HAUSDORFF DIMENSION OF HARMONIC MEASURE FOR SELF-CONFORMAL SETS

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ABSTRACT. Under some technical assumptions it is shown that the Hausdorff dimension of the harmonic measure on the limit set of a conformal infinite iterated function system is strictly less than the Hausdorff dimension of the limit set itself if the limit set is contained in a real-analytic curve, if the iterated function system consists of similarities only, or if this system is irregular. As a consequence of this general result the same statement is proven for hyperbolic and parabolic Julia sets, finite parabolic iterated function systems and generalized polynomial-like mappings. Also sufficient conditions are provided for a limit set to be uniformly perfect and for the harmonic measure to have the Hausdorff dimension less than 1. Some results in flavor of [PUZ] are obtained.

1. Introduction, Preliminaries

The general framework of this paper is provided by the scheme of conformal infinite iterated function systems (see [MU1] and the description below). Conformal infinite iterated function systems appear as natural objects in a number of subfields of dynamical systems. Inducing procedures mentioned briefly at the end of this introduction and thoroughly explored in Section 6 already provide a large class of examples. Conformal infinite iterated function systems emerge also naturally when studying parabolic implosions (see [DSZ], [UZ1] and [UZ2] for example) or transcendental entire or meromorphic functions (see [Ba], [KU1], [KU2], [UZ] for example). The concept of conformal infinite iterated function systems provides methods and tools to treat all this variety of objects with a unified framework. Our paper also contributes in this direction. In particular, we extend in this way the setting from [Vo] and [PV] we focus our attention mainly on the same problem as they did: Is the Hausdorff dimension of the harmonic measure on the limit set (the repeller and the Julia set in their context) of the iterated function system considered strictly less than the Hausdorff dimension of of the limit set? Under some technical assumptions, assuming that the closure of the limit set is uniformly perfect our answer is positive in the following three cases: If the limit set is contained in a real-analytic curve, if the iterated function system consists of similarities only, or if it is irregular.

The research of the first author was supported in part by the NSF Grant DMS 9801583. The research of the second author was supported in part by the Polish KBN Grant 2 P03A 009 17. The first author wishes to thank the Institut des Hautes Études Scientifiques and the Mathematical Institute of the Polish Academy of Sciences in Warsaw. The second author would like to thank the Department of Mathematics of the University of North Texas. All these institutions provided us with warm hospitality and excellent working conditions during preparation of this paper.

The motivation of our approach come essentially from three sources: the paper [Zd2], where a method of constructing invariant measures has been proposed and turned out to be adaptable with some modification to our setting, from [HMU], [MU4] and [MU3], where Hölder families of functions and Hölder potentials on a subshift of finite type with infinite alphabet have been treated from the point of view of thermodynamic formalism and, in our present setting, applied to the Jacobian of harmonic measure, and finally from [MPU], where the Radon-Nikodym derivative of the invariant measure equivalent with conformal measure has been shown to have a real-analytic extension on a neighbourhood of the limit set. This last result enabled us to avoid delicate and difficult considerations concerning the Jacobian of harmonic measure.

The strategy of the proof of our main theorem is the following. By [HMU], equality of dimensions of harmonic measure and the limit set implies equality of invariant harmonic measure and invariant conformal measure. In view of the result for the Radon-Nikodym derivative of the invariant measure conformal measure, this equality of invariant measures yields that the Jacobians of harmonic measure have a real-analytic extensions. Hence, by harmonic rigidity lemma these are constant. Since the invariant harmonic and conformal measures coincide, due to some results from [MPU], this implies that our iterated function system is conformally conjugate with a linear one. And for linear systems we have a separate argument.

Developing various inducing procedures we create suitable infinite iterated function systems to apply our general results for such systems to a large class of "1-dimensional" examples comprising Julia sets of hyperbolic and parabolic rational functions of the Riemann sphere, finite parabolic iterated function systems, and generalized polynomial-like mappings. Note that the main theorem is also true on a sufficiently small open neighbourhood of a "1-dimensional" hyperbolic rational function.

We also provide sufficient conditions for a limit set to be uniformly perfect, for the harmonic measure to have the Hausdorff dimension less than 1, and we obtain some results in flavor of [PUZ].

Remark 1.1. *We would like to end this introduction by emphasizing that if an appropriate version of the harmonic rigidity (Lemma 4.8) is proven, then our results automatically become true for all the systems considered without the "1-dimensionality" assumption.*

Remark 1.2. *We would like also to emphasize that almost a half of our paper (Section 6) is devoted to explore in great detail various non–hyperbolic examples including generalized polynomial–like mappings. Our main goal we achieve in this section is to reduce the problem of inequality between the Hausdorff dimension of the harmonic measure and the Hausdorff dimension of the reference set to the same problem for the limit set of an appropriate infinite hyperbolic iterated function system. Thus, the results from the sections 1–5 apply in this context. We find this reduction step interesting even though actually all real-analytic non-hyperbolic GPL's are critically finite and the critical point is of order* 2.

To start preliminaries, we want to say that throughout the entire paper if $R : X \to Y$ is a measurable map of a measurable space X endowed with a measure η into measurable space Y, then by $\eta \circ R^{-1}$ we mean the measure on Y given by the formula

$$\eta \circ R^{-1}(A) = \eta(R^{-1}(A))$$

for every measurable subset A of Y. If in addition R is injective and ρ is a measure on Y then by $\rho \circ R$ we mean the measure on X given by the formula

$$\rho \circ R(A) = \rho(R(A))$$

for every measurable subset A of X. Passing to iterated function systems let I be a countable index set with at least two elements and let $S = \{\phi_i : X \to X : i \in I\}$ be a collection of injective contractions from a compact metric space X (equipped with a metric ρ) into X for which there exists $0 < s < 1$ such that $\rho(\phi_i(x), \phi_i(y)) \le s\rho(x, y)$ for every $i \in I$ and for every pair of points $x, y \in X$. Thus, the system S is uniformly contractive. Any such collection S of contractions is called an iterated function system. We are particularly interested in the properties of the limit set defined by such a system. We can define this set as the image of the coding space under a coding map as follows. Let $I^* = \bigcup_{n\ge1} I^n$, the space of finite words, and for $\tau \in I^n$, $n \ge 1$, let $\phi_\tau = \phi_{\tau_1} \circ \phi_{\tau_2} \circ \cdots \circ \phi_{\tau_n}$. Let $I^\infty = \{\{\tau_n\}_{n=1}^\infty\}$ be the set of all infinite sequences of elements of I. If $\tau \in I^* \cup I^\infty$ and $n \ge 1$ does not exceed the length of τ, we denote by $\tau|_n$ the word $\tau_1\tau_2 \dots \tau_n$. Since given $\tau \in I^\infty$, the diameters of the compact sets $\phi_{\tau|_n}(X)$, $n \ge 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{n=0}^{\infty} \phi_{\tau|_n}(X)$$

is a singleton and therefore, denoting its only element by $\pi(\tau)$, defines the coding map

$$\pi : I^\infty \to X.$$

The main object in the theory of iterated function systems is the limit set defined as follows.

$$J = \pi(I^\infty) = \bigcup_{\tau\in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\tau|n}(X) = \bigcap_{n\ge1} \bigcup_{|\tau|=n} \phi_\tau(X)$$

Observe that J satisfies the natural invariance equality, $J = \bigcup_{i\in I} \phi_i(J)$. Notice that if I is finite, then J is compact and this property fails for infinite systems. Let $X(\infty)$ be the set of limit points of all sequences $x_i \in \phi_i(X)$, $i \in I'$, where I' ranges over all infinite subsets of I. In [MU1] the following has been proved

Proposition 1.3. *If* $\lim_{i\in I} \operatorname{diam}(\phi_i(X)) = 0$, *then* $\overline{J} = J \cup \bigcup_{\omega\in I^*} \phi_\omega(X(\infty))$.

From now on throughout the whole paper we assume that $d = 2$, more precisely that X is a closed Jordan domain contained in the complex plane $\mathbb{C}$, and that $\{\phi_i\}$ is a conformal iterated function system consisting of holomorphic contractions. As usually in the definition of conformal IFS (see [MU1]), we assume that there exists a topological disc V containing X such that all ϕ_i extend to univalent holomorphic maps defined on V. In addition, and this makes our class of conformal iterated function systems narrower than that in [MU1], we assume that X itself is a closed topological disk and

$$\overline{\bigcup_{i\in I}\phi_i(X)} \subset \mathrm{Int}X \tag{1.1}$$

We also assume that

$$\phi_i(X)\cap\phi_j(X)=\emptyset \tag{1.2}$$

for all $i,j\in I$, $i\neq j$ and that $\overline{J}$ is a topological Cantor set. Note that in general (1.2) does not imply this property; this is however so, if the system S is 1-dimensional, that is if X is contained in a real-analytic curve which is invariant under all maps ϕ_i, $i\in I$.

Notice that due to Koebe's distortion theorem our assumptions imply that the distortion of all ϕ_τ is bounded above and below by a universal constant. More precisely, there exists $K\geq 1$ such that

$$K^{-1}\leq\frac{|\phi_\omega'(y)|}{|\phi_\omega'(x)|}\leq K$$

for all $\omega\in I^*$ and all $x,y\in B(X,\frac{1}{2}\mathrm{dist}(X,\partial V))$. This property is denoted here by (BDP).

Let us now collect some geometric consequences of (BDP). We have for all words $\tau\in I^*$ and all convex subsets C of V

$$\mathrm{diam}(\phi_\tau(C))\leq||\phi_\tau'||\mathrm{diam}(C) \tag{1.3}$$

and

$$\mathrm{diam}(\phi_\tau(V))\leq D||\phi_\tau'||, \tag{1.4}$$

where the norm $||\cdot||$ is the supremum norm taken over V and $D\geq 1$ is a universal constant. Moreover,

$$\mathrm{diam}(\phi_\tau(J))\geq D^{-1}||\phi_\tau'|| \tag{1.5}$$

and

$$\phi_\tau(B(x,r))\supset B(\phi_\tau(x),K^{-1}||\phi_\tau'||r), \tag{1.6}$$

for every $x\in X$, every $0<r\leq\mathrm{dist}(X,\partial V)/2$, and every word $\tau\in I^*$.

The topological pressure function, $\mathrm{P}(t)$, for a conformal iterated function systems is defined as follows.

$$\mathrm{P}(t)=\lim_{n\to\infty}\frac{1}{n}\log\sum_{|\tau|=n}||\phi_\tau'||^t$$

As it was shown in [MU1] there are two natural disjoint classes of conformal iterated function systems, regular and irregular. A system is called regular if there exists $t\geq 0$ such that $\mathrm{P}(t)=0$. In Theorem 1.10 we will provide a different, in a sense more geometric, characterization of regular systems. Otherwise the system is called irregular. Denote by $\mathrm{HD}(A)$ the Hausdorff dimension of a set A (treated as a subset of a metric space) and by H^t the t-dimensional Hausdorff measure. The following result has been proved in [MU1].

Theorem 1.4. *If S is a conformal iterated function system, then*

$$\mathrm{HD}(J) = \sup\{\mathrm{HD}(J_F) :\ F \subset I,\ F \text{ finite}\} = \inf\{t \geq 0 :\ P(t) \leq 0\}.$$

If a system $\mathrm{P}(t) = 0$, then $t = \mathrm{HD}(J)$.

Following [MU4] we will work with the following.

Definition 1.5. *Fix $\beta > 0$ and let $F = \{f^{(i)} : J \to I\!\!R : i \in I\}$ be a family of functions such that defining for each $n \geq 1$,*

$$V_n(F) = \sup_{\tau \in I^n} \sup_{x,y \in J} \{|f^{(\tau_1)}(\phi_{\sigma(\tau)}(x)) - f^{(\tau_1)}(\phi_{\sigma(\tau)}(y))|\} \mathrm{e}^{\beta(n-1)},$$

the following is satisfied:

$$V_\beta(F) = \sup_{n \geq 1}\{V_n(F)\} < \infty$$

The collection F is called then a Hölder family of functions (of order β).

In [HU], [HMU], and [Ur] it was additionally assumed that all the functions $f^{(i)}$, $i \in I$, have continuous extensions to $\overline{J}$. Due to the progress done in [MU4] and [MU3] this requirement is not needed anymore.

Definition 1.6. *If (in addition to Definition 1.5)*

$$\sum_{i \in I} \mathrm{e}^{\sup(f^{(i)})} < \infty,$$

then the family F is called a summable Hölder family of functions of order β.

Remark that in [HMU] and [Ur] instead of summable Hölder families the term strongly Hölder families has been used. Throughout this paper the family F is assumed to be summable Hölder of some order $\beta > 0$. We have made the conventions that the empty word $\emptyset$ is the only word of length 0 and $\phi_\emptyset = \mathrm{Id}_X$. Let $\sigma : I^\infty \cup I^* \to: I^\infty \cup I^*$ be the shift map, i.e. cutting off the first coordinate. Following the classical thermodynamic formalism, we defined the topological pressure of F by setting

$$\mathrm{P}(F) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{|\tau| = n} \exp\left(\sup_J \sum_{j=1}^n f^{\tau_j} \circ \phi_{\sigma^j \tau} \right).$$

Notice that the limit indeed exists since the logarithm of the partition function

$$Z_n(F) = \sum_{|\tau| = n} \exp(\sup(S_\tau(F)))$$

is subadditive, where

$$S_\tau(F) = \sum_{j=1}^n f^{(\tau_j)} \circ \phi_{\sigma^j \tau}.$$

Moreover

$$\mathrm{P}(F) = \inf_{n \geq 1} \left\{ \frac{1}{n} \log Z_n(F) \right\}.$$

Now, a Borel probability measure m_F is said to be F-conformal provided it is supported on J, for every Borel set $A \subset X$

$$m_F(\phi_\tau(A)) = \int_A \exp\Big(S_\tau(F) - \mathrm{P}(F)|\tau|\Big)\, dm_F, \quad \forall \tau \in I^* \tag{1.7}$$

and

$$m(\phi_\tau(X) \cap \phi_\rho(X)) = 0 \tag{1.8}$$

for all incomparable $\tau, \rho \in I^*$. (In our case this last condition is trivially fulfilled.) In [MU4] (comp. also [MU3], [HU], [HMU], and [Ur]) we have proved the following

Theorem 1.7. *If F is a summable Hölder family of functions, then there exists a unique F-conformal measure m_F.*

In addition to Theorem 1.7 we have (see [HMU] for example) the following.

Theorem 1.8. *If F is a summable Hölder family of functions, then*

- **(a):** *There exists a unique Borel probability measure $\tilde m_F$ on I^∞ such that $\tilde m_F \circ \pi^{-1} = m_F$.*
- **(b):** *There exists a unique σ-invariant probability measure $\tilde\mu_F$ absolutely continuous with respect to $\tilde m_F$. Moreover, $\tilde\mu_F$ is equivalent with $\tilde m_F$, $\sup\left\{\log\left(\frac{d\tilde\mu_F}{dm_F}\right)\right\} < \infty$ and the dynamical system $\sigma : I^\infty \to I^\infty$ is completely ergodic with respect to the measure $\tilde\mu_F$.*

By μ_F we denote the measure $\tilde\mu_F \circ \pi^{-1}$. We recall now the definition of the potential function or amalgamated function, f, induced by the family of functions F. Namely, $f : I^\infty \to I\!R$ is defined by setting

$$f(\tau) = f^{(\tau_1)}(\pi(\sigma(\tau))).$$

Our convention will be to use lower case letters for the potential function corresponding to a given family of functions. Frequently instead of $\mathrm{P}(F)$ we will also write $\mathrm{P}(f)$. We say that a function $g : I^\infty \to I\!R$ is Hölder continuous (of some order $\beta > 0$) if

$$V_\beta(g) = \sup_{n \ge 1}\{\sup\{|g(\tau) - g(\rho)|e^{\beta(n-1)} : \tau|_n = \rho|_n\}\} < \infty.$$

Obviously, if F is a Hölder family of functions, then the amalgamated function is Hölder continuous. In order to clarify the situation we would like to mention that in [MU3] we have worked in the abstract (no iterated function system, only the shift space) situation with the functions g as above and in [MU4] we applied the results obtained in [MU3] to geometrical contexts. Given $\tau \in I^*$ we put

$$[\tau] = \{\rho \in I^\infty : \rho|_{|\tau|} = \tau\}.$$

If β is a countable partition of of I^∞ into Borel sets and $\tilde\mu$ is a Borel shift-invariant measure on I^∞, then by

$$\mathrm{H}_{\tilde\mu}(\beta) = -\sum_{B \in \beta} \tilde\mu(B)\log(\tilde\mu(B)).$$

We also write $\mathrm{H}_\mu(\beta) = \mathrm{H}_{\tilde\mu}(\beta)$ if $\mu \circ \pi^{-1} = \tilde\mu$. The following technical result has been proved in [HMU].

Lemma 1.9. *The following four conditions are equivalent:*

(a): $\int_{I^\infty} -f\, d\tilde{\mu}_F < \infty$.
(b): $\sum_{i\in I} \inf(-f|_{[i]}) \exp(\inf f|_{[i]}) < \infty$.
(c): *For every* $q \geq 1$, $\mathrm{H}_{\tilde{\mu}_F}(\alpha^q) < \infty$, *where* $\alpha^q = \{[\tau] : \tau \in I^q\}$.
(d): *There exists* $q \geq 1$ *such that* $\mathrm{H}_{\tilde{\mu}_F}(\alpha^q) < \infty$.

Of special interest are the measures $m_{h\Xi}$, $\mu_{h\Xi}$, $\tilde{m}_{h\Xi}$ and $\tilde{\mu}_{h\Xi}$, where $t\Xi = \{t \log |\phi_i'|\}_{i\in I}$ and $h = \mathrm{HD}(J)$ is the Hausdorff dimension of the limit set J. If the system is regular, meaning that $\mathrm{P}(h\Xi) = 0$, we called in [MU1] the measure $m_{h\Xi}$ simply h-conformal. In the sequel this measure will be denoted by m and the measure $\mu_{h\Xi}$ by μ. The formula (1.7) takes then on the following form

$$m_{h\Xi}(\phi_\tau(A)) = \int_A |\phi_\tau'|^h dm_{h\Xi}. \tag{1.9}$$

The converse is also true. In fact (see [MU1]) we have proved the following.

Theorem 1.10. *The following two conditions are equivalent.*

(a): $\mathrm{P}(h\Xi) = 0$. *and (1.8).*
(b): *There exists a unique Borel probability measure* m *satisfying (1.9) and (1.8).*

If $\tilde{\mu}$ is a Borel shift-invariant measure on I^∞, then by

$$\chi_{\tilde{\mu}} = -\int \log |\phi_{\tau_1}'(\sigma(\tau))| d\tilde{\mu}$$

we denote the Lyapunov exponent of the measure $\tilde{\mu}$. We also write $\chi_\mu = \chi_{\tilde{\mu}}$ if $\mu \circ \pi^{-1} = \tilde{\mu}$. In the sequel we will need the following result proven in [MU4] as Theorem 4.37 in the more general context of conformal graph directed Markov systems (comp. also Theorem 4.1 in [HMU]) and the corollary following it.

