THE DIOPHANTINE ANALYSIS OF CONFORMAL ITERATED FUNCTION SYSTEMS

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Abstract. A formula for the Hausdorff dimension of a Besicovic-Jarnik subset of the limit set of a conformal infinite iterated function system is derived and expressed as a unique zero of the topological pressure of the corresponding strongly Hölder family of functions.

1. Introduction, Preliminaries

The diophantine analysis has its origins in classical papers [Be] and [Ja] by Besicovic and Jarnik respectively. The connections of their work with the theory of dynamical systems is well explained in [HV1]. Also the papers [HV2], [St] and [SU] deal with the subject of Diophantine analysis. Especially interesting phenomena have been discovered in [St] and [SU]. The aim of this paper is to initiate the Diophantine analysis in the general context of conformal infinite iterated function systems. Ultimately we would like to provide a dynamical proof of the classical result by Besicovic and Jarnik and to provide a full explanation of the phenomena (a kind of phase transition of the Hausdorff dimension function) observed in [St] and [SU]. We feel that conformal infinite iterated function systems provide a right setting to do it.

The plan of our approach in this paper is to deal first with finite systems and to prepare some auxiliary technical tools in the setting of infinite systems. Then, assuming an appropriate separate condition, we approximate a conformal infinite iterated function system by finite subsystems, we use a version of the variational principle from [MU2], and we develop the approach of Hill and Velani form [HV] following preatty closely their consierations adapted to the setting of iterated function systems. At the end of this section we formulate our main result and in order to do it we first recall (see [HMU], comp. [HU] and [Ur]) the basic properties of conformal iterated function systems, the general concept of strongly Hölder families of functions and an appropriate version of thermodynamic formalism. So, let I be a countable index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from X into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x, y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We are particularly interested in the properties of the limit set defined by such a system. We can define this set as the image of the coding space under a coding map as follows. Let $I^* = \bigcup_{n\geq 1} I^n$, the space of finite words, and for $\omega \in I^n$, $n \geq 1$, let $\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}$. If $\omega \in I^* \cup I^\infty$ and $n \geq 1$ does not exceed the length of ω, we

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denote by $\omega|_n$ the word $\omega_1\omega_2\ldots\omega_n$. Since given $\omega \in I^\infty$, the diameters of the compact sets $\phi_{\omega|_n}(X)$, $n \geq 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\omega)$, defines the coding map $\pi : I^\infty \to X$. The main object in the theory of iterated function systems is the limit set defined as follows.

$$J = \pi(I^\infty) = \bigcup_{\omega \in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|n}(X) = \bigcap_{n\geq 1} \bigcup_{|\omega|=n} \phi_\omega(X)$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if I is finite, then J is compact and this property fails for infinite systems.
An iterated function system $S = \{\phi_i : X \to X : i \in I\}$ is said to satisfy the Open Set Condition if there exists a nonempty open set $U \subset X$ (in the topology of X) such that $\phi_i(U) \subset U$ for every $i \in I$ and $\phi_i(U) \cap \phi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$.
An iterated function system S satisfying the Open Set Condition is said to be conformal if $X \subset I\!R^d$ for some $d \geq 1$ and the following conditions are satisfied.

(1a): $U = \mathrm{Int}_{I\!R^d}(X)$.

(1b): There exists an open connected set $X \subset V \subset I\!R^d$ such that all maps ϕ_i, $i \in I$, extend to C^1 conformal diffeomorphisms of V into V.

(1c): There exist $\gamma, l > 0$ such that for every $x \in \partial X \subset I\!R^d$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure γ, and altitude l.

(1d): Bounded Distortion Property(BDP). There exists $K \geq 1$ such that

$$|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|$$

for every $\omega \in I^*$ and every pair of points $x, y \in V$, where $|\phi'_\omega(x)|$ means the norm of the derivative.

In fact throughout the whole paper we will need one condition more which (comp. [MU1]) can be considered as a strengthening of (BDP).

(1e): There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\Big||\phi'_i(y)| - |\phi'_i(x)|\Big| \leq L||\phi'_i|||y - x|^\alpha.$$

for every $i \in I$ and every pair of points $x, y \in V$.

Let us now collect some geometric consequences of (BDP). We have for all words $\omega \in I^*$ and all convex subsets C of V

$$\mathrm{diam}(\phi_\omega(C)) \leq ||\phi'_\omega||\mathrm{diam}(C) \tag{1.1}$$

and

$$\mathrm{diam}(\phi_\omega(V)) \leq D||\phi'_\omega||, \tag{1.2}$$

where the norm $||\cdot||$ is the supremum norm taken over V and $D \geq 1$ is a universal constant. Moreover,

$$\operatorname{diam}(\phi_\omega(X)) \geq D^{-1}||\phi'_\omega|| \tag{1.3}$$

and

$$\phi_\omega(B(x,r)) \supset B(\phi_\omega(x), K^{-1}||\phi'_\omega||r), \tag{1.4}$$

for every $x \in X$, every $0 < r \leq \operatorname{dist}(X, \partial V)$, and every word $\omega \in I^*$.

The topological pressure function, $\mathrm{P}(t)$, for a conformal iterated function systems is defined as follows.

$$\mathrm{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} ||\phi'_\omega||^t$$

As it was shown in [MU1] there are two natural disjoint classes of conformal iterated function systems, regular and irregular. A system is regular if there exists $t \geq 0$ such that $\mathrm{P}(t) = 0$. Otherwise the system is irregular. Denote by $\mathrm{HD}(A)$ the Hausdorff dimension of a set A (treated as a subset of a metric space) and by H^t the t-dimensional Hausdorff measure. The following result has been proved in [MU1].

Theorem 1.1. *If S is a conformal iterated function system, then*

$$\mathrm{HD}(J) = \sup\{\mathrm{HD}(J_F) : \ F \subset I, \ F \text{ finite}\} = \inf\{t \geq 0 : \ P(t) \leq 0\}.$$

If a system is regular and $\mathrm{P}(t) = 0$*, then* $t = \mathrm{HD}(J)$.

