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HOLOMORPHIC MAPS FOR WHICH THE UNSTABLE MANIFOLDS DEPEND ON PREHISTORIES

Department of Mathematics
Texas A&M University
College Station, TX 77843-3368, USA
and
Institute of Mathematics of the Romanian Academy
P. O. Box 1-764
Ro 70700, Bucharest, Romania

Department of Mathematics
University of North Texas
P.O. Box 311430
Denton, TX 76203-1430, USA

Abstract. For points x belonging to a basic set Λ of an Axiom A holomorphic endomorphism of $\mathbb{P}^2$, one can construct the local stable manifold $W^s_{\varepsilon_0}(x)$ and the local unstable manifolds $W^u_{\varepsilon_0}(\hat{x})$. A priori, $W^u_{\varepsilon_0}(\hat{x})$ should depend on the entire prehistory $\hat{x}$ of x. However, all known examples have all their local unstable manifolds depending only on the base point x. Therefore a natural problem is to give actual examples where, for different prehistories of points in the basic sets of holomorphic Axiom A maps, we obtain different unstable manifolds. We solve this problem by considering the map $(z^4 + \varepsilon w^2, w^4)$ and then also show that, by perturbing $(z^2 + c, w^2)$, one can get also maps f_ε which are injective on Λ_ε, their corresponding basic sets, hence the cardinality of the set $(f_\varepsilon|_{\Lambda_\varepsilon})^{-1}(x), x \in \Lambda_\varepsilon$, is not stable under perturbation.

1. Introduction.

Throughout the paper we consider an Axiom A holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$ on the 2-dimensional complex projective space. Denote by $\Omega := \{x \in \mathbb{P}^2, \forall\, U \text{ neighbourhood of } x, \exists\, n \geq 1 \ s.t\ f^n(U) \cap U \neq \emptyset\}$, the non-wandering set of f. For the general theory of Axiom A maps, we refer to [2], [8], [9], [10]. If f is an Axiom A map, then Ω is the closure of the periodic points of f, and f is hyperbolic on Ω. First, let us define the space of prehistories of Ω,

$$\hat{\Omega} := \{\hat{x} := (x_n)_{n \leq 0}, f(x_{n-1}) = x_n, x_n \in \Omega, \forall n \leq 0\}$$

and the inverse limit $\hat{f} : \hat{\Omega} \to \hat{\Omega}$ defined by the formula

$$\hat{f}((x_n)_{n \leq 0}) = ((f(x_n))_{n \leq 0}).$$

The tangent bundle over $\hat{\Omega}$ is defined by its fibers:

$$T_{\hat{\Omega}}((x_n)_n) = \{((x_n)_n, v), \text{ with } v \in T_{x_0}\mathbb{P}^2\}.$$

If f is hyperbolic, then there exists a continuous splitting of the tangent bundle, $T_{\hat{\Omega}}((x_n)_n) = E^s(x_0) \oplus E^u((x_n)_n)$ and constants $C > 0$, $\lambda > 1$, such that

$$Df_x(E^u(\hat{x})) \subseteq E^u(\hat{f}\hat{x}), Df_x(E^s(x)) \subseteq E^s(fx),$$

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$$||Df^n(v)|| \le C\lambda^{-n} \cdot ||v||, \ \forall v \in E^s(x_0), \ \forall n \ge 0$$

and

$$||Df^n(w)|| \ge C^{-1}\lambda^n \cdot ||w||, \ \forall w \in E^u(\hat{x}), \ \forall n \ge 0.$$

Like in the case of diffeomorphisms, the above splitting into stable and unstable directions, implies the existence of local stable and unstable manifolds for every $0 < \varepsilon_0 << 1$:

$$W^s_{\varepsilon_0}(x) := \{y \in \mathbb{P}^2, d(f^n x, f^n y) \le \varepsilon_0, \forall n \ge 0\},$$

$$W^u_{\varepsilon_0}(\hat{x}) := \{y \in \mathbb{P}^2, y \text{ has a prehistory } \hat{y} = (y_n)_{n \le 0}, d(x_{-n}, y_{-n}) \le \varepsilon_0, \forall n \ge 0\}$$