Theorem 1.11. *(Volume Lemma) Suppose that* η *is a Borel shift-invariant ergodic probability measure on* I^∞ *such that at least one of the numbers* $\mathrm{H}_\eta(\alpha)$ *or* $\chi_\eta(\sigma)$ *is finite, where* $\mathrm{H}_\eta(\alpha)$ *is the entropy of the partition* α *with respect to the measure* η. *Then*

$$\mathrm{HD}(\eta \circ \pi^{-1}) = \frac{h_\eta(\sigma)}{\chi_\eta(\sigma)}.$$

Since the Hausdorff dimension of any measure is finite and since Lyapunov exponents are positive, as an immediate consequence of this theorem we get the following.

Corollary 1.12. *If* η *is a Borel shift-invariant ergodic probability measure on* I^∞ *and* $\chi_\eta(\sigma)$ *is finite, then also the entropy* $\mathrm{H}_\eta(\alpha)$ *is finite.*

An important tool (see [MU4]) of our approach is given by the following.

Theorem 1.13. *Suppose that* $\{\phi_i\}_{i\in I}$ *is a regular conformal system such that* $\chi_{\tilde{\mu}_{h\Xi}} < \infty$. *Suppose also that* $\tilde{\mu}$ *is a Borel ergodic probability shift-invariant measure on* I^∞ *such that* $\mathrm{H}_{\tilde{\mu}}(\alpha) < \infty$. *If* $\mathrm{HD}(\tilde{\mu} \circ \pi^{-1}) = h := \mathrm{HD}(J)$, *then* $\tilde{\mu} = \tilde{\mu}_{h\Xi}$.

For every $\tau \in I^*$ denote by $D_\mu^\tau = \frac{d\mu\circ\phi_\tau}{d\mu}$ the Jacobian of the map $\phi_\tau : J \to J$ with respect to the measure $\mu = \mu_{h\Xi}$. We will also rely on the following result proved in [MPU].

Theorem 1.14. *For every $i \in I$ the Jacobian D_μ^i has a real-analytic extension on a common neighbourhood of X.*

2. **Harmonic Invariant Measure**

Although the title of this section is not entirely correct it presents well the goal of this section: looking for invariant measures equivalent with harmonic measures.

Throughout this whole section we assume that the domain $\mathbb{C}\setminus \overline{J}$ is regular in the sense of Dirichlet.

Since $\overline{\bigcup_{i\in I} \phi_i(X)} \subset \mathrm{Int}X$, there exists a slightly smaller topological disk W with smooth boundary (denoted in the sequel by γ) such that

$$\overline{\bigcup_{i\in I} \phi_i(X)} \subset W.$$

Obviously

$$J = \bigcap_{n=1}^{\infty} \bigcup_{|\omega|=n} \phi_\omega(W).$$

Let $\mathcal{G}$ be the class of all subharmonic functions defined on W which are harmonic and positive on $W \setminus \overline{J}$ and vanish on $\overline{J}$. Note that $G|_W$, the restriction to W of the Green's function with the pole at ∞ of the domain $\mathbb{C}\setminus \overline{J}$, is a member of $\mathcal{G}$. Recall also that $\frac{1}{2\pi}\Delta G = \omega$. Our first result is the following.

Proposition 2.1. *The formula*

$$L(g) = \sum_{i\in I} g \circ \phi_i$$

defines an operator acting on the space $\mathcal{G}$.

Proof. First of all we check that the sum above is finite. For this end we shall prove the following result interesting itself.

Lemma 2.2. *There exists a constant C such that for every $y \in \partial W$, every $\tau \in I^*$ we have*

$$C^{-1}\omega(\phi_\tau(X)) \le G(\phi_\tau(y)) \le C\omega(\phi_\tau(X)).$$

Proof. Consider the following function

$$F_\tau = G \circ \phi_\tau \cdot \frac{1}{\omega(\phi_\tau(X))}.$$

defined in W. Then F_τ is subharmonic in X, positive and harmonic on $X \setminus \overline{J}$, and vanishes on $\overline{J}$. In addition

$$\Delta F_\tau = \Delta\left(\frac{1}{\omega(\phi_\tau(X))} G \circ \phi_\tau\right) = \frac{1}{\omega(\phi_\tau(X))}\omega \circ \phi_\tau.$$

Since there is some definite space between $\gamma = \partial W$ and ∂X, one can use Harnack's inequality on γ to deduce that there exists a constant $K > 0$ such for all $\tau \in I^*$ and all $x, y \in \gamma$

$$\frac{F_\tau(x)}{F_\tau(y)} < K.$$

If for some $y \in \gamma$, $F_\tau(y) < CG(y)$ with some $C < 1/K^2$, then by above inequalities we would have $F_\tau(x) < cG(x)$ for every $x \in W$ with some constant $c < 1$. This implies that the Radon-Nikodym derivative of ΔF_τ with respect to ΔG is bounded from above by $c < 1$. But since both ΔF_τ and ΔG are probability measures, this is impossible. The conclusion is that for every $y \in \gamma$

$$G(\phi_\tau(y)) > \frac{1}{K^2}\omega(\phi_\tau(\overline{J})).$$

The opposite inequality is obtained in the same way. The proof is complete. □

Now, we continue the proof of Proposition 2.1. Let $F \in \mathcal{G}$. Then (by Maximum Principle) there exist constants c,C such that $cG \le F \le CG$. Thus, the functions

$$P_j(F)(y) = \sum_{i \le j} F(\phi_i(y))$$

satisfy for $y \in \partial W$

$$P_j(F) \le C \sum_{i \le j} G(\phi_i(y)) \le C \cdot \mathrm{const} \sum \omega(\phi_i(X)) \le C \le \mathrm{const} G(y).$$

By Maximum Principle the same inequality holds in the whole domain W. Thus, the sequence F_i is increasing and uniformly bounded, by const$G$; the infinite sum is an element of $\mathcal{G}$. □

If μ is a Borel finite measure on X we define

$$\mu \circ S = \sum_{i \in I} \mu \circ \phi_i.$$

and inductively $\mu \circ S^{n+1} = (\mu \circ S^n) \circ S$. A Borel finite measure μ on X is said to be S-invariant if $\mu \circ S = \mu$. Notice that due to (1.2) a Borel finite measure on J is S-invariant if and only if $\mu \circ \pi$ is shift-invariant. Using the observation that if $H : A \to B$ is a holomorphic homeomorphism between the domains $A, B \subset \mathbb{C}$, then $\Delta(g \circ H) = \Delta g \circ H \cdot |H'|^2$ (for $g \in C_0^\infty$) we (see [Zd2, Prop. 1.2]) get the following.

Proposition 2.3. *If $g \in \mathcal{G}$, then $\Delta(Lg) = \Delta g \circ S$.*

As an immediate consequence of this proposition, we get the following.

Corollary 2.4. *If $g \in \mathcal{G}$, then $\Delta(L^n g) = \Delta g \circ S^n$ for every integer $n \ge 1$.*

Definition 2.5. *The system S is called ω-conservative if $\omega(\overline{J} \setminus J) = 0$, or equivalently, if $\omega(X(\infty)) = 0$.*

Repeating the reasoning from Section 2 in [Zd2] we shall prove the following two results.

Lemma 2.6. *If the system S is ω-conservative then for every $g \in \mathcal{G}$ there exists a constant $C \geq 1$ such that*
$$C^{-1}g(z) \leq L^n g(z) \leq Cg(z)$$
for all $n \geq 1$ and all $z \in W$, a neighbourhood of $\overline{J}$.

Proof. Recall that W is the domain of g. Since ∂W is a compact Jordan curve, using Harnack's inequality, we deduce that there exists a constant $T \geq 1$ such that for all $n \geq 1$ and all $g \in \mathcal{G}$
$$\frac{\sup_{\partial W} L^n g}{\inf_{\partial W} L^n g} \leq T.$$
Let $l = \inf_{\partial W} g$ and $M = \sup_{\partial W} g$. Fix $n \geq 1$. Suppose that at some point $z_0 \in \partial W$ we have $L^n g(z_0) \leq l/2T$. Then $L^n g(z) \leq T L^n g(z_0) \leq l/2 \leq \frac{1}{2}g(z)$ for every $z \in \partial W$. Since in addition $L^n g - \frac{1}{2}g$ vanishes on $\overline{J}$, is continuous on X and harmonic on $X \setminus \overline{J}$, we conclude from the Maximum Principle that $L^n g - \frac{1}{2}g \leq 0$ on $W \setminus \overline{J}$. Since $L^n g - \frac{1}{2}g = 0$ on $\overline{J}$, we therefore obtain $L^n g \leq \frac{1}{2}g$ on the whole set W. By ω-conservativity, $\Delta g(\bigcup_{i\in I} \phi_i(X)) = \Delta g(X)$ and therefore $\Delta g \circ S^n(\overline{J})$ are equal $(n \geq 1)$. Since in addition the measures $\Delta L^n g$ and Δg are supported on $\overline{J}$, we therefore get from Corollary 2.4 and Proposition 2.3 that
$$\Delta g(\overline{J}) = \Delta g \circ S^n(\overline{J}) = \Delta(L^n g)(\overline{J}) \leq \frac{1}{2}\Delta g(\overline{J}).$$
This contradiction shows that $\inf_{\partial W} L^n g \geq l/(2T)$. Thus, for every $z \in \partial W$ we have
$$L^n g(z) \geq \frac{l}{2T} = \frac{l}{2TM}K \geq \frac{l}{2TM}g(z)$$
and applying the Maximum Principle in the same way as above we conclude that $L^n g \geq \frac{l}{2TM}g$ on W. Starting with the hypothesis that there exists a point $z' \in \partial W$ such that $L^n g(z') \geq 2MT$ we could proceed similarly as above to conclude that $L^n g \leq \frac{2MT}{l}g$ on W. The proof is complete. $\square$

Theorem 2.7. *If the system S is ω-conservative, then there exists a Borel probability S-invariant measure ν on J equivalent with the harmonic measure ω. In addition*
$$\sup\left\{\left|\log\frac{d\nu}{d\omega}\right|\right\} < \infty \text{ and } \nu = \Delta\tilde{G}$$
for some function $\tilde{G} \in \mathcal{G}$ such that $C^{-1}G \leq \tilde{G} \leq CG$ on W for some constant $C \geq 1$.

Proof. Recall that G is the Green's function of the domain $\mathbb{C} \setminus \overline{J}$ with the pole at ∞. Consider the sequence
$$G_n = \frac{1}{n}\sum_{i=0}^{n-1} L^i(G),\ n \geq 1,$$
of the functions from $\mathcal{G}$. By Lemma 2.6 there exists a constant $C \geq 1$ such that $C^{-1}G \leq G_n \leq CG$ on W. Thus G_n, $n \geq 1$, are uniformly bounded and one can choose a subsequence

G_{n_k}converging uniformly on compact subsets of W to a function $\hat{G} \in \mathcal{G}$ satisfying

$$C_1^{-1} G \leq \hat{G} \leq C_1 G \tag{2.1}$$

on W. Let $x \in W \setminus \bar{J}$. We fix $\varepsilon > 0$ and choose a compact subset $F \subset W \setminus \bar{J}$ such that $x \in F$ and

$$\sum_{\{i:\phi_i(x) \in W \setminus F\}} G(\phi_i(x)) < \frac{\varepsilon}{C_1} \tag{2.2}$$

Let $M = \#\{i \in I : \phi_i(x) \in F\}$. Since G_n converges to $\hat{G}$ uniformly on compact sets, there exists n_0 such that for $n \geq n_0$

$$M \sup_{z \in F} |\hat{G}(z) - G_n(z)| < \varepsilon. \tag{2.3}$$

By (2.2) and (2.3) we get

$$|L(\hat{G})(x) - L(G_n)(x)| < 2\varepsilon \tag{2.4}$$

and

$$\begin{aligned} L(G_n) &= L\left(\frac{1}{n}\sum_{i=0}^{n-1} L^i(G)\right) = \frac{1}{n}\sum_{i=1}^{n-1} L^{i+1}(G) = \frac{1}{n}\sum_{i=0}^{n-1} L^i(G) + \left(\frac{1}{n}L^{n+1}G - \frac{1}{n}G\right) \\ &= G_n + \left(\frac{1}{n}L^{n+1}G - \frac{1}{n}G\right) \end{aligned}$$

Thus,

$$|L(G_n)(x) - G_n(x)| < \varepsilon \tag{2.5}$$

for all n large enough. Now, by (2.4) and (2.5) we get

$$|L(\hat{G})(x) - G_n(x)| < 3\varepsilon.$$

But by (2.3), $|G_n(x) - \hat{G}(x)| < \varepsilon$. So,

$$|L(\hat{G}(x)) - \hat{G}(x)| < 4\varepsilon.$$

Since ε was arbitrary, we get $L(\hat{G}(x)) = \hat{G}(x)$. Denote $\tilde{G} = \frac{\hat{G}}{\Delta\hat{G}(\overline{J})}$ and $\nu = \Delta\tilde{G}$. Then, applying Proposition 2.3, we get

$$\nu \circ S = \Delta\tilde{G} \circ S = \Delta(L\tilde{G}) = \Delta\tilde{G} = \nu.$$

This means that ν is S-invariant and, in view of (2.1), $C^{-1} \leq \frac{d\nu}{d\omega} \leq C$ for $C = C_1/\Delta\tilde{G}(\overline{J})$. In particular, ν is supported on J. The proof is complete. $\square$

Remark 2.8. *Actually, we have verified (and then used) the fact that the operator L acts continuously on the space $\mathcal{G}$. Namely, if $G^{(n)} \in \mathcal{G}$ and $G^{(n)} \to G^{(0)}$ uniformly on compact sets then $L(G^{(n)}) \to L(G^{(0)})$ uniformly on compact sets.*

We want to close this section with an example of a system which is not ω-conservative. The construction goes as follows. Let $C \subset S^1$ be a closed totally disconnected set of positive 1-dimensional Lebesgue measure. Consider a countable set I and a system S consisting of similarities

$$\phi_i : B(0,2) \to \mathbb{C},\ i \in I$$

such that the images $\phi_i(B(0,2))$ are disjoint, $\phi_i(B(0,2) \subset B = B(0,1)$ and $X(\infty) = C \subset S^1$. If J is the limit set of this system, then $C \subset \bar{J}$. Let ω be the harmonic measure in $\overline{\mathbb{C}} \setminus \bar{J}$ evaluated at ∞. Denote by ω_B the harmonic measure in $\overline{\mathbb{C}} \setminus \bar{B}$. It coincides with the usual Lebesgue measure, thus $\omega_B(C) > 0$. But $\omega(C) \geq \omega_B(C)$, so it is positive and the system is not ω-conservative.

3. **Uniform Perfectness**

In this section we will provide some number of auxiliary results needed to complete the proofs in Section 4. We recall the definition

Definition 3.1. *A compact set $K \subset \mathbb{C}$ is uniformly perfect if there exists a constant $0 < c < 1$ such that for each positive radius small enough and each point $z \in J$ the annulus $A(z, cr, r) := \{w \in \mathbb{C} \colon cr \leq |w - z| < r\}$ intersects K.*

Remark 3.2. *Uniform perfectness (which, itself is an interesting geometric property, see [Po]) guarantees, in particular that the complement of K in the Riemann sphere is a domain regular in the sense of Dirichlet.*

Let us recall from Section 1 that

$$X(\infty) = \overline{\lim}_{i \to \infty} \phi_i(X) = \bigcap_F \overline{\bigcup_{i \in I \setminus F} \phi_i(X)},$$

where the intersection is taken over all finite subsets of I.

If $X(\infty)$ is empty (i.e. if the set I is finite) then our limit set is a (classical) conformal expanding repeller. In this case J is always uniformly perfect (one of possible arguments is provided in Section 6, Theorem 6.2). If the set of symbols I is infinite, this is no longer the case.

Indeed, one can easily define an infinite system which is not uniformly perfect, because already the sets $\phi_i(X)$ can be well separated. So, let $X = \overline{B(0,1)}$ and let a_i be a decreasing sequence of positive numbers quickly converging to 0, so that $\frac{a_i}{a_{i+1}} \to \infty$. Let $\phi_i(z) = a_i + \lambda_i z$ with λ_i small. Then $0 \in \bar{J}$ and, obviously, if λ_i have been chosen small enough, $\bar{J}$ is not uniformly perfect.

So, some condition is certainly necessary: the construction of ϕ_i itself must not violate the uniform perfectness property. More precisely, we need the following property below.

Definition 3.3. *We say that the set J is C-uniformly perfect ($c \geq 1$) at large scale if the following condition (UP) holds:*

$X(\infty)$ is finite and for each index $i \in I$ there exists an infinite sequence $\{i_n\}_{n\geq 1}$ of elements in I ($i_n \neq i_m$ if $n \neq m$) such that $i_1 = i$ and

$$C^{-1} < \frac{\mathrm{diam}(\phi_{i_n}(X))}{\mathrm{diam}(\phi_{i_{n+1}}(X))} < C$$

$$\frac{\mathrm{dist}(\phi_{i_{n+1}}(X), \phi_{i_n}(X))}{\min(\mathrm{diam}(\phi_{i_n}(X)), \mathrm{diam}(\phi_{i_{n+1}}(X)))} < C$$

Remark 3.4. *It can be easily seen that if J is uniformly perfect at large scale then for every $w \in X(\infty)$ there exists a sequence j_n such that $\phi_{j_n}(X) \to w$ and the condition (UP) above is satisfied for the sequence j_n.*

The theorem below says that if the set $X(\infty)$ is finite, then this necessary condition is also sufficient.

Theorem 3.5. *Suppose that J is C–uniformly perfect in large scale for some constant C. Then $\overline{J}$, the closure of the limit set J, is uniformly perfect.*

Proof. The proof below is divided into three steps. First, using the condition (UP) above, we verify the uniform perfectness property at all the points $w \in X(\infty)$. Next, we consider an arbitrary point $z \in J$ (i.e. $z \in \phi_i(J)$ for some $i \in I$) and, using the first step of the proof, we verify the uniform perfectness property for all $r \geq \mathrm{const} \cdot \mathrm{diam}\phi_i(X)$ bounded above by a universal constant. At the last step, we consider an arbitrary point $z \in J$ and an arbitrary radius r with $0 < r < \mathrm{diam}(X)$.

It follows from (1.4) and (1.5) that for every $\tau \in I^*$

$$\mathrm{diam}(\phi_\tau(X)) \leq D||\phi_\tau'|| \leq D^2\mathrm{diam}(\phi_\tau(J)). \tag{3.1}$$

Let the constant $C > 0$ be as in the condition (UP) above.

Step 1: The uniform perfectness of $\overline{J}$ at the points w of $X(\infty)$.

