Conformal, Geometric and Invariant Measures for Transcendental Expanding Functions

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Abstract

We describe the fractal structure of expanding meromorphic maps of the form $H\circ\exp\circ Q$, where H and Q are rational functions whose most transparent examples are among the functions of the form $\frac{A\exp(z^p)+B\exp(-z^p)}{C\exp(z^p)+D\exp(-z^p)}$ with $AD-BC\neq 0$. In particular we show that depending upon whether the Hausdorff dimension of the Julia set is greater or less than 1, the finite non-zero geometric measure is provided by the Hausdorff or packing measure. In order to describe this fractal structure we introduce and explore in detail Walters expanding conformal maps and jump-like conformal maps.

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1 Introduction and preliminaries

The orbits of points under iteration by a meromorphic function fall into three categories: they may be infinite, they may become periodic and hence consist of a finite number of distinct points or they may terminate at a pole of the function. Points in the last category are called *prepoles*. It follows from Picard's theorem that for transcendental meromorphic functions with more than one pole, there are infinitely many prepoles.

The *Fatou set* $F(f)$ of a meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is defined in exactly the same manner as for rational functions; $F(f)$ is the set of points $z \in \mathbb{C}$ such that all the iterates are defined and form a normal family on a neighborhood of z. The *Julia set* $J(f)$ is the complement of $F(f)$ in $\overline{\mathbb{C}}$. Thus, $F(f)$ is open, $J(f)$ is closed, $F(f)$ is completely invariant while $f^{-1}(J(f)) \subset J(f)$ and $f(J(f)\backslash\{\infty\}) = J(f)$. For a general description of the dynamics of meromorphic functions see e.g. (see [3]). We would however like to note that it easily follows from Montel's criterion of normality that if $f : \mathbb{C} \to \overline{\mathbb{C}}$ has at least one pole which is not an omitted value then (see [3])

$$J(f) = \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}.$$

We consider a class of transcendental meromorphic function of the form

$$f(z) = H(\exp(Q(z))) \qquad \tilde{f}(z) = \exp(Q(H(z)))$$

where Q and H are non-constant rational functions. Let $Q^{-1}(\infty) = \{d_j : j = 1, \dots, m\}$ be the set of poles of Q. Then

$$f(z) = H(\exp(Q(z))) : \bar{\mathbb{C}} \setminus \{d_j; j = 1, \dots, m\} \to \bar{\mathbb{C}} \setminus \{H(0), H(\infty)\}$$

and

$$\tilde{f}(z) = \exp(Q(H(z))) : \bar{\mathbb{C}} \setminus H^{-1}(\{d_j : j = 1, \dots, m\}) \to \bar{\mathbb{C}} \setminus \{0, \infty\}.$$

We additionally assume that there is at least one pole d_i of Q such that $d_i \neq H(0), H(\infty)$. We may assume that without loosing generality that $d_i = d_1$. Then the set

$$Ess_\infty(f) := \bigcup_{n=0}^{\infty} f^{-n}(\{d_j : i = 1, \dots, m\})$$

contains infinitely many points. Since $\{0, \infty\} \cap H^{-1}(d_1) = \emptyset$, the set

$$Ess_\infty(\tilde{f}) := \bigcup_{n=0}^{\infty} \tilde{f}^{-n}(H^{-1}(\{d_j := 1, \dots, m\}))$$

contains infinitely many points. The Fatou sets $F(f)$ and $F(\tilde{f})$ are defined in the same manner as for transcendental meromorphic of the complex plane i.e. $F(f)$ (resp. $F(\tilde{f})$) is the set of points $z \in \overline{\mathbb{C}}$ such that all the iterates are defined and form a normal family on a neighborhood of z. The *Julia set* $J(f)$ (resp. $J(\tilde{f})$) is the complement of $F(f)$ (resp. $F(\tilde{f})$) in $\bar{\mathbb{C}}$. Thus, $F(f)$ and $F(\tilde{f})$ are open, $J(f)$ and $J(\tilde{f})$ are closed, $F(f)$ and $F(\tilde{f})$ are completely invariant while $f^{-1}(J(f)) \subset J(f)$ and $f(J(f) \setminus \{d_j : j = 1, \dots, m\}) = J(f)$. Analogously $\tilde{f}^{-1}(J(\tilde{f})) \subset J(\tilde{f})$ and $f(J(\tilde{f}) \setminus H^{-1}(\{d_j : j = 1, \dots, m\}) = J(f)$. It follows from Montel's criterion that

$$J(f) = \overline{Ess_\infty(f)} \quad \text{and} \quad J(\tilde{f}) = \overline{Ess_\infty(\tilde{f})}.$$

Let $Crit(f)$ denote the set of critical points of f i.e.

$$Crit(f) = \{z : f'(z) = 0\}.$$

We also consider the general critical points of f. The point z is called a *general critical point* of f, if f restricted to any neighbourhood of z is not univalent. Thus, a general critical point is either a critical point or a multiple pole. The set of general critical points of f (resp. $\tilde{f}$) we denote by $Crit_G(f)$ (resp. $Crit_G(\tilde{f})$).

The exponential map has two omitted values $0, \infty$. By Iversen's theorem (see [5]) they are asymptotic values. Obviously $\exp(z)$ has no other asymptotic values. Hence f has at least one asymptotic value and maximum two if $H(0) \neq H(\infty)$. Let $Asymp(f) := \{H(0), H(\infty)\}$, $Asymp(\tilde{f}) := \{0, \infty\}$ denote respectively the sets of asymptotic values of f and $\tilde{f}$. We say that f satisfies the assumption (*), if the following conditions are satisfied:

(1) $J(f) \cap \overline{\bigcup_{n=0}^{\infty} f^n \left(Crit(f) \cup Asymp(f)\right)} = \emptyset$,

(2) if $a \in Crit(Q)$, then $\exp(Q(a))$ is not a pole of H,

(3) if H has a multiple pole, then $Q(\infty) \neq \infty$.

Let h denote the Hausdorff dimension of $J(f)$. Let $\mathcal{H}^h$ and $\mathcal{P}^h$ denote respectively $h-$ dimensional Hausdorff and packing measures. The following three theorems constitute the main results of our paper.

Theorem 1. *If f satisfies the assumption* (*), *then:*

(a) *If $h < 1$, then $\mathcal{P}^h(J(f)) > 0$ and $\mathcal{P}^h_{|J(f)}$ is σ-finite, while $\mathcal{H}^h(J(f)) = 0$.*

(b) *If $h = 1$, then $\mathcal{P}^h(J(f)) > 0, \mathcal{H}^h(J(f)) > 0$ and both measures restricted to $J(f)$ are σ-finite.*

(c) *If $h > 1$, then $\mathcal{H}^h(J(f)) > 0$ and $\mathcal{H}^h_{|J(f)}$ is σ-finite, while $\mathcal{P}^h(J(f)) = \infty$,*

where the Hausdorff measure and packing measure are defined by means of Euclidean metric.

Theorem 2. *If f satisfies the assumption* (*), *then:*

(a) *If $h < 1$, then $0 < \mathcal{P}^h(J(f)) < \infty$ and $\mathcal{H}^h(J(f)) = 0$.*

(b) *If $h = 1$, then $0 < \mathcal{P}^h(J(f)),\ \mathcal{H}^h(J(f)) < \infty$.*

(c) *If* $h > 1$, *then* $0 < \mathcal{H}^h(J(f)) < \infty$ *and* $\mathcal{P}^h(J(f)) = \infty$,

where the Hausdorff measure and packing measure are defined by means of spherical metric.

Of course, the Hausdorff measures (resp. packing measures) defined in the Euclidean and spherical metrics are equivalent, despite that the Radon-Nikodym derivatives don't have to be bounded. If the Hausdorff or packing measure is positive and σ-finite (in particular finite) we call it a *geometric measure*.

Theorem 3. *Suppose* f *satisfies the condition* (*). *Then there exists a unique probabilistic invariant measure equivalent to a geometric measure.*

We would like to mention that in the case when Q is the identity map, Theorem 1 easily follows from the results stated in [2]. The idea of the interplay between f and $\tilde{f}$ and an appropriate application of the thermodynamic formalism develops the approach from [2]. Our further methods are based on rigorous application of Walters expanding maps and the development of the theory of Walters expanding conformal maps along with jump-like conformal maps.

The paper is organized as follows. In the second section we recall (see [10]) the concept of Walters expanding maps, we define conformal Walters expanding maps and we prove its geometrical properties concerning the Hausdorff and packing measures. At the end of this section we briefly mention some class of conformal Walters expanding maps coming from meromorphic transcendental maps.

In section 3 we consider more special conformal expanding maps. By analogy to parabolic situation (see [4], [1], [7] and [8]) we call them jump-like conformal maps. This section contains the proof of Theorem 3.3 establishing positivity and finiteness of appropriate Hausdorff and packing measures.

In section 4 we show that $\tilde{f}$ is a jump-like conformal map and we apply the results of previous section in the context of $\tilde{f}$.

In section 5 using a semiconjugacy of f and $\tilde{f}$ we establish our main results namely Theorem 1, Theorem 2 and Theorem 3.

In the last section we describe examples illustrating our main theorems, in particular we provide the examples not covered by [2].

Writing $A \preceq B$ ($A \succeq B$) we mean that the quotient A/B is bounder from above (resp. below) by a finite positive constant independent of an appropriate variable under consideration.

2 Walters expanding conformal maps

We first define Walters expanding mappings and collect their selected properties needed in the sequel. For a full account of Walters theory see [10].

So, let X_0 be an open and dense subset of a compact metric space X endowed with a metric ρ. We call a continuous map $T : X_0 \to X$ Walters expanding provided that the following conditions are satisfied:

(2a) The set $T^{-1}(x)$ is at most countable for each $x \in X$.

(2b) There exists $\delta > 0$ such that for every $x \in X$ and every $n \geq 0$, $T^{-n}(B(x, 2\delta))$ can be written uniquely as a disjoint union of open sets $\{B_y(x)\}_{y \in T^{-n}(x)}$ such that $y \in B_y(x)$ and $T^n : B_y(x) \to B(x, 2\delta)$ is a homeomorphism from $B_y(x)$ onto $B(x, 2\delta)$. The corresponding inverse map from $B(x, 2\delta)$ to $B_y(x)$, $y \in T^{-n}(x)$, will be denoted by T_y^{-n}.

(2c) There exists $\lambda > 1$ and $n \geq 1$ such that for every $x \in X$, every $y \in T^{-n}(x)$ and all $z_1, z_2 \in B_y(x)$

$$d(T^n(z_1), T^n(z_2)) \geq \lambda d(z_1, z_2)$$

(2d) $\forall \epsilon > 0, \exists\ s \geq 1\ \forall x \in X\ T^{-s}(x)$ is ϵ-dense in X.

Recall that a function $g : Y \to \mathbb{R}$, where (Y, ρ) is a metric space, is Hölder continuous if there exist $\beta > 0$ and $L > 0$ such that for all $y_1, y_2 \in Y$, $|g(y_1) - g(y_2)| \leq L|y_1 - y_2|^\beta$. The parameter β is called the Hölder exponent of the function g and L is called its Hölder constant. A function $\phi : X_0 \to \mathbb{R}$ is called dynamically Hölder if there exists $\beta > 0$ and $L > 0$ such that for every $n \geq 1$, every $x \in X$ and every $y \in T^{-n}(x)$, the restriction $\phi|_{T_y^{-n}(B(x,\delta))}$ is Hölder continuous with exponent $\beta > 0$ and constant L. For every $n \geq 1$ put

$$S_n(\phi(x)) = \sum_{j=0}^{n-1} \phi \circ T^j(x).$$

Using (2c), the standard argument in thermodynamic formalism shows that there exists a constant $C > 0$ such that

$\forall_{x \in X}\, \forall_{y,z \in B(x,\delta)}\, \forall_{n \geq 0}\, \forall u \in T^{-n}(x)$

$$|S_n\phi(T_u^{-n}(y)) - S_n\phi(T_u^{-n}(z))| \leq Cd^\beta(y, z). \tag{1}$$

The function $\phi : X_0 \to X$ is called summable if

$$\sup_{x \in X} \left\{ \sum_{y \in T^{-1}(x)} \exp(\phi(y)) \right\} < \infty.$$

As an immediate consequence of (1) we get the following.

Proposition 2.1. *If $T : X_0 \to X$ is a Walters expanding map and $\phi : X_0 \to \mathbb{R}$ is a dynamically Hölder function, then the following conditions are equivalent.*

(a) *The function $\phi : X_0 \to X$ is summable.*

(b) There exists a finite δ-net W of X such that

$$\max_{x\in W}\left\{\sum_{y\in T^{-1}(x)}\exp(\phi(y))\right\}<\infty.$$

(c) For every δ-net W of X we have

$$\sup_{x\in W}\left\{\sum_{y\in T^{-1}(x)}\exp(\phi(y))\right\}<\infty.$$

(d) For every $x\in X$ we have

$$\sum_{y\in T^{-1}(x)}\exp(\phi(y))<\infty.$$

For every $n\geq 1$ and every $x\in X$ let

$$Z_n(\phi,x)=\sum_{y\in T^{-n}(x)}\exp\big(S_n\phi(y)\big).$$

We will need the following key fact.

Lemma 2.2. *If $T:X_0\to X$ is a Walters expanding map and $\phi:X_0\to\mathbb{R}$ is a dynamically Hölder function, then*

$$\exists_{p\geq 1}\,\forall_{q\geq p}\,\forall_{n\geq 1}\,\forall_{x,y\in X}\,\exists_{C_{x,y}>0}Z_n(\phi,x)\leq C_{x,y}Z_{n+q}(\phi,y).$$

Proof: Let $p\geq 1$ be the number provided by condition (2d) with $\epsilon=\delta$ and fix $q\geq p$. This condition implies then that there exists $z\in B(x,\delta)\cap T^{-q}(y)$. By (1) we then get

$$Z_n(\phi,x)\leq \mathrm{e}^M\exp(-S_q\phi(z))Z_{n+q}(\phi,y)$$

and we are done. ■

Given $x\in X$ we set

$$P_x(\phi)=\limsup_{n\to\infty}\frac{1}{n}\log\sum_{y\in T^{-n}(x)}\exp(S_n(\phi(x)).$$

As an immediate consequence of Lemma 2.2 and Proposition 2.1, we immediately get the following two remarkable statements.

