February 5, 2003

THERMODYNAMIC FORMALISM, TOPOLOGICAL PRESSURE, AND ESCAPE RATES FOR CRITICALLY NON-RECURRENT CONFORMAL DYNAMICS

MARIUSZ URBAŃSKI

ABSTRACT. We show that for critically non-recurrent rational functions all the definitions of topological pressure proposed in [12], coincide for all $t \geq 0$. Then we study with detail the Gibbs states corresponding to the potentials $-t\log|f'|$ and their σ-finite invariant versions. In particular we provide a sufficient condition for their finiteness. We determine the escape rates of critically non-recurrent rational functions. In the case of presence of parabolic points we also establish a polynomial rate of appropriately modified escape. This extends the corresponding result from [6] proven in the context of parabolic rational functions. In the last part of the paper we introduce the class of critically tame generalized polynomial-like mappings. We show that if f is a critically tame non-recurrent generalized polynomial-like mapping and g is a Hölder continuous potential (with sufficiently large exponent if f has parabolic points) and the topological pressure $\mathrm{P}(g) > \sup(g)$, then for a sufficiently small $\delta > 0$, the function $t \mapsto \mathrm{P}(tg)$, $t \in (1-\delta, 1+\delta)$, is real-analytic.

1. Introduction and Preliminaries

A rational function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called critically non-recurrent if no critical point contained in its Julia set is recurrent. In [17] and [18] we explored some geometrical and dynamical properties of critically non-recurrent rational functions. In this paper we continue the investigations originated in these two papers. More precisely, in Section 2 we deal with various generalizations of topological pressure $\mathrm{P}(t)$ of the potential $-t\log|f'|$, $t \geq 0$ proposed in [12]. We demonstrate (see Theorem 2.6 that for critically non-recurrent rational functions all these definitions of topological coincide for all $t \geq 0$. In Section 3 we deal with thermodynamic formalism of critically non-recurrent dynamics. We study with detail the Gibbs states corresponding to the potentials $-t\log|f'|$ and their σ-finite invariant versions. In particular we provide a sufficient condition for these invariant measures to be finite. In Section 4 we deal with escape rates. We show that in the critically non-recurrent case this rate is equal to $\mathrm{P}(2)$. In the case of presence of parabolic points we establish a polynomial rate of appropriately modified escape. This extends the corresponding result from [6] proven in the context of parabolic rational functions. Our approach differs from Haydn's and Isola's in that point that we estimate moduli of appropriate annuli and we use McMullen's result relating such moduli with hyperbolic diameters of corresponding sets enclosed by these annuli. In Section 5, the

The research was supported in part by the NSF Grant DMS 0100078.

last section, we deal with generalized polynomial-like maps and Hölder continuous potentials. We show that if f is a tame non-recurrent generalized polynomial-like mapping and g is a Hölder continuous potential (with sufficiently large exponent if f has parabolic points) and the topological pressure $\mathrm{P}(g) > \sup(g)$, then for a sufficiently small $\delta > 0$, the function $t \mapsto \mathrm{P}(tg)$, $t \in (1-\delta, 1+\delta)$, is real-analytic.

2. Various Pressures

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function of the Riemann sphere $\overline{\mathbb{C}}$ of degree ≥ 2. In [12] F. Przytycki has proposed several ways of extending the concept of topological pressure of the potential $-t\log|f'|$, $t \geq 0$ to the entirely general case. Let us describe them briefly:

1. Variational pressure.

$$\mathrm{P}_{var}(t) = \sup\{h_\mu(f) - t\int \log|f'|d\mu\},$$

where the supremum is taken over all ergodic f-invariant measures on $J(f)$.

2. Hyperbolic variational pressure.

$$\mathrm{P}_{hypvar}(t) = \sup\{h_\mu(f) - t\int \log|f'|d\mu\},$$

where the supremum is taken over all ergodic f-invariant measures on $J(f)$ with positive Lyapunov exponent, i.e. such that $\chi_\mu(f) = \int \log|f'|d\mu > 0$.

3. Hyperbolic pressure.

We call a forward invariant compact set $X \subset J(f)$ hyperbolic if there exists $n \geq 1$ such that for every $x \in X$, $|(f^n)'(x)| > 1$. The hyperbolic pressure

$$\mathrm{P}_{hyp}(t) = \sup_X\{\mathrm{P}(f|_X, -t\log|f'|)\},$$

where the supremum is taken over all f-invariant hyperbolic subsets X of $J(f)$ such that an iterate of $f|_X$ is topologically conjugate with a subshift of finite type.

4. DU pressure.

Let V be an open subset of $J(f)$ such that $J(f) \cap \mathrm{Crit}(f) \subset V$ and let

$$K(V) = J(f) \setminus \bigcup_{n\geq 0} f^{-n}(V).$$

Since $K(V)$ is compact, f-invariant and disjoint from the set of critical points, we can consider the standard topological pressure $\mathrm{P}(f|_{K(V)}, -t\int\log|f'|)$. Put

$$\mathrm{P}_{\mathrm{DU}}(t) = \sup_V\{\mathrm{P}(f|_{K(V)}, -t\log|f'|)\},$$

where the supremum is taken over all open sets V considered above.

5. Minimal conformal eigenvalue. The minimal conformal eigenvalue $\lambda(t)$ is defined to be the infimum of all $\lambda > 0$ for which there exists a Borel probability measure m such that

$$\frac{dm \circ f}{dm} = \lambda |f'|^t.$$

We set

$$\mathrm{P}_c(t) = \log \lambda(t).$$

6. Point pressure.
Given $z \in \overline{\mathbb{C}} \setminus \bigcup_{n \geq 0} f^n(\mathrm{Crit}(f))$ and $t \geq 0$ put

$$\mathrm{P}_z(t) = \limsup_{n \to \infty} \frac{1}{n} \sum_{x \in f^{-n}(z)} |(f^n)'(x)|^{-t}.$$

F. Przytycki has proved in [12] that there exists a set $G \subset \overline{\mathbb{C}} \setminus \bigcup_{n \geq 0} f^n(\mathrm{Crit}(f))$ such that $\mathrm{HD}(\overline{\mathbb{C}} \setminus G) = 0$ and $\mathrm{P}_z(t) = \mathrm{P}_w(t)$ for all $z, w \in G$. This common value will be denoted by $\mathrm{P}_p(t)$. It is not difficult to check (see [12]) that the following proposition is true.

Proposition 2.1. *All the pressures defined in the items (1)-(6) are Lipschitz continuous and monotone with respect to the variable t.*

The following fact has been proved in [12] (comp. [4] and [16])

Theorem 2.2. *There exists a number $h = h(f)$ called the Poincaré exponent of the function f in [12] and called the dynamical dimension of the Julia set in [4] such that all the pressures defined in items (1)-(6) coincide on the interval $[0, h]$, are positive on $[0, h)$ and vanish at the point h.*

Our aim in this section is to extend this equality of pressures to the whole set $[0, \infty)$ in the case of critically non-recurrent dynamics. We start with the following.

Lemma 2.3. *If f is critically non-recurrent and $y \in J(f)$ is a periodic point of f, say of period q, then there exists a sequence $\{y_n\}_{n=1}^\infty \subset J(f)$ of periodic points of f, say of period q_n respectively, all different from y such that $\lim_{n \to \infty} y_n = y$ and*

$$\lim_{n \to \infty} \frac{1}{q_n} \log |(f^{q_n})'(y_n)| \leq \frac{1}{q} \log |(f^q)'(y)|.$$

In addition, if there exists no $k \geq 1$ such that $f^{-k}(y) \setminus \{f^j(y) : 0 \leq j \leq q-1\} \subset \mathrm{Crit}(f)$ or if $y \in \Omega$, then this inequality can be replaced by equality.

Proof. Our strategy is to approximate y by periodic points of f^q. Without loss of generality we may assume that $q = 1$. Regardless whether y is repelling or rationally indifferent there exists $\theta > 0$, $x \in J(f) \cap B(y,\theta)$ and $\epsilon > 0$ such that all the local holomorphic inverse branches f_y^{-n} of all iterates of f are defined on $\overline{B}(x, 2\epsilon) \subset B(y,\theta) \setminus \{y\}$, the closed ball centered at x and with radius 2ϵ. Local means here that $\lim_{n\to\infty} f_y^{-n}(x) = y$ and $f_y^{-n}(\overline{B}(x,2\epsilon)) \subset B(y,\theta)$. In the case when y is a repelling point all these branches are defined on the entire ball $B(y,\theta)$ for $\theta > 0$ small enough. Since $\overline{\bigcup_{n\ge 0} f^n(\mathrm{Crit}(f))}$ is by Lemma 5.2 from [17] nowhere dense in $J(f))$, there exists a closed ball $\overline{B} \subset 2B \subset B(x,\epsilon)$ centered at a point $w \in J(f)$ such that $2B \cap \overline{\bigcup_{n\ge 0} f^n(\mathrm{Crit}(f))} = \emptyset$. This implies that for every $n \ge 0$, $f_y^{-n}(2B) \cap \overline{\bigcup_{n\ge 0} f^n(\mathrm{Crit}(f))} = \emptyset$ and since $f : J(f) \to J(f)$ is topologically exact, there thus exists $l \ge 1$ independent of n and a holomorphic branch $f_*^{-l} : f_y^{-n}(2B) \to \overline{\mathbb{C}}$ of f^{-l} sending $f_y^{-n}(w)$ to B. Therefore, for every $n \ge 1$ large enough $f_*^{-l} \circ f_y^{-n}(2B) \subset 2B$. Hence by the Brouwer fixed point theorem there exists $y_{n+l} \in 2B$, a fixed point of $f_*^{-l} \circ f_y^{-n} : 2B \to 2B$. Hence $f^{n+l}(y_{n+l}) = y_{n+l}$ and $y_{n+l} \ne y$ as, by the choice of ϵ, $y \notin 2B$. It is clear that in the repelling case

$$\lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(y_n)| \le \log |f'(y)| \tag{2.1}$$

and in the parabolic case this follows from the fact that $|(f_y^{-n})'(w)| \asymp n^{-\frac{p+1}{p}}$ for all $w \in B(x,\epsilon)$, where $p \ge 1$ is the number of petals of the point y. If there exists no $k \ge 1$ such that $f^{-k}(y) \setminus \{f^j(y) : 0 \le j \le q-1\} \subset \mathrm{Crit}(f)$, then f_*^{-l} extends holomorphically on $B(y,\kappa)$ for some $\kappa > 0$ sufficiently small and the inequality (2.1) becomes an equality. If $y \in \Omega$ and $f^{-k}(y) \setminus \{f^j(y) : 0 \le j \le q-1\} \subset \mathrm{Crit}(f)$, then $|(f_y^{-n})'(y_{n+l})| \asymp n^{-\frac{p+1}{p}}$ and $|f_y^{-n}(y_{n+l}) - y| \asymp n^{-\frac{1}{p}}$. If $l < k$, we conclude the proof as above. If $k \le l$, passing to a subsequence, we may assume that $\lim_{n\to\infty} y_{n+l} = c$, a critical point of f^l belonging to $f^{-l}(y)$. Denote the order of the critical point c of f^l by s. We then obtain the following.

$$|(f_*^{-l} \circ f_y^{-n})'(y_{n+l})| = |(f_*^{-l})'(f_y^{-n}(y_{n+l}))| \cdot |(f_y^{-n})'(y_{n+l})| \asymp \left(n^{-\frac{1}{p}}\right)^{\left(\frac{1}{s}-1\right)} \cdot n^{-\frac{p+1}{p}} = n^{-\left(1+\frac{1}{sp}\right)}.$$

Thus $|(f^{n+l})'(y_{n+l})| \asymp n^{1+\frac{1}{sp}}$ and therefore

$$\lim_{n\to\infty} \frac{1}{n+l} \log |(f^{n+l})'(y_{n+l})| = 0.$$

The proof is complete. ■

Let Ω be the set of all parabolic points of f, i.e.

$$\Omega = \{\omega \in J(f) : \exists_{q\ge 1} f^q(\omega) = \omega \text{ and } (f^q)'(\omega) = 1\}$$