Fix $w \in X(\infty)$ and let $\{j_n\}_{n=1}^{\infty}$ be the sequence claimed in Remark 3.4. It follows from (1.2) that the set $\{n \geq 1 : w \in \phi_{j_n}(X)\}$ is either a singleton or an empty set. Let $n_w \geq 1$ be the least element in the complement of this set. Set

$$R_w = \mathrm{dist}(w, \phi_{j_{n_w}}(X))$$

and consider any radius $0 < r < R_w$. Since $w \in \overline{\lim}_{n\to\infty}\phi_{j_n}(X)$ and since $\lim_{n\to\infty} \mathrm{diam}(\phi_{j_n}(X)) = 0$, there exists an element $k \geq n_w$ such that $\phi_{j_k}(X) \subset B(w, r)$. Let p be such a least index k. If $\mathrm{diam}(\phi_{j_p}(J)) \geq r/8D^2C$, then using the fact that $\phi_{j_p}(J) \subset B(w, r)$, we conclude that $A(z, r/16D^2C, r) \cap \phi_{j_p}(J) \neq \emptyset$. But since $\phi_{j_p}(J) \subset J$, we get

$$A(z, r/16D^2C, r) \cap J \neq \emptyset$$

and we are done in this case with any constant $c \leq 1/16D^2C$. So suppose that $\mathrm{diam}(\phi_{j_p}(J)) \leq r/8D^2C$. Then by (3.1), $\mathrm{diam}(\phi_{j_p}(X)) \leq r/8C$. So, by the definition of w,

$$\mathrm{dist}(\phi_{j_p}(X), \phi_{j_{p-1}}(X)) < \frac{r}{8} \text{ and } \mathrm{diam}(\phi_{j_{p-1}}(X)) \leq \frac{r}{8}.$$

Since $\phi_{j_{p-1}}(X)\cap(\mathbb{C}\setminus B(w,r))\neq\emptyset$, we deduce that

$$\mathrm{dist}(\phi_{j_p}(X),\partial B(w,r))<\frac{r}{8}+\frac{r}{8}=\frac{r}{4}.$$

Since $\phi_{j_p}(X)\subset B(w,r)$ and $\mathrm{diam}(\phi_{j_p}(X))\le r/8C<r/4$, we conclude that $\phi_{j_p}(X)\subset A(w,r-\frac{r}{4}-\frac{r}{4},r)=A(w,r/2,r)$. Since $J\cap\phi_{j_p}(X)\neq\emptyset$, we are done in this case with any constant $c\le 1/2$.

Put

$$R=\min\{R_w:w\in X(\infty)\}>0 \text{ and } c_1=\min\{1/2,1/16D^2C\}.$$

Step 2: Consider now an arbitrary index $i\in I$, an arbitrary point $z\in\phi_i(J)$ and an arbitrary radius r such that $\frac{4}{c_1}\mathrm{diam}(\phi_i(X))<r<R$.

Let $\{i_n\}_{n\ge1}$ be the sequence ascribed to the index i guaranteed by our hypothesis. Suppose that

$$\phi_{i_n}(X)\cap\left(\mathbb{C}\setminus B\left(z,\frac{c_1}{4}r)\right)\right)\neq\emptyset$$

for some $n\ge1$. Then $n\ge2$ and let $q\ge2$ be the least index n with this property. If $\mathrm{diam}(\phi_{i_{q-1}}(J))\ge c_1r/32D^2C$, then using the fact that $\phi_{i_{q-1}}(J)\subset B(z,c_1r/4)$, we conclude that $A(z,c_1r/64D^2C,c_1r/4)\cap\phi_{i_{q-1}}(J)\neq\emptyset$. But since $\phi_{i_{q-1}}(J)\subset J$, we get

$$A(z,c_1r/64D^2C,c_1r/4)\cap J\neq\emptyset$$

and we are done in this case with any constant $c\le c_1/64D^2C$. So suppose that $\mathrm{diam}(\phi_{i_{q-1}}(J))\le c_1r/32D^2C$. Then by (3.1),

$$\mathrm{diam}(\phi_{i_{q-1}}(X))\le c_1r/32C. \tag{3.2}$$

But then

$$\mathrm{dist}(\phi_{i_{q-1}}(X),\phi_{i_q}(X))<\frac{c_1r}{32} \text{ and } \mathrm{diam}(\phi_{i_q}(X))\le\frac{c_1r}{32}.$$

Since $\phi_{i_q}(X)\cap(\mathbb{C}\setminus B(z,c_1r/4))\neq\emptyset$, we deduce then that

$$\mathrm{dist}(\phi_{i_{q-1}}(X),\partial B(z,c_1r/4))<\frac{c_1r}{32}+\frac{c_1r}{32}=\frac{c_1}{16}r.$$

Since $\phi_{i_{q-1}}(X)\subset B(z,c_1r/4)$ and, by (3.2), $\mathrm{diam}(\phi_{i_{q-1}}(X))<c_1r/16$, we conclude that $\phi_{i_{q-1}}(X)\subset A(z,\frac{c_1}{4}r-\frac{c_1}{16}-\frac{c_1}{16},\frac{c_1}{4}r)\subset A(z,\frac{c_1}{8}r,r)$. Since $J\cap\phi_{i_{q-1}}(X)\neq\emptyset$, we are done in this case with any constant $c\le c_1/8$. So, suppose in turn that $\phi_{i_n}(X)\subset B(z,c_1r/4)$ for all $n\ge1$. Let $w\in\mathbb{C}$ be an arbitrary point of $\overline{\lim}_{n\to\infty}\phi_{i_n}(X)$. Then $w\in X(\infty)\cap\overline{B}(z,c_1r/4)$ and, as $r/2<R\le R_w$, we conclude from what has been already proved that $A(w,c_1r/2,r/2)\cap J\neq\emptyset$. Then $A(w,c_1r/2,r/2)\subset\overline{B}(z,(r/2)+(c_1r/4))\subset B(z,r)$ and take an arbitrary point $x\in A(w,c_1r/2,r/2)\cap J$. Then $|x-z|\ge|x-w|-|z-w|\ge c_1r/2-c_1r/4=c_1r/4$ which implies that $A(z,c_1r/4,r)\cap J\neq\emptyset$ and we are done in this case with the constant $c_2=c_1/4=\min\{1/8,1/64D^2C\}$.

Obviously, taking $c_2>0$ appropriately smaller, if necessary, we have now the local uniform perfectness at the point z with the constant c_2 for every radius r satisfying

$$\frac{4}{c_1}\mathrm{diam}(\phi_i(X))<r<8KD^3c_1^{-1}.$$

Step 3: Passing to the last step of this proof, fix an arbitrary point $z = \pi(\tau) \in J$, $\tau \in I^\infty$, and a positive radius $r < 8(KD)^2 c_1^{-1}\mathrm{diam}(X)$. Let $n \geq 1$ be the least integer such that

$$\phi_{\tau|_n}(X) \subset B(z, 8^{-1}K^{-2}D^{-2}c_1 r). \tag{3.3}$$

Consider the ball

$$B(\pi(\sigma^n(\tau)), K^{-1}||\phi'_{\tau|_{n-1}}||^{-1}r)$$

(note that $\pi(\sigma^n(\tau)) = \phi_{\tau|_{n-1}}^{-1}(z)$ and that if $n = 1$, then $\phi_{\tau|_{n-1}}$ is the identity map). Since $8^{-1}K^{-2}D^{-2}c_1 r \leq \mathrm{diam}(\phi_{\tau|_{n-1}}(X))$, using (1.4), we get

$$\begin{aligned} K^{-1}||\phi'_{\tau|_{n-1}}||^{-1}r &\leq 8KD^2c_1^{-1}||\phi'_{\tau|_{n-1}}||^{-1}\mathrm{diam}(\phi_{\tau|_{n-1}}(X)) \\ &\leq 8KD^2c_1^{-1}||\phi'_{\tau|_{n-1}}||^{-1}D||\phi'_{\tau|_{n-1}}|| = 8KD^3c_1^{-1} \end{aligned} \tag{3.4}$$

Using (3.3) and (1.5), we obtain

$$\begin{aligned} K^{-1}||\phi'_{\tau|_{n-1}}||^{-1}r &= K^{-1}||\phi'_{\tau|_{n-1}}||^{-1}4K^2D^2c_1^{-1}(4^{-1}K^{-2}D^{-2}c_1 r) \\ &\geq 4KD^2c_1^{-1}||\phi'_{\tau|_{n-1}}||^{-1}\mathrm{diam}(\phi_{\tau|_n}(X)) \\ &\geq 4KD^2c_1^{-1}||\phi'_{\tau|_{n-1}}||^{-1}D^{-1}||\phi'_{\tau|_n}|| \geq 4Dc_1^{-1}||\phi'_{\tau_n}|| \geq \frac{4}{c_1}\mathrm{diam}(\phi_{\tau_n}(X)). \end{aligned}$$

This inequality and (3.4) enable us to apply the previous step, and as its consequence, we obtain an annulus $A(\pi(\sigma^n(\tau)), c_2K^{-1}||\phi'_{\tau|_{n-1}}||^{-1}r, K^{-1}||\phi'_{\tau|_{n-1}}||^{-1}r)$ having a non-empty intersection with J. Hence

$$\phi_{\tau|_{n-1}}(A(\pi(\sigma^n(\tau)), c_2K^{-1}||\phi'_{\tau|_{n-1}}||^{-1}r, K^{-1}||\phi'_{\tau|_{n-1}}||^{-1}r)) \cap J \neq \emptyset.$$

Assuming K to be so large that $K^{-1}D < \mathrm{dist}(X, \partial V)$, using the Bounded Distortion property, the mean value inequality, and (1.6), we get

$$\phi_{\tau|_{n-1}}(A(\pi(\sigma^n(\tau)), c_2K^{-1}||\phi'_{\tau|_{n-1}}||^{-1}r, K^{-1}||\phi'_{\tau|_{n-1}}||^{-1}r)) \subset A(z, c_2K^{-2}r, r).$$

Hence $A(z, c_2K^{-2}r, r) \cap J \neq \emptyset$ and the proof is complete. □

4. **Properties of invariant harmonic measure ν**

For every $\tau \in I^*$ let $D_\nu^\tau : J \to (0, \infty)$ be the Jacobian of the measure $\nu \circ \phi_\tau$ with respect to the measure ν, e.i. $D_\nu^\tau = \frac{d\nu\circ\phi_\tau}{d\nu}$. Given a function $g : \phi_\tau(J) \to I\!R$, $\tau \in I^*$, let $\mathrm{osc}(g) = \sup(g) - \inf(g)$. We shall prove the following easy

Lemma 4.1. *The collection $\mathcal{D} = \{\log(D_\nu^i)\}_{i\in I}$ forms a Hölder family of functions on J if and only if there exist $C > 0$ and $\beta > 0$ such that*

$$\mathrm{osc}\left(\log\left(\frac{d\nu\circ\phi_{\tau_1}^{-1}}{d\nu}\right)\right) \leq C\mathrm{e}^{-\beta|\tau|}$$

for every $\tau \in I^*$, where the map $\phi_{\tau_1}^{-1}$ is treated as defined on $\phi_\tau(J)$.

Proof. Suppose first that $\mathcal{D}$ is a Hölder family of functions of some order β on $\overline{J}$. Fix $\tau \in I^*$. Then for all $x, y \in \phi_\tau(J)$, say $x = \phi_\tau(x')$, $y = \phi_\tau(y')$, $x', y' \in J$, we have

$$\begin{aligned}\left|\log\left(\frac{d\nu\circ\phi_{\tau_1}^{-1}}{d\nu}(y)\right) - \log\left(\frac{d\nu\circ\phi_{\tau_1}^{-1}}{d\nu}(x)\right)\right| &= |\log((D_\nu^{\tau_1})(\phi_{\sigma\tau}(y'))^{-1}) - \log((D_\nu^{\tau_1})(\phi_{\sigma\tau}(x'))^{-1})| \\ &= |\log((D_\nu^{\tau_1})(\phi_{\sigma\tau}(x'))) - \log((D_\nu^{\tau_1})(\phi_{\sigma\tau}(y')))| \\ &\le V_\beta(\mathcal{D})\mathrm{e}^{-\beta(|\tau|-1)} = \mathrm{e}^\beta V_\beta(\mathcal{D})\mathrm{e}^{-\beta|\tau|}\end{aligned}$$

and we are done with the first implication. The proof of the opposite implication is similar. Suppose that $\mathrm{osc}\left(\log\left(\frac{d\nu\circ\phi_{\tau_1}^{-1}}{d\nu}\right)\right) \le C\mathrm{e}^{-\beta|\tau|}$ for every $\tau \in I^*$. Fix $\tau \in I^*$ and $x, y \in J$. We then have

$$\begin{aligned}&|\log((D_\nu^{\tau_1})(\phi_{\sigma\tau}(x'))) - \log((D_\nu^{\tau_1})(\phi_{\sigma\tau}(y')))| = \\ &\qquad= \left|-\log\left(\frac{d\nu\circ\phi_{\tau_1}^{-1}}{d\nu}\right)(\phi_\tau(x)) + \log\left(\frac{d\nu\circ\phi_{\tau_1}^{-1}}{d\nu}\right)(\phi_\tau(y))\right| \\ &\qquad\le C\mathrm{e}^{-\beta|\tau|} = C\mathrm{e}^\beta\mathrm{e}^{-\beta(|\tau|-1)}\end{aligned}$$

The proof is complete. □

Now, the following proposition extends a known result from [Ca] (comp [MV]) to our situation.

Proposition 4.2. *If the system S is ω-conservative then, there exist β, C such that for every $\tau \in I^*$.*

$$\mathrm{osc}\left(\log\left(\frac{\tilde{G}\circ(\phi_{\tau_1})^{-1}}{\tilde{G}}\right)\right) < Ce^{-\beta|\tau|}.$$

on $\phi_\tau(X)\setminus\overline{J}$. In particular for every $i \in I$ and every $x \in J$ the limit

$$\lim_{z\to x}\frac{\tilde{G}\circ(\phi_i)^{-1}(z)}{\tilde{G}(z)}$$

exists, where z converges to x in $X\setminus\overline{J}$.

Proof. It is enough to check that Carleson's proof in [Ca] of the same statement for finite iterated function system can be extended for our case. For this reason the proof is omitted. □

Corollary 4.3. *There are constants $C, \beta > 0$ such that*

$$\mathrm{osc}\left(\log\left(\frac{d\nu\circ\phi_{\tau_1}^{-1}}{d\nu}\right)\right) \le C\mathrm{e}^{-\beta|\tau|}$$

for every $\tau \in I^*$.

Proof. Fixing $i \in I$ and restricting the domain to $\phi_i(X)$ we have

$$\nu = \Delta(\tilde{G}) \tag{4.1}$$

and

$$\nu \circ \phi_i^{-1} = \Delta(\tilde{G} \circ \phi_i^{-1}) \tag{4.2}$$

To finish the proof of our corollary we notice that the bounds on the ratio $\frac{\tilde{G} \circ \phi_{\tau_1}^{-1}}{\tilde{G}}$ translate automatically to the bounds on densities of corresponding measures (see [LV], appendix). □

Proposition 4.4. *If the system S is ω-conservative then, the collection $\mathcal{D} = \{\log(D_\nu^i)\}_{i \in I}$ forms a summable Hölder family of functions on J.*

Proof. It follows from Lemma 4.3 and Lemma 4.1 that $\mathcal{D}$ satisfied the condition on Hölder family of functions on J, say of order $\beta > 0$. It remains to check that the second condition for strong Hölder continuity is satisfied. And indeed, employing Theorem 2.7 we may estimate as follows.

$$\begin{aligned}\sum_{i \in I} \exp(\sup(\log(D_\nu^i))) &\leq \sum_{i \in I} \exp(\inf(\log(D_\nu^i)) + V_\beta(\mathcal{D})) = \mathrm{e}^{\mathcal{V}_\beta(\mathcal{D})} \sum_{\rangle \in \mathcal{I}} \inf(\mathcal{D}_\nu^\rangle) \\ &\leq \mathrm{e}^{V_\beta(\mathcal{D})} \sum_{i \in I} \int D_\nu^i d\nu = \mathrm{e}^{V_\beta(\mathcal{D})} \sum_{i \in I} \nu(\phi_i(\overline{J})) = \mathrm{e}^{V_\beta(\mathcal{D})} < \infty.\end{aligned}$$

The proof is complete. □

Lemma 4.5. $\mathrm{P}(\mathcal{D}) = 0$

Proof. Indeed, using Proposition 4.4 and Theorem 2.7 we get for every $n \geq 1$

$$\sum_{|\tau|=n} \exp(\sup(S_\tau(\mathcal{D}))) \asymp \sum_{|\tau|=n} \int D_\nu^\tau d\nu = \sum_{|\tau|=n} \nu(\phi_\tau(\overline{J})) = 1.$$

The proof is complete. □

Let $m_\mathcal{D}$ be the measure claimed in Theorem 1.7. Since $S_\tau(\mathcal{D}) = \mathcal{D}_\nu^\tau$ and $\nu(\phi_\tau(A)) = \int_A D_\nu^\tau d\nu$, we obtain the following theorem as an immediate consequence of Lemma 4.5, Lemma 2.11 from [HMU] (saying in our context that a Borel probability measure η on X is F-conformal if and only if $\eta(\phi_\omega(A)) \geq \int_A \exp\Big(S_\omega(F) - \mathrm{P}(F)\Big)\, d\eta$ for all $\omega \in I^*$ and for all Borel subsets A of X) and S-invariantness of ν.

Theorem 4.6. $\nu = m_\mathcal{D} = \mu_\mathcal{D}$.

In the sequel in order to simplify notation we will write $\mathrm{H}_\nu(\alpha)$ for $\mathrm{H}_{\mu_\mathcal{D}}(\alpha)$. We would like to end this section up with the following two technical results, the first proven in Theorem 3.1 of [MPU], the second being a rigidity fact from [PV].

Lemma 4.7. *Suppose that the system $S = \{\phi_i\}_{i \in I}$ is regular. Then the following two conditions are equivalent.*

(a): *For all $i \in I$ the Jacobians D_μ^i are constant on a common neighbourhood of X.*

(b): *The conformal structure on $\overline{J}$ admits a Euclidean isometries refinement so that all maps ϕ_i, $i \in I$, become affine conformal, more precisely there exists an atlas $\{\psi_t : U_t \to \mathbb{C}\}$ with open disks U_t, consisting of conformal injections such that $\bigcup_t U_t \supset \overline{J}$, all $U_t \cap U_s$ and $U_t \cap \phi_i(U_s)$ are connected and the compositions $\psi_t \circ \psi_s^{-1}$ and $\psi_t \circ \phi_i \circ \psi_s^{-1}$, respectively on $\psi_s(U_t \cap U_s)$ and $\psi_s \circ \phi_i^{-1}(U_t \cap \phi_i(U_s))$, are conformal affine with $|(\psi_t \circ \psi_s^{-1})'| \equiv 1$.*

The assumptions of the following lemma are slightly weaker than those in [PV]. However the proof (see below) goes through unchanged.

Lemma 4.8. *(harmonic rigidity) Let u, v be two non-negative subharmonic functions on a topological disk B. Suppose that K is a compact uniformly perfect subset of B contained in a real-analytic curve and that the 1-dimensional Hausdorff measure of K vanishes. Suppose also that u and v are positive and harmonic on $B \setminus K$ and both vanish on K. If $H = \frac{d\Delta u}{d(\Delta v)}$ has a real-analytic extension on B, then H=constant.*

Proof. (sketch following [PV]) One can assume that K is contained in the real line. Also, one can symmetrize u and v to get $u(\bar{z}) = u(z)$, $v(\bar{z}) = v(z)$.

Now, since K is real, H can be extended to a complex-analytic function defined on some ball B containing points of K. We denote this extension again by H. Let $I = B \cap K$.

Consider the function w_1 in B given by the formula

$$w_1 = \partial u - H \partial v.$$

The crucial observation is that w_1 is holomorphic outside K and its distributional derivative $\bar{\partial} w_1 = \Delta u - H \Delta v = 0$. Thus, w_1 is holomorphic in B. Similarly, $w_2 = \bar{\partial} u - \bar{H} \bar{\partial} v$ is anti-holomorphic. Consider now the function $W = u - Hv$. It is obviously continuous. We shall show that $W|_I$ is C^1.

