GEOMETRY AND ERGODIC THEORY OF NON-RECURRENT ELLIPTIC FUNCTIONS

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Abstract. We explore the class of elliptic functions whose critical points all contained in the Julia set are non-recurrent and whose ω-limit sets form compact subsets of the complex plane. In particular, this class comprises hyperbolic, subhyperbolic and parabolic elliptic maps. Let h be the Hausdorff dimension of the Julia set of such an elliptic function f. We construct an atomless h-conformal measure m and we show that the h-dimensional Hausdorff measure of the Julia set of f vanishes unless the Julia set is equal to the entire complex plane $\mathbb{C}$. The h-dimensional packing measure is always positive and it is finite if and only if there are no rationally indifferent periodic points. Furthermore, we prove the existence of a (unique up to a multiplicative constant) σ-finite f-invariant measure μ equivalent to m. The measure μ is then proved to be ergodic and conservative and we identify the set of points whose open neighborhoods all have infinite measure μ. In particular we show that ∞ is not among them.

1. Introduction and General Preliminaries

1.1. Introduction.

First examples of elliptic (in fact $\wp$-Weierstrass) functions with detailed descriptions of their Julia sets appeared in [14]. Our paper dealing with elliptic functions whose critical points all contained in the Julia set are non-recurrent and whose ω-limit sets form compact subsets of the complex plane, basically stems from [26], [27] and [15]. Any such elliptic function will be called non-recurrent. We study geometric properties of the Julia sets ultimately resulting in Theorem 4.1 which says that the h-dimensional Hausdorff measure of the Julia set of f vanishes unless the Julia set is equal to the entire complex plane $\mathbb{C}$. The h-dimensional packing measure is always positive and it is finite if and only if there are no rationally indifferent periodic points. We would like to emphasize that the Hausdorff and packing measures appearing in this theorem are taken with respect to the spherical metric on $\overline{\mathbb{C}}$. The fact that the h-dimensional Hausdorff measure of the Julia set vanishes in the case when $h < 2$ or equivalently when the Julia set is not equal to the entire complex plane (note that due to [15] $h > 1$), dramatically differentiates non-recurrent elliptic functions from the akin class of non-recurrent rational functions. The reason is that for these latter functions the h-dimensional Hausdorff measure of the Julia set is always finite and positive if and only if $h > 1$ (see [26]). Our main

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technical tool employed in this paper is the concept of semi-conformal, almost-conformal and conformal measures. We provide an elaborated proof of the existence, uniqueness and continuity of an h-conformal measure. Another important tool is provided by Proposition 2.23, where, expressed in an appropriate language, all non-singular points are shown to be conical. Although there are some overlaps with rational functions (see [26]), most of the proofs are substantially different, mainly because of the existence of poles in the Julia set.

Our second major theme in this paper is the dynamics of f with respect to the conformal measure m. As the first result in this direction, we prove the existence of a conservative ergodic σ-finite measure μ equivalent to m. Developing this direction, we study points of finite and infinite condensation of the measure μ, the concepts introduced in [27]. After collecting some basic facts about these points we show in Subsection 5.2 that ∞ is always a point of finite condensation, perhaps the most interesting fact about the measure μ. In the next subsection we relate points of infinite condensation with the set $\Omega(f)$ of rationally indifferent periodic points, providing in particular some sufficient conditions ($\Omega(f) = \emptyset$) for the invariant measure μ to be finite. At the end of this section we deal with parabolic points themselves.

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1.2. **General Preliminaries.**

All the points (numbers) appearing in this paper are complex unless it is clear from the context that they are real. In particular x and y are always assumed to be complex numbers and not the real and imaginary parts of a complex number. Given $A \subset \mathbb{C}$ and $r > 0$, the symbol $B(A, r)$ denotes the Euclidean open r-neighbourhood of the set A. Throughout the entire paper f^*, diam_s and $B_s(A, r)$ denote respectively the derivatives, diameters and open balls defined by means of the spherical metric whereas f', diam and $B(A, r)$ are considered in the Euclidean sense. We would like to emphasize that when counting f^*, we consider the spherical metric in the domain and in the codomain.

Definition 1.1. *If $H : D \to \mathbb{C}$ is an analytic map, $z \in \mathbb{C}$, and $r > 0$, then by*

$$\text{Comp}(z, H(z), H, r)$$

we denote the connected component of $H^{-1}(B(H(z), r))$ that contains z.

Suppose now that c is a critical point of an analytic map $H : D \to \mathbb{C}$. Then there exists $R = R(H, c) > 0$ and $A = A(H, c) \geq 1$ such that

$$A^{-1}|z - c|^q \leq |H(z) - H(c)| \leq A|z - c|^q$$

and

$$A^{-1}|z - c|^{q-1} \leq |H'(z)| \leq A|z - c|^{q-1}$$

for every $z \in \mathrm{Comp}(c, H(c), H, R)$ and that

$$H(\mathrm{Comp}(c, H(c), H, R)) = B(H(c), R)$$

where $q = q(H, c)$ is the order of H at the critical point c. Moreover, by taking $R > 0$ sufficiently small, we can ensure that the two above inequalities hold for every $z \in B(c, (R/A)^{1/q})$ and the ball $B(c, (R/A)^{1/q}) \cup \mathrm{Comp}(c, H(c), H, R)$ can be expressed as a union of q closed topological disks with smooth boundaries and mutually disjoint interiors such that the map H restricted to each of these interiors, is injective.

Koebe's $\frac{1}{4}$-Theorem. If $z \in \mathbb{C}$, $r > 0$ and $H : B(z, r) \to \mathbb{C}$ is an arbitrary univalent analytic function, then $H(B(z, r)) \supset B(H(z), 4^{-1}|H'(z)|r)$.

Koebe's Distortion Theorem, I (Euclidean version). There exists a function $k : [0, 1) \to [1, \infty)$ such that for any $z \in \mathbb{C}$, $r > 0, t \in [0, 1)$ and any univalent analytic function $H : B(z, r) \to \mathbb{C}$ we have that

$$\sup\{|H'(w)| : w \in B(z, tr)\} \le k(t) \inf\{|H'(w)| : w \in B(z, tr)\}.$$

We put $K = k(1/2)$.

Koebe's Distortion Theorem, I (spherical version). Given a number $s > 0$ there exists a function $k_s : [0, 1) \to [1, \infty)$ such that for any $z \in \overline{\mathbb{C}}$, $r > 0, t \in [0, 1)$ and any univalent analytic function $H : B(z, r) \to \overline{\mathbb{C}}$ such that the complement $\overline{\mathbb{C}} \setminus H(B(z, r))$ contains a spherical ball of radius s we have

$$\sup\{|H^*(w)| : w \in B(z, tr)\} \le k_s(t) \inf\{|H^*(w)| : w \in B(z, tr)\}.$$

The following is a straightforward consequence of these two Distortion Theorems.

Lemma 1.2. *Suppose that $D \subset \mathbb{C}$ is an open set, $z \in D$ and $H : D \to \mathbb{C}$ is an analytic map which has an analytic inverse H_z^{-1} defined on $B(H(z), 2R)$ for some $R > 0$. Then for every $0 \le r \le R$*

$$B(z, K^{-1}r|H'(z)|^{-1}) \subset H_z^{-1}(B(H(z), r)) \subset B(z, Kr|H'(z)|^{-1}).$$

Lemma 1.3. *Suppose that $D \subset \overline{\mathbb{C}}$ is an open set, $z \in D$ and $H : D \to \overline{\mathbb{C}}$ is an analytic map which has an analytic inverse H_z^{-1} defined on $B(H(z), 2R)$ for some $R > 0$ avoiding a spherical ball of some radius s. Then for every $0 \le r \le R$*

$$B(z, k_s^{-1}(1/2)r|H^*(z)|^{-1}) \subset H_z^{-1}(B(H(z), r)) \subset B(z, k_s(1/2)r|H^*(z)|^{-1}).$$

We shall also use the following more geometric versions of Koebe's Distortion Theorems involving moduli of annuli.

Koebe's Distortion Theorem, II (Euclidean version). There exists a function $w : (0,+\infty) \to [1,\infty)$ such that for any two open topological disks $Q_1 \subset Q_2$ with $\mathrm{Mod}(Q_2 \backslash Q_1) \geq t$ and any univalent analytic function $H : Q_2 \to \overline{\mathbb{C}}$ such that the complement $\overline{\mathbb{C}} \backslash H(Q_2)$ contains a ball of radius s we have

$$\sup\{|H'(\xi)| : \xi \in Q_1\} \leq w(t) \inf\{|H'(\xi)| : \xi \in Q_1\}.$$

Koebe's Distortion Theorem, II (spherical version). Given a number $s > 0$ there exists a function $w_s : (0,+\infty) \to [1,\infty)$ such that for any two open topological disks $Q_1 \subset Q_2$ with $\mathrm{Mod}(Q_2 \setminus Q_1) \geq t$ and any univalent analytic function $H : Q_2 \to \overline{\mathbb{C}}$ such that the complement $\overline{\mathbb{C}} \setminus H(Q_2)$ contains a ball of radius s we have

$$\sup\{|H'(\xi)| : \xi \in Q_1\} \leq w_s(t) \inf\{|H'(\xi)| : \xi \in Q_1\}.$$

Given an analytic function H defined throughout a region $D \subset \mathbb{C}$, we put

$$\mathrm{Crit}(H) = \{z \in D : H'(z) = 0\}.$$

We will need in the sequel the following technical lemma proven in [27] as Lemma 2.11.

Lemma 1.4. *Suppose that an analytic map $Q \circ H : D \to \overline{\mathbb{C}}$, a radius $R > 0$ and a point $z \in D$ are such that*

$$\mathrm{Comp}(H(z), Q(H(z)), Q, 2R) \cap \mathrm{Crit}(Q) = \emptyset \text{ and } \mathrm{Comp}(z, Q \circ H(z), Q \circ H, R) \cap \mathrm{Crit}(H) \neq \emptyset,$$

If c belongs to the last intersection and

$$\mathrm{diam}\Big(\mathrm{Comp}(z, Q \circ H(z), Q \circ H, R)\Big) \leq (AR(H,c))^{1/q}$$

then

$$|z - c| \leq KA^2 |(Q \circ H)'(z)|^{-1} R.$$

1.3. **Preliminaries concerning iteration of meromorphic functions.**
The Fatou set $F(f)$ of a meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is defined in exactly the same manner as for rational functions; $F(f)$ is the set of points $z \in \mathbb{C}$ such that all the iterates are defined and form a normal family on a neighborhood of z. The *Julia set* $J(f)$ is the complement of $F(f)$ in $\mathbb{C}$. Thus, $F(f)$ is open, $J(f)$ is closed, $F(f)$ is completely invariant while $f^{-1}(J(f)) \subset J(f)$ and $f(J(f)) = J(f) \cup \{\infty\}$. For a general description of the dynamics of meromorphic functions see e.g. [6]. We would however like to note that it easily follows from Montel's criterion of normality that if $f : \mathbb{C} \to \overline{\mathbb{C}}$ has at least one pole which is not an omitted value then

$$J(f) = \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}.$$

In further sections we will be dealing with the foolowing set of points escaping to ∞ under iterates of f.

$$I_\infty(f) = \{z \in \mathbb{C} \colon z \in \bigcup_{n\geq 0} f^{-n}(\infty) \text{ or } \lim_{n\to\infty} f^n(z) = \infty\}$$

Let us now provide two related concepts, which play the central role in the approach undertaken in this paper. If $t \geq 0$, then a measure m supported on $J(f)$ is said to be semi t-conformal for $f : \mathbb{C} \to \overline{\mathbb{C}}$, if

$$m(f(A)) \geq \int_A |f^*|^t \, dm \tag{1.1}$$

for every Borel set $A \subset J(f)$ such that $f|_A$ is injective and m is said to be t-conformal for $f : \mathbb{C} \to \overline{\mathbb{C}}$, if

$$m(f(A)) = \int_A |f^*|^t \, dm \tag{1.2}$$

for these sets A.

2. The Dynamics of Non-recurrent Elliptic Functions

2.1. **Preliminary Results Concerning Elliptic Functions.** As we already indicated in the introduction, throughout the entire paper $f : \mathbb{C} \to \overline{\mathbb{C}}$ denotes a non-constant elliptic function. Every such function is doubly periodic and meromorphic. In particular there exist two vectors w_1, w_2, $\mathrm{Im}(\frac{w_1}{w_2}) \neq 0$, such that for every $z \in \mathbb{C}$ and $n, m \in \mathbb{Z}$,

$$f(z) = f(z + mw_1 + nw_2).$$

The set

$$\Lambda = \{mw_1 + nw_2 : m, n \in \mathbb{Z}\}$$

will be called the lattice of the elliptic function f. This object is independent of the choice of its generators w_1 and w_2. Let

$$\mathcal{R} = \{t_1 w_1 + t_2 w_2 : 0 \leq t_1, t_2 \leq 1\},$$

be the basic fundamental parallelogram of f. It follows from the periodicity of f that $f(\mathbb{C}) = f(\mathcal{R})$. Therefore $f(\mathbb{C})$ as a closed and open subset of the connected set $\overline{\mathbb{C}}$ is equal to $\overline{\mathbb{C}}$. This means that each elliptic function is surjective. It also follows from the periodicity of f that

$$f^{-1}(\infty) = \bigcup_{m,n\in\mathbb{Z}} \Big(\mathcal{R} \cap f^{-1}(\infty) + mw_1 + nw_2\Big).$$

For every pole b of f let q_b denote its multiplicity. We define

$$q := \sup\{q_b : b \in f^{-1}(\infty)\} = \max\{q_b : b \in f^{-1}(\infty) \cap \mathcal{R}\}.$$

Let

$$B_R = \{z \in \overline{\mathbb{C}} \colon |z| > R\}.$$

For every pole b of f by $B_b(R)$ we denote the connected component of $f^{-1}(B_R)$ containing b. Recall that $\mathrm{Crit}(f)$ is the set of critical points of f i.e.

$$\mathrm{Crit}(f) = \{z : \ f'(z) = 0\}.$$

Its image, $f(\mathrm{Crit}(f))$, is called the set of critical values of f. Since $\mathcal{R} \cap \mathrm{Crit}(f)$ is finite and since $f(\mathrm{Crit}(f)) = f(\mathcal{R} \cap \mathrm{Crit}(f))$, the set of critical values $f(\mathrm{Crit}(f))$ is also finite. Thus, if $R > 0$ is large enough, say $R \geq R_0$, then B_R contains no critical values of f, all sets $B_b(R)$ are simply connected, mutually disjoint, and for $z \in B_b(R)$

$$f(z) = \frac{G_b(z)}{(z-b)^{q_b}} \tag{2.1}$$

where $G_b : B_b(R) \to \mathbb{C}$ is a holomorphic function taking values out of some neighbourhood of 0. If $U \subset B_R \setminus \{\infty\}$ is an open simply connected set, then all the holomorphic inverse branches $f^{-1}_{b,U,1}, \dots, f^{-1}_{b,U,q_b}$ of f are well-defined on U and for every $1 \leq j \leq q_b$ and all $z \in U$ we have

$$|(f^{-1}_{b,U,j})'(z)| \asymp |z|^{-\frac{q_b+1}{q_b}}. \tag{2.2}$$

Therefore, comp. [15]

$$|(f^{-1}_{b,U,j})^*(z)| \asymp \frac{|z|^{\frac{q_b-1}{q_b}}}{1+|b|^2} \asymp \frac{|z|^{\frac{q_b-1}{q_b}}}{|b|^2}, \tag{2.3}$$

where the second comparability sign we wrote assuming in addition that $|b|$ is large enough, say $|b| \geq R_1 > R_0$. It is shown in [15] that there exists a constant $L \geq 1$ such that for all poles b and all $R \geq R_1$, we have

$$\begin{aligned} L^{-1}R^{-\frac{1}{q_b}} &\leq \mathrm{diam}(B_b(R)) \leq LR^{-\frac{1}{q_b}}, \\ L^{-1}R^{-\frac{1}{q_b}}(1+|b|^2)^{-1} &\leq \mathrm{diam}_s(B_b(R)) \leq LR^{-\frac{1}{q_b}}(1+|b|^2)^{-1}. \end{aligned} \tag{2.4}$$

We will frequently use the following fact proven in [15].

Theorem 2.1. *If $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an arbitrary elliptic function, then*

$$\mathrm{HD}(J(f)) > \frac{2q}{q+1} \geq 1,$$

where $q = \sup\{q_b : b \in \mathrm{inf}^{-1}(\infty)\} = \max\{q_b : b \in \mathcal{R} \cap f^{-1}(\infty)\}$.

2.2. **Local behavior around parabolic fixed points.**
In this section $f : \mathbb{C} \to \overline{\mathbb{C}}$ is an arbitrary elliptic function of degree ≥ 2; in fact all the results stated here are of local character and are true for all meromorphic functions. In particular the map f is not assumed yet to be non-recurrent. In what follows we basically summarize the results concerning local behavior around parabolic fixed points which have been proved in [1], [10], and [11]. Although they were formulated and proved in the context of parabolic rational maps that is assuming that the Julia set contains no critical points, nevertheless they and their proofs are of local character and, in particular, extend to the class of all elliptic functions. Through this section ω is a simple parabolic fixed point of f, that is $f(\omega) = \omega$ and $f'(\omega) = 1$.

First note that on a sufficiently small open neighbourhood V of ω a holomorphic inverse branch $f_\omega^{-1} : V \to \overline{\mathbb{C}}$ of f is well defined which sends ω to ω. Moreover, V can be taken so small that on V the transformation f_ω^{-1} can be expressed in the form

$$f_\omega^{-1}(z) = z - a(z-\omega)^{p+1} + a_2(z-\omega)^{p+2} + a_3(z-\omega)^{p+3} + \dots \tag{2.5}$$

where $a \neq 0$ and $p = p(\omega)$ is a positive integer.

$$f_\omega^{-1}(z) - \omega = z - \omega - a(z-\omega)^{p+1} + a_2(z-\omega)^{p+2} + a_3(z-\omega)^{p+3} + \dots$$

Consider the set $\{z : a(z-\omega)^p \in \mathbb{R} \text{ and } a(z-\omega)^p > 0\}$. This set is the union of p rays beginning in ω and forming angles which are integer multiples of $2\pi/p$. Denote these rays by $L_1, L_2, \dots, L_p$. For $1 \leq j \leq p$, $0 < r \leq \infty$ and $0 \leq \alpha < 2\pi$ let $S_j(r,\alpha) \subset V$ be the set of those points z lying in the open ball $B(\omega, r)$ for which the angle between the rays L_j and the interval which joins the points ω and z does not exceed α. Using (2.5) an easy computation leads to the following

$$\begin{gathered} \forall \alpha > 0 \ \exists r_1(\alpha) > 0 \ \exists 0 < \alpha_0 \leq \alpha \ \forall 1 \leq j \leq p \\ f_\omega^{-1}(S_j(r_1(\alpha), \alpha_0)) \subset S_j(\infty, \alpha) \end{gathered} \tag{2.6}$$

and there are $\beta > 0$ and $\theta_1 > 0$ such that

$$|f_\omega^{-1}(z) - \omega| < |z - \omega| \quad \text{and} \quad |(f_\omega^{-1})'(z)| < 1 \tag{2.7}$$

for every $\omega \neq z \in S_1(\theta_1, \beta) \cup \ldots \cup S_p(\theta_1, \beta)$. The following version of Fatou's flower theorem, (see [5], [21], comp. [1]) shows that the Julia set $J(f)$ approaches the fixed point ω tangentially to the lines $L_1, L_2, \dots, L_p$. This can be precisely formulated as follows.

Lemma 2.2. *(Fatou's flower theorem) For every $\alpha > 0$ there exists $r_2(\alpha) > 0$ such that*

$$J(f) \cap B(\omega, r_2(\alpha)) \subset S_1(r_2(\alpha), \alpha) \cup \ldots \cup S_p(r_2(\alpha), \alpha).$$

Since the Julia set $J(f)$ is fully invariant ($f^{-1}(J(f)) = J(f)$ and $f(J(f)) = J(f) \cup \{\infty\}$, we conclude from this lemma and (2.7) that for every $0 < \theta_2 \leq \min\{\theta_1, r_2(\beta)\}$ we have

$$f_\omega^{-1}(J(f) \cap B(\omega, \theta_2)) \subset J(f) \cap B(\omega, \theta_2).$$

Thus all iterates $f_\omega^{-n} : J(f) \cap B(\omega, \theta_2) \to J(f) \cap B(\omega, \theta_2)$, $n = 0, 1, 2, \ldots$ are well defined. From (2.6), Lemma 2.2, and (2.7) we obtain the following

$$\begin{aligned} &\forall \alpha > 0 \; \exists r_3(\alpha) > 0 \; \forall 1 \le j \le p \\ &\qquad f_\omega^{-1}(S_j(r_3(\alpha), \alpha) \cap J(f)) \subset S_j(r_3(\alpha), \alpha). \end{aligned} \tag{2.8}$$

Put

$$\theta = \theta(f, \omega) = \min\{\theta_2, r_2(\beta), r_3(\beta)\} \tag{2.9}$$

Then, it follows from (2.7), (2.6), and Lemma 2.2 that for every $z \in J(f) \cap B(\omega, \theta)$.

$$\lim_{n \to \infty} f_\omega^{-n}(z) = \omega \tag{2.10}$$

In fact it can be proved that this convergence is uniform on compact subsets of $B(\omega, \theta) \cap J(f) \setminus \{\omega\}$. See (2.11) for even stronger result. By precise computations one can prove the following.

Lemma 2.3. *For every $\tau > 0$ sufficiently small and every $z \in J(f) \cap B(\omega, \theta)$*

$$f_\omega^{-1}(B(z, \tau|z - \omega|)) \subset B(f_\omega^{-1}(z), \tau|f_\omega^{-1}(z) - \omega|).$$

This lemma immediately leads to the following.

Lemma 2.4. *For every $\tau > 0$ sufficiently small, every $z \in J(f) \cap B(\omega, \theta)$ and every $n \ge 0$ there exists a unique holomorphic inverse branch*

$$f_{\omega,z}^{-n} : B(z, 2\tau|z - \omega|) \to B(f_\omega^{-n}(z), 2\tau|f_\omega^{-n}(z) - \omega|)$$

of f^n which sends z to $f_\omega^{-n}(z)$.

The following two results (comp. Lemma 1 and Lemma 2 of [10] and Lemma 4.8 of [11]) can be proved in exactly the same way as in [10] and [11].

$$\lim_{n \to \infty} |f_{\omega,z}^{-n}(z) - \omega| n^{1/p} = (|a|p)^{-1/p} \; \text{ and } n^{-\frac{p+1}{p}} \preceq |(f_{\omega,z}^{-n})'(z)|, |(f_{\omega,z}^{-n})^*(z)| \preceq n^{-\frac{p+1}{p}} \tag{2.11}$$

uniformly on compact subsets of $B(\omega, \theta) \cap J(f) \setminus \{\omega\}$.

Lemma 2.5. *Let m be a semi t-conformal measure for f. Then for every $R > 0$ there exists a constant $C = C(t, \omega, R) \ge 1$ such that for every $0 < r \le R$*

$$\frac{m(B(\omega, r) \setminus \{\omega\})}{r^{\alpha_t(\omega)}}, \frac{m(B_s(\omega, r) \setminus \{\omega\})}{r^{\alpha_t(\omega)}} \le C.$$

where $\alpha_t(\omega) = t + p(\omega)(t-1)$. *If* m *is* t*-conformal, then in addition*

$$\frac{m(B(\omega,r)\setminus\{\omega\})}{r^{\alpha_t(\omega)}}, \frac{m(B_s(\omega,r)\setminus\{\omega\})}{r^{\alpha_t(\omega)}} \geq C^{-1}.$$

2.3. **Basic properties of non-recurrent elliptic functions.** .
We say that the elliptic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is non-recurrent, if the following conditions are satisfied:

(1) If $c \in \text{Crit}(f) \cap J(f)$, then the ω-limit set $\omega(c)$ is a compact subset of $\mathbb{C}$ (i.e. $\infty \notin \omega(c)$) and $c \notin \omega(c)$
(2) If $c \in \text{Crit}(f) \cap F(f)$ then either there exists an attracting periodic point w or a rationally indifferent periodic point w such that $\omega(c) \subset \{w, f(w), \dots, f^{p-1}(w)\}$, p is a period.

From now on, unless otherwise stated, we assume throughout the entire paper that the elliptic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is non-recurrent. We denote for every set $A \subset \mathbb{C}$

$$O_+(A) = \bigcup_{n \geq 0} f^n(A).$$

As an immediate consequence of the definition of non-recurrent elliptic functions and the finiteness of the set $f(\text{Crit}(J(f)))$, we get the following easy but useful fact.

Proposition 2.6. $\overline{O_+(f(\text{Crit}(J(f))))}$ *is a compact subset of* $\mathbb{C}$.

We recall that a periodic point ω of f is called parabolic if there exits $q \geq 1$ such that $f^q(\omega) = \omega$ and $(f^q)'(\omega) = 1$. The set of all parabolic points will be denoted by $\Omega(f)$. Since the set of critical values of f is finite, it follows from transcendental meromorphic version of Fatou's theorem (proven in [3] that $\Omega(f)$ is also finite. In addition, $\Omega(f)$ is contained in the Julia set $J(f)$. A crucial tool for our approach in this paper is the following elliptic counterpart of Mane's theorem proven in [17] for non-recurrent rational functions.

Theorem 2.7. *Let* $f : \mathbb{C} \to \overline{\mathbb{C}}$ *be a non-recurrent elliptic function. If* $X \subset J(f) \setminus \Omega(f)$ *is a closed subset of* $\mathbb{C}$, *then for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for every* $x \in X$ *and every* $n \geq 0$, *all the connected components of* $f^{-n}(B(x,\delta))$ *have Euclidean diameters* $\leq \epsilon$.

Since this theorem forms an extremally important tool in our paper and promptly distinguishes the class of non-recurrent elliptic functions from among all other elliptic functions, we would like to provide a few words of comment. First, Mane's original most general result is this.

