Hausdorff Dimension Estimates for Infinite Conformal Iterated Function Systems

Stefan-M. Heinemann[*][†]
Mariusz Urbański[‡][§]

Institut für Theoretische Physik
TU Clausthal
Arnold-Sommerfeld-Straße 6
38678 Clausthal-Zellerfeld, Germany

Mathematics Department,
University of North Texas,
Denton TX 76203-1430, USA

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Abstract

We show that in order to estimate the Hausdorff dimension of conformal infinite iterated function systems (IFS) it is completely sufficient to consider finite subsystems with $N < \infty$ generators. We give estimates on the accuracy of this approximation and present a simple straightforward algorithm which allows to reduce this setting further to finite iteration time n. Our method allows to calculate the Hausdorff dimension of an IFS with the same speed of convergence as the algorithm proposed by McMullen in [7]

[*]e-mail: `Stefan.Heinemann@TU-Clausthal.DE`

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[‡]e-mail: `urbanski@unt.edu`

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(i.e. the distance of the approximation to the actual value is $\sim 1/n$, where n is the iteration depth, hence corresponding to $\sim N^n$ calculation steps). This is much slower than Jenkinson/Pollicott's method for bounded continued fractions (which gives superexponential accuracy $\sim \exp(-n^{3/2})$ for the same depth) but does not require extensive knowledge about the periodic points of the involved transformations and is applicable in a more general setting. The method is applied in order to perform numerical calculations for certain classes of examples.

1 Preliminaries

In this paragraph we collect some useful facts concerning (infinite) iterated functions systems (cf. [4], [5], [6], [9]).

1.1 Iterated Functions Systems

We consider an *iterated function system* (IFS) on a compact subset X of the d-dimensional Euclidean space $\mathbb{R}^d$, where $d \in \mathbb{N}$. We denote the Euclidean metric on $\mathbb{R}^d$, X, resp. with ϱ. The IFS is generated by a *countable* family S of injective *contractions* $\{\varphi_i : X \to X\}_{i\in I}$, where the *alphabet* I is given by $\mathbb{N} = \{1, 2, \ldots\}$ or by $\{1, \ldots, N\}$ for some $N \in \mathbb{N}$, $N \geq 2$. We shall assume that S is *uniformly contracting* i.e. there is $0 < s < 1$ such that for all $x, y \in X$ and $\varphi_i \in S$ we have that $\{\mathrm{rot}\varrho(\varphi_i(x), \varphi_i(y)) \leq s \cdot \varrho(x, y)$.

An IFS of this type induces a limit set $J \subseteq X$ which can be interpreted as the image of the *coding space* I^∞ under a canonical *coding map* $\pi : I^\infty \to X$ which is obtained as follows. Let $I^* := \bigcup_{n\geq 1} I^n$ denote the *space of finite words*. For $\omega = \omega_1 \cdots \omega_n \in I^n$, $n \in \mathbb{N}$, we define $\varphi_\omega := \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_n}$.

For $\omega \in I^* \cup I^\infty$ and $n \in \mathbb{N}$, $n \leq \mathrm{length}(\omega)$ we set $\omega\lceil n := \omega_1\omega_2 \cdots \omega_n$. We shall also use the notation of the *shift operator* σ which is defined by mapping $\omega = \omega_1\omega_2 \cdots \omega_n \cdots$ to $\sigma(\omega) := \omega_2\omega_3 \cdots \omega_{n+1} \cdots$.

Since the family S is uniformly contracting, we have that for any $\omega \in I^\infty$ and $n \to \infty$ the diameter of $\varphi_{\omega\lceil n}(X)$ is converging to 0, thus the set

$$\pi(\omega) \;:=\; \bigcap_{n\geq 1} \varphi_{\omega\lceil n}(X) \tag{1}$$

is a singleton and we can use (1) in order to define the coding map $\pi : I^\infty \to X$. It is straightforward to see that for finite I the limit set $J = \pi(I^\infty) = \bigcup_{\omega\in I^\infty} \bigcap_{n\geq 1} \varphi_{\omega\lceil n}(X)$ is compact. However, we are mainly interested in infinite systems.