Proposition 2.3. *If $T:X_0\to X$ is a Walters expanding map and $\phi:X_0\to\mathbb{R}$ is a dynamically Hölder function then for all $x,y\in X$, $P_x(\phi)=P_y(\phi)$. This common value is called the topological pressure of ϕ with respect to T and will be denoted by $P(\phi)$.*

and

Proposition 2.4. *If $T : X_0 \to X$ is a Walters expanding map and $\phi : X_0 \to \mathbb{R}$ is a dynamically Hölder function then $P(\phi) < \infty$ if and only if ϕ is summable.*

From the results of P.Walters in [10] one can extract the following:

Theorem 2.5. *If $T : X_0 \to X$ is a Walters expanding map and $\phi : X_0 \to X$ is a dynamically Hölder summable function, then there exist m_ϕ and μ_ϕ, Borel probability measures on X such that*

(a) $\forall n \geq 1,\ \forall x \in X,\ \forall y \in T^{-n}(x)$ and for every Borel set $A \subset T_y^{-n}(B(x,\delta))$

$$m_\phi(T^n(A)) = \int_A e^{P(\phi)-S_n(\phi)} dm_\phi$$

(b) μ_ϕ is T-invariant which means that $\mu_\phi \circ T^{-1} = \mu_\phi$, ergodic and equivalent with with m_ϕ Radon-Nikodym derivative bounded away from zero and infinity.

The reader may notice that the property (a) means that the measure μ_ϕ is an egeinmeasure with eigenvalue $e^{P(\phi)}$ of the corresponding Perron-Frobenius operator. Many additional stochastic properties of the dynamical systems can be found in [10].

A Walters expanding map $F : X_0 \to X$ is called conformal if $X \subset \mathbb{C}$ and if for every $x \in X$, every $n \geq 1$ and every $y \in F^{-n}(x)$ the inverse map $F_y^{-n} : B_X(x, 2\delta) \to X_0$ has a (unique) holomorphic extension to the ball $B_{\mathbb{C}}(x, 2\delta)$. This extension will be denoted by the same symbol F_y^{-n}. From now and throghout this entire section we assume that F is a Walters expanding conformal map. Of special importance will be the following functions $g_t : X_0 \to X, t \geq 0$ given by the formula:

$$g_t(x) = -t \log |F'(x)|.$$

It immediately follows from Koebe's distortion theorem that each function g_t is dynamically Hölder with the Hölder exponent $1/3$. Following [6] we define θ_F to be the infinum of all $t \geq 0$ for which the function g_t is summable. Due to Proposition 2.4, $\theta_F = \inf\{t \geq 0 : P(g_t) < \infty\}$. The following proposition is a straightforward standard consequence of the definition of pressure and property (2c).

Proposition 2.6. *The function $P : (\theta_F, \infty) \to \mathbb{R}$ is convex, continuous, strictly decreasing and $\lim_{t\to\infty} P(t) = -\infty$.*

We define

$$h_F = h = \inf\{t : P(t) \leq 0\}.$$

Obviously $h_F \geq \theta(F)$. Following the terminology of [6] and [9] we call the map F regular if $P(h) = 0$, strongly regular if there exists $t \geq 0$ such that $0 < P(t) < \infty$

and hereditarily regular if $P(\theta_F) = \infty$. In view of Proposition 2.6 each strongly regular map is regular and each hereditarily regular map is strongly regular. If F is regular, then $m = m_{g_h}$ is called the h-conformal measure for F. Its F-invariant version will be denoted by μ. Let

$$X_\infty = \bigcap_{n \ge 0} F^{-n}(X_0).$$

Denote by $\mathrm{HD}(Y)$ the Hausdorff dimension of a metric space Y. The first and most important fact concerning geometry of Walters expanding conformal maps is provided by the following.

Theorem 2.7. *If $F : X_0 \to X$ is a Walters expanding conformal map, then $\mathrm{HD}(X_\infty) \le h$. If in addition, F is strongly regular, then $\mathrm{HD}(X_\infty) = h$ and, in particular, $\mathrm{HD}(X_\infty) > \theta_F$.*

Proof: Fix $t > h$. Then, by Proposition 2.6, $P(t) < 0$. Let $W \subset X$ be a finite δ-net of X (i.e. $\bigcup_{x \in W} B(x, \delta) = X$). Since W is finite, there exists $k \ge 1$ such that for every $n \ge k$ and every $x \in W$,

$$\sum_{y \in F^{-n}(x)} |(F^n)'(y)|^{-t} < \exp\left(\frac{P(t)n}{2}\right) \tag{2}$$

Since in addition for every $n \ge 1$, $\bigcup_{x \in W} \bigcup_{y \in F^{-n}(x)} F^{-n}(B(x, \delta)) \supset X_\infty$, since $diam(F_y^{-n}(B(x, \delta)) \le K\delta|(F^n)'(y)|^{-1}$ where K is the Koebe distortion constant associated with the scale $\frac{1}{2}$ and since $\frac{1}{2}P(t) < 0$, we conclude from (2) that $\mathcal{H}^t(X_\infty) = 0$. Consequently $\mathrm{HD}(X_\infty) \le t$. Hence $\mathrm{HD}(X_\infty) \le h$ and the first part of our theorem is proved.

In order to cope with the other, harder part of this theorem we shall first prove the following.

Claim: If F is strongly regular, then $0 < \int \log |F'| d\mu < \infty$.

Proof of the claim. The inequality $\int \log |F'| d\mu > 0$ follows immediately from (2c), Koebe Distortion Theorem and invariance of μ. We shall show the finiteness of the integral of $\log |F'|$. Since F is strongly regular, there exists $t > 0$ such that $0 < P(t) < \infty$. By Proposition 2.6, $h > t \ge \theta$. In view of Proposition 2.4 and Proposition 2.1 there exists a constant $M > 0$ such that for every $x \in X$

$$\sum_{y \in F^{-1}(x)} |(F)'(y)|^{-t} \le M. \tag{3}$$

Let now $W \subset X$ be a finite δ-net of X. Since for every $T > 0$ the set $\bigcup_{x \in W} \{y \in F^{-1}(x) : |F'(y)| \le T\}$ is finite, we conclude that the set $\bigcup_{x \in W} \{y \in F^{-1}(x) : \log |F'(y)| \ge |F'(y)|^{h-t}\}$ is finite. Therefore, using Theorem 2.5(a) and (b) along with equality $P(h)=0$, bounded distortion property and (3), we get that

$$\int \log|F'|d\mu \leq$$

$$\leq \sum_{x\in W}\sum_{y\in F^{-1}(x)}\int_{F_y^{-1}(B(x,\delta))}\log|F'|d\mu$$

$$\leq const_1 + \sum_{x\in W}\sum_{y\in F^{-1}(x)}\int_{F_y^{-1}(B(x,\delta))}|F'|^{h-t}d\mu$$

$$\leq const_1 + const_2\sum_{x\in W}\sum_{y\in F^{-1}(x)}|F'(y)|^{h-t}\mu(F_y^{-1}(B(x,\delta)))$$

$$\leq const_1 + const_2\sum_{x\in W}\sum_{y\in F^{-1}(x)}|F'(y)|^{h-t}|F'(y)|^{-h}$$

$$\leq const_1 + const_2\sum_{x\in W}\sum_{y\in F^{-1}(x)}|F'(y)|^{-t}$$

$$\leq const_1 + const_2 M < \infty.$$

The proof of the claim is finished.

Fix now $\epsilon > 0$. Since $\mu(X_\infty)=1$, and since (see Theorem 2.5(b)) μ is F-invariant and ergodic, it follows from Birkhoff's ergodic theorem and Jegorov's theorem that there exist a Borel set $Y\subset X_\infty$ and the integer $k\geq 1$ such that $\mu(Y)\geq \frac{1}{2}$ and for every $x\in Y$ and every $n\geq k$

$$\left|\frac{1}{n}\log|(F^n)'(x)|-\chi\right|<\epsilon, \tag{4}$$

where $\chi=\int\log|F'|d\mu$. Put $\nu=\mu_{|Y}$. Given $x\in Y$ and $0<r<\delta$ let $n\geq 0$ be the largest integer such that

$$B(x,r)\subset F_x^{-n}(B(F^n(x),\delta)). \tag{5}$$

Then $B(x,r)$ is not contained in $F_x^{-(n+1)}(B(F^{n+1}(x),\delta))$ and applying Koebe's distortion theorem we get

$$r\geq K^{-1}\delta|(F^{n+1})'(x)|^{-1}. \tag{6}$$

Taking $r>0$ sufficiently small, we may assume that $n\geq k$. Combining now (5), Theorem 2.5(a), (b) along with $P(h)=0$, Koebe Distortion Theorem and (6), we obtain

$$\mu(B(x,r))\leq\mu(F_x^{-n}(B(F^n(x),\delta)))\asymp|(F^{-n})'(x)|^h\preceq r^h\frac{|(F^{n+1})'(x)|^h}{|(F^n)'(x)|^h}$$

Employing now (4), we thus get

$$\mu(B(x,r))\leq r^h e^{(\chi+\epsilon)(n+1)}e^{-(\chi-\epsilon)n}\simeq r^h e^{2\epsilon n} \tag{7}$$

But we deduce from (4) and (6) that $r \succeq e^{-(\chi+\epsilon)(n+1)}$ and therefore $e^{2\epsilon n} \succeq r^{-\frac{2\epsilon}{\chi+\epsilon}}$. This and (7) imply that $\nu(B(x,r)) \le \mu(B(x,r)) \preceq r^{h-\frac{2\epsilon}{\chi+\epsilon}}$. Consequently $\mathrm{HD}(X_\infty) \ge \mathrm{HD}(\nu) \ge h - \frac{2\epsilon}{\chi+\epsilon}$ and letting $\epsilon \to 0$ we finally obtain $\mathrm{HD}(X_\infty) \ge h$. We are done.¡ ■

Our next goal is to provide a formula for the upper ball-counting dimension (occasionally also called box-counting or Minkowski dimension). We denote by $N_r(A)$ the minimal number of balls with centers at the set A and of radii $r > 0$ needed to cover A. The upper ball-counting dimension of A is defined to be

$$\mathrm{BD}(A) = \limsup_{r\to 0} \frac{\log N_r(A)}{-\log r}.$$

We will need the following three auxiliary lemmas first.

Lemma 2.8. *If F is a regular Walters expanding conformal map, then*

$$\mathrm{BD}(X) = \max\{\mathrm{HD}(X_\infty), \sup_{n\ge 1}\sup_{x\in X}\{\mathrm{BD}(F^{-n}(x))\}\}$$

Proof: Since inequality "$\ge$" is obvious we only need to show the opposite one. Fix W, a finite δ-net of X. Fix $t > M$, the right-hand side of the formula appearing in Lemma 2.8. By Theorem 2.7, $P(t) < 0$. Therefore, there exists $u \ge 1$ such that if $q \ge u$, and $x \in W$, then

$$Z_q(t,x) := \sum_{y\in F^{-q}(x)} |(F^q)'(y)|^{-t} < \frac{1}{2}(4K)^{-t}. \tag{8}$$

In view of (2c) we may in addition assume that $|(F^q)'(z)| \ge 4K$ for all $q \ge u$ and all $z \in \bigcap_{j=0}^{q-1} F^{-j}(X)$. Fix $q \ge u$. Since $t > \mathrm{BD}(F^{-q}(x))$, there exists $A > 0$ such that $N_r(F^{-q}(x)) \le Ar^{-t}$ for all $x \in W$ and all $0 < r < 2\delta$. Choose now $B \ge 2 \times 4^t A\#W$ such that if $1 \le r \le 2\delta$, then $N_r(X) \le Br^{-t}$.

We shall show by induction that for each $n \ge 1$, if $1/n \le r \le 2\delta$, then $N_r(X) \le Br^{-t}$. By the definition of B this inequality holds for $n = 1$. Suppose that it holds for some $n \ge 1$ and fix $\frac{1}{n+1} \le r < 1/n$. Let

$$C_{n+1} = \left\{ y \in F^{-q}(W) : \mathrm{diam}\big(F_y^{-q}(B(F^q(y),\delta))\big) \le \frac{1}{2(n+1)} \right\}.$$

Since

$$X_\infty \subset \bigcup_{y\in C_{n+1}} F_y^{-q}(B(F^q(y),\delta)) \cup \bigcup_{y\in F^{-q}(W)\setminus C_{n+1}} F_y^{-q}(B(F^q(y),\delta))$$

and since $\overline{X_\infty} = X$, we get

$$\begin{aligned} N_r(X) &= N_r(X_\infty) \le N_{\frac{1}{n+1}}(X_\infty) \\ &\le N_{\frac{1}{n+1}}\left(\bigcup_{y\in C_{n+1}} F_y^{-q}(B(F^q(y),\delta))\right) + \sum_{y\in F^{-q}(W)\setminus C_{n+1}} N_{\frac{1}{n+1}}\left(F_y^{-q}(B(F^q(y),\delta))\right). \end{aligned} \tag{9}$$

Now, for every $y \in F^{-q}(W) \setminus C_{n+1}$ we have

$$N_{\frac{1}{n+1}}\left(F_y^{-q}(B(F^q(y),\delta))\right) \le N_{\frac{|(F^q)'(y)|}{K(n+1)}}(X \cap B(F^q(y),\delta)) \le N_{\frac{|(F^q)'(y)|}{2K(n+1)}}(X). \tag{10}$$

Since $|(F^q)'(y)| \ge 4K$ and $\frac{1}{4} \le \frac{n}{2(n+1)}$, we get

$$\frac{1}{n} \le \frac{|(F^q)'(y)|}{2K(n+1)}.$$

Since

$$\frac{1}{2(n+1)} < \operatorname{diam}\left(F_y^{-q}(B(F^q(y),\delta))\right) \le 2K\delta|(F^q)'(y)|^{-1},$$

we obtain

$$\frac{|(F^q)'(y)|}{2K(n+1)} \le 2\delta.$$

So, by inductive hypothesis and (10), we get

$$N_{\frac{1}{n+1}}\left(F_y^{-q}(B(F^q(y),\delta))\right) \le B\left(\frac{|(F^q)'(y)|}{2K(n+1)}\right)^{-t}. \tag{11}$$

Next, we claim that

$$N_{\frac{1}{n+1}}\left(\bigcup_{y\in C_{n+1}} F_y^{-q}(B(F^q(y),\delta))\right) \le N_{\frac{1}{2(n+1)}}(F^{-q}(W)). \tag{12}$$

To see this let $\left\{B\left(z_j, \frac{1}{2(n+1)}\right)\right\}_{j\in V}$ be a collection of balls with radii $\frac{1}{2(n+1)}$ covering $F^{-q}(W)$. Take $y \in C_{n+1}$ and $\xi \in F_y^{-q}(B(F^q(y),\delta))$. Then

$$|\xi - y| \le \operatorname{diam}\left(F_y^{-q}(B(F^q(y),\delta))\right) \le \frac{1}{2(n+1)}.$$

Since in addition $|y - z_j| < \frac{1}{2(n+1)}$ for some $j \in V$, we conclude that the balls $\left\{B\left(z_j, \frac{1}{n+1}\right)\right\}_{j\in V}$ cover the set $\bigcup_{y\in C_{n+1}} F_y^{-q}(B(F^q(y),\delta))$. Our claim is therefore proven.