Lemma 2.4. *Assume that f is critically non-recurrent. If μ is a Borel probability f-invariant ergodic measure supported on $J(f)$, then either*

$$\chi_\mu = \int \log |f'| d\mu > 0 \text{ or } \mu(\Omega) = 1.$$

Proof. Suppose that $\mu(\Omega) < 1$. Since Ω consists of periodic points and since μ is ergodic, this implies that $\mu(\Omega) = 0$. Since no critical point contained in the Julia set of f is periodic, we conclude that

$$\mu\left(\bigcup_{n\ge 0} f^{-n}(\Omega \cup (\mathrm{Crit}(f)\cap J(f)))\right) = 0.$$

Hence, by Birkhoff's ergodic theorem there exists $z \in J(f) \setminus \bigcup_{n\ge 0} f^{-n}(\Omega \cup (\mathrm{Crit}(f)\cap J(f)))$ such that

$$\int \log|f'|d\mu = \lim_{n\to\infty} \frac{1}{n}\log|(f^n)'(z)| \tag{2.2}$$

In view of Proposition 6.1 from [17] there exist an increasing to infinity sequence $\{n_j\}_{j=1}^\infty$ (depending on z) and a number $\eta(z) > 0$ such that

$$C_{n_j}(z, B(f^{n_j}(z), \eta(z))) \cap \mathrm{Crit}(f^{n_j}) = \emptyset,$$

where $C_n(z, F)$ is the connected component of $f^{-n}(F)$ containing z. We may assume that $\eta(z) \le \eta$, where $\eta > 0$ is the constant appearing in Lemma 7.7 of [18]. It therefore follows from Koebe's distortion theorem that

$$|(f^{n_j})'(z)| \asymp \mathrm{diam}(C_{n_j}(z, B(f^{n_j}(z), \eta(z)/2)) \tag{2.3}$$

Choose $\theta > 0$ used in the definition of the operation Comp_*^k from [18]. There then for every $n \ge 1$ exists $n^* \le n$, the only number such that $z \in \mathrm{Comp}_*^{n^*}(B(f^{n_j}(z), \eta(z)/2)) = C_n(z, B(f^n(z), \eta(z)))$. Combining (2.3) and Lemma 7.7 of [18] we obtain that

$$\liminf_{j\to\infty} \frac{1}{n_j^*}\log|(f^{n_j})'(z)| > 0 \tag{2.4}$$

Since $\mu(\Omega) = 0$, we may require $\theta > 0$ to be so small that $\mu(B(\Omega, 2\theta)) < 1/2$. Applying now Birkhoff's ergodic theorem we deduce that we could choose z to satisfy the following

$$\liminf_{j\to\infty} \frac{n_j^*}{n_j} \ge \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} 1\!\!1_{J(f)\setminus B(\Omega,2\theta)} \circ f^j(z) > 1/2.$$

Combining this, (2.4) and (2.2), we conclude that $\int \log|f'|d\mu > 0$. The proof is complete. ■

As an immediate consequence of this lemma we get the following.

Corollary 2.5. *If f is semi-hyperbolic (critically non-recurrent and $\Omega = \emptyset$), then $\int \log|f'|d\mu > 0$ for every Borel probability f-invariant ergodic measure μ supported on $J(f)$.*

The main result of this section is the following.

Theorem 2.6. *Assume that f is critically non-recurrent. Then*

(a) *All the pressures defined in items (1)-(6) coincide throughout the whole interval* $[0,\infty)$ *and denote their common value by* $\mathrm{P}(t)$.
(b) *If* $\Omega=\emptyset$, *then* $\mathrm{P}(t)<0$ *for all* $t>h$.
(c) *If* $\Omega\neq\emptyset$, *then* $\mathrm{P}(t)=0$ *for all* $t\geq h$.
(d) $h=\mathrm{HD}(J(f))$, *the Hausdorff dimension of the Julia sets* $J(f)$.

Proof. The item (d) is an immediate consequence of the results obtained in [17]. It follows from the facts established in the course of the proof of Theorem A2.9 in [12] that

$$\mathrm{P}_c(t)=\mathrm{P}_p(t)\geq \mathrm{P}_{hypvar}(t)=\mathrm{P}_{hyp}(t)\leq \mathrm{P}_{var}(t)\geq \mathrm{P}_{\mathrm{DU}}(t).$$

Thus, in order to complete the proof of item (a), it suffices to show that

$$\mathrm{P}_{var}(t)=\mathrm{P}_{hypvar}(t) \tag{2.5}$$

and

$$\mathrm{P}_{\mathrm{DU}}(t)\geq \mathrm{P}_c(t). \tag{2.6}$$

And indeed, if $\Omega=\emptyset$, then (2.5) immediately follows from Lemma 2.4. If $\Omega\neq\emptyset$ and $t>h$, then in view of Proposition 2.1, $\mathrm{P}_{var}(t)=0$. Using in addition Lemma 2.4, we see that also $\mathrm{P}_{hypvar}(t)=0$. Therefore, applying Theorem 2.2, we conclude the proof of formula (2.5). In order to prove (2.6) we shall construct a Borel probability measure m on $J(f)$ such that $\frac{dm\circ f}{dm}=\mathrm{e}^{\hat{\mathrm{P}}(t)}|f'|^t$ for some $\hat{\mathrm{P}}(t)\leq \mathrm{P}_{\mathrm{DU}}(t)$. And indeed, for every $c\in \mathrm{Crit}(f)\cap J(f)$ there exists a point $y_c\in\omega(c)\setminus\bigcup_{n\geq 0}f^{-n}(\mathrm{Crit}(f)\cap J(f))$. For every $n\geq 1$ let

$$V_n=\bigcup_{c\in\mathrm{Crit}(f)\cap J(f)}B(y_c,1/n).$$

Then for all n large enough $V_n\cap\mathrm{Crit}(f)=\emptyset$. In addition, for every $c\in\mathrm{Crit}(f)\cap J(f)$ there exists $k(c)\geq 1$ such that $f^{k(c)}(c)\in V_n$ and

$$K(V_n)\subset K\left(\bigcup_{c\in\mathrm{Crit}(f)\cap J(f)}f^{-k(c)}(V_n)\right).$$

Thus

$$\mathrm{P}_n(t):=\mathrm{P}(f|_{K(V_n)},-t\log|f'|)\leq \mathrm{P}\Big(f|_{K\left(\bigcup_{c\in\mathrm{Crit}(f)\cap J(f)}f^{-k(c)}(V_n)\right)},-t\log|f'|\Big)\leq \mathrm{P}_{\mathrm{DU}}(t) \tag{2.7}$$

Since it is not difficult to see that $f|_{K\left(\bigcup_{c\in\mathrm{Crit}(f)\cap J(f)}f^{-k(c)}(V_n)\right)}$ is expansive and consequently so is $f|_{K(V_n)}$, it follows from Theorem 3.12 in [2] that there exists a Borel probability measure m_n supported on $K(V_n)$ for which

$$m_n(f(A))=\int_A \mathrm{e}^{\mathrm{P}_n(t)}|f'|^t dm_n \tag{2.8}$$

for every Borel set $A \subset K(V_n) \setminus \partial V_n$ (∂V_n is the only set where the map $f|_{K(V_n)}$ may fail to be open) such that $f|_A$ is 1-to-1 and

$$m_n(f(A)) \geq \int_A \mathrm{e}^{\mathrm{P}_n(t)} |f'|^t dm_n \tag{2.9}$$

for every Borel set $A \subset K(V_n)$ such that $f|_A$ is 1-to-1. A straightforward analysis (see [4] for details) shows that (2.8) continues to hold for all sets $A \subset J(f) \setminus V_n$ and (2.9) continues to hold for all sets $A \subset J(f)$ satisfying in each case the requirement that $f|_A$ is 1-to-1. Let m be a weak limit of measures m_n as $n \nearrow \infty$. Since $\mathrm{P}_n(t)$ is an increasing function, the limit $\hat{\mathrm{P}}(t) = \lim_{n\to\infty} \mathrm{P}_n(t)$ exists and by (2.7), $\hat{\mathrm{P}}(t) \leq \mathrm{P}_{\mathrm{DU}}(t)$. Proceeding as in the proof of Lemma 5.5 in [4] (comp. [16]) we conclude that

$$m(f(A)) = \int_A \mathrm{e}^{\hat{\mathrm{P}}(t)} |f'|^t dm \tag{2.10}$$

for every Borel set $A \subset J(f) \setminus \{y_c : c \in \mathrm{Crit}(f)\}$ such $f|_A$ is 1-to-1 and

$$m(f(A)) \geq \int_A \mathrm{e}^{\hat{\mathrm{P}}(t)} |f'|^t dm \tag{2.11}$$

for every Borel set $A \subset J(f)$ such $f|_A$ is 1-to-1. In order to proceed further we need to impose more restrictions on the choice of points y_c. Namely, since for every $c \in \mathrm{Crit}(f)$, $\omega(c)$ is compact and $f(\omega(c)) \subset \omega(c)$, there thus exists a Borel probability f-invariant ergodic measure μ_c supported on $\omega(c)$. Fix an arbitrary point $y_c \in \mathrm{supp}(\omega(c))$ which is recurrent and such that

$$\lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(y_c)| = \chi_{\mu_c} = \int \log |f'| d\mu_c. \tag{2.12}$$

Our aim is to show that (2.10) is also satisfied for the singleton $A = \{y_c\}$. And indeed suppose first that y_c is eventually periodic. Since y_c is recurrent, it must be periodic. Fix $\epsilon > 0$. In view of Lemma 2.3 there exists a periodic point x_c whose periodic orbit is disjoint from $\mathrm{Crit}(f)$ and such that

$$\frac{1}{q} \log |(f^q)'(x_c)| \leq \chi_{\mu_c} + \epsilon,$$

where $q \geq 1$ is the shortest period of x_c. Let μ_q be the atomic probability measure equidistributed on the forward orbit of x_c. Of course μ_q is ergodic and f-invariant. Since for all $n \geq 1$ large enough $x_c \in K(V_n)$, we get

$$\hat{\mathrm{P}}(t) \geq \mathrm{P}_n(t) \geq \mathrm{h}_{\mu_q} - t \log |f'| d\mu_q = -t\frac{1}{q} \log |(f^q)'(x_c)| \geq -t\chi_{\mu_c} - t\epsilon.$$

Letting $\epsilon \searrow 0$ we therefore get $\hat{\mathrm{P}}(t) \geq -t\chi_{\mu_c} = -\frac{t}{p} \log |(f^p)'(y_c)|$, where $p \geq 1$ is the shortest period of periodic point y_c. Equivalently

$$\mathrm{e}^{p\hat{\mathrm{P}}(t)} |(f^p)'(y_c)|^t \geq 1 \quad \text{or} \quad \mathrm{e}^{p\hat{\mathrm{P}}(t)} |(f^p)'(f(y_c))|^t \geq 1.$$

Using this and applying (2.11) consecutively to the sets $\{f^p(y_c)\}, \{f^{p-1}(y_c)\} \dots, \{f(y_c)\}$, we get

$$m(f(y_c)) = m(f(f^p(y_c))) \ge \int_{f^p(y_c)} \mathrm{e}^{\hat{\mathrm{P}}(t)}|f'|^t dm \ge \int_{f^{p-1}(y_c)} \mathrm{e}^{2\hat{\mathrm{P}}(t)}|(f^2)'(f(y_c))|^t dm \ge \dots \ge$$
$$\int_{f(y_c)} \mathrm{e}^{p\hat{\mathrm{P}}(t)}|(f^p)'(f(y_c))|^t dm = \mathrm{e}^{p\hat{\mathrm{P}}(t)}|(f^p)'(f(y_c))m(f(y_c)) \ge m(f(y_c)).$$

So, all the inequalities in this formula are in fact equalities and in particular

$$m(f(y_c)) = \int_{f^p(y_c)} \mathrm{e}^{\hat{\mathrm{P}}(t)}|f'|^t dm = \int_{y_c} \mathrm{e}^{\hat{\mathrm{P}}(t)}|f'|^t dm$$

and we are done in this case. So, suppose that the point y_c is not eventually periodic. Since $y_c \in \mathrm{supp}(\omega(c))$, along with Lemma 2.4 and (2.12) this implies that

$$\lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(y_c)| = \chi_{\mu_c} > 0.$$