$W^s_{\varepsilon_0}(x)$ and $W^u_{\varepsilon_0}(\hat{x})$ are complex disks. If $y \notin W^s_{2\varepsilon_0}(x), W^s_{\varepsilon_0}(x) \cap W^s_{\varepsilon_0}(y) = \emptyset$, and the family $(W^s_{\varepsilon_0}(x))_x$ forms a lamination. $W^u_{\varepsilon_0}(\hat{x})$ depends a priori on the entire prehistory $\hat{x}$, but as will be explained later, in all known examples so far, $W^u_{\varepsilon_0}(\hat{x})$ depends only on the base point x and not on the entire prehistory. Obviously, before one should try to find examples when this is not true, one should obtain examples of families of maps which are not injective on their basic sets. On the other hand we will show that in fact, by perturbing the map $(z^2 + c, w^2)$, one gets in general both maps which are injective and maps which are not injective on their respective basic sets. If (M, d) is a compact Riemannian manifold, then we consider the usual $\mathcal{C}^r, r \ge 1$ topology on the space of $\mathcal{C}^r$ functions on M. In the real case, we would like to mention Przytycki's result ([8]), showing that, if f is any special Anosov endomorphism of a compact manifold M (special meaning that unstable spaces do not depend on prehistories), f not expanding or a diffeomorphism, $\varepsilon > 0$ and $x \in M$ are given, then there exists $g : M \to M$, and G'_x, a set of prehistories of x under g, so that $d(f, g) < \varepsilon$, G'_x is homeomorphic to a Cantor set, and the unstable manifolds $W^u_{\varepsilon_0}(\hat{x}), \hat{x} \in G'_x$ are all mutually distinct. Also in this situation, $G'_x \ni \hat{x} \to E^u(\hat{x})$ is a homeomorphism, so the unstable vector spaces depend on the prehistories from G'_x as well. However, his proof is depending on local C^∞ real perturbations and does not seem to be extendable to the complex case. We will obtain our examples as perturbations of maps of the form (z^4, w^4). Firstly, in order to understand the problem at hand let us look at some classical examples of holomorphic mappings on $\mathbb{P}^2$.

1) $f[z : w : t] = [P(z : t) : Q(w : t) : t^d]$, with P, Q one-dimensional complex polynomials of degree d, hyperbolic on their Julia sets J_P, respectively J_Q. Then, in the set $t = 1, S_1 = \{\text{periodic sinks of } P\} \times J_Q \cup J_P \times \{\text{periodic sinks of } Q\}$. In $t = 0, S_1$ has a basic set given by the Julia set of the map $[P(z : 0) : Q(w : 0) : 0]$. In the example of this form, if $x \in S_1$ is fixed, then all the unstable manifolds $W^u_\varepsilon(\hat{x})$, for $\hat{x} \in \widehat{S}_1$ coincide. So $W^u_\varepsilon(\hat{x})$ does not really depend on $\hat{x}$, it depends only on x.

2) Let ϕ the Serre map, $\phi\colon\ \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$, $\phi([z_0 : w_0], [z_1 : w_1]) = [z_0 z_1 : w_0 w_1 : z_0 w_1 + w_0 z_1]$, a 2-to-1 covering of $\mathbb{P}^2$. If $f_0\colon\ \mathbb{P}^1 \to \mathbb{P}^1$ is holomorphic and hyperbolic on its Julia set, then there exists f holomorphic on $\mathbb{P}^2$ such that $\phi(f_0, f_0) = f \circ \phi$, and f is Axiom A ([2]). The basic sets for S_1 are of the form ϕ ($\{$periodic sink for $f_0\} \times J_{f_0}$) and again it is easy to check the independence on prehistory of the unstable manifolds.

3) There are other examples of Axiom A maps f (even s-hyperbolic) which have basic sets Λ near which the map is injective, like the solenoidal examples ([2], [5]). For these examples the map f behaves like a diffeomorphism, hence the local unstable manifolds depend only on x and they form a lamination on Λ.

So, the question arises whether there are examples of Axiom A holomorphic maps on $\mathbb{P}^2$ for which the unstable manifolds $W^u_\varepsilon(\hat{x})$ truly depend on $\hat{x}$, and not only on x. Also is it possible that for every point $x \in \Lambda$, we have different local unstable manifolds $W^u_\varepsilon(\hat{x})$ and $W^u_\varepsilon(\hat{x}')$ whenever the prehistories of $x, \hat{x}$ and $\hat{x}'$ are different? Obviously, before we attack this problem, there must be found first a non-trivial example of a map g with a basic set Λ_g such that $g|_{\Lambda_g}$ is non-injective, since this would guarantee the existence of at least two different prehistories for a given point $x \in \Lambda_g$.