Combining now (9), (12) and (11) we obtain

$$N_r(X) \le \sum_{y\in C_{n+1}} N_{\frac{1}{2(n+1)}}(F^{-q}(W)) + \sum_{y\in F^{-q}(W)\setminus C_{n+1}} B\left(\frac{|(F^q)'(y)|}{2K(n+1)}\right)^{-t}.$$

Since, $r < 1/n$, we have $n+1 \le 2/r$, and using (8) along with the definitions of A and B, we may continue the above inequality as follows

$$\begin{aligned} N_r(X) &\le \#W A(2(n+1))^t + B(2K)^t(n+1)^t \sum_{w\in W}\sum_{y\in F^{-q}(w)} |(F^q)'(y)|^{-t} \\ &\le 4^t\left(A\#W + BK^t\sum_{w\in W} Z_q(t,w)\right) r^{-t} \\ &\le 4^t\left(A\#W + 2^{-1}B4^{-t}\right) \le Br^{-t}. \end{aligned}$$

The inductive proof is thus finished and therefore $\mathrm{BD}(X) \le t$. We complete the proof of our lemma by letting $t \searrow M$. ■

Lemma 2.9. *If F is a regular Walters expanding conformal map, then for all $w \in X$, all $x, y \in B(w,\delta)$ and for all $n \ge 0$, we have* $\mathrm{BD}(F^{-n}(x)) = \mathrm{BD}(F^{-n}(y))$.

Proof: Fix $n \ge 0$. In view of the expanding property (2c) and Koebe's distortion theorem, a straightforward geometrical argument shows that
$\exists_{M\ge 1}\forall_{r>0}\forall_{z\in X}\forall_{v\in X}$

$$\#\{\xi \in F^{-n}(z) : B(v,r) \cap F_\xi^{-n}(B(z,\delta))) \ne \emptyset$$

and

$$\mathrm{diam}(F_\xi^{-n}(B(z,\delta))) \ge r/2\} \le M. \tag{13}$$

In order to prove the lemma it suffices to demonstrate that $\mathrm{BD}(F^{-n}(x)) \le \mathrm{BD}(F^{-n}(y))$. So, take $0 < r < \delta$ and put

$$I_r = \{\xi \in F^{-n}(w) : \mathrm{diam}(F_\xi^{-n}(B(w,\delta)) < r/2\}.$$

Then $N_r\big(\{F_\xi^{-n}(y) : \xi \in I_r\}\big) \le N_{r/2}\big(\{F_\xi^{-n}(x) : \xi \in I_r\}\big)$. Obviously $N_r\big(\{F_\xi^{-n}(z) : \xi \in F^{-n}(w) \setminus I_r\}\big) \le \#(F^{-n}(w) \setminus I_r)$ for all $z \in B(w,r)$. On the other hand, by (13), for the same z we have $N_r\big(\{F_\xi^{-n}(z) : \xi \in F^{-n}(w) \setminus I_r\}\big) \ge \#(F^{-n}(w) \setminus I_r)/M$. Hence

$$\begin{aligned} N_r(F^{-n}(y)) &\le N_{r/2}\big(\{F_\xi^{-n}(x) : \xi \in I_r\}\big) + N_r\big(\{F_\xi^{-n}(y) : \xi \in F^{-n}(w) \setminus I_r\}\big) \\ &\le N_{r/2}\big(F^{-n}(x)\big) + MN_r\big(\{F_\xi^{-n}(x) : \xi \in F^{-n}(w) \setminus I_r\}\big) \\ &\le (1+M)N_{r/2}\big(F^{-n}(x)\big). \end{aligned}$$

Therefore

$$\mathrm{BD}(F^{-n}(y)) = \limsup_{r\to 0} \frac{\log N_r(F^{-n}(y))}{-\log r} \le \limsup_{r\to 0} \frac{\log N_r(F^{-n}(x))}{-\log r} = \mathrm{BD}(F^{-n}(x)).$$

The proof is complete. ■

Proposition 2.10. *If F is a regular Walters expanding conformal map, and W is a finite δ-net of X, then* $\mathrm{BD}(F^{-1}(W)) \ge \theta_F$ *for all $x \in X$.*

Proof: Fix first an arbitrary $x \in W$. Fix then $t > \mathrm{BD}(F^{-1}(W)) \ge \mathrm{BD}(F^{-1}(x))$ and $u > 0$ so small that $t - u > \mathrm{BD}(F^{-1}(x))$. For every $\epsilon > 0$ consider the set

$$I(\epsilon) = \{y \in F^{-1}(x) : K\epsilon\delta^{-1} \le |F'(y)|^{-1} \le 2\epsilon\delta^{-1}\}.$$

Since by Koebe's distortion theorem the balls $\{B(y,\epsilon) : y \in I(\epsilon)\}$ are mutually disjoint, $N_\epsilon(F^{-1}(x)) \ge \#I(\epsilon)$. Since also for all $\epsilon > 0$ small enough,

$N_\epsilon(F^{-1}(x)) \le \epsilon^{-(t-u)}$, the following estimate hold for all k_0 large enough

$$\begin{aligned}\sum_{k\ge k_0}\sum_{y\in I(2^{-k})}|F'(y)|^{-t} &\le \sum_{k\ge k_0}(2K\delta^{-1})^t 2^{-kt}\#I(2^{-k})\\ &\le (2K\delta^{-1})^t\sum_{k\ge k_0}2^{-kt}N_{2^{-k}}(F^{-1}(x))\\ &\le (2K\delta^{-1})^t\sum_{k\ge k_0}2^{-kt}2^{kt}2^{-ku}\\ &= (2K\delta^{-1})^t\sum_{k\ge k_0}2^{-ku} \le (2K\delta^{-1})^t\frac{1}{1-2^{-u}}.\end{aligned}$$

In view of Proposition 2.1 the function g_t is thus summable and we are done. ■

Lemma 2.11. *If F is a regular Walters expanding conformal map and if W is a finite δ-net of X, then for all $n \ge 0$,*

$$\mathrm{BD}(F^{-n}(W)) \le \mathrm{BD}(F^{-1}(W)) = \max_{w\in W}\{\mathrm{BD}(F^{-1}(w))\}.$$

Proof: Fix $t > \mathrm{BD}(F^{-1}(W))$. We shall show by induction that for every $n \ge 1$ there exists $0 < A_n < \infty$ such that

$$N_r(F^{-n}(W)) \le A_n r^{-t} \tag{14}$$

for all $0 < r \le 2\delta$. And indeed, the existence of A_1 is immediate since $t > \mathrm{BD}(F^{-1}(W))$. So, fix $n \ge 1$ and suppose that A_n satisfying (14) exists. In order to demonstrate the existence of A_{n+1} put

$$I = \left\{y \in F^{-n}(W) : \mathrm{diam}\left(F_y^{-n}(B(F^n(y),\delta))\right) < r/2\right\}.$$

Then

$$\begin{aligned}N_r\left(\bigcup_{y\in I}F_y^{-n}(F^{-1}(W))\right) &= N_r\left(\bigcup_{y\in I}F_y^{-n}(B(F^n(y),\delta)\cap F^{-1}(W))\right)\\ &\le N_r\left(\bigcup_{y\in I}F_y^{-n}(B(F^n(y),\delta))\right)\\ &\le N_{r/2}(I) \le N_{r/2}(F^{-n}(W)) \le 2^t A_n r^{-t}.\end{aligned}$$

If $y \in F^{-n}(W)\setminus I$, then

$$N_r\left(F_y^{-n}(F^{-1}(W))\right) \le N_{K^{-1}|(F^n)'(y)|r}(F^{-1}(W)) \le A_1(K^{-1}|(F^n)'(y)|)^{-t}r^{-t},$$

where the second inequality holds since

$$\begin{aligned}K^{-1}|(F^n)'(y)|r &\le K^{-1}|(F^n)'(y)|2\mathrm{diam}\left(F_y^{-n}(B(F^n(y),\delta))\right)\\ &\le 2K^{-1}|(F^n)'(y)|K\delta|(F^n)'(y)|^{-1} = 2\delta.\end{aligned}$$

Now, by Proposition 2.10, $t > \theta_F$ which implies that the series $\sum_{y \in F^{-n}(W)} |(F^n)'(y)|^{-t}$ converges. Therefore the following estimates complete our inductive proof.

$$\begin{aligned} N_r(F^{-(n+1)}(W)) &\le 2^t A_n r^{-t} + (A_1 K)^t r^{-t} \sum_{y \in F^{-n}(W)\setminus I} |(F^n)'(y)|^{-t} \\ &\le \big(2^t A_n + (A_1 K)^t \sum_{y \in F^{-n}(W)\setminus I} |(F^n)'(y)|^{-t}\big) r^{-t}. \end{aligned}$$

■

Combining Lemma 2.8, Lemma 2.11 and Lemma 2.9, we immediately obtain the following.

Theorem 2.12. *If F is a regular Walters expanding conformal map and if W is a finite δ-net of X, then*

$$\begin{aligned} \mathrm{BD}(X) &= \max\{\mathrm{HD}(X_\infty), \mathrm{BD}(F^{-1}(W))\} \\ &= \max\{\mathrm{HD}(X_\infty), \max\{\mathrm{BD}(F^{-1}(w)) : w \in W\}\}. \end{aligned}$$

Our next geometrical results concern Hausdorff and packing measures. We start with the following.

Theorem 2.13. *If F is a regular Walters expanding conformal map, then $\mathcal{H}^h(X_\infty) < \infty$ and $\mathcal{P}^h(X_\infty) > 0$. In addition $\mathcal{H}^h \ll m$ and $\frac{d\mathcal{H}^h}{dm} < \infty$.*

Proof: Let $W \subset X$ be a finite δ-net of X. Let $A \subset X_\infty$ be a closed subset in the topology relative to X_∞. For every $n \ge 1$ put

$$I_n = \{y \in F^{-n}(W) : F_y^{-n}(B(F^n(y),\delta)) \cap A \ne \emptyset\}.$$

Then the family $\{F_y^{-n}(B(F^n(y),\delta))\}_{y \in I_n}$ covers A and its multiplicity is bounded above by $\sharp W$, the cardinality of W. In addition by the expanding property, for every $\epsilon > 0$ there exists $n \ge 1$ such that

$$\bigcup_{y \in I_n} F_y^{-n}(B(F^n(y),\delta)) \subset B(A,\epsilon).$$

Since this union is open and A is closed, this implies that

$$\lim_{n \to \infty} m\Big(\bigcup_{y \in I_n} F_y^{-n}(B(F^n(y),\delta))\Big) = m(A) \tag{15}$$

Now,

$$\begin{aligned} \sum_{y \in I_n} \mathrm{diam}^h(F_y^{-n}(B(x,\delta))) &\preceq \sum_{y \in I_n} |(F^n)'(y)|^{-h} (2\delta)^{\asymp} \sum_{y \in I_n} |(F^n)'(y)|^{-h} \\ &\asymp \sum_{y \in I_n} m(F_y^{-n}(B(x,\delta))) \\ &\le \sharp W m \left(\sum_{y \in I_n} m(F_y^{-n}(B(x,\delta))) \right). \end{aligned}$$

Thus using (15) , we conclude that $\mathcal{H}^h(A) \preceq m(A)$. Since X_∞ is a separable metric space, and m is regular, this inequality extends to all Borel subsets of X_∞. This implies that $\mathcal{H}^h(X_\infty) \ll m$ and $\frac{d\mathcal{H}^h}{dm} < \infty$ and consequently $\mathcal{H}^h(X_\infty) < \infty$.

Fix now $x \in X_\infty$. Then for every $n \geq 1$

$$F_x^{-n}(B(F^n(x),\delta)) \supset B(x, K^{-1}|(F^n)'(x)|^{-1}\delta)$$

and therefore

$$\begin{aligned} m\big(B(x, K^{-1}|(F^n)'(x)|^{-1})\delta\big) &\leq K^h|(F^n)'(x)|^{-h} m(B(F^n(x),\delta))) \\ &\leq K^{2h}(K^{-1}|(F^n)'(x)|)^{-h}. \end{aligned}$$

Thus $\mathcal{P}^h(X_\infty) > 0$. The proof is complete. ■

We shall now provide a sufficient condition for the h-dimensional Hausdorff measure to be positive and for the packing measure to be finite.

Theorem 2.14. *Suppose that F is a regular Walters expanding conformal map. Assume that there exist $\gamma \geq 1$ and $0 < \xi \leq \delta$ such that for every $x \in X_0$ and for every r satisfying the condition $\gamma \mathrm{diam}(\mathrm{F_x^{-1}(B(F(x),\delta))}) \leq \mathrm{r} \leq \xi$, we have $m(B(x,r)) \geq Lr^h$. Then $\mathcal{P}^h(X_\infty) < \infty$.*

Proof: Fix $x \in X_0$ and r satisfying $\gamma \mathrm{diam}(\mathrm{F_x^{-1}(B(F(x),\delta))}) \leq \mathrm{r} \leq \xi$. Let $n \geq 0$ be the maximal integer such that $B(x,r) \subset F_x^{-n}(B(F^n(x),\delta))$. Then $F_x^{-(n+1)}(B(F^n(x),\delta))$ does not contain the ball $B(x,r)$ and consequently $r \geq \delta K^{-1}|(F^{n+1})'(x)|^{-1}$. Hence

$$B(x,r) \supset B(x, \delta K^{-1}|(F^{n+1})'(x)|^{-1}) \supset F_x^{-(n+1)}(B(F^{n+1}(x), \delta K^{-2}))$$

and thus $m(B(x,r)) \geq M(\delta K^{-2})|(F^{n+1})'(x)|^{-h}K^{-h}$, where

$$M(\beta) = \inf\{m(B(z,\beta)) : z \in X\} > 0$$

since $\mathrm{supp}(\mu) = X$. If $|(F^{n+1})'(x)| \leq r^{-1}\gamma K^2\delta$, then

$$m(B(x,r)) \geq K^{-3h}M(\delta K^{-2})\gamma^{-h}\delta^{-h}r^h. \tag{16}$$

So, suppose that

$$|(F^{n+1})'(x)| \geq r^{-1}\gamma K^2\delta \tag{17}$$

By the Koebe distortion theorem

$$F_x^{-n}(B(F^n(x), K^{-1}|(F^n)'(x)|r)) \subset B(x,r).$$

Now, in view of (17)

$$\begin{aligned} K^{-1}|(F^n)'(x)|r = K^{-1}|(F^{n+1})'(x)|r|F'(F^n(x))|^{-1} &\geq \gamma K\delta|F'(F^n(x))|^{-1} \\ &\geq \gamma \mathrm{diam}(\mathrm{F_{F^n(x)}^{-1}(B(F(F^n(x)),\delta))}). \end{aligned}$$

Therefore, by our assumption and conformality of m

$$\begin{aligned} m(B(x,r)) &\geq K^{-h}|(F^n)'(x)|^{-h} m(B(F^n(x), K^{-1}|(F^n)'(x)|r)) \\ &\geq K^{-h}|(F^n)'(x)|^{-h}(K^{-1}|(F^n)'(x)|r)^h = K^{-2h}r^h. \end{aligned}$$