It is not difficult to demonstrate (see [16] for instance) that there exists $R > 0$ such that for every $n \ge 1$ the holomorphic inverse branch $f_{y_c}^{-n} : B(f^n(y_c), 4R) \mapsto \overline{\mathbb{C}}$ of f^n sending $f^n(y_c)$ to y_c is well-defined. Since by Corollary 6.2 from [17] and by Koebe's distortion theorem, $\lim_{n\to\infty} \frac{1}{n}\mathrm{diam}\Big(f_{y_c}^{-n}(B(f^n(y_c), 2R)\Big) = 0$, and since the point y_c is recurrent, there thus exists an increasing to infinity sequence $\{n_k\}_{k\ge1}$ of positive integers such that

$$f_{y_c}^{-n_k}\Big(\overline{B}(f^{n_k}(y_c), R)\Big) \subset \overline{B}(f^{n_k}(y_c), R).$$

Therefore, by Brouwer's fixed point theorem for every $k \ge 1$ there exists a point $x_k \in \overline{B}(f^{n_k}(y_c), R)$ such that $f_{y_c}^{-n_k}(x_k) = x_k$. This implies that $f^{n_k}(x_k) = x_k$ and by Koebe's distortion theorem,

$$|(f^{n_k})'(x_k)|^{-1} = |(f_{y_c}^{-n_k})'(x_k)| \ge K^{-1}|(f_{y_c}^{-n_k})'(f_{y_c}^{n_k}(y_c))| = K^{-1}|(f^{n_k})'(y_c))|^{-1}, \tag{2.13}$$

where $K \ge 1$ is the Koebe constant corresponding to the Koebe's factor $1/2$. Since $\lim_{k\to\infty} x_k = y_c$ and all the points x_k are different from y_c, infinitely many of them are mutually distinct and (since these are periodic) their forward trajectory is disjoint from $\{y_a : a \in \mathrm{Crit}(f)\}$. Hence, for every $k \ge 1$ there exists $j_k \ge 1$ such that $x_k \in K(V_{j_k})$. Therefore

$$\hat{\mathrm{P}}(t) \ge \mathrm{P}_{j_k}(t) \ge \mathrm{h}_{\nu_k}(f) - t\int \log|f'|d\nu_k = \frac{-t}{n_k}\log|(f^{n_k})'(x_k)|,$$

where ν_k is the ergodic f-invariant probability measure equidistributed on the forward orbit of x_k. Thus, it follows from (2.13) that

$$\mathrm{e}^{n_k\hat{\mathrm{P}}(t)}|(f^{n_k})'(y_c))|^t \ge \mathrm{e}^{n_k\hat{\mathrm{P}}(t)}K^{-t}|(f^{n_k})'(x_k)|^t \ge K^{-t}.$$

So, applying (2.11) we obtain that

$$1 \ge m(\{f^{n_k}(y_c) : k \ge 1\}) \ge \sum_{k=1}^{\infty} K^{-t} m(y_c)$$

which implies that $m(y_c) = 0$. Replacing in the above considerations y_c by $f(y_c)$, we see that also $m(f(y_c)) = 0$, and, in particular

$$m(f(y_c)) = \int_{y_c} \mathrm{e}^{\hat{\mathrm{P}}(t)} |f'|^t dm.$$

Consequently, (2.10) holds for every Borel set $A \subset J(f)$ such $f|_A$ is 1-to-1. Thus $\mathrm{P}_c(t) \leq \hat{\mathrm{P}}(t) \leq \mathrm{P}_{\mathrm{DU}}(t)$ which completes the proof of part (2.6) and the whole item (a) of Theorem 2.6.

In order to prove item (b) notice that it immediately follows from Theorem 2.1 in [1] and Koebe's distortion theorem that if $\Omega = \emptyset$, then there are constants $C > 0$ and $\beta > 1$ such that for every $z \in J(f) \setminus \overline{\bigcup_{n \geq 0} f^n(\mathrm{Crit}(f))}$, every $n \geq 1$ and every $x \in f^{-n}(z)$, we have $|(f^n)'(x)| \geq C\beta^n$. Write t in the form $\delta + \eta$, $\eta > 0$. Then for every $n \geq 1$

$$\sum_{x \in f^{-n}(z)} |(f^n)'(x)|^{-(\delta+\eta)} = \sum_{x \in f^{-n}(z)} |(f^n)'(x)|^{-\eta} |(f^n)'(x)|^{-\delta} \leq C^{-\eta}\beta^{-\eta n} \sum_{x \in f^{-n}(z)} |(f^n)'(x)|^{-\delta}.$$

Hence

$$\begin{aligned} \mathrm{P}_p(t) &\leq \limsup_{n\to\infty} \frac{1}{n} \log(C^{-\eta}) - \eta \log \beta + \lim_{n\to\infty} \frac{1}{n} \sum_{x \in f^{-n}(z)} |(f^n)'(x)|^{-\delta} \\ &= -\eta \log \beta + \mathrm{P}_p(\delta) \leq -\eta \log \beta < 0. \end{aligned}$$

Let us finally prove item (c). If $\omega \in \Omega$, let ν be the purely atomic probability measure equidistributed on the forward orbit of ω. The measure ν is f-invariant and ergodic. It follows from item (a) of our theorem that for each $t \geq 0$, $\mathrm{P}(t) = \mathrm{P}_{var}(t) \geq \mathrm{h}_\nu - t \log |f'| d\mu = 0 - 0 = 0$. The proof of Theorem 2.6 is complete. ■

Remark 2.7. *As Feliks Przytycki has pointed out to me the equality* $\mathrm{P}_p(t) = \mathrm{P}_{var}(t)$ *follows easily for all rational functions and all* $t \geq 0$ *from the results proven in* [13].

3. Conformal and Invariant Measures

Throughout the entire section, similarly as in the previous one, $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is assumed to be a critically non-recurrent rational function. Given $t \geq 0$ a Borel probability measure m_t supported on the Julia set $J(f)$ is called a t-conformal Gibbs state if f is non-singular with respect to m_t and moreover

$$\frac{dm_t \circ f}{dm_t} = \mathrm{e}^{\mathrm{P}(t)} |f'|^t.$$

It follows from Theorem 2.6 that a t-conformal Gibbs state exists for all $t \geq 0$. Since no critical point in the Julia set is periodic, for every $c \in \mathrm{Crit}(J(f)) := \mathrm{Crit}(f) \cap J(f)$ there exists $p(c) \geq 1$ such that $f^{p(c)-1} \in \mathrm{Crit}(J(f))$ and

$$\{f^j(c) : j \geq p(c)\} \cap \mathrm{Crit}(f) = \emptyset. \tag{3.1}$$

Let

$$\chi(c) = \liminf_{k\to\infty} \frac{1}{k} \log \inf_{n\ge 1}\{|(f^k)'(f^n(c))|\}$$

and let

$$\chi := \min\left\{\frac{\chi(c)}{q(c)} : c \in \text{Crit}(f)\right\},$$

where $q(c)$ is the order of the critical point c for the function $f^{p(c)}$. We have the following.

Proposition 3.1. *If $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a critically critically non-recurrent rational function, then we have $m_t\left(\overline{\bigcup_{n\ge 1}(f^n(\text{Crit}(J(f))))}\right) = 0$ for each t-conformal Gibbs state m_t. If in addition $t \in [0, \text{HD}(J(f))]$ (the case $t = \text{HD}(J(f))$ is established in* [17]*), then $m_t(\Omega) = 0$.*

Proof. Since the set

$$\overline{\bigcup_{n\ge 1}(f^n(\text{Crit}(J(f)))}$$

is nowhere dense in $J(f)$, the equality $m_t\left(\overline{\bigcup_{n\ge 1}(f^n(\text{Crit}(f))}\right) = 0$ can be proved in the same way as Corollary 7.2 (and Lemma 7.1 in [17]. For an elaborated argument in a more complexed situation comp. [15].

The same reasoning as in Theorem 4.2 of [18] gives the following.

Theorem 3.2. *If m_t is a t-conformal Gibbs state and if $m_t(\Omega) = 0$, then up to a multiplicative constant there exists a unique f-invariant σ-finite measure μ_t absolutely continuous with respect to m_t. Moreover μ_t is equivalent with m_t and it is conservative and ergodic.*

The measure μ_t will be frequently called an invariant t-conformal Gibbs state. The critical question for our purposes is when the measure μ_t is finite. In order to give at least a partial answer to this question we need to know how the measure μ_t is constructed. This is done in the paper [8]. So, let us describe the relevant theorem and relevant construction from this paper. Suppose that X is a σ-compact metric space, m is a Borel probability measure on X, positive on open sets, and that a measurable map $T : X \to X$ is given with respect to which measure m is quasi-invariant, i.e. $m \circ T^{-1} \prec m$. Moreover we assume the existence of a countable partition $\alpha = \{A_n : n \ge 0\}$ of subsets of X which are all σ-compact and of positive measure m. We also assume that $m(X \setminus \bigcup_{n\ge 0} A_n) = 0$, and if additionally for all $m, n \ge 1$ there exists $k \ge 0$ such that

$$m(T^{-k}(A_m) \cap A_n) > 0,$$

then the partition α is called irreducible. Martens' result comprising Proposition 2.6 and Theorem 2.9 of [8] reads as follows.

Theorem 3.3. *Suppose that $\alpha = \{A_n : n \geq 0\}$ is an irreducible partition for $T : X \to X$. Suppose that T is conservative and ergodic with respect to the measure m. If for every $n \geq 1$ there exists $K_n \geq 1$ such that for all $k \geq 0$ and all Borel subsets A of A_n*

$$K_n^{-1}\frac{m(A)}{m(A_n)} \leq \frac{m(T^{-k}(A))}{m(T^{-k}(A_n))} \leq K_n\frac{m(A)}{m(A_n)},$$

then T has a σ-finite T-invariant measure μ absolutely continuous with respect to m. Additionally μ is equivalent with m, conservative and ergodic, and unique up to a multiplicative constant.

Since in the sequel we will the method in which the invariant measure claimed in Theorem 3.3 is produced we shall also describe this procedure briefly. Following Martens one considers the following sequences of measures

$$S_k(m) = \sum_{i=0}^{n-1} m \circ T^{-i} \quad \text{and} \quad Q_k(m) = \frac{S_k(m)}{S_k(m)(A_0)}.$$

It is proven in [8] that each weak limit μ of the sequence $Q_n(m)$ has the properties required in Theorem 3.3, where a sequence $\{\nu_k : k \geq 1\}$ of measures on X is said to converge weakly if for all $n \geq 1$ the measures ν_k converge weakly on all compact subsets of A_n. In fact it turns out that the sequence $Q_n(m)$ converges and

$$\mu(F) = \lim_{n\to\infty} Q_n(m)(F)$$

for every Borel set $F \subset X$. Let us now describe the construction of partition α in the context of critically non-recurrent rational functions. Indeed, set $Y = J(f) \setminus (\mathrm{PC}(f) \cup \Omega)$, where $\mathrm{PC}(f) = \overline{\bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))}$. For every $y \in Y$ consider a ball $B(y, r(y))$ such that $r(y) > 0$, $m(\partial B(y, r(y))) = 0$, and $r(y) < (1/2)\mathrm{dist}(y, \mathrm{PC}(f) \cup \Omega)$. The balls $B(y, r(y))$, $y \in Y$, cover Y and since Y is a metric separable space, one can choose a countable cover, say $\{\tilde{A}_n : n \geq 0\}$, from them. We may additionally require that the family $\{\tilde{A}_n : n \geq 0\}$ is locally finite that is that each point $x \in Y$ has an open neighborhood intersecting only finitely many balls $\tilde{A}_n$, $n \geq 0$. We now define the family $\alpha = \{A_n : n \geq 0\}$ inductively setting

$$A_0 = \tilde{A}_0 \quad \text{and} \quad A_{n+1} = \tilde{A}_{n+1} \setminus \bigcup_{k=1}^{n} \overline{\tilde{A}_n}$$

(and throwing away empty sets). Obviously α is a disjoint family and

$$\bigcup_{n\geq 1} A_n \supset J(f) \setminus (\mathrm{PC}(f) \cup \Omega) \setminus \bigcup_{n\geq 0} \partial\tilde{A}_n$$

whence in view of Corollary 7.2] in [17], $m\Big(\bigcup_{n\geq 0} A_n\Big) = 1$.