2. **Main results and proofs.** In this section we provide precise formulations and prove all the results announced in the introduction. Let $f(z,w) = (z^4, w^4)$. Then f has a basic set $\Lambda = \{0\} \times J_{(w^4)} = \{0\} \times S^1$. Since f is Axiom A and has no cycles, the Stability Theorem ([8], [9]) will imply that for $\varepsilon > 0$ small enough, $f_\varepsilon(z,w) := (z^4 + \varepsilon w^2, w^4)$ has a basic set Λ_ε close to Λ in the Hausdorff metric and f_ε is hyperbolic on Λ_ε. For more clarity, we shall denote the unstable vector space of a prehistory $\hat{x} \in \widehat{\Lambda_\varepsilon}$ by $E^u_\varepsilon(\hat{x})$. If $(z,w) \in \Lambda_\varepsilon$ then $|w| = 1$.

Theorem 2.1. *For every point $(z,w) \in \Lambda_\varepsilon$, and $\widehat{(z,w)}$, $\widehat{(z,w)}'$, any two different prehistories of (z,w) from $\widehat{\Lambda_\varepsilon}$, we have $E^u_\varepsilon\widehat{(z,w)} \neq E^u_\varepsilon\widehat{(z,w)}'$. In particular $W^u_{\varepsilon_0}\widehat{(z,w)} \neq W^u_{\varepsilon_0}\widehat{(z,w)}'$.*

Proof. In fact we prove more. Namely, we estimate from below the angle between $E^u_\varepsilon\widehat{(z,w)}$ and $E^u_\varepsilon\widehat{(z,w)}'$. Let us consider $(z_0, w_0) \in \Lambda_\varepsilon$ and show that $(-z_0, w_0) \in \Lambda_\varepsilon$ also. Before we go any further let us estimate the following two quantities: $C_\varepsilon := \inf_{(z,w)\in\Lambda_\varepsilon} |z|, D_\varepsilon := \sup_{(z,w)\in\Lambda_\varepsilon} |z|$. Assume D_ε is attained at a point $(z_0, w_0) \in \Lambda_\varepsilon$. Therefore, since $f_\varepsilon : \Lambda_\varepsilon \to \Lambda_\varepsilon$ is onto, there will exist a point $(z,w) \in \Lambda_\varepsilon$ such that $f_\varepsilon(z,w) = (z_0, w_0)$, so:

$$z^4 + \varepsilon w^2 = z_0 \Rightarrow |z_0| = |z^4 + \varepsilon w^2| \leq |z|^4 + \varepsilon \leq |z_0|^4 + \varepsilon,$$

where we used the fact that $|w| = 1$ for any (z,w) in Λ_ε. Hence, from the above:

$$|z_0|^4 - |z_0| + \varepsilon \geq 0.$$

Let us study the function of one real variable $g(t) := t^4 - t + \varepsilon$. We have $g'(t) = 4t^3 - 1$, so g has the critical point $t_0 = (1/4)^{1/3}$. This is a point of minimum and $g(t_0) < 0$, for $\varepsilon > 0$ sufficiently small; also g is strictly decreasing for $t < t_0$ and strictly increasing for $t > t_0$. Now, $g(\varepsilon) = \varepsilon^4 - \varepsilon + \varepsilon = \varepsilon^4 > 0$ and $g(2\varepsilon) = 2^4\varepsilon^4 - 2\varepsilon + \varepsilon = 16\varepsilon^4 - \varepsilon < 0$ for ε small. Also, let us notice that $g(1/2) = (1/2)^4 - 1/2 + \varepsilon < 0$ and $g(1) = 1 - 1 + \varepsilon = \varepsilon > 0$, hence g has two real zeros, one between ε and 2ε and the other between $1/2$ and 1. But $|z_0| = D_\varepsilon$ is close to 0, because f_ε is a small perturbation of (z^4, w^4), therefore we obtain:

$$\varepsilon \leq D_\varepsilon \leq 2\varepsilon.$$

Let us estimate also C_ε. Assume $C_\varepsilon = |z_1|$. Hence there exists $(z,w) \in \Lambda_\varepsilon$ such that $f_\varepsilon(z,w) = (z_1, w_1)$, meaning that

$$z^4 + \varepsilon w^2 = z_1 \Rightarrow |z_1| \geq \varepsilon - |z|^4 \geq \varepsilon - (2\varepsilon)^4 > 0.9\varepsilon$$

for ε small enough. In conclusion, for $(z,w) \in \Lambda_\varepsilon$, we have $0.9\varepsilon < |z| < 2\varepsilon$.