First notice that the function W is smooth in the set $I \setminus K$ and its derivative is equalto $W' = u' - H'v - Hv' = \partial u - H'v - H\partial v = w_1 - H'v$ (since we have assumed that u and v are symmetric). Thus, W' exists a.e in I and extends continuously to I.

In order to check that W is really C^1 it is enough to verify that $W|_I$ is absolutely continuous. In fact, it is even Lipschitz-continuous. This can be verified as follows.

$$W(a + i\varepsilon) - W(b + i\varepsilon) = \int_a^b \partial_x W = \int_a^b \partial W + \bar{\partial} W = \int_a^b w_1 + w_2 + (\bar{H} - H)\bar{\partial} v - \partial H v.$$

In order to see that this last integral can be estimated from above by $\mathrm{const}(b - a)$ observe that

$$|\bar{\partial} v(z)| \leq \frac{\mathrm{const}}{\mathrm{dist}(z, K)}.$$

Letting then $\varepsilon \to 0$ this shows that $W|_I$ is Lipschitz-continuous, thus C_1. Now, since $W|_{|K} = 0$, every point in K can be approximated by points satisfying $W'(x) = 0$ (there is at least one such point in each component of $I \setminus K$). By C^1 property, this implies that $W' = 0$ in K. Since $W' = w_1$ on K, we conclude that $w_1 = 0$ on K which implies $w_1 \equiv 0$. This, in turn, implies that for every point x in $I \setminus K$ such that $W'(x)$=0 we have $H'(x)v(x) = 0$, thus

$H'(x) = \partial H(x) = 0$. Since the set of these points has an accumulation point in B and H is holomorphic, H is constant in B. $\square$

Remark 4.9. *This proof heavily relies on the fact that K is contained in the real line. In general, if K is the limit set of an expanding repeller then using the fact that H is real analytic (in fact: C^∞ is enough) one can prove (in another way!) that W is $C^{1+\varepsilon}$ for some $\varepsilon > 0$ in B. But this is not sufficient to finish the proof in the non-real case.*

5. **Results and Proofs**

Recall that ν is the invariant measure equivalent to ω and μ is the invariant measure equivalent to the h–conformal measure, where $h = HD(J)$. We start our considerations with the linear case.

Theorem 5.1. *If the system S is regular, ω-conservative, $\mathrm{H}_\mu(\alpha), \mathrm{H}_\nu(\alpha) < \infty$, $\overline{J}$ is uniformly perfect, and all the maps $\{\phi_i\}_{i\in I}$ are affine (similarities), then* $\mathrm{HD}(\omega) < \mathrm{HD}(J)$.

Proof. Suppose on the contrary that $\mathrm{HD}(\omega) = \mathrm{HD}(J)$. Then $\mathrm{HD}(\nu) = \mathrm{HD}(J)$, where ν is the invariant measure produced by Theorem 2.7. It then follows from Theorem 1.13 that $\nu = \mu$. Recall that we have built the invariant measure ν as $\nu = \Delta(\tilde{G})$, $\tilde{G} \in \mathcal{G}$. Fix $W_i = \phi_i(W)$ for some $i \in I$. Consider two subharmonic functions in W: $|\phi_i'|^h \cdot \tilde{G}$ and $\tilde{G} \circ \phi_i$. The first function is subharmonic since $|\phi_i'|$ is constant in W. The Riesz measure of the first function is $|\phi_i'|^h \cdot \nu$, while for the second one we get $\nu \circ \phi_i$. By our assumption these two measures coincide. It follows from Riesz representation theorem that there exists a harmonic function H_i in W such that

$$\tilde{G} \circ \phi_i = |\phi_i'|^h \cdot \tilde{G} + H_i \tag{5.1}$$

But both $\tilde{G}$ and $\tilde{G} \circ \phi_i$ are equal to 0 in $\bar{J}$. This means that either

- **(1):** $\tilde{G} \circ \phi_i = |\phi_i'|^h \tilde{G}$ for all $i \in I$ or
- **(2):** there exists i so that $\tilde{G} \circ \phi_i = |\phi_i'|^h \cdot \tilde{G} + H_i$ for some harmonic function H_i and, consequently $\bar{J}$ is contained in the analytic set $H_i = 0$.

The case (1) is impossible, i. e. there is no function $u \in \mathcal{G}$ such that $u \circ \phi_i = |\phi_i'|^h u$. This was already observed in [Vo]. Namely, the equation (1) allows us to extend $\tilde{G}$ to the whole plane. It will satisfy (5.1). The set of zeros of this extended function contains the image of $\bar{J}$ under the group Γ generated by all maps ϕ_i. But (see [Vo]) this group contains arbitrary small shifts. On the other hand, $\tilde{G} \neq 0$ in $W \setminus \bar{J}$. Thus, this case is ruled out.

In the case (2), $\bar{J}$ is contained in l_i, a finite union of analytic curves (the set of solutions to the equation $H_i = 0$). Let $x_i \in J$ be the only fixed point of the contraction ϕ_i. The set l_i forms around x_i an analytic curve since otherwise the branching points of l would spread over a dense subset of J. Therefore for all n large enough $l_i \cap \phi_i^n(X)$ is an analytic curve. But then $\overline{J} \subset \phi_i^{-n}(l_i \cap \phi_i^n(X))$ which is also an analytic curve. So, $(\phi_i^n)'(x_i)$ is a real number and this curve must be a segment of a straight line. Without loss of generality we may assume that this segment is contained in the real line. Let

$$\bar{G}(z) = \tilde{G}(z) + \tilde{G}(\overline{z}).$$

Since by symmetry $H_i(\overline{z}) = -H_i(z)$, we get

$$\bar{G}(\phi_i(z)) = \tilde{G}(\phi_i(z)) + \tilde{G}(\phi_i(\overline{z})) = |\phi_i'|^h\tilde{G}(z) + |\phi_i'|^h\tilde{G}(\overline{z}) + H_i(z) + H_i(\overline{z}) = |\phi_i'|^h\bar{G}(z) \tag{5.2}$$

Notice that the equation 5.2 is satisfied also with ϕ_i replaced by any ϕ_j, $j \in I$, since we have

$$\tilde{G} \circ \phi_j = |\phi_j'|^h \cdot \tilde{G} + H_j$$

and $H_{j_{|l_i}} = 0$. The conclusion is that, in this way, we have reduced the case (2) to the case (1), which has been already excluded. □

If $\tilde{\eta}$ is a Borel probability shift-invariant measure on I^∞, the by $\mathrm{h}_{\tilde{\eta}}$ we mean the Sinai-Kolmogorov entropy of the dynamical system $\sigma : I^\infty \to I^\infty$ with respect to the measure $\tilde{\eta}$. If η is a measure on the limit set such that $\eta\circ\pi^{-1} = \tilde{\eta}$, then we also write $\mathrm{h}_\eta = \mathrm{h}_{\tilde{\eta}}$. In Section 6 we will replace (using some inducing procedure) the original non-hyperbolic dynamics by the infinite iterated function system. We do not know if this new system is regular. Also, there is an invariant measure equivalent with conformal measure, but we do not know if its entropy is finite. For this reason, we prove now two technical theorems below. They will be used in Section 6.

Theorem 5.2. *If the system S is irregular and $\mathrm{H}_\nu(\alpha) < \infty$ or $\chi_\nu < \infty$, then $\mathrm{HD}(\omega) < \mathrm{HD}(J)$.*

Proof. If $\mathrm{H}_\nu(\alpha) < \infty$ and $\chi_\nu = \infty$, then $\mathrm{HD}(\omega) = \mathrm{HD}(\nu) = 0 < h := \mathrm{HD}(J)$ by Theorem 4.3.2 from [MU4]. So, we may assume that $\chi_\nu < \infty$. Since the system S is irregular, then by Theorem 3.21 in [MU1], $\mathrm{P}(h) < 0$. By the former fact, we can apply Theorem 1.5 in [MU3] (comp. Theorem 2.1.6 in [MU4]), a version of variational principle, to get $\mathrm{P}(h) \geq \mathrm{h}_\nu - h\chi_\nu$. By the same reason Theorem 4.3.2 from [MU4] is applicable and we consequently obtain.

$$\mathrm{HD}(\omega) = \mathrm{HD}(\nu) = \frac{\mathrm{h}_\nu}{\chi_\nu} \leq h + \frac{\mathrm{P}(h)}{\chi_\nu} < h.$$

The proof is complete. □

Let us now deal with the next case.

Theorem 5.3. *If the system S is regular, ω-conservative, $\int |\log|\phi_{\tau_1}'(\pi(\sigma(\tau))|^k d\tilde{\nu}(\tau) < \infty$ for some real $k > 2$, but the entropy h_μ is infinite, and the limit set J is uniformly perfect, then $\mathrm{HD}(\omega) < \mathrm{HD}(J)$.*

Proof. Suppose on the contrary that $\mathrm{HD}(\omega) = \mathrm{HD}(J)$ and denote this common value by t. Since, the system S is ω-conservative, it follows from Proposition 4.4 that the collection $\mathcal{D} = \{\log(D_\nu^i)\}_{i\in I}$ forms a summable Hölder family of functions on J and from Theorem 4.6

that $\nu = m_{\mathcal{D}} = \mu_{\mathcal{D}}$. Let ψ be the amalgamated function of the family $\mathcal{D}$, i.e. $\psi(\tau) = \log(D_\nu^{\tau_1}(\pi(\sigma\tau))$. Our assumption imply $h_{\tilde{\nu}}(\sigma) = -\int \log\psi d\tilde{\nu} < \infty$ and since $h_{\tilde{\mu}}(\sigma) = \infty$, the measures $\tilde{\nu}$ and $\tilde{\mu}$ do not coincide. let $\rho : I^\infty \to I\!R$ be given by the formula

$$\rho(\tau) = \psi(\tau) - t\log|\phi'_{\tau_1}(\pi(\sigma\tau))|.$$

Then

$$\int \rho d\tilde{\nu} = \int \psi d\tilde{\nu} - \mathrm{HD}(\nu)\chi_\nu = h_{\tilde{\nu}}(\sigma) - h_{\tilde{\nu}} = 0.$$

Since $\int |\log|\phi'_{\tau_1}(\pi(\sigma(\tau))|^k d\tilde{\nu}(\tau) < \infty$ for some real $k > 2$ and since $\tilde{\nu} \neq \tilde{\mu}$, it follows from Theorem 6.4 in [Ur] that $\hat{\sigma}^2 > 0$, where

$$\hat{\sigma}^2 = \lim_{n\to\infty} \int \left(\sum_{i=0}^{n-1} \rho\circ\sigma^i\right)^2 d\tilde{\nu}.$$

Then (see for ex. Section 5 of [Ur]) the process $\{\rho\circ\sigma^n\}_{n=0}^\infty$ satisfies the central limit theorem meaning that

$$\frac{\sum_{i=0}^{n-1}\rho\circ\sigma^i}{\hat{\sigma}\sqrt{n}} \to \mathcal{N}(0,1)$$

in distribution, where $\mathcal{N}(0,1)$ is the normalized Gaussian distribution. This implies that for every $A > 0$

$$\lim_{n\to\infty} \tilde{\nu}\left(\left\{\tau \in I^\infty : \sum_{i=0}^{n-1} \rho\circ\sigma^i(\tau) < -A\hat{\sigma}\sqrt{n}\right\}\right) = c$$

for some positive constant $c = c(A)$. Since ν is supported on J and $J = \bigcup_{i\in I}\phi_i(J)$ there exists a finite subset F of I such that

$$\nu\left(\bigcup_{i\in F}\phi_i(J)\right) > 1 - \frac{1}{4}c.$$

Thus, for every n large enough there exists at least one $i = i(n) \in F$ such that

$$\tilde{\nu}\left(\left\{\tau \in I^\infty : \tau_n = i \text{ and } \sum_{i=0}^{n-1} \rho\circ\sigma^i(\tau) < -A\hat{\sigma}\sqrt{n}\right\}\right) \geq \frac{1}{2}c\nu(\phi_i(X)).$$

(Indeed, otherwise we would have

$$\tilde{\nu}\left(\left\{\tau \in I^\infty : \sum_{i=0}^{n-1} \rho\circ\sigma^i(\tau) < -A\hat{\sigma}\sqrt{n}\right\}\right) \leq \frac{1}{2}c\sum_{i\in F}\tilde{\nu}\{\tau \in I^\infty : \tau_n = i\} + \frac{1}{4}c$$

Now, the right hand side of the above inequality can be estimated form above by $\frac{1}{2}c\cdot\frac{3}{2} = \frac{3}{4}c$. Since, for large n, the left hand side is close to c, this is a contradiction.)

Denote the set of $\tau's$ appearing in this formula by $Z_{i,n}$. We are going to define a collection of disjoint discs . For every $\tau \in Z_{i,n}$ consider the corresponding map $\phi_{\tau|n} = \phi_{\tau_1}\circ\cdots\circ\phi_{\tau_n} = \phi_{\tau_1}\circ\cdots\circ\phi_i$ (since $\tau_n = i$). Composing with ϕ_i we get the map $\phi_u = \phi_i\circ\phi_{\tau_1}\circ\cdots\circ\phi_i$ Obviously, $\phi_u(X) \subset \phi_i(X)$.

In this way, we have chosen a family of maps ϕ_u. Denote it by $G_{i,n}$. It follows from our construction that

$$\nu\left(\bigcup_{u\in G_{i,n}}\phi_u(X)\right)\ge \frac{1}{2}c\nu(\phi_i(X))\inf D_\nu^i. \tag{5.3}$$

Since $\nu(\phi_u(X))\asymp \exp\left(\sum_{j=0}^n\psi\circ\sigma^j(\tau)\right)$ for any $\tau\in I^\infty$ such that $\tau|_{n+1}=u$, we get for every $u\in G_{i,n}$ and corresponding $\tau\in Z_{i,n}$ that

$$\begin{aligned}\frac{||\phi'_{\sigma(u)}||^t}{\nu(\phi_u(X))}&\ge const\exp\left(-\sum_{j=0}^{n-1}\rho\circ\sigma^j(\tau)\right)\frac{\inf\{|\phi_i'(\tau)|\}^t}{\sup\{D_\nu^i\}}\\&\ge const\exp\left(-\sum_{j=0}^{n-1}\rho\circ\sigma^j(\tau)\right)\min_{i\in F}\left\{\frac{\inf\{|\phi_i'(\tau)|\}^t}{\sup\{D_\nu^i\}}\right\}\\&\ge const\exp\Big(A\hat\sigma\sqrt{n}\Big).\end{aligned}$$

Thus, using this and (5.3) we obtain

$$\begin{aligned}\sum_{u\in G_{i(n),n}}||\phi'_{\sigma u}||^t&\ge const\exp\Big(A\hat\sigma\sqrt{n}\Big)\sum_{u\in G_{i(n),n}}\nu(\phi_u(X))\\&\ge const\frac{1}{2}c\min_{k\in F}\{\nu(\phi_k(X))\inf\{D_\nu^k\}\}\exp\Big(A\hat\sigma\sqrt{n}\Big)\\&=C\exp\Big(A\hat\sigma\sqrt{n}\Big)\end{aligned}\tag{5.4}$$

for an appropriate constant $C>0$. Consider now a new iterated function system $S_n=\{\phi_v:\phi_{i(n)}(X)\to\phi_{i(n)}(X)\}_{v\in\sigma(G_{i(n),n})}$ where n is so large that $C\exp\Big(A\hat\sigma\sqrt{n}\Big)>K^t$ and K is the distortion constant of the system S. Then (comp. the proof of Proposition 10 in [Zd1]) using (5.4) we get

$$\begin{aligned}\mathrm{P}_{S_n}(t)&=\lim_{q\to\infty}\frac{1}{q}\log\sum_{\tau\in\sigma(G_{i(n),n})^q}||\phi'_\tau||^t\\&\ge\lim_{q\to\infty}\frac{1}{q}\log\sum_{\tau\in\sigma(G_{i(n),n})^q}K^{-tq}||\phi_\tau|_n'||^t\cdot||\phi_\tau|'_{n+1,2n}||^t\cdot\ldots\cdot||\phi_\tau|'_{(q-1)n+1,qn}||^t\\&=\lim_{q\to\infty}\frac{1}{q}\log\left(\left(K^{-t}\sum_{v\in\sigma(G_{i(n),n})}||\phi'_v||^t\right)^q\right)\ge\lim_{q\to\infty}\log\left(K^{-t}C\exp\Big(A\hat\sigma\sqrt{n}\Big)\right)\\&=\log\left(K^{-t}C\exp\Big(A\hat\sigma\sqrt{n}\Big)\right)>0.\end{aligned}$$

Hence $\mathrm{HD}(J(f))\ge\mathrm{HD}(J_S)\ge\mathrm{HD}(J_{S_n})>t$ (the system S_n is finite) and the proof is complete. □

Let us recall that the system S is 1-dimensional if X is contained in a closed real-analytic arc $M \subset X$ such that $\phi_i(M) \subset M$ for all $i \in I$. Then of course the limit set J_I is contained in M. We shall now prove our main result about 1-dimensional systems by reducing it to the linear case above.

Theorem 5.4. *If a* 1*-dimensional system* S *is regular,* ω*-conservative,* $\mathrm{H}_\mu(\alpha), \mathrm{H}_\nu(\alpha) < \infty$*, and* $\overline{J}$ *is uniformly perfect, then* $\mathrm{HD}(\omega) < \mathrm{HD}(J)$.

Proof. Suppose on the contrary that $\mathrm{HD}(\omega) = \mathrm{HD}(J)$. Then $\mathrm{HD}(\nu) = \mathrm{HD}(J)$, where ν is the invariant measure produced by Theorem 2.7. It then follows from Theorem 1.13 that $\nu = \mu$. Hence, in view of Theorem 1.14 all the Jacobians $D_\nu^i : J \to (0, \infty)$ have a real-analytic extension on a common open neighbourhood of X. Decreasing it if necessary we may assume that this neighbourhood is a topological disk. From Theorem 2.7 we get the following.

$$\frac{d\Delta(\tilde{G} \circ \phi_i)}{d\Delta\tilde{G}}(z) = \frac{d(\Delta\tilde{G} \circ \phi_i)}{d\Delta\tilde{G}}(z) = \frac{d(\nu \circ \phi_i)}{d\nu}(z) = D_\nu^i(z).$$

Since the system S is conservative, $\omega(\overline{J} \setminus J) = 0$. Denote the harmonic measure in $\mathbb{C} \setminus M$ by ω_M. Obviously $\omega_M(\overline{J} \setminus J) \le \omega(\overline{J} \setminus J) = 0$ and ω_M is equivalent with the 1-dimensional Hausdorff measure H_1 on M. Hence $\mathcal{H}_1(\overline{J} \setminus J) = 0$. Since, $\mathrm{HD}(J) < 1$ by Theorem 4.5 from [MU1], and consequently $\mathcal{H}_1(J) = 0$, we conclude that $\mathcal{H}_1(\overline{J}) = 0$. We therefore infer from Lemma 4.8 with $u = \tilde{G} \circ \phi_i$ and $v = \tilde{G}$ that all the functions D_ν^i, $i \in I$, are constant on X. So, we can apply Lemma 4.7 to obtain an atlas $\{\psi_t : U_t \to \mathbb{C}\}_{t \in T}$ consisting of holomorphic maps with the "affine" properties listed there. Fix now $x \in J$, choose $s \in T$ such that $x \in U_s$ and then $\rho \in I^*$ such that $x \in \phi_\rho(J) \subset \phi_\rho(V) \subset U_s$. Consider now the iterated function system

$$S_\rho = \{\psi_s \circ \phi_\rho \circ \phi_i \circ \phi_\rho^{-1} \circ \psi_s^{-1}\}_{i \in I},$$

where the role of X is played by $\psi_s(\phi_\rho(X))$ and the role of V is played by $\psi_s(\phi_\rho(V))$. It follows from Lemma 4.7 that each map $\psi_s \circ \phi_\rho \circ \phi_i \circ \phi_\rho^{-1} \circ \psi_s^{-1}$, $i \in I$, is affine on each sufficiently small neighbourhood of each point of $\psi_s(\phi_\rho(\overline{J}))$. Hence, as holomorphic, this map must be affine on the whole connected domain $\psi_s(\phi_\rho(V))$. Therefore, due to Proposition 5.1, $\mathrm{HD}(\omega_\rho) < \mathrm{HD}(J_{S_\rho})$, where ω_ρ is the harmonic measure of the domain $\mathbb{C} \setminus \overline{J_{S_\rho}}$. Since $J_{S_\rho} = \psi_s(\phi_\rho(J))$ and since $\overline{J_{S_\rho}} = \psi_s(\phi_\rho(\overline{J}))$, we get $\mathrm{HD}(J) = \mathrm{HD}(J_{S_\rho})$ and $\mathrm{HD}(\omega) = \mathrm{HD}(\omega_\rho)$. Thus $\mathrm{HD}(\omega) < \mathrm{HD}(J)$ and the proof is complete. $\square$

We would like to notice that we needed 1-dimensionality of the system S only to apply Lemma 4.8 (harmonic rigidity).