Passing to Hölder families of functions fix $\beta > 0$ and let $F = \{f^{(i)} : X \to I\!\!R : i \in I\}$ be a family of continuous functions such that defining for each $n \geq 1$,

$$V_n(F) = \sup_{\omega \in I^n} \sup_{x,y \in X} \{|f^{(\omega_1)}(\phi_{\sigma(\omega)}(x)) - f^{(\omega_1)}(\phi_{\sigma(\omega)}(y))|\}\mathrm{e}^{\beta(n-1)},$$

the following is satisfied:

$$V_\beta(F) = \sup_{n\geq 1}\{V_n(F)\} < \infty$$

The collection F is called then a Hölder family of functions (of order β). If in addition

$$\sum_{i \in I} \mathrm{e}^{\sup(f^{(i)})} < \infty \quad \text{or equivalently} \quad \mathcal{L}_F(1\!\!1) \in C(X),$$

where

$$\mathcal{L}_F(g)(x) = \sum_{i\in I} \mathrm{e}^{f^{(i)}(x)} g(\phi_i(x)), \quad g \in C(X),$$

is the associated Perron-Frobenius operator, then the family F is called a strongly Hölder family of functions of order β. Throughout this paper the family F is assumed to be strongly Hölder of some order $\beta > 0$. We have made the conventions that the empty word $\emptyset$ is the only word of length 0 and $\phi_\emptyset = \mathrm{Id}_X$. Following the classical thermodynamic formalism, we defined the topological pressure of F by setting

$$\mathrm{P}(F) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\left(\sup_X \sum_{j=1}^n f^{\omega_j} \circ \phi_{\sigma^j\omega}\right).$$

Notice that the limit indeed exists since the logarithm of the partition function

$$Z_n(F) = \sum_{|\omega|=n} \exp(\sup(S_\omega(F)))$$

is subadditive, where

$$S_\omega(F) = \sum_{j=1}^n f^{(\omega_j)} \circ \phi_{\sigma^j\omega}.$$

and $\sigma : I^\infty \cup I^* \to: I^\infty \cup I^*$ is the shift map, i.e. cutting off the first coordinate. Moreover

$$\mathrm{P}(F) = \inf_{n\ge 1} \left\{ \frac{1}{n} \log Z_n(F) \right\}.$$

Now, if A is a subset of I, then we denote $F|_A := \{f^{(i)} : i \in A\}$ and $\mathrm{P}_A(F|_A)$ is the topological pressure of the family $F|_A$ with resepect to the iterated function system $\{\phi_i\}_{i\in A}$. Combining Theorem 3.1 from [MU2] and the display immediately preceeding (2.18) from [HMU], we get the following version of the variational principle.

Theorem 1.2. *If F is a strongly Hölder family of functions then*

$$\mathrm{P}(F) = \sup\{\mathrm{P}_A(F|_A)\},$$

where the supremum is taken over all finite subsets of I.

Now, a Borel probability measure m_F is said to be F-conformal provided it is supported on J, for every Borel set $A \subset X$

$$m_F(\phi_\omega(A)) = \int_A \exp\Big(S_\omega(F) - \mathrm{P}(F)|\omega|\Big)\, dm_F, \quad \forall \omega \in I^* \tag{1.5}$$

and

$$m(\phi_\omega(X) \cap \phi_\tau(X)) = 0 \tag{1.6}$$

for all incomparable $\omega, \tau \in I^*$. In ([HU], [HMU], and [Ur]) we have proved the following

Theorem 1.3. *If F is a strongly Hölder family of functions, then there exists exactly one F-conformal measure m_F.*

Although we will not use this object too often, we will recall now the definition of the potential function or amalgamated function, f, induced by the family of functions F. namely, $f : I^\infty \to \mathbb{R}$ is defined by setting

$$f(\omega) = f^{(\omega_1)}(\pi(\sigma(\omega))).$$

Our convention will be to use lower case letters for the potential function corresponding to a given family of functions. Frequently instead of $\mathrm{P}(F)$ we will also write $\mathrm{P}(f)$. Using the properties (1e) and (1d) it is not difficult to check that the family

$$\Xi = \{-\log|\phi_i'|\}_{i\in I}$$

is strongly Hoölder with exponent α and the corresponding amalgamated function is given by the formula

$$\xi(\omega) = -\log|\phi_{\omega_1}'(\pi(\sigma\omega))|.$$

We end recalling facts about Hölder families of functions with the following technical but frequently used result.

Lemma 1.4. *Suppose that F is a Hölder family of functions. Then there exists a constant $Q \geq 1$ such that if $x, y \in \phi_\tau(X)$ for some $\tau \in I^*$, then for all $\omega \in I^*$*

$$|S_\omega(F)(x) - S_\omega(F)(y)| \leq Q e^{-\beta|\tau|}$$

Suppose now that $F = \{f^{(i)} : X \to I\!\!R\}$ is a strongly Hölder family of functions such that the amalgamated function $f : I^\infty \to I\!\!R$ satisfies the following inequality

$$f \geq \xi. \tag{1.7}$$

Let

$$\theta(F) = \inf\{t \geq 0 : \mathrm{P}(-tf) < \infty\}.$$

We shall briefly sketch the following easy to prove result.

Lemma 1.5. $\mathrm{P}|_{[0,\theta(F))} = +\infty$. *The function $t \mapsto \mathrm{P}(-tf)$ is convex, continuous and strictly decreasing to $-\infty$ on $(\theta(F), \infty)$.*

Proof. Since $f \geq \xi > 0$, the function $t \mapsto \mathrm{P}(-tf)$ is noincreasing. Hence $\mathrm{P}|_{[0,\theta(F))} = +\infty$ and $\mathrm{P}(-tf) < \infty$ for every $t > \theta(F)$. Using Hölder's inequality it is easy to see that the function $t \mapsto \mathrm{P}(-tf)$ is convex on $(\theta(F), \infty)$, and therefore continuous. In order to see that $\mathrm{P}(-tf)$ is strictly decreasing on $(\theta(F), \infty)$, fix $\theta(F) < u < t$. Then, since $f \geq \xi \geq -\log s > 0$, we get

$$\begin{aligned}
\mathrm{P}(-tf) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\Big(\sup S_n(-tf)|_{[\omega]}\Big) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\Big(\sup S_n(-uf)|_{[\omega]} + \sup S_n((u-t)f)|_{[\omega]}\Big) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{|\omega|=n} \exp\Big(\sup S_n(-uf)|_{[\omega]}\Big)\right) \\
&= \mathrm{P}(-uf) + (t-u)\log s < \mathrm{P}(-uf)
\end{aligned} \tag{1.8}$$

Keeping u fixed, this last display also implies that

$$\liminf_{t\to+\infty} \mathrm{P}(-tf) \leq \mathrm{P}(-uf) + \lim_{t\to+\infty} (t-u)\log s = -\infty$$

The proof is complete. $\square$

We set

$$\delta(F) = \inf\{t \geq 0 : \mathrm{P}(-tf) \leq 0\}.$$

Now we shall define define the main object of our interest, the set $D_x(F,C)$, called the Besicovic-Jarnik set associated with the family F. More precisely, fix $x \in X$, $C > 0$ and define $D_x(F,C)$ as the set of those points $z \in X$ that

$$z \in B\Big(\phi_\omega(x), C\exp(-S_\omega F(x))\Big)$$

for infinitely $\omega \in I^*$. The first obvious observations concerning the sets $D_x(F,C)$ are contained in the following.