In order to provide some sufficient conditions for the measure μ_t to be finite, we need a stronger assumption than critically non-recurrence. Namely, from now on throughout this entire section we assume that $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is parabolically semi-hyperbolic, that is that $c \notin \omega(\mathrm{Crit}(f))$ for all $c \in \mathrm{Crit}(J(f))$ and $\omega(\mathrm{Crit}(J(f))) \cap \Omega(f) = \emptyset$.

Theorem 3.4. *Suppose that $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is a parabolically semi-hyperbolic map and that $\mathrm{P}(t)>-\chi t$. If m_t is the t-conformal Gibbs state and $m_t(\Omega)=0$, then the set of points of infinite condensation of the invariant t-conformal Gibbs state μ_t is contained in $\Omega(f)$.*

Proof. By the standard normal family argument there exists $u\ge 1$ and $\lambda>1$ such that

$$|(f^u)'(z)|>\lambda$$

for all $z\in\omega(\mathrm{Crit}(J(f)))$. Thus

$$|(f^u)'(f^n(c))|\ge\lambda \tag{3.2}$$

for every $c\in\mathrm{Crit}(J(f))$ and all n large enough. Since the conformal measure m_t is positive on non-empty open sets, $\inf\{m_t(B(x,r)):x\in J(f)\}>0$ for every $r>0$; even more, there exists $\alpha(r)\in(0,r)$ such that

$$M(r)=\inf\{m_t(B(x,r)\setminus B(x,\alpha(r)):x\in J(f)\}>0. \tag{3.3}$$

It follows from (3.1) and (3.2) that there exists $\delta>0$ such that for every $c\in\mathrm{Crit}(J(f))$ $k\ge 1$ and every $n\ge p(c)$ the holomorphic inverse branch $f^{-k}_{f^n(c)}:B(f^{n+k}(c),4\delta)\to\mathbb{C}$ sending $f^{n+k}(c)$ to $f^n(c)$ is well-defined. It also follows (3.2) that for all u large enough, all $c\in\mathrm{Crit}(J(f))$, all $k\ge 0$ and all $0\le i\le u-1$

$$f^{-u}_{f^{p(c)+i+ku}}\Big(B(f^{p(c)+i+(k+1)u}(c),2\delta)\Big)\subset B(f^{p(c)+i+ku}(c),\alpha(\delta)). \tag{3.4}$$

For every $c\in\mathrm{Crit}(J(f))$, all $0\le j\le u-1$ and all $i\ge 0$ define now

$$\begin{aligned} R_{i,j}(c)&=f^{-ju}_{f^{p+i}(c)}\Big(B(f^{p+i+ju}(c),2\delta)\Big)\setminus f^{-(j+1)u}_{f^{p+i}(c)}\Big(B(f^{p+i+(j+1)u}(c),2\delta)\Big)\\ &=f^{-ju}_{f^{p+i}(c)}\Big(B(f^{p+i+ju}(c),2\delta)\setminus f^{-u}_{f^{p+i+ju}(c)}\Big(B(f^{(p+i+(j+1)u}(c),2\delta)\Big)\Big), \end{aligned} \tag{3.5}$$

where $p=p(c)$. Applying Koebe's distortion theorem and using (3.3) along with (3.4) we conclude that

$$\begin{aligned} m_t(R_{i,j}(c))&\asymp e^{-\mathrm{P}(t)ju}|(f^{ju})'(f^{p+i}(c))|^{-t}m_t\Big(B(f^{p+iju}(c),2\delta)\setminus f^{-u}_{f^{p+i+ju}(c)}\Big(B(f^{(p+i+(j+1)u}(c),2\delta\Big)\Big)\\ &\asymp e^{-\mathrm{P}(t)ju}|(f^{ju})'(f^{p+i}(c))|^{-t} \end{aligned} \tag{3.6}$$

Fix now a point $\xi\in\overline{\bigcup_{n\ge 0}f^n(\mathrm{Crit}(J(f)))}$. Obviously, because of parabolical semi-hyperbolicity of f, the latter set is disjoint from Ω. Let $x=f^s(\xi)$, where $s\ge 1$ is so large that

$$\{f^n(x):n\ge 0\}\cap\mathrm{Crit}(f)=\emptyset. \tag{3.7}$$

Since $\mathrm{Crit}(J(f))\cap\omega(\mathrm{Crit}(J(f)))=\emptyset$, it follows from Lemma 2.13 in [17] that there exists $0<\gamma<\delta/2$ such that if $n\ge 1$ and $y\in f^{-n}(x)$, then $f^k(C_n(y,B(x,4\gamma)))\cap\mathrm{Crit}(f)$ consists of at most one point for every $0\le k\le n-1$. Without loss of generality we may assume that the set A_0 involved in the definition of the invariant measure μ_t is contained in $B(x,\gamma)$.

Suppose first that $C_n(y,B(x,2\gamma))\cap \mathrm{Crit}(f^n)=\emptyset$. It then follows from Koebe's distortion theorem that

$$\frac{m_t\Big(C_n(y,B(x,\gamma))\Big)}{m_t\Big(C_n(y,B(x,\gamma))\cap f^{-n}(A_0)\Big)}\preceq \frac{m_t(B(x,\gamma)}{m_t(A_0)}\asymp 1. \tag{3.8}$$

Suppose in turn that $C_n(y,B(x,2\gamma))\cap\mathrm{Crit}(f^n)\neq\emptyset$ and let $0\le k\le n-1$ be the least integer such that $f^k(C_n(y,B(x,4\gamma)))\cap\mathrm{Crit}(f)\neq\emptyset$. Denote by c its only element. Note that by (3.7) $k+p(c)\le n$. Put $p=p(c)$. Taking $\gamma>0$ sufficiently small, we may assume that

$$\mathrm{Crit}(f)\cap\bigcup_{k=0}^n f^k\Big(C_n(y,B(x,\gamma)\Big)\subset\{f^j(c):0\le j\le p-1\}.$$

In particular

$$c\in f^k(C_n(y,B(x,2\gamma))). \tag{3.9}$$

We have

$$|(f^{p+i})'(z)|\asymp|z-c|^{q(c)-1} \tag{3.10}$$

for all $0\le i\le u-1$ and all $z\in C_{p+i}(c,B(f^{p+i}(c),2\delta))$ (note that q(c) is also the order of c for the function f^{p+i}). Write $n-k-p=su+r$, $s\ge 0$, $0\le r\le u-1$. Since

$$f^k(C_n(y,B(x,\gamma)))\subset C_{p+r+su}(c,B(f^{p+r+su}(c),\delta)),$$

using (3.6) and (3.10), we get

$$\begin{aligned}m_t\Big(f^k(C_n(y,B(x,\gamma)))\Big)&\preceq\sum_{j\ge s}|(f^{ju})'(f^{p+r}(c))|^{-t}e^{-\mathrm{P}(t)(p+r+ju)}\Big(|(f^{ju})'(f^{p+r}(c))|^{-1}\Big)^{\left(\frac{1}{q(c)}-1\right)t}\\&\asymp\sum_{j\ge s}e^{-\mathrm{P}(t)(p+r+ju)}|(f^{ju})'(f^{p+r}(c))|^{-\frac{t}{q(c)}}.\end{aligned}$$

Since $A_0\subset B(x,\gamma)\subset B(f^{p+r+su}(c),\delta)$, using Koebe's distortion theorem, we get

$$\begin{aligned}&m_t\Big(f^k(C_n(y,B(x,\gamma)))\cap f^{-(n-k)}(A_0)\Big)\asymp\\&\qquad\asymp m_t(A_0)|(f^{su})'(f^{p+r}(c))|^{-t}e^{-\mathrm{P}(t)(p+r+su)}\Big(|(f^{su})'(f^{p+r}(c))|^{-1}\Big)^{\left(\frac{1}{q(c)}-1\right)t}\\&\qquad\asymp e^{-\mathrm{P}(t)(p+r+su)}|(f^{su})'(f^{p+r}(c))|^{-\frac{t}{q(c)}}.\end{aligned}$$

Therefore, using the assumption $\mathrm{P}(t)>-\chi t$, we conclude that

$$\frac{m_t\Big(f^k(C_n(y,B(x,\gamma)))\Big)}{m_t\Big(f^k(C_n(y,B(x,\gamma)))\cap f^{-(n-k)}(A_0)\Big)}\preceq\sum_{j\ge0}e^{-\mathrm{P}(t)(p+r+ju)}|(f^{ju})'(f^{p+r+su}(c))|^{-\frac{t}{q(c)}}\le S(c) \tag{3.11}$$

for some number $S(c)$ depending only on c. Using (3.9) we conclude that

$$\Big(f^k(C_n(y,B(x,4\gamma)))\setminus f^k(C_n(y,B(x,2\gamma)))\Big)\cap\mathrm{Crit}(f^{n-k})=\emptyset$$

and therefore $\mathrm{Mod}\Big(f^k(C_n(y,B(x,4\gamma)))\setminus f^k(C_n(y,B(x,2\gamma)))\Big)\ge(\log 2)/q(c)$. Hence, applying Koebe's distortion theorem and (3.11), we obtain the following.

$$\frac{m_t(C_n(y,B(x,\gamma)))}{m_t\Big(C_n(y,B(x,\gamma)))\cap f^{-n}(A_0)\Big)}\asymp\frac{|(f^k)'(y)|^{-t}e^{-\mathrm{P}(t)k}}{|(f^k)'(y)|^{-t}e^{-\mathrm{P}(t)k}}\cdot\frac{m_t\Big(f^k(C_n(y,B(x,\gamma)))\Big)}{m_t\Big(f^k(C_n(y,B(x,\gamma)))\cap f^{-(n-k)}(A_0)\Big)}$$
$$\preceq S(c).$$

Therefore
$$\frac{m_t(f^{-n}(B(x,\gamma))}{m_t(f^{-n}(A_0)}\preceq\max\{S(c):c\in\mathrm{Crit}(J(f))\}$$
and consequently $Q_n(B(x,\gamma))\preceq\max\{S(c):c\in\mathrm{Crit}(J(f))\}$ for all $n\ge1$. Thus $\mu_t(B(x,\gamma))<\infty$ and x is a point of finite condensation of μ_t. Since μ_t is an invariant measure, we conclude that ξ is also a point of finite condensation of μ_t and we are done. If $\xi\in J(f)\setminus\Big(\overline{\bigcup_{n\ge0}f^n(\mathrm{Crit}(J(f)))}\cup\Omega(f)\Big)$ the argument is easier. ■

A parabolically semi-hyperbolic map $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is called semi-hyperbolic if $\Omega(f)=\emptyset$. As an immediate consequence of Theorem 3.4, we get the following.

Corollary 3.5. *Suppose that $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is semi-hyperbolic. If $\mathrm{P}(t)>-\chi t$, then the invariant t-conformal Gibbs state μ_t is finite.*

Allowing parabolic points we still get the following remarkable.

Theorem 3.6. *Suppose that $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is parabolically semi-hyperbolic. If $t\in[0,\mathrm{HD}(J(f)))$ (then $m_t(\Omega)=0$ is satisfied by Proposition 3.1, $\chi>0$ follows from parabolical semi-hyperbolicity and we have $\mathrm{P}(t)>0>-\chi t$), then the t-conformal Gibbs state μ_t is finite.*

Proof. Since $h=\delta(f)$ and $t\in[0,h)$, it follows from Theorem 2.2 that $\mathrm{P}(t)>0$. Hence, for every $k\ge1$ and every $\omega\in\Omega$
$$m_t\left(B\left(\omega,k^{-\frac{1}{p(\omega)}}\right)\right)\asymp\sum_{j=k^{\frac{1}{p(\omega)}}}^{\infty}\mathrm{e}^{-\mathrm{P}(t)j}j^{-\frac{p(\omega)+1}{p(\omega)}t}\le\sum_{j=k^{\frac{1}{p(\omega)}}}^{\infty}\mathrm{e}^{-\mathrm{P}(t)j}\asymp\mathrm{e}^{-\mathrm{P}(t)k^{\frac{1}{p(\omega)}}}.$$
Therefore, proceeding exactly as in the proof of Proposition 6.2 from [18], instead of (6.2) we would get
$$Q_n(B)\asymp\sum_{k=1}^{n-1}\mathrm{e}^{-\mathrm{P}(t)k^{\frac{1}{p(\omega)}}}\frac{S_{n-k}(m_t)(A)}{S_n(m_t)(A)}\le\sum_{k=1}^{\infty}\mathrm{e}^{-\mathrm{P}(t)k^{\frac{1}{p(\omega)}}}.$$
Since $\mathrm{P}(t)>0$, this last series converges and all the numbers $Q_n(B)$ are bounded above by its sum multiplied by a universal constant. This shows that ω is a point of finite condensation of μ_t which finishes the proof in view of Theorem 3.4. ■

4. Escape Rates

Throughout this section $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a critically non-recurrent rational map. Let λ be the normalized Lebesgue measure on the sphere $\overline{\mathbb{C}}$ (with respect to the spherical metric). Following the reasoning from Lemma 5.3 in [18] we shall prove the following.