Let $\theta(S) = \inf\{t \ge 0 : \mathrm{P}(t) < \infty\}$. Following [MU1] the system S is called hereditarily regular if $\mathrm{P}(\theta(S)) = \infty$. Each system hereditarily regular is regular and it is easy to verify (see the proof of Corollary 3.25 in [MU1]) $\mathrm{H}_\mu(\alpha) < \infty$ for such a system. Therefore, as an immediate consequence of Theorem 5.4, we get the following.

Corollary 5.5. *If a* 1*-dimensional system* S *is hereditarily regular,* ω*-conservative,* $\mathrm{H}_\nu(\alpha) < \infty$ *and* $\overline{J}$ *is uniformly perfect, then* $\mathrm{HD}(\omega) < \mathrm{HD}(J)$.

The remainder of this section is devoted to describe the finer structure of harmonic measure in the spirit of [PUZ,I,II]. In order to do it we need some short preparations. We say that a function $h : [1, \infty) \to (0, \infty)$ is slowly growing if $h(t) = o(t^\alpha)$ for all $\alpha > 0$. A slowly growing function h is said to belong to the lower class if

$$\int_1^\infty \frac{h(t)}{t} \exp\Big(-\frac{1}{2}h(t)^2\Big)dt < \infty$$

and to the upper class if

$$\int_1^\infty \frac{h(t)}{t} \exp\Big(-\frac{1}{2}h(t)^2\Big)dt = \infty.$$

It is known (see the beginning of Section 6 of [Ur]) that if $\int |d_\nu|^k d\tilde\nu < \infty$ for some real $k > 2$, where d_ν is the amalgamated function generated by the summable Hölder family $\{\log D_\nu^i\}_{i\in I}$, then

$$\hat\sigma^2 = \hat\sigma^2(d_\nu) = \int_{I^\infty} (d_\nu - \tilde\nu(d_\nu))^2 \, d\tilde\nu + 2\sum_{n=1}^\infty \int_{I^\infty} (d_\nu - \tilde\nu(d_\nu))(d_\nu \circ \sigma^n - \tilde\nu(d_\nu)) \, d\tilde\nu < \infty.$$

Given a function $h : [1, \infty) \to (0, \infty)$ define then for all sufficiently small $t > 0$

$$\tilde h(t) = t^\kappa \exp\left(\frac{\hat\sigma}{\sqrt{\chi_\nu}} h(-\log t)\sqrt{-\log t}\right),$$

where $\kappa = \mathrm{HD}(\nu) = \mathrm{HD}(\omega)$. Finally given a function $g : [0, \epsilon) \to [0, \infty)$, continuous at 0, by $\mathcal{H}^g$ we denote the generalized Hausdorff measure with the gauge function g. As an immediate consequence of Theorem 6.3 from [Ur] along with [Ur, Theorem 6.4], Theorem 5.4 and Theorem 4.6 we get the following.

Theorem 5.6. *Let S be a regular 1-dimensional ω-conservative system such that $\mathrm{H}_\mu(\alpha) < \infty$, and $\int |d_\nu|^k d\tilde\nu < \infty$ for some real $k > 2$. Suppose in addition that $\overline{J}$ is uniformly perfect. If $h : [1, \infty) \to (0, \infty)$ is a slowly growing function, then*

- **(a):** *If h belongs to the upper class, then the measures ω and $\mathcal{H}^{\tilde h}$ on J are singular.*
- **(b):** *If h belongs to the lower class, then ω is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^{\tilde h}$.*

In particular ω and $\mathcal{H}^\kappa$ are singular.

6. Examples

Definition 6.1. *(conformal expanding repellers) Let $Y \subset \overline{\mathbb{C}}$ be a topological Cantor set and let $W \supset Y$ be an open set. A holomorphic map $f : W \to \overline{\mathbb{C}}$ is said to be a conformal expanding repeller if the following conditions are satisfied.*

- **(6.c1):** *$f(Y) = Y = f^{-1}(Y)$.*
- **(6.c2):** *There exists $p \geq 1$ such that $\inf\{|(f^p)'(z)| : z \in Y\} > 1$.*
- **(6.c3):** *$f : Y \to Y$ is topologically transitive, meaning that there exists a point $y \in Y$ such that $\overline{\{f^n(y) : n \geq 0\}} = Y$.*

Frequently the name conformal expanding repeller is attributed also to the set Y.

The best known examples of conformal expanding repellers are hyperbolic rational functions with Y being the Julia sets. For a systematic treatment of these repellers the reader may consult the book [PU]. The uniform perfectness of Y is a straightforward consequence of the bounded distortion theorem which can be proved in this context. We will however provide a different simple indirect proof whose idea is applicable also in other contexts.

Theorem 6.2. *Each conformal expanding repeller (not necessarily homeomorphic with a Cantor set) is uniformly perfect.*

Proof. Let $h = \mathrm{HD}(Y)$. It is well-known (see [PU] for a proof for instance) that the h-dimensional Hausdorff measure $\mathcal{H}^h$ on Y is finite, positive and there exists a constant $C \geq 1$ such that

$$C^{-1} \leq \frac{\mathcal{H}^h(B(x,r))}{r^h} \leq C \tag{6.1}$$

for every $x \in Y$ and every $0 < r \leq 1$. Fix now $x \in Y$ and $0 < r \leq 1$. We then get

$$\begin{aligned} \mathcal{H}^h(A(x,(2C^2)^{-1/h}r,r)) &= \mathcal{H}^h(B(x,r)) - \mathcal{H}^h(B(x,(2C^2)^{-1/h}r)) \\ &\geq C^{-1}r^h - C((2C^2)^{-1/h}r)^h = C^{-1}r^h - \frac{1}{2}C^{-1}r^h \\ &= \frac{1}{2}C^{-1}r^h > 0. \end{aligned} \tag{6.2}$$

In particular $A(x,(2C^2)^{-1/h}r,r) \cap Y \neq \emptyset$ and the proof is complete. $\square$

We shall now temporarily restrict our attention to the special class of conformal expanding repellers mentioned above, namely to hyperbolic rational functions. Surprisingly enough, checking the requirements of the previous sections for general conformal expanding repellers turns out much more difficult than for the class of hyperbolic rational functions. The crucial fact is provided by the following.

Proposition 6.3. *If $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a hyperbolic rational function such that the Fatou set $\overline{\mathbb{C}} \setminus J(f)$ is connected, then there exists a closed topological Jordan disk $X \supset \mathrm{Int}X \supset J(f)$ with a piecewise smooth boundary such that $f^{-1}(X) \subset \mathrm{Int}X$ and X is disjoint from the closure of the forward orbit of all critical points of f.*

Proof. Since the Fatou set $\overline{\mathbb{C}} \setminus J(f)$ is connected and since f is hyperbolic, $\overline{\mathbb{C}} \setminus J(f)$ is the basin of immediate attraction to an attracting fixed point a. Therefore there exists a closed topological Jordan disk $B \subset \overline{\mathbb{C}} \setminus J(f)$ with a piecewise smooth boundary (in this stage B can be chosen to be a geometric disk) such that

$$f(B) \subset \mathrm{Int}B.$$

Fix c, a critical point of f. Since $\lim_{n\to\infty} f^n(c) = a$, there exists $q \geq 0$ such that $f^n(c) \in \mathrm{Int}B$ for every $n \geq q$. Choose γ_0, a closed topological smooth arc (homeomorphic with a closed segment of the real line) with the following properties.

(a): The initial point of γ_0 is $f^q(c)$ and its terminal point is $f^{q+1}(c)$.

(b): $\gamma_0 \cap \bigcup_{n\ge 1} f^n(\gamma_0) = \emptyset$. (This can be done by taking q large enough and looking at the linearized coordinates.)

(c): Except for $f^q(c)$ and $f^{q+1}(c)$ γ_0 contains no other critical values of any order.

Then, even though c is a critical point, we can define by induction a sequence $\{\gamma_i\}_{i=1}^q$ of topological smooth arcs contained in $\overline{\mathbb{C}}\setminus J(f)$ and such that $f(\gamma_i) = \gamma_{i-1}$ for all $1 \le i \le q$, the initial point of γ_i is $f^{q-i}(c)$ and the terminal point of γ_i is $f^{q-(i+1)}(c)$. Since f is continuous and analytic we can define by a straightforward induction with respect to $i = 1, \dots, q$, a sequence F_i, $i = 1, \dots, q$, of closed Jordan disks with smooth boundary such that

$$\gamma_i \subset \mathrm{Int} F_i \text{ for every } i = 1, \dots, q,$$

$$f(F_1) \subset \mathrm{Int} B,$$

$$f(F_i) \subset \mathrm{Int}(F_{i-1}) \text{ for every } i = 2, \dots, q$$

and $\bigcup_{j=1}^i F_j \cup B$ is a closed topological Jordan disk with a piecewise smooth boundary for every $i = 1, \dots, q$ (this is due to the fact that the family $\{\gamma_i\}_{i=1}^q$ has no points of intersection except endpoints of the curves g_i which in turn follows from property (b). Hence

$$\begin{aligned} f\left(B \cup \bigcup_{i=1}^q F_i\right) &\subset f(B) \cup \bigcup_{i=1}^q f(F_i) \subset \mathrm{Int} B \cup \mathrm{Int} B \cup \bigcup_{i=2}^q \mathrm{Int}(F_{i-1}) \\ &\subset \mathrm{Int} B \cup \bigcup_{i=2}^q \mathrm{Int}(F_{i-1}) \subset \mathrm{Int}\left(B \cup \bigcup_{i=1}^q F_i\right). \end{aligned}$$

Since $B \cup \bigcup_{i=1}^q F_i \subset \overline{\mathbb{C}} \setminus J(f)$ is a closed topological Jordan disk with a piecewise smooth boundary, we can repeat the above construction with another critical point of f and the disk B replaced by $B \cup \bigcup_{i=1}^q F_i$. Moving on with this procedure inductively over all critical points of f, we finally end up with a closed topological Jordan disk $Y \subset \overline{\mathbb{C}} \setminus J(f)$ with a piecewise smooth boundary such that $f(Y) \subset \mathrm{Int} Y$ and $\{f^n(c) : c \in \mathrm{Crit}(f), n \ge 0\} \subset \mathrm{Int} Y$. Therefore

$$(\overline{\mathbb{C}}\setminus \mathrm{Int} Y) \cap \{f^n(c) : c \in \mathrm{Crit}(f), n \ge 0\} = \emptyset,$$

$$\overline{\mathbb{C}}\setminus \mathrm{Int} Y \supset J(f),$$

and

$$f^{-1}(\overline{\mathbb{C}}\setminus \mathrm{Int} Y) = \overline{\mathbb{C}}\setminus f^{-1}(\mathrm{Int} Y) \subset \overline{\mathbb{C}}\setminus Y \subset \mathrm{Int}\Big(\overline{\mathbb{C}}\setminus \mathrm{Int} Y\Big).$$

Since, in addition, $\overline{\mathbb{C}} \setminus \mathrm{Int} Y$ is a closed topological Jordan disk with a piecewise smooth boundary, the proof is complete by taking $X = \overline{\mathbb{C}}\setminus \mathrm{Int} Y$. □

Theorem 6.4. *If $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a hyperbolic rational function such that the Fatou set $\overline{\mathbb{C}}\setminus J(f)$ is connected and the Julia set is contained in a real-analytic curve, then for every rational function g sufficiently close to f, $\mathrm{HD}(\omega) < \mathrm{HD}(J(g))$, where ω is the harmonic measure on $J(g)$.*

Proof. Since $J(f)$ is a conformal expanding repeller, it follows from Theorem 6.2 that $J(f)$ is uniformly perfect (this is in fact known for all Julia sets, see [CG] for ex.). Let $d = \deg(f)$ be the degree of f. In view of Proposition 6.3 all the holomorphic inverse branches $f_1^{-1}, \dots, f_d^{-1} : X \to \mathrm{Int}X$ of f are well-defined on X. Since additionally all the sets $f_i^{-1}(X)$, $i = 1, \dots, d$, are mutually disjoint, $S = \{f_i^{-1}\}_{i=1}^d$ is actually a conformal iterated function system satisfying conditions (1.1) and (1.2). We wrote "actually" since the maps f_i^{-1}, $i = 1, \dots, d$, do not have to be contractions. Due to the Koebe distortion theorem and a simple area argument this can be however remedy by fixing n large enough and considering the system S_n consisting of all compositions of n mappings from the system S. Since the system S_n is finite, the entropies $\mathrm{H}_\mu(\alpha)$ and $\mathrm{H}_\nu(\alpha)$ are finite. Since $J(f)$ obviously coincides with the limit set J_S, the proof is completed for the function f itself by invoking Theorem 5.4. To see this for g sufficiently close to f notice that the Hausdorff dimension depends continuously on g (use Bowen's formula ([Bo] and [PUZ,I] and J-stability of f) and also that the Hausdorff dimension of harmonic measure depends continuously on g (see [B1], [B2]). $\square$

Remark that having Proposition 6.3 one could also use results from [Vo] to establish this theorem for f.

Let us come back to the class of general conformal expanding repellers $f : Y \to Y$. It is known (see [PU]) that the map $f : Y \to Y$ admits Markov partitions of sufficiently small diameters. For us it means that there exists a finite cover $R = \{R_t\}_{t \in T}$ of Y consisting of mutually disjoint closed disks such that $\partial R_t \cap Y = \emptyset$ for every $t \in T$. Moreover, we may require the existence of an integer $q \geq 1$ and $\delta > 0$ such that the following holds:

If $x \in Y$, say $x \in R_s$, and $f^{qn}(x) \in R_t$, then there exists a unique holomorphic inverse branch $f_x^{-qn} : B(R_t, 2\delta) \to \overline{\mathbb{C}}$ of f^{qn} sending $f^{qn}(x)$ to x. Moreover $f_x^{-qn}(R_t) \subset R_s$ and, taking q sufficiently large, we may require, due to (6c.2) that

$$f_x^{-qn}(B(R_t, 2\delta)) \subset \overline{B(Y \cap R_s, \delta)} \subset \mathrm{Int}(R_s). \tag{6.3}$$

For every $t \in T$ we now build recursively our conformal iterated function system S_t as a disjoint union of the families S_t^j, $j \geq 1$, as follows. S_t^1 consists of all the maps f_x^{-q}, where $x, f^q(x) \in Y \cap R_t$. Suppose that the families $S_t^1, S_t^2, \dots, S_t^{n-1}$ have been already constructed. S_t^n is composed then of all the maps f_y^{-qn} such that $y, f^{qn}(y) \in Y \cap R_t$ and $f^{qj}(y) \notin R_t$ for every $1 \leq j \leq n-1$. We shall prove the following.

Theorem 6.5. *For each $t \in T$, $S_t = \{\phi_{t,i}\}_{i \in I_t}$ is a conformal iterated function system, i.e. the conditions (1.1) and (1.2) are satisfied.*

Proof. Condition (1.1) follows immediately from (6.3). In order to prove (1.2) which is a stronger version of the open set conditions (1a), take two distinct maps f_x^{-qm} and f_y^{-qn} belonging to S_t. Without loosing generality we may assume that $m \leq n$. Suppose on the contrary that

$$f_x^{-qm}(R_t) \cap f_y^{-qn}(R_t) \neq \emptyset.$$

Then

$$\emptyset \neq f^{qm}\Big(f_x^{-qm}(R_t) \cap f_y^{-qn}(R_t)\Big) \subset R_t \cap f^{qm}(f_y^{-qn}(R_t)) = R_t \cap f_{f^{qm}(y)}^{-q(n-m)}(R_t).$$

Hence $f_{f^{qm}(y)}^{-q(n-m)}(R_t) \subset R_t$, and therefore $f^{qm}(y) \in R_t$. Due to our construction of the system S_t, this implies that $m = n$. But then $f_x^{-qn}(R_t) \cap f_y^{-qn}(R_t) = \emptyset$ since f_x^{-qn} and f_y^{-qn} are distinct inverse branches of the same map f^{qn}. This contradiction finishes the proof. □

Our next aim is to demonstrate that the systems R_t are regular. By J_t we denote the limit set of the system S_t. We will need the following.

Lemma 6.6. *If η is a Borel probability ergodic f-invariant measure on Y positive on open sets of Y, then $\eta(J_t) = \eta(Y \cap R_t) > 0$ for every $t \in T$.*

Proof. By the construction of the system S_t, the set

$$t(\infty) = \{z \in Y : \#\{n \geq 0 : f^n(z) \in R_t \cap Y\} = \infty\}$$

is contained in J_t. By Birkhoff's ergodic theorem $\eta(t(\infty)) = \eta(R_t \cap Y)$ and this number is positive since $R_t \cap Y$ is an open subset of Y. □

As we have already mentioned, the h-dimensional Hausdorff measure on Y is positive and finite. Its normalized version m (giving mass 1 to Y) is an h-conformal measure on Y in the sense that

$$m(f(A)) = \int_A |f'|^h dm$$

for every Borel set $A \subset Y$ such that $f|_A$ is injective. Moreover (see [PU]), m admits a Borel probability f-invariant ergodic measure μ equivalent with m with bounded Radon-Nikodym derivatives. In view of (6.1) m and μ are positive on open sets. We can give now a simple proof of the following.

Lemma 6.7. *For each $t \in T$ the system S_t is regular. Moreover* $\mathrm{HD}(J_t) = \mathrm{HD}(Y)(= h)$ *and the h-conformal measure m_t for S_t is equal to $\frac{1}{m(J_t)} m|_{J_t}$.*

Proof. By the previous lemma $\mu(J_t) > 0$ and, consequently, $m(J_t) > 0$. Since m is the normalized Hausdorff measure on Y, the rest of this lemma is immediate (existence of a conformal measure means regularity). □

Lemma 6.8. *For every $t \in T$, $\overline{J_t} = Y \cap R_t$.*

Proof. Obviously $\overline{J_t} \subset Y \cap R_t$. The opposite inclusion follows immediately from Theorem 6.6 and positivity of μ on non-empty open sets of Y. □

From Section 1 we know that for every $t \in T$ there exists an S_t-invariant measure equivalent with $m|_{J_t}$ (see Lemma 6.7) with bounded Radon-Nikodym derivatives. Combining this fact and the properties of the measure μ listed before Lemma 6.7, we get the following.

