Conformal families of measures for fibred systems

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Abstract

In this paper, we give sufficient conditions for the existence of conformal families of measures for fibred systems. We describe a general construction principle, modelled on the principle developed by Denker and Urbanski. For those systems that are weakly topologically exact along fibres and fibrewise countably critical, we show that each measure in the constructed family is supported on the whole fibre where it is naturally defined. We illustrate our result with a natural example, namely, a family of non fibrewise expanding fibred polynomials which are weakly topologically exact along fibres and fibrewise countably critical.[1]

1 Introduction

Throughout this paper, a fibred system is a collection $\mathcal{Y} = (Y, T, X, S, \pi)$, where Y and X are compact metrizable spaces, $T : Y \to Y$ and $S : X \to X$ are continuous maps, and $\pi : Y \to X$ is a continuous surjective map which satisfies $\pi \circ T = S \circ \pi$. Thus, T preserves the fibres $Y_x := \pi^{-1}(\{x\})$, $x \in X$. The restriction of T to the fibre over x, Y_x, will be denoted by T_x; so $T_x : Y_x \to Y_{Sx}$.

In 1999, Denker and Gordin [5] (See also [4, 6]) showed that compact fibred systems, whose fibre maps are Ruelle expanding and topologically exact, admit, for every globally continuous, uniformly fibrewise Hölder continuous potential, a unique Gibbs family of conditional measures on their fibres. This result reduces to the celebrated Ruelle-Perron-Frobenius theorem in the non-fibred case (or, in other words, in the one-fibre case, that is, when X consists of only one point).

Recall that a family $\{m_x\}_{x\in X}$ of Borel probability measures on Y is called a measurable system of conditional probabilities for $\mathcal{Y}$ if $m_x(Y_x) = 1$ for all $x \in X$ and the integral $\int g(y)\, dm_x(y)$ is a Borel-measurable function of x for every bounded Borel-measurable function g on Y.

Denker and Gordin [5] defined the notion of Gibbs family for fibred systems in the following manner. A system $\{m_x\}_{x\in X}$ of conditional probabilities for $\mathcal{Y}$ is called a Gibbs family (or conformal family) for a measurable function $\varphi : Y \to I\!R$ if there

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exists a positive Borel-measurable function $A : X \to I\!R_+$ with the following property: For all $x \in X$ the Jacobian of m_x with respect to the map T is given by

$$\frac{d(m_{Sx} \circ T_x)}{dm_x} = A(x) \exp(-\varphi) \quad m_x\text{-a.e.},$$

that is,

$$m_{Sx}(T_x B) = A(x) \int_B \exp[-\varphi(y)])dm_x(y) \tag{1}$$

for every special relative set $B \subseteq Y_x$ (this simply means that B is a measurable set such that $T_x B$ is measurable in Y_{Sx} and $T_x|_B$ is one-to-one).

It is called a weakly conformal family if Equation (1) is satisfied on every special relative set which is disjoint from the closure of the set of singular points.

In his Ph.D. thesis, the first author considered whether such a family exists for systems having fibre maps that are expansive rather than Ruelle expanding. He showed that, whenever the base space X is countable, compact fibred systems, whose fibre maps are uniformly open, expansive and topologically exact, and whose fibres are continuous in the Hausdorff metric, possess, for every continuous potential, a Gibbs family, though this family is generally not unique. Moreover, he proved that, without any restriction on the cardinality of X, these systems admit a unique Gibbs family for every globally continuous, uniformly fibrewise locally constant potential (See [9]). The approach he used is based upon Denker and Gordin's results and the existence of a change of metric on Y, compatible with the topology, that transforms the initially fibre expansive system into a fibre expanding one under the reign of the new metric (See [9, 10]).

In this paper, we generalize the first result mentioned in the previous paragraph. We give sufficient conditions for the existence of weakly conformal and conformal families of measures. However, we do not assume that the systems are submitted to any kind of expansion along their fibres. Neither do we assume that the systems have continuous fibres in the Hausdorff metric. Moreover, we do not impose any restriction upon the cardinality of the fibres (cf. [5]).

When the base space is countable, the weakly conformal and conformal families obtained satisfy all the requirements imposed by Denker and Gordin.

Theorem 1 *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibred system with countable base space X. Assume that $\mathcal{Y}$ is fibrewise surjective and bounded-to-one on fibres great orbits. Then, for every continuous function φ on Y, the system $\mathcal{Y}$ admits a weakly conformal family of measures $\{m_x\}_{x \in X}$. If, furthermore, $\mathcal{Y}$ is weakly topologically exact along fibres and fibrewise countably critical, then supp $m_x = Y_x$ for all $x \in X$.*

Moreover, if $\mathcal{Y}$ is supplementarily a fibrewise local homeomorphism, then the family $\{m_x\}_{x \in X}$ is conformal.

Since we work along individual great orbits, and the partition of a system into its great orbits is usually not measurable, it seems that the measurability of the disintegration obtained is hardly provable, not to say beyond the scope of our method, in the case where X is uncountable. Nevertheless, our method does apply when the base map S is countable-to-one, though it does not permit us to conclude anything about the measurability of the system of measures.

We describe a general construction principle, inspired by the construction Denker and the second author of this paper did concerning conformal measures for dynamical systems. We provide the reader with a natural example of a fibrewise expansive, though not fibrewise expanding, family of fibred polynomials that are fibrewise open and weakly topologically exact along fibres, some of whom have a countable base. These latter hence admit, according to Theorem 1, a conformal family of measures (which may not be unique by the way). Our work also ensures the existence of conformal families for mapping families, as studied by Arnoux and Fischer [1].

2 Construction of Conformal Families of Measures

Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibred system and φ be a continuous function on Y. Assume that $\mathcal{Y}$ is fibrewise surjective, that is, T_x is a surjective map for every $x \in X$. Assume also that $\mathcal{Y}$ is bounded-to-one on fibres great orbits, that is, for every $x \in X$ there is a $M_x \in \mathbb{N}$ such that $T_{\tilde{x}}$ is at most M_x-to-1 for every $\tilde{x} \in [x]$, where $[x]$ denotes the great orbit of x, that is, $[x] = \{\tilde{x} \in X | S^m(\tilde{x}) = S^n(x) \text{ for some } m, n \in \mathbb{Z}_+\}$. Finally, assume that S is countable-to-one.

