a_x^p - b_x^p)| \\
&\le |a_x^p||a_y - b_y|\sum_{i=0}^{q-1}|a_y|^i|b_y|^{q-1-i} + |b_y^q||a_x - b_x|\sum_{i=0}^{p-1}|a_x|^i|b_x|^{p-1-i} \\
&\le \left(q\left(\frac{r}{16}\right)^p\left(\frac{r}{16}\right)^{q-1} + p\left(\frac{r}{16}\right)^q\left(\frac{r}{16}\right)^{p-1}\right)||\lambda' - \lambda|| \qquad (10.21) \\
&\le \frac{16}{r}(p+q)\left(\frac{r}{16}\right)^p\left(\frac{r}{16}\right)^q||\lambda' - \lambda||
\end{aligned}
$$

Combining this, (10.20) and (10.12), we obtain

$$
|\mathrm{Re}\log\psi_z(\lambda') - \mathrm{Re}\log\psi_z(\lambda)| \le \frac{16}{r}||\lambda' - \lambda||\sum_{p,q=0}^{\infty}(p+q)8^{-(p+q)} = \frac{16C_1}{r}||\lambda' - \lambda||,
$$

(where $C_1 = \sum_{p,q=0}^{\infty}(p+q)8^{-(p+q)}$ is finite) for all $t \in B_{\mathbb{C}}(t_0, \rho/2)$ and all $\lambda, \lambda' \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$. Now fix $z, w \in J(F)$ with $|z - w| \le \delta$. It follows from (10.16) and (10.21) that

$$
\begin{aligned}
&|t|\Big|\mathrm{Re}\log\psi_w(\lambda') - \mathrm{Re}\log\psi_w(\lambda) - (\mathrm{Re}\log\psi_z(\lambda') - \mathrm{Re}\log\psi_z(\lambda))\Big| \\
&= |t|\left|\sum_{p,q=0}^{\infty}(c_{p,q}(w) - c_{p,q}(z))\left((\lambda'_x - \mathrm{Re}\lambda_0)^p(\lambda'_y - \mathrm{Im}\lambda_0)^q - (\lambda_x - \mathrm{Re}\lambda_0)^p(\lambda_y - \mathrm{Im}\lambda_0)^q\right)\right| \\
&\le (|t_0| + \rho)\frac{32C}{r}|z - w|^\alpha||\lambda' - \lambda||\sum_{p,q=0}^{\infty}(p+q)2^{-(p+q)} \\
&\le \frac{4M\hat{K}_{\lambda_0}C_2}{r}(|t_0| + \rho)||\lambda' - \lambda|||z - w|^\alpha
\end{aligned}
$$

where $C_2 = \sum_{p,q=0}^{\infty}(p+q)2^{-(p+q)}$ is finite. Thus,

$$
v_\alpha\Big(-t\mathrm{Re}\log\psi_{(\cdot)}(\lambda) - (-t\mathrm{Re}\log\psi_{(\cdot)}(\lambda'))\Big) \le 4M\hat{K}_{\lambda_0}C_2(|t_0| + \rho)r^{-1}||\lambda' - \lambda||.
$$

for all $\lambda, \lambda' \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$ and $t \in B_{\mathbb{C}}(t_0, \rho)$. Thus the proof of the continuity of the function $(\lambda, t) \mapsto -t\mathrm{Re}\log\psi_{(\cdot)}(\lambda)$, $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times B_{\mathbb{C}}(t_0, \rho)$, is complete. The continuity of the function $(\lambda, t) \mapsto \phi_{(\lambda,t)}(\cdot) = \exp\Big(-t\mathrm{Re}\log\psi_{(\cdot)}(\lambda)\Big)$, $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times B_{\mathbb{C}}(t_0, \rho)$ follows now immediately from (10.13) and inequality $|e^b - e^a| = |e^a||e^{b-a} - 1| \le A|b-a|$, where A depends on the upper bound of $|a|$.