1.2 Conditions and Properties

We call an iterated function system *conformal* if $X \subset \mathbb{R}^d$ for some $d \geq 1$, X is connected, and the following conditions are fulfilled.

(1a) the *open set condition*: for each pair $i, j \in I$, $i \neq j$ we have that

$$\varphi_i(\mathrm{int}(X)) \cap \varphi_j(\mathrm{int}(X)) \;=\; \emptyset;$$

(1b) *conformality*: there exists an open connected set $X \subset V \subseteq \mathbb{R}^d$ such that all maps φ_i extend to $\mathcal{C}^1$ conformal diffeomorphisms of V into V (for $d = 1$ this equivalent to the fact that the φ_i are monotone diffeomorphisms, for $d = 2$ this means that the φ_i are holomorphic or antiholomorphic, for $d \geq 3$ the maps must be (restrictions of) Möbius transformations (cf. [1], [10]));

(1c) the *cone condition*: there exists $\alpha, \ell > 0$ such that for every $x \in \partial X \subset \mathbb{R}^d$ there is an open cone $\mathrm{Con}(x, u, \alpha) \subset \mathrm{int}(X)$ with vertex x, symmetry axis $u \in \mathbb{R}^d$ of length ℓ and angle α, i.e.

$$\mathrm{Con}(x, u, \alpha) \;=\; \{y \in X : \; 0 \leq \langle y - x, u\rangle \leq \cos(\alpha) \cdot \|y - x\| < \ell\}.$$

(1d) the *bounded distortion property*: there is $1 \leq K < \infty$ such that

$$|\varphi'_\tau(y))| \;\leq\; K \cdot |\varphi'_\tau(x)|$$

for every $\tau \in I^*$ and each pair $x, y \in V$ (cf. (1b)).

Throughout this paper we shall even assume that the following stronger condition holds.

(1e) There are two constants $L \geq 1$ and $\alpha > 0$ such that

$$\Big| |\varphi'_i(x)| - |\varphi'_i(y)| \Big| \;\leq\; L \cdot \|\varphi'_i\| \cdot |y - x|^\alpha.$$

We note that for $d \geq 3$ the conditions (1d) and (1e) are always satisfied (with $\alpha = 1$), for $d = 2$ at least condition (1d) is always fulfilled. Finally, we set

$$\theta \;=\; \theta_S \;:=\; \inf_{t \in [0,\infty]} \left\{ t : \; \sum_{i \in I} \|\varphi'_i\|^t < \infty \right\}.$$

1.3 Hölder Families

A family $G := \{g^{(i)} : X \to \mathbb{R}\}_{i\in I}$ of continuous functions is called a *Hölder family of order* $\beta > 0$ if

$$V_\beta(G) \;:=\; \sup_{n\in\mathbb{N}} V_\beta^{(n)}(G) < \infty, \tag{2}$$

where $V_\beta^{(n)}(G) := \sup_{\omega\in I^n} \sup_{x,y\in X} \left\{\left|g^{(\omega_1)}(\varphi_{\sigma(\omega)}(x)) - g^{(\omega_1)}(\varphi_{\sigma(\omega)}(y))\right| \cdot \exp(\beta\cdot n)\right\}$.
If in addition we have that $\sum_{i\in I} \exp\left(\left\|g^{(i)}\right\|_X\right) < \infty$, then G is called a *summable Hölder family*. The associated *Perron-Frobenius-operator* $\mathcal{L}_G$ is given by

$$\mathcal{L}_G(\psi)(x) \;:=\; \sum_{i\in I} \exp(g^{(i)}(x)) \cdot \psi(\varphi_i(x))$$

for $\psi \in \mathcal{C}(X)$. Clearly, for a summable Hölder family G, $\mathcal{L}_G$ is well defined, preserves the Banach space $\mathcal{C}(X)$ and is continuous. Its norm is bounded from above by $\sum_{i\in I} \exp\left(\left\|g^{(i)}\right\|_X\right)$, where $\|g\|_X := \sup_{x\in X}\{|g(x)|\}$. In [9] the pressure of G was defined as

$$\begin{aligned} P(G) \;&:=\; \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \left\| \exp\left(\sum_{j=1}^{n} g^{(\omega_j)} \circ \varphi_{\sigma^j(\omega)}\right)\right\|_X \\ &=\; \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\omega|=n} \exp\left(\sup_{x\in X} \sum_{j=1}^{n} g^{(\omega_j)} \circ \varphi_{\sigma^j(\omega)}(x)\right). \end{aligned}$$