In analogy to the construction principle for conformal measures enounced by Denker and Urbanski [3], we will rely on the analytical fact that the transition parameter $c = \limsup_{n\to\infty} a^n/n$ of a sequence of real numbers $(a^n)_{n\in\mathbb{N}}$ is uniquely determined by the fact that $\sum_{n\in\mathbb{N}} \exp(a^n - ns)$ converges for $s > c$, and diverges for $s < c$. For $s = c$, the series may converge or diverge. However, there is a sequence $(b^n)_{n\in\mathbb{N}}$ of positive real numbers satisfying

$$\sum_{n=1}^{\infty} b^n \exp(a^n - ns) \begin{cases} < \infty, & s > c \\ = \infty, & s \leq c \end{cases}$$

and $\lim_{n\to\infty} b^n/b^{n+1} = 1$ (See [3, Lemma 3.1]).

We define sets $(E_x^n)_{n\in\mathbb{N}, x\in X}$ in the following way. For each $x \in X$, choose a finite subset E_x^1 of Y_x. Then, having successively defined the sets $(E_x^k)_{x\in X}$, $1 \leq k \leq n$, for a fixed $n \in \mathbb{N}$, define $E_x^{n+1} = T_x^{-1}(E_{Sx}^n)$ for every $x \in X$. (Since T_x is surjective, we have in particular that $T_x(E_x^{n+1}) = E_{Sx}^n$.) Thereafter, define the sequence $(a_x^n)_{n\in\mathbb{N}}$ for every $x \in X$, where $a_x^n = \log \sum_{y\in E_x^n} \exp[(S_n\varphi)(y)]$ and $S_n\varphi = \sum_{0\leq k<n} \varphi \circ T^k$, and denote by c_x the transition parameter of the sequence $(a_x^n)_{n\in\mathbb{N}}$. We then have the following result.

Lemma 2 *The transition parameter function $c : X \to \mathbb{R}$ is constant over great orbits, that is, $c_{\tilde{x}} = c_x$ for every $\tilde{x} \in [x]$ and every $x \in X$.*

Proof. It is sufficient to show that for every $x \in X$, it holds that $c_x = c_{Sx}$. For this, given $x \in X$ and $s \in \mathbb{R}$, we compare the $(n+1)$st term of the series $\sum_{n\in\mathbb{N}} \exp(a_x^n - ns)$ with the nth term of $\sum_{n\in\mathbb{N}} \exp(a_{Sx}^n - ns)$:

$$\begin{aligned}
\frac{\exp(a_x^{n+1} - (n+1)s)}{\exp(a_{Sx}^n - ns)} &= \exp(-s)\exp(a_x^{n+1} - a_{Sx}^n) \\
&= \exp(-s)\frac{\sum_{y\in E_x^{n+1}} \exp[(S_{n+1}\varphi)(y)]}{\sum_{y\in E_{Sx}^n} \exp[(S_n\varphi)(y)]} \\
&= \exp(-s)\frac{\sum_{y\in E_x^{n+1}} \exp(\varphi(y)) \exp[(S_n\varphi)(Ty)]}{\sum_{y\in E_{Sx}^n} \exp[(S_n\varphi)(y)]}
\end{aligned}$$

As $\mathcal{Y}$ is bounded-to-one on fibres great orbits and $T_x(E_x^{n+1}) \subseteq E_{Sx}^n$, we obtain an upper bound:

$$\begin{aligned}\frac{\exp(a_x^{n+1}-(n+1)s)}{\exp(a_{Sx}^n - ns)} &\leq \exp(-s+\|\varphi\|_\infty)M_x\frac{\sum_{y\in E_{Sx}^n}\exp[(S_n\varphi)(y)]}{\sum_{y\in E_{Sx}^n}\exp[(S_n\varphi)(y)]}\\ &= \exp(-s+\|\varphi\|_\infty)M_x.\end{aligned}$$

Whereas, as $T_x(E_x^{n+1}) \supseteq E_{Sx}^n$, we get a lower bound:

$$\begin{aligned}\frac{\exp(a_x^{n+1}-(n+1)s)}{\exp(a_{Sx}^n - ns)} &\geq \exp(-s-\|\varphi\|_\infty)\frac{\sum_{y\in E_{Sx}^n}\exp[(S_n\varphi)(y)]}{\sum_{y\in E_{Sx}^n}\exp[(S_n\varphi)(y)]}\\ &= \exp(-s-\|\varphi\|_\infty).\end{aligned}$$

The simultaneous existence of these bounds ensures that both series have the same nature for all $s \in \mathbb{R}$ and therefore $c_x = c_{Sx}$ for every $x \in X$.

The following result is an immediate consequence of the previous lemma.

Corollary 3 *There exists a sequence of positive functions $(b^n : X \to \mathbb{R}_+)_{n\in\mathbb{N}}$ which are constant over great orbits and are such that for every $x \in X$, it holds that*

$$\sum_{n=1}^{\infty} b_x^n \exp(a_x^n - ns)\begin{cases} <\infty, & s > c_x\\ =\infty, & s\leq c_x\end{cases}$$

and $\lim_{n\to\infty} b_x^n/b_x^{n+1} = 1$. In particular, the functions $L : X \to \mathbb{R}_+$ and $H : X \to \mathbb{R}_+$, respectively defined by $L_x = \inf\{b_x^{n+1}/b_x^n \mid n \in \mathbb{N}\}$ and $H_x = \sup\{b_x^{n+1}/b_x^n \mid n \in \mathbb{N}\}$, are constant over great orbits.