We shall now show that for every $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16) \times B_{\mathbb{C}}(t_0, \rho)$ the function $z \mapsto |\theta_{\lambda_0}(\cdot)|^{-t}$, $z \in J(F)$, is $\in \mathrm{H}_\alpha$. Indeed, for every $z \in J(F)$, we have

$$|\theta_{\lambda_0}(z)|^{-t} = |1 - e^{-z}|^{-t} = \exp\Big(-t \log|1 - e^{-z}|\Big).$$

Since $\mathrm{Re} z < 0$ and $\mathrm{Re} t > 0$, we see from this formula that the supremum norm M_5 of the function $z \mapsto |\theta_{\lambda_0}(\cdot)|^{-t}$, $z \in J(F)$, is finite. Fix in addition $w \in J(F) \cap B(z, \delta)$. Then, by (10.9), we have

$$\Big|\log|1 - e^{-z}| - \log|1 - e^{-w}|\Big| \le M|z - w|,$$

and therefore, using (10.8) with Tr enlarged to $M\delta(t_0 + \rho)$, we get

$$\begin{aligned}
\Big||\theta_{\lambda_0}(z)|^{-t}| - |\theta_{\lambda_0}(w)|^{-t}\Big| &= \Big|\exp\Big(-t\log|1 - e^{-z}|\Big) - \exp\Big(-t\log|1 - e^{-w}|\Big)\Big| \\
&= |1 - e^{-z}|^{-t}\Big|1 - \exp\Big(t(\log|1 - e^{-z}| - \log|1 - e^{-w}|)\Big)\Big| \\
&\le M_5 E\Big|t(\log|1 - e^{-z}| - \log|1 - e^{-w}|)\Big| \\
&\le M_5 E(t_0 + \rho)M|z - w| \\
&\le M_5 E(t_0 + \rho)M|z - w|^\alpha.
\end{aligned}$$

So, it remains to show that the function $(\lambda, t) \mapsto |\theta_{\lambda_0}(\cdot)|^{-t}| \in \mathrm{H}_\alpha$ is continuous. Since $|\theta_{\lambda_0}(z)|^{-t}$ does not depend on λ, we only need to check its continuity with respect to the variable t. First notice that

$$\nabla(1 - e^{-z}) = e^{-z}(1, i)$$

and

$$\nabla(\overline{1 - e^{-z}}) = \nabla(1 - e^{-\overline{z}}) = e^{-\overline{z}}(1, -i) = \overline{e^{-z}}(1, -i).$$

Hence, using the Leibniz rule, we get

$$\nabla(|1 - e^{-z}|^2) = \nabla\Big((1 - e^{-z})(\overline{1 - e^{-z}})\Big) = (1 - e^{-z})\overline{e^{-z}}(1, -i) + (1 - \overline{e^{-z}})e^{-z}(1, i).$$

Assuming now that $\mathrm{Re} z \le 0$, we therefore obtain

$$||\nabla(|1 - e^{-z}|^2)|| \le 2|e^{-z}\overline{e^{-z}}|||(1, -i)|| + 2|\overline{e^{-z}}e^{-z}|||(1, i)|| = 4\sqrt{2}|e^{-2z}|.$$

Thus

$$||\nabla(|1 - e^{-z}|)|| = ||\nabla\Big((|1 - e^{-z}|^2)^{1/2}\Big)|| = \frac{1}{2}|1 - e^{-z}|^{-1}||\nabla(|1 - e^{-z}|^2)|| \le 2\sqrt{2}|e^{-2z}||1 - e^{-z}|^{-1}.$$

Since

$$\nabla\Big(|\theta_{\lambda_0}(z)|^{-t}\Big) = \nabla\Big(|1 - e^{-z}|^{-t}\Big) = -t|1 - e^{-z}|^{-t-1}\nabla(|1 - e^{-z}|),$$

we therefore get for all $t_1, t_2 \in B(t_0, \rho)$ that

$$\begin{aligned}
\Big\|\nabla\Big(|\theta_{\lambda_0}(z)|^{-t_1} - |\theta_{\lambda_0}(z)|^{-t_2}\Big)\Big\| &= |1 - e^{-z}|^{-1}\Big|t_1|1 - e^{-z}|^{-t_1} - t_2|1 - e^{-z}|^{-t_2}\Big|||\nabla(|1 - e^{-z}|)|| \\
&\le 2\sqrt{2}|e^{-2z}||1 - e^{-z}|^{-2}\Big||1 - e^{-z}|^{-t_1} - t_2|1 - e^{-z}|^{-t_2}\Big|.
\end{aligned}$$

Since $(ta^{-t})' = a^{-t}(1 - t\log a)$, using the Mean Value Inequality, we can continue the above estimate as follows.

$$\begin{aligned}&\left\|\nabla\Big(|\theta_{\lambda_0}(z)|^{-t_1} - |\theta_{\lambda_0}(z)|^{-t_2}\Big)\right\| \le \\ &\qquad \le 2\sqrt{2}|e^{-2z}||1-e^{-z}|^{-2}\sup\Big\{|1-e^{-z}|^{-t}(1-t\log|1-e^{-z}|) : t\in[t_1,t_2]\Big\}|t_2-t_1|.\end{aligned}$$