Lemma 1.6.
(a) $D_x(F,C) \subset \overline{J}$.
(b) If $C < D^{-1}$ or if there exists $\beta > 0$ suc that $f \geq \xi + \beta$ and $x \in \mathrm{Int}X$, then $D_x(F,C) \subset J$.

We call an element $x \in J$ finitely accessible if there exists a finte subset E of I such that $x \in J_E$, the limit set generated by the contractions from E. The main result of this paper is the following.

Theorem 1.7. *Suppose that Φ is a conformal iterated function system such that $J \subset \mathrm{Int}X$ and F is a strongly Hölder family of functions such that $f \geq \xi$. Then for every finitely accessible $x \in J$ and every $C > 0$*

$$\mathrm{HD}(D_x(F,C)) = \delta(F).$$

We would like to end up this section with the remark that $\mathrm{HD}(D_x(F,C)) \leq \delta(F)$ (see Proposition 2.1) without the assumptions that $J \subset \mathrm{Int}X$ and that x is finitely accessible.

2. $\mathrm{HD}(D_x(F,C)) \leq \delta(F)$

In this section we shall prove the following easier part of Theorem 1.7 without assuming that $J \subset \mathrm{Int}X$ and that x is finitely accessible.

Proposition 2.1. *If Φ is a conformal iterated function system and F is a strongly Hölder family of functions such that $f \geq \xi$, then for every $x \in X$ and every $C > 0$, $\mathrm{HD}(D_x(F,C)) \leq \delta(F)$.*

Proof. By definition

$$D_x(F,C) = \bigcap_{q\geq 1}\bigcup_{n\geq q}\bigcup_{|\omega|=n} B\Big(\phi(x), C\exp(-S_\omega F(x))\Big).$$

Fix $\epsilon > 0$. Then

$$\begin{aligned}\mathrm{H}^{\delta(F)+\epsilon}(D_x(F,C)) &\leq \liminf_{q\to\infty}\sum_{n\geq q}\sum_{|\omega|=n}\Big(2C\exp(-S_\omega F(x))\Big)^{\delta(F)+\epsilon}\\ &= (2C)^{\delta(F)+\epsilon}\sum_{n\geq q}\sum_{|\omega|=n}\exp\Big(-(\delta(F)+\epsilon)S_\omega F(x)\Big).\end{aligned}$$

Since $\mathrm{P}(-(\delta(F)+\epsilon)F) < 0$, thereexists $\eta > 0$ and $q_0 \geq 1$ such that for all $n \geq q_0$, $\sum_{|\omega|=n} \exp\Big(-(\delta(F)+\epsilon)S_\omega F(x)\Big) \leq e^{-\eta n}$. Therefore

$$\mathrm{H}^{\delta(F)+\epsilon}(D_x(F,C)) \leq (2C)^{\delta(F)+\epsilon} \liminf_{q\to\infty} \sum_{n\geq q} e^{-\eta n} = (2C)^{\delta(F)+\epsilon} \liminf_{q\to\infty} \frac{e^{-\eta q}}{1-e^{-\eta}} = 0.$$

Hence $\mathrm{H}^{\delta(F)+\epsilon}(D_x(F,C)) = 0$ and consequently $\mathrm{HD}(D_x(F,C)) \leq \delta(F)+\epsilon$. Letting now $\epsilon \searrow 0$, we conclude that $\mathrm{HD}(D_x(F,C)) \leq \delta(F)$. The proof is complete. $\square$

3. Auxiliary results

We recall that a finite measure ν defined on Borel sets of a metric space X satisfies the doubling property if there exists a constant $C \geq 1$ such that for every ball B, $\nu(2B) \leq C\nu(B)$, where $2B$ is the ball centered at the same point as B and with the radius twice as big as the radius of B.

Proposition 3.1. *If Φ is a finite conformal iterated function system, F is a strongly Hölder family of functions, and $J \subset \mathrm{Int}X$, then the corresponding F-conformal measure $m = m_F$ satisfies the doubling property.*

Proof. Fix $y = \pi(\omega) \in J$ and $0 < r < R := \mathrm{dist}(J, \partial X)$. Let $l \geq 0$ be the minimal number such that $\phi_{\omega|_l}(B(\pi(\sigma^l\omega), R)) \subset B(y,r)$, and let $k \geq 0$ be the maximal number such that $\phi_{\omega|_k}(B(\pi(\sigma^k\omega), R)) \supset B(y, 2r)$. By (1.5) and Lemma1.4 we get

$$m(B(y,r)) \geq \int_{B(\pi(\sigma^l\omega),R)} \exp\Big(S_{\omega|_l}F(z) - \mathrm{P}(F)l\Big)dm(z) \geq MQ^{-1}\exp\Big(S_{\omega|_l}F(x) - \mathrm{P}(F)l\Big), \tag{3.1}$$

where x is an arbitrary in X and $M = \inf\{m(B(z,R)) : z \in J\} > 0$ since m is positive on open sets of J. Similarly

$$m(B(y,2r)) \leq \int_{B(\pi(\sigma^k\omega),R)} \exp\Big(S_{\omega|_k}F(z) - \mathrm{P}(F)k\Big)dm(z) \leq Q\exp\Big(S_{\omega|_k}F(x) - \mathrm{P}(F)k\Big). \tag{3.2}$$

Of course $k < l$. On the other and, we have from the definitions of k and l that

$$\mathrm{diam}\Big(\phi_{\omega|_{l-1}}(B(\pi(\sigma^l\omega), R))\Big) \geq r \text{ and } \phi_{\omega|_{k+1}}\Big(\partial B(\pi(\sigma^{k+1}\omega), R))\Big) \cap B(y, 2r) \neq \emptyset.$$