Now, for each $x \in X$ and $s > c_x$, define the normalized measure

$$m_x^s = \frac{1}{M_x^s}\sum_{n=1}^{\infty}\sum_{y\in E_x^n} b_x^n \exp[(S_n\varphi)(y) - ns]\delta_y,$$

where

$$M_x^s = \sum_{n=1}^{\infty}\sum_{y\in E_x^n} b_x^n \exp[(S_n\varphi)(y) - ns]$$

and δ_y is the Dirac measure supported at y. Observe that, for each $x \in X$, all measures $(m_x^s)_{s>c_x}$ satisfy $m_x^s(Y_x) = 1$, and so do their accumulation points. They also share the same support, supp $m_x^s \subseteq Y_x$, and their common support is uniquely determined by the sets $(E_x^n)_{n\in\mathbb{N}}$. Furthermore, note that if there exists $\epsilon > 0$ such that for every $y \in Y$ and $\delta > 0$, there is $n = n(y,\delta) \in \mathbb{Z}_+$ for which

$$T^n\Big(B(y,\delta)\cap Y_{\pi y}\Big) \supseteq B\Big(T^n(y),\epsilon\Big)\cap Y_{S^n\pi y},$$

and if the finite sets $(E_x^1)_{x\in X}$ are chosen so that they are ϵ-dense in Y_x for every $x \in X$, then supp $m_x^s = Y_x$ for all $s > c_x$ and all $x \in X$. This follows from the fact that, by definition, $E_x^n = (T_{S^{n-2}x}\circ\cdots\circ T_{Sx}\circ T_x)^{-1}(E_{S^{n-1}x}^1)$ for all $x \in X$ when $n \geq 2$. This is evidently the case when $\mathcal{Y}$ is weakly topologically exact along fibres, and, consequently, when $\mathcal{Y}$ is topologically exact along fibres in the sense of Denker and Gordin (cf. [4, 5]).

Definition 4 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is said to be weakly topologically exact along fibres if for every $y \in Y$ and $\delta > 0$, there exists $n = n(y, \delta) \in \mathbb{Z}_+$ such that*

$$T^n\Big(B(y,\delta) \cap Y_{\pi y}\Big) = Y_{S^n \pi y}.$$

In summary, we have the following.

Lemma 5 *For each $x \in X$, the measures $(m_x^s)_{s>c_x}$ satisfy $m_x^s(Y_x) = 1$. They further have the same support, and this latter is uniquely determined by the sets $(E_x^n)_{n\in\mathbb{N}}$. Moreover, if the system is weakly topologically exact along fibres, then their support is the whole fibre where they are defined, namely, supp $m_x^s = Y_x$ for all $s > c_x$ and all $x \in X$.*

Now, fix $x \in X$ and $s > c_x$. Let $B \subseteq Y_x$ be a special relative set and let

$$\Delta_B(s) = \left| m_{Sx}^s(T_x B) - \frac{M_x^s}{M_{Sx}^s} \int_B \exp[c_x - \varphi(y)]dm_x^s(y) \right| \tag{2}$$

As $T_x^{-1}(E_{Sx}^n) = E_x^{n+1}$ for every $n \in \mathbb{N}$, we get

$$\begin{aligned}
m_{Sx}^s(T_x B) &= \frac{1}{M_{Sx}^s} \sum_{n=1}^{\infty} \sum_{y \in E_{Sx}^n \cap T_x B} b_{Sx}^n \exp[(S_n\varphi)(y) - ns] \\
&= \frac{1}{M_{Sx}^s} \sum_{n=1}^{\infty} \sum_{y \in B \cap T_x^{-1}(E_{Sx}^n)} b_x^n \exp[(S_n\varphi)(Ty) - ns] \\
&= \frac{1}{M_{Sx}^s} \sum_{n=1}^{\infty} \sum_{y \in B \cap E_x^{n+1}} b_x^n \exp[(S_{n+1}\varphi)(y) - (n+1)s] \cdot \exp[s - \varphi(y)].
\end{aligned}$$

Observe also that

$$M_x^s \int_B \exp[c_x - \varphi(y)]dm_x^s(y) = \sum_{n=1}^{\infty} \sum_{y \in B \cap E_x^n} b_x^n \exp[(S_n\varphi)(y) - ns] \cdot \exp[c_x - \varphi(y)].$$

Therefore

$$\begin{aligned}
\Delta_B(s) &= \frac{1}{M_{Sx}^s} \left| \sum_{n=1}^{\infty} \sum_{y \in B \cap E_x^{n+1}} b_x^{n+1} e^{c_x} \exp[(S_{n+1}\varphi)(y) - (n+1)s] \cdot \exp[-\varphi(y)] \right. \\
&\qquad \left. \cdot \left(\frac{b_x^n}{b_x^{n+1}} e^{s-c_x} - 1 \right) - b_x^1 \sum_{y \in B \cap E_x^1} e^{c_x - s} \right|.
\end{aligned}$$

The second term in the absolute value is clearly bounded. Henceforth, we study the properties of the ratio M_x^s/M_{Sx}^s when $s \searrow c_x$. Both $M_x^s \nearrow \infty$ and $M_{Sx}^s \nearrow \infty$ as $s \searrow c_x = c_{Sx}$. But what can be said of their ratio? Does $\lim_{s \searrow c_x} M_x^s/M_{Sx}^s$ exist? If so, is it greater than 0? Or does M_x^s/M_{Sx}^s remain at least bounded as $s \searrow c_x$?

To answer this latter question, we compare the $(n+1)$st term of M_x^s with the nth term of M_{Sx}^s:

$$\frac{\sum_{y \in E_x^{n+1}} b_x^{n+1} \exp[(S_{n+1}\varphi)(y) - (n+1)s]}{\sum_{y \in E_{Sx}^n} b_{Sx}^n \exp[(S_n\varphi)(y) - ns]}$$

$$= \frac{\sum_{y\in E_x^{n+1}} b_x^{n+1}\exp(-s+\varphi(y))\exp[(S_n\varphi)(Ty)-ns]}{\sum_{y\in E_{Sx}^n} b_{Sx}^n \exp[(S_n\varphi)(y)-ns]}$$

As $\mathcal{Y}$ is bounded-to-one on fibres great orbits and $T_x(E_x^{n+1})\subseteq E_{Sx}^n$, we obtain an upper bound:

$$\begin{aligned}
&\frac{\sum_{y\in E_x^{n+1}} b_x^{n+1}\exp[(S_{n+1}\varphi)(y)-(n+1)s]}{\sum_{y\in E_{Sx}^n} b_{Sx}^n \exp[(S_n\varphi)(y)-ns]}\\
&\qquad\leq \exp(-s+\|\varphi\|_\infty)M_x\frac{\sum_{y\in E_{Sx}^n} b_{Sx}^{n+1}\exp[(S_n\varphi)(y)-ns]}{\sum_{y\in E_{Sx}^n} b_{Sx}^n \exp[(S_n\varphi)(y)-ns]}\\
&\qquad\leq \exp(-s+\|\varphi\|_\infty)M_xH_{Sx}\frac{\sum_{y\in E_{Sx}^n} b_{Sx}^{n}\exp[(S_n\varphi)(y)-ns]}{\sum_{y\in E_{Sx}^n} b_{Sx}^n \exp[(S_n\varphi)(y)-ns]}\\
&\qquad= \exp(-s+\|\varphi\|_\infty)M_xH_{Sx}\\
&\qquad= \exp(-s+\|\varphi\|_\infty)M_xH_x. \qquad (3)
\end{aligned}$$