Theorem 2.8. *Suppose that $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is a rational function of degree $d\ge 1$. Suppose also that $x\in J(f)$ is not a rationally indifferent periodic point nor x belongs to the ω-limit set of any recurrent critical point. Then for every $\epsilon>0$ there exists $\delta>0$ such that for every $n\ge 0$ all the connected components of $f^{-n}(B_s(x,\delta))$ have spherical diameters $\le\epsilon$.*

It is easy to see that in the context of rational functions Theorem 2.7 follows from Theorem 2.8 if $\mathbb{C}$ is replaced by $\overline{\mathbb{C}}$ and Euclidean diameters are replaced by spherical ones. In the Appendix we formulate and prove Theorem 6.3, the elliptic counterpart of Theorem 2.8. Although the proof very closely follows Manne's original arguments and the proof of Przytycki's lemma, we have decided to include it for the following reasons. First, Mane's original paper [17] is not easy accessible. Secondly, the original Mane's proof of Theorem 2.8 contains some minor misprints and one gap which is filled in by Przytycki's lemma from [23]. The last, third reason, is that elliptic functions seem to form the only "reasonable" class of transcendental entire and meromorphic functions for which an appropriate version of Theorem 2.8 or Theorem 2.7 could hold and we do want to check it rigorously. Although we have a bunch of counterexamples of meromorphic functions that do not satisfy Mane's theorem, we feel that our paper is not the right place to describe them. However, we would like to add that a weaker form of Mane's theorem (hyperbolicity on compact forward invariant subsets with some other appropriate technical assumption of Mane's flavors) for fairly large class of meromorphic functions has been proved in [12]. We would also like to point out that Theorem 2.7 easily follows from Theorem 6.3, the elliptic counterpart of Theorem 2.8, as long as the set $X\subset J(f)\setminus\Omega(f)$ is assumed to be a compact subset of the complex plane $\mathbb{C}$. The proof that Theorem 2.7 is also true for closed, not compact, subsets of $\mathbb{C}$ results from its "compact" part as follows. Suppose that $X\subset J(f)\setminus\Omega(f)$ is a closed subset of $\mathbb{C}$. Let $\Delta=\mathrm{dist}(\Omega(f),f^{-1}(\infty))>0$. In view of (2.2) and (2.4) there exists $R>0$ so large that if $|f(z)|\ge R/2$, then for some $b\in f^{-1}(\infty)$, $z\in B_b(R/2)$

$$|f'(z)|\ge 2 \text{ and } \mathrm{diam}(B_b(R/2))\le\Delta/2. \tag{2.12}$$

Consider now the compact set $Y=X\cup(J(f)\setminus B(\Omega(f),\Delta/2))\setminus B_R$ and the corresponding number $0<\delta\le\min\{\epsilon,R/2\}$ ascribed to Y and the number $\min\{\epsilon,R/2\}$ according to the "compact" part of Theorem 2.7. In order to complete the proof it suffices to show that if $x\in B_R$, then the diameter of each connected component $C_n(x)$ of $f^{-n}(B(x,\delta))$ does not exceed ϵ for every $\epsilon>0$. And indeed, fix $w\in f^{-n}(x)\cap C_n(x)$ and let $1\le k\le n$ be the least integer such that $f^{n-k}(w)\notin B_R$ provided it exists. Otherwise, set $k=n$. We shall show by mathematical induction that

$$\mathrm{diam}\Big(f^{n-j}(C_n(x))\Big)\le\delta\le\min\{\epsilon,R/2\} \tag{2.13}$$

for every $0\le j\le k$. For $j=0$ this formula is true since $f^n(C_n(x))=B(x,\delta)$. So, suppose that it is true for some $0\le j\le k-1$. Since $f^{n-j}(w)\in B_R$ and since diam $(f^{n-j}(C_n(x)))\le R/2$, we conclude that

$$f^{n-j}(C_n(x))\subset B_{R/2}. \tag{2.14}$$

It therefore follows from the first part of formula (2.12) that

$$\operatorname{diam}\Big(f^{n-(j+1)}(C_n(x))\Big) \le \frac{1}{2}\operatorname{diam}\Big(f^{n-j}(C_n(x))\Big) \le \delta.$$

This proves formula (2.13). In the case when $k = n$, the result follows from (2.13). Otherwise, it follows from (2.14) and the second part of formula (2.12) that $f^{n-k}(C_n(x)) \subset \mathbb{C} \setminus B(\Omega(f), \Delta/2)$. Since we also know that $f^{n-k}(w) \notin B_R$, we conclude that $f^{n-k}(w) \in Y$, we see that $\operatorname{diam}(C_n(x)) \le \min\{\epsilon, R/2\} \le \epsilon$. We are done.

As a consequence of Theorem 2.7 we shall prove the following.

Corollary 2.9. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a non-recurrent elliptic function. If $X \subset J(f) \cup \{\infty\} \setminus \Omega(f)$ is compact, then for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in X$ and every $n \ge 0$, all connected components of $f^{-1}(B_s(x,\delta))$ have Euclidean diameters $\le \epsilon$.*

Proof. Apply Theorem 2.7 for the set $f^{-1}(\infty)$ and given $\epsilon > 0$. This gives us the corresponding number $\delta_1 > 0$. Taking now $\xi > 0$ so small that each connected component of $f^{-n}(B_s(\infty,\xi))$ is contained in $B(b,\delta_1)$ for some pole $b \in f^{-1}(\infty)$ consider the set $Y = X \setminus B_s(\infty,\xi)$. Since Y is a compact subset of $\mathbb{C}$, it follows from Theorem 2.7 that there exists $\delta_2 > 0$ such that for every $x \in Y$ and every $n \ge 0$ all the connected components of $f^{-n}(B_s(x,\delta))$ have Euclidean diameters $\le \epsilon$. Consider a finite cover $\{B_s(x_1,\delta_2), \dots, B_s(x_k,\delta_2), B_s(\infty,\xi)\}$ of X, where $x_j \in Y$ for all $j = 1, 2, \dots, k$. Taking as δ half of the Lebesgue number of this cover (see [16]) finishes the proof. ■

Let us introduce the following notation.

$$\theta = \theta(f) = \min\Big\{\min\{\theta(f^a,\omega) : \omega \in \Omega(f)\}, \frac{1}{2}\operatorname{dist}(\Omega(f), \operatorname{Crit}(f))\Big\} > 0 \tag{2.15}$$

where $a \ge 1$ is so large that all parabolic points of f^a are simple and the numbers $\theta(f^a,\omega)$ are defined in (2.9). We denote

$$A = A(f) = \max\{A(f,c) : c \in \operatorname{Crit}(f)\}, \tag{2.16}$$

where $A(f,c)$ was defined just after Definition 1.1. We call two points z and w equivalent and we write $z \sim w$ if $w - z \in \Lambda$, the lattice associated with the elliptic function f. Obviously $z \sim w$ implies that $O_+(z) = O_+(w)$ and $\omega(z) = \omega(w)$. Since the set $\operatorname{Crit}(f) \cap \mathcal{R}$ is finite, we conclude that the sets $\omega(\operatorname{Crit}(f)) = \bigcup_{c \in \operatorname{Crit}(f)\cap\mathcal{R}} \omega(c)$ and $\overline{O_+(f(\operatorname{Crit}(f)))} = \bigcup_{c \in \operatorname{Crit}(f)\cap\mathcal{R}} \overline{O_+(f(c))}$ are compact subsets of $\mathbb{C}$. A positive number $\beta < \theta/2$ is now chosen to be less than the following three numbers.

$$\min\{\operatorname{dist}(c, O_+(f(c)) : c \in \operatorname{Crit}(f)\}$$
$$\min\{(A(c)R(f,c))^{1/q(c)} : c \in \operatorname{Crit}(f)\}$$
$$\min\{|c - c'| : c, c' \in \operatorname{Crit}(f) \text{ and } c \ne c'\},$$

where $q(c) = q(f,c)$ is the order of the critical point c of f and $R(f,c)$ was defined just after Definition 1.1. Notice that the first of these numbers is positive since $O_+(f(\mathrm{Crit}(f))$ is a compact subset of $\mathbb{C}$ and $\mathrm{Crit}(f)$ has no accumulation points in $\mathbb{C}$. Since f contains no recurrent critical points, it follows from Theorem 2.7 that there exists $0 < \gamma < 1/4$ such that if $n \geq 0$ is an integer, $z \in J(f)$ and $f^n(z) \notin B(\Omega(f), \theta)$, then

$$\mathrm{diam}\Big(\mathrm{Comp}(z, f^n(z), f^n, 2\gamma)\Big) < \beta. \tag{2.17}$$

From now on fix also $0 < \tau < \theta^{-1}\min\{\beta, 2\gamma\}$ so small as required in Lemma 2.4 for every $\omega \in \Omega(f)$ and so small that for every $z \in J(f)$

$$\mathrm{diam}\Big(\mathrm{Comp}(z, f(z), f, \theta\tau)\Big) < \min\{\beta, 2\gamma\}. \tag{2.18}$$

Lemma 2.10. *If $n \geq 0$ is an integer, $\eta > 0$, $z \in J(f)$ and for every $k \in \{0, 1, \ldots, n\}$*

$$\mathrm{diam}\Big(\mathrm{Comp}(f^k(z), f^n(z), f^{n-k}, \eta)\Big) \leq \beta,$$

then each connected component $\mathrm{Comp}(f^k(z), f^n(z), f^{n-k}, \eta)$ contains at most one critical point of f and the equivalence class of each critical point intersects at most one of these components.

Proof. The first part is obvious by the choice of β. In order to prove the second part suppose that

$$c_1 \in \mathrm{Crit}(f) \cap \mathrm{Comp}(f^{k_1}(z), f^n(z), f^{n-k_1}, \eta), \;\; c_2 \in \mathrm{Comp}(f^{k_2}(z), f^n(z), f^{n-k_2}, \eta)$$

and $c_1 \sim c_2$, where $0 \leq k_1 < k_2 \leq n$. But then

$$f^{k_2-k_1}(c_2) = f^{k_2-k_1}(c_1) \in \mathrm{Comp}(f^{k_2}(z), f^n(z), f^{n-k_2}, \eta)$$

and therefore $|f^{k_2-k_1}(c_2) - c_1| < \beta$, contrary to the choice of β. ■

Let $\kappa = \Big(\Pi_{c \in \mathrm{Crit}(f) \cap \mathcal{R}} q(c)\Big)^{-1}$. We shall prove the following.

Lemma 2.11. *If $z \in J(f)$, $f^n(z) \notin B(\Omega(f), \theta)$, then*

$$\mathrm{Mod}\Big(\mathrm{Comp}(z, f^n(z), f^n, 2\gamma) \setminus \mathrm{Comp}(z, f^n(z), f^n, \gamma)\Big) \geq \kappa \log 2/\#(\mathrm{Crit}(f) \cap \mathcal{R})$$

Proof. By Lemma 2.10 there exists a geometric annulus $R \subset B(f^n(z), 2\gamma) \setminus B(f^n(z), \gamma)$ centered at $f^n(z)$ of modulus $\log 2/\#(\mathrm{Crit}(f) \cap \mathcal{R})$ such that $f^{-n}(R) \cap \mathrm{Comp}(z, f^n(z), f^n, 2\gamma) \cap \mathrm{Crit}(f^n)) = \emptyset$. Since covering maps increase moduli of annuli at most by factors equal to their degrees, we conclude that

$$\begin{aligned}\mathrm{Mod}&\Big(\mathrm{Comp}(z, f^n(z), f^n, 2\gamma) \setminus \mathrm{Comp}(z, f^n(z), f^n, \gamma)\Big)\\ &\geq \mathrm{Mod}(R_n) \geq \Big(\log 2/\#(\mathrm{Crit}(f) \cap \mathcal{R})\Big)/\Pi_{c \in \mathrm{Crit}(f) \cap \mathcal{R}} q(c)\\ &= \frac{\kappa \log 2}{\#(\mathrm{Crit}(f) \cap \mathcal{R})},\end{aligned}$$

where $R_n \subset \mathrm{Comp}(z, f^n(z), f^n, 2\gamma)$ is the connected component of $f^{-n}(B(f^n(z), 2\gamma))$ enclosing $\mathrm{Comp}(z, f^n(z), f^n, \gamma)$. ■

As an immediate consequence of this lemma and Koebe's Distortion Theorem, II (Euclidean version) we get the following.

Lemma 2.12. *Suppose that $z \in J(f)$ and $f^n(z) \notin B(\Omega(f), \theta)$. If $0 \le k \le n$ and $f^k : \mathrm{Comp}(z, f^n(z), f^n, 2\gamma) \to \mathrm{Comp}(f^k(z), f^n(z), f^{n-k}, 2\gamma)$ is univalent, then*

$$\frac{|(f^k)'(y)|}{|(f^k)'(x)|} \le \mathrm{const}$$

for all $x, y \in \mathrm{Comp}(z, f^n(z), f^n, \gamma)$, where const *is a number depending only on $\#(\mathrm{Crit}(f) \cap \mathcal{R})$ and κ.*

For A, B, any two subsets of a metric space put

$$\mathrm{dist}(A, B) = \inf\{\mathrm{dist}(a, b) : a \in A, b \in B\}$$

and

$$\mathrm{Dist}(A, B) = \sup\{\mathrm{dist}(a, b) : a \in A, b \in B\}.$$

We shall prove the following.

Lemma 2.13. *Suppose that $z \in J(f)$ and $f^n(z) \notin B(\Omega(f), \theta)$. Suppose also that $Q^{(1)} \subset Q^{(2)} \subset B(f^n(z), \gamma)$ are connected sets. If $Q_n^{(2)}$ is a connected component of $f^{-n}(Q^{(2)})$ contained in $\mathrm{Comp}(z, f^n(z), f^n, \gamma')$ and $Q_n^{(1)}$ is a connected component of $f^{-n}(Q^{(1)})$ contained in $Q_n^{(2)}$, then*

$$\frac{\mathrm{diam}\left(Q_n^{(1)}\right)}{\mathrm{diam}\left(Q_n^{(2)}\right)} \succeq \frac{\mathrm{diam}\left(Q^{(1)}\right)}{\mathrm{diam}\left(Q^{(2)}\right)}.$$

Proof. Let $1 \le n_1 \le \ldots \le n_u \le n$ be all the integers k between 1 and n such that

$$\mathrm{Crit}(f) \cap \mathrm{Comp}(f^{n-k}(z), f^n(z), f^k, 2\gamma) \ne \emptyset.$$

Fix $1 \le i \le u$. If $j \in [n_i, n_{i+1} - 1]$ (we set $n_{u+1} = n - 1$), then by Lemma 2.11 there exists a universal constant $T > 0$ such that

$$\frac{\mathrm{diam}\left(Q_j^{(1)}\right)}{\mathrm{diam}\left(Q_j^{(2)}\right)} \ge T \frac{\mathrm{diam}\left(Q_{n_i}^{(1)}\right)}{\mathrm{diam}\left(Q_{n_i}^{(2)}\right)} \tag{2.19}$$

Since, in view of Lemma 2.10, $u \le \#(\mathrm{Crit}(f) \cap \mathcal{R})$, in order to conclude the proof is therefore enough to show the existence of a universal constant $E > 0$ such that for every $1 \le i \le u - 1$.

$$\frac{\mathrm{diam}\left(Q_{n_{i+1}}^{(1)}\right)}{\mathrm{diam}\left(Q_{n_{i+1}}^{(2)}\right)} \ge E \frac{\mathrm{diam}\left(Q_{n_i}^{(1)}\right)}{\mathrm{diam}\left(Q_{n_i}^{(2)}\right)}.$$

And indeed, let c be the critical point contained in $\text{Comp}(f^{n-n_{i+1}}(z), f^n(z), f^{n_{i+1}}, 2\gamma)$ and let q denote its order. Since both sets $Q^{(2)}_{n_{i+1}}$ and $Q^{(1)}_{n_{i+1}}$ are connected, we get for $i=1,2$ that

$$\text{diam}\Big(Q^{(i)}_{n_{i+1}-1}\Big) \asymp \text{diam}\Big(Q^{(i)}_{n_{i+1}}\Big) \sup\{|f'(x)| : x \in Q^{(i)}_{n_{i+1}}\} \asymp \text{diam}\Big(Q^{(i)}_{n_{i+1}}\Big)\text{Dist}(c, Q^{(i)}_{n_{i+1}}).$$

Hence, using (2.19), we obtain

$$\begin{aligned}\frac{\text{diam}\Big(Q^{(1)}_{n_{i+1}}\Big)}{\text{diam}\Big(Q^{(2)}_{n_{i+1}}\Big)} &\asymp \frac{\text{diam}\Big(Q^{(1)}_{n_{i+1}-1}\Big)}{\text{diam}\Big(Q^{(2)}_{n_{i+1}-1}\Big)} \cdot \frac{\text{Dist}(c, Q^{(2)}_{n_{i+1}})}{\text{Dist}(c, Q^{(1)}_{n_{i+1}})} \ge \frac{\text{diam}\Big(Q^{(1)}_{n_{i+1}-1}\Big)}{\text{diam}\Big(Q^{(2)}_{n_{i+1}-1}\Big)} \\ &\ge T\frac{\text{diam}\Big(Q^{(1)}_{n_i}\Big)}{\text{diam}\Big(Q^{(2)}_{n_i}\Big)}.\end{aligned}$$

We are done. ■

2.4. **Partial order in** $\text{Crit}(J(f))$ **and stratifications of closed forward-invariant subsets of** $J(f)$**.**
We put

$$\text{Crit}(J(f)) = \text{Crit}(f) \cap J(f).$$

We start with the following.

Lemma 2.14. *The set $\omega(\text{Crit}(J(f)))$ is nowhere dense in $J(f)$.*

Proof. Suppose that the interior (relative to $J(f)$) of $\omega(\text{Crit}(J(f)))$ is nonempty. Then there exists $c \in \text{Crit}(J(f))$ such that $\omega(c)$ has nonempty interior. But then there would exist $n \ge 0$ such that $f^n(\omega(c)) = J(f)$ and consequently $\omega(c) = J(f)$. This however is a contradiction as $c \notin \omega(c)$. ■

Now we introduce in $\text{Crit}(J(f))$ a relation $<$ which, in view of Lemma 2.15 below, is an ordering relation, by putting

$$c_1 < c_2 \iff c_1 \in \omega(c_2). \tag{2.20}$$

Since $c_2 \sim c_3$ implies $\omega(c_2) = \omega(c_3)$, then if $c_1 < c_2$ and $c_2 \sim c_3$, then $c_1 < c_3$

Lemma 2.15. *If $c_1 < c_2$ and $c_2 < c_3$, then $c_1 < c_3$.*

Proof. Indeed, we have $c_1 \in \omega(c_2) \subset \omega(c_3)$. ■

Lemma 2.16. *There is no infinite, linear subset of the partially ordered set* $(\text{Crit}(J(f)), <)$

Proof. Indeed, suppose on the contrary that $c_1 < c_2 < \dots$ is an infinite, linearly ordered subset of $\mathrm{Crit}(J(f))$. Since the number of equivalency classes of relation $\sim$ is equal to $\#(\mathrm{Crit}(J(f)) \cap \mathcal{R})$ which is finite, there exist two numbers $1 \le i < j$ such that $\omega(c_i) = \omega(c_j)$. But this implies that $c_i \in \omega(c_j) = \omega(c_i)$ and we get a contradiction. The proof is finished. ■

The following observation is a reformulation of the condition that $J(f)$ contains no recurrent critical points.

Lemma 2.17. *If $c \in \mathrm{Crit}(J(f))$, then it is not the case that $c < c$.*

Define now inductively a sequence $\{Cr_i(f)\}$ of subsets of $\mathrm{Crit}(J(f))$ by setting $Cr_0(f) = \emptyset$ and

$$Cr_{i+1}(f) = \left\{ c \in \mathrm{Crit}(J(f)) \setminus \bigcup_{j=0}^{i} Cr_j(f) : \text{ if } c' < c, \text{ then } c' \in Cr_0(f) \cup \dots \cup Cr_i(f) \right\} \tag{2.21}$$

Lemma 2.18. *We have*

(a) *If $c \in Cr_i(f)$ and $c' \sim c$, then $c' \in Cr_i(f)$.*
(b) *The sets $\{Cr_i(f)\}$ are mutually disjoint.*
(c) $\exists_{p \ge 1}\ \forall_{i \ge p+1}\quad Cr_i(f) = \emptyset$
(d) $Cr_0(f) \cup \dots \cup Cr_p(f) = \mathrm{Crit}(J(f))$
(e) $Cr_1(f) \neq \emptyset$

Proof. The part (a) follows immediately from the definition of the sets Cr_i and the fact that two equivalent points have the same ω-limit sets. By definition $Cr_{i+1}(f) \cap \bigcup_{j=1}^{i} Cr_j(f) = \emptyset$, so disjointness in (b) is clear. As the number of equivalency classes of the relation $\sim$ is equal to $\#(\mathrm{Crit}(J(f)) \cap \mathcal{R}$ which is finite, (a) and (b) imply (c). Take p to be the minimal number satisfying (c) and suppose that $c \in \mathrm{Crit}(J(f)) \setminus \bigcup_{j=1}^{p} Cr_j(f)$. Since $Cr_{p+1}(f) = \emptyset$, there exists $c' \notin \bigcup_{j=1}^{p} Cr_j(f)$ such that $c' < c$. Iterating this procedure we would obtain an infinite sequence of critical points $c_1 = c > c_2 = c' > c_3 > \dots$. But this contradicts Lemma 2.16 proving (d). Now part (e) follows from (c) and (2.21). ■

As an immediate consequence of the definition of the sequence $\{Cr_i(f)\}$ we get the following simple lemma.

Lemma 2.19. *If $c, c' \in Cr_i(f)$, then it is not the case that $c < c'$.*

For every point $z \in J(f)$ define the set

$$\mathrm{Crit}(z) = \{c \in \mathrm{Crit}(J(f)) : c \in \omega(z)\}$$

We shall prove the following.

Lemma 2.20. *If $z \in J(f) \setminus I_\infty(f)$, then either $z \in \bigcup_{n\ge0} f^{-n}(\Omega(f))$ or $\omega(z) \setminus \{\infty\}$ is not contained in $\overline{O_+(f(\mathrm{Crit}(z))}\cup\Omega(f)$.*

Proof. Suppose that $z \notin \bigcup_{n\ge0} f^{-n}(\Omega(f)) \cup I_\infty(f)$. Then by (2.10) the set $\omega(z)\setminus\{\infty\}$ is not contained in $\Omega(f)$. So, if we suppose that

$$\omega(z)\setminus\{\infty\} \subset \overline{O_+(f(\mathrm{Crit}(z))}\cup\Omega(f), \tag{2.22}$$

then, as $\omega(z)\setminus\{\infty\} \neq \emptyset$, we conclude that $\mathrm{Crit}(z) \neq \emptyset$. Let $c_1 \in \mathrm{Crit}(z)$. It means that $c_1 \in \omega(z)$ and as $c_1 \notin \Omega(f)$, it follows from (2.22) that there exists $c_2 \in \mathrm{Crit}(z)$ such that either $c_1 \in \omega(c_2)$ or $c_1 = f^{n_1}(c_2)$ for some $n_1 \ge 1$. Iterating this procedure we obtain an infinite sequence $\{c_j\}_{j=1}^\infty$ such that for every $j \ge 1$ either $c_j \in \omega(c_{j+1})$ or $c_j = f^{n_j}(c_{j+1})$ for some $n_j \ge 1$. Consider an arbitrary block $c_k, c_{k+1}, \dots, c_l$ such that $c_j = f^{n_j}(c_{j+1})$ for every $k \le j \le l-1$ and suppose that $l-(k-1) \ge \#(\mathrm{Crit}(f)\cap\mathcal{R})$. Then there are two indexes $k \le a < b \le l$ such that $c_a \sim c_b$. Then

$$f^{n_a+n_{a+1}+\ldots+n_{b-1}}(c_a) = f^{n_a+n_{a+1}+\ldots+n_{b-1}}(c_b) = c_a$$

and consequently, as $n_a+n_{a+1}+\ldots+n_{b-1} \ge b-a \ge 1$, c_a is a super-attracting periodic point of f. Since $c_a \in J(f)$, this is a contradiction, and in consequence the length of the block $c_k, c_{k+1}, \dots, c_l$ is bounded above by $\#(\mathrm{Crit}(f)\cap\mathcal{R})$. Hence, there exists an infinite subsequence $\{n_k\}_{k\ge1}$ such that $c_{n_k} \in \omega(c_{n_k+1})$ for every $k \ge 1$ or equivalently $c_{n_k} < c_{n_{k+1}}$ for every $k \ge 1$. This however contradicts Lemma 2.16 and we are done. ■

Recall that the integer p was defined in Lemma 2.18. Define now for every $i = 0, 1, \dots, p$

$$S_i(f) = Cr_0(f) \cup \ldots \cup Cr_i(f) \tag{2.23}$$

and for every $i = 0, 1, \dots, p-1$ consider $c' \in \bigcup_{c\in Cr_{i+1}(f)} \omega(c) \cap \mathrm{Crit}(J(f))$. Then there exists $c \in Cr_{i+1}(f)$ such that $c' \in \omega(c)$ which equivalently means that $c' < c$. Thus, by (2.21) we get $c' \in S_i(f)$. So

$$\bigcup_{c\in Cr_{i+1}(f)} \omega(c) \cap (\mathrm{Crit}(J(f)) \setminus S_i(f)) = \emptyset \tag{2.24}$$

Therefore, since the set $\bigcup_{c\in Cr_{i+1}(f)} \omega(c) \subset \mathbb{C}$ is compact and $\mathrm{Crit}(J(f)) \setminus S_i(f)$ has no accumulation point in $\mathbb{C}$,

$$\delta_i = \mathrm{dist}\Big(\bigcup_{c\in Cr_{i+1}(f)} \omega(c), \mathrm{Crit}(J(f)) \setminus S_i(f)\Big) > 0 \tag{2.25}$$

Set

$$\rho = \min\{\delta_i/2 : i = 0, 1, \dots, p-1\}.$$

Fix a closed forward-invariant subset E of $J(f)$ and for every $i = 0, 1, \dots, p$ define

$$E_i(f) = \{z \in E : \mathrm{dist}\big(O_+(z), \mathrm{Crit}(J(f)) \setminus S_i(f)\big) \ge \rho\}.$$

Let us now prove the following two lemmas concerning the sets $E_i(f)$.

Lemma 2.21. $E_0 \subset E_1 \subset \ldots \subset E_p = E$.

Proof. Since $S_{i+1}(f) \supset S_i(f)$, the inclusions $E_i \subset E_{i+1}$ is obvious. Since $S_p(f) = \mathrm{Crit}(J(f))$, it holds $E_p = E$. We are done. ■

Let
$$\mathrm{PC}(f) = \overline{O_+(\mathrm{Crit}(J(f)))}$$

We shall prove the following.