Consequently, using the first part of (1.1), we get $2R||\phi'_{\omega|_{l-1}}|| \geq r$, and, using (1.3), we obtain $K^{-1}||\phi'_{\omega|_{k+1}}||R \leq 2r$. If $l \leq k+2$ we stop. Otherwise $l-1 \geq k+2$ and we get

$$||\phi'_{\omega|_{k+1}}|| \cdot ||\phi'_{\omega_{[k+2,l-1]}}|| \geq ||\phi'_{\omega|_{l-1}}|| \geq \frac{r}{2R} \geq \frac{1}{4K}||\phi'_{\omega|_{k+1}}||.$$

hence $s^{l-k-2} \geq ||\phi'_{\omega_{[k+2,l-1]}}|| \geq (4K)^{-1}$, and consequently $l-k-2 \leq \frac{-\log(4K)}{\log s}$. Thus, in any case, $l \leq k+p$, where

$$p = 2 + \frac{-\log(4K)}{\log s}.$$

Using this, Lemma 1.4, and combining (3.1) and (3.2), we therefore get

$$\begin{aligned} m(B(y,r)) &\geq MQ^{-1}\exp\Big(S_{\omega|_k}F(\phi_{\sigma^k\omega|_l}(x)+S_{\sigma^k\omega|_l}F(x)-\mathrm{P}(F)l\Big) \\ &= MQ^{-1}\exp\Big(S_{\omega|_k}F(\phi_{\sigma^k\omega|_l}(x)-\mathrm{P}(F)k\Big)\cdot\exp\Big(S_{\sigma^k\omega|_l}F(x)\Big)\exp(\mathrm{P}(F)(k-l)) \\ &\geq MQ^{-2}\exp\Big(S_{\omega|_k}F(x)-\mathrm{P}(F)k\Big)\exp((k-l)||f||_0)\exp(-p\mathrm{P}(F)) \qquad (3.3) \\ &\geq MQ^{-3}\exp(-p||f||_0)\exp(-p\mathrm{P}(F))m(B(y,2r)) \end{aligned}$$

The proof is complete. □

Fix now $x \in J$, $C > 0$, and for for every $\omega \in I^*$ put

$$B_\omega = B\Big(\phi_\omega(x), C\exp(-S_\omega F(x))\Big).$$

Then for every $n \geq 1$ and every set $B \subset X$ put

$$\Sigma(B,n) = \sum_{\omega\in I^n:B_\omega\subset B}\exp(-\phi(F)S_\omega F(x)).$$

Let $m_\delta = m_{-\delta F}$. We shall prove the following.

Lemma 3.2. *If the assumptions of Proposition 3.1 are satisfied, $f \geq \xi$, and $C < D^{-1}$, then*

$$m_\delta(B) \asymp \Sigma(B,n)$$

for every ball B (either closed or open) centered at a point of J and every $n \geq (\log\operatorname{diam}(B) - 2\log 2 - \log D)/\log s$.

Proof. On the one hand we have

$$B\cap J \subset \bigcup_{\omega\in I^n:B\cap\phi_\omega(X)\neq\emptyset}\phi_\omega(X) \subset \bigcup_{\omega\in I^n:\phi_\omega(X)\subset 2B}\phi_\omega(X) \subset \bigcup_{\omega\in I^n:B_\omega\subset 2B}\phi_\omega(X),$$

where the last inclusion we could write by (1.4) since $C < D^{-1}$ and $f \geq \xi$ (and then $B_\omega \subset \phi_\omega(X)$), and the second inclusion is satisfied provided that $D||\phi_\omega'|| \leq \frac{1}{2}\operatorname{diam}(B)$ which is implied by by the requirement $Ds^{|\omega|} \leq \frac{1}{2}\operatorname{diam}(B)$ which in turn means that

$$|\omega| \geq \frac{\log\operatorname{diam}(B) - \log 2 - \log D)}{\log s}.$$

So, if n is greater than or equal to this last number, then

$$m_\delta(B) \leq \sum_{\omega\in I^n:B_\omega\subset 2B} m_\delta\phi_\omega(X) \asymp \sum_{\omega\in I^n:B_\omega\subset 2B}\exp(-\delta S_\omega F(x)) = \Sigma(2B,n).$$

Thus, applyling Proposition 3.1, we get

$$m_\delta(B) << m_\delta(\frac{1}{2}B) << \Sigma(B,n)$$

for all $n \geq (\log\operatorname{diam}(B) - 2\log 2 - \log D)/\log s$. On the other hand

$$B \supset \bigcup_{\omega\in I^n:\phi_\omega(X)\subset B}\phi_\omega(X) \supset \bigcup_{\omega\in I^n:\frac{1}{2}B\cap\phi_\omega(X)\neq\emptyset}\phi_\omega(X) \supset \bigcup_{\omega\in I^n:\frac{1}{2}B\cap B_\omega\neq\emptyset}\phi_\omega(X) \supset \bigcup_{\omega\in I^n:B_\omega\subset\frac{1}{2}B}\phi_\omega(X),$$

where the third inclusion followed from (1.4) along with inequalities $C < D^{-1}$ and $f \ge \xi$ (since then $B_\omega \subset \phi_\omega(X)$), and the second inclusion is, in view of (1.2), satisfied provided that $D||\phi'_\omega|| \le \frac{1}{4}\text{diam}(B)$ which is implied by by the requirement $Ds^{|\omega|} \le \frac{1}{4}\text{diam}(B)$ which in turn means that

$$|\omega| \ge \frac{\log \text{diam}(B) - 2\log 2 - \log D)}{\log s}.$$

So, if n is greater than or equal to this last number, then

$$m_\delta(B) \ge \sum_{\omega \in I^n : B_\omega \subset \frac{1}{2}B} m_\delta \phi_\omega(X) \asymp \sum_{\omega \in I^n : B_\omega \subset \frac{1}{2}B} \exp(-\delta S_\omega F(x)) = \Sigma(\frac{1}{2}B, n).$$

Thus, applyling Proposition 3.1, we get

$$m_\delta(B) >> m_\delta(2B) >> \Sigma(B, n)$$

for all $n \ge (\log \text{diam}(B) - 2\log 2 - \log D)/\log s$. The proof is complete. □

In the last two proofs we have clearly indicated which of the geometric properties of the Bounded Distortion Property we were using. In the sequel we will use these without indicating it. Given now a set $E \subset X$ such that $E \cap J \ne \emptyset$ and $\text{diam}(E) < R := \text{dist}(J, \partial X)$, define $q(E)$ to be the largest integer for which there exists a word $\omega \in I^\infty$ such that $\phi_{\omega|_{q(E)}}(B(\pi(\sigma^{q(E)}\omega, R)) \subset E$. Since $\text{diam}(E) < R$, $q(E) \ge 0$. Since the maps ϕ_i, $i \in I$, are uniformly contracting, $q(E)$ is a finite number. The following two lemmas describe simple but useful properties of the number $q(E)$ and the word ω associated with it.