Since $T_x(E_x^{n+1})\supseteq E_{Sx}^n$, we get a lower bound:

$$\begin{aligned}
&\frac{\sum_{y\in E_x^{n+1}} b_x^{n+1}\exp[(S_{n+1}\varphi)(y)-(n+1)s]}{\sum_{y\in E_{Sx}^n} b_{Sx}^n \exp[(S_n\varphi)(y)-ns]}\\
&\qquad\geq \exp(-s-\|\varphi\|_\infty)\frac{\sum_{y\in E_{Sx}^n} b_{Sx}^{n+1}\exp[(S_n\varphi)(y)-ns]}{\sum_{y\in E_{Sx}^n} b_{Sx}^n \exp[(S_n\varphi)(y)-ns]}\\
&\qquad\geq \exp(-s-\|\varphi\|_\infty)L_{Sx}\\
&\qquad= \exp(-s-\|\varphi\|_\infty)L_x. \qquad (4)
\end{aligned}$$

It follows immediately from (3) that

$$\begin{aligned}
\frac{M_x^s}{M_{Sx}^s} &= \frac{\sum_{n=1}^\infty\sum_{y\in E_x^n} b_x^n\exp[(S_n\varphi)(y)-ns]}{\sum_{n=1}^\infty\sum_{y\in E_{Sx}^n} b_{Sx}^n\exp[(S_n\varphi)(y)-ns]}\\
&= \frac{b_x^1e^{-s}\sum_{y\in E_x^1}\exp[\varphi(y)]}{M_{Sx}^s}\\
&\quad+\frac{\sum_{n=1}^\infty\sum_{y\in E_x^{n+1}} b_x^{n+1}\exp[(S_{n+1}\varphi)(y)-(n+1)s]}{\sum_{n=1}^\infty\sum_{y\in E_{Sx}^n} b_{Sx}^n\exp[(S_n\varphi)(y)-ns]}\\
&\leq \frac{b_x^1e^{-s}\sum_{y\in E_x^1}\exp[\varphi(y)]}{M_{Sx}^s}\\
&\quad+\frac{\sum_{n=1}^\infty \exp(-s+\|\varphi\|_\infty)M_xH_x\sum_{y\in E_{Sx}^n} b_{Sx}^n\exp[(S_n\varphi)(y)-ns]}{\sum_{n=1}^\infty\sum_{y\in E_{Sx}^n} b_{Sx}^n\exp[(S_n\varphi)(y)-ns]}\\
&\leq \exp(-s+\|\varphi\|_\infty)\left\{\frac{b_x^1|E_x^1|}{M_{Sx}^s}+M_xH_x\right\}
\end{aligned}$$

and, from (4), that

$$\begin{aligned}
\frac{M_x^s}{M_{Sx}^s} &= \frac{\sum_{n=1}^\infty\sum_{y\in E_x^n} b_x^n\exp[(S_n\varphi)(y)-ns]}{\sum_{n=1}^\infty\sum_{y\in E_{Sx}^n} b_{Sx}^n\exp[(S_n\varphi)(y)-ns]}\\
&\geq \frac{\sum_{n=1}^\infty\sum_{y\in E_x^{n+1}} b_x^{n+1}\exp[(S_{n+1}\varphi)(y)-(n+1)s]}{\sum_{n=1}^\infty\sum_{y\in E_{Sx}^n} b_{Sx}^n\exp[(S_n\varphi)(y)-ns]}\\
&\geq \frac{\sum_{n=1}^\infty \exp(-s-\|\varphi\|_\infty)L_x\sum_{y\in E_{Sx}^n} b_{Sx}^n\exp[(S_n\varphi)(y)-ns]}{\sum_{n=1}^\infty\sum_{y\in E_{Sx}^n} b_{Sx}^n\exp[(S_n\varphi)(y)-ns]}\\
&= \exp(-s-\|\varphi\|_\infty)L_x.
\end{aligned}$$

Hence, under the assumptions that $\mathcal{Y}$ is bounded-to-one along fibres great orbits and $T_x(E_x^{n+1}) = E_{Sx}^n$ for all $n \in I\!N$, the ratio M_x^s/M_{Sx}^s remains bounded as $s \searrow c_x$. More precisely, we have shown that

$$\exp(-s - \|\varphi\|_\infty)L_x \leq \frac{M_x^s}{M_{Sx}^s} \leq \exp(-s + \|\varphi\|_\infty) \left\{ \frac{b_x^1 |E_x^1|}{M_{Sx}^s} + M_x H_x \right\}.$$

The fact that this ratio is bounded and that $M_x^s \nearrow \infty$ as $s \searrow c_x$ suffice to ensure that

$$\lim_{s \searrow c_x} \Delta_B(s) = 0.$$

Now, for each $[x] \in [X]$, where $[X]$ denotes the set of great orbits of the base system (X, S), choose a sequence $(s_k)_{k \in I\!N}$, with $s_k \searrow c_x$ when $k \nearrow \infty$, such that the sequences $(M_{\tilde{x}}^{s_k}/M_{S\tilde{x}}^{s_k})_{k \in I\!N}$ and $(m_{\tilde{x}}^{s_k})_{k \in I\!N}$ converge for all $\tilde{x} \in [x]$ (the measures are assumed to converge in the weak*-topology). This is possible since the great orbits of (X, S) are all countable, the map S being countable-to-one by assumption. Thereafter, define

$$A'(\tilde{x}) = \lim_{k \to \infty} M_{\tilde{x}}^{s_k}/M_{S\tilde{x}}^{s_k}, \tag{5}$$

and

$$m_{\tilde{x}} = \lim_{k \to \infty} m_{\tilde{x}}^{s_k} \tag{6}$$

for every $\tilde{x} \in [x]$ and each $[x] \in [X]$. Then, given any $x \in X$ and any special relative set $B \subseteq Y_x$, we have

$$\lim_{k \to \infty} \Delta_B(s_k) = 0. \tag{7}$$

Moreover, if B is a m_x-continuity set, that is, $m_x(\partial B) = 0$, we have, according to the Portmanteau Theorem (see [2, page 11]),

$$\lim_{k \to \infty} \int_B \exp[c_x - \varphi(y)] dm_x^{s_k}(y) = \int_B \exp[c_x - \varphi(y)] dm_x(y). \tag{8}$$