If we denote with $\mathcal{L}_G^*$ the dual of $\mathcal{L}_G$ then equation (2.2) and lemma 2.2 in [9] guarantee that there exists a Borel measure m_G supported on the limit set J such that

$$\mathcal{L}_G^*(m_G) \;=\; \exp(P(G)) \cdot m_G.$$

Applying theorem 2.4 from [9] one shows in the same way as in lemma A.6 in [5] that $\mathcal{L}_G : \mathcal{C}(X) \to \mathcal{C}(X)$ is almost periodic, i.e. for every $\psi \in \mathcal{C}(X)$ the family $\{\mathcal{L}_G^n(\psi) : X \to \mathbb{R}\}_{n\in\mathbb{N}}$ is equicontinuous. Using this fact one proves the analogue of theorem 4 in [6].

Theorem 1.1
There exists a (uniquely determined) continuous function $\varrho_G : X \to (0,\infty)$ such that $\mathcal{L}_G(\varrho_G) = \exp(P(G))\varrho_G$ and $\int \varrho_G \mathrm{d}m_G = 1$. Moreover, $\mathcal{C}(X)$ splits into

$$\mathcal{C}(X) \;=\; \mathbb{R}\varrho_G \oplus \mathcal{C}_G^0(X), \tag{3}$$

where $\mathcal{C}_G^0(X) := \left\{\psi \in \mathcal{C}(X) : \int \psi \mathrm{d}m_G = 0\right\}$. Obviously, this splitting is $\mathcal{L}_G$-invariant as the operator $\exp(-P(G))\cdot\mathcal{L}_G$ preserves integrals w.r.t. the measure m_G.
□

2 The First Reduction Step: Finite Time

In order to be able to perform numerical calculations, we have to reduce the setting from infinitely many calculations to finitely many. In this paragraph (cf. (4) below) we shall show that one can reduce the number of iterates to finitely many in order to obtain estimates for the Hausdorff dimension of the limit set.

We have been assuming that our system is contracting, more precisely, we have that

$$\beta_n \;:=\; \max_{\substack{|\omega|=n \\ x\in X}} |\varphi_\omega'(x)| \;\leq\; s^n.$$

We define α_ω and β_ω as the extremal values of the derivatives of φ_ω

$$\alpha_\omega \;:=\; \min_{x\in X} |\varphi_\omega'(x)|$$

and

$$\beta_\omega \;:=\; \max_{x\in X} |\varphi_\omega'(x)|.$$

We should note here, that it is particularly easy to determine β_n for maps which fulfill some monotonicity condition such that the chain rule also holds for extrema of derivatives (cf. [5]), thus the number of steps does not increase the order N^n. However, this estimate does not apply to the general case. Using the above notation we set

$$\Psi(n,s) \;:=\; \sum_{|\omega|=n} \alpha_\omega^s$$

and

$$\Phi(n,t) \;:=\; \sum_{|\omega|=n} \beta_\omega^t.$$

Recall (see [5]) that the Hausdorff dimension h of the limit set of the IFS is given by the uniquely determined value h such that, for all $n \in \mathbb{N}$ we have that

$$1 \;\leq\; \Phi(n,h) \;\leq\; K^d. \tag{4}$$

We know that for any $t > \theta$ we have that $\Phi(1,t) < \infty$.

2.1 The full set of generators

We define t_n as the solution of $\Phi(n,t) = 1$ and analogously s_n as the solution of $\Psi(n,s) = 1$.