Combining this and (16) completes the proof. ■

Theorem 2.15. *Suppose F is a regular Walters expanding conformal map. Assume there exist $\gamma \geq 1$ and $L > 0$ such that for every $x \in X_0$ and for every r satisfying the condition $r \geq \gamma \mathrm{diam}(F_x^{-1}(B(F(x), \delta)))$, we have $m(B(x), r)) \leq Lr^h$. Then $\mathcal{H}^h(X_\infty) > 0$.*

Proof: Fix $x \in X_0$ and $\delta \geq r \geq \gamma \mathrm{diam}(F_x^{-1}(B(F(x), \delta)))$. Let $n \geq 0$ be the maximal integer such that

$$B(x, K^2 r) \subset F_x^{-n}(B(F^n(x), \delta)) \tag{18}$$

Then $B(x, K^2 r)$ is not contained in the set $F_x^{-(n+1)}(B(F^{n+1}(x), \delta))$, and as $F_x^{-(n+1)}(B(F^{n+1}(x), \delta)) \supset B(x, K^{-1}\delta|(F^{n+1})'(x)|^{-1})$, we conclude that $K^2 r \geq K^{-1}\delta|(F^{n+1})'(x)|^{-1}$. Hence $\gamma K^4 r|(F^n)'(x)| \geq \gamma K\delta|F'(F^n(x))|^{-1} \geq \gamma \mathrm{diam}(F_{F^n(x)}^{-1}(B(F(F^n(x), \delta))$ and therefore, by our assumptions

$$m(B(F^n(x), K^4 r|(F^n)'(x)|)) \leq LK^{4h}r^h|(F^n)'(x)|^h \tag{19}$$

By (18) $K^2 r \leq K\delta|(F^n)'(x)|^{-1}$ or equivalently $Kr|(F^n)'(x)| \leq \delta$. Hence $B(x,r) \subset F_x^{-n}(B(F^n(x), Kr|(F^n)'(x)|)$ and using (19) we get that

$$\begin{aligned} m(B(x,r)) &\leq K^h|(F^n)'(x)|^{-h} m(B(F^n(x), Kr|(F^n)'(x)|)) \\ &\leq K^h|(F^n)'(x)|^{-h} m(B(F^n(x), K^4 r|(F^n)'(x)|)) \leq LK^{5h}r^h. \end{aligned}$$

The proof is complete. ■

Theorem 2.16. *Suppose F is a regular Walters expanding conformal map. Let $h = \mathrm{HD}(X_\infty)$ and let m be the corresponding conformal measure. If there exist a sequence of points $z_j \in X, j \geq 1$, and a sequence of positive reals $\{r_j\}_{=1}^{\infty}$ such that $r_j \leq \delta/2$ and*

$$\overline{\lim}_{j\to\infty} \frac{m(B(z_j, r_j))}{r_j^h} = \infty,$$

then $\mathcal{H}^h(X_\infty) = 0$.

Proof: Let X_t be the set of transitive points of F. Since μ, the F-invariant version of m is ergodic and positive on non-empty sets of X, it follows from Birkhoff's ergodic theorem that $m(X_t) = 1$. Fix $x \in X_t$ and then $\epsilon > 0$. By our assumptions there exists $j \geq 1$ such that $m(B(z_j, r_j))r_j^{-h} \geq \epsilon^{-1}$. Since $x \in X_t$, there exists $n \geq 0$ such that $F^n(x) \in B(z_j, r_j)$. It implies that

$$x \in F_x^{-n}(B(z_j, r_j)) \subset B(F_x^{-n}(z_j), K|(F_x^{-n})'(z_j)|r_j).$$

Hence $|x - F_x^{-n}(z_j)| < K|(F_x^{-n})'(z_j)|r_j$ and therefore

$$B(x, 2K|(F_x^{-n})'(z_j)|r_j) \supset B(F_x^{-n}(z_j), r_j K|(F_x^{-n})'(z_j)|) \supset F_x^{-n}(B(z_j, r_j)).$$

Hence

$$\begin{aligned} m(B(x, 2r_j K|(F_x^{-n})'(z_j)|)) &\geq m(F_x^{-n}(B(z_j, r_j))) \\ &\geq K^{-h}|(F_x^{-n})'(z_j)|^h m(B(z_j, r_j)) \\ &\geq K^{-h}|(F_x^{-n})'(z_j)|^h \epsilon^{-1} r_j^h \\ &= (2K^2)^{-h}\epsilon^{-1}(2r_j K|(F_x^{-n})'(z_j)|)^h. \end{aligned}$$

Letting $\epsilon \searrow 0$, we conclude that

$$\overline{\lim}_{r\to 0}\frac{m(B(x,r))}{r^h} = \infty$$

and consequently $\mathcal{H}^h(X_t) = 0$. Since $m(X \backslash X_t) = 0$ it follows from Theorem 2.13 that $\mathcal{H}^h(X \setminus X_t) = 0$. We are done. ■

Theorem 2.17. *Suppose F is a regular Walters expanding conformal map. Let $h = \mathrm{HD}(X_\infty)$ and let m be the corresponding conformal measure. If there exists sequence of points $z_j \in X, j \geq 1$ and a sequence of positive reals $\{r_j\}_{=1}^{\infty}$ such that*

$$\underline{\lim}_{j\to\infty}\frac{m(B(z_j, r_j))}{r_j^h} = 0,$$

then $\mathcal{P}^h(X_\infty) = \infty$.

Proof: Let, similarly as in Theorem 2.16, X_t be the set of transitive points of F. Fix $x \in X_t$ and then $\epsilon > 0$. By the assumptions there exists $j \geq 1$ such that $m(B(z_j, r_j)) \leq \epsilon r_j^h$. Since x is a transitive point, there exists $n \geq 1$ such that $F^n(x) \in B(z_j, r_j/2)$. Therefore

$$m(B(x, K^{-1}|(F^n)'(x)|^{-1} r_j/2)) \leq m(F_x^{-n}(B(F^n(x), r_j/2)))$$

$$\leq K^h|(F^n)'(x)|^{-h} m(B(F^n(x), r_j/2)) \leq K^h|(F^n)'(x)|^{-h} m(B(z_j, r_j))$$

$$\leq K^h|(F^n)'(x)|^{-h}\epsilon r_j^h = (2K^2)^h \epsilon (1/2K^{-1}|(F^n)'(x)^{-1} r_j)^h.$$

Thus letting $\epsilon \to 0$ we see that

$$\underline{\lim}_{r\to 0}\frac{m(B(x,r))}{r^h} = 0$$

Since $m(X_t) = 1$ this implies that $\mathcal{P}^h(X_\infty) \geq \mathcal{P}^h(X_t) = \infty$. ■

If ν is a Borel measure on a metric space Y, then ν is said to be locally finite at a point $y \in Y$ if there exists an open set $U \subset Y$ such that $y \in U$ and $\nu(U) < \infty$. As a supplement to the last theorem we shall prove the following.

Proposition 2.18. *If F is a Walters expanding conformal map and $\mathcal{P}^h(X_\infty) < \infty$, then $\mathcal{P}^h|_{X_\infty}$ is locally finite at every point of X_∞.*

Proof: Suppose that $\mathcal{P}^h|_{X_\infty}$ is locally finite at some point $y \in X_\infty$. It means that there exists $\epsilon > 0$ such that $\mathcal{P}^h(B(y, 2\epsilon)) < \infty$. Fix an arbitrary point $x \in X_\infty$. In view of (2c) and (2d) there exists an integer $n \geq 1$ such that $B(y, \epsilon) \cap F^{-n}(x) \neq \emptyset$ and if z is in this intersection, then $\text{diam}\big(F_z^{-n}(B(x, \delta))\big) < \epsilon$. Employing conformality of the packing measure $\mathcal{P}^h$ and Koebe's distortion theorem, we get the following.

$$\begin{aligned}\mathcal{P}^h(B(x,\delta)) &\leq K^h |(F^n)'(z)|^h \mathcal{P}^h\big(F_z^{-n}(B(x,\delta))\big) \\ &\leq K^h |(F^n)'(z)|^h \mathcal{P}^h(B(y,2\epsilon)) < \infty.\end{aligned}$$

Thus, covering X_∞ by finitely many balls with radii δ, we conclude that $\mathcal{P}^h(X_\infty) < \infty$. The proof is finished. ■

Remark. We would like to mention that if $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a meromorphic mapping for which

$$J(f) \cap \overline{\bigcup_{n=0}^{\infty} f^n(Crit_G(f) \cup Asymp(f))} = \emptyset,$$

where $J(f)$ is the Julia set of f, then it is not difficult to prove (see the proof of Theorem 4.7 for more details than we need here) that if $M : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a Möbius transformation such that $M(\infty) \notin J(f)$, then the Julia set of the map $\tilde{f} : M^{-1} \circ f \circ M : \overline{\mathbb{C}} \setminus M^{-1}(\infty) \to \overline{\mathbb{C}}$ is a compact subset of $\mathbb{C}$ and $\tilde{f}$ restricted to its Julia set is a conformal Walters expanding map. In particular all the theorems proven in this section apply to $\tilde{f}$.

3 Jump-like conformal maps

Let (X, d) be a compact metric space. For every $A, B \subset X$ we define

$$\text{dist}(A, B) := \inf\{d(a, b) : a \in A,\ b \in B\}$$

$$\text{Dist}(A, B) := \sup\{d(a, b) : a \in A,\ b \in B\}$$

We call a Walters expanding conformal map $F : X_0 \to X$ jump-like if the following requirements are met. There exists $C \geq 1, p \geq 1, A \geq 2, b_j \in X$ and $q_j \geq 1$ for every $j = 1, \ldots, p$ such that the following conditions are satisfied:

(3a) $\{b_1, \ldots, b_p\} \cap F^{-1}(X) = \emptyset$

(3b) For every $x \in X$, the set $F^{-1}(x)$ can be uniquely represented as $\{x_{j,a,n} : n \in \mathbb{Z}, 1 \leq j \leq p, 1 \leq a \leq q_j\}$

(3c) $\max_{1 \leq j \leq p} \max_{1 \leq a \leq q_j} \sup_{x \in X} \{\lim_{n \to \infty} \text{Dist}(b_j, F_{x_{j,a,n}}^{-1}(B(x, \delta)))\} = 0$

(3d) $\forall_{x\in X}$, $\forall_{1\le j\le p}$, $\forall_{1\le a\le q_j}$, $\forall_{n\in\mathbb{Z},\ |n|\ge A}$

$$C^{-1}|n|^{\frac{q_j+1}{q_j}} \le |F'(x_{j,a,n})| \le C|n|^{\frac{q_j+1}{q_j}}$$

(3e) $\forall_{y,z\in X}$, $\forall_{1\le j\le p}$, $\forall_{a,b\in\{1,\dots,q_j\}}$, $\forall_{k,n\in\mathbb{Z},\ ||k|-|n||\ge A,\ |n|\ge A,\ |k|\ge A}$

$$\mathrm{dist}(F^{-1}_{y_{j,a,k}}(B(y,\delta)), F^{-1}_{z_{j,b,n}}(B(z,\delta))) \ge C^{-1}\left||k|^{-\frac{1}{q_j}} - |n|^{-\frac{1}{q_j}}\right|$$

(3f) $\forall_{y,z\in X}$, $\forall_{1\le j\le p}$, $\forall_{a\in\{1,\dots,q_j\}}$, $\forall_{k,n\in\mathbb{Z},\ kn>0,\ ||k|-|n||\ge A,\ |n|\ge A,\ |k|\ge A}$

$$\mathrm{Dist}(F^{-1}_{y_{j,a,k}}(B(y,\delta)), F^{-1}_{z_{j,a,n}}(B(z,\delta))) \le C\left||k|^{-\frac{1}{q_j}} - |n|^{-\frac{1}{q_j}}\right|.$$

As an immediate consequence of (3d), with a bigger constant C perhaps, we get the following $\forall_{x\in X}$, $\forall_{1\le j\le p}$, $\forall_{1\le a\le q_j}$, $\forall_{n\in\mathbb{Z},\ |n|\ge A}$

$$C^{-1}|n|^{-\frac{q_j+1}{q_j}} \le \mathrm{diam}(F^{-1}_{x_{j,a,n}}(B(x,\delta))) \le C|n|^{-\frac{q_j+1}{q_j}}. \tag{20}$$

Letting $k\to\infty$, it immediately follows from (3c), (3e) and (3f) that $\forall_{1\le j\le p}$, $\forall_{x\in X}$, $\forall_{1\le a\le q_j}$, $\forall_{|n|\ge 2A}$

$$C^{-1}|n|^{-\frac{1}{q_j}} \le \mathrm{dist}(b_j, F^{-1}_{x_{j,a,n}}(B(x,\delta))| \le \mathrm{Dist}(b_j, F^{-1}_{x_{j,a,n}}(B(x,\delta))| \le C|n|^{-\frac{1}{q_j}} \tag{21}$$

Our first result concerning geometry of jump-like conformal maps is the following.

Proposition 3.1. *Suppose that $F: X_0 \to X$ is a jump-like conformal map and let $q = \max\{q_j : 1\le j\le p\}$. Then the map F is hereditarily regular and $\theta_F = \frac{q}{q+1}$.*

Proof: Fix $x\in X$, ≥ 0 and consider the series

$$S_t(x) = \sum_{y\in F^{-1}(x)} |F'(y)|^{-t}.$$

In order to complete the proof of this proposition it suffices to show that $S_{\frac{q}{q+1}}(x) = \infty$ and that for all $t > \frac{q}{q+1}$ there exists $M_t > 0$ such that for all $x\in X$ we have $S_t(x) \le M_t$. And indeed, putting

$$\Sigma_1(t,x) = \sum_{n=-A+1}^{A-1}\sum_{j=1}^{p}\sum_{a=1}^{q_j} |(F^{-1}_{x_{j,a,n}})'(x)|^t$$

in view of (3d) we get

$$\begin{aligned} S_t(x) &= \sum_{n\in\mathbb{Z}}\sum_{j=1}^{p}\sum_{a=1}^{q_j} |(F^{-1}_{x_{j,a,n}})'(x)|^t \asymp \Sigma_1(t,x) + \sum_{|n|\ge A}\sum_{j=1}^{p}\sum_{a=1}^{q_j} |n|^{-\frac{q_j}{q_j+1}t} \\ &\asymp \Sigma_1(t,x) + \sum_{|n|\ge A}\sum_{j=1}^{p} q_j|n|^{-\frac{q_j}{q_j+1}t} \end{aligned} \tag{22}$$

Since for every $t \geq 0$, $\sup\{\Sigma_1(t,x) : x \in X\} < \infty$ and since the series $\sum_{n\geq 1} n^{-s}$ converges if and only if $s > 1$, our proof is completed by applying (22) ■

Since it is easy to see that if F is a jump-like conformal map, then for every $x \in X$, $\mathrm{BD}(F^{-1}(X)) = \frac{q}{q+1}$, where $q = \max\{q_j : 1 \leq j \leq p\}$, as an immediate consequence of Proposition 3.1 and Theorem 2.12 we get the following.