Let us calculate now the iterates of f_ε.

$$f^2_\varepsilon(z,w) = ((z^4 + \varepsilon w^2)^4 + \varepsilon w^8, w^{16}) =$$

$$= (\varepsilon^4 w^8 + \varepsilon w^8 + C_2(z,w)\varepsilon^4, w^{16}) = (w^8(\varepsilon + \varepsilon^4) + C_2(z,w)\varepsilon^4, w^{16}),$$

where $C_2(z,w)$ is a function which is bounded everywhere by a positive constant C and which is independent of ε, and where we used that $0.9\varepsilon < C_\varepsilon \le |z| \le D_\varepsilon < 2\varepsilon, \forall (z,w) \in \Lambda_\varepsilon$. Also, $f_\varepsilon^3(z,w) = ((w^8(\varepsilon+\varepsilon^4) + C_2(z,w)\varepsilon^4)^4 + \varepsilon w^{32}, w^{64}) = (w^{32}(\varepsilon + (\varepsilon+\varepsilon^4)^4) + C_3(z,w)\varepsilon^6, w^{64})$, where again $|C_3(z,w)| \le C$. In general, for any n, it can be showed by induction that

$$f_\varepsilon^n(z,w) = (w^{\frac{4^n}{2}} E_n(\varepsilon) + C_n(z,w)\varepsilon^{2n}, w^{4^n}),$$

with $C_n(z,w) < C$ a bounded function coming from the iterated development of the power in the expression. Consider now an arbitrary point (z_0,w_0) from Λ_ε and let us prove that $(-z_0,w_0)$ belongs to Λ_ε. Denote in general by $D(z_0,\eta)$ the disk of center z_0 and radius $\eta > 0$ in the complex plane. Take $U = D(z_0,\eta) \times D(w_0,\eta)$ a small neighborhood of (z_0,w_0); then, from the fact that (z_0,w_0) is a non-wandering point, we get that there exists an increasing sequence of integers n_k such that $f_\varepsilon^{n_k}(U) \cap U \ne \emptyset$. In consequence there exists a point (z,w) η-close to (z_0,w_0) such that $|w^{4^{n_k}/2} - \sqrt{w_0}| < \eta/3$ and $z_{n_k} := p_1 \circ f_\varepsilon^{n_k}(z,w)$ is also $\eta/2$-close to z_0, where p_1 is the projection on the first component and $\sqrt{w_0}$ is a fixed determination of the square root of w_0. Therefore $|E_{n_k}(\varepsilon) - \frac{z_0}{\sqrt{w_0}}| < \eta/2$, for k large enough. Let now another point $(z',w') \in U$ such that $|(w')^{4^{n_k}/2} - (-\sqrt{w_0})| < \eta/2$. Then $|(w')^{4^{n_k}/2} E_{n_k}(\varepsilon) + z_0| < \eta$, if k is large enough. But this would imply that $f_\varepsilon^{n_k}(z',w')$ belongs to a neighbourhood $D(-z_0,\eta) \times D(w_0,\eta)$ of $(-z_0,w_0)$. Hence we showed that $(-z_0,w_0)$ belongs also to Λ_ε. Since obviously $f_\varepsilon(z_0,w_0) = f_\varepsilon(-z_0,w_0)$ we get that the map $f_\varepsilon|_{\Lambda_\varepsilon}$ is at least 2-to-1. We will now look closer at the definition of hyperbolicity for $f_\varepsilon|_{\Lambda_\varepsilon}$. According to the definition (for example from [2]), there exists a continuous splitting of the tangent bundle

$$T_{\widehat{\Omega_\varepsilon}}(\hat{x}) = E_\varepsilon^u(\hat{x}) \oplus E_\varepsilon^s(x_0)$$