Lemma 2.1
For the values s_n and t_n we have that

$$s_n \;\leq\; h \;\leq\; t_n.$$

Proof: The left hand side follows from relation (4) and the fact that $\Phi(n,t)$ is decreasing in t. For the right hand side we note that

$$\Psi(n+k,s_n) \;\geq\; \Psi(n,s_n)\cdot\Psi(k,s_n)$$

which implies that

$$\Psi(kn,s_n) \;\geq\; \Psi(n,s_n)^k \;=\; 1,$$

hence

$$\Phi(kn,s_n) \;\geq\; K^{-s_n}.$$

It follows that

$$P(s_n) \;=\; \lim_{k\to\infty}\frac{1}{kn}\log\Phi(kn,s_n) \;\geq\; 0,$$

which implies that

$$s_n \;\leq\; h.$$

□

Proposition 2.2
We have that

$$t_n - s_n \;\leq\; \frac{\Phi(n,s_n)-\Phi(n,t_n)}{-\log(\beta_n)}. \tag{5}$$

Proof: We have

$$\begin{aligned}
\Phi(n,s_n)-\Phi(n,t_n) &= -\int_{s_n}^{t_n}\frac{\partial}{\partial\tau}\Phi(n,\tau)\mathrm{d}\tau \\
&= \int_{s_n}^{t_n}\sum_{|\omega|=n}\left(-\log(\beta_\omega)\cdot\beta_\omega^\tau\right)\mathrm{d}\tau \\
&\geq \int_{s_n}^{t_n}\min_{|\omega|=n}-\log(\beta_\omega)\cdot\Phi(n,\tau)\mathrm{d}\tau \\
&\geq \int_{s_n}^{t_n}-\log(\beta_n)\cdot 1\mathrm{d}\tau \\
&= -\log(\beta_n)\cdot(t_n-s_n),
\end{aligned}$$

which implies the assertion of the proposition. □

Combining the preceding results we obtain the following fact.

Corollary 2.3
The sequences s_n and t_n converge to the Hausdorff dimension h of the limit set, more precisely,

$$t_n - h \;\leq\; \frac{K^{t_n}-1}{-n\log(s)}$$

and

$$h - s_n \;\le\; \frac{K^{s_n}-1}{-n\log(s)} \;\le\; \frac{K^d-1}{-n\log(s)}, \tag{6}$$

where s is the contraction constant defined in paragraph **1.1**. □

Thus, we are able to calculate estimates for the Hausdorff dimension of the limit set with prescribed accuracy by performing iterations with finite iteration time.

Remark 2.4
We should note here that for particular finite systems (see section **4**) there are algorithms which yield much faster convergence than (6) to the actual dimension (cf. e.g. [3] for continued fractions with bounded entries). However, we were mainly interested in proving a statement valid for general IFS without any additional knowledge about the distribution of periodic points.

3 The Second Reduction Step: Finitely Many Generators

Evidently, condition (1e) implies that for every $t \ge 0$ the family

$$\zeta_t \;:=\; \{t \cdot \log|\varphi_i'|\}_{i\in I}$$

is Hölder in the sense of (2) and for $t > \theta$ it is even summable. For simplicity we shall write $\mathcal{L}_t$, $P(t)$, m_t, ϱ_t, and $C_t^0(X)$, resp. instead of $\mathcal{L}_{\zeta_t}$, $P(\zeta_t)$, m_{ζ_t}, ϱ_{ζ_t}, and $C^0_{\zeta_t}(X)$, resp.

In the previous paragraph we showed that it is possible to reduce the setting to finitely many iterations if one wants to obtain estimates for the Hausdorff dimension of the limit set J. Still, if the system has infinitely many generators, there are infinitely many calculation steps to perform. Thus, we shall investigate if it suffices to consider suitably chosen finite subsets $F \subset I$. We denote the terms obtained from these *reduced sets of generators* by $\mathcal{L}_{Ft}$, $P_F(t)$, m_{Ft}, ϱ_{Ft}, and $C^0_{Ft}(X)$. The distortion property (1d) immediately implies that for all $F \subset I$, $\omega \in I^*$ and all $x, y \in X$

$$\frac{\exp(P_F(t))\cdot\varphi'(y)}{\exp(P_F(t))\cdot\varphi'(x)} \;\ge\; K^t.$$

In the sequel we shall use the abbreviations $\lambda_t := \exp(P(t))$, $\lambda_{Ft} := \exp(P_F(t))$, resp. With theorems 7 from [6] and 2.4 from [8] we deduce the following proposition.