Lemma 2.22. *There exists $l = l(f)$ such that for every $i = 0, 1, \ldots, p-1$*
$$\bigcup_{c \in Cr_{i+1}(f)} \omega(c) \subset \overline{O_+(f^l(Cr_{i+1}(f)))} \subset \mathrm{PC}(f)_i$$

Proof. The left-hand inclusion is obvious regardless whatever $l(f)$ is. In order to prove the right-hand one fix $i \in \{0, 1, \ldots, p-1\}$. By the definition of ω-limit sets there exists $l_i \geq 1$ such that for every $c \in Cr_{i+1}(f)$ we have $\mathrm{dist}\Big(O_+(f^{l_i}(c)), \bigcup_{c \in Cr_{i+1}(f)} \omega(c)\Big) < \delta_i/2$. Thus, by (2.25), $\mathrm{dist}\Big(\overline{O_+(f^{l_i}(c))}, \mathrm{Crit}(J(f)) \setminus S_i(f)\Big) > \delta_i/2$. Since $\rho \leq \delta_i/2$ and since for every $z \in \overline{O_+(f^{l_i}(c))}$ also $O_+(z) \subset \overline{O_+(f^{l_i}(c))}$, we therefore get $O_+(f^l(Cr_{i+1}(f))) \subset \mathrm{PC}(f)_i$. So, putting $l(f) = \max\{l_i : i = 0, 1, \ldots, p-1\}$ the proof is completed. ■

2.5. **Holomorphic inverse branches.** In this section we prove the existence of suitable holomorphic inverse branches-our basic tools in the next section. Set
$$\mathrm{Sing}^-(f) = \bigcup_{n \geq 0} f^{-n}\Big(\Omega(f) \cup \mathrm{Crit}(J(f)) \cup f^{-1}(\infty)\Big) \text{ and } I_-(f) = \bigcup_{n \geq 1} f^{-n}(\infty).$$
We start with the following.

Proposition 2.23. *If $z \in J(f) \setminus \mathrm{Sing}^-(f)$, then there exist a positive number $\eta(z)$, an increasing sequence of positive integers $\{n_j\}_{j \geq 1}$, and a point $x = x(z) \in \omega(z) \setminus (\Omega(f) \cup \omega(\mathrm{Crit}(z)))$ such that $x \neq \infty$ if $z \notin I_\infty(f)$, $\lim_{j \to \infty} f^{n_j}(z) = x$ and*
$$\mathrm{Comp}(z, f^{n_j}(z), f^{n_j}, \eta(z)) \cap \mathrm{Crit}(f^{n_j}) = \emptyset$$
for every $j \geq 0$.

Proof. Suppose first that $z \in I_\infty(f) \setminus \mathrm{Sing}^-(f)$. Since $\overline{O_+(f(\mathrm{Crit}(f)))}$ is a compact subset of $\mathbb{C}$, we conclude that for all n large enough $\mathrm{dist}(f^n(z), O_+(f(\mathrm{Crit}(f)))) \geq 1$. We are therefore done taking $x = \infty$ and $\eta(z) = 1$. So, suppose that $\underline{z \notin I_\infty(f)}$. This means that $\omega(z) \setminus \{\infty\} \neq \emptyset$. Suppose that $\omega(z) \setminus \{\infty\}$ is unbounded. Since $\overline{O_+(f(\mathrm{Crit}(f)))}$ is a compact subset of $\mathbb{C}$, there thus exists $x \in \omega(z) \setminus \{\infty\}$ such that $\mathrm{dist}(x, \overline{O_+(f(\mathrm{Crit}(f)))}) \geq 2$ and we are done fixing a sequence $\{n_j\}_{j=1}^\infty$ such $|f^{n_j}(z) - x| \leq 1$ and taking $\eta(z) = 1$. So, assume that $\omega(z) = F \cup \{\infty\}$

where $F \subset \mathbb{C}$ is a compact set. Then $F \cap f^{-1}(\infty) \neq \emptyset$ and so we can fix $x \in F \cap f^{-1}(\infty)$. Again, since $\overline{O_+(f(\mathrm{Crit}(f)))}$ is a compact subset of $\mathbb{C}$ and since $f\left(\overline{O_+(f(\mathrm{Crit}(f)))}\right) \subset \overline{O_+(f(\mathrm{Crit}(f)))}$, we see that $x \notin \overline{O_+(f(\mathrm{Crit}(f)))}$ and we are done taking $\eta(z) = \frac{1}{2}\mathrm{dist}(x, \overline{O_+(f(\mathrm{Crit}(f)))})$. So suppose finally that $\omega(z)$ is a compact subset of $\mathbb{C}$. In view of Lemma 2.20 there exists $x \in \omega(z) \setminus (\Omega(f) \cup \overline{O_+(f(\mathrm{Crit}(z))} \cup \{\infty\})$. The number $\eta = \frac{1}{2}\mathrm{dist}(x, \Omega(f) \cup \overline{O_+(f(\mathrm{Crit}(z)))})$ is positive since $\omega(\mathrm{Crit}(z))$ is a compact subset of $\mathbb{C}$ and $\Omega(f)$ is finite. Then there exists an infinite increasing sequence $\{m_j\}_{j\geq 1}$ such that

$$\lim_{j\to\infty} f^{m_j}(z) = x \tag{2.26}$$

and

$$B(f^{m_j}(z), \eta) \cap \bigcup_{n\geq 1} f^n(\mathrm{Crit}(z)) = \emptyset. \tag{2.27}$$

Now we claim that there exists $\eta(z)$ such that for every $j \geq 1$ large enough

$$\mathrm{Comp}(z, f^{m_j}(z), f^{m_j}, \eta(z)) \cap \mathrm{Crit}(f^{m_j}) = \emptyset. \tag{2.28}$$

Otherwise we would find an increasing to infinity subsequence $\{m_{j_i}\}$ of $\{m_j\}$ and a decreasing to zero sequence of positive numbers η_i such that $\eta_i < \eta$ and

$$\mathrm{Comp}(z, f^{m_{j_i}}(z), f^{m_{j_i}}, \eta_i) \cap \mathrm{Crit}(f^{m_{j_i}}) \neq \emptyset$$

Let $\tilde{c}_i \in \mathrm{Comp}(z, f^{m_{j_i}}(z), f^{m_{j_i}}, \eta_i) \cap \mathrm{Crit}(f^{m_{j_i}})$. Then there exists $c_i \in \mathrm{Crit}(f)$ such that $f^{p_i}(\tilde{c}_i) = c_i$ for some $0 \leq p_i \leq m_{j_i} - 1$. Since the set $f^{-1}(x)$ is at a positive distance from $\Omega(f)$ and since $\eta_i \to 0$, it follows from Theorem 2.7 that $\lim_{i\to\infty} \tilde{c}_i = z$. Since $z \notin \bigcup_{n\geq 0} f^{-n}(\mathrm{Crit}(f))$, it implies that $\lim_{i\to\infty} p_i = \infty$. But then using Theorem 2.7 again and the formula $f^{p_i}(\tilde{c}_i) = c_i$ we conclude that the set of all accumulation points of the sequence $\{c_i\}$ is contained in $\omega(z)$. Hence, passing to a subsequence, we may assume that the limit $c = \lim_{i\to\infty} c_i$ exists. But since $c \in \omega(z)$, since $\omega(z)$ is a compact subset of $\mathbb{C}$ and since ∞ is the only accumulation point of $\mathrm{Crit}(f)$, we conclude that the sequence c_i is eventually constant. Thus, dropping some finite number of initial terms, we may assume that this sequence is constant. This means that $c_i = c$ for all $i = 1, 2, \ldots$. Since $c = f^{p_i}(\tilde{c}_i)$, we get

$$|f^{m_{j_i}}(z) - f^{m_{j_i}-p_i}(c)| = |f^{m_{j_i}}(z) - f^{m_{j_i}}(\tilde{c}_i)| < \eta_i.$$

Since $\lim_{i\to\infty} \eta_i = 0$ and since $\omega(z)$ is a compact subset of $\mathbb{C}$, we conclude that $\lim_{i\to\infty} |f^{m_{j_i}}(z) - f^{m_{j_i}-p_i}(c)| = 0$. Since $c \in \mathrm{Crit}(z)$, in view of (2.27) this implies that $m_{j_i} - p_i \leq 0$ for all i large enough. So, we get a contradiction as $0 \leq p_i \leq m_{j_i} - 1$ and (2.28) is proved. We are done. ■

It is well-known in meromorphic dynamics that if $z \in J(f) \setminus (\mathrm{Sing}^-(f) \cup I_\infty(f))$, then the limit points of the normal family

$$f_z^{-n_j} : B(x(z), \eta(z)/2) \to \overline{\mathbb{C}}$$

consist only of constant functions. Therefore we get the following.

Corollary 2.24. *If $z \in J(f) \setminus (\mathrm{Sing}^-(f) \cup I_\infty(f))$ and the sequence $\{n_j\}_{j=1}^\infty$ is taken from Proposition 2.23, then*

$$\limsup_{n\to\infty} |(f^n)^*(z)| = \limsup_{n\to\infty} |(f^n)'(z)| = \lim_{n\to\infty} |(f^{n_j})'(z)| = +\infty.$$

In addition, if we assume only that $z \in J(f) \setminus \mathrm{Sing}^-(f)$, then

$$\limsup_{n\to\infty} |(f^n)'(z)| = \infty.$$

3. Conformal Measures

In this section we deal in detail with the existence, uniqueness and some geometrical properties of conformal measures. We start with the subsection describing some basic facts from the geometric measure theory.

3.1. **Preliminaries from Geometric Measure Theory.** Given a subset A of a metric space (X, d), a countable family $\{B(x_i, r_i)\}_{i=1}^\infty$ of open balls centered at the set A is said to be a packing of A if and only if for any pair $i \neq j$

$$d(x_i, x_j) > r_i + r_j.$$

Given $t \geq 0$, the t-dimensional outer Hausdorff measure $\mathrm{H}^t(A)$ of the set A is defined as

$$\mathrm{H}^t(A) = \sup_{\epsilon>0} \inf\Big\{\sum_{i=1}^\infty r_i^t\Big\}$$

where infimum is taken over all covers $\{B(x_i, r_i)\}_{i=1}^\infty$ of the set A by open balls centered at A with radii which do not exceed ϵ.

The t-dimensional outer packing measure $\Pi^t(A)$ of the set A is defined as

$$\Pi^t(A) = \inf_{\cup A_i = A}\Big\{\sum_i \Pi_*^t(A_i)\Big\}$$

(A_i are arbitrary subsets of A), where

$$\Pi_*^t(A) = \sup_{\epsilon>0} \sup\Big\{\sum_{i=1}^\infty r_i^t\Big\}.$$

Here the second supremum is taken over all packings $\{B(x_i, r_i)\}_{i=1}^\infty$ of the set A by open balls centered at A with radii which do not exceed ϵ. These two outer measures define countable additive measures on Borel σ-algebra of X.

The definition of the Hausdorff dimension $\mathrm{HD}(A)$ of the set A is the following

$$\mathrm{HD}(A) = \inf\{t : \mathrm{H}^t(A) = 0\} = \sup\{t : \mathrm{H}^t(A) = \infty\}.$$

Let ν be a Borel probability measure on X which is positive on open sets. Define the function $\rho = \rho_t(\nu) : X \times (0, \infty) \to (0, \infty)$ by

$$\rho(x, r) = \frac{\nu(B(x, r))}{r^t}$$

The following two theorems (see [DU5]) are for our aims the key facts from geometric measure theory. Their proofs are an easy consequence of Besicovič covering theorem (see [G]).

Theorem 3.1. *Let $X = I\!R^n$ for some $n \geq 1$. Then there exists a constant $b(n)$ depending only on n with the following properties. If A is a Borel subset of $I\!R^n$ and $C > 0$ is a positive constant such that*

(1) *for all (but countably many) $x \in A$*

$$\limsup_{r \to 0} \rho(x,r) \geq C^{-1},$$

then for every Borel subset $E \subset A$ we have $\mathrm{H}^t(E) \leq b(n)C\nu(E)$ and, in particular, $\mathrm{H}^t(A) < \infty$.
or

(2) *for all $x \in A$*

$$\limsup_{r \to 0} \rho(x,r) \leq C^{-1},$$

then for every Borel subset $E \subset A$ we have $\mathrm{H}^t(E) \geq C\nu(E)$.

Theorem 3.2. *Let $X = I\!R^n$ for some $n \geq 1$. Then there exists a constant $b(n)$ depending only on n with the following properties. If A is a Borel subset of $I\!R^n$ and $C > 0$ is a positive constant such that*

(1) *for all $x \in A$*

$$\liminf_{r \to 0} \rho(x,r) \leq C^{-1},$$

then for every Borel subset $E \subset A$ we have $\Pi^t(E) \geq Cb(n)^{-1}\nu(E)$,
or

(2) *for all $x \in A$*

$$\liminf_{r \to 0} \rho(x,r) \geq C^{-1},$$

then $\Pi^t(E) \leq C\nu(E)$ and, consequently, $\Pi^t(A) < \infty$.

(1') *If ν is non–atomic then (1) holds under the weaker assumption that the hypothesis of part (1) is satisfied on the complement of a countable set.*

3.2. **Support of Conformal Measure.** From now on throughout this section and the entire paper we set

$$h = \mathrm{HD}(J(f)).$$

We begin with the following.

Lemma 3.3. *If m is a t-conformal measure for $f : J(f) \to J(f) \cup \{\infty\}$, then $t \geq \mathrm{HD}(J(f))$ and $\mathrm{H}^t|_{J(f)}$ is absolutely continuous with respect to m.*

Proof. Fix $z \in J(f) \setminus (\mathrm{Sing}^-(f) \cup I_\infty(f))$. Let $\eta(z) > 0$, $x \in \omega(z) \setminus \{\infty\}$ and the sequence $\{n_j\}_{j \geq 1}$ be taken from Proposition 2.23. It then follows from this proposition and Koebe's Distortion Theorem, I(spherical version) that

$$f_z^{-n_j}(B(f^{n_j}(z), \eta(z)/2)) \asymp B(z, |(f^{n_j})^*(z)|^{-1}\eta(z)/2).$$

Applying again this Koebe's Distortion Theorem and conformality of the measure m, we get for all $j \geq 1$ large enough

$$\begin{aligned} m(B(z, |(f^{n_j})'(z)|^{-1}\eta(z)/2)) &\asymp |(f^{n_j})^*(z)|^{-t} m(B(f^{n_j}(z), \eta(z)/2)) \\ &\succeq |(f^{n_j})^*(z)|^{-t} m(B(x, \eta(z)/4)) \\ &\asymp |(f^{n_j})'(z)|^{-t} m(B(x, \eta(z)/4)) \\ &= (2\eta(z)^{-1})^t m(B(x, \eta(z)/4)) \Big(|(f^{n_j})'(z)|^{-1}\eta(z)/2) \Big)^t, \end{aligned}$$

where the second comparability sign depends on $|z|$ and holds for all $j \geq 1$ large enough so that $f^{n_j}(z)$ is sufficiently close to x. In particular

$$\limsup_{r \to 0} \frac{m(B(z,r))}{r^t} \geq R(z) > 0,$$

where $R(z) = (2\eta(z)^{-1})^t m(B(x, \eta(z)/4))$. Therefore, putting

$$X_k = \{z \in J(f) \setminus (\mathrm{Sing}^-(f) \cup I_\infty(f))) : |z| \leq k \text{ and } R(z) \geq 1/k\}$$

we have $\bigcup_{k \geq 1}^\infty X_k = J(f) \setminus (\mathrm{Sing}^-(f) \cup I_\infty(f))$, and in view of Theorem 3.1(1), $d\mathrm{H}^t/dm \leq b(2)k$ on X_k. In particular $\mathrm{H}^t \ll m$ on $J(f) \setminus (\mathrm{Sing}^-(f) \cup I_\infty(f))$. Hence $\mathrm{HD}\Big(J(f) \setminus (\mathrm{Sing}^-(f) \cup I_\infty(f))\Big) \leq t$. By Theorem 1 and Theorem 2 in [15]), $\mathrm{HD}(J(f)) > \mathrm{HD}(I_\infty(f))$. Of course $\mathrm{HD}(\mathrm{Sing}^-(f)) = 0$ as $\mathrm{Sing}^-(f)$ is a countable set. Thus $\mathrm{HD}(J(f)) = \mathrm{HD}\Big(J(f) \setminus (\mathrm{Sing}^-(f) \cup I_\infty(f))\Big) \leq t$ and $\mathrm{H}^t \ll m$ on $J(f)$. ■

We will need in the sequel the following result which is interesting itself.

Lemma 3.4. *If m is a t-conformal measure for $f : \mathbb{C} \to \overline{\mathbb{C}}$, then $m(I_\infty(f) \setminus I_-(f)) = 0$. Even more, there exists $R > 0$ such that*

$$m(\{z : \liminf_{n \to \infty} |f^n(z)| > R\}) = 0.$$

Proof. Let b be a pole of $f : \mathbb{C} \to \overline{\mathbb{C}}$. We shall obtain first an upper estimate on $m(B_b(R))$ similar to the second inequality in (2.4). And indeed, covering $B_R \setminus \{\infty\}$ by two simply connected domains

$$B_R^+ = \{z \in B_R \setminus \{\infty\} : \mathrm{Im} z > 0\} \quad \text{and} \quad B_R^1 = \{z \in B_R \setminus \{\infty\} : \mathrm{Im} z < 1\}$$

we obtain

$$m(B_b(R)\setminus\{b\}) \le \sum_{j=1}^{q_b}\int_{B_R^+}|(f^{-1}_{b,B_R^+,j})^*|^t dm + \sum_{j=1}^{q_b}\int_{B_R^1}|(f^{-1}_{b,B_R^-,j})^*|^t dm.$$

Using now (2.3), we obtain

$$\int_{B_R^+}|(f^{-1}_{b,B_R^+,j})^*|^t dm \asymp \int_{B_R^+}\left(\frac{1}{1+|b|^2}|z|^{\frac{q_b-1}{q_b}}\right)^t dm(z) = \frac{1}{(1+|b|^2)^t}\int_{B_R^+}|z|^{\frac{q_b-1}{q_b}t}dm(z)$$
$$\le (1+|b|^2)^{-t}\int_{B_R^+}|z|^{\frac{q-1}{q}t}dm(z).$$

Looking at the first line of this formula with a pole b of maximal multiplicity, we see that the integral $\int_{B_R^+}|z|^{\frac{q-1}{q}t}dm(z)$ is finite and even more:

$$\lim_{R\to\infty}\int_{B_R^+}|z|^{\frac{q-1}{q}t}dm(z) = 0. \tag{3.1}$$

Similarly, the integral $\int_{B_R^1}|z|^{\frac{q-1}{q}t}dm(z)$ is finite and it also converges to 0 as $R\to\infty$. Putting

$$\Sigma_R = \max\left\{\int_{B_R^+}|z|^{\frac{q-1}{q}t}dm(z), \int_{B_R^1}|z|^{\frac{q-1}{q}t}dm(z)\right\}$$

we therefore conclude that

$$m(B_b(R)\setminus\{b\}) \le 2q\Sigma_R(1+|b|^2)^{-t} \le 2q\Sigma_R|b|^{-2t}. \tag{3.2}$$

Now the argument goes essentially in the same way as in [15]. We present it here for the sake of completeness. We take $R_2 \ge R_1$ defined in Section 2.1 so large that

$$LR^{-\frac{1}{q_b}} < R_0 \tag{3.3}$$

for all poles $b\in B_{R_2}$ and all $R\ge R_2$. Given two poles $b_1, b_2 \in B_{2R_2}$ we denote by $f^{-1}_{b_2,b_1,j}$: $B(b_1,R_0)\to \mathbb{C}$ all the holomorphic inverse branches $f^{-1}_{b_2,B(b_1,R_0),j}$. It follows from (2.4) and (3.3) that

$$f^{-1}_{b_2,b_1,j}\Big(B(b_1,R_0)\Big) \subset B_{b_2}(2R_2-R_0) \subset B_{b_2}(R_2) \subset B(b_2,R_0) \tag{3.4}$$

Set

$$I_R(f) = \{z\in\mathbb{C} : \forall_{n\ge 0}|f^n(z)| > R\}.$$

Since the series $\sum_{b\in f^{-1}(\infty)\setminus\{0\}}|b|^{-s}$ converges for all $s>2$ and since by Lemma 3.3 and Theorem 3 from [15], $t\ge h > \frac{2q}{q+1}$ there exists $R_3\ge R_2$ such that

$$qM^t\sum_{b\in B_{R_3}\cap f^{-1}(\infty)}|b|^{-\frac{q+1}{q}t} \le 1/2. \tag{3.5}$$

Consider $R\ge 4R_3$. Put

$$I = f^{-1}(\infty)\cap B_{(R/2)}$$

Since $R/2 + R_0 \le R/2 + R_3 < R/2 + R/2 = R$, it follows from (3.4), (2.4) and (3.3) that for every $l \ge 1$ the family W_l defined as

$$\left\{ f^{-1}_{b_l, b_{l-1}, j_l} \circ f^{-1}_{b_{l-1}, b_{l-2}, j_{l-1}} \circ \ldots \circ f^{-1}_{b_2, b_1, j_2} \circ f^{-1}_{b_1, b_0, j_1}\Big(B_{b_0}(R/2) \setminus f^{-1}(\infty)\Big)\right\},$$

where $b_i \in I : 1 \le j_i \le q_{b_i}, i = 0, 1, \ldots, l$, is well-defined and covers $I_R(f)$. Applying (2.3) and (2.4) we may now estimate as follows.

$$\begin{aligned}
& m(I_R(f)) \le \\
& \le \sum_{b_l \in I}\sum_{j_l=1}^{q_{b_l}} \cdots \sum_{b_1 \in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0 \in I} m\left(f^{-1}_{b_l, b_{l-1}, j_l} \circ f^{-1}_{b_{l-1}, b_{l-2}, j_{l-1}} \circ \ldots \circ f^{-1}_{b_2, b_1, j_2} \circ f^{-1}_{b_1, b_0, j_1}\Big(B_{b_0}(R/2)\Big)\right) \\
& \le \sum_{b_l \in I}\sum_{j_l=1}^{q_{b_l}} \cdots \sum_{b_1 \in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0 \in I} \Big|\Big|\Big(f^{-1}_{b_l, b_{l-1}, j_l} \circ f^{-1}_{b_{l-1}, b_{l-2}, j_{l-1}} \circ \ldots \circ f^{-1}_{b_2, b_1, j_2} \circ f^{-1}_{b_1, b_0, j_1}\Big)^*|_{B_{b_0}(R/2)}\Big|\Big|_\infty^t m\Big(B_{b_0}(R/2)\Big) \\
& \le \sum_{b_l \in I}\sum_{j_l=1}^{q_{b_l}} \cdots \sum_{b_1 \in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0 \in I} M^{lt}\left(\frac{|b_{l-1}|^{\frac{q_{b_l}-1}{q_{b_l}}}}{|b_l|^2}\right)^t \cdot \left(\frac{|b_{l-2}|^{\frac{q_{b_{l-1}}-1}{q_{b_{l-1}}}}}{|b_{l-1}|^2}\right)^t \cdots \left(\frac{|b_0|^{\frac{q_{b_1}-1}{q_{b_1}}}}{|b_1|^2}\right)^t (2q\Sigma_R)^t \frac{1}{|b_0|^{2t}} \\
& = (2q\Sigma_R)^t M^{lt} \sum_{b_l \in I}\sum_{j_l=1}^{q_{b_l}} \cdots \sum_{b_1 \in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0 \in I} |b_l|^{-2t}\Big(|b_{l-1}|^{-\frac{q+1}{q}t} \ldots |b_0|^{-\frac{q+1}{q}t}\Big) \\
& \le (2q\Sigma_R)^t M^{lt} \sum_{b_l \in I}\sum_{j_l=1}^{q_{b_l}} \cdots \sum_{b_1 \in I}\sum_{j_1=1}^{q_{b_1}}\sum_{b_0 \in I} \Big(|b_l|^{-\frac{q+1}{q}t}|b_{l-1}|^{-\frac{q+1}{q}t} \ldots |b_0|^{-\frac{q+1}{q}t}\Big) \\
& \le (2q\Sigma_R)^t M^{lt}\left(\sum_{b \in I}|b|^{-\frac{q+1}{q}t}\right)^l q^l \\
& \le (2q\Sigma_R)^t \left(qM^t \sum_{b \in B_{R_3} \cap f^{-1}(\infty)} |b|^{-\frac{q+1}{q}t}\right)^l
\end{aligned}$$

Applying (3.5) we therefore get $m(I_R(f)) \le (2q\Sigma_R)^t 2^{-l}$. Letting $l \to \infty$ we therefore get $m(I_R(f)) = 0$. Since $m \circ f^{-1} \ll m$ and since $\{z : \liminf_{n\to\infty}|f^n(z)| > R\} = \bigcup_{j=0}^\infty f^{-j}(I_R(f))$, we conclude that $m\Big(\{z : \liminf_{n\to\infty}|f^n(z)| > R\}\Big) = 0$. We are done. ■

3.3. **The Existence of h-Conformal Measure.** Developing the general scheme from [9] we shall now prove in several steps the existence of an h-conformal measure. In order to begin we call $Y \subset \{\infty\} \cup \Omega(f) \cup \bigcup_{n\ge1} f^n(\mathrm{Crit}(J(f)))$ a crossing set if Y is finite and the following four conditions are satisfied.

- (y1) $\infty \in Y$.
- (y2) $Y \cap \{f^n(x) : n \ge 1\}$ is a singleton for all $x \in \mathrm{Crit}(J(f))$.
- (y3) $Y \cap \mathrm{Crit}(f) = \emptyset$.
- (y4) $\Omega(f) \subset Y$.