such that $(Df_\varepsilon)_{x_0}(E_\varepsilon^u(\hat{x})) \subseteq E_\varepsilon^u(\hat{f}_\varepsilon \hat{x})$, $(Df_\varepsilon)_{x_0}(E_\varepsilon^s(x_0)) \subseteq E_\varepsilon^s(f_\varepsilon x_0)$, and there exists $\lambda > 1$ with $\|Df_\varepsilon^n(v)\| \ge \lambda^n \cdot \|v\|$ and $\|Df_\varepsilon^n w\| \le \lambda^{-n} \cdot \|w\|$ for all $v \in E_\varepsilon^u(\hat{x})$ and all $w \in E_\varepsilon^s(x)$. So, a priori the unstable spaces $E_\varepsilon^u(\hat{x})$ should depend on $\hat{x}$, and in our situation we show they really do. We have

$$(Df_\varepsilon)_{(z,w)} = \begin{pmatrix} 4z^3 & 2\varepsilon w \\ 0 & 4w^3 \end{pmatrix}$$

and assume $E_\varepsilon^u(\widehat{z,w}) = \{(v_1,v_2) \in \mathbb{C}^2,\ v_1 = \beta(\widehat{z,w}) \cdot v_2\}$, for $(\widehat{z,w}) \in \hat{\Lambda}_\varepsilon$; $E_\varepsilon^s(z,w)$ is close to the v_1-axis, and $E_\varepsilon^u(\widehat{z,w})$ is close to the v_2-axis, so $\beta(\widehat{z,w}) < \beta_0 < 1$, where β_0 is independent of $\varepsilon > 0$ small enough and of $(\widehat{z,w}) \in \hat{\Lambda}_\varepsilon$. Now

$$(Df_\varepsilon)_{(z,w)} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 4z^3 & 2\varepsilon w \\ 0 & 4w^3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 4z^3 v_1 + 2\varepsilon w v_2 \\ 4w^3 v_2 \end{pmatrix}.$$

Denote by $(z_{-i}, w_{-i}) = (\widehat{z,w})_{-i}$, $(z_{-i}, w_{-i}) \in \Lambda_\varepsilon$. We know that

$$(Df_\varepsilon)_{(z_{-1},w_{-1})}(E_\varepsilon^u(\widehat{z_{-1},w_{-1}})) = E_\varepsilon^u(\widehat{z,w})$$

and

$$4z_{-1}^3 v_1 + 2\varepsilon w_{-1} v_2 = \beta(\widehat{z,w}) \cdot 4w_{-1}^3 v_2,$$

where $(v_1,v_2) \in E_\varepsilon^u(\widehat{z_{-1},w_{-1}})$. Since $v_1 = \beta(\widehat{z_{-1},w_{-1}}) \cdot v_2$, we get

$$4z_{-1}^3 \beta(\widehat{z_{-1},w_{-1}}) v_2 + 2\varepsilon w_{-1} v_2 = \beta(\widehat{z,w}) \cdot 4w_{-1}^3 v_2,$$

where $v_2 \neq 0$. Thus

$$\beta(\widehat{z,w}) = \varepsilon/2\frac{1}{w_{-1}^2} + (\frac{z_{-1}}{w_{-1}})^3 \cdot \beta(\widehat{z_{-1},w_{-1}}).$$

Applying recursively the above identity, gives

$$\beta(\widehat{z,w}) = \varepsilon/2\frac{1}{w_{-1}^2} + (\frac{z_{-1}}{w_{-1}})^3\beta(\widehat{z_{-1},w_{-1}})$$

$$= \varepsilon/2\frac{1}{w_{-1}^2} + \varepsilon/2\frac{1}{w_{-2}^2}(\frac{z_{-1}}{w_{-1}})^3 + (\frac{z_{-1}}{w_{-1}}\cdot\frac{z_{-2}}{w_{-2}})^3\beta(\widehat{z_{-2},w_{-2}}) = ... =$$

$$= \varepsilon/2(\frac{1}{w_{-1}^2}+(\frac{z_{-1}}{w_{-1}})^3\cdot\frac{1}{w_{-2}^2}+...+(\frac{z_{-1}...z_{-n+1}}{w_{-1}...w_{-n+1}})^3\frac{1}{w_{-n}^2})+(\frac{z_{-1}...z_{-n}}{w_{-1}...w_{-n}})^3\beta(\widehat{z_{-n},w_{-n}}).$$