Proposition 3.1
For $\theta_S < t \le d$ and any $F \subset I$ we have that

$$K^{-d} \;\le\; \varrho_F(t) \;\le\; K^d.$$

□

We are now in a position to prove the following theorem.

Theorem 3.2
For $F \subset I$ and $t \in (\theta_S, d]$ we have that

$$|\lambda_t - \lambda_{Ft}| \ \leq \ K^{3d}\left(2 + K^d\right) \cdot \|\mathcal{L}_t - \mathcal{L}_{Ft}\|$$

Proof: Consider a function ψ of form

$$\psi \ = \ r \cdot \varrho_t + u$$

where $r \in \mathbb{R}$ and $u \in C_t^0(X)$ and assume that $\|\psi\| = 1$. This implies that

$$1 \ = \ \|\psi\| \ \geq \ \left|\int \psi \mathrm{d}m_t\right| \ = \ |r| \cdot \int \varrho_t \mathrm{d}m_t \ = \ |r|.$$

Thus, using proposition **3.1**, we conclude that

$$\|u\| \ \leq \ \|\psi\| + |r| \cdot \|\varrho_t| \ \leq \ 1 + K^d.$$

Hence we have that

$$\|r\varrho_t + u\| \ = \ 1 \ = \ \left(1 + K^d + 1\right)/\left(2 + K^d\right) \ \geq \ \left(\|u\| + |r|\right)/\left(2 + K^d\right).$$

Since (7) is linear in the components of ψ, this relation holds for arbitrary ψ. According to (3) the function ϱ_{Ft} has a unique representation

$$\varrho_{Ft} \ = \ r \cdot \varrho_t + u$$

with $r \in \mathbb{R}$ and $u \in \mathcal{C}_t^0(X)$. Again, using proposition **3.1**, we see that

$$K^{-d} \ \leq \ \int \varrho_{Ft} \mathrm{d}m_t \ = \ \int (r\varrho_t + u)\mathrm{d}m_t \ = \ r\int \varrho_d \mathrm{d}m_t = r. \tag{7}$$

Now application of proposition **3.1** and theorem **1.1** together with $\mathcal{L}\_$-invariance of $\mathcal{C}_t^0(X)$, and relations (7) and (7) give that

$$\begin{aligned}
K^d \cdot \|\mathcal{L}_t - \mathcal{L}_{Ft}\| \ &\geq \ \|(\mathcal{L}_t - \mathcal{L}_{Ft})\,\varrho_{Ft}\| \\
&= \ \|r\mathcal{L}_t(\varrho_t) + \mathcal{L}_t(u) - \lambda_{Ft}\varrho_{Ft}\| \\
&= \ \|r(\lambda_t - \lambda_{Ft})\varrho_t + \mathcal{L}_t(u) - \lambda_{Ft}u\| \\
&\geq \ \left(r|\lambda_t - \lambda_{Ft}| \cdot \|\varrho_t\| + \|\mathcal{L}_t(u) - \lambda_{Ft}u\|\right)/\left(2 + K^d\right) \\
&\geq \ \left(K^{-d}|\lambda_t - \lambda_{Ft}| \cdot \|\varrho_t\|\right)/\left(2 + K^d\right) \\
&\geq \ \left(K^{-d}|\lambda_t - \lambda_{Ft}| \cdot K^{-d}\right)/\left(2 + K^d\right)
\end{aligned}$$

□

We call a system S *strongly regular* if there is $t \geq 0$ such that we have that $0 < P(t) < \infty$. For example, each finite system is strongly regular. For a strongly regular system there exists a unique $h = h_S$ such that $P(h) = 0$ and (see e.g. [5]) equals the Hausdorff dimension of the limit set J. We define

$$\chi := \sup_{n\geq 1} \inf_{\omega\in I^n} \{-\log(\beta_\omega)/n\} = -\inf_{n\geq 1} \{\log(\beta_n)/n\}.$$