Lemma 4.1. *If $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is critically non-recurrent, then for every $\theta > 0$ there exists $\epsilon > 0$ such that if D is an open ball centered in $J(f) \setminus B(\Omega, \theta)$ with radius ϵ and B is an open ball such that $2B \subset D \setminus \overline{\bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))}$, then*

$$\inf\left\{\frac{\lambda(B_n)}{\lambda(D_n)}\right\} > 0,$$

where the infimum is taken over all integers $n \geq 0$, all D_n, the connected components of $f^{-n}(D)$ and all connected components B_n of $f^{-n}(B)$ contained in D_n.

Proof. Using Koebe's distortion theorem we obtain

$$\lambda(B_n) \asymp \mathrm{diam}^2(B_n) = \left(\frac{\mathrm{diam}(B_n)}{\mathrm{diam}(D_n)}\right)^2 \mathrm{diam}^2(D_n) \geq \lambda(D_n)\left(\frac{\mathrm{diam}(B_n)}{\mathrm{diam}(D_n)}\right)^2$$

and applying Lemma 3.3 in [18] we get

$$\lambda(B_n) \succeq \lambda(D_n)\left(\frac{\mathrm{diam}(B)}{\mathrm{diam}(D)}\right)^2.$$

The proof is complete. ■

Theorem 4.2. *If f is critically non-recurrent, then for every $\epsilon > 0$ sufficiently small*

$$\limsup_{n\to\infty}\frac{1}{n}\log\lambda\Big(f^{-n}(B(J(f),\epsilon))\Big) = \mathrm{P}(2).$$

Proof. Fix $\epsilon > 0$ so small as required in Lemma 4.1 and such that there exists $z \in G \setminus \overline{\bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))}$ for which $B(z, 2\epsilon) \cap \overline{\bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))} = \emptyset$ (by topological exactness of $f : J(f) \to J(f)$, $\overline{\bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))}$ is obviously a nowhere-dense subset of $J(f)$ in the critically non-recurrent case), where G is the set coming from item (6) of the definition of topological pressures. Applying Koebe's distortion theorem we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\frac{1}{n}\log\lambda\Big(f^{-n}(B(J(f),\epsilon))\Big) &\geq \limsup_{n\to\infty}\frac{1}{n}\log\lambda\Big(f^{-n}(B(z,\epsilon)\Big) \\
&= \limsup_{n\to\infty}\frac{1}{n}\log\sum_{y\in f^{-n}(z)}\lambda\Big(f_y^{-n}(B(z,\epsilon))\Big) = \limsup_{n\to\infty}\frac{1}{n}\log\sum_{y\in f^{-n}(z)}|(f^n)'(y)|^{-2} \qquad (4.1)\\
&= \mathrm{P}(2).
\end{aligned}$$

Since obviously

$$\limsup_{n\to\infty}\frac{1}{n}\log\lambda\Big(f^{-n}(B(J(f),\epsilon))\Big)\le 0,$$

combining this, (4.1) and Theorem 2.6(c), concludes the proof in the case when $\Omega\ne\emptyset$. If $\Omega=\emptyset$, cover $J(f)$ by balls $\{D_i\}_{i=1}^k$ centered respectively at some points $\{x_i\}_{i=1}^k$ in G and with radii so small as required in Lemma 4.1. Since, as we have already observed, $\overline{\bigcup_{n\ge1}f^n(\text{Crit}(f))}$ is nowhere-dense in the Julia set $J(f)$, each ball D_i, $i=1,2,\dots k$ contains a non-empty open ball B_i such that $2B_i\subset D_i\setminus\overline{\bigcup_{n\ge1}f^n(\text{Crit}(f))}$. Thus, applying Lemma 4.1 and Koebe's distortion theorem, we get for every $i=1,2,\dots k$

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\lambda\Big(f^{-n}(D_i)\Big)&=\limsup_{n\to\infty}\frac{1}{n}\log\lambda\Big(f^{-n}(B_i)\Big)\\&=\limsup_{n\to\infty}\frac{1}{n}\log\sum_{y\in f^{-n}(x_i)}|(f^n)'(y)|^{-2}=\mathrm{P}(2).\end{aligned}$$

Hence, assuming $\epsilon>0$ to be so small that $B(J(f),\epsilon)\subset D_1\cup D_2\cup\ldots\cup D_k$, we get

$$\limsup_{n\to\infty}\frac{1}{n}\log\lambda\Big(f^{-n}(B(J(f),\epsilon))\Big)\le\limsup_{n\to\infty}\frac{1}{n}\log\lambda\left(\bigcup_{i=1}^k f^{-n}(D_i)\right)=\mathrm{P}(2).$$

The proof is complete. ■

In particular, it therefore follows from Theorem 2.6(b) that if $\Omega=\emptyset$, then the rate of escape is exponential. In the case when $\Omega\ne\emptyset$, something more can be said about a modified rate of escape. Let $p(\omega)\ge1$ be the number of petals of a parabolic point ω and let

$$p=\max\{p(\omega):\omega\in\Omega\}.$$

Let

$$\xi=\left(\prod_{c\in\text{Crit}(f)}q(c)\right)^{-1},\tag{4.2}$$

where $q(c)\ge2$ is the order of the critical point c of f. Let θ have the same meaning as in [18]. We will need the following.

Lemma 4.3. *If f is critically non-recurrent, then there exists a constant $B\ge1$ such that if $\epsilon>0$ is small enough, $n\ge1$ is an integer, $z\in J(f)$, and $f^n(z)\notin B(\Omega,\theta)$, then*

$$\operatorname{diam}(C_n(z,B(f^n(z),\epsilon)))\le Bn^{-\xi\frac{p+1}{p}}.$$

Proof. Since

$$\lim_{n\to\infty}\sup\Big\{\operatorname{diam}\Big(C_n(w,B(f^n(w)),\epsilon))\Big):w\in J(f),\ f^n(w)\notin B(\Omega,\theta)\Big\}=0,$$

there exists $q\ge1$ such that for every $n\ge q$, $w\in J(f)$, $f^n(w)\notin B(\Omega,\theta)$, we have

$$\operatorname{diam}\Big(C_n(w,B(f^n(w)),\epsilon))\Big)<\epsilon/4.$$

Suppose now that $n \geq q+1$. We shall inductively define the sequence

$$x_0 = f^{k_0}(z), x_1 = f^{k_1}(z), \ldots, x_l = f^{k_l}(z)$$

$(l \leq n)$ of points from the set $\{z, f(z), \ldots, f^n(z)\}$ as follows. We declare $k_0 = n$ and $x_0 = f^n(z)$. If all other points from $\{z, f(z), \ldots, f^n(z)\}$ are contained in $B(\Omega, \theta)$, we put $x_1 = z$ and stop the inductive procedure. Assume now that $x_j = f^{k_j}(z) \in \{z, f(z), \ldots, f^n(z)\} \setminus B(\Omega, \theta)$ has been defined for all $j \in \{0, 1, \ldots, n\}$. If $k_j < q$, we stop the inductive procedure. If $\{z, f(z), \ldots, f^{k_j - q}(z)\} \subset B(\Omega, \theta)$, we stop the inductive procedure by setting $x_{j+1} = z$. Otherwise we define $x_{j+1} = f^k(z)$, where $0 \leq k \leq k_j - q$ is the largest integer such that $f^k(z) \notin B(\Omega, \theta)$. For every $0 \leq j \leq l-1$ define first the sets

$$C_j = C_{k_j - k_{j+1}}(x_{j+1}, B(x_j, \epsilon)) \text{ and } B_j = C_{k_j}(z, B(x_j, \epsilon)).$$

Since $k_j - k_{j+1} \geq q$, we have $C_j \subset B(x_{j+1}, \epsilon/2)$ and define the set

$$A_j = C_{k_{j+1}}(z, B(x_{j+1}, \epsilon)) \setminus C_{k_{j+1}}(z, C_j) = B_{j+1} \setminus B_j.$$

Since $C_j \subset B(x_{j+1}, \epsilon/2)$, $B_j \subset B_{j+1}$ if $j \leq l-1$ and

$$A_j \supset C_{k_{j+1}}(B(x_{j+1}, \epsilon) \setminus C_j) \tag{4.3}$$

where $C_{k_{j+1}}(B(x_{j+1}, \epsilon) \backslash C_j)$ is this connected component of $f^{-k_{j+1}}(B(x_{j+1}, \epsilon) \backslash C_j)$ that encloses the set B_j in B_{j+1}. Improving slightly Lemma 3.1 in [18] we get the following.

$$\mathrm{Mod}(A_j) \geq \xi \log 2. \tag{4.4}$$

It follows from the local behaviour of f around parabolic points and the definition of the sequence $\{x_j\}_{j=0}^{l}$ that there exists a constant $L \leq 1$ such that for all $0 \leq j \leq l-1$

$$\mathrm{diam}(C_j) \leq (L(k_j - k_{j+1}))^{-\frac{p+1}{p}}. \tag{4.5}$$

Fix now an integer $u \geq 1$ so large that for every $t \geq 2$

$$\frac{te^{Mt}}{u^{t-1}} \leq 1, \tag{4.6}$$

where $M = -\left(\log L + \frac{p}{p+1} \log \epsilon\right)$. We define

$$R = \{j \in \{0, 1, \ldots, l-1 : k_j - k_{j+1} \geq u\}.$$

It then follows from (4.5) that for all $j \in R$,

$$\begin{aligned}
\mathrm{Mod}(B(x_{j+1}, \epsilon) \setminus C_j) &\geq \log \epsilon - \log \mathrm{diam}(C_j) \\
&\geq \log \epsilon + \frac{p+1}{p} \log L + \frac{p+1}{p} \log\Big(k_j - k_{j+1}\Big) \\
&= \frac{p+1}{p} \left(\log\Big(k_j - k_{j+1}\Big) - M\right).
\end{aligned}$$

Proceeding in the same way as in the proof of Lemma 3.1 in [18] and using (4.3), we therefore obtain that if $j \in R$, then

$$\mathrm{Mod}(A_j) \geq \xi\frac{p+1}{p}\left(\log\Big(k_j - k_{j+1}\Big) - M\Big)\right. . \tag{4.7}$$

Combining this, (4.4) and using Grötzsch inequality, we conclude that

$$\begin{aligned}\mathrm{Mod}&\Big(B(z,\epsilon) \setminus C_n(z, B(f^n(z),\epsilon))\Big) \\ &\geq \xi \log 2(l-1-\#R) + \xi\frac{p+1}{p}\left(\sum_{j\in R}\log\Big(k_j - k_{j+1}\Big) - M\#R\right). \end{aligned}\tag{4.8}$$

If $\#R \geq 2$, then using (4.6) we obtain

$$\begin{aligned}&\frac{\exp\left(-\xi\frac{p+1}{p}\left(\sum_{j\in R}\log\Big(k_j - k_{j+1}\Big) - M\#R\right)\right)}{\left(\sum_{j\in R}\Big(k_j - k_{j+1}\Big)\right)^{-\xi\frac{p+1}{p}}} = \\ &= \left(\mathrm{e}^{M\#R}\frac{\sum_{j\in R}\Big(k_j - k_{j+1}\Big)}{\prod_{j\in R}\Big(k_j - k_{j+1}\Big)}\right)^{\xi\frac{p+1}{p}} = \left(\mathrm{e}^{M\#R}\sum_{j\in R}\frac{1}{\prod_{i\in R\setminus\{j\}}\Big(k_i - k_{i+1}\Big)}\right)^{\xi\frac{p+1}{p}} \\ &\leq \left(\frac{\#R\mathrm{e}^{M\#R}}{u^{\#R-1}}\right)^{\xi\frac{p+1}{p}} \leq 1\end{aligned}\tag{4.9}$$