Lemma 3.3. *If an infinite word $\tau \in I^\infty$ and $p \ge 0$ are such that $\phi_{\tau|_p}(B(\pi(\sigma^p\tau, R)) \subset E$, then $p \le q(E)$ and $\tau|_p = \omega|_p$, where $\omega \in I^\infty$ is the word involved in the definition of the number $q(E)$. In particular $\omega|_{q(E)}$ is determined uniquely.*

Proof. Put $q = q(E)$. By maximality of q, $q \ge p$. Since both balls $B(\pi(\sigma^q\omega), R)$ and $B(\pi(\sigma^p\tau), R)$ are contained in Int$X$ and since $\phi_{\omega|_q}(B(\pi(\sigma^q\omega), R)) \cap \phi_{\tau|_p}(B(\pi(\sigma^p\tau), R)) \supset E$, it follows from the open set condition that either $\omega|_q$ extends $\tau|_p$ or vice-versa. Since $p \le q$, we thus conclude that $\omega|_q$ extends $\tau|_p$ and the proof is complete. □

Lemma 3.4. *If $D \subset E$ are such that $D \cap J \ne \emptyset$ and $\text{diam}(E) > R$, then $q(D) \ge q(E)$ and $\omega_D|_{q(E)} = \omega_E|_{q(E)}$. Moreover, for every $0 < C \le 1$ there exists an integere $k_C \ge 1$ such that if $\text{diam}(D) \ge C\text{diam}(E)$, then $q(D) \le q(E) + k_C$.*

Proof. The first part of this lemma is an immediate consequence of Lemma 3.3. Denote ω_D by τ and ω_E by ρ. Since $\phi_{\rho|_{q(E)}}(B(\pi(\sigma^{q(E)}\rho), R)) \supset E$, $E \cap J \ne \emptyset$ and the open set condition is satisfied, there exists $\beta \in I^\infty$ such that $\pi(\beta) \in B(\pi(\sigma^{q(E)}\rho), R)$ and $\phi_{\rho|_{q(E)}}(\pi(\beta)) \in E \cap J$. Since $\phi_{\rho|_{q(E)}\beta_1}(\pi(\sigma\beta)) = \phi_{\rho|_{q(E)}}(\pi(\beta)) \in E$, it follows from the definition of $q(E)$ that $\phi_{\rho|_{q(E)}\beta_1}\Big(B(\pi(\sigma\beta), R)\Big)$ does not contain E and that there exists a point $y \in \partial B(\pi(\sigma\beta), R)$

such that $|\phi_{\rho|_{q(E)}\beta_1}(\pi(\sigma\beta)) - \phi_{\rho|_{q(E)}\beta_1}(y)| \le \mathrm{diam}(E)$. Hence

$$\begin{aligned} K^{-1}||\phi'_{\rho|_{q(E)}\beta_1}||R &= K^{-1}||\phi'_{\rho|_{q(E)}\beta_1}|||\pi(\sigma\beta)-y| \\ &\le |\phi_{\rho|_{q(E)}\beta_1}(\pi(\sigma\beta)) - \phi_{\rho|_{q(E)}\beta_1}(y)| \\ &\le \mathrm{diam}(E). \end{aligned}$$

And consequently $||\phi'_{\rho|_{q(E)}}||K^2R^{-1}||\Phi'||^{-1}\mathrm{diam}(E)$, where $||\Phi'|| = \min\{||\phi'_i|| : i \in I\}$. Writing now $\tau|_{q(D)} = \rho|_{q(E)}\eta$, $\eta \in I^\infty$, we thus obtain

$$\begin{aligned} C\mathrm{diam}(E) \le \mathrm{diam}(D) &\le \phi_{\rho|_{q(E)}\eta}\Big(B(\pi(\sigma^{q(D)}\tau), R)\Big) \\ &\le ||\phi'_{\rho|_{q(E)}}|| \cdot ||\phi'_\eta||R \le K^2||\Phi'||^{-1}s^{|\eta|}\mathrm{diam}(E). \end{aligned}$$

Therefore $s^{|\eta|} \ge C||\Phi'||K^{-2}$, and taking logarithms we get $|\eta| \le \log(C||\Phi'||K^{-2})/\log s$. Thus, taking $k_C = \log(C||\Phi'||K^{-2})/\log s$ finishes the proof. $\square$

Define now $G = \{g_i : i \in I\} = G - \Xi$, a new strongly Hölder family of functions, by setting

$$g_i = f_i - \xi_i = f_i + \log|\phi'_i|.$$

Our last result in this section is the following.

Lemma 3.5. *If $B = B(y,r) \subset X$ and $y \in J$, then*

$$\frac{r^{\delta(F)}}{m_\delta(B)} \asymp \exp\Big(\delta(F)S_{\omega_B|_{q(B)}}G(x)\Big).$$

Proof. Put $\rho = \omega_B|_{q(B)}$. In view of te definition of ω_B, $q(B)$, conformality of measure m_δ, and Lemma 1.4 we can estimate as follows

$$\begin{aligned} m_\delta(B) &\le \int_{B(\pi(\sigma^{q(B)}\omega_B),R)} \exp\Big(-\delta(F)S_\rho F\Big)dm_\delta(z) \\ &\le Qm_\delta\Big(B(\pi(\sigma^{q(B)}\omega_B), R)\Big)\exp\Big(-\delta(F)S_\rho F(x)\Big) \\ &= Q\exp\Big(-\delta(F)S_\rho G(x)\Big)\exp\Big(-\delta(F)S_\rho \Xi(x)\Big) \asymp \exp\Big(-\delta(F)S_\rho G(x)\Big)|\phi'_\rho(x)|^{\delta(F)}. \end{aligned} \tag{3.4}$$

We also have

$$2r = \mathrm{diam}(B) \le \mathrm{diam}\Big(\phi_\rho(B(\pi(\sigma^{q(B)}\omega_B), R))\Big) \le ||\phi'_\rho||2R \le K|\phi'_\rho(x)|2R$$

and consequently

$$|\phi'_\rho(x)| \ge (KR)^{-1}r. \tag{3.5}$$

We shall now prove inequalities opposite to (3.4) and (3.5). Writing $y = \pi(\tau)$, $\tau \in I^\infty$, let $p = p(B)$ be the least non-negative integer such that

$$\phi|_{\tau|_p}\Big(B(\pi(\sigma^p\tau), R)\Big) \subset B(y,r). \tag{3.6}$$

Then $\operatorname{diam}\Big(\phi|_{\tau|_{p-1}}(B(\pi(\sigma^p\tau),R))\Big)\geq r$, which implies that

$$r\leq ||\phi|'_{\tau|_{p-1}}||2R. \tag{3.7}$$

Put now

$$u=2+\left[\frac{-\log(2K)}{\log s}\right],$$

where [] denotes the integer part. Then $u\geq 1-\frac{\log(2K)}{\log s}$ which means that $s^{1-u}\geq 2K$. Using thus inequality $||\phi'_{\tau|_{p-1}}||\leq ||\phi'_{\tau|_{p-u}}||s^{1-u}$ and (3.7) we can estimate as follows

$$K^{-1}||\phi'_{\tau|_{p-u}}||R\geq K^{-1}R||\phi'_{\tau|_{p-1}}||s^{1-u}\geq K^{-1}\frac{r}{2}s^{1-u}\geq r.$$