Fix now a point $(z,w) \in \Lambda_\varepsilon$. From our discussion at the beginning of the Example, we know that $(-z_{-1},w_{-1}) \in \Lambda_\varepsilon$ too and denote by $(\widehat{z,w})'$ any prehistory of (z,w) such that $(\widehat{z,w})'_{-1} = (-z_{-1},w_{-1})$. We will show that $\beta(\widehat{z,w}) \neq \beta(\widehat{z,w})'$. With the notation $(z'_n,w'_n) = (\widehat{z,w})'_n$, we get:

$$\beta(\widehat{z,w})' = \varepsilon/2\frac{1}{w_{-1}^2} - \varepsilon/2(\frac{z_{-1}}{w_{-1}})^3\frac{1}{w_{-2}^2} + (\frac{z_{-1}z_{-2}}{w_{-1}w_{-2}})^3\beta(\widehat{z_{-2},w_{-2}}).$$

Hence

$$|\beta(\widehat{z,w}) - \beta(\widehat{z,w})'| = |\varepsilon\frac{1}{w_{-2}^2}\cdot(\frac{z_{-1}}{w_{-1}})^3 + O(\varepsilon^6)| \geq \varepsilon C_\varepsilon^3 + O(\varepsilon^6) \geq 0.5\varepsilon^4 \qquad (2.1)$$

In the equation above we used that $C_\varepsilon > 0.9\varepsilon$ and $D_\varepsilon < 2\varepsilon$. Moreover, we can estimate the difference $\beta(\widehat{z,w}) - \beta(\widehat{z,w})'$ in the same way as before also for other prehistories.

For example, if $(\widehat{z,w})_{-i} = (\widehat{z,w})'_{-i}$, $0 \leq i < j$, and $(\widehat{z,w})'_{-j} = (-z_{-j},w_{-j})$, then, since $C_\varepsilon := \inf_{(z,w)\in\Lambda_\varepsilon}|z| > 0.9\varepsilon$, and $D_\varepsilon := \sup_{(z,w)\in\Lambda_\varepsilon}|z| < 2\varepsilon$, we get

$$0.5\varepsilon^{3j+1} < |\beta(\widehat{z,w}) - \beta(\widehat{z.w})'| < 5\cdot\varepsilon^{3j+1} \qquad (2.2)$$

for $\varepsilon > 0$ small enough. We see therefore that for these maps f_ε, there pass an uncountable collection of local unstable manifolds through every point of their basic sets Λ_ε.

□

For $|c|$ small, the map $f_\varepsilon(z,w) = (z^4 + c + \varepsilon w^2, w^4)$ has the same characteristics on one of its basic sets $\Lambda_\varepsilon(c)$ as the function $(z^4+\varepsilon w^2, w^4)$. A similar calculation as before will show that $\sup_{(z,w)\in\Lambda_\varepsilon(c)}|z| < 3|c|$ and $\inf_{(z,w)\in\Lambda_\varepsilon(c)}|z| \geq \frac{3}{4}|c|$, if $0 < \varepsilon < \varepsilon(c)$ and then for $|c|$ small, (2.1) will give also that $\beta(\widehat{z,w}) \neq \beta(\widehat{z,w})'$. The results in [4] and [5] applied to the map $f_\varepsilon(z,w) = (z^4 + c + \varepsilon w^2, w^4)$ give that $W^u(\widehat{\Lambda_\varepsilon(c)})$ has empty interior and estimate $HD(W^u_\varepsilon(\widehat{\Lambda_\varepsilon(c)}))$ and $HD(\widehat{W^u_\varepsilon(\hat{\Lambda})})$, where $HD(\cdot)$ denotes the Hausdorff dimension.

In the course of working on this article, we also came upon the following question: If $f|_\Lambda$ as d'-to-1 and f_ε is a small perturbation of f, then is $f_\varepsilon|_{\Lambda_\varepsilon} d'$-to-1? Of course the first case to check would be that of $f(z,w) = (z^2 + c, w^2), c$ small, $c \neq 0$ on its basic set $\Lambda = \{p_0(c)\} \times S^1$, with $p_0(c)$ a fixed attracting point of $z^2 + c$. If c is small enough, f is Axiom A (see also [1]); and $f|_\Lambda$ is 2-to-1.