Since, by our inductive construction,

$$l - 1 - \#R \geq \frac{1}{q}\left(n - q - \sum_{j\in R}\Big(k_j - k_{j+1}\Big)\right) \geq \frac{1}{2q}\left(n - \sum_{j\in R}\Big(k_j - k_{j+1}\Big)\right)$$

for all n large enough, we therefore obtain

$$\begin{aligned}\exp&\Big(-\mathrm{Mod}\Big(B(z,\epsilon) \setminus C_n(z, B(f^n(z),\epsilon))\Big)\Big) \\ &\leq \exp\left(\frac{-\xi\log 2}{2q}\left(n - \sum_{j\in R}\Big(k_j - k_{j+1}\Big)\right)\right)\left(\sum_{j\in R}(k_j - k_{j+1})\right)^{-\xi\frac{p+1}{p}} \\ &\leq B_1 n^{-\xi\frac{p+1}{p}}\end{aligned}\tag{4.10}$$

for a sufficiently large constant B_1. Since in the case when $\#R \leq 1$, (4.10) follows immediately from (4.8), the formula (4.10) is always true. In view of Theorem 2.4 in [11], the hyperbolic diameter of $C_n(z, B(f^n(z),\epsilon))$ in $B(z,\epsilon)$ is bounded above by

$$B_2 \exp\Big(-\mathrm{Mod}\Big(B(z,\epsilon) \setminus C_n(z, B(f^n(z),\epsilon))\Big)\Big) \leq B_2 B_1 n^{-\xi\frac{p+1}{p}}$$

for some universal constant B_2 and the inequality sign was written due to (4.10). Since $z \in C_n(z, B(f^n(z),\epsilon))$ and since $\lim_{n\to\infty} B_2 B_1 n^{-\xi\frac{p+1}{p}} = 0$, the euclidean and hyperbolic diameters of $C_n(z, B(f^n(z), \epsilon))$ become comparable, and consequently there exists a constant $B \geq 1$ such that

$$\operatorname{diam}\Big(C_n(z, B(f^n(z), \epsilon))\Big) \leq Bn^{-\xi\frac{p+1}{p}}$$

for all $n \geq 1$. The proof is complete. ■

Remark 4.4. *In the proof of Lemma 4.3 one shows in fact that* $\mathrm{Mod}(A_j) \geq \xi_n$*, where* ξ_n *is the degree of* f^n *restricted to* $C_n(z, B(f^n(z), \epsilon))$*. In particular if* $C_n(z, B(f^n(z), \epsilon))$ *contains no critical points of* f^n*, then* $\xi_n = 1$ *and*

$$\operatorname{diam}\Big(C_n(z, B(f^n(z), \epsilon))\Big) \leq Bn^{\frac{p+1}{p}}.$$

So, in the parabolic case (no critical points in the Julia sets), we get the result proven in ([6]).

Recall that in view of Theorem 2.6 $h = \mathrm{HD}(J(f))$ and $\mathrm{P}(h) = 0$. Developing now the approach from ([6]) we shall prove the following.

Theorem 4.5. *For every* $0 < \delta < \operatorname{diam}(J(f))$ *there exist constants* $\epsilon > 0$ *and* $C \geq 1$ *such that*

$$C^{-1}n^{-2\frac{p+1}{p}} \leq \lambda\Big(f^{-n}\Big(B(J(f) \setminus B(\Omega, \delta), \epsilon)\Big)\Big) \leq Cn^{-\xi(h-2)\frac{p+1}{p}}$$

Proof. Fix $\omega \in \Omega$ such that $p(\omega) = p$. Let $q \geq 1$ be the period of ω. Fix also $y \in B(J(f) \setminus B(\Omega, \delta), \epsilon) \setminus \overline{\bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))}$ and then $x \in f^{-l}(y)$ for some $l \geq 1$, so close to ω that the inverse branches f_ω^{-qk} of f^{qk} are well-defined on some some neighbourhood $B(x, r)$ of x and $\lim_{k\to\infty} f_\omega^{-qk}(B(x, r)) = \omega$. Take then $0 < \eta < \epsilon$ so small that $B(y, \eta) \subset B(J(f) \setminus B(\Omega, \delta), \epsilon)$ and $f_x^{-l}(B(y, \eta)) \subset B(x, r)$. It follows from the local behaviour of f around parabolic points that for every $k \geq 1$,

$$\lambda\left(f^{-(l+qk)}\Big(B(J(f) \setminus B(\Omega, \delta), \epsilon)\Big)\right) \geq \lambda\Big(f_\omega^{-qk}(B(x, r)\Big) \asymp |(f_\omega^{-qk})'(x)|^2 \asymp (qk)^{-2\frac{p+1}{p}}$$

and the first inequality of Theorem 4.5 easily follows. In order to prove the opposite inequality fix $\kappa > 0$ ascribed to $\delta > 0$ so small as required in Lemma 4.1 and cover $J(f) \setminus B(\Omega, \delta)$ with finitely many open balls $D_1, D_2, \ldots, D_k$ with radii κ centered at the points of the set $J(f) \setminus B(\Omega, \delta)$. Since $J(f) \setminus B(\Omega, \delta)$ is a compact set, there exists $\epsilon > 0$ such that

$$B(J(f) \setminus B(\Omega, \delta), \epsilon) \subset D_1 \cup D_2 \cup \ldots D_k. \tag{4.11}$$

For every $i \in \{1, 2, \ldots, k\}$ fix then an open ball B_i such that $2B_i \subset D_i \setminus \overline{\bigcup_{n\geq 1} f^n(\mathrm{Crit}(f))}$ and denote its center by z_i. Now, applying Koebe's distortion theorem and Lemma 4.3 (with

$\delta=\theta$), for every $i\in\{1,2,\ldots,k\}$ and for every $n\ge 1$ we obtain

$$
\begin{aligned}
\lambda\Big(f^{-n}(B_i)\Big) &= \lambda\left(\bigcup_{x\in f^{-n}(z_i)} f_x^{-n}(B_i)\right) \le \sum_{x\in f^{-n}(z_i)} \lambda\Big(f_x^{-n}(B_i)\Big) \asymp \sum_{x\in f^{-n}(z_i)} |(f^n)'(x)|^{-2}\lambda(B_i) \\
&\asymp \lambda(B_i)\sum_{x\in f^{-n}(z_i)} |(f^n)'(x)|^{h-2}|(f^n)'(x)|^{-h} \\
&\asymp \lambda(B_i)\sum_{x\in f^{-n}(z_i)} \mathrm{diam}\,(C_n(x,B(z_i,r)))^{2-h}\,|(f^n)'(x)|^{-h} \\
&\preceq \lambda(B_i)\sum_{x\in f^{-n}(z_i)} n^{\xi(h-2)\frac{p+1}{p}}|(f^n)'(x)|^{-h} \\
&= \lambda(B_i)n^{\xi(h-2)\frac{p+1}{p}}\sum_{x\in f^{-n}(z_i)} |(f^n)'(x)|^{-h} \\
&\asymp \lambda(B_i)n^{\xi(h-2)\frac{p+1}{p}}\sum_{x\in f^{-n}(z_i)} m_h(f_x^{-n}(B(z_i,r))) \asymp \lambda(B_i)n^{\xi(h-2)\frac{p+1}{p}}m_h(f^{-n}(B(z_i,r))) \\
&\preceq \lambda(B_i)n^{\xi(h-2)\frac{p+1}{p}}
\end{aligned}
\tag{4.12}
$$

Applying now Lemma 4.1 and (4.11) we obtain for every $n\ge 1$ the following

$$
\begin{aligned}
\lambda\left(f^{-n}\Big(B(J(f)\setminus B(\Omega,\delta),\epsilon)\Big)\right) &\le \sum_{i=1}^k \lambda(f^{-n}(D_i)) \preceq \sum_{i=1}^k \lambda(f^{-n}(B_i)) \preceq \sum_{i=1}^k \lambda(B_i)n^{\xi(h-2)\frac{p+1}{p}} \\
&\le k\max\{\lambda(B_i): i\in\{1,2,\ldots,k\}\}n^{\xi(h-2)\frac{p+1}{p}}
\end{aligned}
$$

We are done. ■

Since in the parabolic case (no critical points in the Julia sets), $\xi=1$, as an immediate consequence of Theorem 4.5 we get the corresponding result proven in [6].

5. Real-Analyticity of Topological Pressure

This section differs from the previous sections in two points. We consider the generalized polynomial-like mappings (defined below) instead of rational functions and we consider Hölder continuous potentials instead of the functions $-t\log|f'|$. Notice that each generalized polynomial-like mapping with one critical point was proved in [7] to be quasiconformally conjugate to a polynomial. The case of a bigger number of critical points can be treated similarly. Since such a conjugacy is Hölder continuous, it might turn out to be helpful in some parts of this section if one wants to deal with maps without parabolic points only. Since this conjugacy is usually not Lipschitz continuous, it is rather useless if parabolic points are present. In order to define the generalized polynomial-like mappings let $U\subset\mathbb{C}$ be an open Jordan domain with smooth boundary and let $\{U_j\}_{j=1,2\dots,l}$ be a finite family of Jordan domains contained in U and with mutually disjoint closures. A map

$$f:\bigcup_{j=1}^{l}U_j\to U$$

is called a generalized polynomial-like mapping (GPL) if f extends holomorphically to an open neighbourhood of $\bigcup_{j=1}^{l}U_j$ and for all $j=1,2,\dots,l$, the restriction $f|_{U_j}:U_j\to U$ is a surjective branched covering map. The branched points of f coincide of course with $\mathrm{Crit}(f)$, the set of all critical points of f, and we denote by Br the set of such indexes $i\in\{1,2,\dots,l\}$ that $U-i$ contains a critical point of f. We call Br the set of branched inducers. If $j\in\mathrm{Br}$, then by C_j we denote the set of critical points of f contained in U_j. We also assume that if $\partial U_j\cap\partial U\neq\emptyset$, for some $j\in\{1,2,\dots,l\}$, then this intersection is a singleton consisting of a periodic parabolic point. Following tradition we call the branched conformal GPL f critically non-recurrent if for all $j\in\mathrm{Br}$

$$\overline{U_j}\cap\bigcup_{n\geq 1}f^n(C_j)=\emptyset.$$

We finally call a GPL critically tame if there exists $1\leq j\leq l$ such that

$$U_j\cap\bigcup_{n\geq 1}f^n(\mathrm{Crit}(f))=\emptyset,\ \overline{U_j}\subset U,\ \text{ and }\ U_j\cap J(f)\neq\emptyset,$$

where $J(f)$ is the Julia set of f, i.e. the boundary of the set of those points in U whose all forward iterates under f are well-defined. In the context of critically tame GPLs we will always assume without loss of generality that the distinguished index j is equal to 1. We would like to notice that each critically non-recurrent GPL with one critical point is critically tame if for the branched index j, $\overline{U_j}\subset U$. There are of course critically tame GPLs which are not recurrent. It can be easily verified that everything proven so far for rational critically non-recurrent functions can be also proven for critically non-recurrent GPLs. Let $g:\overline{U}\to I\!\!R$ be a Hölder continuous function such that

$$\mathrm{P}(g)=\mathrm{P}(g,f|_{J(f)})>\sup\{g|_{J(f)}\}.$$

Similarly, as in [3] only easier due to the fact that the disk U is already given, we can prove the following.

Theorem 5.1. *Let $g:\overline{U}\to I\!\!R$ be a Hölder continuous function such that $\mathrm{P}(g)=\mathrm{P}(g,f|_{J(f)})>\sup\{g|_{J(f)}\}$. Then*

(a) *There exists a unique Borel probability measure m supported on $J(f)$ such that*

$$m(f(A))=\int_A e^{\mathrm{P}(g)-g}\,dm$$

for every Borel set $A\subset\bigcup_{j=1}^{l}U_j$ such that $f|_A$ is 1-to-1. The measure m is atomless.

(b) *There exists a unique Borel probability f-invariant measure μ absolutely continuous with respect to m. In addition, μ is ergodic and the Radon-Nikodym derivative $\psi = \frac{d\mu}{dm} : J(f) \to [0,\infty)$ has in $L^1(\mu)$ a continuous version which is bounded away from zero.*

We will need the following technical mixing type result.