Therefore $\phi_{\tau|_{p-1}}(B(\pi(\sigma^{p-u}\tau),R))\Big)\supset B(\pi(\tau),r)=B$, and consequently, due to Lemma 3.3, $p-u\leq q(B)$ and $\tau|_{p-u}=\rho|_{p-u}$. Using (3.5) and Lemma 1.4 we therefore conclude that

$$\begin{aligned}
m_\delta(B) &\geq \int_{B(\pi(\sigma^p\tau),R)}\exp\Big(-\delta(F)S_{\tau|_p}F\Big)dm_\delta\geq Q^{-1}\exp\Big(-\delta(F)S_{\tau|_p}F(x)\Big)m_\delta\Big(B(\pi(\sigma^p\tau),R)\Big)\\
&\geq MQ^{-1}\exp\Big(-\delta(F)S_{\tau|_p}G(x)\Big)Q^{-1}\exp\Big(-\delta(F)S_{\tau|_p}\Xi(x)\Big)\\
&\geq MQ^{-2}\exp\Big(-\delta(F)S_{\tau|_p}G(x)\Big)|\phi|'_{\tau_{p-1}}(x)|^{\delta(F)}\exp(-u||g||_0)\\
&\geq MQ^{-3}K^{-\delta(F)}\exp(-u||g||_0)\exp\Big(-\delta(F)S_\rho G(x)\Big)|\phi|'_{\tau_{p-1}}(x)|^{\delta(F)}\\
&\geq MQ^{-3}K^{-\delta(F)}\exp(-u||g||_0)\exp\Big(-\delta(F)S_\rho G(x)\Big)r^{\delta(F)}.
\end{aligned}$$

So, the proof of the "<<" of Lemma 3.5 is finished. In order to prove the ">>" part notice that due to (3.6) $r\geq K^{-1}||\phi|'_{\tau|_p}||R$ and therefore, as $p-u\leq q(B)$ and $\tau|_{p-u}=\rho|_{p-u}$, we obtain $|\phi'_\rho(x)|\leq ||\phi|'_{\tau_{p-u}}||\leq ||\phi'_{\tau|_p}||\cdot||\Phi'||^{-u}\leq KR^{-1}||\Phi'||^{-u}r$. Hence, we can continue (3.4) as follows

$$m_\delta(B)\leq (KR^{-1}||\Phi'||^{-u})^{\delta(F)}r^{\delta(F)}.$$

So, the proof of Lemma 3.5 is complete. □

4. Proof of the Main Theorem

Our first aim in this section is to prove the "$\geq$" part of Theorem 1.7 assuming that the system Φ is finite. Formally we will prove the following.

Lemma 4.1. *If Φ is a finite conformal iterated function system such that $J\subset \operatorname{Int}X$ and F is a strongly Hölder family of functions such that $f\geq\xi$, then for every $x\in J$ and every $C>0$, $\operatorname{HD}(D_x(F,C))\geq\delta(F)$.*

Proof. Take a constant $0<C_1\leq\min\{C,D^{-1}\}$ so small that $\forall_{n\geq1}$, $\forall_{\omega,\tau\in I^n}$, if $\omega\neq\tau$, then

$$\overline{B}(\phi_\omega(x),C_1||\phi'_\omega||)\cap\overline{B}(\phi_\tau(x),C_1||\phi'_\tau||)=\emptyset.$$

It is possible to fulfill this requirement since $x \in \mathrm{Int}X$ and the sets $\phi_\omega(\mathrm{Int}X)$ and $\phi_\tau(\mathrm{Int}X)$ are disjoint. We will understand and treat the objects introduced in the previous sections (as $\Sigma(B,n)$ for ex.) as associated with the constant C_1. Notice that $D_x(F,C_1) \subset D_x(F,C)$. Fix now a rapidly increasing sequence $\{n_l\}_{l\ge 1}$ of non-negative integers which will be gradually required to satisfy more and more conditions. We then define the sets $\{K(l)\}_{l\ge1}$ inductively as follows.

$$K(1) = X,\ K(l+1) = \bigcup \Big\{\overline{B}\Big(\phi_\omega(x), C_1 \exp\Big(-S_\omega F(x)\Big)\Big)\Big\}$$

where the summation is taken over all words ω of length n_{l+1} such that

$$\overline{B}\Big(\phi_\omega(x), C_1 \exp\Big(-S_\omega F(x)\Big)\Big) \subset K(l).$$

For every $\omega \in I^*$ put

$$B_\omega = \overline{B}\Big(\phi_\omega(x), C_1 \exp\Big(-S_\omega F(x)\Big)\Big)$$

and define

$$I_l := \{\omega \in I^{n_l} : B_\omega \subset K(l)\}.$$

Notice that by our choice of C_1, $B_\omega \cap B_\tau = \emptyset$ if $\omega \ne \tau$ and $\omega, \tau \in I^{n_l}$. Since the sets $\{K(l)\}_{l\ge1}$ form a descending sequence of compact sets, the intersection

$$K = \bigcap_{l=1}^\infty K(l)$$

is a compact subset of X. Since each point $\phi_\omega(x) \in J$, there exists $\tau \in I^*$ longer than ω such that $B_\tau \subset B_\omega$. In particular, letting n_l grow fast enough we will have $K(l) \ne \emptyset$ for all $l \ge 1$, and consequently $K \ne \emptyset$. Since $C_1 \le C$, it immediately follows from the definition of K that $K \subset D_x(F,C)$. We shall now define a Borel probability measure on K as a weak limit of the sequence $\{\mu_l\}_{l\ge1}$ of Borel probability measures on X which will be constructed inductively below as follows. Let μ_1 be any Borel probability measure on $K(1) = X$. Suppose that μ_l has been defined for some $l \ge 1$ on the set $K(l)$. For every $\omega \in I_{l+1}$ fix an arbitrary Borel probability measure ν_ω on I_ω We then define μ_{l+1} on $K(l+1)$ by setting for every $\omega \in I_{l+1}$ and every $A \subset B_\omega$

$$\mu_{l+1}(A) = \frac{\exp\Big(-\delta(F)S_\omega F(x)\Big)\nu_\omega(A)}{\sum_{\tau\in I^{n_{l+1}-n_l}:\omega|_{n_l}\tau\in I_{l+1}} \exp\Big(-\delta(F)S_{\omega|_{n_l}}F(x)\Big)}\mu|_l\Big(B_{\omega|_{n_l}}\Big).$$