Since $J(F) \subset \{z \in Q : \mathrm{Re} z \le -\epsilon_{\lambda_0}\}$, we get that

$$M_6 := \sup\left\{\left|\frac{e^{-z}}{1-e^{-z}}\right| : z \in J(F)\right\} < \infty.$$

And since obviously

$$M_7 := \sup\{|1-e^{-z}|^{-t}(1-t\log|1-e^{-z}|) : z \in J(F),\, t\in[t_1,t_2]\} < \infty$$

we therefore obtain

$$\left|\nabla\Big(|\theta_{\lambda_0}(z)|^{-t_1} - |\theta_{\lambda_0}(z)|^{-t_2}\Big)\right| \le 2\sqrt{2}M_6M_7|t_2-t_1|.$$

Thus, for every $z \in J(F)$ and every $w \in B(z,\delta)$, we get that

$$\begin{aligned}\Big||\theta_{\lambda_0}(w)|^{-t_1} - |\theta_{\lambda_0}(w)|^{-t_2} - \Big(|\theta_{\lambda_0}(z)|^{-t_1} - |\theta_{\lambda_0}(z)|^{-t_2}\Big)\Big| &\le \\ &\le \int_z^w \left|\nabla\Big(|\theta_{\lambda_0}(\xi)|^{-t_1} - |\theta_{\lambda_0}(\xi)|^{-t_2}\Big)\right||d\xi| \\ &\le 2\sqrt{2}M_6M_7|t_2-t_1||w-z| \\ &\le 2\sqrt{2}M_6M_7|t_2-t_1||w-z|^\alpha.\end{aligned}$$

Utilizing the Mean Value inequality we also obtain that

$$\begin{aligned}&\Big||\theta_{\lambda_0}(z)|^{-t_1} - |\theta_{\lambda_0}(z)|^{-t_2}\Big| \le \\ &\qquad \le \log|1-e^{-z}|\sup\{|1-e^{-z}|^{-t} : z\in J(F),\, t\in[t_1,t_2]\}|t_2-t_1| \\ &\qquad \le M_7|t_2-t_1|.\end{aligned}$$

These last two estimates complete the proof of the item (c) that the function $(\lambda,t) \mapsto |\theta_{\lambda_0}(\cdot)|^{-t} \in \mathrm{H}_\alpha$ is continuous. So, it remains to check item (f), the last assumption in Corollary 8.7. In order to do it fix arbitrary $\lambda_2 \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$ and $t_2 \in B_{\mathbb{C}}(t_0,\rho)$. Take $\gamma > 0$ so small that $\mathbb{D}_{\mathbb{C}^2}(\lambda_2,\gamma) \subset \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$ and $B_{\mathbb{C}}(t_2, 2\gamma) \subset B_{\mathbb{C}}(t_0,\rho)$. Then fix arbitrary $\lambda_1 \in \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)$ and $t_1 \in (1, \mathrm{Re}(t_2) - \gamma)$ Then for every $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^2}(\lambda_2,\gamma) \times B_{\mathbb{C}}(t_2,\gamma)$, we have

$$\begin{aligned}\left|\frac{\phi_{(\lambda,t)}(z)}{\phi_{(\lambda_1,t_1)}(z)}\right| &= \frac{e^{-t\mathrm{Re}\log\psi_z(\lambda)}}{e^{-t_1\mathrm{Re}\log\psi_z(\lambda_1)}}\,|\theta_{\lambda_0}(z)|^{-(t-t_1)} = e^{t_1\mathrm{Re}\log\psi_z(\lambda_1) - t\mathrm{Re}\log\psi_z(\lambda)}\cdot|\theta_{\lambda_0}(z)|^{-(t-t_1)} \\ &= e^{t_1(\mathrm{Re}\log\psi_z(\lambda_1) - \mathrm{Re}\log\psi_z(\lambda))}\cdot e^{(t_1-t)\mathrm{Re}\log\psi_z(\lambda)}\cdot|\theta_{\lambda_0}(z)|^{-(t-t_1)}\end{aligned}$$