Lemma 5.2. *There exists $0 < \eta < 1$ such that*

$$\mu\left(\bigcap_{j=0}^{n-1} f^{-j}(U \setminus U_1)\right) \le \eta^n$$

for all $n \ge 1$.

Proof. It follows from Theorem 5.1 that

$$0 < \underline{\psi} := \inf\{\psi(z) : z \in J(f)\} \le \overline{\psi} := \sup\{\psi(z) : z \in J(f)\} < \infty.$$

Consider a partition $D_1, \dots, D_q$ (modulo a set of μ measure zero) of U such that all holomorphic inverse branches of f are well-defined on each of the sets D_j, $j = 1, \dots, q$. Such a partition exists since, due to Theorem 5.1, measures m and μ are atomless (in particular $m(f(\mathrm{Crit}(f)) = \mu(f(\mathrm{Crit}(f)) = 0)$. Notice also that due to the same theorem the map f is nonsingular with respect to both m and μ. For every $i \in \{1, \dots, q\}$ choose one holomorphic inverse branch f_i^{-1} of f mapping D_i into U_1. Applying Theorem 5.1 we get for every Borel set $A \subset U$ that

$$\begin{aligned}
\mu(U_1 \cap f^{-1}(A)) &\ge \mu\left(\bigcup_{i=1}^{q} f_i^{-1}(A \cap D_i)\right) = \sum_{i=1}^{q} \mu(f_i^{-1}(A \cap D_i)) \\
&\ge \underline{\psi} \sum_{i=1}^{q} m(f_i^{-1}(A \cap D_i)) = \underline{\psi} \sum_{i=1}^{q} \int_{A \cap D_i} e^{g \circ f_i^{-1} - \mathrm{P}(g)} dm \\
&\ge \underline{\psi} e^{-\mathrm{P}(g)} e^{-||g||_\infty} \sum_{i=1}^{q} m(A \cap D_i) = \underline{\psi} e^{-(\mathrm{P}(g) + ||g||_\infty)} m(A) \\
&= \gamma \mu(A),
\end{aligned}$$

where $\gamma = \underline{\psi}\,\overline{\psi}^{-1} e^{-(\mathrm{P}(g)+||g||_\infty)} > 0$. Therefore

$$\begin{aligned}
\mu\Big((U \setminus U_1) \cap f^{-1}(A)\Big) &= \mu\Big(f^{-1}(A)\Big) - \mu\Big(U_1 \cap f^{-1}(A)\Big) = \mu(A) - \mu\Big(U_1 \cap f^{-1}(A)\Big) \\
&\le \mu(A) - \gamma\mu(A) = (1-\gamma)\mu(A).
\end{aligned}$$

Thus

$$\mu\left(\bigcap_{j=0}^{n-1} f^{-j}(U \setminus U_1)\right) = \mu\left((U \setminus U_1) \cap f^{-1}\left(\bigcap_{j=0}^{n-2} f^{-j}(U \setminus U_1)\right)\right) \le (1-\gamma)\mu\left(\bigcap_{j=0}^{n-2} f^{-j}(U \setminus U_1)\right)$$

and the lemma follows by induction taking $\eta = 1 - \gamma$. ■

From now on we assume that

$$f : \bigcup_{j=1}^{l} U_j \to U$$

is a critically tame GPL. Our first aim is to associate with f a conformal infinite (hyperbolic) iterated function system satisfying all the requirements from [9]. And indeed, since f is critically tame, there exists an open topological disk $V \supset \overline{U_1}$ whose closure is contained in U and is disjoint from $\overline{U_2} \cup \overline{U_3} \cup \ldots \cup \overline{U_l}$. In particular

$$V \cap \bigcup_{n\geq 1} f^n(\mathrm{Crit}(f)) = \emptyset. \tag{5.1}$$

For every $n \geq 1$ let

$$R_n = \{z \in U_1 : f^n(z) \in U_1 \text{ and } U_1 \cap \{f^k(z) : k = 1, 2, \ldots, n-1\} = \emptyset\}.$$

That is R_n is the set of those points in U_1 whose first return time to U_1 is equal to n. Let now $x \in R_n$. In view of (5.1) there exists a unique holomorphic inverse branch $f_x^{-n} : V \to \overline{\mathbb{C}}$ of f^n sending $f^n(x)$ to x. Notice that $f_x^{-n}(V) \subset U_1 \subset V$. Since if y is another point in R_n, then either $f_y^{-n} = f_x^{-n}$, or $f_x^{-n}(V) \cap f_x^{-n}(V) = \emptyset$, we conclude that there exists a finite set $A_n \subset R_n$ such that $f_x^{-n}(V) \cap f_y^{-n}(V) = \emptyset$ for all $x, y \in A_n$, $x \neq y$ and for every $z \in R_n$ there exists $w \in A_n$ such that $f_z^{-n} = f_w^{-n}$. Therefore, the countable family

$$S = \{f_x^{-n} : V \to V, f_x^{-n} : \overline{U_1} \to \overline{U_1}\}_{n\geq 1, x\in A_n}$$

forms a conformal iterated function system in the sense of [9]. We will frequently denote the elements of S by ϕ_i, $i \in I$. Given $\omega = \omega_1\omega_2 \ldots \omega_n \in I^n$, we put

$$\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \ldots \phi_{\omega_n}.$$

Let $N(i)$ denote the only integer satisfying $\phi_i = f_x^{-N_i}$ for some $x \in A_{N_i}$. Given $t \geq 0$ and $s \in \mathbb{R}$ we introduce the family $G_{t,s} = \{g_{i,t,s} : V \to \mathbb{R}\}_{i\in I}$ by the formulas

$$g_{i,t,s}(x) = t\sum_{j=1}^{N_i} g \circ f^j(\phi_i(x)) - sN(i).$$

Recall that $p = \max\{p(\omega) : \omega \in \Omega\}$. Let us prove the following.

Lemma 5.3. *Suppose that f is a critically tame critically non-recurrent GPL. If the GPL f has no parabolic points, then all the functions $g_{i,t,s}$, $i \in I$, are Hölder continuous with the same Hölder exponent and the same Hölder constant. If the GPL f has parabolic points, then the same is true assuming additionally that the Hölder exponent of g is greater than $\frac{p}{p+1}$.*

Proof. Notice first that U_1 has no parabolic points. In the case of the lack of parabolic points, it immediately follows from Lemma 7.7 in [18] (the stars can be dropped there), comp. [1]. So, suppose that $\Omega \neq \emptyset$. It then follows from (5.1) and Remark 4.4 that $\mathrm{diam}(V_n) \leq Bn^{-\frac{p+1}{p}}$ for all $n \geq 1$ and all connected components V_n of $f^{-n}(V)$. Hence, it follows from

Koebe's distortion theorem that $|(f_*^{-n})'(x)| \le \tilde{B}(n+1)^{-\frac{p+1}{p}}$ for all $n \ge 0$, all holomorphic inverse branches $f_*^{-n} : V \to U$ of f^n, all $x \in \overline{U_1}$ and some $\tilde{B} > 0$. In particular if $y, z \in \overline{U_1}$, then for all $i \in I$

$$\begin{aligned}\left|\sum_{j=1}^{N_i} g \circ f^j(\phi_i(z)) - \sum_{j=1}^{N_i} g \circ f^j(\phi_i(y))\right| &\le \sum_{j=1}^{N_i} |g \circ f^j(\phi_i(z)) - g \circ f^j(\phi_i(y))| \\ &\le \sum_{j=1}^{N_i} C|f^j(\phi_i(z)) - f^j(\phi_i(y))|^\alpha \\ &\le \sum_{j=1}^{N_i} C\tilde{B}(N_i + 1 - j)^{-\frac{p+1}{p}}|z-y|^\alpha \\ &\le \sum_{j=1}^{\infty} k^{-\alpha\frac{p+1}{p}}|z-y|^\alpha,\end{aligned}$$

where α is the Hölder exponent of g and C is the Hölder constant. We are done. ■

Lemma 5.4. *For all $t \ge 0$ and $s \in I\!R$, $G_{t,s}$ is a Hölder family of functions in the sense of* [5] *and* [10].

Proof. By Montel's theorem the family $\{\phi_i : V \to U_1\}_{i \in I}$ is normal, and since $U_1 \cap J(f) \neq \emptyset$, all its limit functions are constant. Therefore all the limit functions of derivatives ϕ_i' of ϕ_i are equal to the constant function 0. Thus, there exists $q \ge 1$ such that $||\phi_\omega'||_\infty \le 1/2$ for all $\omega \in I^n$ with $n \ge q$, where $||\cdot||_\infty$ is the supremum norm on $\overline{U_1}$. Notice also that by Koebe's distortion theorem (and since $\overline{U_1} \subset V$),

$$Q = \sup\{||\phi_\omega'||_\infty : \omega \in I^n,\ 1 \le n \le q-1\} < \infty.$$

Fix now $n \ge 1$, $\omega \in I^n$, and two points $x, y \in \overline{U_1}$. Write $n - 1 = kq + r$, where $0 \le r \le q-1$. Then $||\phi_\omega'||_\infty \le (1/2)^k Q \le Q(1/2)^{\frac{n-1}{q}}$ and therefore

$$\mathrm{diam}(\phi_\omega(\overline{U_1})) \le \overline{Q}\left(2^{-\frac{1}{q}}\right)^{n-1},$$

where $\overline{Q}$ is the constant depending on Q, the diameter of U_1, and the maximal number of segments needed to join each point in $\overline{U_1}$ with an arbitrarily frozen point in U_1 (note that U_1 is not assumed to be convex nor of the star shape). Using Lemma 5.3, with the Hölder exponent α following from Lemma 5.3, we conclude that

$$|g_{\omega_1,t,s}(\phi_{\sigma(\omega)}(y)) - g_{\omega_1,t,s}(\phi_{\sigma(\omega)}(x))| \le C_1|\phi_{\sigma(\omega)}(y) - \phi_{\sigma(\omega)}(x)|^\alpha \le C_1\overline{Q}\left(2^{-\frac{\alpha}{q}}\right)^{n-1}$$

for some universal constant C_1 depending on t. We are done. ■

We recall (see [10], comp [5], where in the latter paper a different terminology was used) that a Hölder family $\{q_i : \overline{U_1} \to I\!R\}_{i\in I}$ is called summable if

$$\sum_{i\in I} e^{\sup(q_i)} < \infty.$$

Using Lemma 5.2 let us prove the following.