In particular

$$\mu_{l+1}\Big(B_\omega\Big) = \frac{\exp\Big(-\delta(F)S_\omega F(x)\Big)}{\sum_{\tau\in I^{n_{l+1}-n_l}:\omega|_{n_l}\tau\in I_{l+1}} \exp\Big(-\delta(F)S_{\omega|_{n_l}}F(x)\Big)}\mu_l\Big(B_{\omega|_{n_l}}\Big). \tag{4.1}$$

Obviously μ_{l+1} is a Borel probability measure on $K(l+1)$ and by a straightforward calculation we see that for every $p \ge 1$, every $\omega \in I_p$ and every $q \ge p$, $\mu_q(B_\omega) = \mu_p(B_\omega)$. Since aditionally $B_\omega \cap K$ are clopen subsets of K, we conclude that the weak limit $\mu = \lim_{l\to\infty}\mu_l$ exists, is supported on K, and for every $l \geq 1$ and every $\omega \in I_l$, $\mu(B_\omega) = \mu_l(B_\omega)$. Coming back to formula (4.1) we may transform it as follows.

$$\mu_{l+1}\Big(B_\omega\Big) = \frac{\exp\Big(-\delta(F)S_\omega F(x)\Big)}{\sum_{\rho\in I^{n_{l+1}}:B_\rho\subset B_{\omega|_{n_l}}}\exp\Big(-\delta(F)S_\rho F(x)\Big)}\mu_l\Big(B_{\omega|_{n_l}}\Big) = \frac{\exp\Big(-\delta(F)S_\omega F(x)\Big)}{\Sigma\Big(B_{\omega|_{n_l}}, n_{l+1}\Big)}\mu_l\Big(B_{\omega|_{n_l}}\Big).$$

Iterating this formula we get for all $\omega \in I_l$

$$\mu(B_\omega) = \mu_l(B_\omega) = \exp\Big(-\delta(F)S_\omega F(x)\Big)\prod_{i=1}^{l-1}\frac{\exp\Big(-\delta(F)S_{\omega|_{n_i}}F(x)\Big)}{\Sigma\Big(B_{\omega|_{n_i}}, n_{i+1}\Big)}.$$

Since the system Φ is finite, and since $C_1 \leq D^{-1}$, letting the sequence $\{n_l\}$ growing sufficiently fast and applying Lemma 3.2 and Lemma 3.5, we can write

$$\begin{aligned}\mu(B_\omega) &= \exp\Big(-\delta(F)S_\omega F(x)\Big)\prod_{i=1}^{l-1}\frac{\exp\Big(-\delta(F)S_{\omega|_{n_i}}F(x)\Big)}{m_\delta\Big(B_{\omega|_{n_i}}}\Big)\exp(O(l)) \\ &\asymp \exp\Big(-\delta(F)S_\omega F(x)\Big)\prod_{i=1}^{l-1}\exp\Big(\delta(F)S_{\rho_i}G(x)\Big)\times\exp(O(l)),\end{aligned} \tag{4.2}$$

where $\rho_i = \omega_{B_{\omega|_{n_i}}}|_{q\left(B_{\omega|_{n_i}}\right)}$. Fix now a point $y \in D_x(F, C)$, a radius $0 < r < R$ and the ball $B = B(y, r)$. Consider the family $\mathcal{F}_1$ of all the words $\omega \in \bigcup_{l\geq 1} I_l$ such that $B_\omega \cap B \cap J \neq \emptyset$, $\mathrm{diam}(B_\omega) < 4r$ and $\mathrm{diam}(B_\omega|_{n_{l-1}}) \geq 4r$. Next, consider the family $\mathcal{F}_2$ consisting of all balls of the form $B_{\omega|_{n_l-1}}$, where $\omega \in \mathcal{F}_1 \cap I_l$ for some $l \geq 1$. Since all the balls of the family $\mathcal{F}_2$ have radii $\geq 2r$, intersect the ball $B(y, r)$ and the complement of the ball $B(y, 2r)$, and any two of them are either disjoint or contained one in the other, a straightforward area argument shows that the number of elements of the family $\mathcal{F}_2$ is bounded from above by a constant M independent of y and r. Fix now an element $B_\tau \in \mathcal{F}_2$, $\tau \in I_{l-1}$ for some $l \geq 1$. Then using (4.2)we get

$$\Sigma(\tau) := \sum_{\omega\in T_1(\tau)}\mu(B_\omega) << \sum_{\omega\in T_2(\tau)}\exp\Big(-\delta(F)S_\omega F(x)\Big)\prod_{i=1}^{l-1}\exp\Big(\delta(F)S_{\rho_i}G(x)\Big)\times\exp(O(l)),$$

where

$$T_1(\tau) = \{\omega \in I_{l+1} : B_\omega \cap B \cap J \neq \emptyset, \omega|_{n_l} \in \mathcal{F}_1, B_{\omega|_{n_l}} \subset B_\tau \text{ and } \mathrm{diam}(B_\omega) < 4r\}$$

and

$$T_2(\tau) = \{\omega \in I_{l+1} : B_\omega \cap B \cap J \neq \emptyset, \omega|_{n_l} \in \mathcal{F}_1, B_{\omega|_{n_l}} \subset B_\tau \text{ and } B_{\omega|_{n_l}} \subset B(y, 5r) = 5B\}.$$

Take now $\omega \in T_2(\tau)$. Then for every $1 \le i \le l-1$, $\rho_i = \omega_{B_{\omega|_{n_i}}}|_{q\left(B_{\omega|_{n_i}}\right)} = \omega_{B_{\tau|_{n_i}}}|_{q\left(B_{\tau|_{n_i}}\right)}$ is independent of ω. Therefore, we can write

$$\begin{aligned}
\Sigma(\tau) &<< \prod_{i=1}^{l-1} \exp\Big(\delta(F) S_{\rho_i} G(x)\Big) \sum_{\omega \in T_2(\tau)} \exp\Big(-\delta(F) S_\omega F(x)\Big) \times \exp(O(l)) \\
&\le \prod_{i=1}^{l-1} \exp\Big(\delta(F) S_{\rho_i} G(x)\Big) \sum_{\omega \in I^{n_{l+1}} : B_\omega \subset 5B} \exp\Big(-\delta(F) S_\omega F(x)\Big) \times \exp(O(l)) \\
&= \prod_{i=1}^{l-1} \exp\Big(\delta(F) S_{\rho_i} G(x)\Big) \Sigma(5B, n_{l+1}) \times \exp(O(l)).
\end{aligned}$$