Theorem 3.2. *If $F : X_0 \to X$ is a jump-like conformal map, then*

$$\mathrm{BD}(X) = \mathrm{HD}(X_\infty).$$

The main result of this section is the following.

Theorem 3.3. *Suppose that $F : X_0 \to X$ is a jump-like conformal map. Then*

(a) *If $h < 1$, then $0 < \mathcal{P}^h(X_\infty) < \infty$ and $\mathcal{H}^h(X_\infty) = 0$.*

(b) *If $h = 1$, then $0 < \mathcal{P}^h(X_\infty)$, $\mathcal{H}^h(X_\infty) < \infty$.*

(c) *If $h > 1$, then $0 < \mathcal{H}^h(X_\infty) < \infty$ and $\mathcal{P}^h(X_\infty) = \infty$.*

The proof of this theorem will be contained in the following four lemmas.

Lemma 3.4. *Suppose that $F : X_0 \to X$ is a jump-like conformal map. If $h = \mathrm{HD}(X_\infty) < 1$, then $\mathcal{H}^h(X_\infty) = 0$.*

Proof: Let $1 \leq i \leq p$, $1 \leq a \leq q_i$ and $x \in X$. Fix $0 < r < (\frac{C}{A})^{\frac{1}{q_i}}$. Then $F^{-1}_{x_{i,a,n}}(B(x,\delta)) \subset B(b_i, r))$ iff $\mathrm{Dist}(b_i, F^{-1}_{x_{i,a,n}}(B(x,\delta))) < r$. By (21) this is true if $C|n|^{-\frac{1}{q_i}} < r$. Thus

$$r^{-h} m(B(b_i,r)) \geq r^{-h} \sum_{|n|>(Cr^{-1})^{q_i}} m(F^{-1}_{x_{i,a,n}}(B(x,\delta)))$$

Since we provide estimates from below it is enough for us to consider only positive values of n. We continue

$$\begin{aligned} r^{-h} m(B(b_i,r)) &\geq \sum_{|n|>(Cr^{-1})^{q_i}} r^{-h} K^{-h} |(F^{-1}_{x_{i,a,n}})'(x)|^h m(B(x,\delta)) \\ &\succeq r^{-h} \sum_{n>(Cr^{-1})^{q_i}} n^{-\frac{q_i+1}{q_i}h} \asymp r^{-h} (Cr^{-1})^{(1-\frac{q_i+1}{q_i}h)q_i} \\ &\asymp r^{q_i(h-1)} \to \infty. \end{aligned}$$

Thus the assumption of Theorem 2.16 are satisfied putting $z_j = b_i$ for all $j \geq 1$. Consequently $\mathcal{H}^h(X_\infty) = 0$. ■

Lemma 3.5. *Suppose that $F : X_0 \to X$ is a jump-like conformal map. If $h = \mathrm{HD}(X_\infty) \leq 1$ then $\mathcal{P}^h(X_\infty) < \infty$.*

Proof: Fix $n \in \mathbb{Z}$, $1 \le i \le p$, $1 \le a \le q_i$. For $x \in X$ let $y = F^{-1}_{x_{i,a,n}}(x)$. Since we want to apply Theorem 2.14 we may disregard finitely many branches of F^{-1} and in particular, we may assume that $|n| \ge 2A$. Take $r \ge \mathrm{diam} F^{-1}_{x_{i,a,n}}(B(x,\delta)) \asymp n^{-\frac{q_i+1}{q_i}} \asymp (n-A)^{-\frac{q_i+1}{q_i}}$. Since the considerations in the case $n < 0$ are analogous to those with $n > 0$ with obvious modification, we assume throughout this proof that $n > 0$. Take $A \le k \le n - A$ and such that $C(k^{-\frac{1}{q_i}} - n^{-\frac{1}{q_i}}) < r$. Then, by (3f) $B(y,r) \supset F^{-1}_{x_{i,a,k}}(B(x,\delta))$. Hence

$$
\begin{aligned}
m(B(y,r)) &\ge \sum_{k=E((C^{-1}r+n^{-\frac{1}{q_i}})^{-q_i})+1}^{n-A} m(F^{-1}_{x_{i,a,k}}(B(x,\delta)) \\
&\succeq \sum_{k=E((C^{-1}r+n^{-\frac{1}{q_i}})^{-q_i})+1}^{n-A} K^{-h}|F^{-1}_{x_{i,a,k}}(x)|^h m(B(x,\delta)) \\
&\succeq \sum_{k=E((C^{-1}r+n^{-\frac{1}{q_i}})^{-q_i})+1}^{n-A} k^{-\frac{q_i+1}{q_i}h} \\
&\succeq \left(C^{-1}r + (n-A)^{-\frac{1}{q_i}}\right)^{-q_i(1-\frac{q_i+1}{q_i}h)} - (n-A)^{1-\frac{q_i+1}{q_i}h}\Big) \\
&\asymp \left((C^{-1}r + (n-A)^{-\frac{1}{q_i}})^{(q_i+1)h-q_i} - ((n-A)^{-\frac{1}{q_i}})^{(q_i+1)h-q_i}\right)
\end{aligned}
$$

By the Mean Value Theorem there exists $(n-A)^{-\frac{1}{q_i}} \le \eta \le C^{-1}r + (n-A)^{-\frac{1}{q_i}}$ such that

$$
\left((C^{-1}r + (n-A)^{-\frac{1}{q_i}})^{(q_i+1)h-q_i} - ((n-A)^{-\frac{1}{q_i}h})^{(q_i+1)h-q_i}\right) \asymp
$$
$$
\asymp C^{-1}r((q_i+1)h - q_i)\eta^{(q_i+1)h-q_i-1}
$$

Since by our assumptions $r \ge \mathrm{diam}(F^{-1}_{x_{i,a,n}}(B(x,\delta)))$ and by (20)

$$
\mathrm{diam}(F^{-1}_{x_{i,a,n}}(B(x,\delta)) \asymp n^{-\frac{q_i+1}{q_i}} \asymp (n-A)^{-\frac{q_i+1}{q_i}},
$$

we can estimate $m(B(y,r))$ as follows

$$
\begin{aligned}
m(B(y,r)) &\succeq ((q_i+1)h - q_i)C^{-1}r\eta^{(q_i+1)(h-1)} \\
&\succeq ((q_i+1)h - q_i)r(C^{-1}r + (n-A)^{-\frac{1}{q_i}})^{(q_i+1)(h-1)} \\
&\succeq r(C^{-1}r + C_1 r^{\frac{1}{q_i+1}})^{(q_i+1)(h-1)} \\
&\asymp r r^{\frac{1}{q_i+1}(h-1)(q_i+1)} = r r^{h-1} = r^h.
\end{aligned}
$$

Thus, by Theorem 2.14, $\mathcal{P}^h(X_\infty) < \infty$. ■

Lemma 3.6. *Suppose that $F : X_0 \to X$ is a jump-like conformal map. If $h = \mathrm{HD}(X_\infty) > 1$, then $\mathcal{P}^h(X_\infty) = \infty$.*

Proof: Fix $1 \le i \le p$ and $r > 0$. Let W be a δ-net of X. Then

$$m(B(b_i, r)) \le \sum_{x \in W} \sum_{1 \le a \le q_i} \sum_{n} m(F_{x_{i,a,n}}^{-1}(B(x,\delta))),$$

where the summation is taken over all triples (i, a, n) such that $F_{x_{i,a,n}}^{-1}(B(x,\delta)) \cap B(b_i, r) \ne \emptyset$. If $r > 0$ is taken sufficiently small, then it follows from (3c) that $|n| > A$. Then by (21) we have $C^{-1}|n|^{-\frac{1}{q_i}} \le \mathrm{dist}(b_i, F_{x_{i,a,n}}^{-1}(B(x,r))) < r$. Thus

$$\begin{aligned} m(B(b_i, r)) &\preceq \sum_{x \in W} \sum_{a=1}^{q_i} \sum_{|n| > (Cr)^{-q_i}} |(F_{x_{i,a,n}}^{-1})'(x)|^h m(B(x,\delta)) \\ &\preceq \sum_{x \in W} \sum_{a=1}^{q_i} \sum_{|n| > (Cr)^{-q_i}} |n|^{-\frac{q_i+1}{q_i}h} \\ &\preceq \sharp W q_i (Cr)^{-q_i(1-\frac{q_i+1}{q_i}h)} \preceq r^{-q_i + (q_i+1)h} \end{aligned}$$

Thus

$$\frac{m(B(b_i, r))}{r^h} \preceq r^{-q_i+(q_i+1)h} r^{-h} = r^{q_i(h-1)}$$

This implies that

$$\overline{\lim}_{r \to 0} \frac{m(B(b_i, r))}{r^h} = 0.$$

So by Theorem 2.17, $\mathcal{P}^h(X_\infty) = \infty$. ■

Lemma 3.7. *Suppose that $F : X_0 \to X$ is a jump-like conformal map. If $h = \mathrm{HD}(X_\infty) \ge 1$, then $\mathcal{H}^h(X_\infty) > 0$.*

Proof: Fix $n \in \mathbb{Z}$, $1 \le i \le p$, $1 \le u \le q_i$. Since the considerations in the case $n < 0$ are analogous to those with $n > 0$ with obvious modification, we assume throughout this proof that $n > 0$. For $x \in X$ let $y = F_{x_{i,u,n}}^{-1}(x)$. Since we want to apply Theorem 2.15 we may disregard finitely many branches of F^{-1} and in particular, we may assume that $n \ge 2A$. Take $r \ge 2\mathrm{diam} F_{x_{i,u,n}}^{-1}(B(x,\delta)) \asymp$

$n^{-\frac{q_i+1}{q_i}} \asymp (n-A)^{-\frac{q_i+1}{q_i}}$. We have

$$\begin{aligned}
\sum\nolimits_1 := & \sum_{w\in W}\sum_{a=1}^{q_i}\sum_{k=n-A}^{n+A} m(F^{-1}_{w_{i,a,k}}(B(w,\delta)) \\
& + \sum_{w\in W}\sum_{a=1}^{q_i}\sum_{k=-n-A}^{-n+A} m(F^{-1}_{w_{i,a,k}}(B(w,\delta)) \\
& \preceq \sum_{w\in W}\sum_{a=1}^{q_i}\sum_{k=n-A}^{n+A} |(F^{-1}_{w_{i,a,k}})'(w)|^h m(B(w,\delta)) \\
& + \sum_{w\in W}\sum_{a=1}^{q_i}\sum_{k=-n-A}^{-n+A} |(F^{-1}_{w_{i,a,k}})'(w)|^h m(B(w,\delta)) \\
& \preceq \sum_{w\in W}\sum_{a=1}^{q_i}\sum_{k=n-A}^{n+A} |k|^{-\frac{q_i+1}{q_i}h} + \sum_{w\in W}\sum_{a=1}^{q_i}\sum_{k=-n-A}^{-n+A} |k|^{-\frac{q_i+1}{q_i}h} \\
& \preceq \sharp W q_i A (n-A)^{-\frac{q_i+1}{q_i}h} \\
& \preceq \left(\frac{n}{n-A}\right)^{\frac{q_i+1}{q_i}h} n^{-\frac{q_i+1}{q_i}h} \\
& \preceq \left(2^{\frac{q_i+1}{q_i}h}\right) \mathrm{diam}(F^{-1}_{x_{i,u,n}}(B(x,\delta)))^h \preceq r^h
\end{aligned}$$

and

$$\sum\nolimits_2 := \sum_{w\in W}\sum_{a=1}^{q_i} \sum_{k:\, |n^{-\frac{1}{q_i}} - |k|^{-\frac{1}{q_i}}| < Cr,\, |k|\geq A\, |n-|k||\geq A} m(F^{-1}_{w_{i,a,k}}(B(w,\delta)))$$

Let

(i) $l = E((n^{\frac{-1}{q_i}} - Cr)^{-q_i}) + 1$ if $Cr < n^{-\frac{1}{q_i}}$

(ii) $l = \infty$ otherwise

In the case (i) we have

$$\begin{aligned}
\sum\nolimits_2 & \preceq \sum_{w\in W}\sum_{a=1}^{q_i} \sum_{k=E((n^{\frac{-1}{q_i}}+Cr)^{-q_i})+1}^{l} m(F^{-1}_{w_{i,a,k}}(B(w,\delta))) + m(F^{-1}_{w_{i,a,-k}}(B(w,\delta))) \\
& \preceq \sum_{w\in W}\sum_{a=1}^{q_i} \sum_{k=E((n^{\frac{-1}{q_i}}+Cr)^{-q_i})+1}^{l} (|(F^{-1}_{w_{i,a,k}})'(w)|^h + |(F^{-1}_{w_{i,a,-k}})'(w)|^h) \\
& \leq 2\sharp W q_i \sum_{k=E\left((n^{\frac{-1}{q_i}}+Cr)^{-q_i}\right)+1}^{l} k^{-\frac{q_i+1}{q_i}h} \\
& \preceq \left((Cr + n^{-\frac{1}{q_i}})^{-q_i+(q_i+1)h} - (n^{-\frac{1}{q_i}} - Cr)^{-q_i+(q_i+1)h}\right)
\end{aligned}$$

By the Mean Value Theorem there exists $\eta \in (n^{-\frac{1}{q_i}} - Cr, n^{-\frac{1}{q_i}} + Cr)$ such that

$$\left((Cr + n^{-\frac{1}{q_i}})^{-q_i+(q_i+1)h} - (n^{-\frac{1}{q_i}} - Cr)^{-q_i+(q_i+1)h}\right)$$

$$\asymp r\eta^{-q_i+(q_i+1)h-1} = r\eta^{(q_i+1)(h-1)} \leq r(n^{-\frac{1}{q_i}} + Cr)^{(q_i+1)(h-1)}$$

But the relation $n^{-\frac{1}{q_i}} \asymp \text{diam}^{\frac{1}{q_i+1}} \preceq r^{\frac{1}{q_i+1}}$ then $n^{-\frac{1}{q_i}} \leq r^{\frac{1}{q_i+1}}$ implies that

$$\sum\nolimits_2 \preceq r(r^{\frac{1}{q_i+1}})^{(q_1+1)(h-1)} = rr^{h-1} = r^h.$$

In the case (ii) $l = \infty$ we get

$$\begin{aligned}\sum\nolimits_2 &\preceq 2\sharp W q_i \sum_{k=E((n^{\frac{-1}{q_i}}+Cr)^{-q_i})+1}^{\infty} k^{-\frac{q_i+1}{q_i}h} \\ &\preceq (Cr + n^{\frac{-1}{q_i}})^{-q_i(1-\frac{q_i+1}{q_i}h)} \\ &\preceq r^{(q_i+1)h-q_i} = r^h r^{q_i(h-1)} \leq r^h.\end{aligned}$$

Now we notice that if $F^{-1}_{w_{i,a,k}}(B(w,\delta)) \cap B(y,r) \neq \emptyset$, then $\text{dist}(F^{-1}_{x_{i,u,n}}(B(x,\delta)), F^{-1}_{w_{i,a,k}}(B(w,\delta))) < r$. Therefore using (3e) we conclude that for $r > 0$ small enough the sets $F^{-1}_{w_{i,a,k}}(B(w,\delta))$ involved in $\sum_1$ and $\sum_2$ cover $B(y,r)$. Hence $m(B(y,r)) \leq \sum_1 + \sum_2$. By Theorem 2.15 this gives that $\mathcal{H}^h(X_\infty) > 0$. ■

Theorem 3.3 follows now from Lemma 3.4, Lemma 3.5, Lemma 3.6 and Lemma 3.7.