Lemma 5.5. *Assume the same as in Lemma 5.3. If* $\mathrm{P}(g) > \sup(g)$*, then there exists* $\delta > 0$ *such that if* $(t,s) \in (1-\delta, 1+\delta) \times (\mathrm{P}(g) - \delta, +\infty)$*, then* $G_{t,s}$ *is a summable Hölder family of functions.*

Proof. The fact that the families $G_{t,s}$ are Hölder, has been proved in Lemma 5.4. Since

$$R_n = U_1 \cap \bigcap_{j=1}^{n-1} f^{-j}(U \setminus U_1) \cap f^{-n}(U_1) \subset f^{-1}\left(\bigcap_{j=0}^{n-2} f^{-j}(U \setminus U_1)\right)$$

and since the measure μ is f-invariant, it follows from Lemma 5.2 that $\mu(R_n) \le \eta^{n-1}$. Applying now Theorem 5.1(b), we infer that there exists a constant $C_1 > 0$ such that

$$m(R_n) \le C_1 \eta^n$$

for all $n \ge 1$. Combining this, Lemma 5.3 and Theorem 5.1, we conclude that there exists a constant $C_2 > 0$ such that

$$\sum_{w\in A_n} \exp\left(\sup_{\overline{U_1}}\left(\sum_{j=1}^n g\circ f^j \circ f_w^{-1} - \mathrm{P}(g)n\right)\right) \le C_2\eta^n$$

for all $n \ge 1$. Hence, for every $t \ge 0$ and all $s \in I\!R$, we get the following.

$$\begin{aligned}
&\sum_{w\in A_n} \exp\left(\sup_{\overline{U_1}}\left(\sum_{j=1}^n tg\circ f^j \circ f_w^{-1} - sn\right)\right) = \\
&\quad= \sum_{w\in A_n} \exp\left(\sup_{\overline{U_1}}\left(\sum_{j=1}^n g\circ f^j \circ f_w^{-1} - \mathrm{P}(g)n + (t-1)\sum_{j=1}^n g\circ f^j \circ f_w^{-1} + (\mathrm{P}(g)n - sn\right)\right) \\
&\quad\le \exp(|t-1|||g||_\infty n)\exp\big((\mathrm{P}(g)-s)n\big)\sum_{w\in A_n}\exp\left(\sup_{\overline{U_1}}\left(\sum_{j=1}^n g\circ f^j\circ f_w^{-1} - \mathrm{P}(g)n\right)\right) \\
&\quad\le C_2\exp(-\kappa n)\exp\big((\mathrm{P}(g)-s)n\big)\exp(|t-1|||g||_\infty n),
\end{aligned}$$

where $\kappa = -\log\eta$. Taking now $\delta = \kappa(4(1+||g||_\infty)^{-1}$, we conclude that for all $(t,s) \in (1-\delta, 1+\delta) \times (\mathrm{P}(g)-\delta, +\infty)$ and all $n \ge 1$,

$$\sum_{w\in A_n} \exp\left(\sup_{\overline{U_1}}\left(\sum_{j=1}^n tg\circ f^j\circ f_w^{-1} - sn\right)\right) \le C_2\exp\left(-\frac{1}{2}\kappa n\right)$$

and therefore

$$\sum_{n\ge1}\sum_{w\in A_n}\exp\left(\sup_{\overline{U_1}}\left(\sum_{j=1}^n tg\circ f^j\circ f_w^{-1}-sn\right)\right)<\infty.$$

We are done. ■

Since both functions $g\mapsto \mathrm{P}(g)$ and $g\mapsto \sup(g)$ are continuous, we get the following.

Lemma 5.6. *If* $\mathrm{P}(g)>\sup(g)$, *then there exists* $\delta>0$ *such that* $\mathrm{P}(tg)>\sup(tg)$ *for all* $t\in(1-\delta,1+\delta)$.

Given $\omega=\omega_1\omega_2\ldots\omega_n\in I^n$, $n\ge1$, let $\sigma(\omega)=\omega_1\omega_2\ldots\omega_{n-1}$. Recall from [5] and [10] that the topological pressure $\mathrm{P}(G_{t,s})$ of the family $G_{t,s}$ is defined as follows.

$$\begin{aligned}\mathrm{P}(G_{t,s})&=\lim_{n\to\infty}\frac1n\log\sum_{|\omega|=n}\left\|\exp\left(\sum_{j=1}^n g_{\omega_j,t,s}\circ\phi_{\sigma^j\omega}\right)\right\|_0\\&=\lim_{n\to\infty}\frac1n\log\sum_{|\omega|=n}\exp\left(\sup_{\overline{U_1}}\sum_{j=1}^n g_{\omega_j,t,s}\circ\phi_{\sigma^j\omega}\right).\end{aligned}$$

Let J_S be the limit set of the iterated function system

$$S=\{f_x^{-n}:V\to V, f_x^{-n}:\overline{U_1}\to\overline{U_1}\}_{n\ge1,x\in A_n}.$$

Its limit set is defined as follows

$$J_S=\bigcap_{n=1}^\infty\bigcup_{|\tau|=n}\phi_\tau(\overline{U_1}).$$

For an alternative definition of the limit set J_S and its further properties see [9] and [10] for example. Let us also recall ([5], [10]) that the $G_{t,s}$-conformal measure $m_{G_{t,s}}$ supported on J_S is uniquely determined by the following two properties.

$$m_{G_{t,s}}(\phi_i(A))=\int_A\exp\Big(g_{i,t,s}-\mathrm{P}(F)\Big)\,dm_{G_{t,s}},\quad \forall i\in I$$

and

$$m(\phi_i(\overline{U_1})\cap\phi_j(\overline{U_1}))=0$$

for all $i\ne j$.

Let $\delta(g)>0$ be the minimum of the numbers δ from Lemma 5.5 and Lemma 5.6. For $t\in(1-\delta,1+\delta)$ let m_t be the measure produced by Theorem 5.1(a) for the potential $t\phi$.

We shall prove now the following.

Lemma 5.7. *If $t \in (1-\delta(g), 1+\delta(g))$, then*

$$\mathrm{P}(G_{t,\mathrm{P}(tg)}) = 0 \text{ and } m_{G_{t,\mathrm{P}(tg)}} = m_t|_{J_S}.$$

Proof. Denote $\mathrm{P}(G_{t,\mathrm{P}(tg)})$ by $\hat{\mathrm{P}}(t)$ and $m_{G_{t,\mathrm{P}(t)}}$ by $\hat{m}_t$. By the definitions of measures $\hat{m}_t$, m_t and by Lemma 5.4, Lemma 5.6 and Theorem 5.1, for every $\tau \in I^* = \bigcup_{n\geq 1} I^n$, we have

$$\frac{\hat{m}_t(\phi_\tau(U_1))}{m_t(\phi_\tau(U_1))} = \frac{\int \exp\left(\sum_{j=1}^{|\tau|} g_{\tau_j,t,\mathrm{P}(tg)}\right)\exp\Big(-\hat{\mathrm{P}}(t)|\tau|\Big)d\hat{m}_t}{\int \exp\left(\sum_{j=1}^{|\tau|} g_{\tau_j,t,\mathrm{P}(tg)}\right)dm_t} \asymp \exp\Big(-\hat{\mathrm{P}}(t)|\tau|\Big). \tag{5.2}$$

So, if $\hat{\mathrm{P}}(t) > 0$, then $\hat{m}_t(J_S) = 0$ and if $\hat{\mathrm{P}}(t) < 0$, then $m_t(J_S) = 0$ which contradicts the fact that $m_t(J_S) = m_t(U_1) > 0$ which follows by a straightforward induction from the formula $m_t\left(\bigcup_{i\in I}\phi_i(U_1)\right) = m_t(U_1)$ resulting from Poincare's recurrence theorem. Thus $\hat{\mathrm{P}}(t) = 0$ and the first part of our lemma is proven. The equality $\hat{m}_t = m_t|_{J_S}$ follows now from (5.2) considered as two formulas: one for the numerator and one for the denominator along with either Theorem 3.1.7 from [10] or Corollary 2.12 from [5]. The proof is complete. ■

Following [10] and [5] we call $g_{t,s}: I^\infty \to I\!\!R$, defined by the formula

$$g_{t,s}(\tau) = g_{\tau_1,t,s}(\pi(\sigma\tau))$$

the amalgamated function of the family $G_{t,s}$. Here $\pi: I^\infty \to \mathbb{C}$ is the projection associated with the conformal iterated function system S and $\sigma: I^\infty \to I^\infty$ is the shift map. We are now in position to prove the following.

Theorem 5.8. *Assume that f is a critically tame non-recurrent GPL and that ϕ is Hölder continuous with the exponent greater than $\frac{p}{p+1}$ if f has parabolic points. Then the function $t \mapsto \mathrm{P}(tg)$, $t \in (1-\delta, 1+\delta)$ is real-analytic.*

Proof. Let $\hat{m}_t = m_{G_{t,\mathrm{P}(tg)}}$. In view of Theorem 2.6.12 from [10] (see also [19] and [5]) and Lemma 5.5, the function

$$(t,s) \mapsto \mathrm{P}(g_{t,s}) = \mathrm{P}(G_{t,s}),\ (t,s) \in (1-\delta, 1+\delta) \times (\mathrm{P}(g)-\delta, \mathrm{P}(g)+\delta)$$

is real-analytic. In view of Lemma 5.7 and the implicit function theorem, it is therefore now enough to demonstrate that $\frac{\partial \mathrm{P}}{\partial s}(t,s) \neq 0$ at the point $(t, \mathrm{P}(tg))$ for every $t \in (1-\delta, 1+\delta)$. And indeed, let $\tilde{\mu}_t$, $t \in (1-\delta, 1+\delta)$ be the shift invariant measure on I^∞ equivalent to $\tilde{\hat{m}}_t$, the lift of the measure $\hat{m}_t$ to the coding space I^∞. Since μ_t, the measure appearing in Theorem 5.1(b) for the potential tg, is f-invariant and the system S is defined according to the first-return time, μ_t is S-invariant in the sense that

$$\mu_t\left(\bigcup_{i\in I}\phi_i(A)\right) = \sum_{i\in I}\mu_t(\phi_i(A)) = \mu_t(A)$$

for every Borel set $A \subset J_S$. Hence, using the last part of Lemma 5.7, we deduce that $\mu_t|_{J_S} = \tilde{\mu}_t \circ \pi^{-1}$. Therefore applying Proposition 2.6.13 in [10] (see also [19] and [5]) along with Kac's lemma, we obtain that

$$\frac{\partial \mathrm{P}}{\partial s}(t, \mathrm{P}(tg)) = -\int_{I^\infty} N(\tau_1)\, d\tilde{\mu}_t = -\int_{J_S} N d\mu_t|_{J_S} = -\frac{1}{\mu_t(J_S)} \neq 0,$$

where after the second equality sign we treated slightly informally the function N as defined on J_S. The proof is complete. ■

After this paper has been written the analogous result for potentials of the form $-t\log|f'|$ was established in [14].

Acknowledgments. The author wishes to thank the referee and Feliks Przytycki whose comments and suggestions improved the final version of this paper.

References

[1] L. Carleson, P. Jones, C. Yoccoz, Julia and John, Bol. Soc. Bras. Mat. 25 (1994), 1-30.
[2] M. Denker, M. Urbański, On the existence of conformal measures, Trans. A.M.S. 328 (1991), 563-587.
[3] M. Denker, M. Urbański, Ergodic theory of equilibrium states for rational maps, Nonlinearity 4 (1991), 103-134.
[4] M. Denker, M. Urbański, On Sullivan's conformal measures for rational maps of the Riemann sphere, Nonlinearity 4 (1991), 365-384.
[5] P. Hanus, D. Mauldin, M. Urbański, Thermodynamic formalism and multifractal analysis of conformal infinite iterated function systems, Acta Math. Hungarica, 96 (2002), 27-98.
[6] N. Haydn, S. Isola, Parabolic rational maps, Preprint 1999.
[7] G. Levin, S. van Strien, Local connectivity of the Julia set of real polynomials Ann. of Math. 147 (1998), 471-541.
[8] M. Martens, The existence of σ-finite invariant measures, Applications to real one-dimensional dynamics, Preprint.
[9] D. Mauldin, M. Urbański, Dimensions and measures in infinite iterated function systems, Proc. London Math. Soc. (3) 73 (1996) 105-154.
[10] D. Mauldin, M. Urbański, Graph Directed Markov Systems, to appear, available on author's web-page.
[11] C. McMullen, Complex dynamics and renormalization, Annals of Math. Studies 135. Princeton University Press, 1994.
[12] F. Przytycki, Conical limit set and Poincaré exponent for iterations of rational functions, Trans. A.M.S. 351 (1999), 2081-2099.
[13] F. Przytycki, J. Rivera-Letelier, S. Smirnov, Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps, Inv. Math. 151 (2003), 29-63.
[14] B. Stratmann, M. Urbański, Real Analyticity of Topological Pressure for parabolically semihyperbolic generalized polynomial-like maps, Preprint 2001, accepted for publication in Indagationes Math.
[15] B. Stratmann, M. Urbański, Multifractal Analysis of parabolically semi-hyperbolic generalized polynomial-like maps, in preparation.
[16] F. Przytycki, M. Urbański, Fractals in the Plane - Ergodic Theory Methods, to appear.
[17] M. Urbański, Rational functions with no recurrent critical points, Ergod. Th. and Dynam. Sys. 14 (1994), 391-414.

[18] M. Urbański, Geometry and ergodic theory of conformal nonrecurrent dynamics, Ergod. Th. and Dynam. Sys. 17 (1997), 1449-1476.
[19] M. Urbański, M. Zinsmeister, Geometry of hyperbolic Julia-Lavaurs sets, Indagationes Math. 12 (2001) 273 - 292.

Mariusz Urbański; Department of Mathematics, University of North Texas, Denton, TX 76203-1430, USA
urbanski@unt.edu, http://www.math.unt.edu/~urbanski