We want now to apply Lemma 3.2 to the term $\Sigma(5B, n_{l+1})$. In order to do it we require the sequence $\{n_l\}$ to grow so rapidly that for every l, $-n_{l+1}\log s \ge -\log 5 + 3\log 2 + \log D - \log C + ||f||_0 n_l$. Fix now one element $\omega \in T_2(\tau)$. Then

$$\operatorname{diam}(B) \ge \frac{1}{2}\operatorname{diam}\Big(B_{\omega|_{n_l}}\Big) = \frac{C}{2}\exp\Big(-S_{\omega|_{n_l}} F(x)\Big)$$

and consequently

$$\begin{aligned}
\log(5\operatorname{diam}(B)) - 2\log 2 - \log D &\ge \log 5 - 3\log 2 - \log D + \log C - S_{\omega|_{n_l}} F(x) \\
&\ge \log 5 - 3\log 2 - \log D + \log C - ||f||_0 n_l \\
&\ge n_{l+1}\log s.
\end{aligned}$$

Therefore

$$n_{l+1} \ge \frac{\log(5\operatorname{diam}(B)) - 2\log 2 - \log D}{\log s}$$

and Lemma 3.2 is applicable to the sum $\Sigma(5B, n_{l+1})$. Hence, we obtain

$$\Sigma(\tau) << m_\delta(B) \prod_{i=1}^{l-1} \exp\Big(\delta(F) S_{\rho_i} G(x)\Big) \exp(O(l)).$$

Using now Proposition 3.1 and Lemma 3.5, we can write

$$\Sigma(\tau) << r^{\delta(F)} \exp\Big(-\delta(F) S_{\omega_B|_{q(B)}} G(x)\Big) \prod_{i=1}^{l-1} \exp\Big(\delta(F) S_{\rho_i} G(x)\Big) \exp(O(l)). \tag{4.3}$$

Now, since $2B \cap B_\tau \cap J \ne \emptyset$ and since $2B \cap B_\tau$ contains a ball of radius $r/2$, it follows from Lemma 3.4 that $\omega_B|_{q(B)} = \omega_{B\cap B_\tau}|_{q(B)}$ and

$$q(B) \ge q(2B) \ge q(2B \cap B_\tau) - k_{1/4} \ge q(B_\tau) - k_{1/4}.$$

Therefore, applying Lemma 3.4 again, we get

$$\omega_B|_{q(B_\tau)-k_{1/4}} = \omega_{B\cap B_\tau}|_{q(B_\tau)-k_{1/4}} = \omega_{B_\tau}|_{q(B_\tau)-k_{1/4}}.$$

Hence, we can continue (4.3) as follows

$$
\begin{aligned}
\Sigma(\tau) &<< r^{\delta(F)} \exp\Big(-\delta(F) S_{\omega_B|_{q(B_\tau)-k_{1/4}}} G(x)\Big) \prod_{i=1}^{l-1} \exp\Big(\delta(F) S_{\rho_i} G(x)\Big) \exp(O(l)) \\
&= r^{\delta(F)} \exp\Big(-\delta(F) S_{\omega_{B_\tau}|_{q(B_\tau)-k_{1/4}}} G(x)\Big) \prod_{i=1}^{l-1} \exp\Big(\delta(F) S_{\rho_i} G(x)\Big) \exp(O(l)) \\
&\asymp r^{\delta(F)} \exp\Big(-\delta(F) S_{\omega_{B_\tau}|_{q(B_\tau)}} G(x)\Big) \prod_{i=1}^{l-1} \exp\Big(\delta(F) S_{\rho_i} G(x)\Big) \exp(O(l)) \qquad (4.4) \\
&= r^{\delta(F)} \prod_{i=1}^{l-2} \exp\Big(\delta(F) S_{\rho_i} G(x)\Big) \exp(O(l)).
\end{aligned}
$$

Fix now $\epsilon > 0$. We may require the sequence $\{n_l\}$ grow so fast that

$$
\begin{aligned}
&\prod_{i=1}^{l-2} \exp\Big(\delta(F)||g||_0 \max\{q(B(\rho)) : \rho \in I^{n_i}\}\Big) \exp(O(l)) \le \\
&\qquad \le \exp\Big(\delta(F)||g||_0 (l-2) \max\{q(B(\rho)) : \rho \in I^{n_{l-2}}\}\Big) \exp(O(l)) \\
&\qquad \le \min\Big\{\Big(C \exp\Big(-S_\eta F(x)\Big)\Big)^{-\epsilon} : \eta \in I^{n_{l-1}}\Big\}
\end{aligned}
$$

Coming now back to our ball B and corresponding element $\tau \in I_{l-1}$, we therefore see that

$$
\begin{aligned}
r^{-\epsilon} &\ge 4^\epsilon \Big(C \exp\Big(-S_\tau F(x)\Big)\Big)^{-\epsilon} \\
&\ge \min\Big\{\Big(C \exp\Big(-S_\eta F(x)\Big)\Big)^{-\epsilon} : \eta \in I^{n_{l-1}}\Big\} \ge \prod_{i=1}^{l-2} \exp\Big(\delta(F) S_{\rho_i} G(x)\Big) \exp(O(l)).
\end{aligned}
$$

So, combining this and (4.4), we obtain $\Sigma(\tau) << r^{\delta(F)-\epsilon}$. Hence $\mu(B) \le \sum_{\tau \in \mathcal{F}_2} \Sigma(\tau) \le M r^{\delta(F)-\epsilon}$. Thus $\mathrm{HD}(D_x(F,C)) \ge \mathrm{HD}(K) \ge \delta(F) - \epsilon$ and letting $\epsilon \searrow 0$ we conclude that $\mathrm{HD}(D_x(F,C)) \ge \delta(F)$. The proof is complete. $\square$

Now, in order to conclude the proof of Theorem1.7 suppose that $x \in J_A$ for some finite subset A of I. Take then any finite subset B of I containing A. Then $x \in J_B$ and $D_x(F,C;B) \subset D_x(F,C)$, where writeing $D_x(F,C;B)$ we indicate that this set is made up only with the help of the alphabet B. Thus, in view of Lemma 4.1, to complete the proof it is enough to demonstrate that

$$
\sup_{B \supset A}\{\delta_B(F|_B)\} = \sup_B\{\delta_B(F|_B)\} \ge \delta(F),
$$

where the suprema are taken over finite subsets of I. The "equality" part in this formula is obvious. To prove the "inequality" part put $u = \sup_B\{\delta_B(F|_B)\}$. Then by Theorem1.2 and Lemma 1.5 we get $\mathrm{P}(u) = \sup_B\{\mathrm{P}_B(-uF|_B)\} \le 0$, where the supremum is taken over all finite subsets of I. This, again in view of Lemma 1.5 implies that $u \ge \delta(F)$. The proof is complete. $\square$

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