4 Geometry of $\tilde{\mathbf{f}}$

The goal of this section is to introduce the map $\tilde{f}$ derived from f and to prove that the restriction of this map to its Julia set $J(\tilde{f})$ is a jump-like conformal map.

We recall that $f(z) = H(\exp(Q(z)))$ while $\tilde{f}(z) = \exp(Q \circ H(z))$ where H, Q are non-constant rational functions. We additionally assume that there is at least one pole d of Q diffrent than $H(0)$ and $H(\infty)$. Let $Ess(R)$ denote the set of essential singularities of a meromorphic function R. Then $Ess(f)$ is the set of poles of Q and $Ess(\tilde{f})$ is the set of poles of $Q \circ H$. We have $H^{-1}(Ess(f)) = Ess(\tilde{f})$. The following lemma follows from a straightforward calculation.

Lemma 4.1. *The maps H and $\exp(Q)$ are semiconjugacies of f and $\tilde{f}$ i.e.*

$$f \circ H(z) = H \circ \tilde{f}(z) \qquad z \notin Ess(\tilde{f})$$

$$\exp(Q) \circ f = \tilde{f} \circ \exp(Q) \qquad z \notin Ess(f)$$

Proof: Since $f \circ H(z) = H(\exp(Q(H(z)))$ and $H \circ \tilde{f} = H(\exp(Q(H(z)))$, we get the first equality appearing in our lemma. Analogously,

$$\exp(Q) \circ f(z) = \exp(Q(H(\exp(Q(z)))) = \tilde{f} \circ \exp(Q(z)).$$

■

As an immediate consequence of Lemma 4.1 we get the following.

Corollary 4.2. *For evry $n \geq 1$ we have $\tilde{f}$ i.e.*

$$f^n \circ H(z) = H \circ \tilde{f}^n(z) \qquad z \notin Ess(\tilde{f}^n) \tag{23}$$

$$\exp(Q) \circ f^n = \tilde{f}^n \circ \exp(Q) \qquad z \notin Ess(f^n) \tag{24}$$

We recall from the Section 1 that

$$Ess_\infty(f) = \bigcup_{n\geq 1} Ess(f^n) \text{ and } Ess_\infty(\tilde{f}) = \bigcup_{n\geq 1} Ess(\tilde{f}^n).$$

Let us prove the following.

Proposition 4.3. *We have*

$$H^{-1}(Ess_\infty(f)) = Ess_\infty(\tilde{f}) \text{and } Ess_\infty(f) \supset (\exp \circ Q)^{-1}(Ess_\infty(\tilde{f})).$$

Proof: The equality $H^{-1}(Ess_\infty(f)) = Ess_\infty(\tilde{f})$ follows immediately from Corollary 4.2 and the remark that H is defined everywhere in $\overline{\mathbb{C}}$. In order to prove the inclusion $(\exp \circ Q)^{-1}(Ess_\infty(\tilde{f})) \subset Ess_\infty(f)$ consider an arbitrary point $z \in (\exp \circ Q)^{-1}(Ess_\infty(\tilde{f}))$. This means that $\exp \circ Q(z) \in Ess_\infty(\tilde{f})$. It now follows from Corollary 4.2 that either $z \in Ess_\infty(f)$ or $z \notin Ess_\infty(f)$ and $f^n(z)$ is a pole of Q for some $n \geq 1$. In the former case we are done immediately and in the latter one we conclude from the definition of f that z is an essential singularity of $H \circ (\exp \circ Q) \circ f^n = f^{n+1}$. This means that $z \in Ess(f^{n+1}) \subset Ess_\infty(f)$ which is a contradiction and we are done. ■

Theorem 4.4. *Suppose f satisfies the assumption (*). Then there exists $0 < \kappa < \infty$ such that*

$$J(\tilde{f}) \subset \{z : e^{-\kappa} \leq |z| \leq e^{\kappa}\}$$

Proof: By Proposition 4.3 $Ess_\infty(\tilde{f}) \subset \exp(Q(Ess_\infty(f)))$. First we show that there exists a constant $0 < \kappa < \infty$ such that for every $z \in J(f)$

$$|\Re Q(z)| \leq a \tag{25}$$

Suppose there exists a sequence $y_n \in J(f), n \in \mathbb{N}$ such that $|\Re Q(y_n)| \to \infty$. Since f is not defined in a finite set of points, we may assume that f is defined at y_n. Then $f(y_n) \in J(f)$ and $f(y_n) = H(\exp(Q(y_n)))$ tends either to $H(\infty)$ or to $H(0)$ depending on whether $\Re Q(y_{n_k}) \to +\infty$ or $\Re Q(y_{n_k}) \to -\infty$. It implies that at least one of the asymptotic values of f (i.e. either $H(0)$ or $H(\infty)$) belongs to $J(f)$, what contradicts our assumptions. Let $z \in Ess_\infty(\tilde{f})$. By Proposition 4.3 there exists $w \in Ess_\infty(f)$ such that $z = \exp(Q(w))$. It follows from (25) that $z = \exp(Q(w))$ satisfies $e^{-\kappa} \leq |\exp(Q(z))| \leq e^{\kappa}$. But $J(\tilde{f}) = \overline{Ess_\infty(\tilde{f})}$, so we obtain that $J(\tilde{f}) \subset \{z : e^{-\kappa} \leq |z| \leq e^{\kappa}\}$. ■

Proposition 4.5. $H(J(\tilde{f})) = J(f)$

Proof: It follows from the proof of Theorem 4.4 $Ess_\infty(\tilde{f}) \subset \{z : e^{-a} \leq |z| \leq e^a\}$. Since $\overline{Ess_\infty(\tilde{f})}$ is a compact subset of $\mathbb{C}$ and H is continuous $H(\overline{Ess_\infty(\tilde{f})}) = \overline{Ess_\infty(f)}$. But $J(f) = \overline{Ess_\infty(f)}$ and $J(\tilde{f}) = \overline{Ess_\infty(\tilde{f})}$ we obtain that $H(J(\tilde{f})) = J(f)$. ■

Theorem 4.6. *Suppose f satisfies the assumption (*). Then*

$$(**) \quad J(\tilde{f}) \cap \overline{\bigcup_{n=0}^{\infty} \tilde{f}^n \left(Crit_G(\tilde{f}) \cup Asymp(\tilde{f}) \right)} = \emptyset$$

Proof: Let α be an asymptotic value of $\tilde{f}$ and suppose that there are a sequence $n_k \to \infty$ and $x \in J(\tilde{f})$ such that $\tilde{f}^{n_k}(\alpha) \to x$. Then $H(\tilde{f}^{n_k}(\alpha)) \to H(x)$. Since for every $n \geq 1$, $f^n(H(\alpha))$ is defined, then by Corollary 4.2 $f^n(H(\alpha)) = H(\tilde{f}^n(\alpha))$. Thus $H(\tilde{f}^{n_k}(\alpha)) = f^{n_k}(H(\alpha)) \to H(x) \in J(f)$. But $H(\alpha)$ is an asymptotic value of f. This implies that $f^{n_k}(H(\alpha)) \to J(f)$ and contradicts the condition (1) of assumption (*).

Let $c \in Crit_G(\tilde{f})$. Then either c is a multiple pole or a critical point of $\tilde{f}$. Since $\tilde{f}$ has no poles and $(\exp)'(z)$ never vanishes, we conclude that c must be a general critical point of $Q \circ H$. This leads to the following cases:

(i) $H'(c) = 0$

(ii) $Q'(H(c)) = 0$

(iii) c is a multiple pole of H and $Q(\infty) \neq \infty$

In the case (i) we can assume that $c \neq 0, \infty$ since $0, \infty \in Asymp(\tilde{f})$. Thus there exists u such that $c = \exp(Q(u))$ and $f(u) = H(\exp(Q(u))) = H(c)$. Then $f'(u) = H'(c)\exp(Q(u))Q'(u) = 0$. Hence $u \in Crit(f)$. If there is a sequence n_k of non-negative integers (possibly bounded) such that $\tilde{f}^{n_k}(c) \to x \in J(\tilde{f})$, then $H(\tilde{f}^{n_k}(c)) = f^{n_k}(H(c)) \to H(x) \in J(f)$. But $f^{n_k}(H(c)) = f^{n_k+1}(u)$, so $f^{n_k+1}(u) \to H(x) \in J(f)$. Since u is a critical point of f, we again obtain a contradiction with condition (1) of assumption (*).

In the case (ii) we have two possibilities. Either $\exp(Q(H(c)))$ is a pole of H, what is excluded by the condition (2), or else $\exp(Q(H(c)))$ is not a pole of H. In the latter case there exists $f'(H(c))$ is well-defined and $f'(H(c)) = H'(\exp(Q(H(c))\exp(Q(H(c))Q'(H(c)) = 0$. It implies that $H(c)$ is a critical point of f, so repeating the arguments from (i) we can show that the $\tilde{f}$-forward trajectory of c stays away from $J(\tilde{f})$.

The case (iii) is excluded by the assumption (3). Thus condition (**) is satisfied by $\tilde{f}$. ■

Let
$$\{b_j : j = 1, \ldots, p\} = (Q \circ H)^{-1}(\infty).$$

Theorem 4.7. *$\tilde{f} : J(\tilde{f}) \setminus \{b_j : j = 1, \dots, p\} \to J(\tilde{f})$ is a Walters expanding conformal map.*

Proof: Since $Q \circ H$ is a rational function, the set $\{b_j : j = 1, \dots, p\}$ is finite. We take $X = J(\tilde{f})$ and $X_0 = X \setminus \{b_j : j = 1, \dots, p\}$. By Theorem 4.6 and Theorem 4.4 there exists $\beta > 0$ such that for every $x \in J(\tilde{f})$ and every $n \geq 1$ there exists a unique holomorphic inverse branch $\tilde{f}_x^{-n} : B(\tilde{f}^n(x), 2\beta) \to \overline{\mathbb{C}}$ of $\tilde{f}^n$ sending $\tilde{f}^n(x)$ to x. Since, by Theorem 4.4, the set $J(\tilde{f})$ is bounded and since $\tilde{f}^{-1}(J(\tilde{f})) \subset J(\tilde{f})$, it follows from Koebe's distortion theorem combined with the standard area argument that

$$\lim_{n \to \infty} \sup\{|(\tilde{f}_y^{-n}(z))'| : y \in \tilde{f}^{-n}(x), z \in B(x, \beta)\} = 0.$$

In particular there exists n_x such that for every $n \geq n_x$, every $z \in B(x, \eta)$ and every $y \in \tilde{f}^{-n}(x)$, we have $|(\tilde{f}_y^{-n})'(z)| \leq \frac{1}{2}$. Since $J(\tilde{f})$ is a compact set, there exists a finite δ-net W of $J(\tilde{f})$. Let $2\delta \leq \beta$ be a Lebesgue number of the cover $\{B(x, \beta) : x \in W\}$. Let $n_0 = \max\{n_x : x \in W\}$. Since for every $z \in J(\tilde{f})$, $B(z, 2\delta) \subset B(x, \beta)$ for some $x \in W$, the requirements (2a)-(2c) of Walters expanding maps along with conformality requirement are satisfied with $\lambda = 2$ and $u = n_0$. In order to see that the condition (2d) is satisfied fix $\epsilon > 0$ and consider V, a finite ϵ-net of $J(\tilde{f})$. Since $J(\tilde{f}) = \overline{Ess_\infty(\tilde{f})}$, it follows from Picard's theorem that for every $x \in V$ there exists $s_x \geq 1$ such that for every $k \geq s_x$, $\tilde{f}^k(B(x, \epsilon) \setminus Ess(\tilde{f}^k))$ contains the entire sphere $\overline{\mathbb{C}}$ except for at most two points which lie in the complement of $J(\tilde{f})$. In particular $B(x, \epsilon) \cap \tilde{f}^{-k}(w) \neq \emptyset$ for all $w \in J(\tilde{f})$. Putting $s = \max\{s_x : x \in V\}$ completes the proof of condition (2d) and simultaneously the proof of our theorem. ■

Frequently, in order to simplify notation we will slightly informally write $\tilde{f}_{|J(\tilde{f})}$ for $\tilde{f} : J(\tilde{f}) \setminus \{b_j : j = 1, \dots, p\} \to J(\tilde{f})$

Theorem 4.8. *$\tilde{f}_{|J(\tilde{f})}$ is a jump-like conformal map i.e. there exist $C \geq 1$ and $A \geq 2$ such that the following conditions are satisfied:*

(4a) $\{b_j : j = 1, \dots, b_p\} \cap \tilde{f}^{-1}(J(\tilde{f})) = \emptyset$

(4b) For every $x \in J(\tilde{f})$ the set $\tilde{f}^{-1}(x)$ can be uniquely represented as $\{x_{j,a,n} : n \in \mathbb{Z}, 1 \leq j \leq p, 1 \leq a \leq q_j\}$

(4c) $\max_{1 \leq j \leq p} \max_{1 \leq a \leq q_j} \sup_{x \in J(\tilde{f})} \{\lim_{n \to \infty} \mathrm{Dist}(b_j, \tilde{f}_{j,a,n}^{-1}(B(x, \delta)))\} = 0$

(4d) $\forall_{z \in J(\tilde{f})}, \ \forall_{1 \leq j \leq p}, \ \forall_{1 \leq a \leq q_j}, \ \forall_{n \in \mathbb{Z}, \ |n| \geq A}$

$$C^{-1}|n|^{-\frac{q_j+1}{q_j}} \leq |(\tilde{f}_{j,a,n}^{-1})'(z)| \leq C|n|^{-\frac{q_j+1}{q_j}}$$

(4e) $\forall_{w,z \in J(\tilde{f})}, \forall_{1 \leq j \leq p}, \forall_{a,b \in \{1,\dots,q_j\}}, \forall_{k,n \in \mathbb{Z}, \ ||k|-|n|| \geq A, \ |n| \geq A, \ |k| \geq A}$

$$\mathrm{dist}(\tilde{f}_{j,a,k}^{-1}(B(w, \delta)), \tilde{f}_{j,b,n}^{-1}(B(z, \delta))) \geq C^{-1} \left||k|^{-\frac{1}{q_j}} - |n|^{-\frac{1}{q_j}}\right|$$