Theorem 3.3
Let $\gamma > \theta_S$ and for some finite $F \subset I$ assume that $h_F \geq \gamma$. Then we have that

$$0 \leq h - h_F \leq K^{3d}(2 + K^d) \sum_{i\in I\setminus F} \beta_i^\gamma \Big/ \chi. \tag{8}$$

Proof: Application of proposition 6.5 from [2] (cf. proposition 2.6.13 from [4]) together with Birkhoff's ergodic theorem applied to the functions $tf(\omega) := \log|\varphi_{\omega_1}(\pi(\sigma(\omega)))|$ for $\omega \in I^\infty$ yields that

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = \int \log|\varphi_{\omega_1}(\pi(\sigma(\omega)))|\mathrm{d}\tilde{\mu}_t \leq -\chi,$$

where $\tilde{\mu}_t$ denotes the lift of $\varrho_t m_t$ to the coding space I^∞ (cf. [2],[4] or [8]). Thus, on the hand hand we have that

$$\begin{aligned}
\exp(P(h_F)) - 1 &= \exp(P(h_F)) - \exp(P(h)) \\
&= \int_h^{h_F} \frac{\mathrm{d}}{\mathrm{d}t} \exp(P(t))\mathrm{d}t \\
&= \int_h^{h_F} P'(t)\exp(P(t))\mathrm{d}t \\
&= -\int_h^{h_F} -P'(t)\exp(P(t))\mathrm{d}t \\
&\geq \int_h^{h_F} \chi\mathrm{d}t \\
&= \chi\,(h - h_F).
\end{aligned} \tag{9}$$

On the other hand, theorem **3.2** implies that

$$\begin{aligned}
\exp(P(h_F)) - 1 &= \exp(P(h_F)) - \exp(P(h)) \\
&\leq K^{3d}(2 + K^d) \cdot \|\mathcal{L}_{h_F} - \mathcal{L}_{Fh_F}\| \\
&= K^{3d}(2 + K^d) \cdot \sum_{i\in I\setminus F} \beta_i^{h_F} \\
&\leq K^{3d}(2 + K^d) \cdot \sum_{i\in I\setminus F} \beta_i^{\gamma}.
\end{aligned} \tag{10}$$

Comparison of (9) and (10) finishes the proof. □

Let us note that the s_n are of course canonical values for γ.

4 Examples

Combining the results from the two preceding paragraphs, in order to calculate the Hausdorff dimension of an 'actual' limit set with precision $\varepsilon > 0$, we can proceed as follows. Given ε we choose a reduced set of generators $F \subset I$ such that (10) is strictly smaller than ε, then using corollary **2.3** we can find a finite iteration depth n such the difference between s_{Fn} for the reduced system and the Hausdorff dimension h of the original system is smaller than ε, more precisely, $0 \leq h - h_F + h_F - s_{Fn} \leq \varepsilon$. Let us note that one might want to apply more sophisticated (and faster) algorithms (cf. e.g. [3]) than the one described in paragraph **2** in order to calculate the Hausdorff dimension of the reduced system. In this section we show that it is possible to apply this procedure to two classes of examples.

4.1 Continued fractions with even entries

We consider the IFS on the unit interval $[0,1]$ generated by $\varphi_i(x) : x \mapsto 1/(2i+x)$ for $i \in \mathbb{N}$. Clearly we have that $K = 9/4$. The α_n are given by

$$\alpha_n \;=\; 4 \Big/ \left((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right)^2 ,$$

the β_n are obtained as

$$\beta_n \;=\; 8 \Big/ \left((1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}\right)^2 .$$