We deduce from (5) and (8) that

$$\lim_{k \to \infty} \frac{M_x^{s_k}}{M_{Sx}^{s_k}} \int_B \exp[c_x - \varphi(y)] dm_x^{s_k}(y) = A'(x) \int_B \exp[c_x - \varphi(y)] dm_x(y). \tag{9}$$

It then follows from (7), (2) and (9) that

$$\lim_{k \to \infty} m_{Sx}^{s_k}(T_x B) = A'(x) \int_B \exp[c_x - \varphi(y)] dm_x(y). \tag{10}$$

From (10), we obtain, for those B such that, supplementarily, $T_x B$ is a m_{Sx}-continuity set,

$$\begin{aligned} m_{Sx}(T_x B) &= \lim_{k \to \infty} m_{Sx}^{s_k}(T_x B) \\ &= A'(x) \int_B \exp[c_x - \varphi(y)] dm_x(y) \\ &= A'(x) e^{c_x} \int_B \exp[-\varphi(y)] dm_x(y) \\ &= A(x) \int_B \exp[-\varphi(y)] dm_x(y), \end{aligned} \tag{11}$$

where $A(x) = A'(x)e^{c_x}$.

Remark. The construction of this family suggests that it may not be unique, as expected.

Remark. Given any $x \in X$ and any measurable set $B \subseteq Y_x$ such that $T_x^n B$ is measurable in $Y_{S^n x}$, $T_x^n|_B$ is one-to-one and $m_{S^j x}(\partial T_x^j B) = 0$ for all $0 \leq j \leq n$, we obtain by induction

$$m_{S^n x}(T_x^n B) = A_n(x) \int_B \exp[-(S_n \varphi)(y)] dm_x(y)$$

where $A_n(x) = A(S^{n-1}x) \cdot \dots \cdot A(Sx) \cdot A(x)$.

In the same spirit as Denker and Urbanski (cf. [3]), for each $x \in X$ we denote by Y_x^{no} the set of all points $y \in Y_x$ at which T_x fails to be open, that is, for which there is a sequence $\{V_n(y)\}$ of open neighbourhoods of y such that $\operatorname{diam} V_n(y) \to 0$ and $T_x(V_n(y) \cap Y_x)$ is not open in Y_{Sx}.

On the other hand, we denote by $\mathrm{Crit}(T_x)$ the set of critical points of T_x, that is, the set of points $y \in Y_x$ which do not have any open neighbourhood $V(y)$ such that $V(y) \cap Y_x$ is a special relative or, equivalently, the set of points $y \in Y_x$ that do not have any neighbourhood on which T_x is injective.

Note that $\mathrm{Crit}(T_x)$ is a compact subset of Y_x for every $x \in X$. Indeed, fix $x \in X$. If $(y_n)_{n \in I\!N}$ is a sequence in $\mathrm{Crit}(T_x)$ which converges to y, then, given $\epsilon > 0$, there exists $N = N(\epsilon) \in I\!N$ such that $y_n \in B(y, \epsilon/2)$ for every $n \geq N$. Therefore $B(y_n, \epsilon/2) \subseteq B(y, \epsilon)$ for all $n \geq N$. Since $y_n \in \mathrm{Crit}(T_x)$ for every $n \in I\!N$, the map T_x is not injective on $B(y_n, \epsilon/2)$ for each $n \in I\!N$. Hence T_x is not injective on $B(y, \epsilon)$ for every $n \geq N$. This shows that $y \in \mathrm{Crit}(T_x)$ and $\mathrm{Crit}(T_x)$ is thereby closed, and compact, in Y_x.

Finally, we define $\mathrm{Sing}(T_x) := Y_x^{no} \cup \mathrm{Crit}(T_x)$ for every $x \in X$. A point $y \in \mathrm{Sing}(T_x)$ is called singular. Note that this set is generally not compact.

Lemma 6 *Let $x \in X$, μ be a Borel probability measure on Y_x, ν a Borel probability measure on Y_{Sx} and $f \in L^1(\mu)$. Assume that there is $\alpha(x) \in I\!R_+$ such that the relation*

$$\nu(T_x C) = \alpha(x) \int_C f(y) d\mu(y) \tag{12}$$

holds for every special relative set $C \subseteq Y_x$ with $C \cap \overline{Sing(T_x)} = \emptyset$ and $\mu(\partial C) = \nu(\partial T_x C) = 0$. Then the aforementioned relation (12) holds for any special relative set $A \subseteq Y_x$ with $A \cap \overline{Sing(T_x)} = \emptyset$.

Proof. The proof is essentially the same as in [3, Lemma 2.4], so we will just give some additional guidelines. Let $\epsilon > 0$. Let $A \subseteq Y_x$ be a special set satisfying $A \cap \overline{\mathrm{Sing}(T_x)} = \emptyset$. Since μ is regular on Y_x, there is an open set $A \subseteq V \subseteq Y_x \backslash \overline{\mathrm{Sing}(T_x)}$ such that

$$\alpha(x) \int_{V \backslash A} f(y) d\mu(y) < \epsilon.$$

For each $y \in A$, there exists a nonempty open ball $B(y, r_y) \cap Y_x \subseteq V$ such that $T_x|_{B(y,r_y) \cap Y_x}$ is a homeomorphism and $\mu(\partial(B(y, r_y) \cap Y_x)) = \nu(\partial T_x(B(y, r_y) \cap Y_x)) = 0$. Indeed, since $y \in A$ and $A \cap \overline{\mathrm{Sing}(T_x)} = \emptyset$, the point y is nonsingular. This means that there exists an open neighbourhood $N(y)$ of y such that $T_x|_{N(y) \cap Y_x}$ is a homeomorphism. Since $N(y)$ is an open neighbourhood of y, there is $r_0 > 0$ such that

$B(y, r_0) \subseteq N(y)$ and $B(y, r_0) \cap Y_x \subseteq V$. Observe that the sets $\{\partial(B(y, r) \cap Y_x) : r < r_0\}$ and $\{\partial T_x(B(y, r) \cap Y_x) : r < r_0\}$ form two uncountable families of mutually disjoint sets (this fact holds for the second family since $T_x|_{N(y)\cap Y_x}$ is injective). Therefore all but countably many of these sets have measure zero and we can hence find a r_y satisfying the above mentioned assertion.