Since $f(\mathrm{Crit}(f))$ is finite, crossing sets do exist. Let $V \subset \overline{\mathbb{C}}$ be an open neighbourhood of Y such that

$$\mathrm{Crit}(J(f)) \cap \partial V = \emptyset. \tag{3.6}$$

Notice that there exists a decreasing to zero sequence $\{r_n\}_{n=1}^{\infty}$ of positve radii such that the sets V of the form $B_s(Y, r_n)$ satisfy all the above requirements. We define

$$K(V) = \{z \in J(f) : f^n(z) \notin V \ \forall (n \geq 0)\}.$$

Obviously $f(K(V)) \subset K(V)$ and since $f : \mathbb{C} \to \overline{\mathbb{C}}$ is continuous and V is open, we see that $K(V)$ is a closed subset of $\mathbb{C}$. Since in addition $K(V) \subset \overline{\mathbb{C}} \setminus V$, we conclude that $K(V)$ is a compact subset of $\mathbb{C}$. Fix $w \in K(V)$ and $t \geq 0$. For all $n \geq 1$ consider the sets

$$E_n = \left(f|_{K(V)}\right)^{-n}(w)$$

and the number

$$c(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in E_n} |(f^n)^*(x)|^{-t}.$$

Since the continuous map $f|_{K(V)} : K(V) \to K(V)$ has no critical points, all the sets E_n are (n, δ)-separated (meaning that $\max\{|f^j(x) - f^j(y)| : 0 \leq j \leq n-1\} \geq \delta$ for all different points $x, y \in E_n$), where

$$\delta = \inf_{y \in K(V)} \{\min\{|z - x| : x, z \in \left(f|_{K(V)}\right)^{-1}(y) \text{ and } x \neq z\}\} > 0.$$

Therefore

$$c(t) \leq \mathrm{P}\left(f|_{K(V)}, -t \log |f^*|\right), \tag{3.7}$$

where the right-hand side of this inequality is the topological pressure (see [28], [7], [25] or [24] for its definition and an extensive amount of useful properties) of the potential $-t \log |f^*|$ with respect to the dynamical system $f|_{K(V)} : K(V) \to K(V)$. Denote this pressure simply by $\mathrm{P}(f, V)$. We call a Borel set $A \subset \mathbb{C}$ special if $f|_A$ is injective. Having in mind (3.6, Lemma 3.1 and 3.2 from [9] (comp. [8]) enlarged by the reasoning started from the second paragraph of the proof of Lemma 5.3 in [9] can now be formulated together as follows.

Lemma 3.5. *For every $t \geq 0$ there exists a Borel probability measure $m_{V,t}$ supported on $K(V)$ such that*

(a) $m_{V,t}(f(A)) \geq \int_A e^{c(t)}|f^*|^t dm_{V,t}$ *for every special set $A \subset \mathbb{C}$ and*
(b) $m_{V,t}(f(A)) = \int_A e^{c(t)}|f^*|^t dm_{V,t}$ *for every special set $A \subset \mathbb{C} \setminus \overline{V}$.*

We will need the following technical lemma.

Lemma 3.6. *The function $t \mapsto c(t)$ is continuous, $c(0) > 0$ and $c^{-1}(0) \cap (0, h] \neq \emptyset$ if V has a sufficiently small diameter.*

Proof. Continuity of the function $c(t)$ follows from the fact that $0 < \inf_{K(V)}\{|f^*|\} \le \sup_{K(V)}\{|f^*|\} < \infty$. Since periodic points of f are dense in $J(f)$, $K(V) \ne \emptyset$ for all V sufficiently small. Also if V is sufficiently small and $w \in K(V)$, then $\#E_n \ge 2^n$ and consequently $c(0) \ge \log 2 > 0$. Since $c(0) > 0$ and since the function $c(t)$ is continuous, in order to prove the last claim of our lemma, it suffices to show that $c(t) \le 0$ for all $t \ge h$. So, suppose on the contrary that $c(t) > 0$ for some $t \ge h$. It follows from (3.7) that

$$\mathrm{P}(f, V) > 0. \tag{3.8}$$

Since the proof of Lemma 4.1 and Corollary 4.2 from [9] go word by word in our context, we conclude that the Lyapunov exponent $\chi_\mu = \int \log|f^*|d\mu \ge 0$ for every Borel probability f-invariant measure μ supported on $K(V)$. It follows from (3.8) and the variational principle for topological pressure that there exists a Borel probability f-invariant measure μ supported on $K(V)$ such that $\mathrm{h}_\mu(f) - t\chi_\mu > 0$. Since $\chi_\mu \ge 0$, this implies that $\mathrm{h}_\mu(f) > 0$ and due to Ruelle's inequality $\chi_\mu > 0$. Hence, applying Przytycki's-Manne volume lemma (see [22], comp. [18]), we can write

$$t < \frac{\mathrm{h}_\mu(f)}{\chi_\mu} = \mathrm{HD}(\mu) \le h$$

and this contradiction finishes the proof. ■

Let

$$s(V) = \min\{c^{-1}(0) \cap (0, h]\} > 0.$$

Combining Lemma 3.5 and Lemma 3.6 we get the following.

Lemma 3.7. *There exists a Borel probability measure m_V supported on $K(V)$ such that*

(a) $m_V(f(A)) \ge \int_A |f^*|^{s(V)} dm_V$ *for every special set* $A \subset \mathbb{C}$ *and*
(b) $m_V(f(A)) = \int_A |f^*|^{s(V)} dm_V$ *for every special set* $A \subset \mathbb{C} \setminus \overline{V}$.

Fix now a descending sequence $\{\mathrm{Crit}\}_{n\ge 1}$ of neighbourds of Y satisfying (3.6) and such that $\mathrm{diam}_s(V_n) \le 1/n$. Since the sequence $n \mapsto s(V_n)$ is monotonically non-decreasing, proceeding similarly as in the proof of Lemma 5.4 from [9] (note that in the place where Lemma 3.3 from [9] is invoked, only the first inequality in (d) is needed; in particular $m_Y(\infty) = 0$, where m_Y is an arbitrary weak accumulation point of the sequence $\{m_{V_n}\}_{n=1}^\infty$, we obtained the following.

Lemma 3.8. *For every $s(Y)$, an accumulation point of the sequence $s(B_s(Y, 1/n))$, $s(Y) \in (0, h]$ and there exists a Borel probability measure m_Y (an appropriate weak accumulation point of the sequence $\{m_{B_s(Y,1/n)}\}_{n\ge 1}$) supported on $J(f)$ such that*

(a) $m_Y(f(A)) \ge \int_A |f^*|^{s(Y)} dm_Y$ *for every special set* $A \subset \mathbb{C}$ *and*
(b) $m_Y(f(A)) = \int_A |f^*|^{S(Y)} dm_Y$ *for every special set* $A \subset \mathbb{C} \setminus Y$.

The next fact proven in this section is provided by the following.

Lemma 3.9. *For every crossing set Y, $m = m_Y$ is an $s(Y)$-conformal measure for $f : J(f) \to J(f) \cup \{\infty\}$, $s(Y) = h$, and all atoms of m are contained in $I_-(f) \cup \bigcup_{n\ge 0} f^{-n}(\mathrm{Crit}(J(f))$.*

Proof. Since we already know that $m(\infty) = 0$ (see the paragraph proceeding Lemma 3.8) and since $Y \cap (\mathrm{Sing}^-(f) \cup I_\infty(f)) \subset \Omega(f) \cup \{\infty\}$, it follows from Lemma 3.8(b) and Corollary 2.24 that

$$m(Y \setminus \Omega(f)) = 0. \tag{3.9}$$

We shall show now that $m(\Omega(f)) = 0$. And indeed, fix $\omega \in \Omega(f)$. Take $a \ge 1$ so large that $f^a(\omega) = \omega$ and $(f^a)'(\omega) = 1$. It then follows from (2.11) that there exist a compact set $F_\omega \subset B(\omega, \theta) \setminus \{\omega\}$ and a constant $C \ge 1$ such that for every $k \ge 1$ and all $z \in F_\omega$, we have

$$C^{-1}k^{-\frac{p(\omega)+1}{p(\omega)}} \le |(f_\omega^{-ak})^*(z)| \le Ck^{-\frac{p(\omega)+1}{p(\omega)}} \tag{3.10}$$

and for every $n \ge 1$ there exists $k_n \ge 1$ such that

$$B(\omega, 1/n) \subset \bigcup_{j=k_n}^{\infty} f_\omega^{-aj}(F_\omega) \;\text{ and }\; \lim_{n\to\infty} k_n = \infty. \tag{3.11}$$

It follows from Lemma 3.8(b), (3.10) and the fact that the family $\{f_\omega^{-an}(F_\omega)\}_{n\ge 1}$ is of bounded multiplicity, that

$$\sum_{n\ge 1} n^{-\frac{p(\omega)+1}{p(\omega)}s(Y)} < \infty.$$

In particular $\frac{p(\omega)+1}{p(\omega)}s(Y) > 1$. Denote $m|_{V_n}$ by m_n and $s(V_n)$ by s_n. Since $\lim_{n\to\infty} s_n = s(Y)$, we see that for every $n \ge 1$ large enough, say $n \ge n_0$,

$$\frac{p(\omega)+1}{p(\omega)}s_n > 1 + \sigma.$$

for some $\sigma > 0$. It therefore follows from Lemma 3.8(a), (3.11) and (3.10) that for all $n \ge n_0$ and all $l \ge 1$

$$\begin{aligned} m_n(B(\omega, 1/l)) &\le \sum_{j=k_l}^{\infty} m_n\Big(f_\omega^{-aj}(F_\omega)\Big) \le C^{\frac{p(\omega)+1}{p(\omega)}s_n} \sum_{j=k_l}^{\infty} j^{-\frac{p(\omega)+1}{p(\omega)}s_n} \\ &\le C^{\frac{p(\omega)+1}{p(\omega)}s(Y)} \sum_{j=k_l}^{\infty} j^{-(1+\sigma)}. \end{aligned}$$

Consequently

$$m(B(\omega, 1/l)) \le C^{\frac{p(\omega)+1}{p(\omega)}s(Y)} \sum_{j=k_l}^{\infty} j^{-(1+\sigma)}.$$

Since $\lim_{l\to\infty} k_l = \infty$, we infer

$$m(\Omega(f)) = 0.$$

Combining this and (3.9), we see that $m(Y) = 0$. Since $f(\Omega(f)) = \Omega(f)$, in order to prove $s(Y)$-conformality of the measure m, it therefore suffices to show that $m(f(Y \setminus \Omega(f))) = 0$. But if $y \in Y \setminus (\Omega(f) \cup \{\infty\})$, then due to our definition of Y, $y \notin \mathrm{Sing}^-(f)$ and the formula $m(f(y)) = 0$ immediately follows from Corollary 2.24, the formula $m(f^n(f(y))) \geq |(f^n)^*(f(y))|^{s(Y)} m(f(y))$ and inequality $s(Y) > 0$ stated in Lemma 3.8. Thus the $s(Y)$-conformality of m is proven and in addition all the atoms of m must be contained in $J(f) \setminus \Omega$. In view of Lemma 3.8 and Lemma 3.3, $s(Y) = h$. Applying now Lemma 3.4 and Corollary 2.24 we see that all atoms of m must be contained in $I_-(f) \cup \bigcup_{n \geq 0} f^{-n}(\mathrm{Crit}(J(f))$. The proof is complete. ■

4. Hausdorff and Packing Measures

In this section H^h and Π^h denote respectively the Hausdorff and packing measures considered with respect to the spherical metric on $\overline{\mathbb{C}}$. Our aim here is to prove first that the conformal measure m produced in Lemma 3.9 is atomless and then the following main result.

Theorem 4.1. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a non-recurrent elliptic function. If $h = \mathrm{HD}(J(f)) = 2$, then $J(f) = \mathbb{C}$. So suppose that $h < 2$. Then*

(a) $\mathrm{H}^h(J(f)) = 0$.
(b) $\Pi^h(J(f)) > 0$.
(c) $\Pi^h(J(f)) = \infty$ *if and only if* $\Omega(f) \neq \emptyset$.

As an immediate consequence of this theorem we get the following.

Corollary 4.2. *If $\Omega(f) = \emptyset$, then the Euclidean h-dimensional packing measure Π_e^h is finite on each bounded subset of $J(f)$.*

4.1. **Special Facts from the Geometric Measure Theory.** We list in this subsection without proofs some more technical facts taken from Section 2, Section 3 and Section 4 of [26].

Definition 4.3. *Given $r > 0$ and $L > 0$ a point $x \in \mathbb{C}$ is said to be (r, L)-t-upper estimable if $\rho(x, r) \leq L$ and is said to be (r, L)-t-lower estimable if $\rho(x, r) \geq L$. We will frequently abbreviate the notation writing (r, L)-t-u.e. for (r, L)-t-upper estimable and (r, L)-t-l.e. for (r, L)-t-lower estimable. We also say that the point x is t-upper estimable (t-lower estimable) if it is (r, L)-t-upper estimable ((r, L)-t-lower estimable) for some $L > 0$ and all $r > 0$ sufficiently small.*

We will also need the following more technical notion.

Definition 4.4. *Given $r > 0$, $\sigma > 0$ and $L > 0$ the point $x \in X$ is said to be (r, σ, L)-t-strongly lower estimable, or shorter (r, σ, L)-t-s.l.e. if $\nu(B(y, \sigma r)) \geq Lr^t$ for every $y \in B(x, r)$.*

We collect now from [26] the technical facts about the notions defined above.

Lemma 4.5. *If z is (r, σ, L)-t-s.l.e., then every point $x \in B(z, r/2)$ is $(r/2, 2\sigma, 2^t L)$-t-s.l.e..*

Lemma 4.6. *If x is (r, σ, L)-t-s.l.e., then for every $0 < u \leq 1$ it is $(ur, \sigma/u, Lu^{-t})$-t-s.l.e..*

Lemma 4.7. *If ν is positive on nonempty open sets, then for every $r > 0$ there exists $E(r) \geq 1$ such that every point $x \in X$ is $(r, E(r))$-t-u.e. and $(r, E(r)^{-1})$-t-l.e..*

Passing to conformal maps we consider now the situation where $H : U_1 \to U_2$ is an analytic map of open subsets U_1, U_2 of the complex plane $\mathbb{C}$. We say that given $t \geq 0$, the Borel measure ν finite on bounded sets of $\mathbb{C}$ is a Euclidean semi t-conformal measure if and only if

$$\nu(H(A)) \geq \int_A |H'|^t \, d\nu$$

for every Borel subset A of U_1 such that $H|_A$ is one-to-one and is called t-conformal if the "$\geq$" sign can be replaced by an "$=$" sign.

Lemma 4.8. *Let ν be a Euclidean semi t-conformal measure. Suppose that $D \subset \mathbb{C}$ is an open set, $z \in D$ and $H : D \to \mathbb{C}$ is an analytic map which has an analytic inverse H_z^{-1} defined on $B(H(z), 2R)$ for some $R > 0$. Then for every $0 \leq r \leq R$*

$$K^{-t}\nu(B(z, K^{-1}r|H'(z)|^{-1})) \leq |H'(z)|^{-t}\nu((B(H(z), r))).$$

If, in addition, ν is t-conformal, then also

$$|H'(z)|^{-t}\nu((B(H(z), r))) \leq K^t\nu(B(z, Kr|H'(z)|^{-1})).$$

Lemma 4.9. *Suppose that ν is a Euclidean t-conformal measure. If the point $H(z)$ is (r, σ, L)-t-s.l.e., where $r \leq R/2$ and $\sigma \leq 1$, then the point z is $(K^{-1}|H'(z)|^{-1}r, K^2\sigma, L)$-$t$-s.l.e..*

Lemma 4.10. *Suppose that ν is a Euclidean t-conformal measure. Let c be a critical point of an analytic map $H : D \to \mathbb{C}$. If $0 < r \leq R(H, c)$ and $H(c)$ is (r, L)-t-l.e., then c is $((Ar)^{1/q}, A^{-2t}L)$-t-l.e..*

Lemma 4.11. *Let c be a critical point of an analytic map $H : D \to \mathbb{C}$. Let ν be a Euclidean semi t-conformal measure such that $\nu(c) = 0$. If $0 < r \le R(H,c)$ and $H(c)$ is (s,L)-t-u.e. for all $0 < s \le r$, then c is $\Big((A^{-1}r)^{1/q}, q(2A^2)^t(2^{t/q}-1)^{-1}L\Big)$-$t$-u.e..*

Note that the proof of this lemma is the same as the proof of Lemma 3.4 in [26]. The only modification is that the equality sign in the first line of the first displayed formula of this proof is to be replaced by the "$\ge$" sign.

Lemma 4.12. *Suppose that ν is a Euclidean t-conformal measure. Let c be a critical point of an analytic map $H : D \to \overline{\mathbb{C}}$. If $0 < r \le \frac{1}{3}R(H,c)$, $0 < \sigma \le 1$ and $H(c)$ is (r,σ,L)-t-s.l.e, then c is $((A^{-1}r)^{1/q}, \tilde{\sigma}, \tilde{L})$-$t$-s.l.e, where $\tilde{\sigma} = (2^{q+1}KA^2\sigma)^{1/q}$ and $\tilde{L} = L\min\{K^{-t}, (A^2\sigma)^{\frac{1-q}{q}t}\}$.*

Notice now that if m is a semi t-conformal measure for $f : J(f) \to J(f) \cup \{\infty\}$, then the measure $m_e = (1+|z|^2)^t m$ is Euclidean semi t-conformal, i.e.

$$m_e(f(A)) \ge \int_A |f'|^t dm_e$$

for every Borel set $A \subset J(f)$ such that $f|_A$ is 1-to-1. If m is t-conformal, then so is m_e in the obvious sense. The measure m_e is called the Euclidean version of m. Obviously m_e is equivalent to m and is finite on bounded subsets of $\mathbb{C}$. From now on throughout the entire paper we fix a crossing set Y and we consider an open neighbourhood $V \subset \overline{\mathbb{C}}$ of Y such that the closure of V is disjoint from at least one fundamental parallelogram of f. A semi t-conformal measure m is said to be almost t-conformal if

$$m(f(A)) = \int_A |f'|^t dm$$

for every Borel set $A \subset J(f)$ such that $f|_A$ is 1-to-1 and $A \cap \overline{V} = \emptyset$. Hence for every Borel set A such that $f|_A$ is 1-to-1 and $A \cap \overline{V} = \emptyset$ and for every $w \in \Lambda$, we have

$$\int_A |f'|^t dm_e = m_e(f(A)) = m_e(f(A+w)) \ge \int_{A+w} |f'|^t dm_e$$

and the last inequality sign becomes an equality either if in addition $(A+w) \cap \overline{V} = \emptyset$ or if m is a t-conformal measure and we assume only that $f|_A$ is 1-to-1. Since f' is periodic with respect to the lattice Λ, all the above statements and assumptions lead to the following.

Lemma 4.13. *For every $w \in \Lambda$, every Borel set $A \subset \mathbb{C}$ such that $A \cap \overline{V} = \emptyset$ and every almost t-conformal measure m*

$$m_e(A+w) \le m_e(A).$$

If either in addition $(A+w) \cap \overline{V} = \emptyset$ or if m is h-conformal and we assume only that $f|_A$ is 1-to-1, then this inequality becomes an equality. For every $r > 0$ there exists $M(r) \in (0,\infty)$

independent of any almost t-conformal measure m such that

$$m_e(F) \le M(r). \tag{4.1}$$

for every Borel set $F \subset \mathbb{C}$ with the diameter $\le r$. If in addition m is h-conformal, then for every $R > 0$ there exist constants $Q(R)$ and $Q_h(R)$ such that

$$m_e(B_e(x,r)) \ge Q(R)r^2 \ge Q_h(R)r^h \tag{4.2}$$

for all $x \in J(f)$ and all $r \ge R$.

The following lemma is proven in the same way as the corresponding lemma from Section 4 of [26].

Lemma 4.14. *Suppose that m_e is a Euclidean t-conformal measure. Then for every $R > 0$ and every $0 < \sigma \le 1$ there exists $L = L(\omega, R, \sigma) > 0$ such that for every $0 < r \le R$ every point $\omega \in \Omega(f)$ is (r, σ, L)-$\alpha_t(\omega)$-s.l.e. with respect to the measure m_e.*

4.2. **Conformal Measure and Holomorphic Inverse Branches.**
In this subsection we prove two technical propositions modeled on Proposition 6.3 and Proposition 6.4 from [26]. The proofs we present also follow those in [26]. However, in our context, unchanged formulations of Proposition 6.3 and Proposition 6.4 fail to be meaningful since $||f'||_{J(f)} = +\infty$. The remedy, sufficient for our purposes, is to replace $J(f)$ by an arbitrary closed forward invariant subset of $J(f)$ on which the modulus of the derivative of f is uniformly bounded above. For the convenience of the reader and the sake of completness we check below that appropriate modifications of Proposition 6.3 and Proposition 6.4 from [26] turn out to be true. Let m be an almost t-conformal measure and let m_e be its Euclidean version. The upper estimability and strongly lower estimability will be considered in this section with respect to the measure m_e. When we speak about lower estimability we assume more, that the measure m is t-conformal. Since the number of parabolic points is finite, passing to an appropriate iteration, we assume in this and the next section without loosing generality that all parabolic points of f are simple. Consider a forward f-invariant closed subset E of $\mathbb{C}$ such that

$$||f'||_E := \sup\{|f'(z)| : z \in E\} < \infty.$$

Such sets will be called f-pseudo-compact. Obviously, each f-invariant compact subset E of $\mathbb{C}$ is f-pseudo-compact. Recall that θ was defined in (2.9) and that $\tau > 0$ is so small as required in Lemma 2.3.

Proposition 4.15. *Fix an f-pseudo-compact subset E of $\mathbb{C}$. Let $z \in E$, $\lambda > 0$ and let $0 < r \le \tau\theta||f'||_E^{-1}\lambda^{-1}$ be a real number. Suppose that at least one of the following two* *conditions is satisfied:*

$$z \in E \setminus \bigcup_{n\geq 0} f^{-n}(\mathrm{Crit}(J(f))$$

or

$$z \in E \quad \text{and} \quad r > \tau\theta||f'||_E^{-1}\lambda^{-1}\inf\{|(f^n)'(z)|^{-1} : n = 1, 2, \dots\}.$$

Then there exists an integer $u = u(\lambda, r, z) \geq 0$ *such that* $r|(f^u)'(z)| \leq \lambda^{-1}\theta\tau$ *and the following four conditions are satisfied*

$$\mathrm{diam}\Big(\mathrm{Comp}(f^j(z), f^u(z), f^{u-j}, r|(f^u)'(z)|)\Big) \leq \beta \tag{4.3}$$

for every $j = 0, 1, \dots, u$. *For every* $\eta > 0$ *there exists a continuous function* $t \mapsto B_t = B_t(\lambda, \eta) > 0$, $t \in [0, \infty)$, *(independent of* z, n, *and* r*) and such that if* $f^u(z) \in B(\omega, \theta)$ *for some* $\omega \in \Omega(f)$, *then*

$$f^u(z) \quad \textit{is} \quad (\eta r|(f^u)'(z)|, B_t) - \alpha_t(\omega)\text{-}u.e. \tag{4.4}$$

and there exists a function $W_t = W_t(\lambda, \eta) : (0, 1] \to (0, 1]$ *(independent of* z, n, *and* r*) such that if* $f^u(z) \in B(\omega, \theta)$ *for some* $\omega \in \Omega(f)$, *then for every* $\sigma \in (0, 1]$

$$f^u(z) \quad \textit{is} \quad (\eta r|(f^u)'(z)|, \sigma, W_t(\sigma)) - \alpha_t(\omega)\text{-}s.l.e. \tag{4.5}$$

If $f^u(z) \notin B(\Omega(f), \theta)$, *then formulas (4.4) and (4.5) are also true with* $\alpha_t(\omega)$ *replaced by* t. (4.6)

Proof. Suppose first that $\sup\{\lambda r|(f^j)'(z)| : j \geq 1\} > \theta\tau||f'||_E^{-1}$ and let $n = n(\lambda, z, r) \geq 0$ be a minimal integer such that

$$\lambda r|(f^n)'(z)| > \theta\tau||f'||_E^{-1}. \tag{4.7}$$

Then $n \geq 1$ (due to the assumption imposed on r) and also

$$\lambda r|(f^n)'(z)| \leq \theta\tau \tag{4.8}$$

If $f^n(z) \notin B(\Omega(f), \theta)$ set $u = u(\lambda, r, z) = n$. The items (4.4), (4.5) and (4.6) are obvious in view of our assumptions imposed on E.

So suppose that $f^n(z) \in B(\Omega(f), \theta)$, say $f^n(z) \in B(\omega, \theta)$, $\omega \in \Omega(f)$. Let $0 \leq k = k(\lambda, z, r) \leq n$ be the smallest integer such that $f^j(z) \in B(\Omega(f), \theta)$ for every $j = k, k+1, \dots, n$. Consider all the numbers

$$r_i = |f^i(z) - \omega||(f^i)'(z)|^{-1}$$

where $i = k, k+1, \dots, n$. By (4.7) we have

$$r_n = |f^n(z) - \omega||(f^n)'(z)|^{-1} \leq \theta||f'||_E\theta^{-1}\tau^{-1}\lambda r = ||f'||_E\tau^{-1}\lambda r$$

and therefore there exists a minimal $k \leq u = u(\lambda, r, z) \leq n$ such that $r_u \leq ||f'||_E\tau^{-1}\lambda r$. In other words

$$|f^u(z) - \omega| \leq ||f'||_E\tau^{-1}\lambda r|(f^u)'(z)| \leq ||f'||_E\tau^{-1}\eta^{-1}\lambda\eta r|(f^u)'(z)| \tag{4.9}$$

If $\sup\{\lambda r|(f^j)'(z)| : j \geq 1\} \leq \theta\tau||f'||_E^{-1}$, then it follows from Corollary 2.24 that $z \in \bigcup_{j\geq 0} f^{-j}(\Omega(f))$. Define then $u(\lambda, z, r) = k(\lambda, z, r)$ to be the minimal integer $j \geq 0$ such that $f^j(z) \in \Omega(f)$ and put $\omega = f^u(z)$. Notice that in this case formulas (4.8) and (4.9) are also satisfied. Our further considerations are valid in both cases. First note that by (4.9) we have

$$B(f^u(z), \eta r|(f^u)'(z)|) \subset B(\omega, (1 + ||f'||_E \tau^{-1}\eta^{-1}\lambda)\eta r|(f^u)'(z)|) \tag{4.10}$$

and in view of Lemma 2.5 and (4.8)

$$m_e\Big(B(f^u(z),\eta r|(f^u)'(z)|)\Big) \leq$$
$$\leq C(\omega, (1 + ||f'||_E\tau^{-1}\eta^{-1}\lambda)\theta\tau\eta\lambda^{-1})(1 + ||f'||_E\tau^{-1}\eta^{-1}\lambda)^{\alpha_t(\omega)}(\eta r|(f^u)'(z)|)^{\alpha_t(\omega)}$$

So, item (4.4) is proved. Also applying (4.9), Lemma 4.14, Lemma 4.5 and 4.8) we see that the point $f^u(z)$ is

$$\Big(||f'||_E\tau^{-1}\lambda r|(f^u)'(z)|, \sigma\tau||f'||_E^{-1}\eta\lambda^{-1}, 2^{\alpha_t(\omega)}L(\omega, 2||f'||_E\theta, \sigma\tau(2||f'||_E)^{-1}\eta\lambda^{-1})\Big) - \alpha_t(\omega)\text{-s.l.e.}$$

So, if $||f'||_E\tau^{-1}\lambda \geq \eta$, then by Lemma 4.6, $f^u(z)$ is

$$\Big(\eta r|(f^u)'(z)|, \sigma, (2||f'||_E\tau^{-1}\lambda\eta^{-1})^{\alpha_t(\omega)}L(\omega, 2||f'||_E\theta, \sigma\tau(2||f'||_E)^{-1}\eta)\lambda^{-1}\Big) - \alpha_t(\omega)\text{-s.l.e}$$

If instead $||f'||_E\tau^{-1}\lambda \leq \eta$, then again it follows from (4.9), Lemma 4.14, Lemma 4.5 and (4.8) that the point $f^u(z)$ is $\Big(\eta r|(f^u)'(z)|, \sigma, 2^{\alpha_t(\omega)}L(\omega, 2\theta\tau\lambda^{-1}\eta, \sigma/2)\Big) - \alpha_t(\omega)$-s.l.e.. So, part (4.5) is also proved.