Lemma 6.9. *For every $t \in T$ the measures μ_t and $\mu|_{J_t}$ are equivalent, and even more*

$$\sup_{J_t}\left\{\left|\log \frac{d\mu_t}{d\mu}\right|\right\} < \infty.$$

The same reasoning as in Lemma 4.1 and Proposition 4.2 shows that for every $z \in J(f)$ there exists the limit

$$\psi(z) = \lim_{x \to z} \frac{G \circ f(x)}{G(x)}, \quad x \in \overline{\mathbb{C}} \setminus J(f),$$

its logarithm is Hölder continuous and this limit is the Jacobian of the harmonic measure ω. So, its Gibbs state provides us with a unique f-invariant probability measure ν equivalent with the harmonic measure ω. On the other hand, it follows from Theorem 2.7 that for every $t \in T$ there exists an S_t-invariant Borel probability measure ν_t on J_t equivalent with the harmonic measure on J_t. Since the map f is conformal and T is finite, we get the following.

Lemma 6.10. *For every $t \in T$ the measures ν_t and $\nu|_{J_t}$ are equivalent, and even more*

$$\sup_{J_t}\left\{\left|\log \frac{d\nu_t}{d\nu}\right|\right\} < \infty.$$

Now, we want to use Theorem 5.4. Below we check that the assumptions of this theorem are satisfied in our case.

Lemma 6.11. *If η is a Gibbs state for the map $f : Y \to Y$ and a Hölder continuous potential $\rho : Y \to I\!R$, then $\mathrm{H}_\eta(\alpha_t) < \infty$, where α_t is the partition of R_t into the sets $\{\phi_{t,i}(R_t)\}_{i \in I_t}$.*

Proof. Adding an additive constant ($= -\mathrm{P}(\rho)$), we may assume that $\mathrm{P}(\rho) = 0$. Since ρ is a Hölder continuous function, $f : Y \to Y$ is expanding and η is the Gibbs state for ρ, there exists a constant $C \geq 1$ such that for every $i \in I_t$

$$\eta(f_{x_i}^{-n_i}(R_t)) \geq C \exp\left(\sum_{j=0}^{n_i - 1} \rho \circ f^j(x_i)\right) \geq C \exp\Big(-||\rho||_\infty n_i\Big).$$

Hence

$$\begin{aligned}\mathrm{H}_\eta(\alpha_t) &= \sum_{i \in I_t} -\eta(f_{x_i}^{-n_i}(R_t)) \log\Big(\eta(f_{x_i}^{-n_i}(R_t))\Big)\\ &\leq \sum_{i \in I_t} \eta(f_{x_i}^{-n_i}(R_t))\Big(\log C + ||\rho||_\infty n_i\Big)\\ &= \log C + ||\rho||_\infty \sum_{i \in I_t} \eta(f_{x_i}^{-n_i}(R_t)) n_i.\end{aligned}$$

But, by the construction of the partition $\{f_{x_i}^{-n_i}(R_t)\}_{i \in I_t}$, each integer n_i is the first return time to the set R_t of all the points in $f_{x_i}^{-n_i}(R_t)$ under iterations of f. Hence, applying Kac's formula we get

$$\mathrm{H}_\eta(\alpha_t) \leq \log C + \frac{||\rho||_\infty}{\eta(R_t)}.$$

The proof is complete. $\square$

Recall that a conformal expanding repeller $f : Y \to Y$ is called 1-dimensional if there exists a real-analytic curve M such that $Y \subset M$. It is clear that then the systems S_t defined above are all 1-dimensional. We have now all the ingredients needed to provide a short proof of the following.

Theorem 6.12. *If $f : Y \to Y$ is a 1-dimensional conformal expanding repeller, then $\mathrm{HD}(\omega) < \mathrm{HD}(Y)$, where ω is the harmonic measure on Y.*

Proof. Since f is conformal and transitive, $\mathrm{HD}(\nu_t) = \mathrm{HD}(\omega)$ and, by Lemma 6.7, $\mathrm{HD}(J_t) = \mathrm{HD}(Y)$ for all $t \in T$. By Theorem 6.2 Y is uniformly perfect. Fix $t \in T$. It then follows from Lemma 6.8 that $\overline{J_t}$ is uniformly perfect. By Lemmas 6.10 and 6.11, the entropies $\mathrm{H}_{\nu_t}(\alpha_t)$ and $\mathrm{H}_{\mu_t}(\alpha_t)$ are both finite. Since, by Lemma 6.7, the system S_t is regular, it follows from Theorem 5.4 that $\mathrm{HD}(\omega) = \mathrm{HD}(\nu_t) < \mathrm{HD}(J_t) = \mathrm{HD}(Y)$. The proof is complete. □

We have already seen that hyperbolic rational functions with connected Fatou set provide good examples of conformal expanding repellers. Another natural class of examples is given by the limit sets of Kleinian groups of Schottky type. Here (see [Be], [Ts]) a finite set of generators $\{g_1, \dots, g_k\}$ can be found along with finitely many mutually disjoint geometric disks $D_1, \dots, D_k$ covering the limit set and such that

$$\min_{i \le k}\{\inf\{|g_i'(z)| : z \in B_i\}\} > 1.$$

Hence, as an immediate consequence of Theorem 6.12 we get the following.

Corollary 6.13. *If G is a Fuchsian group of Schottky type, and its limit set $L(G)$ is contained in a real-analytic curve, then $\mathrm{HD}(\omega) < \mathrm{HD}(L(G))$, where ω is the harmonic measure on $L(G)$.*

Following [MU2] we shall now recall the definition of parabolic iterated function systems slightly modified to fit better into our needs. We shall then prove that they satisfy the assumptions of Theorem 5.4 and afterwards, we shall show that parabolic rational functions and parabolic Fuchsian groups (respectively with the Julia sets and limit set homeomorphic with the Cantor set) can be treated as parabolic iterated function systems.

Definition 6.14. *Let X be a compact topological disk in $\overline{\mathbb{C}}$ with a piecewise smooth boundary. Suppose that we have finitely many conformal maps $\phi_i : X \to X$, $i \in I$, where I has at least two elements and the following conditions are satisfied.*

- **(6pa):** *(Strong Open Set Condition) $\phi_i(X) \cap \phi_j(X) = \emptyset$ for all $i \neq j$.*
- **(6pb):** *$|\phi_i'(x)| < 1$ everywhere except for finitely many pairs (i, x_i), $i \in I$, for which x_i is the unique fixed point of ϕ_i and $|\phi_i'(x_i)| = 1$. Such pairs and indices i will be called parabolic and the set of parabolic indices will be denoted by Ω. All other indices will be called hyperbolic.*
- **(6pc):** *$\forall n \ge 1$ $\forall \omega = (\omega_1, ..., \omega_n) \in I^n$ if ω_n is a hyperbolic index or $\omega_{n-1} \neq \omega_n$, then ϕ_ω extends conformally to an open topological disk $V \subset \overline{\mathbb{C}}$ with a piecewise smooth boundary and ϕ_ω maps V into itself.*

(6pd): *If i is a parabolic index, then $\bigcap_{n\geq 0}\phi_{i^n}(X)=\{x_i\}$ and the diameters of the sets $\phi_{i^n}(X)$ converge to 0.*

(6pe): *(Bounded Distortion Property) $\exists K\geq 1$ $\forall n\geq 1$ $\forall \omega=(\omega_1,...,\omega_n)\in I^n$ $\forall x,y\in V$ if ω_n is a hyperbolic index or $\omega_{n-1}\neq\omega_n$, then*

$$\frac{|\phi_\omega'(y)|}{|\phi_\omega'(x)|}\leq K.$$

(6pf): $\exists s<1$ $\forall n\geq 1$ $\forall\omega\in I^n$ *if ω_n is a hyperbolic index or $\omega_{n-1}\neq\omega_n$, then $||\phi_\omega'||\leq s$.*

(6pg): *(Cone Condition) There exist $\alpha,l>0$ such that for every $x\in\partial X\subset\mathbb{C}$ there exists an open cone $Con(x,\alpha,l)\subset\mathrm{Int}(X)$ with vertex x, central angle of Lebesgue measure α, and altitude l.*

(6ph): *There are two constants $L\geq 1$ and $\alpha>0$ such that*

$$\Big||\phi_i'(y)|-|\phi_i'(x)|\Big|\leq L||\phi_i'|||y-x|,$$

for every $i\in I$ and every pair of points $x,y\in V$.

(6pi): $\phi_i(X)\subset\mathrm{Int}(X)$ *for every hyperbolic element $i\in I$.*

Any system S satisfying the above conditions (6pa)-(6pi) will be called a parabolic iterated function system. Notice that conditions (6pe) and (6ph) are satisfied because of Koebe's distortion theorem.

We shall now recall from [MU2] how to associate with any parabolic iterated function system S a canonical, infinite but hyperbolic, iterated function system S^* which essentially has the same limit set as S.

Definition 6.15. *The system S^* is by definition generated by the set of maps of the form ϕ_{i^nj}, where $n\geq 1$, $i\in\Omega$, $i\neq j$, and the maps ϕ_k, where $k\in I\setminus\Omega$. The corresponding alphabet $\{i^nj:i\in\Omega,i\neq j,n\geq 1\}\cup(I\setminus\Omega)$ will be denoted by I_*.*

The following fact has been proved in [MPU] as Theorem 5.1.

Theorem 6.16. *The system S^* is a (hyperbolic) conformal iterated function system in the sense of Section 1. Increasing X a little bit to X^* we may make condition (1.1) satisfied for the system S^* (but not S) and to keep condition (1.2) satisfied for S^*.*

Note that $J_{S^*}=J_S\setminus\{\phi_\omega(x_i):i\in\Omega,\omega\in I^*\}$. By (6pa), $\overline{J_{S^*}}=\overline{J_S}=J_S$ is a topological Cantor set. Since each finite (parabolic or hyperbolic) iterated function system is regular, the following is an immediate consequence of Corollary 5.8 from [MU2].

Proposition 6.17. *The hyperbolic system S^* is regular.*

In view of Lemma 2.4 in [MU2], every parabolic point ρ lies on the boundary of X. It is easy to see that $\phi_i'(\rho)=1$ (i is the corresponding parabolic element of I) and the Taylor's series expansion of ϕ_i at ρ has the form

$$\phi_i(z)=z+a(z-\rho)^{p+1}+\dots$$

for some integer $p \ge 1$. Changing the system of coordinates via the map $\frac{1}{z-\rho}$ sending ρ to ∞, one can easily deduce that for every $j \ne i$ and for every $n \ge 1$

$$\operatorname{diam}(\phi_{i^n j}(X^*)) \asymp \operatorname{dist}\Big(\phi_{i^{n+1} j}(X^*), \phi_{i^n j}(X^*)\Big) \asymp ||\phi'_{i^n j}|| \asymp n^{-\frac{p+1}{p}}. \tag{6.4}$$

It immediately follows from (6.4) that the assumptions of Theorem 3.5 are satisfied for the system S^*. Thus, we get the following.

Theorem 6.18. *If S is a parabolic iterated function system, then $\overline{J_S}$, the closure of the limit set J_S, is uniformly perfect.*

Let
$$\alpha = \{\phi_{i^n j}(J) : i \in \Omega, i \ne j, n \ge 1\} \cup \{\phi_i(J) : i \in I \setminus \Omega\}.$$
We shall prove the following.

Lemma 6.19. *The entropy $\mathrm{H}_\mu(\alpha)$ is finite, where μ is the unique S^*-invariant measure equivalent with the h-conformal measure m for S^*.*

Proof. Since μ and m are equivalent with bounded Radon-Nikodym derivatives and since I is finite, it suffices to demonstrate that for every $i \in \Omega$ and every $j \in I \setminus \{i\}$,

$$\sum_{n\ge1} -m(\phi_{i^n j}(J)) \log\Big(m(\phi_{i^n j}(J))\Big) < \infty.$$

Since the sets $\phi_{i^n j}(J)$, $n \ge 1$, are mutually disjoint, using (6.4) and conformality of m for the system S^*, we get
$$\sum_{n\ge1} n^{-\frac{p+1}{p}h} \asymp \sum_{n\ge1} m\Big(\phi_{i^n j}(J)\Big) \le 1.$$
Hence $\frac{p+1}{p}h > 1$ and therefore

$$\sum_{n\ge1} -m(\phi_{i^n j}(J)) \log\Big(m(\phi_{i^n j}(J))\Big) \asymp \sum_{n\ge1} n^{-\frac{p+1}{p}h} \frac{p+1}{p} h \log n < \infty.$$

The proof is complete. □

Developing the calculation done in [PSV] we shall prove the following.

Lemma 6.20. *The entropy $\mathrm{H}_\nu(\alpha)$ and the Lyapunov exponent χ_ν is finite, where ν is the unique S^*-invariant measure equivalent with the harmonic measure ω on J_S.*

Proof. In view of Corollary 1.12 it is sufficient to demonstrate that the Lyapunov exponent χ_ν is finite. A straightforward calculation shows that the Lyapunov exponent χ_ν is equal to

$$\sum_{b\in I_*} \int_{\phi_b(X)} \log|(\phi_b^{-1})'| d\nu.$$

Using (6.4) we can estimate this integral from above by

$$
\begin{aligned}
&\sum_{i\in\Omega}\sum_{j\neq i}\sum_{n\geq 1}\sup_{\phi_{i^n j}(X)}\{\log|(\phi_{i^n j}^{-1})'|\}\omega(\phi_{i^n j}(J)) + const\\
&\leq \sum_{i\in\Omega}\sum_{j\neq i}\sum_{n\geq 1}\left(const + \frac{p_i}{p_i+1}\log n\right)\omega(\phi_{i^n j}(J)) + const\\
&\leq \sum_{i\in\Omega}\sum_{j\neq i}\sum_{n\geq 1}\frac{p_i}{p_i+1}\left(\sum_{k=1}^{n-1}(\log(k+1)-\log k)\omega(\phi_{i^n j}(J))\right) + const\\
&= \sum_{i\in\Omega}\sum_{j\neq i}\frac{p_i}{p_i+1}\left(\sum_{k=1}^{\infty}\left((\log(k+1)-\log k)\Big(\sum_{q=k+1}^{\infty}\omega(\phi_{i^q j}(J))\Big)\right)\right) + const\\
&= \sum_{i\in\Omega}\sum_{j\neq i}\frac{p_i}{p_i+1}\left(\sum_{k=1}^{\infty}\left(\log\left(\frac{k+1}{k}\right)\Big(\sum_{q=k+1}^{\infty}\omega(\phi_{i^q j}(J))\Big)\right)\right) + const\\
&\leq const\sum_{i\in\Omega}\sum_{j\neq i}\sum_{k=1}^{\infty}\left(\log\left(\frac{k+1}{k}\right)\omega(B(\rho_i, const\,k^{-p}))\right) + const\\
&\leq const\sum_{i\in\Omega}\sum_{j\neq i}\sum_{k=1}^{\infty}\frac{1}{k}\omega(B(\rho_i, const\,k^{-p})) + const,
\end{aligned}
$$

where ρ_i is the parabolic fixed point associated with the parabolic index i. Since $\overline{J}$ is uniformly perfect and Ω is finite, there exists $0<\kappa<1$ such that $\omega(B(\rho_i,r)) = O(r^\kappa)$. Therefore the last series in the above display converges, and consequently the Lyapunov exponent χ_ν is finite. □

We would like to remark that another method of estimating the entropy for parabolic rational functions with Julia sets contained in the real line has been proposed in [PV].

Combining now Proposition 6.17, Theorem 6.18 and Lemmas 6.19 and 6.20, as an immediate consequence of Theorem 5.4, we get the following.

Theorem 6.21. *If S is a parabolic 1-dimensional iterated function system, then* $\mathrm{HD}(\omega) < \mathrm{HD}(J_S)$*, where ω is the harmonic measure on J.*

Recall from [DU] that a rational function $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is said to be parabolic if the Julia set $J(f)$ contains no critical points but it contains at least one rationally indifferent periodic (abbr. parabolic) point. If the Fatou set $\overline{\mathbb{C}}\setminus J(f)$ is connected, then $\overline{\mathbb{C}}\setminus J(f)$ is the basin of immediate attraction to a unique parabolic point a which, in fact, is a fixed point of f. We shall prove the following.

Proposition 6.22. *If $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is a parabolic rational function such that the Fatou set $\overline{\mathbb{C}}\setminus J(f)$ is connected, then there exists a closed topological Jordan disk $X\supset \mathrm{Int}(X)\cup\{a\}\supset J(f)$ with a piecewise smooth boundary such that $f^{-1}(X)\subset \mathrm{Int}(X)\cup\{a\}$ and X is disjoint* *from the forward orbit of all critical points of f. The point a appearing here is a unique parabolic fixed point of f.*

Proof. Since $\overline{\mathbb{C}}\setminus J(f)$ is connected, f has only one petal. By the Fatou's flower theorem there exists a closed topological disk B such that $a\in\partial B$, ∂B is a Jordan curve smooth everywhere except at the point a,

$$f(B)\subset\{a\}\cup \mathrm{Int}(B),$$

and if c is a critical point of f, then $\lim_{n\to\infty}f^n(c)=a$ and the intersection $\{f^n(c):n\geq 0\}\cap(\overline{\mathbb{C}}\setminus B)$ is finite. Following now step by step the inductive construction from the proof of Proposition 6.3, we end up with $Y\subset(\overline{\mathbb{C}}\setminus J(f))\cup\{a\}$, a closed topological Jordan disk with a piecewise smooth boundary such that

$$f(Y)\subset\{a\}\cup \mathrm{Int}(Y)$$

and

$$\{f^n(c):c\in Crit(f),n\geq 0\}\subset \mathrm{Int}(Y).$$

Therefore

$$(\overline{\mathbb{C}}\setminus \mathrm{Int}(Y))\cap\{f^n(c):c\in Crit(f),n\geq 0\}=\emptyset,$$
$$\overline{\mathbb{C}}\setminus \mathrm{Int}(Y)\supset J(f),$$

and

$$\begin{aligned}f^{-1}(\overline{\mathbb{C}}\setminus \mathrm{Int}(Y))&=\overline{\mathbb{C}}\setminus f^{-1}(\mathrm{Int}(Y))\subset\overline{\mathbb{C}}\setminus(Y\setminus f^{-1}(\{a\})=\overline{\mathbb{C}}\setminus(Y\setminus\{a\})\\&=(\overline{\mathbb{C}}\setminus Y)\cup\{a\}\subset \mathrm{Int}(\overline{\mathbb{C}}\setminus \mathrm{Int}(Y))\cup\{a\}.\end{aligned}$$

Since, in addition $\overline{\mathbb{C}}\setminus \mathrm{Int}(Y)$ is a closed topological Jordan disk with a piecewise smooth boundary, the proof is complete by taking $X=\overline{\mathbb{C}}\setminus \mathrm{Int}(Y)$. □

Since X is a closed topological Jordan disk disjoint from the forward orbit of all critical points of f, all the holomorphic inverse branches $f_1^{-1},\ldots,f_d^{-1}:X\to\overline{\mathbb{C}}$ of f are well-defined, where f_d^{-1} is the inverse branch fixing the parabolic point a. By Proposition 6.22, $f_j^{-1}(X)\subset X$ for every $j=1,\ldots,d$; so we get an iterated function system. Since $f_1^{-1},\ldots,f_d^{-1}$ are all the analytic inverse branches of the same analytic map f, all the images $f_i^{-1}(X)$, $i=1,\ldots,d$, are mutually disjoint. It also follows from Proposition 6.22 that $f_i^{-1}(X)\subset \mathrm{Int}(X)$ for every $i=1,\ldots,d-1$. Thus the iterated function system $S=\{f_1^{-1},\ldots,f_d^{-1}\}$ actually satisfies all the requirements (6pa)-(6pi). We wrote "actually" since the maps f_i^{-1}, $i=1,\ldots,d-1$, do not have to be contractions. Due to the Koebe distortion theorem, the observation that $\lim_{k\to\infty}\mathrm{diam}(f_d^{-k}(X))=0$, and a simple area argument, this can be however remedy by fixing n large enough and considering the system S_n consisting of all compositions of n mappings from the system S. Since $J_S=J(f)\setminus\bigcup_{n\geq 0}f^{-n}(\{a\})$, we finally obtain the following result as an immediate consequence of Theorem 6.21.