Theorem 2.2. *If $|c| \neq 0$, small, the map $f_\varepsilon(z,w) = (z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2, w^2)$ is injective on its basic set Λ_ε close to $p_0(c) \times S^1$, as long as $|c| < c(a,b,d,e)$, $b \neq 0$, and $\varepsilon < \varepsilon(a,b,c,d,e)$.*

Proof. Assume that $f_\varepsilon(z,w) = f_\varepsilon(z',w')$ for two points $(z,w),(z',w') \in \Lambda_\varepsilon$. Then

$$z^2 + \varepsilon(az + bw + dzw + ew^2) + c = z'^2 + \varepsilon(az' + bw' + dz'w' + ew^2) + c$$

and consequently

$$(z^2 - z'^2) + \varepsilon(a(z-z') + b(w-w') + d(zw - z'w') + e(w^2 - w'^2)) = 0.$$

Assume that $w \neq w'$. Then $w' = -w$ since $w^2 = w'^2$. Hence

$$z^2 - z'^2 + \varepsilon(a(z-z') + 2bw + dw(z+z')) = 0$$

and therefore

$$(z-z')(z+z'+\varepsilon a) = -2bw \cdot \varepsilon - d\varepsilon w(z+z'). \tag{2.3}$$

Let $p_0(c)$ be a fixed attracting point for $z \to z^2 + c$, c small. And consider $\alpha := \sup\limits_{(z,w)\in\Lambda_\varepsilon} |z - p_0(c)|$; let (z_0,w_0) a point in Λ_ε where the supremum in Λ_ε is attained so $\alpha = |z_0 - p_0(c)|$, and $p_0^2(c) + c = p_0(c)$. Now, we can find a point $(z,w) \in \Lambda_\varepsilon$ such that $f_\varepsilon(z,w) = (z_0,w_0)$. Then $z_0 = z^2 + a\varepsilon z + b\varepsilon w + c + d\varepsilon zw + e\varepsilon w^2$. So,

$$\begin{aligned} z_0 - p_0(c) &= z^2 - p_0^2(c) + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2 \\ &= (z - p_0(c))^2 + 2zp_0(c) - 2p_0^2(c) + a\varepsilon z + b\varepsilon w + d\varepsilon zw + e\varepsilon w^2. \end{aligned}$$

Consequently, $\alpha \leq \alpha^2 + 2|p_0(c)|\alpha + K\varepsilon$ with $K = K(a,b,d,e)$, a suitable majoration constant. Now $\alpha^2 + \alpha(2|p_0(c)| - 1) + K\varepsilon \geq 0$, and since $\alpha \ll 1$ (since Λ_ε is very close to $\{p_0(c)\} \times S^1$), we obtain:

$$0 \leq \alpha \leq \frac{2K\varepsilon}{1 - 2|p_0(c)| + \sqrt{(1 - 2|p_0(c)|)^2 - 4K\varepsilon}} \leq K' \cdot \varepsilon, \qquad K' = K'(a,b,c,d,e).$$

From (2.3) and since $|z - z'| \leq 2\alpha$ and $|z + z'| \leq 3|p_0(c)|$, one gets

$$2|b|\varepsilon \leq 2K'\varepsilon \cdot (\varepsilon a + 3|p_0(c)|) + 3d\varepsilon|p_0(c)|$$

for $\varepsilon < \varepsilon(c)$, since z, z' are ε-close to $p_0(c)$. We used also here the fact that $|w|$ is close to 1, for $(z,w) \in \Lambda_\varepsilon$.

So $0 < |b| \leq K'(\varepsilon a + 3|p_0(c)|) + 3d|p_0(c)|$, which is a contradiction if $b \neq 0$, and c is small in comparison to b, $c < c(a,b,d,e)$ (c small will imply that $|p_0(c)|$ is also small, and we can always reduce ε accordingly). Hence we proved $w' = w$. Then, from $f_\varepsilon(z,w) = f_\varepsilon(z',w')$ it follows that

$$z^2 - z'^2 + \varepsilon(a(z-z') + dw(z-z')) = 0$$

If $z \neq z'$, we would then get $z + z' + \varepsilon(a + dw) = 0$, hence

$$|z + z'| = \varepsilon|a + dw| \leq \varepsilon \cdot (|a| + |d|)$$

But z, z' are both close to $p_0(c) \neq 0$, so if we choose $\varepsilon < \varepsilon(a,b,c,d)$ small enough, the above inequality gives a contradiction. In conclusion $z = z', w = w'$ hence $f_\varepsilon|_{\Lambda_\varepsilon}\colon\ \Lambda_\varepsilon \to \Lambda_\varepsilon$ is an injective map.