In order to prove (4.3) suppose first that $u = k$. In particular this is the case if $z \in \bigcup_{j\geq 0} f^{-j}(\Omega(f))$. Then

$$\text{Comp}(f^{k-1}(z), f^k(z), f, r|(f^u)'(z)|) \subset \text{Comp}(f^{k-1}(z), f^k(z), f, \theta\tau)$$

and by the choice of k and (2.7) we have $f^{k-1}(z) \notin B(\Omega(f), \theta)$. Therefore (4.3) follows from the choice of τ (see (2.18)) and (2.17).
If $u > k$ (so the first case holds), then $r_{u-1} > ||f'||_E\tau^{-1}\lambda r$ and by (2.17) we get

$$r_u = \frac{|f^u(z) - \omega|}{|f^{u-1}(z) - \omega|}|f'(f^{u-1}(z))|^{-1}r_{u-1} \geq \|f\|^{-1}r_{u-1} \geq \tau^{-1}\lambda r.$$

So, $\lambda r|(f^u)'(z)| \leq \tau|f^u(z) - \omega|$ and applying Lemma 2.4 and (2.7) $u - k$ times we conclude that for every $k \leq j \leq u$

$$\text{diam}\Big(\text{Comp}(f^j(z), f^u(z), f^{u-j}, \lambda r|(f^u)'(z)|)\Big) \leq \theta\tau < \beta$$

And now for $j = k-1, k-2, \ldots, 1, 0$, the same argument applies as in the case $u = k$. ■

Proposition 4.16. *Fix an f-pseudo-compact subset E of $\mathbb{C}$. Let ϵ and λ be both positive numbers such that $\epsilon < \lambda\min\{1, \tau^{-1}, \theta^{-1}\tau^{-1}\gamma\}$. If $0 < r < \tau\theta||f'||_E^{-1}\lambda^{-1}$ and $z \in E \setminus \text{Crit}(J(f))$, then there exists an integer $s = s(\lambda, \epsilon, r, z) \geq 1$ with the following three properties.*

$$|(f^s)'(z)| \neq 0. \tag{4.11}$$

If $= u(\lambda, r, z)$ *is well-defined, then* $s \le u(\lambda, r, z)$. *If either* u *is not defined or* $s < u$, *then there exists a critical point* $c \in Crit(f)$ *such that*

$$|f^s(z) - c| \le \epsilon r|(f^s)'(z)|. \tag{4.12}$$

In any case

$$\mathrm{Comp}\Big(z, f^s(z), f^s, (KA^2)^{-1}2^{-\#(\mathrm{Crit}(f)\cap\mathcal{R})}\epsilon r|(f^s)'(z)|\Big) \cap \mathrm{Crit}(f^s) = \emptyset. \tag{4.13}$$

Proof. Since $z \notin \mathrm{Crit}(f)$ and in view of Proposition 4.15, there exists a minimal number $s = s(\lambda, \epsilon, r, z)$ for which at least one of the following two conditions is satisfied

$$|f^s(z) - c| \le \epsilon r|(f^s)'(z)| \tag{4.14}$$

for some $c \in \mathrm{Crit}(J(f))$ or

$$u(\lambda, r, z) \text{ is well-defined} \quad \text{and} \quad s(\lambda, \epsilon, r, z) = u(\lambda, r, z) \tag{4.15}$$

Since $|(f^s)'(z)| \ne 0$, the parts (4.11) and (4.12) are proved.

In order to prove (4.13) notice first that no matter which of the two numbers s is, in view of Proposition 4.15 we always have

$$\epsilon r|(f^s)'(z)| \le \epsilon\lambda^{-1}\theta\tau \tag{4.16}$$

Let us now argue that for every $0 \le j \le s$

$$\mathrm{diam}\Big(\mathrm{Comp}(f^{s-j}(z), f^s(z), f^j, \epsilon r|(f^s)'(z)|)\Big) \le \beta \tag{4.17}$$

Indeed, if $s = u$, it follows immediately from Proposition 4.15 and (4.3) since $\epsilon \le \lambda$. Otherwise $|f^s(z) - c| \le \epsilon r|(f^s)'(z)| \le \epsilon\lambda^{-1}\theta\tau < \theta$ and therefore, by (2.15), $f^s(z) \notin B(\Omega(f), \theta)$. Thus (4.17) follows from (2.17).

Now by (4.17) and Lemma 2.10, there exists $0 \le p \le \#(\mathrm{Crit}(f) \cap \mathcal{R})$, an increasing sequence of integers $1 \le k_1 < k_2 < \ldots < k_p \le s$ and mutually distinct critical points $c_1, c_2, \ldots, c_p$ of f such that

$$\{c_l\} = \mathrm{Comp}(f^{s-k_l}(z), f^s(z), f^{k_l}, \epsilon r|(f^s)'(z)|) \cap \mathrm{Crit}(f). \tag{4.18}$$

for every $l = 1, 2, \ldots, p$ and if $j \notin \{k_1, k_2, \ldots, k_p\}$, then

$$\mathrm{Comp}(f^{s-j}(z), f^s(z), f^j, \epsilon r|(f^s)'(z)|) \cap \mathrm{Crit}(f) = \emptyset. \tag{4.19}$$

Setting $k_0 = 0$ we shall show by induction that for every $0 \le l \le p$

$$\mathrm{Comp}(f^{s-k_l}(z), f^s(z), f^{k_l}, (KA^2)^{-1}2^{-l}\epsilon r|(f^s)'(z)|) \cap \mathrm{Crit}(f^{k_l}) = \emptyset. \tag{4.20}$$

Indeed, for $l = 0$ there is nothing to prove. So, suppose that (4.20) is true for some $0 \le l \le p - 1$. Then by (4.19)

$$\mathrm{Comp}(f^{s-(k_{l+1}-1)}(z), f^s(z), f^{k_{l+1}-1}, (KA^2)^{-1}2^{-l}\epsilon r|(f^s)'(z)|) \cap \mathrm{Crit}(f^{k_{l+1}-1}) = \emptyset.$$

So, if

$$c_{l+1} \in \mathrm{Comp}(f^{s-k_{l+1}}(z), f^s(z), f^{k_{l+1}}, (KA^2)^{-1}2^{-(l+1)}\epsilon r|(f^s)'(z)|)$$

then by Lemma 1.4 applied for holomorphic maps $H = f$, $Q = f^{k_{l+1}-1}$ and the radius $R = (KA^2)^{-1}2^{-(l+1)}\epsilon r|(f^s)'(z)| < \gamma$ we get

$$\begin{aligned}|f^{s-k_{l+1}}(z) - c_{l+1}| &\le KA^2|(f^{k_{l+1}})'(f^{s-k_{l+1}}(z))|^{-1}(KA^2)^{-1}2^{-(l+1)}\epsilon r|(f^s)'(z)| \\ &= 2^{-(l+1)}\epsilon r|(f^{s-k_{l+1}}(z))'| \\ &\le \epsilon r|(f^{s-k_{l+1}}(z))'|\end{aligned}$$

which contradicts the definition of s and proves (4.20) for $l+1$. In particular it follows from (4.20) that

$$\text{Comp}(z, f^s(z), f^s, (KA^2)^{-1}2^{-\#(\text{Crit}(f)\cap\mathcal{R})}\epsilon r|(f^s)'(z)|) \cap \text{Crit}(f^s) = \emptyset$$

The proof is finished. ■

4.3. **Hausdorff and Conformal Measure.**

Let m be a Borel probability measure on $\mathbb{C}$ and let m_e be its Euclidean version, i.e. $\frac{dm_e}{dm}(z) = (1+|z|^2)^t$. We will need in this and the next section the following.

Lemma 4.17. *If $z \in J(f)$, $r_n \searrow 0$ and $M = \lim_{n\to\infty} r_n^{-t} m_e(B(z,r_n))$, then*

$$\limsup_{n\to\infty} \frac{m\Big(B_s(z,(2(1+|z|^2))^{-1}r_n\Big)}{((2(1+|z|^2))^{-1}r_n)^t} \le 2^t M$$

and

$$\liminf_{n\to\infty} \frac{m\Big(B_s(z,2(1+|z|^2)^{-1}r_n\Big)}{(2(1+|z|^2)^{-1}r_n)^t} \ge 2^{-t} M$$

Proof. Since for every $r > 0$ sufficiently small

$$B(z, 2^{-1}(1+|z|^2)r) \subset B_s(z,r) \subset B(z, 2(1+|z|^2)r)$$

and since

$$\lim_{r\searrow 0} \frac{m_e(B(z,r))}{m(B(z,r))} = (1+|z|^2)^t,$$

we get

$$\limsup_{n\to\infty} \frac{m\Big(B_s(z,(2(1+|z|^2))^{-1}r_n\Big)}{((2(1+|z|^2))^{-1}r_n)^t} \le \lim_{n\to\infty} \frac{m(B(z,r_n))}{2^{-t}(1+|z|^2)^{-t}r_n^t} = 2^t M$$

and

$$\liminf_{n\to\infty} \frac{m\Big(B_s(z,2(1+|z|^2)^{-1}r_n\Big)}{(2(1+|z|^2)^{-1}r_n)^t} \ge \lim_{n\to\infty} \frac{m(B(z,r_n))}{2^{t}(1+|z|^2)^{-t}r_n^t} = 2^{-t} M.$$

We are done. ■

Our first goal is to show that the h-conformal measure m proven to exist in Lemma 3.9 is atomless and that $\mathrm{H}^h(J(f)) = 0$ if $h < 2$. We will consider almost t-conformal measures ν with $t \geq 1$. The notion of upper estimability introduced in Definition 4.3 is considered with respect to the Euclidean almost t-conformal measure ν_e. Recall that $l = l(f) \geq 1$ is the integer claimed in Lemma 2.22 and put

$$\begin{aligned} R_l(f) &= \inf\{R(f^j, c) : c \in \mathrm{Crit}(f) \text{ and } 1 \leq j \leq l(f)\} \\ &= \min\{R(f^j, c) : c \in \mathrm{Crit}(f) \cap \mathcal{R} \text{ and } 1 \leq j \leq l(f)\} < \infty \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} A_l(f) &= \sup\{A(f^j, c) : c \in \mathrm{Crit}(f) \text{ and } 1 \leq j \leq l(f)\} \\ &= \max\{A(f^j, c) : c \in \mathrm{Crit}(f) \cap \mathcal{R} \text{ and } 1 \leq j \leq l(f)\} \end{aligned} \tag{4.22}$$

where the numbers $R(f^j, c)$ and $A(f^j, c)$ are defined just above Definition 1.1. Since $\overline{O_+(f(\mathrm{Crit}(J(f))))}$ is a compact f-invariant subset of $\mathbb{C}$ (so disjoint from $f^{-1}(\infty)$) and since $\mathrm{PC}(f) = \overline{O_+(\mathrm{Crit}(J(f)))} = \mathrm{Crit}(J(f)) \cup \overline{O_+(f(\mathrm{Crit}(J(f))))}$, we have the following straightforward but useful fact.

Lemma 4.18. *The set* $\mathrm{PC}(f)$ *is* f*-pseudo-compact.*

Recall for the needs of the two next lemmas that the sequence $\{Cr_i(f)\}$ was defined inductively by the formula (2.23) and the sequence $S_i(f)$ was defined by the formula (2.23).

Since the number of equivalence classes of the relation $\sim$ is finite, looking at Lemma 2.22 and Lemma 4.13, the following lemma follows immediately from Lemma 4.11.

Lemma 4.19. *If* $R_i^{(u)} > 0$ *is a positive constant and* $t \mapsto C_{t,i}^{(u)} \in (0, \infty)$, $t \in [1, \infty)$, *is a continuous function such that all points* $z \in \mathrm{PC}(f)_i$ *are* $(r, C_{t,i}^{(u)})$*-t-u.e. with respect to any Euclidean almost t-conformal measure* ν_e *(with* $t \geq 1$*) for all* $0 < r \leq R_i^{(u)}$, *then there exists a continuous function* $t \mapsto \tilde{C}_{t,i}^{(u)} > 0$, $t \in [1, \infty)$, *such that all critical points* $c \in Cr_{i+1}(f)$ *are* $(r, \tilde{C}_{t,i}^{(u)})$*-t-u.e. with respect to any Euclidean almost t-conformal measure* ν_e *for all* $0 < r \leq A_l^{-1} R_i^{(u)}$.

In the above lemma the superscript u stands for "upper". In the lemma below it has the same connotation. The number u is also used to denote the value of the function $u(\lambda, r, z)$ defined in Proposition 4.15. This should not cause any confusion.

Lemma 4.20. *If* $R_{i,1}^{(u)} > 0$ *is a positive constant and* $t \mapsto C_{t,i,1}^{(u)} \in (0, \infty)$, $t \in [1, \infty)$, *is a continuous function such that all critical points* $c \in S_i(f)$ *are* $(r, C_{t,i,1}^{(u)})$*-t-u.e. with respect to any Euclidean almost t-conformal measure* ν_e *(with* $t \geq 1$*) for all* $0 < r \leq R_{i,1}^{(u)}$, *then there* *exist a continuous function* $t \mapsto \tilde{C}^{(u)}_{t,i,1} > 0$, $s \in [1,\infty)$, *and* $\tilde{R}^{(u)}_{i,1} > 0$ *such that all points* $z \in \mathrm{PC}(f)_i$ *are* $(r, \tilde{C}^{(u)}_{t,i,1})$*-t-u.e. with respect to any Euclidean almost t-conformal measure* ν_e *(with* $t \geq 1$*) for all* $0 < r \leq \tilde{R}^{(u)}_{i,1}$.

Proof. We shall show that one can take

$$\tilde{R}^{(u)}_{i,1} = \min\Big\{\tau\theta||f'||^{-1}_{\mathrm{PC}(f)}\lambda^{-1}, R^{(u)}_{i,1}, 1\Big\} \ \text{ and } \ \tilde{C}^{(u)}_{t,i,1} = \max\{K^2 2^t C^{(u)}_{t,i,1}, K^{2t}B_s\}.$$

Indeed, denote $\#(\mathrm{Crit}(f) \cap \mathcal{R})$ by $\#$. Put $\epsilon = 2K(KA^2)2^{\#}$ and then choose $\lambda > 0$ so large that

$$\epsilon < \lambda \min\Big\{1, \tau^{-1}, \theta^{-1}\tau^{-1}\min\{\gamma, \rho, R^{(u)}_{i,1}/2\}\Big\}. \tag{4.23}$$

Consider $0 < r \leq \tilde{R}^{(u)}_{i,1}$ and $z \in \mathrm{PC}(f)_i$. If $z \in \mathrm{Crit}(J(f))$, then $z \in S_i(f)$ and we are done. Thus, we may assume that $z \notin \mathrm{Crit}(J(f))$. Let $s = s(\lambda, \epsilon, r, z)$. By the definition of ϵ,

$$2Kr|(f^s)'(z)| = (KA^2)^{-1}2^{-\#}\epsilon r|(f^s)'(z)| \tag{4.24}$$

Suppose first that $u(\lambda, r, z)$ is well defined and $s = u(\lambda, r, z)$. Then by Proposition 4.15(4.4) or Proposition 4.15(4.6), applied with $\eta = 2K$, we see that the point $f^s(z)$ is $(2Kr|(f^s)'(z)|, B_t)$-t-u.e.. Using (4.24), it follows from Proposition 4.16(4.13) and Lemma 4.8 that the point z is $(r, K^{2h}B_h)$-h-u.e..
If either u is not defined or $s < u(\lambda, r, z)$, then in view of Proposition 4.16(4.13), there exists a critical point $c \in \mathrm{Crit}(J(f))$ such that $|f^s(z) - c| \leq \epsilon r|(f^s)'(z)|$. Since $s \leq u$, by Proposition 4.15 and (4.23) we get

$$2Kr|(f^s)'(z)| \leq \epsilon r|(f^s)'(z)| < \epsilon\tau\theta\lambda^{-1}\min\{\rho, R^{(u)}_{i,1}/2\} \tag{4.25}$$

Since $z \in \mathrm{PC}(f)_i$, it implies that $c \in S_i(f)$. Therefore using (4.25), the assumptions of Lemma 4.20, and (4.24) and then applying Proposition 4.16(4.13) (remember that by Lemma 4.18 the set $\mathrm{PC}(f)$ is f-pseudo-compact) and Lemma 4.8, we conclude that z is $(r, K^2 2^t C^{(u)}_{t,i,1})$-$t$-u.e.. The proof is complete. ■

Recall that for any pole b of f, the number q_b denotes its multiplicity and $B_b(R)$ is the connected component of $f^{-1}(B_R)$ containing b.

Lemma 4.21. *If* $b \in f^{-1}(\infty)$, *if* ν *is a Euclidean almost t-conformal measure with* $t > \frac{2q_b}{q_b+1}$ *such that* $\nu(b) = 0$, *and if* m *is the h-conformal measure proven to exist in Lemma 3.9, then*

$$\nu(B_b(R)) \preceq R^{2-\frac{q_b+1}{q_b}t}$$

and

$$m_e(B(b,r)) \succeq r^{(q_b+1)h-2q_b}$$

for all $0 < r \preceq 1$.

Proof. It follows from Lemma 4.13 that $m_e(\{z \in \mathbb{C} : R \le |z| < 2R\}) \asymp R^2$ and $\nu(\{z \in \mathbb{C} : R \le |z| < 2R\}) \preceq R^2$ for all $R > 0$ large enough. It therefore follows from (2.2) that

$$m_e\Big((B_b(R) \setminus \overline{B_b(2R)}\Big) \asymp R^2 R^{-\frac{q_b+1}{q_b}h}. \tag{4.26}$$

and

$$\nu\Big((B_b(R) \setminus \overline{B_b(2R)}\Big) \preceq R^2 R^{-\frac{q_b+1}{q_b}t}. \tag{4.27}$$

Fix now $r > 0$ so small that $R = (r/L)^{-q_b}$ is large enough for the formula (4.26) and (4.27) to hold. Using (2.4) and (4.27) we therefore get

$$\begin{aligned}
\nu(B_b(R)) &= \nu\left(\bigcup_{j\ge 0}\Big(B_b(2^jR) \setminus \overline{B_b(2^{j+1}R)}\Big)\right) = \sum_{j=0}^{\infty} \nu\Big(B_b(2^jR) \setminus \overline{B_b(2^{j+1}R)}\Big) \\
&\preceq \sum_{j=0}^{\infty} (2^jR)^2 (2^jR)^{-\frac{q_b+1}{q_b}t} = R^{2-\frac{q_b+1}{q_b}t} \sum_{j=0}^{\infty} 2^{j\left(2-\frac{q_b+1}{q_b}t\right)} \\
&= L^{q_b\left(2-\frac{q_b+1}{q_b}t\right)} r^{(q_b+1)t-2q_b} \sum_{j=0}^{\infty} 2^{j\left(2-\frac{q_b+1}{q_b}t\right)} \asymp r^{(q_b+1)t-2q_b},
\end{aligned}$$

where the last comparability sign was written since $\frac{q_b+1}{q_b}t > 2$. We are done with the first part of our lemma. Replace now in the above formula ν by m_e and t by h, which is greater than $\frac{2q_b}{q_b+1}$ due to Theorem 2.1. Since in this case the "$\preceq$" sign can be, due to (4.26), replaced by the comparability sign "$\asymp$", since the first equality sign becomes "$\ge$" (we do not rule out the possibility that $m_e(b) > 0$ yet), and since $m_e(B(b,r)) \ge \nu(B_b(R))$, we are also done in this case. ■

We shall prove now the following.

Lemma 4.22. *The h-conformal measure m for $f : J(f) \to J(f) \cup \{\infty\}$ proven to exist in Lemma 3.9 is atomless.*

Proof. Using the induction on $i = 0, 1, \dots, p$, it follows immediately from Lemma 4.20 (this lemma provides the base of induction as $S_0(f) = \emptyset$ and simultaneously contributes to the inductive step), Lemma 4.19, and Lemma 2.21 that there exists a continuous function $t \mapsto C_t \in (0, \infty)$, $t \in [1, \infty)$, such that if ν is an arbitrary almost t-conformal measure on $J(f)$, then

$$\nu_e(B(x,r)) \le C_t r^t \tag{4.28}$$

for all $x \in \mathrm{PC}(f)$ and all $r \le r_0$ for some $r_0 > 0$ sufficiently small. Consider now the almost t_n-conformal measures $m_n = m_{B_s(Y,1/n)}$ (n is assumed to be so large that $B_s(Y, 1/n) \subset V$), where $t_n = S(B_s(Y, 1/n))$. Letting $n \to \infty$ and recalling that m is a weak limit of measures

m_n, formula (4.28) gives

$$m_e(B(x,r)) \le C_h r^h \tag{4.29}$$

for all $x \in \mathrm{PC}(f)$ and all $r \le r_0$. It now follows from Lemma 4.17 that

$$\limsup_{r \searrow 0} \frac{m(B(x,r))}{r^h} \le 2^h C_h.$$

for all $x \in \mathrm{PC}(f)$. In particular $m(\mathrm{Crit}(f)) = 0$ and consequently

$$m\left(\bigcup_{n\ge 0} f^{-n}(\mathrm{Crit}(f))\right) = 0. \tag{4.30}$$

Fix now $b \in f^{-1}(\infty)$. Fix $t \in \left(\frac{2q_b}{q_b+1}, h\right)$. Consider all integers $n \ge 1$ so large that $t_n \ge t$. Since $m_n(f^{-1}(\infty)) \le m_n(f^{-1}(B_s(Y,1/n)) = 0$, it then follows from Lemma 4.21 that

$$m_n(B_b(R)) \preceq R^{2-\frac{q_b+1}{q_b}t_n} \le R^{2-\frac{q_b+1}{q_b}t}.$$

Hence $m_e(b) = 0$. Since m and m_e are equivalent on $\mathbb{C}$, this gives $m(b) = 0$. Since $\bigcup_{n\ge 0} f^{-n}(b) \cap \mathrm{Crit}(f) = \emptyset$, this implies that $m\left(\bigcup_{n\ge 0} f^{-n}(b)\right) = 0$. Invoking now (4.30) and Lemma 3.9 finishes the proof. ■

Denote by $\mathrm{Tr}(f) \subset J(f)$ the set of all transitive points of f, that is the set of points in $J(f)$ such that $\overline{O_+(z)} = J(f)$.