(4f) $\forall_{w,z\in J(\tilde f)}$, $\forall_{1\le j\le p}$, $\forall_{a\in\{1,\dots,q_j\}}$, $\forall_{k,n\in\mathbb{Z},\ kn>0\ \ ||k|-|n||\ge A,\ |n|\ge A,\ |k|\ge A}$

$$\mathrm{Dist}(\tilde f^{-1}_{j,a,k}(B(w,\delta)),\tilde f^{-1}_{j,b,n}(B(z,\delta)))\le C\left||k|^{-\frac{1}{q_j}}-|n|^{-\frac{1}{q_j}}\right|$$

Similarly as in Section 3, as an immediate consequence of (4d), with a bigger constant C perhaps, we get the following $\forall_{x\in J(\tilde f)}$, $\forall_{1\le j\le p}$, $\forall_{1\le a\le q_j}$, $\forall_{n\in\mathbb{Z},\ |n|\ge A}$

$$C^{-1}|n|^{-\frac{q_j+1}{q_j}}\le diam(\tilde f^{-1}_{j,a,n}(B(x,\delta)))\le C|n|^{-\frac{q_j+1}{q_j}} \tag{26}$$

Letting $k\to\infty$, it immediately follows from (4c), (4e) and (4f) that $\forall_{1\le j\le p}$, $\forall_{x\in J(\tilde f)}$, $\forall_{1\le a\le q_j}$, $\forall_{|n|\ge 2A}$

$$C^{-1}|n|^{-\frac{1}{q_j}}\le \mathrm{dist}(b_j,\tilde f^{-1}_{j,a,n}(B(x,\delta)))\le \mathrm{Dist}(b_j,\tilde f^{-1}_{j,a,n}(B(x,\delta)))\le C|n|^{-\frac{1}{q_j}} \tag{27}$$

Theorem 4.8 will follow by combining Proposition 4.9, Corollary 4.11 and Corollary 4.13. Note that every $z\in J(\tilde f)$, each holomorphic branch of $\tilde f^{-1}$ defined on the ball $B(z,2\delta)$ can be expressed in the form

$$\tilde f^{-1}_{j,a,n}(w)=(Q\circ H)^{-1}_{j,a}(\log(w)+2\pi in), \tag{28}$$

where $\log w$ is the value of the logarithm of w lying in the rectangle $[-\kappa,\kappa]\times[0,2\pi]$ and $(Q\circ H)^{-1}_{j,a}$ is a local holomorphic inverse branch of $Q\circ H$. For n with sufficiently large modulus each such inverse branch can be interpreted as a branch of $Q\circ H$ defined on some vertical strip either of the form $[-\kappa,\kappa]\times[T,+\infty]$ or $[-\kappa,\kappa]\times[-\infty,-T]$, $T>>1$, depending up on whether n is positive or negative and sending ∞ to a pole a of $Q\circ H$. Given a, all such inverse branches are parametrized by the numbers $1,2,\dots,q_a$.

Proposition 4.9.

$\exists_{C\ge 1}$, $\forall_{w\in J(\tilde f)}$, $\forall_{1\le j\le p}$, $\forall_{1\le a\le q_j}$, $\forall_{n\in\mathbb{Z},\ |n|\ge A}$

$$C^{-1}|n|^{-\frac{q_j+1}{q_j}}\le|(\tilde f^{-1}_{j,a,n})'(w)|\le C|n|^{-\frac{q_j+1}{q_j}}$$

Proof: The inverse branch $\tilde f^{-1}_{j,a,n}(w)$ is equal to $(Q\circ H)^{-1}_{j,a}(\log(w)+2\pi in)$. For z close to the pole b_j the function $Q\circ H$ has a form

$$(Q\circ H)(z)=\frac{P(z)}{(z-b_j)^{q_j}},$$

where $P(z)$ is a holomorphic map such that $P(b_j)\ne 0$. Since in addition $z=\tilde f^{-1}_{j,a,n}(w)$ is arbitrarily close to b_j for all n with sufficiently large moduli, we obtain

$$\begin{aligned}(\tilde f^{-1}_{j,a,n})'(w)&=\frac{((Q\circ H)'(z))^{-1}}{w}=\frac{(z-b_j)^{q_j+1}}{w[P'(z)(z-b_j)-q_jP(z)]}\\ &\asymp\frac{(z-b_j)^{q_j+1}}{w}\asymp(z-b_j)^{q_j+1},\end{aligned}$$

where writing the last comparability sign we have applied Theorem 4.4. Applying again the facts that $P(b_j) \neq 0$ and that z is arbitrarily close to b_j for all n with sufficiently large moduli, we get

$$(z-b_j)^{q_j} = \frac{P(z)}{(Q\circ H)(z)} = \frac{P(z)}{\log w + 2\pi i n} \asymp \frac{1}{|n|}$$

for all n with $|n| \geq A$ for some universal constant A. Consequently

$$|(\tilde{f}_{j,a,n}^{-1})'(w)| \asymp |n|^{-\frac{q_j+1}{q_j}}$$

for all n with $|n| \geq A$. ■

Proposition 4.10.

$\exists_{C\geq 1}\quad \exists_{A\geq 2}\quad \forall_{w,z\in J(\tilde{f})},\ \forall_{1\leq j\leq p},\ \forall_{a,s\in\{1,\dots,q_j\}},\ \forall_{k,n\in\mathbb{Z},\ ||k|-|n||\geq A,\ |n|\geq A,\ |k|\geq A}$

$$\mathrm{dist}(\tilde{f}_{j,a,k}^{-1}(w), \tilde{f}_{j,s,n}^{-1}(z)) \geq C^{-1}\left||k|^{-\frac{1}{q_j}} - |n|^{-\frac{1}{q_j}}\right|.$$

Proof: Since $(Q\circ H)(z) = \frac{P(z)}{(z-b_j)^{q_j}}$ on some neighbourhood of b_j and since $P(b_j) \neq 0$, there exists a biholomorphic function defined in a sufficiently small neighbourhood of b_j such that $G(b_j) = b_j$ and

$$(Q\circ H\circ G)(z) = \frac{1}{(z-b_j)^{q_j}}. \tag{29}$$

Put

$$\hat{f} = \exp(Q\circ H\circ G) = \tilde{f}\circ G.$$

Since for all $w \in B(J(\tilde{f}),\delta)$, $\hat{f}_{j,a,k}^{-1}(w) = (Q\circ H\circ G)_{j,a}^{-1}(\log(w) + 2\pi i n)$ (comp. formula (28) and the discussion following it) and since for all k with sufficiently large moduli (29) applies with $z = \hat{f}_{j,a,k}^{-1}(w)$, we therefore get

$$(\hat{f}_{j,a,k}^{-1}(w) - b_j)^{q_j} = \frac{1}{Q\circ H\circ G(\hat{f}_{j,a,k}^{-1}(w))} = \frac{1}{\log w + 2\pi i k} \tag{30}$$

In the formulae below we drop the index j i.e. $b = b_j$ and $q = q_j$. In view of (30) for all $w, z \in B(J(\tilde{f}),\delta)$ and all n, k with sufficiently large moduli and with $|n| \neq |k|$, we can estimate as follows.

$$\begin{aligned}
&|\big(\hat{f}_{j,a,k}^{-1}(w) - b\big)^q - \big(\hat{f}_{j,s,n}^{-1}(z) - b\big)^q| \leq \\
&\quad \leq |\hat{f}_{j,a,k}^{-1}(w) - \hat{f}_{j,s,n}^{-1}(z)| \sum_{i=0}^{q-1} |\hat{f}_{j,a,k}^{-1}(w) - b|^{q-1-i} |\hat{f}_{j,s,n}^{-1}(z) - b|^i \\
&\quad \asymp |\hat{f}_{j,a,k}^{-1}(w) - \hat{f}_{j,s,n}^{-1}(z)| \sum_{i=0}^{q-1} |k|^{-\frac{q-1-i}{q}} |n|^{-\frac{i}{q}} \\
&\quad = |\hat{f}_{j,a,k}^{-1}(w) - \hat{f}_{j,s,n}^{-1}(z)| \left(\frac{|n|^{-1} - |k|^{-1}}{|n|^{-\frac{1}{q}} - |k|^{-\frac{1}{q}}}\right)
\end{aligned}$$

So,

$$|\hat{f}^{-1}_{j,a,k}(w)-\hat{f}^{-1}_{j,s,n}(z)| \geq \frac{|(\hat{f}^{-1}_{j,a,k}(w)-b)^q-(\hat{f}_{j,s,n}(z)-b)^q|\left(|n|^{-\frac{1}{q}}-|k|^{-\frac{1}{q}}\right)}{(|n|^{-1}-|k|^{-1})}.$$

Combining this and (30) and assuming that $||k|-|n||$ is large enough, we obtain

$$\begin{aligned}|\hat{f}^{-1}_{j,a,k}(w)-\hat{f}^{-1}_{j,s,n}(z)| &\geq \frac{|\log w+2\pi ik-\log z-2\pi in|}{|(\log w+2\pi ik)(\log z+2\pi in)|}\frac{|k||n|}{(|k|-|n|)}\left(|n|^{-\frac{1}{q}}-|k|^{-\frac{1}{q}}\right)\\ &\asymp \left|\frac{\log w-\log z}{|k|-|n|}+i\frac{2\pi(k-n)}{|k|-|n|}\right|\left(|n|^{-\frac{1}{q}}-|k|^{-\frac{1}{q}}\right)\\ &\succeq \left||n|^{-\frac{1}{q}}-|k|^{-\frac{1}{q}}\right|\end{aligned}$$

Since $\tilde{f}^{-1}_{j,a,k}=G\circ\hat{f}^{-1}_{j,a,k}$ and since G is biholomorphic, with n and k as above, we obtain that

$$\begin{aligned}|\tilde{f}^{-1}_{j,a,k}(w)-\tilde{f}^{-1}_{j,s,n}(z)| &= |G\circ\hat{f}^{-1}_{j,a,k}(w)-G\circ\hat{f}^{-1}_{j,s,n}(z)| \geq L_G^{-1}||\hat{f}^{-1}_{j,a,k}(w)-\hat{f}^{-1}_{j,s,n}(z)|\\ &\succeq \left(|n|^{-\frac{1}{q}}-|k|^{-\frac{1}{q}}\right),\end{aligned}$$

where $L_{G^{-1}}$ is a Lipschitz constant of G^{-1}. ■

Corollary 4.11.

$\exists_{C\geq 1},\ \ \exists_{A\geq 2}\ \ \forall_{w,z\in J(\tilde{f})},\ \forall_{1\leq j\leq p},\ \forall_{a,s\in\{1,\dots,q_j\}},\ \forall_{k,n\in\mathbb{Z},\ ||k|-|n||\geq A,\ |n|\geq A,\ |k|\geq A}$

$$\operatorname{dist}(\tilde{f}^{-1}_{j,a,k}(B(w,\delta)),\tilde{f}^{-1}_{j,s,n}(B(z,\delta)))\geq C^{-1}||k|^{-\frac{1}{q_j}}-|n|^{-\frac{1}{q_j}}|.$$

Proposition 4.12.

$\exists_{C\geq 1},\ \exists_{A\geq 2}\ \forall_{w,z\in J(\tilde{f})},\ \ \forall_{1\leq j\leq p},\ \ \forall_{a\in\{1,\dots,q_j\}},\ \ \forall_{k,n\in\mathbb{Z},\ kn>0,\ ||k|-|n||\geq A,\ |n|\geq A,\ |k|\geq A}$

$$\operatorname{Dist}(\tilde{f}^{-1}_{j,a,k}(w),\tilde{f}^{-1}_{j,a,n}(z))\leq C\left||k|^{-\frac{1}{q_j}}-|n|^{-\frac{1}{q_j}}\right|$$

Proof: The objects G and $\tilde{f}$ appearing in this proof have the same meaning as in the proof of Proposition 4.10. In particular (30) holds. Since the considerations in the case $n<0$ are analogous to those with $n>0$ with obvious modifications, we assume troughout this proof that $n>0$. In the formulae below we drop the index j i.e. $b=b_j$ and $q=q_j$. We assume that $k,n\geq A$ and $n-k\geq n+A$, where A comes from Proposition 4.10. First we shall prove that

$$|\hat{f}^{-1}_{j,a,k}(w)-\hat{f}^{-1}_{j,a,n}(z)|\leq ||\hat{f}^{-1}_{j,a,k}(w)-b)|-|\hat{f}^{-1}_{j,a,n}(z)-b||+C(k^{-\frac{q+1}{q}}+n^{-\frac{q+1}{q}}) \tag{31}$$

And indeed, in view of (30)

$$\hat{w}_k:=\hat{f}^{-1}_{j,a,k}(w)=b+\frac{1}{(\log w+2\pi ik)^{\frac{1}{a}}_{a}} \text{ and } \hat{z}_n:=\hat{f}^{-1}_{j,a,n}(z)=b+\frac{1}{(\log w+2\pi in)^{\frac{1}{q}}_{a}}$$

We also choose additional points w'_k, z'_n which belong to $(Q\circ H\circ G)^{-1}_{j,a}(\mathbb{R}i)$ such that $|w'_k - b| = |\hat{w}_k - b|$ and $|z'_n - b| = |\hat{z}_n - b|$. Then

$$|\hat{w}_k - \hat{z}_n| \le |\hat{w}_k - w'_k| + |w'_k - z'_n| + |\hat{z}_n - z'_n|. \tag{32}$$

Since the points w'_k, z'_n and b are collinear and since b does not lie between w'_k and z'_n, we get

$$|w'_k - z'_n| = \left| |\hat{f}^{-1}_{j,a,k}(w) - b)| - |\hat{f}^{-1}_{j,a,n}(z) - b| \right|. \tag{33}$$

Our aim is to estimate $|\hat{z}_n - z'_n|$. Since

$$\hat{z}_n = b + \frac{1}{(\log z + 2\pi i n)^{\frac{1}{q}}_a} \text{ and } z'_n = b + \frac{1}{(2\pi i t)^{\frac{1}{q}}_a},\ t \in \mathbb{R},$$

for some $t \in \mathbb{R}$, and, since $|z'_n - b| = ||\hat{z}_n - b|$, we have

$$|\hat{z}_n - z'_n| = \left| \frac{1}{(\log z + 2\pi i n)^{\frac{1}{q}}_a} - \frac{1}{(2\pi i t)^{\frac{1}{q}}_a} \right| \asymp \frac{\left| (2\pi i t)^{\frac{1}{q}}_a - (\log z + 2\pi i n)^{\frac{1}{q}}_a \right|}{n^{2/q}}. \tag{34}$$

By the Mean Value Inequality there exists η lying in the segment joining $2\pi i t$ and $\log z + 2\pi i n$ such that

$$\frac{\left| (2\pi i t)^{\frac{1}{q}}_a - (\log z + 2\pi i n)^{\frac{1}{q}}_a \right|}{n^{2/q}} \le \frac{\frac{1}{q}|\eta|^{\frac{1}{q}-1} |2\pi i t - (\log z + 2\pi i n)|}{n^{2/q}}.$$

Since $|z'_n - b| = |z_n - b|$ and since $\log z \in [-\kappa, \kappa] \times [0, 2\pi]$, we conclude that $|t - n| \le M$ for some constant M and all n large enough. Therefore, we may continue (34) as follows

$$|\hat{z}_n - z'_n| \preceq n^{-2/q} |\eta|^{\frac{1}{q}-1} \asymp n^{-2/q} n^{\frac{1}{q}-1} = n^{-\frac{q+1}{q}}$$

Analogously we can prove that $|\hat{w}_k - w'_k| \le k^{-\frac{q+1}{q}}$. Substituting these estimates to (32) and employing (33), we obtain (31).