Thus we have that

$$\chi \;=\; 2\log(1+\sqrt{2}).$$

Moreover, we have that $\theta = 1/2$, and we calculate for $\gamma > \theta$ and $N \in \mathbb{N}$ that

$$\begin{aligned}
\sum_{i>N} \beta_i^\gamma &= \sum_{n>N} 1/(2n)^{2\gamma} \\
&= \frac{1}{4^\gamma} \sum_{n>N} 1/n^{2\gamma} \\
&= \frac{\zeta(2\gamma) - \sum_{n=1}^{N} 1/n^{2\gamma}}{4^\gamma} \\
&\leq \frac{1}{4^\gamma} \int_N^\infty \frac{\mathrm{d}k}{k^{2\gamma}} \\
&= \frac{N^{1-2\gamma}}{4^\gamma(2\gamma-1)}.
\end{aligned}$$

First let us note that an s_{Fn} is always a *lower estimate* for h. Thus, it remains to apply (8) in order to calculate estimates from above (for some numerical results, cf. table **1** on page 12). We conclude that the Hausdorff dimension of the attractor is between .7084 . . . and .7458 . . .

4.2 Complex continued fractions

We consider the IFS on the ball $\{z \in \mathbb{C} : |z - 1/2| \leq 1/2\}$ induced by the maps $\varphi_{m,n} : z \mapsto \frac{1}{m+n\mathbf{i}+z}$ for $m \in \mathbb{N}^*$ and $n \in \mathbb{Z}$. For some information about the geometry of the limit set of this system cf. [5]. Here we have that $K = 4$ and $\theta = 1$. Moreover, we calculate the following estimate.

$$\begin{aligned}\sum_{\|(m,n)\|>N} \frac{1}{|m+n\mathbf{i}+z|} &\leq \int_{R\geq N-1} \int_{\varphi\in[-\pi/2,\pi/2]} R^{1-2\gamma} \mathrm{d}\varphi \mathrm{d}R \\ &= \frac{\pi(N-1)^{2-2\gamma}}{2\gamma-2}.\end{aligned}$$

Combining the upper estimate $h < 1.9$ from Theorem 6.6 in [5] with the numerical results from table **2** on page 12 we conclude that the Hausdorff dimension of the attractor is between $1.7267\ldots$ and 1.9.

n	N	s_{Fn}	t_{Fn}	resulting interval for h
1	10	0.5753...	0.6242...	[0.5753...,1.0000...]
1	100	0.6712...	0.7069...	[0.6712...,1.0000...]
1	1000	0.6939...	0.7249...	[0.6939...,1.0000...]
1	10000	0.7009...	0.7300...	[0.7009...,1.0000...]
1	100000	0.7033...	0.7316...	[0.7033...,0.9671...]
1	1000000	0.7042...	0.7322...	[0.7042...,0.8217...]
1	10000000	0.7046...	0.7324...	[0.7046...,0.7669...]
1	100000000	0.7047...	0.7324...	[0.7047...,0.7458...]
2	1000	0.7028...	0.7164...	[0.7028...,1.0000...]
2	2000	0.7055...	0.7189...	[0.7055...,1.0000...]
2	3000	0.7068...	0.7199...	[0.7068...,1.0000...]
2	4000	0.7075...	0.7206...	[0.7075...,1.0000...]
2	5000	0.7080...	0.7210...	[0.7080...,1.0000...]
2	6000	0.7084...	0.7213...	[0.7084...,1.0000...]

Table 1: Numerical results for even continued fractions

n	N	s_{Fn}	t_{Fn}
2	10	1.6752...	>2
2	20	1.7013...	>2
2	30	1.7082...	>2
2	40	1.7112...	>2
2	50	1.7128...	>2
3	1	0.9061...	1.2350...
3	2	1.3996...	1.7435...
3	3	1.5499...	1.8728...
3	4	1.6171...	1.9239...
3	5	1.6537...	1.9498...
3	6	1.6762...	1.9647...
3	7	1.6912...	1.9740...
3	8	1.7017...	1.9803...
3	9	1.7095...	1.9847...
3	10	1.7154...	1.9880...
3	11	1.7200...	1.9904...
3	12	1.7237...	1.9923...
3	13	1.7267...	1.9938...

Table 2: Numerical results for complex continued fractions

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