Theorem 4.23. *There exists a unique atomless t-conformal measure m for $f : J(f) \to J(f) \cup \{\infty\}$. Then $t = h$, m is ergodic conservative and all other conformal measures are purely atomic, supported on $\mathrm{Sing}^-(f)$ with exponents larger than h. Consequently $m(\mathrm{Tr}(f)) = 1$.*

Proof. In view of Lemma 4.22 there exists an atomless h-conformal measure m for $f : J(f) \to J(f) \cup \{\infty\}$. Suppose that ν is an arbitrary t-conformal measure for f and some $t \ge 0$. By Lemma 3.3, $t \ge h$. Fix $z \in J(f) \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f))$. Then in view of Proposition 2.23 there exist a point $x = x(z) \in J(f)$ and an increasing sequence $\{n_k\}_{k=1}^\infty$ such that $x(z) = \lim_{k\to\infty} f^{n_k}(z)$. Define for every $l \ge 1$

$$Z_l = \{z \in J(f) \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f)) : |x(z)| \le l \text{ and } \eta(z) \ge 1/l\},$$

fix $l \ge 1$ and $z \in Z_l$. Consider for k large enough the sets $f_z^{-n_k}(B(x, \frac{1}{4l}))$ and $f_z^{-n_k}(B(x, \frac{1}{4Kl})))$, where $f_z^{-n_k}$ is the holomorphic inverse branch of f^{n_k} defined on $B(x, \frac{1}{2l})$ and sending $f^{n_k}(z)$ to z. Using conformality of the measure ν along with Koebe's distortion theorem, we easily deduce now that

$$B(\nu,l)^{-1} c|(f^{n_k})^*(z)|^{-h} \le \nu\Big(B_s(z, c|(f^{n_k})^*(z)|^{-1})\Big) \le B(\nu,l)c|(f^{n_k})^*(z)|^{-h} \tag{4.31}$$

for all $k \ge 1$ large enough, where $K \ge 1$ is the constant appearing in the Koebe's distortion theorem and ascribed to the scale $1/2$ and $c > 0$ is some constant comparable with 1. Fix now E, an arbitrary bounded Borel set contained in Z_l. Since m is regular, for every $x \in E$ there exists a radius $r(x) > 0$he form from (4.31) such that

$$m(\bigcup_{x \in E} B_s(x, r(x)) \setminus E) < \epsilon. \tag{4.32}$$

Now by the Besicovič theorem (see [G]) we can choose a countable subcover $\{B_s(x_i, r(x_i))\}_{i=1}^\infty$, $r(x_i) \le \epsilon$, from the cover $\{B_s(x, r(x))\}_{x \in E}$ of E, of multiplicity bounded by some constant $C \ge 1$, independent of the cover. Therefore by (4.31) and (4.32), we obtain

$$\begin{aligned} \nu(E) &\le \sum_{i=1}^\infty \nu(B_s(x_i, r(x_i))) \le B(\nu, l) \sum_{i=1}^\infty r(x_i)^t \\ &\le B(\nu, l) B(m, l) \sum_{i=1}^\infty r(x_i)^{t-h} m(B_s(x_i, r(x_i))) \\ &\le B(\nu, l) B(m, l) C \epsilon^{t-h} m(\bigcup_{i=1}^\infty B_s(x_i, r(x_i))) \\ &\le C B(\nu, l) B(m, l) \epsilon^{t-h} (\epsilon + m(E)). \end{aligned} \tag{4.33}$$

In the case when $t > h$, letting $\epsilon \searrow 0$ we obtain $\nu(Z_l) = 0$. Since $J(f) \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f)) = \bigcup_{l=1}^\infty Z_l$, we therefore get $\nu\Big(J(f) \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f))\Big) = 0$ which by Lemma 3.4 implies that $\nu(\mathrm{Sing}^-(f)) = 1$ and the last part of our theorem is proved . Suppose now that $t = h$. Since, in view of Lemma 3.4, $\nu(I_\infty(f) \setminus I_-(f)) = m(I_\infty(f)) = 0$, using (4.33) and letting $l \nearrow \infty$, we conclude that $\nu|_{J(f) \setminus \mathrm{Sing}^-(f)} << m|_{J(f) \setminus \mathrm{Sing}^-(f)}$. Exchanging the roles of m and ν we infer that the measures $\nu|_{J(f) \setminus \mathrm{Sing}^-(f)}$ and $m|_{J(f) \setminus \mathrm{Sing}^-(f)}$ are equivalent. Suppose that $\nu(\mathrm{Sing}^-(f)) > 0$. Then there exists $y \in \mathrm{Crit}(J(f)) \cup \Omega(f) \cup f^{-1}(\infty)$ such that $m(y) > 0$. But then

$$\sum_{\xi \in y^-} |(f^{n(\xi)})^*(\xi)|^{-h} < \infty,$$

where $y^- = \bigcup_{n \ge 0} f^{-n}(y)$ and for every $\xi \in y^-$, $n(\xi)$ is the least integer $n \ge 0$ such that $f^n(\xi) = y$. Hence,

$$\nu_y = \frac{\sum_{\xi \in y^-} |(f^{n(\xi)})^*(\xi)|^{-h} \delta_\xi}{\sum_{\xi \in y^-} |(f^{n(\xi)})^*(\xi)|^{-h}}$$

is an h-conformal measure supported on $y^- \subset \mathrm{Sing}^-(f)$. This contradicts the proven fact that the measures $\nu_y|_{J(f) \setminus \mathrm{Sing}^-(f)}$ and $m|_{J(f) \setminus \mathrm{Sing}^-(f)}$ are equivalent and $m(J(f) \setminus \mathrm{Sing}^-(f)) = 1$. Thus ν and m are equivalent.

Let us now prove that any h-conformal measure ν is ergodic. Indeed, suppose to the contrary that $f^{-1}(G) = G$ for some Borel set $G \subset J(f)$ with $0 < m(G) < 1$. But then the two conditional measures ν_G and $\nu_{J(f) \setminus G}$

$$\nu_G(B) = \frac{\nu(B \cap G)}{\nu(G)}, \quad \nu_{J(f) \setminus G}(B) = \frac{\nu(B \cap J(f) \setminus G)}{\nu(J(f) \setminus G)}$$

would be h-conformal and mutually singular; a contradiction.

If now ν is again an arbitrary h-conformal measure, then by a simple computation based on the definition of conformal measures we see that the Radon-Nikodyn derivative $\phi = d\nu/dm$ is constant on grand orbits of f. Therefore by ergodicity of m we conclude that ϕ is constant m-almost everywhere. As both m and ν are probability measures, it implies that $\phi = 1$ a.e., hence $\nu = m$.

Let us show now that m is conservative. We shall prove first that every forward invariant ($f(E) \subset E$) subset E of $J(f)$ is either of measure 0 or 1. Indeed, suppose to the contrary that $0 < m(E) < 1$. Since $m(I_\infty(f) \cup \mathrm{Sing}^-(f)) = 0$, it suffices to show that

$$m(E \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f))) = 0.$$

Denote by Z the set of all points $z \in E \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f)))$ such that

$$\lim_{r\to 0} \frac{m(B(z,r) \cap (E \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f))))}{m(B(z,r))} = 1. \tag{4.34}$$

In view of the Lebesgue density theorem (see for example Theorem 2.9.11 in [Fe]), $m(Z) = m(E)$. Since $m(E) > 0$ we find at least one point $z \in Z$. Since $z \in J(f) \setminus (I_\infty(f) \cup \mathrm{Sing}^-(f))$, let $x \in J(f)$, $\eta(z) > 0$, and an increasing sequence $\{n_k\}_{k=1}^\infty$ be given by Proposition 2.23. Put

$$\delta = \eta(z)/8.$$

Suppose that $m(B(x,\delta) \setminus E) = 0$. By conformality of m, $m(f(Y)) = 0$ for all Borel sets Y such that $m(Y) = 0$. Hence,

$$\begin{aligned} 0 = m\Big(f^n(B(x,\delta) \setminus E)\Big) &\geq m\Big(f^n(B(x,\delta)) \setminus f^n(E)\Big) \\ &\geq m\Big(f^n(B(x,\delta)) \setminus E\Big) \geq m\Big(f^n(B(x,\delta)\Big) - m(E) \end{aligned} \tag{4.35}$$

for all $n \geq 0$. Since $J(f) = \overline{\bigcup_{n\geq 1} f^{-n}(\infty)}$, for some $p \geq 2$, the image $f^{p-1}(B(x,\delta))$ contains an open neighbourhood of ∞. Thus, it contains at least one (in fact infinitely many) copy of the fundamental parallelogram $\mathcal{R}$ and consequently $f^p(B(x,\delta)) = \overline{\mathbb{C}}$. In particular $m\Big(f^p(B(x,\delta))\Big) = 1$. Then (4.35) implies that $0 \geq 1 - m(E)$ which is a contradiction. Consequently $m(B(x,\delta) \setminus E) > 0$. Hence for every $j \geq 1$ large enough, $m\Big(B(f^{n_j}(z), 2\delta) \setminus E\Big) \geq m\Big(B(x,\delta) \setminus E\Big) > 0$. Therefore, as $f^{-1}(J(f) \setminus E) \subset J(f) \setminus E$, the standard application of Koebe's Distortion Theorem shows that

$$\limsup_{r\to 0} \frac{m(B(z,r) \setminus E)}{m(B(z,r))} > 0$$

which contradicts (4.34). Thus either $m(E) = 0$ or $m(E) = 1$.

Now conservativity is straightforward. One needs to prove that for every Borel set $B \subset J(f)$ with $m(B) > 0$ one has $m(G) = 0$, where

$$G = \{x \in J(f) : \sum_{n\geq 0} \chi_B(f^n(x)) < +\infty\}.$$

Indeed, suppose that $m(G) > 0$ and for all $n \geq 0$ let

$$G_n = \{x \in J(f) : \sum_{k \geq n} \chi_B(f^n(x)) = 0\} = \{x \in J(f) : f^k(x) \notin B \text{ for all } k \geq n\}.$$

Since $G = \bigcup_{n \geq 0} G_n$, there exists $k \geq 0$ such that $m(G_k) > 0$. Since all the sets G_n are forward invariant we conclude that $m(G_k) = 1$. But on the other hand all the sets $f^{-n}(B)$, $n \geq k$, are of positive measure and are disjoint from G_k. This contradiction finishes the proof of conservativity of m. Consequently $m(\mathrm{Tr}(f)) = 1$. We are done. ■

The proof of part (a) of Theorem 4.1. Let m be the unique h-conformal atomless measure proven to exist in Theorem 4.23. Consider an arbitrary point $z \in \mathrm{Tr}(f)$. Fix a pole $b \in f^{-1}(\infty)$. Since $b \notin \overline{O_+(\mathrm{Crit}(f))}$, there exists $\gamma > 0$ such that

$$B(b, \gamma) \cap O_+(\mathrm{Crit}(f)) = \emptyset. \tag{4.36}$$

Since $z \in \mathrm{Tr}(f)$, there exists an infinite increasing sequence $\{n_j\}_{j=0}^{\infty}$ such that

$$\lim_{j \to \infty} f^{n_j}(z) = b \text{ and } |f^{n_j}(z) - b| < \gamma/4 \tag{4.37}$$

for every $j \geq 1$. It follows from this and (4.36) that for every $j \geq 1$ there exists a holomorphic inverse branch $f_z^{-n_j} : B(f^{n_j}(z), 3\gamma/4) \to \mathbb{C}$ of f^{n_j} sending $f^{n_j}(z)$ to z. Using now Koebe's Distortion Theorem (Euclidean version) and Lemma 4.21, we conclude that

$$\begin{aligned}
m_e\Big(z, B\Big(K|(f^{n_j})'(z)|^{-1}2|f^{n_j}(z) - b|\Big)\Big) &\geq m_e\Big(B\Big(f^{n_j}(z), 2|f^{n_j}(z) - b|\Big)\Big)|(f^{n_j})'(z)|^{-h} \\
&\geq m_e\Big(B(b, |f^{n_j}(z) - b|)\Big)|(f^{n_j})'(z)|^{-h} \\
&\succeq |f^{n_j}(z) - b|^{(q_b+1)h - 2q_b}|(f^{n_j})'(z)|^{-h} \\
&= \Big(K|(f^{n_j})'(z)|^{-1}|f^{n_j}(z) - b|\Big)^h K^{-h}|f^{n_j}(z) - b|^{q_b(h-2)}.
\end{aligned}$$

Since $h < 2$, using (4.37), this implies that $\overline{\lim}_{r \to 0} r^{-h} m_e(B(z, r)) = \infty$. Hence $\mathrm{H}^h(\mathrm{Tr}(f)) = 0$ in view of Theorem 3.1. Since by Theorem 4.23 $m_e(J(f) \setminus \mathrm{Tr}(f)) = 0$, it follows from Lemma 3.3 that $\mathrm{H}^h(J(f) \setminus \mathrm{Tr}(f)) = 0$. In conclusion $\mathrm{H}^h(J(f)) = 0$ and the proof is complete. ■

Proposition 4.24. *The conformal measure m is absolutely continuous with respect to the packing measure Π^h and moreover, the Radon-Nikodym derivative $dm/d\Pi^h$ is uniformly bounded away from infinity. In particular $\Pi^h(J(f)) > 0$.*

Proof. Since $J(f) \cap \omega\Big(\mathrm{Crit}(f) \setminus \mathrm{Crit}(J(f))\Big) = \Omega(f)$, we conclude from Lemma 2.14 that there exists $y \in J(f)$ at a positive distance, say 8η, from $O_+(\mathrm{Crit}(f))$. Fix $z \in \mathrm{Tr}(f)$. Then there exists an infinite sequence $n_j \geq 1$ of increasing integers such that $f^{n_j}(z) \in B(y, \eta)$. Therefore $B(f^{n_j}(z), 4\eta) \cap O_+(\mathrm{Crit}(f)) = \emptyset$ and consequently

$$\mathrm{Comp}(z, f^{n_j}(z), f^{n_j}, \eta/2) \cap \mathrm{Crit}(f^{n_j}) = \emptyset$$

Hence, it follows from Lemma 1.2 and Lemma 4.8 that

$$\liminf_{r\to 0}\frac{m_e(B(z,r))}{r^h}\le B$$

for some constant $B\in(0,\infty)$ and all $z\in \mathrm{Tr}(f)$. Applying Lemma 4.17 we therefore get that

$$\liminf_{r\to 0}\frac{m(B_s(z,r))}{r^h}\le 2^hB.$$

Hence, by Theorem 3.2(1), the measure $m|_{\mathrm{Tr}(f)}$ is absolutely continuous with respect to $\Pi^h|_{\mathrm{Tr}(f)}$. Since, by Theorem 4.23, $m(J(f)\setminus \mathrm{Tr}(f))=0$, we are done. ■

Lemma 4.25. *If* $\Omega(f)\ne\emptyset$, *then* $\Pi^h(J(f))=+\infty$.

Proof. Fix $\xi\in\Omega$. Since $\bigcup_{n\ge 0}f^{-n}(\xi)$ is dense in $J(f)$ and, by Lemma 2.14, $\omega(\mathrm{Crit}(f))$ is nowhere dense in $J(f)$, there exist an integer $s>0$, a real number $\eta>0$, and a point $y\in f^{-s}(\xi)\setminus B\Big(\bigcup_{n\ge 0}f^n(\mathrm{Crit}(f)),\eta\Big)$. Since by Theorem 2.1, $h>1$, it follows from Lemma 2.5 and Lemma 4.11 (y may happen to be a critical point of f^s!) that

$$\liminf_{r\to 0}\frac{m_e(B(y,r))}{r^h}=0. \tag{4.38}$$

Consider now a transitive point $z\in J(f)$, i.e. $z\in \mathrm{Tr}(f)$. Then there exists an infinite increasing sequence $n_j=n_j(z)\ge 1$ of positive integers such that

$$\lim_{j\to\infty}|f^{n_j}(z)-y|=0 \quad\text{ and }\quad r_j=|f^{n_j}(z)-y|<\eta/7$$

for every $j=1,2,\ldots$. By the choice of y, for all $j\ge 1$ there exist holomorphic inverse branches $f_z^{-n_j}:B(f^{n_j}(z),6r_j)\to\overline{\mathbb{C}}$ sending $f^{n_j}(z)$ to z. So, applying Lemma 1.2 and Lemma 4.8 with $R=3r_j$, we conclude from (4.38) that

$$\liminf_{r\to 0}\frac{m_e(B(z,r))}{r^h}=0.$$

Applying Lemma 4.17, we conclude that the same formulas remain true with m_e replaced by m and $B(z,r)$ by $B_s(z,r)$. Therefore, it follows from Theorem 4.23 ($m(\mathrm{Tr}(f))=1$) and Theorem 3.2(1) that $\Pi^h(J(f))=+\infty$. We are done. ■

From now on let m denote the unique atomless h-conformal measure m proven to exist in Theorem 4.23.

Recall that the numbers $R_l(f)$ and $A_l(f)$ have been defined by formulas (4.21) and (4.22) respectively. Since the number of equivalence classes of the relation $\sim$ is finite, looking at Lemma 2.22 and Lemma 4.13, the following lemma (where the superscript l indicates that we mean the “lower” estimates) follows immediately from Lemma 4.12.

Lemma 4.26. *If $C_i^{(l)} > 0$, $0 < R_i^{(l)} \le R_l(f)/3$, and $0 < \sigma \le 1$ are three real numbers such that all points $z \in \mathrm{PC}(f)_i$ are $(r, \sigma, C_i^{(l)})$-h-s.l.e. with respect to the measure m_e, then there exists $\tilde{C}_i^{(l)} > 0$ such that all critical points $c \in Cr_{i+1}(f)$ are $(r, \tilde{\sigma}, \tilde{C}_i^{(l)})$-$h$-s.l.e. with respect to the measure m_e for all $0 < r \le A_l(f)^{-1} R_i^{(l)}$, where $\tilde{\sigma}$ was defined in Lemma 4.12.*

Let us prove the following.

Lemma 4.27. *Suppose that $\Omega(f) = \emptyset$. Assume that $C_{i,1}^{(l)} > 0$, $R_{i,1}^{(l)} > 0$ and $0 < \sigma \le 1$ are three real numbers such that all critical points $c \in S_i(f)$ are $(r, \sigma, C_{i,1}^{(l)})$-$h$-s.l.e. with respect to the measure m_e for all $0 < r \le R_{i,1}^{(l)}$. Then there exist $\tilde{C}_{i,1}^{(l)} > 0$, $\tilde{R}_{i,1}^{(l)} > 0$ and such that all points $z \in \mathrm{PC}(f)_i$ are $(r, 8K^3A^2 2^{\#(\mathrm{Crit}(f)\cap\mathcal{R})}\sigma, \tilde{C}_{i,1}^{(l)})$-$h$-s.l.e. with respect to the measure m_e for all $0 < r \le \tilde{R}_{i,1}^{(l)}$.*

Proof. Recall that by Lemma 4.18 the set $\mathrm{PC}(f)$ is f-pseudo-compact. We shall show that this time one can take

$$\tilde{R}_{i,1}^{(l)} = \min\{\tau\theta ||f'||_F^{-1}\lambda^{-1}, R_{i,1}^{(l)}, 1\} \quad \text{and} \quad \tilde{C}_{i,1}^{(l)} = (8(KA^2)2^{\#})^h C_{i,1}^{(l)},$$

where $||f'|| = ||f'||_{\mathrm{PC}(f)_i}$. Indeed, denote again $\#(\mathrm{Crit}(f) \cap \mathcal{R})$ by $\#$. Take $\epsilon = 4K(KA^2)2^{\#}$ and then choose $\lambda > 0$ so large that

$$\epsilon < \lambda \min\left\{1, \tau^{-1}, \theta^{-1}\tau^{-1}\min\{\gamma, \rho, R_{i,1}^{(l)}/2\}\right\}. \tag{4.39}$$

Consider $0 < r \le \tilde{R}_{i,1}^{(l)}$ and $z \in \mathrm{PC}(f)_i$. If $z \in \mathrm{Crit}(J(f))$, then $z \in S_i(f)$ and we are done. Thus, we may assume that $z \notin \mathrm{Crit}(J(f))$. Let $s = s(\lambda, \epsilon, r, z)$. By the definition of ϵ

$$4Kr|(f^s)'(z)| = (KA^2)^{-1}2^{-\#}\epsilon r|(f^s)'(z)|. \tag{4.40}$$

Suppose first that $u(\lambda, r, z)$ is well defined and $s = u(\lambda, r, z)$. Then by Proposition 4.15(4.5) or Proposition 4.15(4.6), applied with $\eta = K$, we see that the point

$$f^s(z) \text{ is } (Kr\|(f^s)'(z)\|, \sigma/K^2, W_h(\sigma/K^2)) - h\text{-s.l.e.}.$$

Using (4.40) it follows from Proposition 4.16(4.13) and Lemma 4.9 that the point z is $(r, \sigma, W_h(\sigma/K^2))$-$h$-s.l.e.. If either u is not defined or $s \le u(\lambda, r, z)$, then in view of Proposition 4.16(4.12), there exists a critical point $c \in \mathrm{Crit}(f)$ such that $|f^s(z) - c| \le \epsilon r|(f^s)'(z)|$. Since $s \le u$, by Proposition 4.15 and (4.39) we get

$$4Kr|(f^s)'(z)| \le \epsilon r|(f^s)'(z)| < \epsilon\tau\theta\lambda^{-1}\min\{\rho, R_{i,1}^{(l)}/2\}. \tag{4.41}$$

Since $z \in \mathrm{PC}(f)_i$, it implies that $c \in S_i(f)$. Therefore, by the assumptions of Lemma 4.27 and by (4.41) we conclude that c is $(2\epsilon r|(f^s)'(z)|, \sigma, C_{i,1}^{(l)})$-$h$-s.l.e.. Consequently, in view of Lemma 4.5, the point $f^s(z)$ is $(\epsilon r|(f^s)'(z)|, 2\sigma, 2^h C_{i,1}^{(l)})$-$h$-s.l.e.. So, by Lemma 4.6 this point is

$$(Kr|(f^s)'(z)|, 2\sigma\epsilon/K, (2\epsilon K^{-1})^h C_{i,1}^{(l)}) - h\text{-s.l.e.}$$

Using now formula (4.40) and Proposition 4.16(4.13) it follows from Lemma 4.9 that the point z is $(r, 2K\epsilon\sigma, (2\epsilon K^{-1})^h C_{i,1}^{(l)})$-$h$-s.l.e.. If $z \in \mathrm{Crit}(J(f))$, then by the definition of $\mathrm{PC}(f)_i$ we see that $z \in S_i(f)$ and we are done in view of the assumption of the lemma and in view of the definitions of $\tilde{R}_{i,1}^{(l)}$ and $\tilde{C}_{i,1}^{(l)}$). The proof is completed. ■

Lemma 4.28. *If $\Omega(f) = \emptyset$, then $\Pi_e^h(F) < \infty$ for every bounded Borel set $F \subset \mathbb{C}$.*

Proof. Recall that for any pole b of f, the number q_b denotes its multiplicity. Let

$$q_{\min} = \min\{q_b : b \in f^{-1}(\infty)\}.$$

Take $\kappa \in (0,1)$ so small that if $z \in \mathbb{C}$, then $f|_{B(z,d)}$ is 1-to-1 for every $d \le \kappa \mathrm{dist}(z, \mathrm{Crit}(f) \cup f^{-1}(\infty))$. Using induction on $i = 0, 1, \dots, p$, it follows immediately from Lemma 4.27 (this lemma provides the base of induction as $S_0(f) = \emptyset$ and simultaneously contributes to the inductive step), Lemma 4.26, and Lemma 2.21 that each point $z \in \mathrm{PC}(f)$ is (r, σ, G)-h-s.l.e. for some $\sigma \in (0,1)$, $G > 0$, $R > 0$ and all $r \in (0, R)$. Without loss of generality we may assume $R \in (0,1)$ to be so small that

$$5R < \mathrm{dist}(\mathrm{PC}(f), f^{-1}(\infty)) \tag{4.42}$$

and

$$\xi^{-1}|z - b|^{-q_b} \le |f(z)| \le \xi |z - b|^{-q_b} \tag{4.43}$$

for all $b \in f^{-1}(\infty)$, all $z \in B(b, R)$ and some $\xi \ge 1$. Fix a point $z \in J(f) \setminus \mathrm{Sing}^-(f)$ and $r \in (0, R)$. In view of Corollary 2.24 there exists the least $n \ge 1$ such that either

$$\mathrm{dist}(f^n(z), \mathrm{PC}(f)) \le 8(K\kappa)^{-1} r |(f^n)'(z)| \quad \text{or} \quad K^{-1} r |(f^n)'(z)| \ge \frac{1}{8}\kappa R.$$

There are the following three possibilities.

1^0

$$K^{-1} r |(f^n)'(z)| < \frac{1}{8}\kappa R.$$

This in particular implies that

$$\mathrm{dist}(f^n(z), \mathrm{PC}(f)) \le 8(K\kappa)^{-1} r |(f^n)'(z)|.$$

2^0

$$K^{-1} r |(f^n)'(z)| \ge \frac{1}{8}\kappa R \quad \text{and} \quad \mathrm{dist}(f^n(z), \mathrm{PC}(f)) > 8(K\kappa)^{-1} r |(f^n)'(z)|.$$

3^0

$$K^{-1} r |(f^n)'(z)| \ge \frac{1}{8}\kappa R \quad \text{and} \quad \mathrm{dist}(f^n(z), \mathrm{PC}(f)) \le 8(K\kappa)^{-1} r |(f^n)'(z)|.$$

Let us consider the case 1^0. Since

$$8(K\kappa)^{-1}r|(f^{n-1})'(z)| < \operatorname{dist}(f^{n-1}(z), \mathrm{PC}(f)), \tag{4.44}$$

we get

$$8K^{-1}r|(f^{n-1})'(z)| < \kappa\operatorname{dist}(f^{n-1}(z), \operatorname{Crit}(f)). \tag{4.45}$$

Suppose now that

$$8K^{-1}r|(f^{n-1})'(z)| \geq \kappa\operatorname{dist}(f^{n-1}(z), f^{-1}(\infty)).$$

This implies that there exists $b \in f^{-1}(\infty)$ such that

$$|f^{n-1}(z) - b| \leq 8(K\kappa)^{-1}r|(f^{n-1})'(z)| < R. \tag{4.46}$$

Hence

$$B(f^{n-1}(z), 32(K\kappa)^{-1}r|(f^{n-1})'(z)|) \subset B(b, 40(K\kappa)^{-1}r|(f^{n-1})'(z)|) \subset B(b, 5R).$$

In view of (4.42) this implies that $B(f^{n-1}(z), 32(K\kappa)^{-1}r|(f^{n-1})'(z)|) \cap \mathrm{PC}(f) = \emptyset$, and consequently there exists a unique holomorphic inverse branch

$$f_z^{-(n-1)} : B(f^{n-1}(z), 32(K\kappa)^{-1}r|(f^{n-1})'(z)|) \to \mathbb{C}$$

sending $f^{n-1}(z)$ to z. Since $8(K\kappa)^{-1}r|(f^{n-1})'(z)| \leq R$, it follows from Lemma 4.21 that

$$m_e(B(b, 8(K\kappa)^{-1}r|(f^{n-1})'(z)|)) \geq C(8(K\kappa)^{-1}r|(f^{n-1})'(z)|)^{(q_b+1)h-2q_b} \tag{4.47}$$

with some universal constant $C > 0$. Since, it follows from (4.46) that

$$B(f^{n-1}(z), 16(K\kappa)^{-1}r|(f^{n-1})'(z)|) \supset B(b, 8(K\kappa)^{-1}r|(f^{n-1})'(z)|),$$

applying (4.47) and Koebe's Distortion Theorem, I (Euclidean version), we obtain

$$\begin{aligned}
m_e(B(z, 16\kappa^{-1}r)) &\geq m_e(f_z^{-(n-1)}(B(f^{n-1}(z), 16(K\kappa)^{-1}r|(f^{n-1})'(z)|))) \\
&\geq K^{-h}|(f^{n-1})'(z)|^{-h} m_e(B(f^{n-1}(z), 16(K\kappa)^{-1}r|(f^{n-1})'(z)|)) \\
&\geq K^{-h}|(f^{n-1})'(z)|^{-h} m_e(B(b, 8(K\kappa)^{-1}r|(f^{n-1})'(z)|)) \\
&\geq CK^{-h}|(f^{n-1})'(z)|^{-h}(8(K\kappa)^{-1}r|(f^{n-1})'(z)|)^{(q_b+1)h-2q_b} \\
&\geq CK^{-h}(8(K\kappa)^{-1}r|(f^{n-1})'(z)|)^{q_b(h-2)}r^h \\
&\geq CK^{-h}R^{q(h-2)}r^h.
\end{aligned} \tag{4.48}$$

So, we may assume that

$$8K^{-1}r|(f^{n-1})'(z)| < \kappa\operatorname{dist}(f^{n-1}(z), f^{-1}(\infty)).$$