We can then choose a countable family of relative balls that covers A, construct recursively a partition and go along similar lines to Denker and Urbanski's proof from now on.

Corollary 7 *The collection $\{m_x\}_{x\in X}$ constitutes a weakly conformal family of measures whenever the base space X is countable.*

Proof. Fix $x \in X$ and let $\mu = m_x$, $\nu = m_{Sx}$, $f = \exp[-\varphi]$ and $\alpha(x) = A(x)$. Relation (11) shows that the hypothesis of the previous lemma is satisfied and it follows from this that

$$\mu_{Sx}(T_x B) = A(x) \int_B \exp[-\varphi(y)] dm_x(y)$$

holds for every special set $B \subsetneq Y_x$ with $B \cap \overline{\mathrm{Sing}(T_x)} = \emptyset$ or, in other words, that the family of measures $\{m_x\}_{x\in X}$ is weakly conformal, but for its measurability aspect.

When X is countable, the functions A, c and $x \mapsto \int \varphi(y)\, dm_x(y)$ are measurable functions for every measurable function φ on Y, for they are defined on a countable space X, and the Borel σ-algebra of a countable metrizable space coincides with the trivial σ-algebra of all subsets of this space.

The measurability of the family is then trivial.

Remark. If X is uncountable, then the method still applies but does lead to a *pseudo* weakly conformal family of measures, in the sense that we cannot say whether this family forms a measurable system of conditional probabilities or not.

Corollary 8 *Assume that $\mathcal{Y}$ is a fibrewise local homeomorphism, that is, each T_x is a local homeomorphism. Then $\{m_x\}_{x\in X}$ constitutes a conformal family of measures whenever the base space X is countable.*

Proof. Simply observe that $\mathrm{Sing}(T_x) = \emptyset$ for every $x \in X$.

2.1 Sufficient conditions for the conformal family to satisfy supp $m_x = Y_x$ for all $x \in X$

We begin this section with a definition.

Definition 9 *A fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is said to be fibrewise countably critical if for every $x \in X$, the set $Crit(T_x)$ is countable.*

We now prove the following result.

Lemma 10 *If a fibred system $\mathcal{Y} = (Y, T, X, S, \pi)$ is weakly topologically exact along fibres and fibrewise countably critical, then the support of the measures belonging to the previously constructed (weakly) conformal family is the whole fibre where they are defined, namely, supp $m_x = Y_x$ for all $x \in X$.*

Proof. Fix $x \in X$. For every $n \in I\!N$, it holds that

$$\mathrm{Crit}(T_x^n) = \bigcup_{k=0}^{n-1} (T_x^k)^{-1}(\mathrm{Crit}(T_{S^k x})).$$

Since $\mathcal{Y}$ is bounded-to-one on fibres great orbits, the set $\mathrm{Crit}(T_x^n)$ is countable. We can thereby write $\mathrm{Crit}(T_x^n) = \{c_1, c_2, \ldots\}$. On the other hand, for each $y \in Y_x \backslash \mathrm{Crit}(T_x^n)$, let $r_y > 0$ be such that $T_x^n|_{B(y,r_y)}$ is injective. Since $\{B(y, r_y)\}_{y \in Y_x \backslash \mathrm{Crit}(T_x^n)}$ form an open cover of the secound-countable set $Y_x \backslash \mathrm{Crit}(T_x^n)$, one can extract, according to Lindelöf's Theorem, a countable subcover $\{B(y_i, r_i)\}_{i=1}^{\infty}$. Then

$$Y_x = \bigcup_{i=1}^{\infty} B(y_i, r_i) \cup \mathrm{Crit}(T_x^n)$$

Since $\mathcal{Y}$ is weakly topologically exact along fibres, given any relative open set $U_x \subseteq Y_x$, there is a $n = n(U_x) \in Z\!\!Z_+$ such that $T^n(U_x) = Y_{S^n x}$, and therefore

$$\begin{aligned}
1 &= m_{S^n x}(Y_{S^n x}) \\
&= m_{S^n x}(T_x^n U_x) \\
&= m_{S^n x}\left(\bigcup_{i=1}^{\infty} T_x^n (B(y_i, r_i) \cap U_x) \cup \bigcup_{i: c_i \in U_x} \{T_x^n c_i\} \right) \\
&\le \sum_{i=1}^{\infty} m_{S^n x}(T_x^n (B(y_i, r_i) \cap U_x)) + \sum_{i: c_i \in U_x} m_{S^n x}(\{T_x^n c_i\}) \\
&= \sum_{i=1}^{\infty} m_{S^n x}(T_x^n (B(y_i, r_i) \cap U_x)) + \sum_{i: c_i \in U_x} m_{S^n x}(\{T_x^n c_i\}) \\
&= A_n(x) \left[\sum_{i=1}^{\infty} \int_{B(y_i, r_i) \cap U_x} \exp[-(S_n \varphi)(y)] dm_x(y) \right. \\
&\qquad \left. + \sum_{i: c_i \in U_x} \exp[-(S_n \varphi)(c_i)] m_x(\{c_i\}) \right] \\
&\le A_n(x) \exp(n \|\varphi\|_\infty) \left[\sum_{i=1}^{\infty} m_x(B(y_i, r_i) \cap U_x) + \sum_{i: c_i \in U_x} m_x(\{c_i\}) \right]
\end{aligned}$$

This implies in particular that, either there is some y_j with $m_x(B(y_j, r_j) \cap U_x) > 0$, or some $c_k \in U_x$ such that $m_x(\{c_i\}) > 0$. In either case, it follows that $m_x(U_x) > 0$. This completes the proof.

Remark. Theorem 1 is a straightforward rewrite of Corollaries 7 and 8, and Lemma 10.

3 A Family of Non Fibrewise Expanding Fibred Polynomials Admitting Conformal Measures

Fibred polynomials constitute an interesting family of fibred systems. They have been introduced by Sester [11], following earlier work done in the same spirit by Heinemann [7] and Jonsson [8] on Julia sets for maps of several complex variables. Some work in this direction has also been more recently accomplished by Sumi [13] and the first author of this paper [10].

We first recall some basic definitions (cf. [12]).

Definition 11 *Let X be a compact metric space. A continuous map $P : X \times \mathbb{C} \to X \times \mathbb{C}$ is called fibred polynomial if*

$$P(x, z) = (f(x), P_x(z)),$$

where $f : X \to X$ is a continuous map and $P_x : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree at least two for each $x \in X$.