Theorem 6.23. *If $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is a parabolic rational function such that the Fatou set $\overline{\mathbb{C}}\setminus J(f)$ is connected and the Julia set is contained in a real-analytic curve, then* $\mathrm{HD}(\omega)<\mathrm{HD}(J(f))$, *where ω is the harmonic measure on $J(f)$).*

Remark that the maps $f_{a,b}(z) = z - a + \frac{b}{z}$ with big positive a and small real b provide examples of maps satisfying the assumptions of Theorem 6.23 which are not conjugate to Blaschke products, the case explored in [PV].

We shall now turn our attention to the class of examples generated by continued fractions with restricted entries. So, we fix $I = \{n_i\}_{i\geq 1}$, an infinite subset of positive integers represented as an increasing to infinity sequence of positive integers. We will assume that I has bounded gaps. More precisely, we assume that there exists a positive integer $b \geq 2$ such that

$$2 \leq n_{i+1} - n_i \leq b \tag{6.5}$$

for all $i \geq 1$. Consider the iterated function system

$$S = S_I = \left\{\phi_i(x) = \frac{1}{n_i + x}\right\}_{i\geq 1}$$

defined on the closed disk $X = \overline{B}(1/2, 3/4)$. It is easy to see that S_I (actually the system of compositions of sufficiently long length) is a conformal system in the sense of Section 1, satisfying conditions (1.1) and (1.2) (this is the place where we need the left-hand side of inequality (6.5). Observe that the interval $[0,1] \subset I\!R$ is invariant under all the maps ϕ_i, $i \geq 1$, and the limit set J_I consists of all numbers in [0,1] whose continued fraction expansions have all entries in the set I. We shall prove the following.

Theorem 6.24. *If I is an infinite sequence of positive integers satisfying the right-hand side of (6.5), then the corresponding iterated function system S_I is hereditarily regular and the limit set J_I is uniformly perfect.*

Proof. A straightforward calculation shows that

$$|\phi_i'(x)| \asymp \frac{1}{n_i^2} \tag{6.6}$$

independently of $i \geq 1$ and $x \in \overline{B}(1/2, 3/4)$. For every $t \geq 0$ consider the series

$$\psi(t) = \sum_{i\geq 1} ||\phi_i'||^t \asymp \sum_{i\geq 1} \frac{1}{n_i^{2t}}.$$

Since the right-hand side of (6.5) is satisfied, $\theta := \inf\{t : \psi(t) < \infty\} = 1/2$ and $\psi(1/2) = \infty$. Hence, in view of Theorem 3.20 from [MU1], the system S_I is hereditarily regular. It follows from (6.6) that

$$\mathrm{diam}\Big(\phi_i(X)\Big) \asymp \frac{1}{n_i^2},$$

$$\frac{1}{(4b^2+1)n_i^2} \leq \left(\frac{1}{n_i + b}\right)^2 \asymp \mathrm{diam}\Big(\phi_{i+1}(X)\Big) \asymp \frac{1}{n_{i+1}^2} \leq \frac{1}{n_i^2}$$

and

$$\mathrm{dist}\Big(\phi_{i+1}(X), \phi_i(X)\Big) \leq \frac{1}{n_i + 1} - \frac{1}{n_{i+1}} \leq \frac{1}{n_i} - \frac{1}{n_i + b} = \frac{b}{n_i(n_i + b)} \leq \frac{b}{n_i^2}.$$

Hence, the assumptions of Theorem 3.5 are satisfied and it implies that the limit set J_I is uniformly perfect. $\square$

We shall prove the following.

Lemma 6.25. *If I is an infinite sequence of positive integers satisfying (6.5), then both* $\mathrm{H}_\nu(\alpha)$ *and* χ_ν *are positive, where* ν *is the* S*-invariant probability measure equivalent with the harmonic measure on* J_I *and* α *is the partition into sets* $\{\phi_i(J)\}_{i\geq 1}$

Proof. Our assumptions imply that $2i \leq n_i \leq bi$. Thus

$$||\phi_i'|| \asymp \mathrm{diam}(\phi_i(X)) \asymp \frac{1}{n_i^2} \asymp \frac{1}{i^2} \text{ and } \mathrm{dist}(\phi_i(X), 0) \asymp \frac{1}{i}.$$

Hence we can proceed similarly as in the proof of Lemma 6.20 Again, in view of Corollary 1.12 it is sufficient to demonstrate that the Lyapunov exponent χ_ν is finite.
A straightforward calculation shows that the Lyapunov exponent χ_ν is equal to

$$\sum_{b\in I_*} \int_{\phi_b(X)} \log |(\phi_b^{-1})'| d\nu.$$

Using (6.5) we can estimate this integral from above by

$$\begin{aligned}
&\sum_{i\geq 1} \sup_{\phi_i(X)} \{\log \phi_i'\} \omega(\phi_i(J)) + const \\
&\quad \leq \sum_{i\geq 1} (const + 2\log i)\omega(\phi_i(J)) + const \\
&\quad \leq \sum_{i\geq 1} 2\left(\sum_{k=1}^{i-1} (\log(k+1) - \log k)\omega(\phi_i(J))\right) + const \\
&\quad = 2\sum_{k=1}^{\infty} \left((\log(k+1) - \log k)\Big(\sum_{q=k+1}^{\infty} \omega(\phi_q(J))\Big)\right) + const \\
&\quad = 2\sum_{k=1}^{\infty} \log\left(\frac{k+1}{k}\right)\Big(\sum_{q=k+1}^{\infty} \omega(\phi_q(J))\Big) + const \\
&\quad \leq const \sum_{k=1}^{\infty} \log\left(\frac{k+1}{k}\right) \omega(B(0, const\, k^{-1})) + const \\
&\quad \leq const \sum_{k=1}^{\infty} \frac{1}{k} \omega(B(0, const\, k^{-1})) + const.
\end{aligned}$$

Since $\overline{J}$ is uniformly perfect, there exists $0 < \kappa < 1$ such that $\omega(B(0,r)) = O(r^\kappa)$. Therefore the last series in the above display converges, and consequently the Lyapunov exponent χ_ν is finite. ☐

Notice that in order to prove this lemma one could employ the much more complicated reasoning taken from [PV].

Combining now Theorem 6.24 and Lemma 6.25, we obtain the following result as an immediate consequence of Corollary 5.5.

Theorem 6.26. *If I is an infinite sequence of positive integers satisfying (6.5), then* $\mathrm{HD}(\omega) < \mathrm{HD}(J_I)$*, where ω is the harmonic measure on J_I.*

Our last class of examples is provided by all generalized polynomial-like (non-hyperbolic) maps. We follow the definitions, notation and terminology from [Zd2]. So, let

$$f : \bigcup_{i=1}^{q} U_i \to W$$

be a generalized polynomial-like map, i.e. W and U_i, $i = 1, 2 \dots, q$, are open topological disks with smooth boundaries and $f|_{U_i}$ are proper holomorphic maps. We show how to associate with f a conformal infinite iterated function system $S = \{\phi_i\}$ such that $\overline{J_S} = J(f)$ and S has other useful properties which will be discussed later. Next, we show how to answer the question about the Hausdorff dimension of harmonic measure on $J(f)$ using the new dynamics $S = \{\phi_i\}$. It has been proved in [Zd2] that there exists an ergodic invariant measure ν on $J(f)$ equivalent with harmonic measure ω and with positive entropy. Let $\gamma = \partial W$. Let

$$\tilde{J} = \{(x_n)_{n=-\infty}^{+\infty} : x_n \in J(f) \text{ and } f(x_n) = x_{n+1} \text{ for all } n \in \boldsymbol{Z}\}$$

be the natural extension of $J(f)$ associated with the map $f : J(f) \to J(f)$ and let $\tilde{f} : \tilde{J} \to \tilde{J}$ defined by the formula

$$\tilde{f}((x_n)_{n=-\infty}^{+\infty}) = ((x_{n+1})_{n=-\infty}^{+\infty})$$

be the canonical lift of $f : J(f) \to J(f)$ to the natural extension $\tilde{J}$. Finally let $\pi_0 : \tilde{J} \to J(f)$ be the projection onto 0^{th} coordinate, that is

$$\pi_0((x_n)_{n=-\infty}^{+\infty}) = x_0.$$

It is well-known (see Chapter 1 of [PU] for ex.) that there exists a unique $\tilde{f}$-invariant measure $\tilde{\nu}$ such that $\nu = \tilde{\nu} \circ \pi^{-1}$. From now on throughout the entire section we assume that $J(f)$ is not connected. Following the notation of [Zd2] we choose the curve γ_n, a component of $f^{-n}(\gamma)$. Let $X(\gamma_n)$ be the part of $J(f)$ surrounded by γ_n and let D_n be the domain (a topological disk) bounded by γ_n. Lemma 4.4 from [Zd2] yields the following.

Lemma 6.27. *There exist a set $\tilde{F} \subset \tilde{J}$, an integer $n_0 \geq 1$, and a constant $0 < \lambda < 1$ such that*

- **(a):** $\tilde{\nu}(\tilde{F}) > 0$.
- **(b):** *There exists a curve γ_{n_0} defined above such that $\pi(\tilde{F}) \subset X(\gamma_{n_0})$.*
- **(c):** *If $\tilde{x} \in \tilde{F}$, then for every $n \leq 0$ there exists a holomorphic branch $f^{nn_0}_{x_{nn_0}}$ of f^{nn_0} defined on D_{n_0} and sending x_0 to x_{nn_0}.*
- **(d):** *All the holomorphic inverse branches of f^{n_0} are well defined on the disk D_{n_0}.*

Proof.(Sketch) Since $J(f)$ is not connected, the number of "cylinders" $X(\gamma_n)$ grows exponentially fast with n. Indeed, there exists $N \geq 1$ such that there are at least two disjoint disks D_N bounded by curves γ_N and each of them is mapped onto W by some positive iterate of f. Then

$$\#X(\gamma_{nN}) \geq 2^n. \tag{6.7}$$

Since the number of critical values of f^n is $\leq (d-1)n$, where $d \geq 2$ is the degree of f, and $2^n > (d-1)n$, for all n large enough, it follows from (6.7) that there exists a cylinder $X_{n_0} = X(\gamma_{n_0})$ such that there are no critical values of f^{n_0} in D_{n_0}. All holomorphic branches of f^{-n_0} are then well-defined on D_{n_0}. Consider now the collection of all topological disks $f_\eta^{-n_0}(D_{n_0})$, the images of D_{n_0} under all holomorphic branches of f^{-n_0}. Next we remove from this collection those disks which contain critical values of f^{n_0}. In view of Proposition 4.3. in [Zd2] (which says precisely that there exists $\kappa > 0$ such that for every component $X(\gamma_n)$, $\nu(f(X(\gamma_n))) > \exp(\kappa n)\nu(X(\gamma_n))$ and consequently $\nu(X(\gamma_n)) < \exp(-\kappa n))$ the total measure of the removed set can be estimated from above by

$$\delta_1 := (d-1)n_0 \mathrm{e}^{-\beta n_0}\nu(D_{n_0})$$

for some $b > 0$. We call the remaining disks and remaining branches of f^{-n_0}, admissible. Now, one can apply all holomorphic branches of f^{-n_0} to all admissible disks, and remove again those components of f^{-2n_0} that contain critical values of f^{n_0}. In this way we obtain a collection of admissible components of $f^{-2n_0}(D_{n_0})$ and admissible holomorphic branches of f^{-2n_0}. At this step of the construction the measure of removed components can be estimated from above by

$$\delta_2 := (d-1)n_0(\mathrm{e}^{-\beta n_0} + \mathrm{e}^{-2\beta n_0})\nu(D_{n_0}).$$

We continue this procedure inductively. After the nth step one obtains the set

$$F_n \subset \pi_0^{-1}(D_{n_0}) \subset \tilde{J}$$

consisting of sequences $\{x_j\}_{j=-\infty}^{\infty} \in \tilde{J}$ such that $x_0 \in D_{n_0}$ and for every $k \leq n$, x_{-kn_0} is the image of x_0 under an admissible holomorphic branch of f^{-kn_0} defined on D_{n_0}. Put

$$\tilde{F} = \bigcap_{n=1}^{\infty} F_n$$

Since the sets F_n, $n \geq 1$ form a descending family and for every $n \geq 1$, $\tilde{\nu}(F_n) \geq \nu(D_{n_0}) - \delta_n$, where $\delta_n = (d-1)n_0 \sum_{k=1}^n \mathrm{e}^{-k\beta n_0}$, we have

$$\tilde{\nu}(\tilde{F}) \geq \nu(D_{n_0}) - (d-1)n_0 \sum_{k=1}^{\infty} \mathrm{e}^{-k\beta n_0}\nu(D_{n_0})$$

and the above series is less than 1 for n_0 large enough. This takes care of properties (a) and (b) of our lemma. The properties (c) and (d) are obviously satisfied by the way our construction was carried out. □

Now, for all $m, n \geq 1$ fix arbitrary sets of the form $X(\gamma_m)$ and $X(\gamma_n)$ and consider Y_{n+m}, the union of all sets of the form $X(\gamma_{n+m})$ such that $X(\gamma_{n+m}) \subset X(\gamma_n)$ and $f^n(X(\gamma_{n+m})) = X(\gamma_m)$. For reader's convenience let us state now Proposition 4.1 from [Zd2].

Proposition 6.28. *For all n and m*

$$\omega(X(\gamma_m)) \asymp \frac{\omega(Y_{n+m})}{\omega(X(\gamma_n))}$$

Now, all the inverse branches described in part (c) of the previous lemma will be called admissible. Denote

$$F^{(n)} = \{\tilde{x} \in \tilde{F} : \tilde{f}^{-in_0} \notin \tilde{F} \text{ for all } i = 1, 2, \cdots, n\}.$$

Lemma 6.29. *There exists $\gamma > 0$ such that for all $n \geq 0$*

$$\tilde{\nu}(F^{(n)}) \leq \exp(-\gamma n).$$

Proof. Fix $k \geq 1$, take $F^{(k)}$ and consider the collection of all topological disks $Y = f_\eta^{-kn_0}(D_{n_0})$, where we use all the admissible branches of f^{-kn_0} corresponding to the points from $F^{(k)}$. Thus all the inverse branches of f^{n_0} are well-defined on the disks Y. Since $f^{n_0}(D_{n_0}) = W$, for every disk Y some components of $f^{-n_0}(Y)$ fall into D_{n_0}. Denote them by $Y_1, Y_2, \dots, Y_s$ (comp. the paragraph proceeding Proposition 6.28). Using this Proposition one deduces that

$$\nu\Big((Y_1 \cup Y_2 \cup \ldots \cup Y_s) \cap D_{n_0}\Big) \asymp \nu(Y)\nu(D_{n_0}). \tag{6.8}$$

Composing now appropriate branches of f^{-n_0} and f^{-kn_0} we get a collection of holomorphic branches $f_\eta^{-(k+1)n_0}$ mapping D_{n_0} into some $Y_i \subset D_{n_0}$. But we have the starting collection $\tilde{F}$ of (infinite) sequences of backward branches which are defined on the whole disk D_{n_0}. This gives a subset $\tilde{G} \subset F^{(k)} \setminus F^{(k+1)}$ consisting of backward branches which are built as follows. Each inverse branch $f_\eta^{-(k+1)n_0}$ defined above is continued using all sequences of backward trajectories belonging to $\tilde{F}$. Moreover

$$\tilde{\nu}\Big(F^{(k)} \setminus F^{(k+1)}\Big) \geq \tilde{\nu}(\tilde{G}) \geq \delta \sum \nu(Y) \geq \delta \tilde{\nu}(F^{(k)}),$$

where the second inequality is a combined consequence of (6.8) and Proposition 6.28; δ is independent of k. This gives

$$\tilde{\nu}(F^{(k+1)}) \leq (1-\delta)\tilde{\nu}(F^{(k)})$$

and we are done. □

Fix D to be an arbitrary closed disk with smooth boundary containing in its interior $X(\gamma_{n_0})$ and contained in D_{n_0}. By Poincaré's recurrence theorem for $\tilde{\nu}$ a.e. $\tilde{x} \in \tilde{F}$ there exists a least $n(\tilde{x}) \geq 1$ such that $\tilde{f}^{-n(\tilde{x})n_0}(\tilde{x}) \in \tilde{F}$. After removing a set of $\tilde{\nu}$ measure 0 from $\tilde{F}$ we may assume that this holds for every $\tilde{x} \in \tilde{F}$. Denote by T the first return map, i.e. $T(\tilde{x}) = \tilde{f}^{-n(\tilde{x})n_0}(x)$. For every $\tilde{x} \in \tilde{F}$ consider the map $f_{x_{-n(\tilde{x})n_0}}^{-n(\tilde{x})n_0} : D \to W$. It is easy to see that for any two points $\tilde{x}, \tilde{y} \in \tilde{F}$, the images $f_{x_{-n(\tilde{x})n_0}}^{-n(\tilde{x})n_0}(D)$ and $f_{y_{-n(\tilde{y})n_0}}^{-n(\tilde{y})n_0}(D)$ either are equal or are disjoint. Since in addition each set $f_{x_{-n(\tilde{x})n_0}}^{-n(\tilde{x})n_0}(D)$ is contained in D, we obtain in this way the following new family of maps

$$S = \{\phi_i : D \to D\}_{i\in I}$$

composed of (countably many) all maps of the form $f_{x_{-n(\tilde{x})}}^{-n(\tilde{x})n_0}$, $\tilde{x} \in \tilde{F}$. We shall prove the following.

Lemma 6.30. *The family $S = \{\phi_i : D \to D\}_{i\in I}$ (actually the family $\{\phi_\omega : \omega \in I^n\}$ for all n large enough) is a hyperbolic conformal iterated function system.*

Proof. By the Koebe distortion theorem and the choice of D we have uniformly bounded distortion for all maps ϕ_ω, $\omega \in I^n$. Thus, in order to demonstrate that an iterate of the system S is hyperbolic, it suffices to show that

$$\lim_{n\to\infty} \sup\{\mathrm{diam}(\phi_\omega(D)) : \omega \in I^n\} = 0.$$

And indeed, by the choice of D, for every $\omega \in I^*$, the disk $\phi_\omega(D)$ is enclosed by a nested family of $|\omega|$ annuli conformally equivalent with the annulus $D_{n_0} \setminus D$. Thus, by the Grötzsch's inequality the modulus $D \setminus \phi_\omega(D) \ge \mathrm{Mod}(D_{n_0} \setminus D)|\omega|$. We are done.