Along with (4.45) and the definition of κ, this implies that the map f restricted to the ball $B(f^{n-1}(z), 8K^{-1}r|(f^{n-1})'(z)|)$, is univalent. It therefore follows from Koebe's $\frac{1}{4}$-theorem that

$$f\Big(B(f^{n-1}(z), 8K^{-1}r|(f^{n-1})'(z)|)\Big) \supset B(f^n(z), 2K^{-1}r|(f^n)'(z)|). \tag{4.49}$$

Thus, there exists a unique holomorphic inverse branch $f_*^{-1} : B(f^n(z), 2K^{-1}r|(f^n)'(z)|) \to B(f^{n-1}(z), 8K^{-1}r|(f^{n-1})'(z)|)$ of f sending $f^n(z)$ to $f^{n-1}(z)$. Since

$$B(f^{n-1}(z), 8K^{-1}r|(f^{n-1})'(z)|) \cap \mathrm{PC}(f) = \emptyset$$

there exists a unique holomorphic inverse branch

$$f_z^{-(n-1)} : B(f^{n-1}(z), 8K^{-1}r|(f^{n-1})'(z)|) \to \mathbb{C}$$

of f^{n-1} sending $f^{n-1}(z)$ to z. Therefore, the composition

$$f_z^{-n} = f_z^{-(n-1)} \circ f_*^{-1} : B(f^n(z), 2K^{-1}r|(f^n)'(z)|) \to \mathbb{C}$$

is a well-defined holomorphic inverse branch of f^n sending $f^n(z)$ to z. As $\mathrm{dist}(f^n(z), \mathrm{PC}(f)) < R$, since $K^{-1}r|(f^n)'(z)| < R$ and since each point $z \in \mathrm{PC}(f)$ is (r, σ, G)-h-s.l.e., we obtain that

$$m_e\Big(B(f^n(z), K^{-1}r|(f^n)'(z)|)\Big) \geq G(K^{-1}r|(f^n)'(z)|)^h.$$

Using now Koebe's Distortion Theorem, I (Euclidean version), we conclude that

$$\begin{aligned} m_e(B(z,r)) &\geq m_e\Big(f_z^{-n}\Big(B(f^n(z), K^{-1}r|(f^n)'(z)|)\Big)\Big) \\ &\geq K^{-h}|(f^n)'(z)|^{-h} m_e(B(f^n(z), K^{-1}r|(f^n)'(z)|)) \\ &\geq K^{-h}|(f^n)'(z)|^{-h} G^h K^{-h} r^h |(f^n)'(z)|^h = (GK^{-2})^h r^h. \end{aligned} \tag{4.50}$$

Let us now deal with the case 2^0. In this case the holomorphic inverse branch $f_z^{-n} : B(f^n(z), 2K^{-1}r|(f^n)'(z)|) \to \mathbb{C}$ of f^n sending $f^n(z)$ to z is well-defined. Using Koebe's distortion theorem and Lemma 4.13, we get

$$\begin{aligned} m_e(B(z,r)) &\geq m_e\Big(f_z^{-n}\Big(B(f^n(z), K^{-1}r|(f^n)'(z)|)\Big)\Big) \\ &\geq K^{-h}|(f^n)'(z)|^{-h} m_e(B(f^n(z), K^{-1}r|(f^n)'(z)|)) \\ &\geq K^{-h}|(f^n)'(z)|^{-h} C_h\left(\frac{1}{8}R\kappa\right)(K^{-1}r|(f^n)'(z)|)^h \\ &= C_h\left(\frac{1}{8}R\kappa\right) K^{-2h} r^h \end{aligned} \tag{4.51}$$

Case 3^0. Suppose first that

$$|f^{n-1}(z) - b| \leq \frac{1}{2}K^{-1}r|(f^{n-1})'(z)| \tag{4.52}$$

for some pole $b \in f^{-1}(\infty)$. The argument to be presented now is very similar to that used in the very first part of the Case 1^0. By (4.52) we get

$$B(f^{n-1}(z), K^{-1}r|(f^{n-1})'(z)| \supset B\left(b, \frac{1}{2}K^{-1}r|(f^{n-1})'(z)|\right). \tag{4.53}$$

Since $\frac{1}{2}K^{-1}r|(f^{n-1})'(z)| \leq \frac{1}{16}\kappa R < R$, similarly as (4.47), we obtain

$$m_e\Big(B(b, \frac{1}{2}K^{-1}r|(f^{n-1})'(z)|)\Big) \geq C\Big(\frac{1}{2}K^{-1}r|(f^{n-1})'(z)|\Big)^{(q_b+1)h-2q_b}. \tag{4.54}$$

Since $2K^{-1}r|(f^{n-1})'(z)| < 8K^{-1}\kappa^{-1}r|(f^{n-1})'(z)| \le \mathrm{dist}(f^{n-1}(z), \mathrm{PC}(f))$, we see that there exists a unique holomorphic inverse branch $f_z^{-(n-1)} : B(f^{n-1}(z), 2K^{-1}r|(f^{n-1})'(z)|) \to \mathbb{C}$ of f^{n-1} sending $f^{n-1}(z)$ to z. Therefore, applying (4.53), (4.54) and Koebe's Distortion Theorem, I (Euclidean version), we obtain

$$
\begin{aligned}
m_e(B(z,r)) &\ge m_e\Big(f_z^{-(n-1)}\Big(B(f^{n-1}(z), K^{-1}r|(f^{n-1})'(z)|)\Big)\Big) \\
&\ge K^{-h}|(f^{n-1})'(z)|^{-h} m_e\Big(B(f^{n-1}(z), K^{-1}r|(f^{n-1})'(z)|)\Big) \\
&\ge K^{-h}|(f^{n-1})'(z)|^{-h} m_e\Big(B(b, \frac{1}{2}K^{-1}r|(f^{n-1})'(z)|)\Big) \\
&\ge K^{-h}C|(f^{n-1})'(z)|^{-h}\Big(\frac{1}{2}K^{-1}r|(f^{n-1})'(z)|\Big)^{(q_b+1)h-2q_b} \\
&\ge CK^{-h}\Big(K^{-1}r|(f^{n-1})'(z)|\Big)^{q_b(h-2)} r^h \\
&\ge CK^{-h}\left(\frac{1}{8}\kappa R\right)^{q_{\max}(h-2)} r^h.
\end{aligned}
\tag{4.55}
$$

So, suppose finally that

$$|f^{n-1}(z) - b| > \frac{1}{2}K^{-1}r|(f^{n-1})'(z)|.$$

for all poles $b \in f^{-1}(\infty)$. Since also

$$\mathrm{dist}(f^{n-1}(z), \mathrm{PC}(f)) > 4K^{-1}\kappa^{-1}r|(f^{n-1})'(z)|, \tag{4.56}$$

we conclude that the map $f : \mathbb{C} \to \overline{\mathbb{C}}$, restricted to the ball $B(f^{n-1}(z), \frac{1}{2}\kappa K^{-1}r|(f^{n-1})'(z)|)$, is univalent. It therefore follows from Koebe's $\frac{1}{4}$-theorem that

$$f\Big(B(f^{n-1}(z), \frac{1}{2}\kappa K^{-1}r|(f^{n-1})'(z)|)\Big) \supset B\Big(\frac{1}{8}\kappa K^{-1}r|(f^n)'(z)|\Big).$$

Hence, there exists a unique holomorphic inverse branch $f_*^{-1} : B\Big(f^n(z), \frac{1}{8}\kappa K^{-1}r|(f^n)'(z)|\Big) \to B(f^{n-1}(z), \frac{1}{2}\kappa K^{-1}r|(f^{n-1})'(z)|)$ of f sending $f^n(z)$ to $f^{n-1}(z)$. In view of (4.56) there exists a unique holomorphic inverse branch $f_z^{-(n-1)} : B(f^{n-1}(z), \frac{1}{2}\kappa K^{-1}r|(f^{n-1})'(z)|) \to \mathbb{C}$ of f^{n-1} sending $f^{n-1}(z)$ to z. Hence, the composition

$$f_z^{-n} = f_z^{-(n-1)} \circ f_*^{-1} : B\Big(f^n(z), \frac{1}{8}\kappa K^{-1}r|(f^n)'(z)|\Big) \to \mathbb{C}$$

is a well-defined holomorphic inverse branch of f^n sending $f^n(z)$ to z. Since $\frac{1}{16}\kappa K^{-1}r|(f^n)'(z)| > 2^{-7}\kappa^2 R$, applying Koebe's Distortion Theorem, I (Euclidean version) and Lemma 4.13, we get

$$
\begin{aligned}
m_e(B(z,r)) &\ge m_e\Big(f_z^{-n}\Big(B(f^n(z),\frac{1}{16}\kappa K^{-1}r|(f^n)'(z)|)\Big)\Big)\\
&\ge K^{-h}|(f^n)'(z)|^{-h}m_e\Big(B(f^n(z),(16K)^{-1}\kappa r|(f^n)'(z)|)\Big)\\
&\ge K^{-h}C_h(2^{-7}\kappa^2R)|(f^n)'(z)|^{-h}\Big((16K)^{-1}\kappa r|(f^n)'(z)|\Big)^h\\
&= (16)^{-1}K^{-2h}\kappa C_h(2^{-7}\kappa^2R)r^h.
\end{aligned}
$$

Combining this inequality and (4.48), (4.50), (4.51), (4.55), we see that $m_e(B(z,16\kappa^{-1}r)) \ge \hat{C}r^h$ for all $r \in (0,R)$ and some universal connstant $\hat{C} > 0$. Hence, for all $r \in (0,R)$, we obtain $m_e(B(z,r)) = m_e(B(z,16\kappa^{-1}(2^{-4}\kappa r)) \ge \hat{C}(2^{-4}\kappa r)^h = \hat{C}(2^{-4}\kappa r)^h r^{-h}$. Thus, applying Theorem 3.2(2), we see that $\Pi_e^h(F) < \infty$. We are done. ■

Our last lemma in this section is this.

Lemma 4.29. *If $\Omega(f) = \emptyset$, then the spherical packing measure $\Pi^h(J(f))$ is finite.*

Proof. Since the packing measure Π^h is Λ-invariant (recall that Λ is the lattice associated with our elliptic function $f : \mathbb{C} \to \overline{\mathbb{C}}$), it follows from Lemma 4.28 and Proposition 4.24 that $\Pi_e^h\Big(J(f)\cap(B(0,2R)\setminus B(0,R)\Big) \preceq R^2$ for all $R \ge 1$. Since in addition $\frac{d\Pi^h}{d\Pi_e^h}(z) = (1+|z|^2)^{-h}$ and since $h > 1$, we get

$$
\begin{aligned}
\Pi^h\Big(J(f)\cap(\mathbb{C}\setminus B(0,1))\Big) &= \sum_{n=0}^\infty \Pi^h\Big(J(f)\cap(B(0,2^{n+1})\setminus B(0,2^n))\Big)\\
&\asymp \sum_{n=0}^\infty 2^{-2hn}\Pi_e^h\Big(J(f)\cap(B(0,2^{n+1})\setminus B(0,2^n))\Big)\\
&\preceq \sum_{n=0}^\infty 2^{-2hn}2^{2n} = \sum_{n=0}^\infty 2^{(2-2h)n} < \infty.
\end{aligned}
$$

We are done. ■

The proof of Theorem 4.1 is therefore complete. ■

5. Invariant Measures

In this section we deal with σ-finite invariant measures equivalent to the conformal measure m. We prove their existence, ergodicity, conservativity and we detect the points around which these measures are finite or infinite. This allows us to provide sufficient conditions for their finiteness.

5.1. **σ-finite invariant measures equivalent to the conformal measure m.** In order to prove Theorem 5.2 below we apply a general sufficient condition for the existence of σ-finite absolutely continuous invariant measure proven in [19]. In order to formulate this condition suppose that X is a σ-compact metric space, ν is a Borel probability measure on X, positive on open sets, and that a measurable map $f : X \to X$ is given with respect to which measure ν is quasi-invariant, i.e. $\nu \circ f^{-1} << \nu$. Moreover we assume the existence of a countable partition $\alpha = \{A_n : n \geq 0\}$ of subsets of X which are all σ-compact and of positive measure ν. We also assume that $\nu(X \setminus \bigcup_{n\geq 0} A_n) = 0$, and if additionally for all $m, n \geq 1$ there exists $k \geq 0$ such that

$$\nu(f^{-k}(A_m) \cap A_n) > 0,$$

then the partition α is called irreducible. Martens' result comprising Proposition 2.6 and Theorem 2.9 of [19] reads as follows.

Theorem 5.1. *Suppose that $\alpha = \{A_n : n \geq 0\}$ is an irreducible partition for $T : X \to X$. Suppose that T is conservative and ergodic with respect to the measure ν. If for every $n \geq 1$ there exists $K_n \geq 1$ such that for all $k \geq 0$ and all Borel subsets A of A_n*

$$K_n^{-1}\frac{\nu(A)}{\nu(A_n)} \leq \frac{\nu(f^{-k}(A))}{\nu(f^{-k}(A_n))} \leq K_n\frac{\nu(A)}{\nu(A_n)},$$

then T has a σ-finite T-invariant measure μ that is absolutely continuous with respect to ν. In addition, μ is equivalent to ν, conservative and ergodic, and unique up to a multiplicative constant. Moreover, for every Borel set $A \subset X$

$$\mu(A) = \lim_{n\to\infty}\frac{\sum_{k=0}^{n}\nu(f^{-k}(A))}{\sum_{k=0}^{n}m(f^{-k}(A_0))}.$$

The first result of this section is the following.

Theorem 5.2. *There exists a σ-finite f-invariant measure μ that is absolutely continuous with respect to the h-conformal measure m. In addition, μ is equivalent to m and ergodic.*

Proof. Let $\xi \in \mathbb{C}$ be a periodic point of f with some period $p \geq 3$. We put

$$P_3(f) = \overline{O_+(f(\mathrm{Crit}(f)))} \cup \{\xi, f(\xi), \dots, f^{p-1}(\xi)\}.$$

Since $\overline{O_+(f(\mathrm{Crit}(f))}$ is a forward-invariant nowhere-dense subset of $J(f)$ and since the h-conformal measure m is positive on nonempty open subsets of $J(f)$, it follows from ergodicity and conservativity of m (see Theorem 4.23) that $m(\overline{O_+(f(\mathrm{Crit}(f)))}) = 0$. Since m has no atoms (see Theorem 4.23) we therefore obtain that $m(P_3(f)) = 0$. We shall now construct the partition α of the set $J(f) \setminus P_3(f)$. We shall check next that it satisfies the assumptions of Theorem 5.1. We first define the family of balls

$$\left\{B\left(z, \frac{1}{2}\mathrm{dist}(z, P_3(f))\right)\right\}_{z\in\mathbb{C}\setminus P_3(f)}.$$

This family obviously covers $\mathbb{C}\setminus P_3(f)$. Since $\mathbb{C}\setminus P_3(f)$ is an open set, it is a Lindelöf space, and therefore we can choose a countable subcover of $\mathbb{C}\setminus P_3(f)$, which we denote by

$$\left\{B\left(z_i, \frac{1}{2}\mathrm{dist}\Big(z_i, P_3(f)\Big)\right)\right\}_{i=1}^{\infty}.$$

We inductively define a partition $\mathcal{A} = \{A_i\}_{i=0}^{\infty}$ of $\mathbb{C}\setminus P_3(f)$ as follows. Let

$$A_0 = B\left(z_0, \frac{1}{2}\mathrm{dist}(z_0, P_3(f))\right).$$

Assume that we have defined the sets $A_1, \dots, A_n$ such that

$$A_j \subset B\left(z_j, \frac{1}{2}\mathrm{dist}(z_j, P_3(f))\right)$$

and

$$\mathrm{Int}A_j \neq \emptyset.$$

Then A_{n+1} we define as

$$A_{n+1} = B\left(z_{n+1}, \frac{1}{2}\mathrm{dist}(z_{n+1}, P_3(f))\right) \setminus \bigcup_{j=1}^{n} A_j.$$

The set A_{n+1} is disjoint from the sets $A_1, \dots, A_n$ and

$$A_{n+1} \subset B\left(z_{n+1}, \frac{1}{2}\mathrm{dist}(z_{n+1}, P_3(f))\right) \setminus \bigcup_{j=1}^{n} B\left(z_j, \frac{1}{2}\mathrm{dist}(z_j, P_3(f))\right).$$

Thus either $A_{n+1} = \emptyset$ or $\mathrm{Int}A_{n+1} \neq \emptyset$ and we remove all the empty sets.
We shall now check that the partition is irreducible. And indeed, it follows from the construction of the sets $\{A_i\}_{i=0}^{\infty}$ and continuity of the measure m that it suffices to demonstrate that if $z \in \mathbb{C}$, $r > 0$ and $K \subset \mathbb{C}$ is a compact set, then there exists $n \geq 1$ such that

$$f^n\left(B(z,r) \setminus \bigcup_{k\geq 0} f^{-k}(\infty)\right) \supset K \setminus \bigcup_{k\geq 0} f^{-k}(\infty).$$

Since the set of repelling periodic points is dense in the Julia set ([2], comp. [6]), there thus exists a periodic point $x \in B(z,r)$, say of period $q \geq 1$. Since x is repelling there exists $s > 0$ so small that $B(x,s) \subset B(z,r)$ and $f^q(B(x,s)) \supset B(x,s)$. Since $\bigcup_{j\geq 1} f^{qj}(B(x,s)) \supset \mathbb{C}$, since K is a compact subset of $\mathbb{C}$ and since $\{f^{qj}(B(x,s))\}_{j=1}^{\infty}$ is an increasing family of open sets, there thus exists $k \geq 1$ such that $f^{qk}(B(x,s)) \supset K$.
Let us check now the distortion assumption of Theorem 5.1. And indeed, in view of Koebe's distortion theorem there exists a constant $K \geq 1$ such that if $f_*^{-n} : B\Big(z_i, \mathrm{dist}(z_i, P_3(f))\Big) \to \mathbb{C}$ is a holomorphic branch of f^{-n}, then for every $k \geq 0$ and all $x, y \in A_k \subset B\left(z_i, \frac{1}{2}\mathrm{dist}\,(z_i, P_3(f))\right)$ we have

$$\frac{|(f_*^{-n})'(y)|}{|(f_*^{-n})'(x)|} \leq K. \tag{5.1}$$

We therefore obtain for all Borel sets $A, B \subset A_k$ with $m(B) > 0$ and all $n \geq 0$ that

$$\frac{m(f_*^{-n}(A))}{m(f_*^{-n}(B))} = \frac{\int_A |(f_*^{-n})'|^h dm}{\int_A |(f_*^{-n})'|^h dm} \leq \frac{\sup_{A_k}\{|(f_*^{-n})'|^h\} m(A)}{\inf_{A_k}\{|(f_*^{-n})'|^h\} m(B)} \leq K^h \frac{m(A)}{m(B)}.$$

and similarly

$$\frac{m(f_*^{-n}(A))}{m(f_*^{-n}(B))} \geq K^{-h} \frac{m(A)}{m(B)}.$$

Since by Theorem 4.23 the measure is conservative ergodic, all the assumptions of Theorem 5.1 have been checked and we are done. ■

The following lemma easily follows from Theorem 5.1.

Lemma 5.3. *For every $n \geq 0$ we have $0 < \mu(A_n) < \infty$.*

We say that the f-invariant measure μ produced in Theorem 5.2 is of finite condensation at $x \in J(f)$ if and only if there exists an open neighborhood V of x such that $\mu(V) < \infty$. Otherwise μ is said to be of infinite condensation at x. We respectively say that x is a point of finite or infinite condensation of μ. We end this subsection with the following obvious results.

Lemma 5.4. *If x is a point of infinite condensation of μ, then each point of the closure $\overline{\{f^n(x) : n \geq 0\}}$ is also of infinite condensation of μ.*

Lemma 5.5. *The set of points of infinite condensation of the measure μ is contained in the union $\overline{O_+(\mathrm{Crit}(f))} \cup \Omega \cup \{\infty\}$.*

Proof. If $z \notin \overline{O_+(\mathrm{Crit}(f))} \cup \Omega \cup \{\infty\}$, then by local finiteness of the family $\{A_n : n \geq 0\}$ there exist an open neighborhood V of z and an integer $k \geq 0$ such that $m\Big(V \setminus \bigcup_{j=0}^k A_j\Big) = 0$. Hence, in view of Lemma 5.3 and Theorem 5.2 ($\mu \prec m$) we get $\mu(V) \leq \sum_{j=0}^k \mu(A_j) < \infty$. The proof is finished. ■

5.2. ∞ **is a Point of Finite Condensation of** μ**.**
Recall given $R > 0$

$$B_R = \{z \in \overline{\mathbb{C}} : |z| > R\}$$

and given in addition a pole $b \in f^{-1}(\infty)$ by $B_b(R)$ we denote the connected component of $f^{-1}(B_R)$ containing b. The goal of this subsection is to prove that ∞ is a point of finite condensation of the measure μ. We start with the following.

Lemma 5.6. *For every $R > 1$ large enough there exists a constant $C_1(R) > 0$ such that $m(B_b(R)) \leq C_1(R)\mathrm{diam}_s^h(B_b(R))$.*

Proof. Recall that for every pole $b \in f^{-1}(\infty)$, the number q_b stands for its multiplicity. For every $k \geq 0$ let $A_{k,R} = \{z \in \mathbb{C} \colon 2^k R \leq |z| < 2^{k+1}R\}$. As in the proof of Lemma 3.4 let

$$B_R^+ = \{z \in B_R \setminus \{\infty\} : \mathrm{Im} z > 0\} \text{ and } B_R^1 = \{z \in B_R \setminus \{\infty\} : \mathrm{Im} z < 1\} \text{ and } .$$

We also put $A_{k,R}^+ = A_{k,R} \cap B_R^+$ and $A_{k,R}^- = A_{k,R} \cap B_R^-$. Using formula (2.3) we can write for all $b \in f^{-1}(\infty)$, all $j \in \{1, \dots, q_b\}$ and all $k \geq 0$ that

$$m(f_{b,B_R^+,j}^{-1}(A_{k,R}^+)) = \int_{A_{k,R}^+} |(f_{b,B_R^+,j}^{-1})^*|^h dm \asymp (1+|b|^2)^{-h}(2^kR)^{\frac{q_b-1}{q_b}h} m(A_{k,R}^+)$$

and similarly

$$m(f_{b,B_R^+,j}^{-1}(A_{k,R}^-)) \asymp (1+|b|^2)^{-h}(2^kR)^{\frac{q_b-1}{q_b}h} m(A_{k,R}^-)$$

Thus

$$m(f_{b,R,j}^{-1}(A_{k,R})) = m(f_{b,R,j}^{-1}(A_{k,R}^+)) + m(f_{b,R,j}^{-1}(A_{k,R}^-)) \asymp (1+|b|^2)^{-h}(2^kR)^{\frac{q_b-1}{q_b}h} m(A_{k,R}).$$

Summing now over all $j \in \{1, \dots, q_b\}$, we get

$$m(A_{k,R,b}) \asymp (1+|b|^2)^{-h}(2^kR)^{\frac{q_b-1}{q_b}h} m(A_{k,R}) \tag{5.2}$$

where $A_{k,R,b} = B_b(R) \cap f^{-1}(A_{k,R})$. Therefore, putting $S = \sum_{w \in \Lambda}(1+|w|^2)^{-h} < \infty$ (since by Theorem 2.1 $h > 1$), we obtain

$$\begin{aligned}
m(f^{-1}(A_{k,R})) &= \sum_{b \in f^{-1}(\infty)} m(A_{k,R,b}) \\
&= \sum_{b \in \mathcal{R} \cap f^{-1}(\infty)} \sum_{w \in \Lambda} m(A_{k,R,b+w}) \\
&\asymp \sum_{b \in \mathcal{R} \cap f^{-1}(\infty)} \sum_{w \in \Lambda} \left(1+|b+w|^2\right)^{-h} (2^kR)^{\frac{q_b-1}{q_b}h} m(A_{k,R}) \\
&\asymp m(A_{k,R}) \sum_{b \in \mathcal{R} \cap f^{-1}(\infty)} (2^kR)^{\frac{q_b-1}{q_b}h} \sum_{w \in \Lambda} (1+|b+w|^2)^{-h} \\
&\asymp m(A_{k,R}) S (2^kR)^{\frac{q-1}{q}h}.
\end{aligned}$$

Hence $m(A_{k,R}) \asymp (2^kR)^{\frac{1-q}{q}h} S^{-1} m(f^{-1}(A_{k,R}))$ where $q = \max\{q_b : b \in \mathcal{R} \cap f^{-1}(\infty)\}$. Combining this and (5.2), we get for every $b \in f^{-1}(\infty)$ that

$$\begin{aligned}
m(A_{k,R,b}) &\asymp (1+|b|^2)^{-h}(2^kR)^{(1-\frac{1}{q})h}(2^kR)^{(\frac{1}{q}-1)h} S^{-1} m(f^{-1}(A_{k,R})) \\
&\leq (1+|b|^2)^{-h} S^{-1} m(f^{-1}(A_{k,R})) \asymp (1+|b|^2)^{-h} m(f^{-1}(A_{k,R}))
\end{aligned}$$

Summing now over all $k \geq 0$ we get $m(B_b(R)) \preceq (1+|b|^2)^{-h} m(f^{-1}(B_R)) \leq (1+|b|^2)^{-h}$. Combining in turn this with (2.4) we get

$$m(B_b(R)) \preceq L^h R^{\frac{h}{q}} \mathrm{diam}_s^h(B_b(R)) \tag{5.3}$$

The proof is complete. ∎

Lemma 5.7. *Fix $R > 2$ sufficiently large. Re-numerating the elements of the partition $\{A_j\}_{j=0}^{\infty}$, we may assume that $A_0 \subset B_R$ and $\mathrm{diam}_s(A_0) = 1$. For every $b \in f^{-1}(\infty)$ and every $n \geq 0$ let $A^{(n)} = f^{-n}(A_0) \cap B_n$, where B_n is a connected component of $f^{-n}(B_R)$. Then there exists a constant $C_2 > 0$ such that $m(B_n) \leq C_2(R)m(A^{(n)})$.*

Proof. It follows from the construction of the partition $\{A_n\}_{n\geq 0}$ that

$$m(A^{(n)}) \asymp \mathrm{diam}_s^h(A^{(n)}) \tag{5.4}$$

Since $\mathrm{dist}(0, A_0) \geq R > 2$ and since $\mathrm{diam}(A_0) = 1$ using (2.3), and (2.4), we get for every pole $b \in f^{-1}(\infty)$ that

$$\frac{\mathrm{diam}_s(A_{0,b})}{\mathrm{diam}_s(B_b(R))} \asymp \frac{(1 + |b|^2)^{-1}\mathrm{dist}(0, A_0)^{\frac{q_b-1}{q_b}}\mathrm{diam}_s(A_0)}{(1 + |b|^2)^{-1}R^{\frac{-1}{q_b}}} \geq R^{\frac{q_b-1}{q_b}}R^{\frac{1}{q_b}} = 1, \tag{5.5}$$

where $A_{0,b} = f^{-1}(A_0) \cap B_b(R)$. Since $\omega(\mathrm{Crit}(f))$ is a compact subset of the complex plane $\mathbb{C}$, $\mathrm{dist}(\omega(\mathrm{Crit}(f)), f^{-1}(\infty)) > 0$. Therefore there exists $r > 0$ such that for all $R > 1$ large enough $B_b(R) \subset B(b, r)$ and $B(b, 2r) \cap O_+(\mathrm{Crit}(f)) = \emptyset$. Since $B_n = f_*^{-(n-1)}(B_b(R))$ for an appropriate holomorphic inverse branch $f_*^{-(n-1)} : B(b, 2r) \to \mathbb{C}$ of $f^{(n-1)}$, it follows from Koebes's distortion theorem and (5.5)

$$\frac{\mathrm{diam}_s(A^{(n)})}{\mathrm{diam}_s(B_n)} = \frac{\mathrm{diam}_s(f_*^{-(n-1)}(A_{0,b}))}{\mathrm{diam}_s(f_*^{-(n-1)}(B_b(R))} \asymp \frac{\mathrm{diam}_s(A_{o,b})}{\mathrm{diam}_s(B_b(R))} \succeq 1$$

and that

$$\frac{\mathrm{diam}_s^h(B_n)}{m(B_n)} = \frac{\mathrm{diam}_s^h(f_*^{-(n-1)}(B_b(R)))}{m(f_*^{-(n-1)}(B_b(R)))} \asymp \frac{\mathrm{diam}_s^h(B_b(R))}{m(B_b(R))}.$$

Combining the last two formulas and (5.4) we get

$$m(A^{(n)}) \succeq \mathrm{diam}_s^h(B_n) \asymp \left(\frac{\mathrm{diam}_s^h(B_b(R))}{m(B_b(R))}\right) m(B_n) \succeq m(B_n)$$

The proof is complete. ■

We are ready now to prove the main result of this section.