Using (10.13) we can estimate

$$|e^{t_1(\mathrm{Re}\log\psi_z(\lambda_1) - \mathrm{Re}\log\psi_z(\lambda))}| = e^{\mathrm{Re}(t_1(\mathrm{Re}\log\psi_z(\lambda_1) - \mathrm{Re}\log\psi_z(\lambda)))} \le e^{|t_1(\mathrm{Re}\log\psi_z(\lambda_1) - \mathrm{Re}\log\psi_z(\lambda))|} \le e^{8t_1M_1}.$$

and
$$\left|e^{(t_1-t)\mathrm{Re}\log\psi_z(\lambda)}\right| = e^{\mathrm{Re}((t_1-t)\mathrm{Re}\log\psi_z(\lambda))} \le e^{|(t_1-t)\mathrm{Re}\log\psi_z(\lambda)|} \le e^{4\rho M_1}.$$
Since $A = \inf_{z\in J(F)}|\theta_{\lambda_0}(z)|$ is positive, since $\mathrm{Re}(t_1-t)<0$ and since $\mathrm{Re}(t_1-t)>-\rho$, we can write
$$|\theta_{\lambda_0}(z)|^{-(t-t_1)} = |\theta_{\lambda_0}(z)|^{\mathrm{Re}(t_1-t)} \le \min\{1,|\theta_{\lambda_0}(z)|\}^{\mathrm{Re}(t_1-t)} \le \min\{1,|\theta_{\lambda_0}(z)|\}^{-\rho} \le \min\{1,A\}^{-\rho}.$$
Therefore
$$\left|\frac{\phi_{(\lambda,t)}(z)}{\phi_{(\lambda_1,t_1)}(z)}\right| \le \exp\Big(8t_1M_1+4\rho M_1\Big)\min\{1,A\}^{-\rho}$$
and the item (e) is verified. The assumptions of Corollary 8.7 are therefore checked with $G = \mathbb{D}_{\mathbb{C}^2}(\lambda_0, r/16)\times B_{\mathbb{C}}(t_0,\rho)$. ■

We are now in a position to conclude the proof of the following main result of our paper.

Theorem 10.3. *The function* $\lambda\mapsto \mathrm{HD}(J_r(F_\lambda))$, $\mathrm{Re}\lambda>1$, *is real-analytic.*

Proof. Fix $\lambda_0\in\mathbb{C}$ with $\mathrm{Re}\lambda>1$ and $t_0\in(1,\infty)$. Since by Proposition 9.6, $h_\lambda : J(F_{\lambda_0})\to J(F_\lambda)$ is Hölder continuous with the Hölder exponent $\alpha(\lambda)$ depending on λ but converging to 1 as $\lambda\to\lambda_0$, we get that for every $r>0$ sufficiently small
$$\alpha = \inf\{\alpha_\lambda : \lambda\in B(\lambda_0,r)\}>0.$$
Recall now that for every $\lambda\in B(\lambda_0,r)$ and every $t>1$, $\mathcal{L}^0_{\lambda,t} : \mathrm{H}_\alpha(J(F_{\lambda_0}))\to \mathrm{H}_\alpha(J(F_{\lambda_0}))$ is the operator induced by the weight function $|F'_\lambda\circ h_\lambda|^{-t} : J(F_{\lambda_0})\to\mathbb{R}$, i.e.
$$\mathcal{L}^0_{\lambda,t}g(z) = \sum_{x\in F^{-1}_{\lambda_0}(z)} |F'_\lambda(h_\lambda(x))|^{-t}g(x).$$
Proposition 10.2 says that there exist $R>0$ and a holomorphic function
$$L : \mathbb{D}_{\mathbb{C}^3}((\lambda_0,t_0),R)\to L(\mathrm{H}_\alpha(J(F\lambda_0))$$
(λ_0 is treated here as elements of $\mathbb{C}^2$ and t_0 as an element of $\mathbb{C}$) such that for every $(\lambda,t)\in B(\lambda_0,R)\times B(t_0,\rho)\subset\mathbb{C}\times\mathbb{R}$
$$L(\lambda,t) = \mathcal{L}^0_{\lambda,t}. \tag{10.22}$$
Now, in view of Theorem 5.4 and Proposition 4.9, $e^{\mathrm{P}_{\lambda_0}(t)}$ ($t\in B(t_0,R)$ is a simple isolated eigenvalue of the operator $L(\lambda_0,t) = \mathcal{L}^0_{\lambda_0,t} : \mathrm{H}_\alpha(J(F_{\lambda_0})\to\mathrm{H}_\alpha(J(F_{\lambda_0}))$. Applying now the perturbation theory for linear operators (see [18]), we see that there exists $0<R_1\le R$ and a holomorphic function $\gamma : \mathbb{D}_{\mathbb{C}^3}((\lambda_0,t),R_1)\to\mathbb{C}$ such that $\gamma(\lambda_0,t_0) = e^{P_{\lambda_0}(t_0)}$ and for every $(\lambda,t)\in\mathbb{D}_{\mathbb{C}^3}((\lambda_0,t),R_1)$ the number $\gamma(\lambda,t)$ is a simple isolated eigenvalue of $L(\lambda,t)$ with the remainder part of the spectrum uniformly separated from $\gamma(\lambda,t)$. In particular there exist $0<R_2\le R_1$ and $\kappa>0$ such that
$$\sigma(L(\lambda,t))\cap B(e^{\mathrm{P}_{\lambda_0}(t_0)},\kappa) = \{\gamma(\lambda,t)\} \tag{10.23}$$