Let $P : X \times \mathbb{C} \to X \times \mathbb{C}$ be a fibred polynomial. For each $n \in \mathbb{N}$ and $x \in X$, set $P_x^n = P_{f^{n-1}(x)} \circ \ldots \circ P_x$ and $P_x^0 = Id_{\mathbb{C}}$. The set

$$F_x = \left\{ z \in \mathbb{C} \,\middle|\, \{P_x^n\}_{n \in \mathbb{N}} \text{ is normal at } z \right\}$$

is called Fatou set for P in x, and $J_x = \mathbb{C} \setminus F_x$ Julia set for P in x. Moreover, the Julia set for P is defined as

$$J(P) = \overline{\bigcup_{x \in X} \{x\} \times J_x},$$

whereas

$$F(P) = (X \times \mathbb{C}) \setminus J(P)$$

is called Fatou set for P. The filled-in Julia set for P in x is defined by

$$K_x = \left\{ z \in \mathbb{C} \,\middle|\, \{P_x^n(z)\}_{n \in \mathbb{N}} \text{ is a bounded subset of } \mathbb{C} \right\}$$

and the filled-in Julia set for P is

$$K(P) = \overline{\bigcup_{x \in X} K_x}.$$

Finally, the set

$$Crit(P) = \left\{ (x, z) \in X \times \mathbb{C} \,\middle|\, P_x'(z) = 0 \right\}$$

is called critical set for P and

$$Post(P) = \overline{\bigcup_{n \in \mathbb{N}} P^n(Crit(P))}$$

postcritical set for P.

The following lemma regroups the most fundamental properties of $J(P)$ (See [12, Proposition 2.9]).

Lemma 12 *Let $P : X \times \mathbb{C} \to X \times \mathbb{C}$ be a fibred polynomial with $P(x, z) = (f(x), P_x(z))$. The following statements hold.*

(i) For $x \in X$, we have $P_x^{-1}(F_{f(x)}) = F_x$ and $P_x^{-1}(J_{f(x)}) = J_x$;

(ii) $J(P)$ is forward invariant, that is, $P(J(P)) \subseteq J(P)$;

(iii) If f is surjective, then P is surjective, too;

(iv) If f is an open surjective map, then $J(P)$ is completely invariant, that is, it holds that $P^{-1}(J(P)) = J(P) = P(J(P))$;

(v) The map $x \mapsto J_x$ is lower semicontinuous with respect to the Hausdorff metric for the space of compact subsets of $\mathbb{C}$.

An important family of fibred polynomials are those whose postcritical sets stay away from their Julia sets.

Definition 13 *Let $P : X \times \mathbb{C} \to X \times \mathbb{C}$ be a fibred polynomial. We say that P is hyperbolic along fibres if $Post(P) \subseteq F(P)$.*

Another important family of fibred polynomials are those that are coined expanding along fibres.

Definition 14 *Let $P : X \times \mathbb{C} \to X \times \mathbb{C}$ be a fibred polynomial. We say that P is expanding along fibres if there exist constants $C > 0$ and $\lambda > 1$ such that for each $n \in \mathbb{N}$*

$$\inf_{(x,z) \in J(P)} |(P_x^n)'(z)| \geq C\lambda^n.$$

In fact, Sester has shown that the previous two families are the same (See [12, Theorem 1.1]).

Theorem 15 *A fibred polynomial $P : X \times \mathbb{C} \to X \times \mathbb{C}$ is hyperbolic along fibres if and only if it is expanding along fibres.*

Sester further proved the following (See [12, Proposition 4.1]).

Proposition 16 *If $P : X \times \mathbb{C} \to X \times \mathbb{C}$ is a fibred polynomial which is hyperbolic along fibres, then P has continuous fibres, that is, the application $x \mapsto J_x$ is continuous with respect to the Hausdorff metric on the space of compact subsets of $\mathbb{C}$.*

Let us finally recall two other results (See [12, Corollary 2.6] and [12, Theorem 5.2]).

Theorem 17 *The set K_x is connected for all $x \in X$ if and only if $Crit(P) \subseteq K(P)$.*

Theorem 18 *If $P : X \times \mathbb{C} \to X \times \mathbb{C}$ is a fibred polynomial which is hyperbolic along fibres and if the interior of K_x is a non-empty connected set for all $x \in X$, then each K_x is a κ-quasidisk (with κ independent of x) and therefore each J_x is a κ-quasicircle.*

In [10], the first author of this paper showed that some fibred polynomials, despite not being expanding along fibres, are fibrewise expansive.

Theorem 19 *Let $P : X \times \mathbb{C} \to X \times \mathbb{C}$ be a fibred polynomial on the unit interval $X = [0, 1] \subseteq \mathbb{C}$. Assume that $P(x, z) = (f(x), P_x(z))$, where f has the following properties:*

1. *f is a homeomorphism of the interval;*

2. $f(x) < x$ *for all* $0 < x < 1$*;*

3. $f(x) \leq x/8$ *for all* $x \leq x_*$ *and some* $x_* > 0$*,*

and that $P_x(z) = z^2 + x/4$*. Then* $\mathcal{Y} = (J(P), P, X, f, \pi_X)$ *is a fibrewise expansive, compact system whose fibres are continuous in the Hausdorff metric. Furthermore,* $\mathcal{Y}$ *is uniformly fibrewise open, but is neither fibrewise expanding nor expansive with respect to the metric induced on* $J(P)$ *by the usual euclidean norm* $\|(x,z)\| = \sqrt{|x|^2 + |z|^2}$*.*

In the course of the proof of this result, it was established that:

Lemma 20 J_0 *is the unit circle and* J_1 *the cauliflower.*

Lemma 21 *For each* $x \in X$*, we have* $J_x \subseteq \overline{A_{\frac{1}{2}(1+\sqrt{1-x}),\frac{1}{2}(1+\sqrt{1+x})}}$*, where* $\overline{A_{\alpha,\beta}} = \{z \in \mathbb{C} \mid \alpha \leq |z| \leq \beta\}$*.*

Lemma 22 *Given* $\tilde{x} < 1$*, the fibred polynomial* $P_{\leq \tilde{x}} := P|_{[0,\tilde{x}]\times\mathbb{C}}$ *is expanding along fibres and the fibred system* $\mathcal{Y}_{\leq \tilde{x}} := (J(P_{\leq \tilde{x}}), P_{\leq \tilde{x}}, [0,\tilde{x}], f, \pi_X)$ *generated by* $P_{\leq \tilde{x}}$ *is fibrewise expanding. Moreover,* $\mathcal{Y}_{\leq \tilde{x}}$ *has continuous fibres and is not expansive. Furthermore, each* J_x *(*$x \leq \tilde{x}$*) is a quasicircle.*

We now prove that these fibred polynomials are weakly topologically exact along fibres.