□

In order to conclude our paper we need the concept of preimage entropy. Its definition requires some preparations. Let us call a *branch* of length ℓ (or *prehistory* of length ℓ) in X, a sequence of preimages, $\beta = (z_0, z_{-1}, \ldots, z_{-\ell})$, with $z_i \in X$, $-\ell \leq i \leq 0$, such that $f(z_{i-1}) = z_i$, $-\ell + 1 \leq i \leq 0$. Given another branch $\beta' = (z'_0, \ldots, z'_{-\ell})$ of same length, define their *branch distance* to be $d^b(\beta, \beta') = \max_{0\leq j\leq \ell} d(z_{-j}, z'_{-j})$. d^b measures the growth of inverse iterates. Using this, we now define a *branch metric* on X:

$$d^b_\ell(x, x') < \varepsilon,$$

if for every branch β of length ℓ with $z_0 = x$, there exists a branch β' of length ℓ with $z'_0 = x'$ such that $d^b(\beta, \beta') < \varepsilon$, and vice versa. Denote by $N_{\mathrm{span}}(\varepsilon, d^b_\ell, X)$ the smallest cardinality of an ε-spanning set for X in the d^b_ℓ metric. Hence, if A is an ε-spanning set with $\#A = N_{\mathrm{span}}(\varepsilon, d^b_\ell, X)$, then, for all $x \in X$, there is $y \in A$ with $d^b_\ell(x, y) < \varepsilon$. Let also $N_{\mathrm{sep}}(\varepsilon, d^b_\ell, X)$ be the largest cardinality of an ε-separated set for X. So, if A is ε-separated, then for all $x, y \in A$, $x \neq y$, $d^b_\ell(x, y) > \varepsilon$. The following proposition gives the definition of the preimage entropy $h_i(f)$ and two ways to calculate it.

Proposition 2.3 ([7]). *For $f\colon X \to X$ continuous, (X, d) compact metric space, we have*

$$\lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log N_{sep}(\varepsilon, d^b_n, X) = \lim_{\varepsilon\to 0} \overline{\lim} \frac{1}{n} \log N_{\mathrm{span}}(\varepsilon, d^b_n, X)$$

and the common value is called the preimage (branch) entropy*, denoted by $h_i(f)$.* □

In general there is no relation between $h_i(f)$ and the topological entropy $h_{top}(f)$ but, as can be easily proved, they coincide if f is a homeomorphism. We will also use one result about the preimage entropy from [7]. To formulate it we need some additional terminology. A *finite graph* is a compact metric space K with a distinguished finite set of points called *vertices*, whose complement has finitely many connected components, *edges*, homeomorphic to the open interval (0,1). We fix the metric on K by assigning length 1 to each edge and the distance between two points in K is the length of the shortest path connecting them.

Theorem 2.4. *[Nitecki-Przytycki, [7]] Let K be a finite graph and $f\colon K \to K$ continuous map. Then $h_i(f) = 0$.* □

We can give now our concluding result, showing that even a small perturbation can produce significant changes in the character of the basic set.

Corollary 2.5. *If f_ε satisfies the assumptions of Theorem 2, i.e f_ε is injective on its basic set Λ_ε, then Λ_ε is not a Jordan curve.*

Proof. Suppose on the contrary that Λ_ε is a Jordan curve. It then follows from the theorem above that $h_i(f_\varepsilon|_{\Lambda_\varepsilon}) = 0$. But since $f_\varepsilon|_{\Lambda_\varepsilon}$ is a homeomorphism of Λ_ε, $h_i(f_\varepsilon|\Lambda_\varepsilon) = h_{top}(f_\varepsilon|_{\Lambda_\varepsilon}) = h_{top}(\hat{f}_\varepsilon|_{\widehat{\Lambda_\varepsilon}}) = h_{top}(\hat{f}|_{\hat{\Lambda}}) = h_{top}(f|_\Lambda) = \log 2$, since $\hat{f}_\varepsilon|_{\widehat{\Lambda_\varepsilon}}$ and $\hat{f}|_{\hat{\Lambda}}$ are conjugate. This contradiction finishes the proof. □

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Addresses:

Eugen Mihailescu:
Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA
and Institute of Mathematics of the Romanian Academy, P. O. Box 1-764, Ro 70700,
Bucharest, Romania
Email: Eugen.Mihailescu@imar.ro

Mariusz Urbański:
Department of Mathematics
University of North Texas
P.O. Box 311430
Denton, TX 76203-1430
USA
Email:urbanski@unt.edu