for all $(\lambda, t) \in \mathbb{D}_{\mathbb{C}^3}((\lambda_0, t_0), R_2)$. Consider now for each $(\lambda, t) \in B(\lambda_0, t_0) \times (t_0 - R, t_0 + R)$ the operator $\mathcal{L}_{\lambda,t} : \mathrm{H}_1(J(F_\lambda)) \to \mathrm{H}_1(J(F_\lambda))$ (see Lemma 5.1) given by the formula

$$\mathcal{L}_{\lambda,t} g(z) = \sum_{x \in F^{-1}(z)} |F_\lambda'(x)|^{-t} g(x).$$

It is easy to see that the map $T_\lambda : C_b(J(F_\lambda)) \to C_b(J(F_{\lambda_0}))$ defined by the formula $T_\lambda(g) = g \circ h_\lambda$ establishes a bounded linear conjugacy between $\mathcal{L}_{\lambda,t} : C_b(J(F_\lambda)) \to C_b(J(F_\lambda))$ and $\mathcal{L}^0_{\lambda,t} : C_b(J(F_{\lambda_0})) \to C_b(J(F_{\lambda_0}))$. Since the map $h_\lambda : J(F_{\lambda_0}) \to J(F_\lambda)$ is Hölder continuous with the Hölder exponent α, we obtain

$$T_\lambda(\mathrm{H}_1(J(F_\lambda))) \subset H_\alpha(J(F_{\lambda_0})).$$

Hence $e^{P_\lambda(t)}$ is an eigenvalue of the operator

$$\mathcal{L}^0_{\lambda,t} : \mathrm{H}_\alpha(J(F_{\lambda_0})) \to \mathrm{H}_\alpha(J(F_{\lambda_0}))$$

and, by Lemma 10.1, $e^{P_\lambda(t)} \in B(e^{P_\lambda(t_0)}, \kappa)$ for all $\lambda \in B(\lambda_0, R_3)$ and all $t \in (t_0 - \rho, t_0 + \rho)$ if $\rho \in (0, \min\{t_0, R_2\}$ and $R_3 \in (0, I\!\!R_2)$ are sufficiently small. Combining this, (10.22) and (10.23) we see that $\gamma(\lambda, t) = e^{\mathrm{P}_\lambda(t)}$ for λ, t as above. Therefore the function $(\lambda, t) \to P_\lambda(t)$, $(\lambda, t) \in B(\lambda_0, R_3) \times (t_0 - \rho, t_0 + \rho)$, is real-analytic. Since, by Theorem 7.2, $\mathrm{P}_\lambda(s_\lambda) = 0$, where $s_\lambda = \mathrm{HD}(J(F_\lambda))$, it follows from the Implicite Function Theorem that in order to conclude the proof it suffices to show that

$$\frac{\partial P_\lambda(t)}{\partial t} \neq 0$$

for all $(\lambda, t) \in B(\lambda_0, R_3) \times (t_0 - \rho, t_0 + \rho)$. So fix such λ and t. Fix $z \in J(F_\lambda)$. Since for every $u \geq 0$ and every $n \geq 1$

$$\sum_{x \in F_\lambda^{-n}(z)} |(F_\lambda^n)'(x)|^{-(t+u)} = \sum_{x \in F_\lambda^{-n}(z)} |(F_\lambda^n)'(x)|^{-u} |(F_\lambda^n)'(x)|^{-t} \leq L^{-u} \beta^{un} \sum_{x \in F_\lambda^{-n}(z)} |(F_\lambda^n)'(x)|^{-t},$$

we conclude that $\mathrm{P}_\lambda(t+u) - \mathrm{P}_\lambda(t) \leq u \log \beta$ which implies that $\frac{\partial \mathrm{P}_\lambda(t)}{\partial t}(\lambda, t) \leq \log \beta < 0$. We are done. ■

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