Lemma 23 *All fibred polynomials described in Theorem 19 are weakly topologically exact along fibres.*

Proof. Notation: For each $x \in X$ and $n \in \mathbb{Z}_+$, let $x_n = f^n(x)$.

Fix $\tilde{x} < 1$ and choose an arc C in $J_{\tilde{x}}$. Since the fibred polynomial $P_{\leq \tilde{x}} := P|_{[0,\tilde{x}]\times\mathbb{C}}$ is expanding along fibres according to Lemma 22, there exists $a > 0$ and $Q \in \mathbb{Z}_+$ such that $\operatorname{diam}(P^q_{\tilde{x}}(C)) \geq a$ for all $q \geq Q$. Choosing Q so large that $d^H(J_x, J_0) < a/2$ for all $x \leq \tilde{x}_Q$ (the second property of f and the continuity of the fibres in the Hausdorff metric, d^H, ensure the existence of such a Q), there is $z_q \in J_0$ such that $\emptyset \neq B(z_q, a/2) \cap J_{\tilde{x}_q} \subseteq P^q_{\tilde{x}}(C)$ for all $q \geq Q$.

For every open set $U \subseteq \mathbb{C}$ with $U \cap J_0 \neq \emptyset$, there is a smallest $k(U) \in \mathbb{Z}_+$ such that $P_0^{k(U)}(U) \supseteq J_0$. Observe that the family $\{B(z, a/2)\}_{z\in J_0}$ forms an open cover of J_0. Let $0 < \delta < a/4$ be a Lebesgue number for this cover. Then the family $\{B(z,\delta)\}_{z\in J_0}$ constitutes an open cover of J_0. By definition of δ, each element of $\{B(z,\delta)\}_{z\in J_0}$ is contained in at least one element of $\{B(z, a/2)\}_{z\in J_0}$. Choose a finite subcover $\{B(z_j,\delta)\}_{j=1}^{l}$ of $\{B(z,\delta)\}_{z\in J_0}$ and let $k = \max_{1\leq j\leq l} k(B(z_j,\delta))$. Then, for each $z \in J_0$, there exists $1 \leq j(z) \leq l$ such that $B(z_{j(z)},\delta) \subseteq B(z, a/2)$. It thereby follows that $k(B(z, a/2)) \leq k(B(z_{j(z)},\delta)) \leq k < \infty$. This implies in particular that $P_0^k(B(z, a/2)) \supseteq J_0$ for every $z \in J_0$.

Moreover, since P is an open map and $\{B(z_j,\delta) \times \mathbb{C}\}_{j=1}^{l}$ are open sets in $X \times \mathbb{C}$, the sets $\{P^k(B(z_j,\delta) \times \mathbb{C})\}_{j=1}^{l}$ are open in $X \times \mathbb{C}$ and $P^k(B(z_j,\delta) \times \mathbb{C}) \supseteq \{0\} \times J_0$ for every $1 \leq j \leq l$. The set $\{0\} \times J_0$ being compact in $X \times \mathbb{C}$, there exists $\epsilon > 0$ and $x'' > 0$ such that $P^k(B(z_j,\delta) \times \mathbb{C}) \supseteq [0, x''] \times \overline{A_{1-\epsilon,1+\epsilon}}$ for each $1 \leq j \leq l$. This implies in particular that $P^k(B(z, a/2) \times \mathbb{C}) \supseteq [0, x''] \times \overline{A_{1-\epsilon,1+\epsilon}}$ for every $z \in J_0$. We deduce from the first two properties of f and Lemma 21 that there is $0 < x' \leq x''$ such that

$P_x^k(B(z,a/2)) \supseteq J_{x^k}$ for every $z \in J_0$ and all $x \le x'$ (by simply choosing $0 < x' \le x''$ so that $\overline{A_{\frac{1}{2}(1+\sqrt{1-x'}),\frac{1}{2}(1+\sqrt{1+x'})}} \subseteq \overline{A_{1-\epsilon,1+\epsilon}}$). It follows that $P_x^k(B(z,a/2)\cap J_x) = J_{x^k}$ for every $z \in J_0$ and all $x \le x'$.

Now, if necessary, enlarge Q so that $\tilde{x}_Q \le x'$. Then

$$P_{\tilde{x}}^{Q+K}(C) = P_{\tilde{x}_Q}^K(P_{\tilde{x}}^Q(C)) \supseteq P_{\tilde{x}_Q}^K(B(z_Q,a/2)\cap J_{\tilde{x}_Q}) = J_{\tilde{x}_{Q+K}}.$$

Finally, it follows from standard properties of Julia sets (in complex dynamics) that P_1 is weakly topologically exact along fibres on the cauliflower.

An example of fibred polynomial with a countable base can easily be found by restricting f to the orbit of a single point (or even countably many points) of X.

Example 24 *Fix* $0 < x_0 < 1$. *Let* $P : [0,1]\times\mathbb{C} \to [0,1]\times\mathbb{C}$ *be one of the fibred polynomials described in Theorem 19 and* $X = \{f^n(x_0) | n \in \mathbb{Z}\}$. *Then* $P : X\times\mathbb{C} \to X\times\mathbb{C}$ *is a fibred polynomial, and the fibred system* $\mathcal{Y} = (J(P), P, X, f, \pi_X)$ *it induces admits a conformal family of measures* $\{m_x\}_{x\in X}$ *such that supp* $m_x = Y_x$ *for all* $x \in X$.

Indeed, the above system is compact metric, fibrewise surjective and two-to-one on fibres. Its base space is obviously countable. It is weakly topologically exact along fibres and trivially fibrewise countably critical, as Sing$(T_x) = \emptyset$ *for all* $x \in X$. *This is a consequence of the fact that it is a fibrewise local homeomorphism, because of its fibrewise openness and expansiveness.*

It thereby satisfies all the hypotheses of Theorem 1 and the conclusion follows.

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