Continuing the proof of our proposition, notice that

$$||\hat{f}^{-1}_{j,a,k}(w) - b|^q - |\hat{f}^{-1}_{j,a,n}(z) - b|^q| =$$

$$= ||\hat{f}^{-1}_{j,a,k}(w) - b| - |\hat{f}^{-1}_{j,a,n}(z) - b|| \sum_{i=0}^{q-1} |\hat{f}^{-1}_{j,a,k}(w) - b|^{q-1-i} |\hat{f}^{-1}_{j,a,n}(z) - b|^i$$

and therefore, applying (31), we obtain

$$||\hat{f}^{-1}_{j,a,k}(w) - b|^q - |\hat{f}^{-1}_{j,a,n}(z) - b|^q|$$

$$\geq \left(|\hat{f}^{-1}_{j,a,k}(w) - \hat{f}^{-1}_{j,a,n}(z)| - C(k^{-\frac{q+1}{q}} + n^{-\frac{q+1}{q}})\right) \sum_{i=0}^{q-1} |\hat{f}^{-1}_{j,a,k}(w) - b|^{q-1-i} |\hat{f}^{-1}_{j,a,n}(z) - b|^i$$

$$= |\hat{f}^{-1}_{j,a,k}(w) - \hat{f}^{-1}_{j,a,n}(z)| \sum_{i=0}^{q-1} |\hat{f}^{-1}_{j,a,k}(w) - b|^{q-1-i} |\hat{f}^{-1}_{j,a,n}(z) - b|^i -$$

$$-C(k^{-\frac{q+1}{q}} + n^{-\frac{q+1}{q}}) \sum_{i=0}^{q-1} |\hat{f}^{-1}_{j,a,k}(w) - b|^{q-1-i} |\hat{f}^{-1}_{j,a,n}(z) - b|^i$$

This implies that

$$|\hat{f}^{-1}_{j,a,k}(w) - \hat{f}^{-1}_{j,a,n}(z)| \leq$$
$$\leq \frac{||\hat{f}^{-1}_{j,a,k}(w) - b|^q - |\hat{f}^{-1}_{j,a,n}(z) - b|^q|}{\sum_{i=0}^{q-1} |\hat{f}^{-1}_{j,a,k}(w) - b|^{q-1-i} |\hat{f}^{-1}_{j,a,n}(z) - b|^i} + C\big(k^{-\frac{q+1}{q}} + n^{-\frac{q+1}{q}}\big) \tag{35}$$

We shall prove that

$$C(k^{-\frac{q+1}{q}} + n^{-\frac{q+1}{q}}) \preceq (n^{-\frac{1}{q}} - k^{-\frac{1}{q}}) \tag{36}$$

Since $k \geq n + A$, we have

$$C(k^{-\frac{q+1}{q}} + n^{-\frac{q+1}{q}}) \preceq n^{-\frac{q+1}{q}} \tag{37}$$

Now we shall show that there exists a constant $C_2 > 0$ such that

$$C_2 n^{-\frac{q+1}{q}} \leq (n^{-\frac{1}{q}} - k^{-\frac{1}{q}}) \tag{38}$$

By the Mean Value Theorem there exists $\eta \in (n, n+1)$ such that

$$n^{-\frac{1}{q}} - k^{-\frac{1}{q}} \geq n^{-\frac{1}{q}} - (n+1)^{-\frac{1}{q}} = (-1)(-\frac{1}{q})\eta^{-\frac{1}{q}-1} \asymp n^{-\frac{q+1}{q}}$$

Thus (36) follows from (37) and (38). By (30)

$$\sum_{i=0}^{q-1} |\hat{f}^{-1}_{j,a,k}(w) - b|^{q-1-i} |\hat{f}^{-1}_{j,a,n}(z) - b|^i \asymp \sum_{i=0}^{q-1} k^{-\frac{q-1-i}{q}} n^{-\frac{i}{q}} \asymp \left|\frac{k^{-1} - n^{-1}}{k^{-\frac{1}{q}} - n^{-\frac{1}{q}}}\right|.$$

Therefore, using (30) again, we get

$$\frac{||\hat{f}^{-1}_{j,a,k}(w) - b|^q - |\hat{f}^{-1}_{j,a,n}(z) - b|^q|}{\sum_{i=0}^{q-1} |\hat{f}^{-1}_{j,a,k}(w) - b|^{q-1-i} |\hat{f}^{-1}_{j,a,n}(z) - b|^i} =$$

$$
\begin{aligned}
&\asymp \left|\frac{1}{|\log w + 2\pi i k|} - \frac{1}{|\log z + 2\pi i n|}\right| \frac{n^{-\frac{1}{q}} - k^{-\frac{1}{q}}}{n^{-1} - k^{-1}} \\
&= \left|\frac{|\log w + 2\pi i k| - |\log z + 2\pi i n|}{|\log w + 2\pi i k||\log z + 2\pi i n|}\right| \frac{kn\left(n^{-\frac{1}{q}} - k^{-\frac{1}{q}}\right)}{k-n} \\
&\preceq \left|\frac{2\pi k + |\log w| - 2\pi n + |\log z|}{k-n}\right| \left(n^{-\frac{1}{q}} - k^{-\frac{1}{q}}\right) \\
&\preceq \left(2\frac{\kappa}{A} + 2\pi\right)\left(n^{-\frac{1}{q}} - k^{-\frac{1}{q}}\right)
\end{aligned} \tag{39}
$$

Substituting (36) and (39) to (35) we obtain

$$|\hat{f}^{-1}_{j,a,k}(w) - \hat{f}^{-1}_{j,a,n}(z)| \preceq \left(n^{-\frac{1}{q}} - k^{-\frac{1}{q}}\right)$$

Since $\tilde{f}^{-1}_{j,a,k}(w) = G \circ \hat{f}^{-1}_{j,a,k}(w)$ we thus have

$$
\begin{aligned}
|\tilde{f}^{-1}_{j,a,k}(w) - \tilde{f}^{-1}_{j,a,n}(z)| &= |G \circ \hat{f}^{-1}_{j,a,k}(w) - G \circ \hat{f}^{-1}_{j,a,n}(z)| \\
&\le L_G |\hat{f}^{-1}_{j,a,k}(w) - \hat{f}^{-1}_{j,a,n}(z)| \preceq n^{-\frac{1}{q}} - k^{-\frac{1}{q}}
\end{aligned}
$$

where L_G is a Lipschitz constant of G. ■

Corollary 4.13.

$\exists_{C\ge 1},\ \exists_{A\ge 2}\ \forall_{w,z\in J(\tilde{f})},\ \forall_{1\le j\le p},\ \forall_{a\in\{1,\dots,q_j\}}\ \forall_{k,n\in\mathbb{Z},\ kn>0\ \ ||k|-n|\ge A,\ n\ge A,\ |k|\ge A}$

$$\mathrm{Dist}(\tilde{f}^{-1}_{j,a,k}(B(w,\delta)), \tilde{f}^{-1}_{j,a,n}(B(z,\delta))) \le C\left||k|^{-\frac{1}{q_j}} - n^{-\frac{1}{q_j}}\right|.$$

So, we can end this section that all the results proven in the previous section for jump-like maps apply to the map $\tilde{f}$

5 Geometry and dynamics of f

Let ν denote either the Hausdorff measure $\mathcal{H}^h$ or the packing measure $\mathcal{P}^h$ defined by means of the spherical or Euclidean metric. Since $H(J(\tilde{f})) = J(f)$ and since H is biholomorphic except for a finite number of points, the following implications are obvious.

(a) If $\nu(J(\tilde{f})) = 0$, then $\nu(J(f)) = 0$.

(b) If $\nu(J(\tilde{f})) > 0$, then $\nu(J(f)) > 0$.

(c) If $\nu|_{J(\tilde{f})}$ is not locally finite at any point, then $\nu|_{J(f)}$ is locally finite at at most finitely many points. In particular $\nu(J(f)) = \infty$.

(d) If $\nu(J(\tilde{f})) < \infty$, then $\nu(J(f))$ is σ-finite.

Thus Theorem 1 follows from Theorem 4.8, Theorem 3.3 and Proposition 2.18. And by the same token, in order to complete the proof of Theorem 2, we only need to demonstrate the following.

Lemma 5.1. *Let f satisfy the assumption (*). If $\mathcal{H}^h(J(\tilde{f})) < \infty$ (resp. $\mathcal{P}^h(J(\tilde{f})) < \infty$), then $\mathcal{H}^h(J(f)) < \infty$ (resp. $\mathcal{P}^h(J(f)) < \infty$), where the Hausdorff measure and packing measure are defined by means of spherical metric.*

Proof: Let, as above, ν denote either the Hausdorff measure $\mathcal{H}^h$ or the packing measure $\mathcal{P}^h$ defined by means of the spherical depending up on which of these two measures is finite. Since $J(\tilde{f})$ contains no critical points of H, we only need to check that ν is finite on some neighbourhood of ∞. Since $J(f) = H(J(\tilde{f}))$, it therefore suffices to show that each pole of H lying in $J(\tilde{f})$ has a neighbourhood whose image under H has a finite ν measure. And indeed, fix $b \in H^{-1}(\infty)$. Take $r > 0$ so small that $H(z) = \frac{P(z)}{(z-b)^q}$ for $z \in B(b, r)$, where $P : B(b, r) \to \mathbb{C}$ is a holomorphic function omitting some open neighbourhood of 0. Thus $H'(z) \asymp |z - b|^{-(q+1)}$ and in the spherical metric

$$H^\sharp(z) = \frac{H'(z)(1 + |z|^2)}{1 + |H(z)|^2} \asymp |z - b|^{q-1}.$$

for $z \in B(b, r)$. For every $n \geq 0$ we consider annulus

$$A_n = \left\{ z : \frac{r}{2^{n+1}} \leq |z - b| < \frac{r}{2^n} \right\}.$$

Thus

$$\begin{aligned}
\nu(H(J(\tilde{f}) \cap B(b,r))) &= \nu\left(H\left(\bigcup_{n=0}^{\infty} J(\tilde{f}) \cap A_n\right)\right) \leq \sum_{n=1}^{\infty} \nu(H(J(\tilde{f}) \cap A_n)) \\
&\leq \sum_{n=1}^{\infty} \sup\{(H^\sharp(z))^h : z \in J(\tilde{f}) \cap A_n\} \nu(J(\tilde{f}) \cap A_n) \\
&\asymp \sum_{n=1}^{\infty} 2^{-n(q-1)h} \nu(J(\tilde{f}) \cap A_n) \\
&\leq \sum_{n=1}^{\infty} \nu(J(\tilde{f}) \cap A_n) \leq \nu(J(\tilde{f})) < \infty.
\end{aligned}$$

■

Since $H \circ \tilde{f} = f \circ H$ and since $\tilde{\mu}$ is $\tilde{f}$-invariant, the measure $\mu := \tilde{\mu} \circ H^{-1}$ is f-invariant. So, Theorem 3 is proven.

6 Examples

If $Q(z) = z$, then $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a periodic meromorphic function and this case is covered by Barański's paper [2]. We now distinguish some cases not covered by [2]. If $Q(z)$ is a polynomial different than identity, we have a transcendental meromorphic function f of the complex plane with one essential singularity at ∞. The most transparent class of examples is provided by the following.

Example 1. Let

$$f(z) = \frac{A\exp(z^p) + B\exp(-z^p)}{C\exp(z^p) + D\exp(-z^p)}, \quad AD - BC \neq 0.$$

Thus $\mathrm{Crit}(f) = \{0\}$, $\mathrm{Crit}_G(f) = Crit(f)$ and $Asymp(f) = \{\frac{A}{C}, \frac{B}{D}\}$. If $\frac{A}{C}, \frac{B}{D} \neq \infty$, then f is not entire. If additionally f satisfies the assumption (*) which in this context means that only condition (1) is satisfied (note that in the case $p \geq 2$, the function f has a non-empty set of critical points whereas there is no critical point if $p = 1$), then the results stated in the introduction apply.

If $Q(z)$ is a rational function which is not a polynomial, then f is a meromorphic function with more than one essential singularity. We illustrate this situation by the following.

If $Q(z)$ is not a polynomial, then f has more than 1 essential singularity and belongs to the class considered by Bolsch.

Example 2. Let $H(z) = z, Q(z) = \frac{z-1}{z+1}$. Then $f : \mathbb{C} \setminus \{-1\} \to \mathbb{C} \setminus \{0, \infty\}$,

$$f(z) = \exp\left(\frac{z-1}{z+1}\right).$$

and $\tilde{f} = f(z)$. Comparing with [2], f does not belong to the class considered there, but it has the form of functions $\tilde{f}$ studied by Barański in [2]. Since the pole of Q is not an omitted value of f, we see that $\bigcup_{n=0}^{\infty} f^{-n}(-1)$ contains infinitely many points and consequently

$$J(f) = \overline{\bigcup_{n=0}^{\infty} f^{-n}(-1)}.$$

Since $f^{-1}(S^1) \subset S^1$, we have $f^{-n}(-1) \in S^1$ for all $n \in \mathbb{N}$. Therefore $J(f) \subset S^1$. We shall prove that f satisfies the assumption (*) and its Julia set $J(f)$ is a topological Cantor set. Note that $\mathrm{Crit}_G(f) = \emptyset$ and $Asymp(f) = \{0, \infty\}$. One can check that $f(1) = 1$ and $f'(1) = 1/2$, so the number $= 1$ is an attracting fixed point of f. Thus $J(f)$ is a topological Cantor set contained in the circle S^1. In order to conclude the proof it is now sufficient to demonstrate that 1 attracts both asymptotic values 0 and ∞. Since $f'(x) > 0$ for $x \in \mathbb{R} \setminus \{-1\}$, the function f is strictly increasing on $(-\infty, -1)$ and $(-1, +\infty)$. Now, if $x \in (1; \infty)$, then $f(1) < f(x) < x$. This implies that $\lim_{n\to\infty} f^n(x) = 1$ for all $x \in (1; \infty)$. In particular $\lim_{n\to\infty} f^n(\infty) = 1$ since $f(\infty) = e \in (1; \infty)$. If $x \in (-1, 1)$, then $x < f(x) < f(1) = 1$.This implies that $\lim_{n\to\infty} f^n(x) = 1$ for all $x \in (-1; 1)$. In particular $\lim_{n\to\infty} f^n(0) = 1$ since $f(0) = 1/e \in (-1; 1)$. We are done.

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