Corollary 6.31. *There exist constants $C, \beta > 0$ such that for every $q \ge 1$*

$$\nu\left(\bigcup_{\{i\in I:n(i)=q\}} \phi_i(D)\right) \le C\exp(-\beta q).$$

Proof. For every $k \ge 1$ we have

$$\bigcup_{\{i\in I:n(i)=k\}} \phi_i(D) = \pi_{-kn_0}\Big(F^{(k-1)} \setminus F^{(k)}\Big), \tag{6.9}$$

where for every $n \in \boldsymbol{Z}$, $\pi_n : \tilde{J} \to J$ denotes the projection onto nth coordinate, i.e. $\pi_n(\{x_j\}_{j\in\boldsymbol{Z}}) = x_n$. By the definition of the measure $\tilde{\nu}$

$$\nu\Big(\pi_{-kn_0}\Big(F^{(k-1)} \setminus F^{(k)}\Big) \ge \tilde{\nu}\Big(F^{(k-1)} \setminus F^{(k)}\Big).$$

But what we need is the opposite inequality

$$\tilde{\nu}\Big(F^{(k-1)} \setminus F^{(k)}\Big) \ge \mathit{const}\, \nu\Big(\pi_{-kn_0}\Big(F^{(k-1)} \setminus F^{(k)}\Big)\Big).$$

In order to prove it, notice that for all disks $Y = f_\rho^{-kn_0}(D) \subset \pi_{-kn_0}\Big(F^{(k-1)} \setminus F^{(k)}\Big)$, taken over all admissible branches $f_\rho^{-kn_0}$ corresponding to elements in $F^{(k-1)} \setminus F^{(k)}$ one can "attach" all (infinite) backward trajectories belonging to $\tilde{F}$ producing as the result a set $\tilde{Y} \subset \pi^{-1}(Y) \cap F^{(k-1)} \setminus F^{(k)}$. Now

$$\tilde{\nu}(\tilde{Y}) = \lim_{n\to\infty} \nu\left(\bigcup_{\eta_{A_n}} \Big(f_\eta^{-nn_0}(Y)\Big)\right),$$

where the union is taken over A_n, all admissible branches of length n. Since from Proposition 6.28 and Lemma 6.27 we have for every $n \geq 1$

$$\nu\left(\bigcup_{\eta A_n}\left(f_\eta^{-nn_0}(Y)\right)\right) \asymp \frac{\nu\left(\bigcup_{\eta A_n}\left(f_\eta^{-nn_0}(D)\right)\right)}{\nu(D)}\nu(Y) \geq \frac{\tilde{\nu}(\tilde{F})}{\nu(D)}\nu(Y),$$

we conclude that

$$\tilde{\nu}(\tilde{Y}) \geq \frac{\tilde{\nu}(\tilde{F})}{\nu(D)}\nu(Y).$$

Taking now the union over all defined above disks $Y \subset \pi_{-kn_0}\Big(F^{(k-1)} \setminus F^{(k)}\Big)$, we get

$$\tilde{\nu}\Big(F^{(k-1)} \setminus F^{(k)}\Big) \geq \tilde{\nu}(\cup\tilde{Y}) \geq \frac{\tilde{\nu}(\tilde{F})}{\nu(D)}\nu(\cup Y) = \frac{\tilde{\nu}(\tilde{F})}{\nu(D)}\nu\Big(\pi_{-kn_0}\Big(F^{(k-1)} \setminus F^{(k)}\Big)\Big).$$

So, applying (6.9) and Lemma 6.29 completes the proof. □

We shall prove the following.

Proposition 6.32. *We have*

- **(a):** $\nu\left(\bigcup_{i\in I}\phi_i(D)\right) = \nu(D)$.
- **(b):** $\overline{J_S} = J(f) \cap D$.
- **(c):** *If ω_S is the harmonic measure on J_S and ω is the harmonic measure on J, then the measures ω_S and $\omega|_{J_S}$ are equivalent and the system S is ω conservative. Thus there exists η, the S-invariant measure equivalent with ω_S.*
- **(d):** *The Jacobian $\psi : J_S \to (0,\infty)$, $i \in I$, defined by the formula*

  $$\psi(x) = \lim_{z\to x}\frac{G\circ\phi_i^{-1}(z)}{G(z)} \quad (x \in \phi_i(J_S))$$

  satisfies $\int_{J_S}|\log\psi|^k d\eta < \infty$ for every integer $k \geq 0$.
- **(e):** *The system (S,η) has finite entropy.*
- **(f):** *$\chi_\eta = \int \xi d\eta < \infty$, where ξ is the amalgamated function of the family Ξ (ascribed to the system $S = \{\phi_i\}_{i\in I}$ introduced just after Lemma 1.9.*

Proof. Since $\bigcup_{i\in I}\phi_i(D) = \pi_0(T(\tilde{F}))$, it suffices to show that $\nu(B) = 0$, where $B = D \setminus pi_0(T(\tilde{F}))$. If, on the contrary, this measure is positive, then the inequality $\tilde{\nu}(\pi_0^{-1}(B)\cap\tilde{F}) > 0$ follows from the construction of the set $\tilde{F}$. But $\pi_0^{-1}(B)\cap\tilde{F} \subset \tilde{F}\setminus T(\tilde{F})$ and $\tilde{\nu}(\tilde{F}\setminus T(\tilde{F})) = 0$ since T, as an induced map preserves the measure ν. This contradiction finishes the proof of (a). Properties (b) and (c) are immediate consequence of (a) and the remark that ν is positive on open sets. Since (e) follows from (d) in order to prove both we only need to verify (d). Since the measures ν, ω and η are mutually equivalent on J_S with Radon-Nikodym derivatives bounded away from zero and infinity, we can write

$$\int_{J_S}|\log\psi|^k d\eta \asymp \int_{J_S}|\log\psi|^k d\nu \leq \sum_{i\in I}\nu(\phi_i(D))\sup_{\phi_i(D)}|\log\psi|^k$$

Now, by Lemma 2.2

$$\sup_{\partial(\phi_i(D))}\left|\log\frac{G\circ\phi_i^{-1}}{G}\right| \le \sup_{\partial(\phi_i(D))}|\log G| + M \asymp |\log(\nu(\phi_i(D))| + M,$$

where $M = \sup_{\partial V}|\log G| < \infty$ and the "$\asymp$" sign in the formula above has the additive meaning. So, using Corollary 6.31 we get

$$\begin{aligned}\int_{J_S}|\log\psi|^k d\nu &\le const\sum_{i\in I}\nu(\phi_i(D))|\log(\nu(\phi_i(D))|^k + const\\ &\le const\sum_{q\ge 1}\sum_{\{i\in I:n(i)=q\}}\nu(\phi_i(D))q^k + const\\ &\le const\sum_{q\ge 1}C\exp(-\beta q)q^k + const < \infty.\end{aligned}$$

Since Lemma 6.30 gives that $||\phi_i'|| \le const\,\lambda^{n(i)n_0}$ for every $i\in I$ and some $\lambda<1$, the same calculation proves the part (f). $\square$

Now to compare the Hausdorff dimension of harmonic measure and the Hausdorff dimension of $J(f)$ is a straightforward consequence of the results obtained in Section 5. Combining Theorem 5.2 and Proposition 6.32 we get the following.

Proposition 6.33. *If the system S is irregular, then* $\mathrm{HD}(\omega) < \mathrm{HD}(J(f))$.

As a corollary from Theorem 5.3 and Proposition 6.32 we get the following.

Proposition 6.34. *If the system S is regular and the entropy of μ, the invariant measure equivalent with the conformal measure m is infinite, then* $\mathrm{HD}(\omega) < \mathrm{HD}(J_S) \le \mathrm{HD}(J(f))$.

Propositions 6.32 - 6.34 allow us to reduce the question of whether $\mathrm{HD}(\omega) < \mathrm{HD}(J(f))$ for an arbitrary generalized polynomial-like mapping to the corresponding result for regular infinite iterated function systems. We find this reduction interesting itself. In order to make use of Theorem 5.4 we need the additional assumption that the generalized polynomial-like mapping $f:\bigcup_{i=1}^n U_i\to W$ is 1-dimensional. Thus, at the moment we only have the following.

Theorem 6.35. *If the Julia set $J(f)$ of the generalized polynomial-like mapping $f:\bigcup_{i=1}^n U_i\to W$ is 1-dimensional, then* $\mathrm{HD}(\omega) < \mathrm{HD}(J(f))$.

7. Hausdorff dimension of harmonic measure <1

Although the subject of this section is closely related to the contents of the previous sections, it is however out of the mainstream of this paper which is the relation between the Hausdorff dimension of harmonic measure and the Hausdorff dimension of the limit set. This is why we have decided to locate it as the last section. We deal here with the problem of under which conditions the Hausdorff dimension of harmonic measure of a uniformly perfect limit set of a conformal IFS (not necessarily 1-dimensional - just conversely) is strictly less than 1. Our approach is to formulate checkable conditions for the assumptions of Theorem 2 from [JW] to be satisfied.

Following [JW] we recall that given $\epsilon > 0$ and $r_0 > 0$ a point x in a compact set $F \subset \mathbb{C}$ satisfies the (ϵ, r_0)-annulus condition if for every $r \geq r_0$ the annulus $A_\epsilon(x,r) = \{z \in \mathbb{C} : r < |z-x| \leq \epsilon^{-1} r\}$ contains a topological annulus $T_\epsilon(x,r) \subset \mathbb{C} \setminus F$ such that x belongs to the bounded component of $\mathbb{C} \setminus F$ and the modulus of $T_\epsilon(x,r) \subset \mathbb{C} \setminus F$ is greater than or equal to ϵ. We simply say that the point x satisfies the ϵ-annulus condition (with respect to the compact set F) if $r_0 = 0$. If each point of the set F satisfies the ϵ-annulus condition for some common $\epsilon > 0$, then the set F is said to satisfy the annulus condition. Jones and Wolff have proved in [JW] that if a uniformly perfect set F satisfies the annulus condition, then the Hausdorff dimension of its harmonic measure is strictly less than 1 (comp. the article [Wo] where although this result is not explicitly stated, however its methods lead to the proof). As a matter of fact Jones and Wolff were assuming so called capacity density condition instead of uniform perfectness but it is well-known that these two conditions are equivalent. We shall provide now some sufficient conditions for the annulus condition to be satisfied by $\overline{J}$.

Lemma 7.1. *If there exists $\epsilon > 0$ and $\gamma \geq 1$ such that for all $i \in I$ there exists $x_i \in \phi_i(X)$ such that x_i satisfies the $(\epsilon, \gamma \mathrm{diam}(\phi_i(X))$-annulus condition, then the set J satisfies the annulus condition.*

Proof. Fix $0 < \delta \leq \epsilon$ so small that

$$\frac{\epsilon\delta^{-1} - K}{1+\epsilon} \geq \frac{K}{\gamma}, \tag{7.1}$$

where, let us recall, K is the Koebe distortion constant. Rescaling, if necessary, the system by a sufficiently big factor, we may assume that

$$D^{-1}\delta\epsilon^{-1}\mathrm{dist}(X, \partial V) \geq \gamma, \tag{7.2}$$

where D comes from (1.4) and (1.5). In order to prove the lemma it obviously suffices to demonstrate that for every $r > 0$ sufficiently small, each point $z = \pi(\tau) \in J$, $\tau \in I$, satisfies the δ-annulus condition. So, fix $0 < r < \delta\epsilon^{-1}\mathrm{dist}(X, \partial V)$ and consider the least $n \geq 0$ such that

$$Kr||\phi'_{\tau|_n}||^{-1} \geq \gamma \mathrm{diam}(\phi_{\tau_{n+1}}(X)) \tag{7.3}$$

Using (7.1) this implies that $(1+\epsilon)\mathrm{diam}(\phi_{\tau_{n+1}}(X)) \leq (\epsilon\delta^{-1} - K)r||\phi'_{\tau|_n}||^{-1}$ or equivalently

$$Kr||\phi'_{\tau|_n}||^{-1} + \mathrm{diam}(\phi_{\tau_{n+1}}(X)) \leq \epsilon\delta^{-1}r||\phi'_{\tau|_n}||^{-1} - \epsilon\mathrm{diam}(\phi_{\tau_{n+1}}(X)).$$

Fix an arbitrary R such that

$$Kr||\phi'_{\tau|_n}||^{-1} + \mathrm{diam}(\phi_{\tau_{n+1}}(X)) \leq R \leq \epsilon\delta^{-1}r||\phi'_{\tau|_n}||^{-1} - \epsilon\mathrm{diam}(\phi_{\tau_{n+1}}(X)). \tag{7.4}$$

Then

$$\begin{aligned}\phi_{\tau|_n}(B(x_{\tau_{n+1}}, \epsilon^{-1}R) &\subset \phi_{\tau|_n}(B((\pi(\sigma^n\tau), \epsilon^{-1}R + \mathrm{diam}(\phi_{\tau_{n+1}}(X))) \\ &\subset B\Big(z, ||\phi'_{\tau|_n}||\Big(\epsilon^{-1}R + \mathrm{diam}(\phi_{\tau_{n+1}}(X))\Big)\Big) \subset B(z, \delta^{-1}r)\end{aligned} \tag{7.5}$$

Suppose now that $R \ge \mathrm{dist}(X, \partial V)$. Then $\epsilon\delta^{-1}r||\phi'_{\tau|_n}||^{-1} \ge \mathrm{dist}(X, \partial V)$. Hence, by our choice of r, $n \ge 1$ and using (7.2), we get

$$\begin{aligned} Kr||\phi'_{\tau|_{n-1}}||^{-1} &\ge r||\phi'_{\tau|_n}||^{-1}||\phi'_{\tau_n}|| \ge D^{-1}\delta\epsilon^{-1}(\epsilon\delta^{-1}r||\phi'_{\tau|_n}||^{-1}||)\mathrm{diam}(\phi_{\tau_n}(X)) \\ &\ge D^{-1}\delta\epsilon^{-1}\mathrm{dist}(X, \partial V)\mathrm{diam}(\phi_{\tau_n}(X)) \\ &\ge \gamma\mathrm{diam}(\phi_{\tau_n}(X)). \end{aligned}$$

This however contradicts the definition of n and shows that $R - \mathrm{diam}(\phi_{\tau_{n+1}}(X)) < R < \mathrm{dist}(X, \partial V)$. Therefore, using (7.4), we get

$$\begin{aligned} \phi_{\tau|_n}(B(x_{\tau_{n+1}}, R) &\supset \phi_{\tau|_n}(B((\pi(\sigma^n\tau), R - \mathrm{diam}(\phi_{\tau_{n+1}}(X))) \\ &\supset B\Big(z, ||\phi'_{\tau|_n}||K^{-1}\Big(R - \mathrm{diam}(\phi_{\tau_{n+1}}(X))\Big)\Big) \supset B(z, r). \end{aligned}$$

Combining this and (7.5) we deduce that

$$\phi_{\tau|_n}(A(x_{\tau_{n+1}}, R, \epsilon^{-1}R) \subset A(z, r, \delta^{-1}r)$$

and the bounded component of $\mathbb{C}\setminus\phi_{\tau|_n}(A(x_{\tau_{n+1}}, R)$ contains $B(z, r)$. Now, by our assumptions there exists a topological anuulus

$$T_\epsilon(x_{\tau_{n+1}}, R) \subset A(x_{\tau_{n+1}}, R, \epsilon^{-1}R) \setminus \overline{J}$$

with modulus $\ge \epsilon$ and such that $x_{\tau_{n+1}}$ belongs to the bounded component of $\mathbb{C}\setminus T_\epsilon(x_{\tau_{n+1}}, R)$. Then $\phi_{\tau|_n}(T_\epsilon(x_{\tau_{n+1}}, R)) \subset A(z, r\delta^{-1}r) \setminus \overline{J}$ is a topological annulus with modulus equal to $\mathrm{Mod}(T_\epsilon(x_{\tau_{n+1}}, R)) \ge \epsilon \ge \delta$. In addition z belongs to the bounded component of $\mathbb{C} \setminus \phi_{\tau|_n}(T_\epsilon(x_{\tau_{n+1}}, R))$. The proof is complete. □

Theorem 7.2. *Suppose that the (UP) condition from Theorem 3.5 holds, that each point in $X(\infty)$ satisfies the annulus condition with some common $\epsilon > 0$, and that*

$$\inf_{i\in I}\left\{\frac{\mathrm{diam}(\phi_i(X))}{\mathrm{dist}(\phi_i(X), X(\infty))}\right\} > 0. \tag{7.6}$$

Then $\mathrm{HD}(\omega) < 1$.

Proof. Since, by Theorem 3.5, the (UP) condition implies uniform perfectness of $\overline{J}$, in view of Theorem 2 from [JW] it suffices to prove that $\overline{J}$ satisfies the annulus condition. And in order to check this condition it suffices to verify the assumptions of Lemma 7.1 for some $\delta > 0$, corresponding to ϵ appearing in this lemma, and $\gamma = 1$. And indeed, let T be the infimum appearing in formula (7.6). Fix $\delta > 0$ so small that

$$\delta^{-1} \ge 4T + \epsilon^{-1}(1 + 4T) \tag{7.7}$$

and for every $i \in I$ choose an arbitrary $x_i \in \phi_i(X)$. Fix then $z_i \in X(\infty)$ so close to x that $|z_i - x_i| \le 2\mathrm{dist}(x_i, X(\infty))$. If $r \ge \mathrm{diam}(\phi_i(X))$, then using (7.6) we get

$$\begin{aligned} B(x_i, r) &\subset B(z_i, r + |x_i - z_i|) \subset B(z_i, r + 2\mathrm{dist}(x_i, X(\infty))) \subset B(z_i, r + 4T\mathrm{diam}(\phi_i(X))) \\ &\subset B(z_i, r + 4Tr) = B(z_i, (1 + 4T)r). \end{aligned} \tag{7.8}$$

Using (7.6) again and and (7.7) (when writing the last inclusion), we also get

$$\begin{aligned} B(x_i, \delta^{-1}r) &\supset B(z_i, \delta^{-1}r - |x_i - z_i|) \supset B(z_i, \delta^{-1}r - 2\mathrm{dist}(x_i, X(\infty))) \\ &\supset B(z_i, \delta^{-1}r - 4T\mathrm{diam}(\phi_i(X))) \supset B(z_i, \delta^{-1}r - 4Tr) \\ &\supset B(z_i, (\delta^{-1} - 4T)r) \supset B(z_i, \epsilon^{-1}(1 + 4T)r). \end{aligned} \tag{7.9}$$

By our assumptions there exists a topological annulus separating the balls $B(z_i, (1+4T)r)$ and $B(z_i, \epsilon^{-1}(1+4T)r)$, disjoint from $\overline{J}$ and of modulus $\geq \epsilon \geq \delta$. Since, by (7.8) and (7.9), this annulus separates also $B(x_i, r)$ and $B(x_i, \epsilon^{-1}r)$, we are done. □

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