Theorem 5.8. *∞ is a point of finite condensation of the measure μ.*

Proof. Take $R > 0$ so large as required in Lemma 5.7. It follows from this lemma that $m(f^{-k}(B_R)) \leq C_2(R)m(f^{-k}(A_0))$ for every $k \geq 0$. Thus, applying Theorem 5.1 , we get

$$\mu(B_R) = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} m(f^{-k}(B_R))}{\sum_{k=0}^{n} m(f^{-k}(A_0))} \leq C_2(R) < \infty.$$

We are done. ■

5.3. **All Points of Finite and Infinite Condensation.** We say that $z \in J(f) \setminus \Omega$ is geometrically good if

$$m(B_n) \preceq \operatorname{diam}^h(B_n) \tag{5.6}$$

for every set B of sufficiently small diameter containing z, every $n \geq 0$ and every connected component B_n of $f^{-n}(B)$. The direction of the inequality above means that when checking geometrical goodness one can assume the sets B to be balls centered at z. The most general sufficient condition for finite condensation is the following.

Lemma 5.9. *If $z \in J(f) \setminus \Omega$ is geometrically good, then z is a point of finite condensation of measure μ.*

Proof. Since $z \notin \Omega$, taking $\theta > 0$ (defined in (2.15)) sufficiently small, $z \notin B(\Omega, \theta)$. Set $B = B(z, \gamma)$, where γ was defined in (2.17). Since $m(B) > 0$ and $m(\bigcup_{n \geq 0} A_n) = 1$, there exists $i \geq 0$ such that $m(B \cap A_i) > 0$. Since $B \cap A_i \cap J(f)$ has a non-empty interior relative to $J(f)$, there exists an open ball $F \subset B \cap A_i$ having nonempty intersection with $J(f)$. Of course $m(F) > 0$. For every $n \geq 0$ let B_n be a connected component of $f^{-n}(B)$ and let $F_n \subset B_n$ be some connected component of $f^{-n}(F)$ contained in B_n. Using Koebe's Distortion Theorem, I (Euclidean version) and the fact that the point z is geometrically good, we get

$$m(F_n) \asymp \operatorname{diam}^h(F_n) = \left(\frac{\operatorname{diam}(F_n)}{\operatorname{diam}(B_n)}\right)^h \operatorname{diam}^h(B_n) \succeq m(B_n) \left(\frac{\operatorname{diam}(F_n)}{\operatorname{diam}(B_n)}\right)^h$$

Applying now Lemma 2.13 to the connected sets F and B we obtain

$$m(F_n) \succeq m(B_n) \left(\frac{\operatorname{diam}(F)}{\operatorname{diam}(B)}\right)^h .$$

Thus

$$\sum_{k=0}^{n} m(f^{-k}(B)) \preceq \sum_{k=0}^{n} m(f^{-k}(F)) \preceq \sum_{k=0}^{n} m(f^{-k}(A_i)).$$

Hence, using Lemma 5.3, we get $\mu(B) \preceq \mu(A_i) < \infty$ and therefore z is a point of finite condensation of μ. ■

In order to make use of this lemma we need to provide sufficient conditions for points to be geometrically good. This is done below.

Lemma 5.10. *If μ is h-upper estimable at every point $z \in J(f)$ with the same estimability constant, then every point $z \in J(f)$ is geometrically good.*

Proof. The proof of this lemma follows by a straightforward inductive argument incorporating Koebe's Distortion Theorem, Lemma 4.11, finiteness of the equivalence classes of the relation $\sim$ on the set of critical points of f, Lemma 2.10, and equivalently (2.17). ■

Theorem 5.11. *The set of points of infinite condensation of μ is contained in the set of parabolic points $\Omega(f)$.*

Proof. The proof of Lemma 4.22 shows that each point $z \in J(f)$ is upper estimable with respect to the Euclidean h-conformal measure m_e and so, also with respect to the measure m. Therefore the proof of Theorem 5.11 is completed by applying Lemma 5.10 and Lemma 5.9. ■

Corollary 5.12. *If $\Omega = \emptyset$, then there exists an f-invariant probability measure μ equivalent to m.*

Since the case $J(f) = \mathbb{C}$, rules out parabolic points, as an immediate consequence of this corollary we get

Corollary 5.13. *If $J(f) = \mathbb{C}$, then there is a unique probability measure μ equivalent to the Lebesgue's measure on $\mathbb{C}$.*

5.4. **Invariant Measure - Parabolic points.** From what we have shown in the previous section, it is clear that in order to identify the points of infinite condensation of μ we have to look at the parabolic points. Proceeding in exactly the same way as in Section 6 of [27], we can prove the following results.

Proposition 5.14. *If $\omega \in \Omega \setminus \overline{O_+(\mathrm{Crit}(f))}$, then μ has infinite condensation at ω if and only if $h \le \frac{2p(\omega)}{p(\omega)+1}$.*

As an immediate consequence of this proposition and Theorem 2.1, we get the following remarkable corollary.

Corollary 5.15. *If*

$$\max\{q_b : b \in \mathcal{R} \cap f^{-1}(\infty)\} \ge \max\{p(\omega) : \omega \in \Omega(f)\},$$

then the invariant measure μ is finite.

Proposition 5.16. *If $\omega \in \Omega$ and $h \le \frac{2p(\omega)}{p(\omega)+1}$, then μ has infinite condensation at ω.*

Theorem 5.17. *If $c \in J(f)$ is a critical point of f of order s, $\omega = f(c) \in \Omega$, and $h \le \frac{2sp(\omega)}{p(\omega)+1}$, then μ has infinite condensation at ω.*

6. Appendix

The goal of this appendix is to provide a proof of Theorem 6.3, the elliptic version of Mane's theorem from [17]. We have justified this decision in the introduction. We first prove a version of Przytycki's lemma from [23] for the sake of completeness, since it forms the first step in the proof of Theorem 6.3 and since there are places, where one has to proceed more subtly than in the case of rational functions.

Lemma 6.1. *For every integer $K \ge 0$ and every $0 < \lambda < 1$ the following holds. For every $\epsilon > 0$ and every $\kappa > 0$ there exists $\delta_0 = \delta_0(K, \epsilon, \lambda, \kappa) > 0$ such that for every disk $B(x, \delta)$ with $\delta \le \delta_0$ and every $x \in \mathbb{C}$ a distance at least κ apart from the set of parabolic points and attracting points, for every $n \ge 0$ and every connected component $W = \mathrm{Comp} f^{-n}(B(x, \delta))$ such that $f^n_{|W}$ has at most K critical points counted with multiplicities, for every component $W' = \mathrm{Comp}(f^{-n}(B'))$ in W, for the disc $B' = B(x, \lambda\delta)$ we have*

$$\mathrm{diam} W' \le \epsilon$$

$\mathrm{diam} W' \to 0$ *for $n \to \infty$ uniformly (i.e. independently of the choices of B and W').*

Proof. Suppose on the contrary that there exist a sequence $\{x_n\}_{n=1}^{\infty}$ of points in a distance at least κ apart from the set of parabolic points and attracting points, a sequence $\delta_n \searrow 0$, a sequence of components $W_n = \mathrm{Comp} f^{-k_n}(B_n)$ with $k_n \to \infty$, as $n \to \infty$ such that the number of critical points of each map f^{k_n} on W_n is bounded by K and W'_n, the sequence associate to W_n as in the statement of the lemma, such that $\lim_{n\to\infty} \mathrm{diam}(W'_n) \ne 0$. Then for each n there exists $L = L(n) : 0 \le L \le K$ such that there is no critical value of $f^{k_n}_{|W_n}$ in

$$P(n) := B(x_n, \delta_n(\lambda + (1-\lambda)\frac{L+1}{K+1})) \setminus B(x_n, \delta_n(\lambda + (1-\lambda)\frac{L}{K+1})).$$

Without loosing generality we may assume that all the components W'_n intersect the fundamental region $\mathcal{R}$. Put

$$W_n^{(1)} := \mathrm{Comp} f^{-k_n} B(x_n, \delta_n(\lambda + (1-\lambda)\frac{L(n)}{K+1})))$$

$$W_n^{(2)} := \mathrm{Comp} f^{-k_n} B(x_n, \delta_n(\lambda + (1-\lambda)\frac{L(n)+1}{K+1})))$$

the components containing W_n',

$$P_n := W_n^{(2)} \setminus W_n^{(1)}$$

and for every $0 \le m \le k_n$, $i = 1, 2$,

$$W_{n,m}^{(i)} = f^{k_n - m}(W_n^{(i)}), \quad P_{n,m} := f^{k_n-m}(P_n) = W_{n,m}^{(2)} \setminus W_{n,m}^{(1)}.$$

Let, for each n, the number $m = m(n) \le k_n$ be the least integer such that

$$\mathrm{diam} W_{n,m}^{(1)} \ge \inf_{c_1, c_2 \in \mathrm{Crit}(f), c_1 \ne c_2\}} \mathrm{dist}(c_1, c_2)),$$

So for every $0 \le t < m(n)$ the set $P_{n,t}$ is a topological annulus. That is so because at each step back by f^{-1} from $P_{n,t-1}$ to $P_{n,t}$ there is at most one branch point for f^{-1} from $W_{n,t-1}^{(i)}$ to $W_{n,t}^{(i)}, i = 1, 2$. Now, all the annuli $P_{n,m(n)-1}$'s have moduli bounded below by $2^{-K}(1-\lambda)\frac{1}{K+1}$. Since in addition all the components W_n' intersect the fundamental region $\Re$, it follows from Montel's Theorem that there exists a topological (maybe not geometric) annulus P contained in all $P_{n,m(n)-1}$'s for a subsequences n_s, which bounds a topological disk D. So $D \subset W_{n_s, m(n_s)-1}^{(2)}$. Hence $f^{m(n_s)-1}(D) \subset B(x, \delta_n)$. Passing to yet another subsequence we may assume that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point $y \in \mathbb{C}$ at the distance at least κ apart from the set of parabolic points and attracting points. Since $\delta_n \to 0$, we have also $m(n) \to \infty$. Thus D cannot intersect the Julia set $J(f)$. If they were contained in a preimage of a Siegel disk or a Herman ring, the limit of diameters of iterates $f^{m(n_s)-1}(D)$ would be positive. Thus D is either contained in the basin of attraction to an attracting periodic orbit or a parabolic periodic orbit. In either case the limit of the sets $f^{m(n_s)-1}(D)$ would be contained in either an attracting periodic orbit or a parabolic periodic orbit. Since this limit would coincide with y, we get a contradiction. The proof is complete. ■

Remark 6.2. *Obviously this lemma remains true (with the proof requiring only minor modifications) if instead of the disk $B(x, \delta)$ one takes the square centered at x and with edges of length δ. This is the version we will need in the next theorem.*

Theorem 6.3. *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be an elliptic function. If a point $x \in J(f) \setminus \Omega(f)$ is not contained in the ω-limit set of a recurrent critical point, then for all $\epsilon > 0$ there exists a neighbourhood U of x such that:*

(a) *For all $n \ge 0$, every connected component of $f^{-n}(U)$ has Euclidean diameter $\le \epsilon$;*

(b) *There exists $N > 0$ such that for all $n \ge 0$ and every connected component V of $f^{-n}(U)$, the degree of $f^n_{|V}$ is $\le N$;*

Proof. The core of the theorem is (a), from which the property (b) will easily follow. Given an open set $U \subset \overline{\mathbb{C}}$ denote $c(U, n)$ the set of connected components of $f^{-n}(U)$. Observe that $V \in c(U, n)$ implies $f^j(V) \in c(U, n-j)$ for all $0 \le j \le n$. If $V \in c(U, n)$ define $\Delta(V, n) = \#\{\xi \in V; (f^n)'(\xi) = 0\}$ counted with algebraic multiplicity. A square is the set S of the form $S = \{z \in \mathbb{C} : |\mathrm{Re}(z-p)| < \delta, |\mathrm{Im}(z-p)| < \delta\}$. The point p is the center and

δ its radius. Given a square S with center p and radius δ, then, given $k > 0$, denote by S^k the square with center p and radius $k\delta$. If S is a square with radius δ, denote by $\mathcal{L}(S)$ the family of squares contained in $S^{\frac{3}{2}} - S$ and having radius $\delta/4$. Denote by $\mathcal{L}^*(S)$ the family of squares $S_0^{\frac{3}{2}}$ with $S_0 \in \mathcal{L}(S)$. Suppose that x is not a parabolic point and does not belong to the ω-limit set of recurrent critical point. Then there exists $\delta_0 > 0$ such that

(1) There is no critical point c of f such that there exists $0 \le n_1 \le n_2$ satisfying
$$|f^{n_1}(c) - c| < \delta_0$$
and
$$|f^{n_2}(c) - x| < \delta_0.$$
(2) $|x - p| > 10\delta_0$ for every parabolic or attracting periodic point p.

Given $\epsilon > 0$ take $\epsilon_1 > 0$ satisfying

(3) $0 < \epsilon_1 < \min\{\epsilon/10, \delta_0/10\}$
(4) if U is an open connected set with $\mathrm{diam}(U) \le 2\epsilon_1$ then $\mathrm{diam}(W) \le \delta_0$ for all $W \in c(U,1)$

Let N_0 be the number of equivalence classes of the relation $\sim$ between critical points of f. Take $N_1 > 2$ such that

(5) If S is a square and $V \in c(S,n)$ satisfies $\Delta(V,n) \le N_0 + 1$ then the number of connected components of $f^{-n}(S^{\frac{2}{3}})$ contained in V is $\le N_1$.

Finally, take δ given by

(6) $\delta = \min\{\delta_0/10, \epsilon_1/10, \delta(2N_0, \frac{\epsilon_1}{20N_1}, \frac{2}{3}, \delta_0)\}$, where $\delta(2N_0, \frac{\epsilon_1}{20N_1}, \frac{2}{3}, \delta_0)$ was produced in Lemma 6.1.

Let S_0 be the square of center x and radius δ. Suppose that Theorem 6.3(a) fails for $U = S_0$. Then there exists $n > 0$ and $V \in c(S_0, n)$ with $\mathrm{diamV} \ge \epsilon \ge 10\epsilon_1$. On the other hand, by (1), $\mathrm{diam}S_0 = 2\sqrt{2}\delta < 3\delta < \epsilon_1$. Hence there exists an integer $n_0 \ge 0$ such that there exists $V_0 \in c(S_0^{\frac{3}{2}}, n_0)$ satisfying

(7) $\mathrm{diam}(f^{-(n_0-i)}(S_0) \cap f^i(V_0)) \le \epsilon_1$ for all $1 \le i \le n_0$, and
(8) $\mathrm{diam}(f^{-n_0}(S_0) \cap V_0) > \epsilon_1$

Since $\mathrm{diam}S_0 < \epsilon_1$ it follows that $n_0 > 0$. Now, starting with S_0 we shall construct a sequence of squares $S_0, S_1, S_2, \dots$ and strictly positive integers $n_0 \ge n_1 \ge n_2 \dots$ satisfying

(9) $S_{j+1} \in \mathcal{L}^*(S_j)$
(10) there exists $V_j \in c(S_j^{\frac{3}{2}}, n_j)$ such that
$$\mathrm{diam}(f^{(-n_j-i)}(S_j) \cap f^i(V_j)) \le \epsilon_1$$
for all $1 \le i \le n_j$ and
$$\mathrm{diam}(f^{-n_j}(S_j) \cap V_j) > \epsilon_1.$$

From (7) and (8), it follows that S_0 satisfies (10). If we construct such a sequence of squares and integers, then Theorem 6.3 will be proved by contradiction because the condition $n_0 \geq n_1 \geq n_2 \ldots \geq n_m \geq \ldots > 0$ implies that $n_j = n_i$ for all $\geq i$ for a certain i. But (a) implies that the radius of S_j is $(\frac{3}{8})^j\delta$; in particular $\operatorname{diam}(S_j) \to 0$ when $j \to +\infty$. But by (10),

$$\epsilon_1 < \operatorname{diam}(f^{-n_j}(S_j) \cap V_j) = \operatorname{diam}(f^{-n_i}(S_j) \cap V_j),$$

$$V_j \in c(S_j^{\frac{3}{2}}, n_j) = c(S_j^{\frac{3}{2}}, n_i)$$

Taking $j \to +\infty$, and recalling that i is contained and $\lim_{j\to\infty} \operatorname{diam} S_j = 0$, we conclude that the inequality above cannot hold.

The sequence $\{S_j\}$ and $\{n_j\}$ will be constructed by induction starting with S_0. Suppose S_i and n_i are constructed for $0 \leq i \leq j$. To find S_{j+1} and n_{j+1} we begin by observing that from (a) it follows that if $p \in S \in \mathcal{L}^*(S_j)$, then, by the contraction of the squares S_i

$$|p - x| \leq \sum_{i=0}^{j} \operatorname{diam}(S_i) = \sum_{i=0}^{j+1} (\frac{3}{8})^i \operatorname{diam}(S_0) = 2\sqrt{2} \sum_{i=0}^{j+1} (\frac{3}{8})^i \delta \leq 4\sqrt{2}\delta.$$

Hence, if a point q satisfies $\operatorname{dist}(q, S) \leq \delta_0$, we have

$$|q - x| \leq 4\sqrt{2}\delta + \delta_0 \leq 2\delta_0.$$

By (2), this means that

(11) $\operatorname{dist}(q, S) > \delta_0$ for all $S \in \mathcal{L}^*(S_j)$ and all parabolic or attracting periodic point q.

For the induction step (i.e. the construction of S_{j+1} and n_{j+1}), we shall use the following lemma.

Lemma 6.4. *If $U \subset \mathbb{C}$ is an open neighbourhood of x and $V \in c(U, n)$ satisfies*

$$\operatorname{diam} f^i(V) \leq \delta_0, \quad 0 \leq i \leq n,$$

then

$$\Delta(V, n) \leq N_0.$$

Proof. If $\Delta(V, n) \geq N_0 + 1$, there exists $N_0 + 1$ different points x_i, $1 \leq i \leq N_0 + 1$, in V such that $(f^n)'(x_j) = 0$. This means that for each $1 \leq i \leq N_0 + 1$ there exist $1 \leq m_i < n$, such that $f^{m_i}(x_i)$ is a critical point. Recalling that N_0 is the number of the equivalence classes of the equivalence relation $\sim$, it follows that there exists two different points in the set $\{x_i;\ 1 \leq i \leq N_0 + 1\}$, that we shall denote by x_1, x_2, and two critical points c_1 and c_2 in the same equivalence class of the equivalence relation $\sim$, such that $f^{m_1}(x_1) = c_1$ and $f^{m_2}(x_2) = c_2$. Assume without loss of generality that $0 \leq m_1 \leq m_2$. Then by the choice of δ_0, $m_1 < m_2$ and

$$|f^{m_2 - m_1}(c_2) - c_2| = |f^{m_2 - m_1}(c_1) - c_2| = |f^{m_2}(x_1) - f^{m_2}(x_2)| \leq \operatorname{diam} f^{m_2}(V) \leq \delta_0$$

and

$$|f^{n - m_2}(c_2) - x| = |f^{n - m_2}(f^{m_2}(x_2)) - x| = |f^n(x_2) - x| \leq \delta_0$$

contradicting property (1) of δ_0. ■

Now, to find S_{j+1} and n_{j+1} we first claim that there exists $S \in \mathcal{L}(S_j)$ that for some $0 < n \le n_j$ has $V \in c(S,n)$ with $\mathrm{diam} V \ge \frac{\epsilon_1}{10N_1}$. Suppose that the claim is false. Then, for all $1 \le i \le n_j$,

$$\begin{aligned}\mathrm{diam}(f^i(V_j)) &\le \mathrm{diam}\Big(f^{-(n_j-i)}(S_j) \cap f^i(V_j)\Big)\\ &\quad + \sup\{\mathrm{diam(W); W \in c(S, n_j - i), S \in \mathcal{L}(S_j)}\}\\ &\le \epsilon_1 + \frac{\epsilon_1}{10N_1} \le 2\epsilon_1\end{aligned}$$

From this inequality applied to $i = 1$ and property (4), we have

$$\mathrm{diam}(V_j) \le \delta_0.$$

Moreover, since $2\epsilon_1 \le \delta_0$ (by (3)),

$$\mathrm{diam}(f^i(V_j)) \le \delta_0$$

for all $1 \le i \le n_j$, hence for all $0 \le i \le n_j$. By Lemma 6.4, this proves that $\Delta(V_j, n_j) \le N_0$. Then, since $V_j \in c(S_j^{\frac{2}{3}}, n_j)$ it follows from (5), (11) and Lemma 6.4 that

$$[W \in c(S_j, n_j), W \subset V_j] \Rightarrow \mathrm{diam}(W) \le \frac{\epsilon_1}{10N_1}.$$

Moreover, by the way N_1 was chosen, we have

$$\#\{W \in c(S_j, n_j); W \subset V_j\} \le N_1$$

and we assume that

$$[S \in \mathcal{L}(S_j), G \in C(S, n_j)] \Rightarrow \mathrm{diam}(G) \le \frac{\epsilon}{10N_1}.$$

Now observe that V_j is the union of sets $G \in c(S, n_j), G \subset V_j, S \in \mathcal{L}(S_j)$ and the sets $W \in c(S_j, n_j), W \subset V_j$. Moreover, for any two sets W', W'' in this family there exist $W' = W_0, W_1, \dots, W_k = W''$ in $c(S_j, n_j)$ and contained in V_j such that for all $0 \le i < k$ there exist $S_i \in \mathcal{L}(S_j)$ and $U_i \in c(S_i, n_j)$ such that $\overline{U}_i \cap \overline{W}_i \neq \emptyset$ and $\overline{U}_i \cap \overline{W}_{i+1} \neq \emptyset$. Then

$$\mathrm{diam}(V_j) \le N_1 \left(\frac{\epsilon_1}{10N_1} + \frac{\epsilon_1}{10N_1}\right) = \frac{\epsilon_1}{5}$$

contradicting the last inequality in condition (10). This completes the proof of the claim. Now we can take $S \in \mathcal{L}(S_j)$ such that $\mathrm{diam}(V) \ge \frac{\epsilon}{10N_1}$ for some $V \in c(S, n), 0 \le n \le n_j$. Take $\tilde{V} \in c(S^{\frac{3}{2}}, n)$ containing V. Suppose that $\Delta(\tilde{V}, n) \le N_0$. Then by Lemma 6.1 and condition (6)

$$\mathrm{diam}(V) \le \frac{\epsilon_1}{20N_1}$$

since $V \in c((S^{\frac{3}{2}})^{\frac{2}{3}}, n)$ and is contained in $\tilde{V}$. This contradicts the fact that

$$\mathrm{diam}(V) \ge \frac{\epsilon_1}{10N_1}$$

and proves $\Delta(\tilde{V}, n) \geq N_0 + 1$. From Lemma 6.4, it follows that

$$\mathrm{diam}(f^i(\tilde{V})) > \delta_0$$

for some $0 \leq i \leq n$. Now we define $S_{j+1} = S^{\frac{3}{2}}$. Then $f^i(\tilde{V}) \in c(S^{\frac{3}{2}}, n-i)$ and $\mathrm{diam}(f^i(\tilde{V})) > \delta_0 \geq 10\epsilon_1$. Moreover $\mathrm{diam} S_{j+1} \leq 2\delta < \epsilon_1$. Then there exists $0 \leq n_{j+1} \leq n - i \leq n_j - i$ and $V_{j+1} \in c(S_{j+1}, n_{j+1})$ such that

$$\mathrm{diam}(f^{-n_{j+1}}(S_{j+1}) \cap V_{j+1}) > \epsilon_1$$

and

$$\mathrm{diam}(f^{-n_{j+1}+i}(S_{j+1}) \cap f^i(V_{j+1})) \leq \epsilon_1.$$

Observe that $n_{j+1} > 0$ since $\mathrm{diam}\left(\mathrm{S_{j+1}}\right) < 2\delta < \epsilon_1$. This completes the construction of the sequence $\{S_j\}$ and $\{n_j\}$ and the proof of part (a) of Theorem 6.3. Property (b) of Theorem 6.3 follows from (a) and Lemma 6.4. ■

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